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On the extended spectrum of some quasinormal operators

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Abstract: In this paper we consider some extended eigenvalue problems for some quasinormal operators. The spectrum of an algebra homomorphism defined by a compact normal operator is also investigated.

Key words: Quasinormal operators, extended eigenvalue, extended spectrum

1. Introduction

Let H be an infinite separable complex Hilbert space and denote by $L(H)$ the set of bounded linear operators on H. A complex number λ is said to be an extended eigenvalue of a bounded operator A if there exists a nonzero operator T such that

$$TA = \lambda AT.$$

T is called a λ eigenoperator for A and the set of extended eigenvalues is represented by $\sigma_{ext}(A)$. This condition takes place in quantum mechanics and analysis for their spectra [6]. Moreover, there is a nonzero operator Y such that

$$XA = AY \tag{1.1}$$

and ε_A is the set of all X satisfying (1.1), and then it is easily seen that ε_A is an algebra. When A has dense range, one can define the map $\Phi_A : \varepsilon_A \to L(H)$ by $\Phi_A(X) = Y$ and verify that Φ_A is an algebra homomorphism. This homomorphism is a closed (generally unbounded) linear transformation. Biswas et al. defined an eigenvalue of Φ_A as an extended eigenvalue of A and proved that the set of extended eigenvalues of the Volterra operator V is equal to the interval $(0, +\infty)$ in [2]. Karaev gave the set of extended eigenvectors of the Volterra operator V on $L^2[0,1]$ in [11]. However, the problem is open as to the other spectrum parts of Φ_V. Furthermore, Biswas and Petrovic derived the following result as

$$\sigma_{ext}(A) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset\}$$

by using the Rosenblum theorem [3] where $\sigma(A)$ is the set of spectrum of A.

An operator A is called quasinormal if A and A^*A are commutative. The purpose of this paper is to exploit a few facts about the extended eigenvalues for a quasinormal operator. Also, if A is a compact normal operator and has dense range, then the spectrum of Φ_A has been given. Note that Cassier and Alkanjo described the extended spectrum and extended eigenspace for any pure quasinormal operator [5].

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Throughout this work $\sigma_p(A)$, $\sigma_c(A)$, and $\sigma_{ess}(A)$ are denoted as the point spectrum, the continuous spectrum, and the essential spectrum of A, respectively.

2. Extended eigenvalues for some quasinormal operators

Lemma 2.1 *Let $A \in L(H)$ be a quasinormal operator such that $0 \in \sigma_p(A)$; then $\sigma_{ext}(A) = \mathbb{C}$.*

Proof Let $A = U|A|$, where U is a partial isometry and $|A|$ is the square root of A^*A such that $KerU = Ker|A|$, be the polar decomposition of A . Since A is a quasinormal operator, $U|A| = |A|U$ is true [9]. Because $0 \in \sigma_p(A)$, there exists a nonzero element x_0 in H such that $Ax_0 = 0$ and for every $x \in H$

$$(x_0 \otimes x_0) U|A|x = (U|A|x, x_0)x_0 = (x, x_0)U|A|x_0 = U|A| (x_0 \otimes x_0) x = 0$$

is obtained. This means that $\sigma_{ext}(A) = \mathbb{C}$.

□

Theorem 2.2 *If $A : H \to H$ is a quasinormal operator but not a normal operator and $0 \notin \sigma_p(A)$, then*

$$\left\{ \frac{\lambda_i}{\lambda_j} \in \mathbb{C} : \ \lambda_i, \lambda_j \in \sigma_p(A) \right\} \cup \{0\} \subset \sigma_{ext}(A).$$

Proof Because A is a quasinormal and not a normal operator, the equality $AA^*A = A^*AA$ is correct. Hence,

$$(AA^* - A^*A) A = 0 = 0A (AA^* - A^*A),$$

i.e. $0 \in \sigma_{ext}(A)$. On the other hand, if a complex number λ is in $\sigma_p(A)$, then $\overline{\lambda} \in \sigma_p(A^*)$. Therefore, for $\lambda_i, \lambda_j \in \sigma_p(A)$ such that $Ax_j = \lambda_j x_j$ and $A^*x_i = \overline{\lambda_i} x_i$,

$$(x_j \otimes x_i) A = \frac{\lambda_i}{\lambda_j} A (x_j \otimes x_i)$$

is provided. □

Theorem 2.3 *Letting $A \in L(H)$ be a self-adjoint operator and the essential spectrum $\sigma_{ess}(A) = \emptyset$, then $\sigma_{ext}(A) = \{\lambda \in \mathbb{C} : \sigma_p(A) \cap \sigma_p(\lambda A) \neq \emptyset\}$.*

