PERSPECTIVES IN LOGIC

Peter G. Hinman

RECURSION-THEORETIC HIERARCHIES

ASL

CAMBRIDGE

Recursion-Theoretic Hierarchies

Since their inception, the Perspectives in Logic and Lecture Notes in Logic series have published seminal works by leading logicians. Many of the original books in the series have been unavailable for years, but they are now in print once again.

The theory set out in this volume, the 9th publication in the Perspectives in Logic series, is the result of the meeting and common development of two currents of mathematical research: Descriptive Set Theory and Recursion Theory. Both are concerned with notions of definability and with the classification of mathematical objects according to their complexity. These are the common themes which run through the topics discussed here. The author develops a general theory from which the results of both areas can be derived, making these common threads clear.

Peter G. Hinman works in the Department of Mathematics at the University of Michigan, Ann Arbor.

PERSPECTIVES IN LOGIC

The *Perspectives in Logic* series publishes substantial, high-quality books whose central theme lies in any area or aspect of logic. Books that present new material not now available in book form are particularly welcome. The series ranges from introductory texts suitable for beginning graduate courses to specialized monographs at the frontiers of research. Each book offers an illuminating perspective for its intended audience.

The series has its origins in the old *Perspectives in Mathematical Logic* series edited by the Ω-Group for "Mathematische Logik" of the Heidelberger Akademie der Wissenschaften, whose beginnings date back to the 1960s. The Association for Symbolic Logic has assumed editorial responsibility for the series and changed its name to reflect its interest in books that span the full range of disciplines in which logic plays an important role.

For more information, see www.aslonline.org/books_perspectives.html

PERSPECTIVES IN LOGIC

Recursion-Theoretic Hierarchies

PETER G. HINMAN
University of Michigan

ASSOCIATION FOR SYMBOLIC LOGIC

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107168244
10.1017/9781316717110

Association for Symbolic Logic
Richard A. Shore, Publisher
Department of Mathematics, Cornell University, Ithaca, NY 14853
http://www.aslonline.org

A catalogue record for this publication is available from the British Library.

ISBN 978-1-107-16824-4 Hardback

For *M* and *m*

who make life fun

Preface to the Series

On Perspectives. *Mathematical logic arose from a concern with the nature and the limits of rational or mathematical thought, and from a desire to systematise the modes of its expression. The pioneering investigations were diverse and largely autonomous. As time passed, and more particularly in the last two decades, interconnections between different lines of research and links with other branches of mathematics proliferated. The subject is now both rich and varied. It is the aim of the series to provide, as it were, maps or guides to this complex terrain. We shall not aim at encyclopaedic coverage; nor do we wish to prescribe, like Euclid, a definitive version of the elements of the subject. We are not committed to any particular philosophical programme. Nevertheless we have tried by critical discussion to ensure that each book represents a coherent line of thought; and that, by developing certain themes, it will be of greater interest than a mere assemblage of results and techniques.*

The books in the series differ in level: some are introductory some highly specialised. They also differ in scope: some offer a wide view of an area, others present a single line of thought. Each book is, at its own level, reasonably self-contained. Although no book depends on another as prerequisite, we have encouraged authors to fit their book in with other planned volumes, sometimes deliberately seeking coverage of the same material from different points of view. We have tried to attain a reasonable degree of uniformity of notation and arrangement. However, the books in the series are written by individual authors, not by the group. Plans for books are discussed and argued about at length. Later, encouragement is given and revisions suggested. But it is the authors who do the work; if, as we hope, the series proves of value, the credit will be theirs.

History of the Ω-Group. *During 1968 the idea of an integrated series of monographs on mathematical logic was first mooted. Various discussions led to a meeting at Oberwolfach in the spring of 1969. Here the founding members of the group (R. O. Gandy, A. Levy, G. H. Müller, G. Sacks, D. S. Scott) discussed the project in earnest and decided to go ahead with it. Professor F. K. Schmidt and Professor Hans Hermes gave us encouragement and support. Later Hans Hermes joined the group. To begin with all was fluid. How ambitious should we be? Should* *we write the books ourselves? How long would it take? Plans for authorless books were promoted, savaged and scrapped. Gradually there emerged a form and a method. At the end of an infinite discussion we found our name, and that of the series. We established our centre in Heidelberg. We agreed to meet twice a year together with authors, consultants and assistants, generally in Oberwolfach. We soon found the value of collaboration: on the one hand the permanence of the founding group gave coherence to the over-all plans; on the other hand the stimulus of new contributors kept the project alive and flexible. Above all, we found how intensive discussion could modify the authors' ideas and our own. Often the battle ended with a detailed plan for a better book which the author was keen to write and which would indeed contribute a perspective.*

Acknowledgements. *The confidence and support of Professor Martin Barner of the Mathematisches Forschungsinstitut at Oberwolfach and of Dr. Klaus Peters of Springer-Verlag made possible the first meeting and the preparation of a provisional plan. Encouraged by the Deutsche Forschungsgemeinschaft and the Heidelberger Akademie der Wissenschaften we submitted this plan to the Stiftung Volkswagenwerk where Dipl. Ing. Penschuck vetted our proposal; after careful investigation he became our adviser and advocate. We thank the Stiftung Volkswagenwerk for a generous grant (1970–73) which made our existence and our meetings possible.*

Since 1974 the work of the group has been supported by funds from the Heidelberg Academy; this was made possible by a special grant from the Kultusministerium von Baden-Württemberg (where Regierungsdirektor R. Goll was our counsellor). The success of the negotiations for this was largely due to the enthusiastic support of the former President of the Academy, Professor Wilhelm Doerr. We thank all those concerned.

Finally we thank the Oberwolfach Institute, which provides just the right atmosphere for our meetings, Drs. Ulrich Felgner and Klaus Gloede for all their help, and our indefatigable secretary Elfriede Ihrig.

Oberwolfach
September 1975

R. O. Gandy *H. Hermes*
A. Levy *G. H. Müller*
G. Sacks *D. S. Scott*

Author's Preface

At a recent meeting of logicians, one speaker complained — mainly, but perhaps not wholly, in jest — that logic is tightly controlled by a small group of people (the cabal) who exercise careful control over the release of new ideas to the general public (especially students) and indeed suppress some material completely. The situation is surely not so grim as this, but any potential reader of this book must have felt at some time that there is at least a minor conspiracy to keep new ideas inaccessible until the "insiders" have worked them over thoroughly. In particular he might well feel this way about the whole subject of Generalized Recursion Theory, which developed in the second half of the 1960s. The basic definitions and results on recursion involving functionals of higher type appeared in the monumental but extremely difficult paper Kleene [1959] and [1963]. Gandy [1967] gave another presentation *ab initio*, but the planned part II of this paper, as well as several other major advances in the subject, never appeared in print. For the theory of recursion on ordinals, the situation was even worse. Much of the basic material had appeared only in the *abstracts* Kripke [1964, 1964a], and although certain parts of the theory had been worked out in papers such as Kreisel-Sacks [1965] and Sacks [1967], there was no reasonably complete account of the basic facts of the subject in print.

When I first contemplated doing something about this situation in the spring of 1971, I planned to write a short monograph on recursion relative to type-2 functionals with enough background on ordinary Recursion Theory to show how the theories fit together. Before I had done much about it, however, the invitation of the Ω-Group to write a volume for this series stimulated me to think in more ambitious terms and my plan expanded gradually to include functionals of types 3 and higher, ordinal recursion, and a more thorough presentation of the material on definability (Chapters III–V). The constant encouragement of the Ω-Group, collective and individual, was essential to the completion of the task.

The original plan arose from a course I gave at the University of Michigan in the Fall Term of 1970. Thanks to Jens-Erik Fenstad and the University of Oslo I had the opportunity to lecture on much of the material during the academic year 1971–1972. Other occasions to lecture on parts of the material were provided by the University of Michigan in 1972–73 and the Winter Term of 1975, the Warsaw

Logic Semester in May, 1973, and the Michigan–Ohio Logic Seminar. The majority of the actual writing was done in the summers of 1973–75 under grants from the National Science Foundation.

