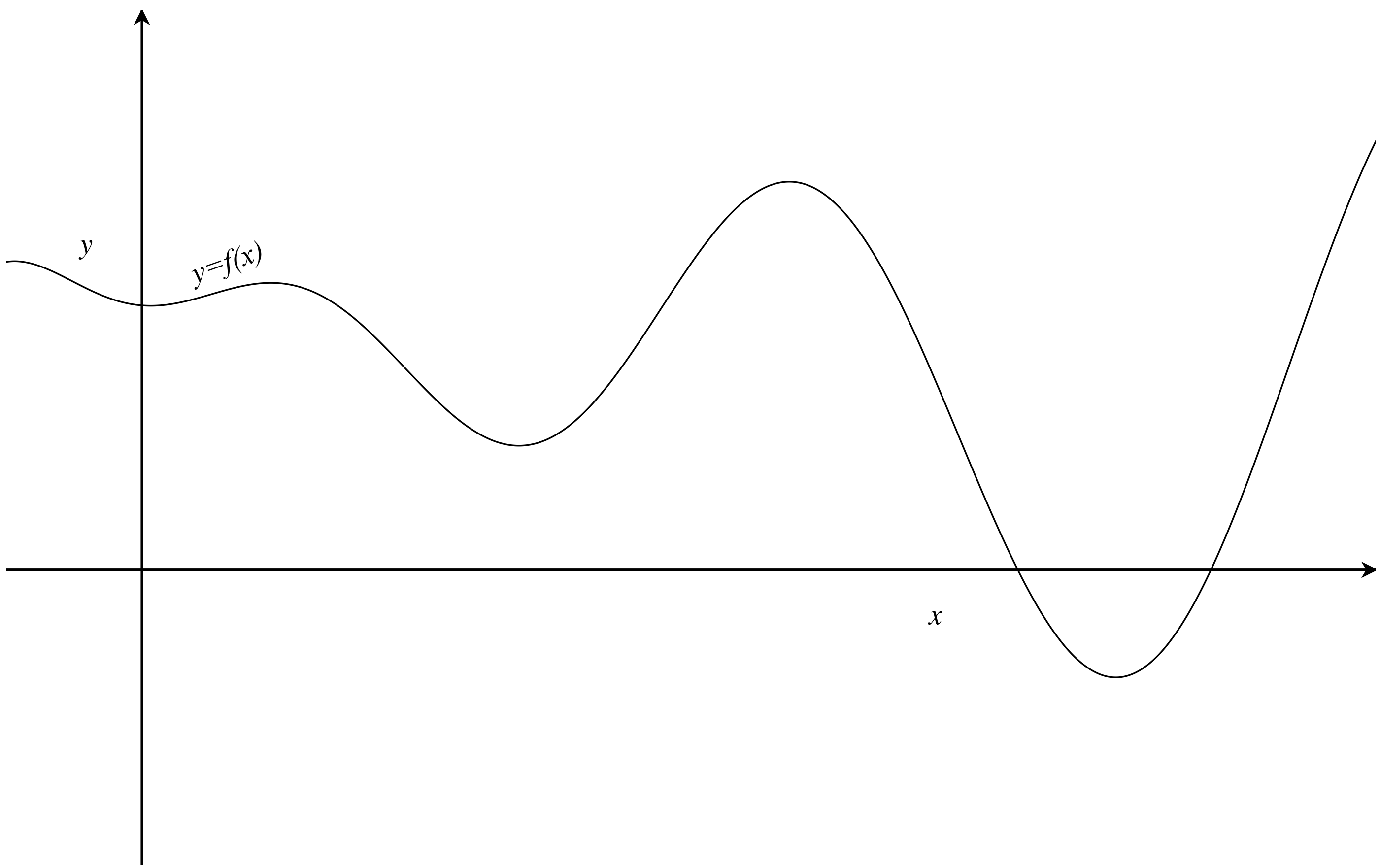