Proof If A is a self-adjoint operator on H, then $\sigma_{ess}(A)$ consists precisely of all points in $\sigma(A)$ except the isolated eigenvalues of finite multiplicity [7]. Since $\sigma_{ess}(A) = \emptyset$, the spectral problem for self-adjoint operators shows that

$$A = \sum_{n=1}^{\infty} \lambda_n P_n$$

with mutually orthogonal finite rank projection P_n, $n \in \mathbb{N}$ [12]. This fact and the proof of the previous theorem give the relation $\sigma_{ext}(A) = \{\lambda \in \mathbb{C} : \sigma_p(A) \cap \sigma_p(\lambda A) \neq \emptyset\}$. □

The following result is obtained from the spectrum structure of a compact normal operator[10]:

Corollary 2.4 *Letting* $A \in L(H)$ *be a compact normal operator, then*

$$\sigma_{ext}(A) = \{\lambda \in \mathbb{C} : \sigma_p(A) \cap \sigma_p(\lambda A) \neq \emptyset\}.$$

Theorem 2.5 *Assume that* $A : H \to H$ *is a compact normal operator and* $0 \in \sigma_c(A)$*. For the algebraic homomorphism* $\Phi_A : \varepsilon_A \to L(H)$*,*

$$\sigma(\Phi_A) = \overline{\sigma_p(\Phi_A)}.$$

Proof Since A is a completely continuous normal operator with dense range, the spectral decomposition theorem implies that

$$A = \sum_{i \geq 1} \lambda_i x_i \otimes x_i,\ \lambda_i \to 0,\ i \to +\infty,$$

where the set $\{x_1, x_2, x_3, \dots\}$ is an orthonormal basis of H and $\{\lambda_n\} \subset \mathbb{C}$[1]. It is well known that $\sigma(\Phi_A)$ is a closed set. Now we consider that $\lambda \in \mathbb{C} \setminus \overline{\sigma_p(\Phi_A)}$ and $Y : H \to H$ is any bounded linear operator on H. An operator $X : H \to H$ defined by

$$X = \sum_{n=1}^{+\infty} A(\lambda_n - \lambda A)^{-1}(Yx_n \otimes x_n)$$

is bounded since for all $n \in \mathbb{N}$

$$\left\| A(\lambda_n - \lambda A)^{-1} \right\| \leqslant \sup\left\{ \frac{\lambda_m}{\lambda_n - \lambda\lambda_m} :\ \lambda_n, \lambda_m \in \sigma_p(A) \right\} < +\infty.$$

Moreover, $\Phi_A(X) = \sum\limits_{n=1}^{+\infty} \lambda_n(\lambda_n - \lambda A)^{-1}(Yx_n \otimes x_n)$ and

$$(\Phi_A - \lambda) X = Y$$

and it means that $\Phi_A - \lambda$ is surjective. From the last result and Corollary 2.4, λ is in the resolvent set of Φ_A. □

Corollary 2.6 *If* $A : H \to H$ *is a compact operator with* $0 \in \sigma_c(A)$*, then* $0 \in \sigma(\Phi_A)$*.*

Proof Because $A : H \to H$ has dense range, it is obvious that $0 \notin \sigma_p(\Phi_A)$. Besides, there exist two orthonormal sequences $\{x_n\}$ and $\{y_n\}$ in H and scalars $\{\lambda_n\}$ such that $\lambda_n \to 0$ and A can be represented as follows:

$$A = \sum_{n=1}^{+\infty} \lambda_n x_n \otimes y_n.$$

In addition, it can be chosen as two subsequences $\{\lambda_{i(n)}\}, \{\lambda_{j(n)}\} \subset \{\lambda_n\}$ satisfying

$$\lim_{n \to +\infty} \frac{\lambda_{i(n)}}{\lambda_{j(n)}} = 0,$$

and a linear bounded operator $Y = \sum\limits_{n=1}^{+\infty} y_{j(n)} \otimes y_{i(n)}$ on H. If Φ_A is surjective, then for the operator Y there is a linear bounded operator $X : H \to H$ in ε_A and $\Phi_A(X) = Y$. However, for all $n \in \mathbb{N}$,

$$Xx_{i(n)} = \frac{\lambda_{j(n)}}{\lambda_{i(n)}} x_{j(n)},$$

which means that X is not a bounded operator on H, so Φ_A is not surjective. We have $0 \in \sigma(\Phi_A)$ and the theorem is proved. □