Of my many teachers, formal and informal, who have personally helped me to form my conception of this subject, I want especially to mention John Addison, Jens-Erik Fenstad, Robin Gandy, Yiannis Moschovakis, and Joe Shoenfield. Andreas Blass read much of the first draft and made many helpful comments. Mm. Bocuse and Haberlin provided inspiring models of excellence. The boldness of the section and subsection headings in the first third of the book is due to the careful work of Monica Scott and her brown crayon. Barbara Perkel did a superb job of typing. Finally, the person to whom the reader should be most grateful is Anne Zalc. In reading carefully the entire final draft she caught hundreds of errors, serious and minor. More importantly, she was an unrelenting enemy of that peculiar brand of obfuscation which results from an author's implicit assumption that the reader has perfectly understood and remembered every detail of what has preceded any given point. Without her the book would be a denser jungle.

January 30, 1978 Peter G. Hinman
Ann Arbor

Table of Contents

Introduction 1

Part A. Basic Notions of Definability 5

Chapter I. Groundwork 7

1. Logic and Set Theory 7
2. Topology and Measure 15
3. Inductive Definitions 22

Chapter II. Ordinary Recursion Theory 27

1. Primitive Recursion 28
2. Recursive Functionals and Relations 37
3. Normal Forms 46
4. Semi-Recursive Relations 52
5. Relativization 62

Chapter III. Hierarchies and Definability 69

1. The Arithmetical Hierarchy 69
2. The Analytical Hierarchy 80
3. Inductive Definability 89
4. Implicit Definability and Bases 106
5. Definability in Formal Languages for Arithmetic 114
6. Arithmetical Forcing 124

Part B. The Analytical and Projective Hierarchies 133

Chapter IV. The First Level 135

1. Π_1^1 and Well-Orderings 135
2. The Boundedness Principle and Other Applications 143

3. The Borel Hierarchy . . . 156
4. The Effective Borel and Hyperarithmetical Hierarchies . . . 163
5. Cardinality, Measurability and Category . . . 180
6. Continuous Images . . . 188
7. Uniformization . . . 194

Chapter V. Δ_2^1 and Beyond . . . 201

1. The Pre-Wellordering Property . . . 202
2. The Hypothesis of Constructibility . . . 214
3. The Hypothesis of Projective Determinacy . . . 221
4. Classical Hierarchies in Δ_r^1 . . . 236
5. Effective Hierarchies in Δ_r^1 . . . 246
6. A Hierarchy for Δ_2^1 . . . 251

Part C. Generalized Recursion Theories . . . 257

Chapter VI. Recursion in a Type-2 Functional . . . 259

1. Basic Properties . . . 259
2. Substitution Theorems . . . 271
3. Ordinal Comparison . . . 284
4. Relations Semi-Recursive in a Type-2 Functional . . . 291
5. Hierarchies of Relations Recursive in a Type-2 Functional . . . 307
6. Extended Functionals . . . 315
7. Recursive Type-3 Functionals and Relations . . . 335

Chapter VII. Recursion in a Type-3 Functional . . . 343

1. Basic Properties . . . 343
2. Relations Semi-Recursive in a Type-3 Functional . . . 350
3. Hierarchies of Relations Recursive in a Type-3 Functional . . . 360
4. Higher Types . . . 363

Chapter VIII. Recursion on Ordinals . . . 371

1. Recursive Ordinal Functions . . . 372
2. Recursively Regular Ordinals . . . 383
3. Ordinal Recursion and the Analytical Hierarchy . . . 393
4. Ordinal Recursion and Type-2 Functionals . . . 403
5. Stability . . . 412
6. Recursively Large Ordinals . . . 419
7. Ordinal Recursion and Constructible Sets . . . 430

Epilogue . . . 445
References . . . 459
Global Notational Conventions . . . 467
Special Notations . . . 469
Index . . . 473

Introduction

The theory set out in this book is the result of the confluence and common development of two currents of mathematical research, Descriptive Set Theory and Recursion Theory. Both are concerned with notions of definability and, more broadly, with the classification of mathematical objects according to various measures of their complexity. These are the common themes which run through the topics discussed in this book.

Descriptive Set Theory arose around the turn of the century as a reaction among some of the mathematical analysts of the day against the free-wheeling methods of Cantorian set theory. People such as Baire, Borel, and Lebesgue felt uneasy with constructions which required the Axiom of Choice or the set of *all* countable ordinals and began to investigate what part of analysis could be carried out by more explicit and constructive means. Needless to say, there was vigorous disagreement over the meaning of these terms. Some of the landmarks in the early days of Descriptive Set Theory are the construction of the Borel sets, Suslin's Theorem that a set of real numbers is Borel just in case both it and its complement are analytic ($\mathbf{\Sigma}_1^1$), and the discovery that analytic sets have many pleasant properties — they are Lebesgue measurable, have the Baire property, and satisfy the Continuum Hypothesis.

A natural concomitant of this interest in the means necessary to effect mathematical constructions is the notion of hierarchy. Roughly speaking, a hierarchy is a classification of a collection of mathematical objects into levels, usually indexed by ordinal numbers. Objects appearing in levels indexed by larger ordinals are in some way more complex that those at lower levels and the index of the first level at which an object appears is thus a measure of the complexity of the object. Such a classification serves both to deepen our understanding of the objects classified and as a valuable technical tool for establishing their properties.

A familiar example, and one which was an important model in the development of the theory, is the hierarchy of Borel sets of real numbers. This class is most simply characterized as the smallest class of sets containing all intervals and closed under the operations of complementation and countable union. In the ρ-th level of the hierarchy are put sets which require a sequence of ρ

applications of complementation and countable union to families of intervals for their construction. The open and closed sets make up the first level, the F_σ (countable union of closed) and G_δ (countable intersection of open) sets the second, etc. This yields an analysis of the class of Borel sets into a strictly increasing sequence of $\aleph_1$ levels (see § V.3).

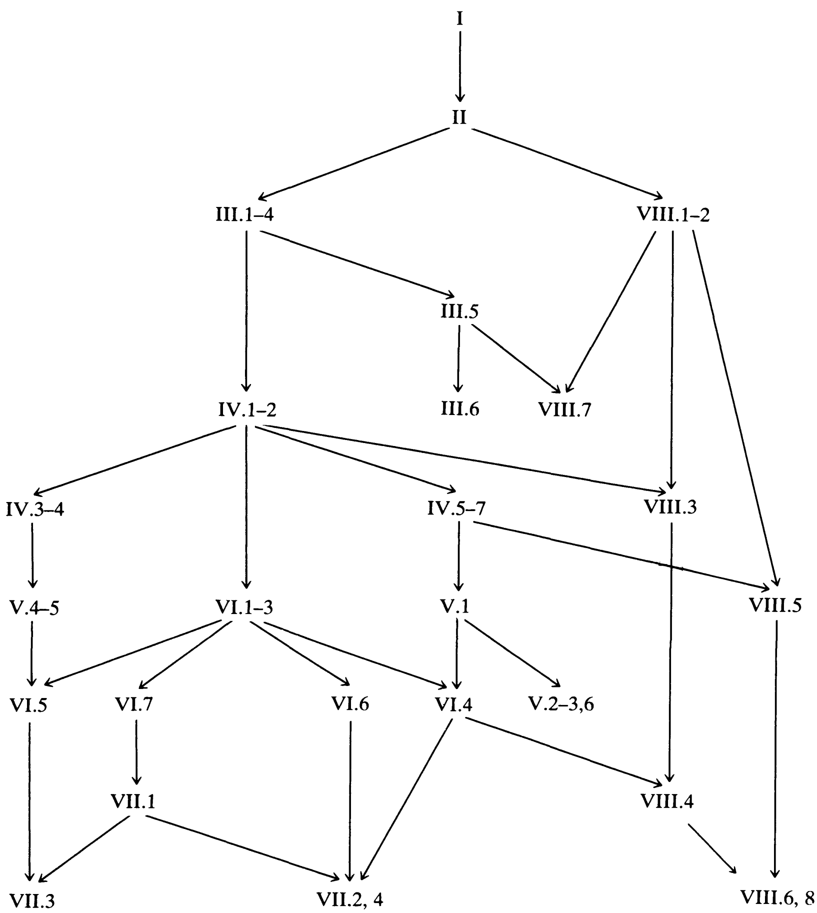
Recursion Theory developed in the 1930's as an attempt to give a rigorous meaning to the notion of a mechanically or algorithmically calculable function. Such a function is, in an obvious sense, more constructive and less complex than an arbitrary function. The first great success of Recursion Theory was Gödel's application of it in his incompleteness theorems in 1931. The diverse characterizations of the class of recursive functions by Church, Kleene, and Turing suggested strongly that this is a natural class of functions. Other related notions of (relative) complexity developed in the 1940's and 1950's — the notion of one function being recursive in another, the arithmetical and analytical hierarchies of Kleene and Mostowski, various sorts of definability in formal languages, and inductive definability.