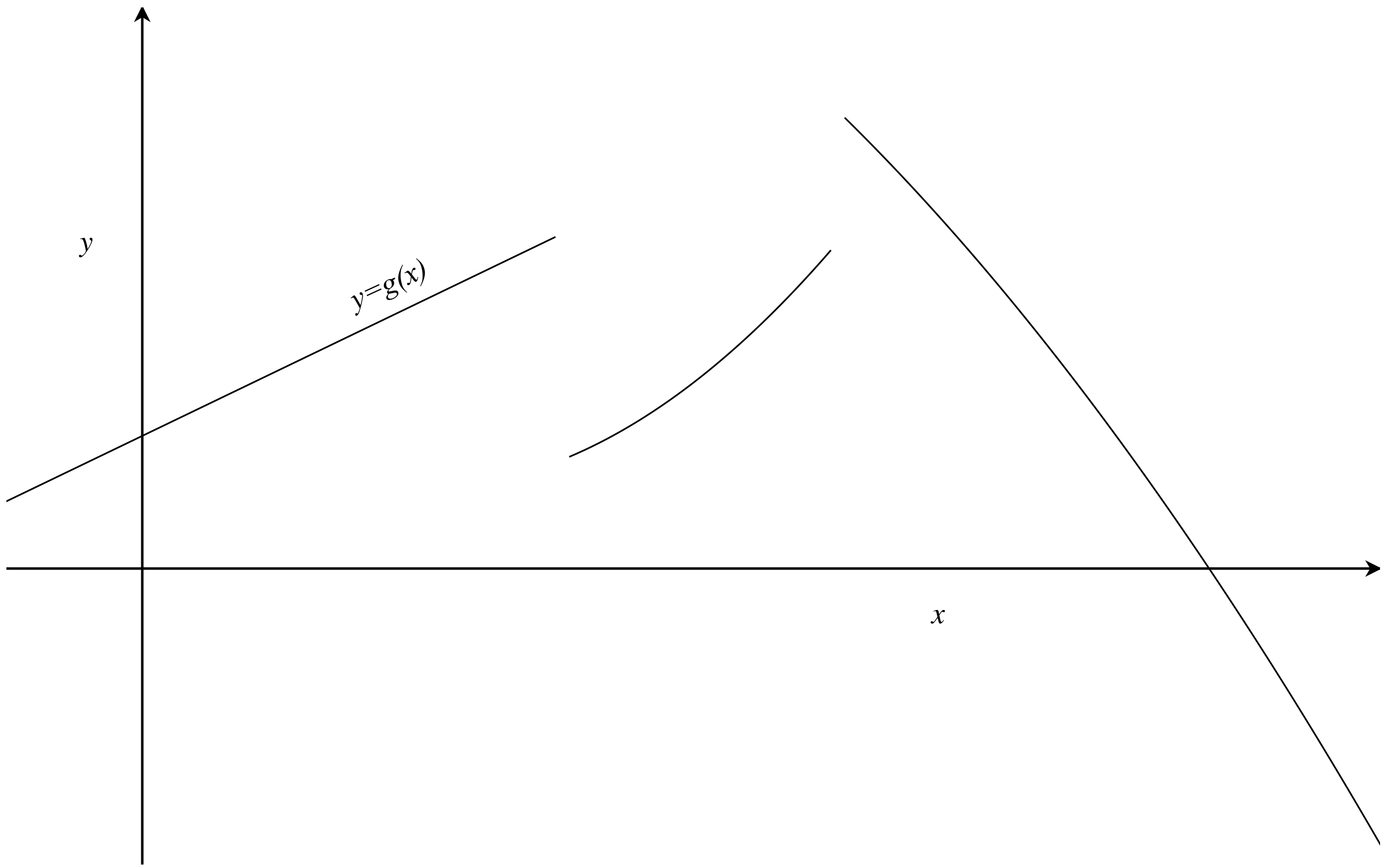
Continuity