Theorem 2.7 *Let $A \in L(H)$ be a quasinormal operator but not normal; then*

$$\overline{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\} \subset \sigma_{ext}(A).$$

Proof In this case, $A : H \to H$ can be written as $A = A_n \oplus A_p$ where A_n is a normal part and A_p is a pure quasinormal part. Therefore, the assertion of the theorem can be directly derived from Corollary 2.6 of [5]. □

Lemma 2.8 *Let A be a bounded operator on any Hilbert space H and S be a unilateral shift operator on $H^{(\infty)} = H \oplus H \oplus \dots$. If $T = [T_{ij}]_{i,j=1}^{\infty}$, $T_{ij} : H \to H$ and $T(S \otimes A) = \lambda (S \otimes A) T$, then*

$$i)\, T_{ij} = 0 \quad for \quad j > i \quad and$$

$$ii)\, T_{ij} A = \lambda A T_{i-1,j-1} \quad for \;\; i \geqslant j.$$

Conversely, if $T = [T_{ij}]_{i,j=1}^{\infty}$ is a bounded operator on $H^{(\infty)}$ satisfying two conditions, T is an eigenoperator of $S \otimes A$.

It is easily seen that A and B are bounded operators and unitary equivalent, and then A and B have the same extended eigenvalues, i.e. $\sigma_{ext}(A) = \sigma_{ext}(B)$.

Theorem 2.9 *Letting $A \in L(H)$ be a pure quasinormal operator, then*

$$\sigma_{ext}(|A|) \subset \sigma_{ext}(A).$$

Proof Let $A = U|A|$ be the polar decomposition of the pure quasinormal operator A. Because A is pure quasinormal, U is an isometry. Also, the equality

$$H = KerU^* \oplus U(KerU^*) \oplus U^2(KerU^*) \oplus \dots$$

is verified and subspaces $U^n(KerU^*)$, $n \in \mathbb{N}$ are invariant under $|A|$[4, 8]. We claim that there exist eigenoperators for all extended eigenvalues of $|A|$ such that they are nonzero on $KerU^*$ and $KerU^*$ invariant under eigenoperators. Supposing that λ is any extended eigenvalue of $|A|$, then there exists a nonzero operator such that

$$T|A| = \lambda |A| T.$$

Moreover, where P_i are projection operators on $U^i(kerU^*)$ for all $i \in \mathbb{N}$, there are two projection operators P_n and P_m such that the operator $P_n T P_m$ is nonzero. We define $X = (U^*)^n P_n T P_m U^m$. This operator is nonzero on $KerU^*$ and $KerU^*$ invariant under X and since $|A|$ and U are commutative, then the equality is

$$X|A| = \lambda |A| X.$$

According to [4], A is unitary equivalent $B : (KerU^*)^{(\infty)} \to (KerU^*)^{(\infty)}$

$$B := \begin{bmatrix} 0 & 0 & 0 & \cdots \\ |A| & 0 & 0 & \cdots \\ 0 & |A| & 0 & \cdots \\ \cdot & \cdot & |A| & \cdots \end{bmatrix}.$$

From Lemma 2.8, $\sigma_{ext}(|A||_{KerU^*}) \subset \sigma_{ext}(A)$ and the operator

$$W := \begin{bmatrix} X & 0 & 0 & \cdots \\ 0 & X & 0 & \cdots \\ \cdot & \cdot & X & \cdots \end{bmatrix}$$

is nonzero. Also, $WB = \lambda BW$ holds. The last result completes the proof of the theorem. □

Corollary 2.10 *If A is a pure quasinormal operator, then*

$$\{\lambda\mu : \lambda \in \sigma_{ext}(|A|), |\mu| \leqslant 1\} \subset \sigma_{ext}(A).$$

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