The two theories developed essentially independently until the middle 1950's when, largely through the work of Addison, it was realized that to a great extent they are both special cases of a single general theory of definability. The results and methods of Recursion Theory are based on a more restrictive notion of constructivity and in many instances may in hindsight be viewed as refinements of their counterparts in Descriptive Set Theory. Since this is a book of mathematics rather than history, I shall develop here the general theory from which the results of both areas can be derived. As a result, many of the earlier parts of Descriptive Set Theory appear to depend on recursion-theoretic techniques. This dependence is explained by the fact that many of these techniques were known in some form before the advent of Recursion Theory. In a few cases (e.g. the Borel hierarchy) where the older theory is much simpler and more elegant than its recursion-theoretic refinement, I adopt a historical approach and present the classical version first. Roughly half of the material of the book comes from the period when the theories were separate; the other half is a product of the marriage.

In accord with the aims of this series, this book is a perspective on Recursion and Hierarchy Theory and not an encyclopedic treatment. Certain approaches are stressed heavily and other equally valid ones are omitted entirely. My definition of recursive function(al) in § II.2 is non-standard and is chosen for the ease with which it can be adapted to the definition of various classes of generalized recursive function(al)s in Chapters VI–VIII. Inductive definability is portrayed (correctly!) as the cornerstone of almost every aspect of the theory. Many topics closely related to those included are omitted. Some of the most notable of these are degree theory (in both ordinary and generalized recursion theory), abstract recursion theory (over other than the "natural" structures), axiomatic recursion theory, and subrecursive hierarchies. Most of these will be treated in other books in this series.

The book is intended for a variety of audiences. As a whole, it is aimed at a student with some general background in abstract mathematics — at least a smattering of topology, measure theory, and set theory — who has finished a course in logic covering the completeness and incompleteness theorems. In general, those parts which are more recursion-theoretic rely more heavily on a background in logic, while the more descriptive set-theoretic parts use more topology. Of course, as always, lack of formal experience in any area is compensated for by that elusive "mathematical maturity". The book may be used for a variety of formal courses of study under titles such as (Generalized) Recursion Theory, Descriptive Set Theory, or Theory of Definability. Students with sufficient background to skim Chapters I and II quickly can cover most of the book in a full-year course, but otherwise some judicious pruning will be required.

In general, the sections of the book depend on each other as indicated in the following diagram; some individual results may presuppose more or less background.



In designing the proofs of results in this book, I have in general made the assumption that the reader has access to pen and paper and will not mind working out a few points for him/herself, but in the main I have tried to give sufficient detail so that anyone who has mastered the prerequisite sections will find this an easy exercise. The subject is full of proofs which require a rather intricate construction followed by a tedious but straightforward inductive verification that the object constructed does the job it was designed for. Furthermore, one often has need in later parts of the theory for a construction quite similar to an earlier one but with one or two additional twists. Indeed, I have often arranged a sequence of lemmas and theorems exactly so that the constructions become increasingly complex in stages in preference to giving at once the most general case. In presenting these I have tried to strike a middle ground between putting the reader to sleep by constant repetition and overly free use of that attractive term "obviously". In general, the construction, or the modification of an earlier construction, is given in full detail, but a good part of the verification is left to the reader. I suspect that the average reader will usually be content to know that the rest of the proof runs "similarly", while the devoted reader (if any!) who seeks a firm grasp on the methods of the subject as well as the results will benefit from the process of working out the details.

The exercises are of two main types. The more routine among them are designed to give the student some experience in handling the methods and techniques of the text and usually require few new ideas. These occasionally include a proof of a lemma from the section. Many of the exercises, however, present results which might well have been included in a larger or more specialized book and constitute a do-it-yourself supplement to the book. (Indeed, one mathematician was somewhat offended that his favorite theorem achieved only the status of an exercise!) I have provided hints and suggestions where they seemed necessary, but many of these exercises will be quite challenging even to the experienced student. The more casual reader should at least glance over the exercises for statements of results.

With respect to the history of the subject, I have taken a middle course between suppressing it altogether and trying to document and credit each minute advance. The primary purpose of the Notes at the end of most sections is to give some idea how, when, and by whom the subject (was) developed, but in the interest of brevity I have omitted mention of many significant contributors. I apologize to those slighted and hope that they will recognize that their sacrifice is for a good cause.

The References similarly contain only a fraction of the articles and books in which our subject matured. Many older references are in a style so different from the current one that they are of little practical use to the working mathematician. I have included a few of these for their historical importance, but in the main the works cited are ones I feel might be of interest to the serious student. In some cases they contain material beyond that of the text, in others they will provide further insight into origins and motivations. In the Epilogue I discuss the current literature and give some guidance for reading which goes beyond the confines of this book.

Part A

Basic Notions of Definability

Chapter I Groundwork

In this introductory chapter we review some of the prerequisites to the theory to be studied in this book. In the first section we discuss the kinds of objects we shall study — functions and relations with natural numbers and functions of natural numbers as arguments. The second section outlines the application of topological and measure-theoretic notions to these objects. In the third section we discuss inductive definability, a notion which plays a dual role in our theory. Most of our fundamental definitions are given inductively, but in addition we shall *study* inductive definability as a means of classifying sets and relations.

The reader need not master all of Chapter I before going on to the theory proper. Subsections 1.1–1.5, 2.1–2.4, and 3.1–3.11 will suffice for a reading of Chapter II and most of Chapter III, and the other subsections may be used for reference. A list of all of the global notational conventions is on page 467.

1. Logic and Set Theory

1.1 Functions and Sequences. A function φ is a set of ordered pairs (x, y). The *domain* of φ is the set $\mathrm{Dm}\,\varphi = \{x\colon \text{for some } y\,(x, y) \in \varphi\}$, and the *image* of φ is the set $\mathrm{Im}\,\varphi = \{y\colon \text{for some } x\,(x, y) \in \varphi\}$. We often write $\varphi(x)\downarrow$ or say $\varphi(x)$ *is defined* to mean that $x \in \mathrm{Dm}\,\varphi$. Similarly, $\varphi(x)\uparrow$ means $x \notin \mathrm{Dm}\,\varphi$ and is read $\varphi(x)$ *is undefined.* If φ and ψ are two functions, we write $\varphi(x) \simeq \psi(x')$ to mean that *either* both $\varphi(x)$ and $\psi(x')$ are undefined *or* both are defined and have the same value ($\varphi(x) = \psi(x')$). In particular, $\varphi(x) \simeq y$ means that $\varphi(x)$ is defined and has value y — that is, $(x, y) \in \varphi$. We write $\varphi(x) = y$ only in contexts where it is clear that $\varphi(x)$ is defined. If $\mathrm{Dm}\,\varphi$ and $\mathrm{Dm}\,\psi$ are both subsets of a set X, then the statement (for all $x \in X$) $\varphi(x) \simeq \psi(x)$ means simply that φ and ψ denote the same function. If the set X is clear from context, this may be written simply $\varphi(x) \simeq \psi(x)$. The *restriction of φ to X* is the function $\varphi \restriction X = \{(x, y)\colon x \in X \text{ and } \varphi(x) \simeq y\}$. The *image of X under φ* is the set $\varphi''\,X = \mathrm{Im}(\varphi \restriction X)$.