Intuition: *The function $f(x)$ is continuous if the graph $y = f(x)$ is an unbroken (i.e., continuous) curve.*

The graph of a continuous function:



The graph of a function that is *not* continuous (at some points):



Observation: The intuition about continuity (unbroken graph) is useful for the visual guidance it provides, but it is not useful in determining which functions are in fact continuous, or where a given function might fail to be continuous.

For that, we need a more precise definition...

Definitions:

A. *The function $f(x)$ is* ***continuous at the point $x = x_0$*** *if*

(i) $f(x)$ is defined at $x = x_0$, and

(ii) $\lim_{x \to x_0} f(x) = f(x_0)$.

B. *The function $f(x)$ is* ***continuous in the interval I=(a,b)*** *if $f(x)$ is continuous at every point x_0 in I.*

Basic continuous functions:

1. Constant functions are continuous for all real x

Because if $f(x) = C$ for all x and x_0 is any real number, then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} C = C = f(x_0).$$

2. If k is a positive integer, then $f(x) = x^k$ is continuous for all real x

Because if x_0 is any real number, then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x^k = x_0^k = f(x_0).$$

3. If α is any real number, then $f(x) = x^\alpha$ is continuous for all $x > 0$

Because if x_0 is any real number, then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x^\alpha = x_0^\alpha = f(x_0).$$

4. The function $f(x) = e^x$ is continuous for all x.

I.e., if x_0 is any real number, then

$$\lim_{x \to x_0} e^x = e^{x_0}.$$

5. The function $f(x) = \ln x$ is continuous for all $x > 0$.

I.e., if x_0 is any *positive* real number, then

$$\lim_{x \to x_0} \ln x = \ln x_0.$$

Combinations:

If $f(x)$ and $g(x)$ are both continuous, then

6. $f(x) + g(x)$ and $f(x) - g(x)$ are both continuous,

7. $f(x)g(x)$ is continuous and

8. $f(x)/g(x)$ is continuous at every point where $g(x) \neq 0$.

Composite functions:

9. If $g(x)$ is continuous in the interval I and $f(x)$ is continuous in

$$g(I) = \{g(x) : x \in I\}$$

then $f(g(x))$ is continuous in I.

Examples:

(a) If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial, then $f(x)$ is continuous for all x because

(i) x^k is continuous for all x if k is a positive integer;

(ii) a_k is continuous for each k, since it is constant;

(iii) $a_k x^k$ is continuous, as a product of continuous functions;

(iv) so $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous as a sum of continuous functions.

(b) If $Q(x) = \dfrac{f(x)}{g(x)}$ is a rational function (i.e., $f(x)$ and $g(x)$ are both polynomials), then $Q(x)$ is continuous at all points where $g(x) \neq 0$.

(c) The function $f(x) = \sqrt{x}$ is continuous for all $x > 0$ and $g(x) = x^2+2$ is continuous for all x. Furthermore, $g(x) \geq 2 > 0$ for all x, so $f(g(x)) = \sqrt{x^2+2}$ is continuous for all x.

(d) The function $|x|$ is continuous for all x. Because...

$$|x| = \begin{cases} x & : \quad x \geq 0 \\ -x & : \quad x < 0 \end{cases}$$

so $|x|$ is continuous for $x > 0$ and for $x < 0$ and it just remains to show that $|x|$ is continuous at $x = 0$. Furthermore,

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0 \quad \text{and} \quad \lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0,$$

so $\lim\limits_{x \to 0} |x| = 0 = |0|$ (since both one-sided limits exist and are equal), hence $|x|$ is also continuous at $x = 0$.

(e) The function $\ln x$ is continuous for all $x > 0$, $|x|$ is continuous for all x and $|x| > 0$ for all $x \neq 0$, so the function $\ln|x|$ is continuous for all $x \neq 0$.

(f) **Question:** on what interval(s) is the function

$$h(x) = \sqrt{x^2 - x - 2}$$

continuous?

Answer: The function $f(x) = \sqrt{x}$ is continuous for all $x > 0$, and the function $g(x) = x^2 - x - 2$ is continuous for all x, so $h(x) = f(g(x))$ will be continuous at all points where $g(x) > 0$. Now $g(x) = x^2 - x - 2 = (x+1)(x-2)$, so $g(x) > 0$ if both factors are positive or if both factors are negative. Both factors are positive when $x > 2$ and both factors are negative when $x < -1$, so $h(x)$ is continuous on the intervals $(-\infty, -1)$ and $(2, \infty)$.