We write $\varphi\colon X \to Y$ to mean φ is a function, $\mathrm{Dm}\,\varphi \subseteq X$, and $\mathrm{Im}\,\varphi \subseteq Y$. If $\mathrm{Dm}\,\varphi = X$ we say φ is *total*; otherwise, φ is *partial.* The set of all total functions $\varphi\colon X \to Y$ is denoted by ${}^{X}Y$.

If y_x denotes an element of Y whenever $x \in Z \subseteq X$, we use any of the expressions $x \mapsto y_x$, $\lambda x . y_x$ and $\langle y_x : x \in Z \rangle$ to denote the function $\{(x, y_x) : x \in Z\}$.

The *natural number* m is the set $\{0, 1, \ldots, m-1\}$ of all smaller natural numbers. The set of all natural numbers is denoted by ω.

For any set X, a *finite sequence* from X is a function $\mathbf{x}$ with domain a natural number k called the *length of* $\mathbf{x}$ ($\lg(\mathbf{x})$) and image a subset of X. Hence $\mathbf{x} \in {}^kX$. For $i < \lg(\mathbf{x})$, $\mathbf{x}(i)$ is called the i-th *component* of $\mathbf{x}$ and is usually denoted by x_i. To exhibit all the components we write $(x_0, x_1, \ldots, x_{k-1})$ for $\mathbf{x}$. Note that the *empty sequence* $\emptyset$ is the unique sequence of length 0. We make no distinction in general between X and ${}^1X$. Note that if $\mathbf{x} = (x_0, \ldots, x_{k-1})$ and $\mathbf{y} = (y_0, \ldots, y_{l-1})$ are two finite sequences from X, then $\mathbf{x} \subseteq \mathbf{y}$ just in case $\mathbf{y}$ extends $\mathbf{x}$; that is, $k \leqslant l$ and for all $i < k$, $x_i = y_i$. The operation $\mathbf{x} * \mathbf{y}$ produces the sequence $(x_0, \ldots, x_{k-1}, y_0, \ldots, y_{l-1})$. If $\varphi \in {}^\omega X$, then $\mathbf{x} * \varphi$ is the function $\psi \in {}^\omega X$ such that for $i < \lg(\mathbf{x})$, $\psi(i) = x_i$, and for $i \geqslant \lg(\mathbf{x})$, $\psi(i) = \varphi(i - \lg(\mathbf{x}))$. If $Z \subseteq X$, we sometimes write $\mathbf{x} \in Z$ to mean that for all $i < \lg(\mathbf{x})$, $x_i \in Z$. Similarly, $\mathbf{m} < n$ means that for all $i < \lg(\mathbf{m})$, $m_i < n$. If $\varphi : X \to Y$ and $\mathbf{x} \in {}^kX$, then $\varphi(\mathbf{x})$ denotes the sequence $(\varphi(x_0), \ldots, \varphi(x_{k-1}))$.

1.2 Functionals and Relations. For $k, l \in \omega$ we set ${}^{k,l}\omega = {}^k\omega \times {}^l({}^\omega\omega)$. A function $\mathsf{F} : {}^{k,l}\omega \to \omega$ is called a *functional of rank* (k, l). A functional of rank $(k, 0)$ is also called a *function of rank* k and is identified with the corresponding function $F : {}^k\omega \to \omega$. Elements of ${}^\omega\omega$ are thus total functions of rank 1.

Elements of ${}^{k,l}\omega$ are ordered pairs of the form $(\mathbf{m}, \boldsymbol{\alpha})$. However, if F is a functional of rank (k, l), we write $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha})$ instead of $\mathsf{F}((\mathbf{m}, \boldsymbol{\alpha}))$ and think of $\mathbf{m}, \boldsymbol{\alpha}$ as a *list* of arguments $m_0, \ldots, m_{k-1}, \alpha_0, \ldots, \alpha_{l-1}$. Thus, for example, we write $\mathsf{F}(m_0, \ldots, m_{k-1}, \boldsymbol{\alpha})$ instead of $\mathsf{F}((m_0, \ldots, m_{k-1}), \boldsymbol{\alpha})$ and $\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}, \beta, \gamma)$ instead of of $\mathsf{F}((p) * \mathbf{m}, \boldsymbol{\alpha} * (\beta, \gamma))$. If F is a total functional of rank $(k+1, l)$, then F may also be thought of as a function from ${}^{k,l}\omega$ into ${}^\omega\omega$ whose values are given by:

$$\mathsf{F}[\mathbf{m}, \boldsymbol{\alpha}] = \langle \mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) : p \in \omega \rangle = \lambda p . \mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}).$$

A subset R of ${}^{k,l}\omega$ is called a *relation of rank* (k, l). We usually write $\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha})$ instead of $(\mathbf{m}, \boldsymbol{\alpha}) \in \mathsf{R}$. A relation of rank $(k, 0)$ is called a *relation (on numbers) of rank* k and is identified with the corresponding subset R of ${}^k\omega$. In accord with the list notation for functionals, we write, for example, $\mathsf{R}(\mathbf{m}, \alpha_0, \ldots, \alpha_{l-1})$ for $\mathsf{R}(\mathbf{m}, (\alpha_0, \ldots, \alpha_{l-1}))$ and $\mathsf{R}(p, q, \mathbf{m}, r, \beta, \boldsymbol{\alpha})$ for $\mathsf{R}((p, q) * \mathbf{m} * (r), (\beta) * \boldsymbol{\alpha})$. For $\mathsf{R} \subseteq {}^{k,l}\omega$, the *complement* of R (with respect to ${}^{k,l}\omega$) is the relation $\sim\mathsf{R} = \{(\mathbf{m}, \boldsymbol{\alpha}) : (\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$ and $(\mathbf{m}, \boldsymbol{\alpha}) \notin \mathsf{R}\}$.

With each relation R of rank (k, l) is associated its *characteristic functional* of rank (k, l) defined by

$$\mathsf{K}_{\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) = \begin{cases} 0, & \text{if } \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}); \\ 1, & \text{otherwise.} \end{cases}$$

Conversely, to each functional F of rank (k, l) corresponds it graph Gr_{F} (occasionally $\mathrm{Gr}(\mathsf{F})$), a relation of rank $(k+1, l)$ defined by

$$\mathrm{Gr}_{\mathsf{F}}(n, \mathbf{m}, \boldsymbol{\alpha}) \quad \text{iff} \quad \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n.$$

By the identification of functionals of rank (1, 0) with functions from ω into ω and of relations of rank (1, 0) with subsets of ω, if $A \subseteq \omega$, then $\mathsf{K}_A \in {}^{\omega}\omega$. We write, for example, $\mathsf{F}(\mathbf{m}, A, \boldsymbol{\alpha}, B)$ instead of $\mathsf{F}(\mathbf{m}, \mathsf{K}_A, \boldsymbol{\alpha}, \mathsf{K}_B)$ and thus extend functionals to admit subsets of ω as arguments.

Compositions of partial functionals and relations are taken to be defined whenever possible. For example, $\mathsf{F}(\mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha}) \simeq n$ just in case for some $p \in \omega$, $\mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}) \simeq p$ and $\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq n$. Similarly, $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\,\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})) \simeq n$ just in case for some $\beta \in {}^{\omega}\omega$, $\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \beta(p)$ for all $p \in \omega$, and $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}, \beta) \simeq n$. Note that $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p\,.\,\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha}))$ is undefined for any $\mathbf{m}$ and $\boldsymbol{\alpha}$ for which $\lambda p\,.\,\mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})$ is not total. For relations, we have, for example, $\mathsf{R}(\mathbf{m}, \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}), \boldsymbol{\alpha})$ is true iff for some $p \in \omega$, $\mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}) \simeq p$ and $\mathsf{R}(\mathbf{m}, p, \boldsymbol{\alpha})$, and false otherwise (not undefined).

Natural numbers are said to be objects of *type* 0. Functions from ${}^{k}\omega$ into ω and subsets of ${}^{k}\omega$ are objects of *type* 1; functionals from ${}^{k,l}\omega$ into ω and subsets of ${}^{k,l}\omega$ $(l > 0)$ are objects of *type* 2. In general a function with natural number values or a relation is of *type* $n+1$ iff its arguments are objects of types at most n. In practice, the arguments of types >0 will almost always be total unary functions. Thus the objects of type 3 discussed in § VI.7 and Chapter VII are functions and relations on ${}^{k,l,l'}\omega = {}^{k}\omega \times {}^{l}({}^{\omega}\omega) \times {}^{l'}({}^{{}^{\omega}\omega}\omega)$. Elements of ${}^{k,l,l'}\omega$ are written $(\mathbf{m}, \boldsymbol{\alpha}, \mathbf{I})$, where $\mathbf{I} = (\mathsf{I}_1, \ldots \mathsf{I}_{l'-1})$. Functionals of type 3 are denoted by letters $\mathbb{F}, \mathbb{G}, \mathbb{H}, \ldots$ and relations of type 3 by $\mathbb{R}, \mathbb{S}, \mathbb{T}, \ldots$.

1.3 Logical Notation. We shall use the logical symbols $\wedge$, $\vee$, $\neg$, $\rightarrow$, and $\leftrightarrow$ as abbreviations for the expressions 'and', 'or', 'not', 'implies', and 'if and only if', respectively. Although we are not, for the most part, dealing with formalized languages, these connectives are to be understood in their usual truth-functional sense. Thus, for example, an expression of the form ——— $\rightarrow$ .... is true just in case ——— is false, or .... is true (or both).

The symbols $\exists$ and $\forall$ will be used as abbreviations for 'there exists' and 'for all', respectively. In most cases the range of the quantifier will be indicated by the type of variable following it in accord with the conventions listed on page 467. For example, an expression of the form $\exists m\ [— m —]$ is true just in case — m — is true for some natural number m. Similarly, the condition for equality of partial functionals is written

$$\mathsf{F} = \mathsf{G} \leftrightarrow \forall \mathbf{m} \forall \boldsymbol{\alpha}\, [\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha})].$$

Further restrictions on the range of a quantifier may be indicated by use of a *bounded quantifier.* For example, $(\exists m < p)$— m — is true iff — m — is true for some m among $0, 1, \ldots, p-1$; $(\forall \gamma \in \mathsf{W}) \ldots . \gamma \ldots .$ is true iff $\ldots . \gamma \ldots .$ is true for all γ belonging to W. We write $\exists! m[— m —]$ to mean that — m — is true for exactly one m.

Parentheses, (and), and brackets, [and], are used interchangeably in sufficient quantity to ensure unique readability of expressions. In addition, a single dot . may be used to set off two parts of an expression — for example, $(\forall p < q) . R(p, \mathbf{m})$ or $\lambda p . f(p, \mathbf{m}) + g(p)$.

1.4 Sequence Coding. For each k, ${}^k\omega$ is a countable set and may thus be put in one-one correspondence with a subset of ω. Similarly for $l > 0$, ${}^l({}^\omega\omega)$ may be put in one-one correspondence with a subset of ${}^\omega\omega$. We define here some particularly simple such correspondences which we call *coding functions.*

Temporarily we let p_i denote the i-th prime number: $p_0 = 2, p_1 = 3, \ldots$. For each k we define a total function $\langle \cdot \rangle^k$ of rank k by:

$$\langle\ \rangle^0 = 1, \text{ and for any } k > 0 \text{ and any } \mathbf{m} \in {}^k\omega,$$

$$\langle \mathbf{m} \rangle^k = \langle m_0, \ldots, m_{k-1} \rangle^k = p_0^{m_0+1} \cdot p_1^{m_1+1} \cdot \cdots \cdot p_{k-1}^{m_{k-1}+1}.$$

The unique factorization theorem of arithmetic ensures that if $\langle \mathbf{m} \rangle^k = \langle \mathbf{n} \rangle^l$, then $k = l$ and $\mathbf{m} = \mathbf{n}$. As the superscript is usually clear from the context, we shall usually omit it.

For any s, t, and $i \in \omega$, let

$$(s)_i = \text{least } m\ [m < s \wedge p_i^{m+2} \text{ does not divide } s];$$

$$\lg(s) = \text{least } k\ [k < s \wedge p_k \text{ does not divide } s];$$

$$s * t = s \cdot t', \text{ where } t' \text{ arises from } t \text{ by replacing each}$$

$$\text{factor } p_i^n \text{ in the prime decomposion of } t \text{ by } p_{\lg(s)+i}^n.$$

Then it is an arithmetical exercise to verify that for all k, all $\mathbf{m} \in {}^k\omega$, all $\mathbf{n}$, and all $i < k$, $(\langle \mathbf{m} \rangle)_i = m_i$, $\lg(\langle \mathbf{m} \rangle) = k$, and $\langle \mathbf{m} \rangle * \langle \mathbf{n} \rangle = \langle \mathbf{m} * \mathbf{n} \rangle$. We denote by Sq the set of all s such that $s = \langle \mathbf{m} \rangle$ for some $\mathbf{m}$. Note that for any k and s, s may be regarded as coding a sequence of length k, namely $((s)_0, \ldots, (s)_{k-1})$. We often regard $\mathbf{m}$ and $\langle \mathbf{m} \rangle$ as interchangeable and write, for example, $\mathbf{p} \subseteq \langle \mathbf{m} \rangle$ instead of $\mathbf{p} \subseteq \mathbf{m}$. In particular, $s \subseteq t$ iff for some $\mathbf{m}$ and $\mathbf{n}$, $s = \langle \mathbf{m} \rangle$, $t = \langle \mathbf{n} \rangle$, and $\mathbf{m} \subseteq \mathbf{n}$. For any $\beta \in {}^\omega\omega$, $\bar{\beta}(k)$ denotes the code for the sequence $\beta \restriction k$ — that is, $\bar{\beta}(k) = \langle \beta(0), \ldots, \beta(k-1) \rangle$.

We next define coding functions from ${}^l({}^\omega\omega)$ into ${}^\omega\omega$:

$$\langle\ \rangle^0 = \lambda m . 1, \text{ and for any } l > 0 \text{ and any } \boldsymbol{\alpha} \in {}^l({}^\omega\omega),$$

$$\langle \boldsymbol{\alpha} \rangle^l = \langle \alpha_0, \ldots, \alpha_{l-1} \rangle^l = \lambda m . \langle \alpha_0(m), \ldots, \alpha_{l-1}(m) \rangle.$$

Again it is obvious that if $\langle\boldsymbol{\alpha}\rangle^l = \langle\boldsymbol{\beta}\rangle^k$, then $l = k$ and $\boldsymbol{\alpha} = \boldsymbol{\beta}$, and that we may omit the superscript without ambiguity.

For any γ and $\delta \in {}^\omega\omega$ and any $j \in \omega$, let

$$(\gamma)_j = \lambda m\,.\,(\gamma(m))_j;$$

$$\lg(\gamma) = \lg(\gamma(0));$$

$$\gamma * \delta = \lambda m\,.\,\gamma(m) * \delta(m).$$

Then for all l, all $\boldsymbol{\alpha} \in {}^l({}^\omega\omega)$, all $\boldsymbol{\beta}$, and all $j < l$, $(\langle\boldsymbol{\alpha}\rangle)_i = \alpha_i$, $\lg(\langle\boldsymbol{\alpha}\rangle) = l$, and $\langle\boldsymbol{\alpha}\rangle * \langle\boldsymbol{\beta}\rangle = \langle\boldsymbol{\alpha} * \boldsymbol{\beta}\rangle$. We denote by Sq_1 the set of all γ such that $\gamma = \langle\boldsymbol{\alpha}\rangle$ for some $\boldsymbol{\alpha}$. For any l and γ, γ may be regarded as coding the sequence $((\gamma)_0, \ldots, (\gamma)_{l-1})$.

It will also be useful occasionally to code ω-sequences of functions. We set

$$\langle\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\rangle = \lambda m\,.\,\alpha_{(m)_0}((m)_1), \quad \text{and}$$

$$(\gamma)^n(m) = \gamma(\langle n, m\rangle).$$

Then clearly $(\langle\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\rangle)^n(m) = \alpha_n(m)$.

1.5 Set Theory. Except where we specify otherwise, the results of this book are all theorems of ZFC, Zermelo-Fraenkel Set Theory with the Axiom of Choice. The (Generalized) Continuum Hypothesis is not assumed. We shall occasionally want to replace the full Axiom of Choice (AC) by the weaker Axiom of Dependent Choices:

(DC) $\quad \forall x\, \exists y\,.\,(x, y) \in X \rightarrow \exists\varphi\, \forall m\,.\,(\varphi(m), \varphi(m+1)) \in X.$

We recall that this implies the principle of choice for countable families of non-empty sets:

(AC_ω) $\quad \forall m\,.\,Y_m \neq \varnothing \rightarrow \exists\psi\, \forall m\,.\,\psi(m) \in Y_m.$

Most of our set-theoretic conventions are standard and we refer the reader to (for example) Lévy [1978] for further background. A set x is transitive iff $\forall y(y \in x \rightarrow y \subseteq x)$. x is an *ordinal* (*number*) iff x and all of its elements are transitive. For ordinals π and ρ, $\pi < \rho$ iff $\pi \in \rho$; the relation $\leqslant$ is a well-ordering on any set of ordinals. For any ordinal π, $\pi + 1$ is the set $\pi \cup \{\pi\}$, the *ordinal successor* of π. ρ is a *successor ordinal* iff $\rho = \pi + 1$ for some π; ρ is a *limit ordinal* iff $\forall\pi(\pi < \rho \rightarrow \pi + 1 < \rho)$. Every ordinal is either 0, a limit, or a successor. The natural numbers are exactly the finite ordinals and ω is the smallest limit ordinal. Or is the class of all ordinals.

For any set X of ordinals we denote by $\inf X$ the $\leqslant$-least element of X.

Although X need not have a $\leqslant$-greatest element, there is always an ordinal greater than or equal to all elements of X and we denote by $\sup X$ the least such ordinal. In fact, $\sup X$ is exactly the union of the members of X. We set also $\sup^+ X = \sup\{\pi + 1: \pi \in X\}$. Then $\sup^+ X$ is the least ordinal strictly greater than all elements of X and is the same as $\sup X$ if X has no greatest element; otherwise, $\sup X$ is the greatest element and $\sup^+ X = \sup X + 1$. If φ is a function from ordinals to ordinals and $X \subseteq \operatorname{Dm} \varphi$, then

$$\inf_{\pi \in X} \varphi(\pi) = \inf\{\varphi(\pi): \pi \in X\}, \quad \text{and}$$

$$\sup^{(+)}_{\pi \in X} \varphi(\pi) = \sup^{(+)}\{\varphi(\pi): \pi \in X\}.$$

An ordinal ρ is a *limit of members of* $X \subseteq \mathrm{Or}$ iff $(\forall \pi < \rho)(\exists \sigma \in X) . \pi < \sigma < \rho$. Any limit of members of any set X is a limit ordinal. If also $\rho \in X$, then ρ is called a *limit point of* X. We denote by $\operatorname{Lim} X$ the set of limit points of X. A subset $Y \subseteq X$ is *cofinal in* X iff $(\forall \sigma \in X)(\exists \tau \in Y)\sigma \leqslant \tau$.

If $\mathfrak{A}$ is a proposition which may be true ($\mathfrak{A}(\sigma)$) or false ($\neg \mathfrak{A}(\sigma)$) of each ordinal σ, then to prove $\forall \sigma\, \mathfrak{A}(\sigma)$ we may use the method of proof by *transfinite induction*: if $\forall \sigma([(\forall \tau < \sigma)\mathfrak{A}(\tau)] \to \mathfrak{A}(\sigma))$, then $\forall \sigma\, \mathfrak{A}(\sigma)$. We use frequently also the parallel method of *definition by transfinite recursion*: for any total $k + 2$-place function ψ, there exists a $k + 1$-place function φ such that for all ρ and $\mathbf{x}$,

$$\varphi(\rho, \mathbf{x}) = \psi(\varphi \restriction_{\mathbf{x}} \rho, \rho, \mathbf{x})$$

where

$$\varphi \restriction_{\mathbf{x}} \rho = \{((\pi, \mathbf{x}), z): \pi < \rho \wedge \varphi(\pi, \mathbf{x}) = z\}.$$

φ is not unique, but any other function φ' which satisfies this equation has $\varphi'(\sigma, \mathbf{x}) = \varphi(\sigma, \mathbf{x})$ for all σ and $\mathbf{x}$.

Since any set X of ordinals is well ordered by the relation $\leqslant$, it is uniquely order-isomorphic to an ordinal which we denote by $\|X\|$, the *order-type* of X. The function φ_X which realizes this isomorphism is recursively defined by:

$$\varphi_X(\rho) = \sup^+\{\varphi_X(\pi): \pi < \rho \wedge \pi \in X\}.$$

We list here some elementary properties of φ_X which will be needed in Chapter VIII:

(1) $\pi, \rho \in X \wedge \pi < \rho \to \varphi_X(\pi) < \varphi_X(\rho)$;

(2) $\rho \in X \wedge \sigma < \varphi_X(\rho) \to \exists \pi[\pi \in X \wedge \pi < \rho \wedge \sigma = \varphi_X(\pi)]$;

(3) $\rho \in X \to \varphi_X(\rho + 1) = \varphi_X(\rho) + 1$;

(4) $\rho \subseteq X \rightarrow \varphi_X(\rho) = \rho;$

(5) X is an ordinal $\leftrightarrow (\forall \rho \in X) . \varphi_X(\rho) = \rho.$

An ordinal κ is an *initial ordinal* or *cardinal* (*number*) iff there is no one-one correspondence between κ and any $\tau < \kappa$. From the axiom of choice it follows that for every set X there is a unique cardinal κ such that there exists a one-one correspondence between X and κ. We denote this κ by Card(X), the *cardinal of* X. Then there is a one-one correspondence between two sets X and Y just in case Card(X) = Card(Y). The natural numbers are exactly the finite cardinals, and ω is the least infinite cardinal. A set X is *countable* iff Card(X) $\leq \omega$ and denumerable or countably infinite iff Card(X) $= \omega$. The infinite cardinals are enumerated by the function $\aleph$ defined by:

$$\aleph_0 = \omega;$$

$$\aleph_{\sigma+1} = \{\rho : \text{Card}(\rho) \leq \aleph_\sigma\};$$

$$\aleph_\sigma = \bigcup \{\aleph_\pi : \pi < \sigma\}, \quad \text{for limit } \sigma.$$

In particular, $\aleph_1$ is the set of countable ordinals.

For any X, $\mathbf{P}(X)$ denotes the power set of X, the set of all subsets of X. If Card(X) $= \kappa$, then Card $\mathbf{P}(X)$ is denoted by 2^κ. If X is infinite, then Card$({}^X 2) =$ Card$({}^X\omega) = 2^\kappa$. In particular, Card$({}^\omega\omega) =$ Card$({}^{k,l}\omega) = 2^{\aleph_0}$ for all k and all $l > 0$. By *Cantor's Theorem*, $\kappa < 2^\kappa$ for all cardinals κ. The *Continuum Hypothesis* is the statement that $2^{\aleph_0} = \aleph_1$.

1.6 Ordering Relations. For any set X and any $Z \subseteq {}^2X$, the *field* of Z is the set

$$\text{Fld}(Z) = \{x : \exists y\, [(x, y) \in Z \vee (y, x) \in Z]\}.$$

Z is a *pre-partial-ordering* iff

(1) $(\forall x \in \text{Fld}(Z))[(x, x) \in Z]$, ($Z$ is *reflexive*),

and

(2) $\forall x\, \forall y\, \forall z\, [(x, y) \in Z \wedge (y, z) \in Z \rightarrow (x, z) \in Z]$ (Z is *transitive*).

Z is a *pre-linear-ordering* iff (1), (2), and

(3) $\forall x\, \forall y\, [x, y \in \text{Fld}(Z) \wedge x \neq y \rightarrow (x, y) \in Z \vee (y, x) \in Z]$ (Z is *connected*).

Z is a *pre-wellordering* iff (1), (2), (3), and

(4) $\forall Y\,(Y \subseteq \mathrm{Fld}(Z) \wedge Y \neq \varnothing \rightarrow (\exists x \in Y)(\forall y \in Y)[(y, x) \in Z \rightarrow (x, y) \in Z])$

(Z is *well founded*).

Z is a *partial-* (*linear-*, *well-*) *ordering* iff Z is a pre-partial- (linear-, well-) ordering and

(5) $\forall x\, \forall y\, [(x, y) \in Z \wedge (y, x) \in Z \rightarrow x = y]$ (Z is *antisymmetric*).

From the Axiom of Dependent Choice (DC) it follows that (4) is equivalent to

(4′) $\forall \varphi\, [\forall m\,.\,(\varphi(m+1), \varphi(m)) \in Z \rightarrow \exists m\,.\,(\varphi(m), \varphi(m+1)) \in Z]$.

If Z is a pre-wellordering, then there is a unique function $|\cdot|_Z$, the ***norm associated with*** Z, from $\mathrm{Fld}(Z)$ onto an ordinal such that for all $x, y \in \mathrm{Fld}(Z)$,

(6) $(x, y) \in Z \leftrightarrow |x|_Z \leqslant |y|_Z$.

In fact, for any $y \in \mathrm{Fld}(Z)$,

$$|y|_Z = \sup{}^+\{|x|_Z : (x, y) \in Z \wedge (y, x) \notin Z\}.$$

Conversely, if $|\ \ |$ is any function from a set Y into the ordinals, the relation $Z_{|\ |}$ defined by

(7) $(x, y) \in Z_{|\ |} \leftrightarrow |x| \leqslant |y|$

is a pre-wellordering. If the image of $|\ \ |$ is an ordinal, then $|\ \ |$ is the norm associated with $Z_{|\ |}$.

Z is a well-ordering just in case $|\ \ |_Z$ is injective (one-one). The image of $|\ \ |_Z$ is called the *(pre-)order-type* of Z and is denoted by $\|Z\|$. Clearly $\|Z\| < \kappa$, where κ is the least cardinal greater than Card(X). In the context of of set theory without the Axiom of Choice, a useful measure of the size of a set X in terms of ordinals is $o(X) = \sup^+\{\|Z\| : Z$ is a pre-wellordering and $\mathrm{Fld}(Z) \subseteq X\}$.

Orderings will generally be denoted by symbols $\leqslant$ or $\preccurlyeq$ with various sub- and superscripts. In any such context, the symbols $<$ or $\prec$ always denote the associated *strict ordering* defined by:

$$x < y \leftrightarrow x \leqslant y \wedge y \nleqslant x.$$

With any $\gamma \in {}^\omega\omega$ we associate a binary relation $\leqslant_\gamma$ by

$$m \leqslant_\gamma n \leftrightarrow \gamma(\langle m, n \rangle) = 0.$$

We shall mainly be interested in γ such that $\leqslant_\gamma$ is a partial ordering. We then set

$$\mathrm{Fld}(\gamma) = \mathrm{Fld}(\leqslant_\gamma);$$

$$\mathsf{W} = \{\gamma : \leqslant_\gamma \text{ is a well-ordering}\};$$

$$\|\gamma\| = \begin{cases} \|\leqslant_\gamma\|, & \text{if } \gamma \in \mathsf{W}; \\ \aleph_1, & \text{otherwise;} \end{cases}$$

$$(\gamma \restriction p)(\langle m, n\rangle) = \begin{cases} 0, & \text{if } m \leqslant_\gamma n \wedge n <_\gamma p; \\ 1, & \text{otherwise;} \end{cases}$$

$$|p|_\gamma = \|\gamma \restriction p\|.$$

The following facts are easily verified: for any $\gamma \in \mathsf{W}$ and any p,

(8) $\gamma \restriction p \in \mathsf{W}$, and if $\|\gamma\| > 0$, then $\|\gamma \restriction p\| < \|\gamma\|$;

(9) for any $\sigma < \|\gamma\|$, there is a unique $p \in \mathrm{Fld}(\gamma)$ such that $\|\gamma \restriction p\| = \sigma$;

(10) $\|\gamma \restriction p\| = \sup^+\{\|\gamma \restriction q\| : q <_\gamma p\}$;

(11) $\aleph_1 = \{\|\gamma\| : \gamma \in \mathsf{W}\}$.

1.7 Notes. The idea of coding finite sequences of natural numbers by prime powers goes back (at least) to Gödel [1931]. For readers less familiar with set theory we recommend Lévy [1978], Zuckerman [1974], or the handiest recent text.

2. *Topology and Measure*

We begin our study of the spaces ${}^{k,l}\omega$ by defining a natural topology and measure theory for them. We define first a topology based on viewing ${}^{k,l}\omega$ as a product of copies of ω, show that with this topology ${}^\omega\omega$ is homeomorphic to the set of binary irrational numbers between 0 and 1 with the topology induced from the reals, and using this homeomorphism, carry Lebesgue measure over to ${}^\omega\omega$.

The set ${}^\omega\omega$ may be viewed as a product $\omega \times \omega \times \cdots \times \omega \times \cdots$ of denumerably many copies of ω. To ω we assign the *discrete topology*: all sets are open (and hence all are also closed). Then to ${}^\omega\omega$ we assign the induced *product topology*: a set $\mathsf{A} \subseteq {}^\omega\omega$ is a *basic open set* iff for some n and some (open) subsets $B_0, \ldots, B_{n-1}$ of ω,

$$\mathsf{A} = B_0 \times B_1 \times \cdots \times B_{n-1} \times \omega \times \cdots \times \omega \times \cdots.$$

In other words, for all α,

$$\alpha \in \mathsf{A} \leftrightarrow (\alpha(0) \in B_0) \wedge (\alpha(1) \in B_1) \wedge \cdots \wedge (\alpha(n-1) \in B_{n-1}).$$

The *open subsets* of ${}^\omega\omega$ are then, of course, arbitrary unions of basic open sets. Finally to ${}^{k,l}\omega$ we again assign the product topology: $\mathsf{R} \subseteq {}^{k,l}\omega$ is a *basic open relation* iff for some $A_0, \ldots, A_{k-1} \subseteq \omega$ and some open sets $\mathsf{A}_0, \ldots, \mathsf{A}_{l-1} \subseteq {}^\omega\omega$,

$$\mathsf{R} = (A_0 \times \cdots \times A_{k-1}) \times (\mathsf{A}_0 \times \cdots \times \mathsf{A}_{l-1}).$$

For any finite sequence $\mathbf{m} = (m_0, \ldots, m_{n-1})$, the *interval* $[\mathbf{m}]$ is defined by:

$$\alpha \in [\mathbf{m}] \leftrightarrow (\alpha(0) = m_0) \wedge \cdots \wedge (\alpha(n-1) = m_{n-1}).$$

If $\mathbf{m} \subseteq \mathbf{n}$, then $[\mathbf{n}] \subseteq [\mathbf{m}]$ is a *subinterval* of $[\mathbf{m}]$. Clearly each interval is a basic open subset of ${}^\omega\omega$. Conversely, if A is the basic open set determined by $B_0, \ldots, B_{n-1}$, then

$$\mathsf{A} = \bigcup \{[\mathbf{m}]: m_0 \in B_0 \wedge \cdots \wedge m_{n-1} \in B_{n-1}\}.$$

Thus the set of intervals is also a base for the topology on ${}^\omega\omega$.

A partial functional is *partial continuous* iff for all n,

$$\mathsf{F}^{-1}(\{n\}) = \{(\mathbf{m}, \boldsymbol{\alpha}): \mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq n\}$$

is open. This is equivalent to the more usual condition that $\mathsf{F}^{-1}(B)$ be open for any open set $B \subseteq \omega$. F is *continuous* iff it is partial continuous and total.

2.1 Lemma. *For any* $\mathsf{R} \subseteq {}^{k,l}\omega$,

(i) R *is open iff* R *is the domain of some partial continuous functional*;

(ii) R *is closed-open iff* K_R *is continuous.*

Proof. If F is partial continuous, then $\mathrm{Dm}\,\mathsf{F} = \bigcup \{\mathsf{F}^{-1}(\{n\}): n \in \omega\}$ is a union of open sets and hence is open. Conversely, if R is open, let

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \begin{cases} 0, & \text{if } \mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}); \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Clearly F is partial continuous and $\mathsf{R} = \mathrm{Dm}\,\mathsf{F}$. (ii) follows immediately from the definitions. □

A set $\mathsf{A} \subseteq {}^\omega\omega$ is *dense in* an interval $[\mathbf{m}]$ iff for every subinterval $[\mathbf{m} * \mathbf{n}] \subseteq [\mathbf{m}]$, $\mathsf{A} \cap [\mathbf{m} * \mathbf{n}] \neq \varnothing$. A is *dense* iff it is dense in the interval $[\varnothing] = {}^\omega\omega$. A is *nowhere dense* iff it is dense in no interval. A is *meager* (*first category*) iff it is a countable union of nowhere dense sets. A is *non-meager* (*second category*) iff it is not meager. A is *comeager* (*residual*) iff $\sim A$ is meager. An element α of A is *isolated in* A iff there exists a neighborhood $[\alpha \restriction k]$ of α such that $[\alpha \restriction k] \cap A = \{\alpha\}$. A is called *perfect* iff it is closed, non-empty, and has no isolated elements.

We mention first some simple direct consequences of these definitions. A singleton is nowhere dense so any countable set is meager. A countable union of meager sets is meager. A subset of a meager set is meager. A set is nowhere dense iff its closure includes no interval. The complement of an open dense set is nowhere dense. A perfect set has power $2^{\aleph_0}$.

2.2 Baire Category Theorem. *No non-empty open set is meager; no comeager set is meager.*

Proof. Both statements follow from the assertion that no interval is meager. Suppose to the contrary that for some $\mathbf{p}$ and some nowhere dense sets A_n, $[\mathbf{p}] = \bigcup \{A_n : n \in \omega\}$. A_0 is not dense in $[\mathbf{p}]$, so for some sequence $\mathbf{m}^0$, $A_0 \cap [\mathbf{p} * \mathbf{m}^0] = \emptyset$. A_1 is not dense in $[\mathbf{p} * \mathbf{m}^0]$, so for some $\mathbf{m}^1$, $A_1 \cap [\mathbf{p} * \mathbf{m}^0 * \mathbf{m}^1] = \emptyset$. In this way we construct $\mathbf{m}^n$ such that for all n,

$$(A_0 \cup \cdots \cup A_n) \cap [\mathbf{p} * \mathbf{m}^0 * \cdots * \mathbf{m}^n] = \emptyset.$$

There is a function $\alpha \in \bigcap \{[\mathbf{p} * \mathbf{m}^0 * \cdots * \mathbf{m}^n] : n \in \omega\}$ and $\alpha \in [\mathbf{p}]$ but $\alpha \notin A_n$ for all n, a contradiction. □

We shall have occasion to consider the subspace ${}^{\omega}2$ consisting of all α which assume only the values 0 and 1. ${}^{\omega}2$ is just the set of characteristic functions of subsets of ω and thus in a natural one-one correspondence with $\mathbf{P}(\omega)$. The interval $[\mathbf{m}]$ has a non-empty intersection with ${}^{\omega}2$ iff $\mathbf{m}$ is a *binary sequence* — all m_i are either 0 or 1. If X is a set of finite sequences we say X is *closed downward* iff whenever $\mathbf{n} \subseteq \mathbf{m} \in X$, also $\mathbf{n} \in X$.

2.3 Infinity Lemma. *For any set X of binary sequences which is closed downward, if X is infinite, then X contains an infinite branch — that is, for some $\alpha \in {}^{\omega}2$, $\alpha \restriction k \in X$ for all k.*

Proof. Let X satisfy the hypotheses and consider the set

$$Y = \{\mathbf{m} : \mathbf{m} \in X \text{ and } \{\mathbf{n} : \mathbf{m} \subseteq \mathbf{n} \wedge \mathbf{n} \in X\} \text{ is infinite}\}.$$

By hypothesis $\emptyset \in Y$. For any $\mathbf{m}$ and any $\mathbf{n} \neq \mathbf{m}$,

$$\mathbf{m} \subseteq \mathbf{n} \leftrightarrow (\mathbf{m} * (0) \subseteq \mathbf{n})) \vee (\mathbf{m} * (1) \subseteq \mathbf{n}).$$

Hence if $\mathbf{m} \in Y$, then at least one of $\mathbf{m} * (0)$ and $\mathbf{m} * (1)$ also belongs to Y. Thus there exists a unique function α such that for all k,

$$\alpha(k) = \text{least}\, p\,.\,(\alpha \restriction k) * (p) \in Y.$$

Since $Y \subseteq X$, this α satisfies the conclusion of the lemma. □

2.4 Theorem. (i) ${}^{\omega}2$ *is a compact subspace of* ${}^{\omega}\omega$;
(ii) *for any* k *and* l, $({}^{k}2) \times {}^{l}({}^{\omega}2)$ *is a compact subspace of* ${}^{k,l}\omega$.

Proof. We prove (i) by showing that any open cover $\mathcal{F}$ of ${}^{\omega}2$ has a finite subcover. Let X be the set of all finite binary sequences $\mathbf{m}$ such that $[\mathbf{m}]$ is included in no member of $\mathcal{F}$. Clearly X is closed downward; suppose X is infinite. Then by the Infinity Lemma, X contains an infinite branch α. Since $\mathcal{F}$ is a cover, $\alpha \in \mathsf{A}$ for some $\mathsf{A} \in \mathcal{F}$. As A is open, for some k, $[\alpha \restriction k] \subseteq \mathsf{A}$, a contradiction. Hence X is finite, so for some k all members of X have length less than k. Let $\mathbf{m}^0, \ldots, \mathbf{m}^{2^k-1}$ be a list of all binary sequences of length k. For each $i < 2^k$ we may choose an $\mathsf{A}_i \in \mathcal{F}$ such that $[\mathbf{m}^i] \subseteq \mathsf{A}_i$. Then $\mathcal{F}_0 = \{\mathsf{A}_i \colon i < 2^k\}$ is the required finite subcover. The proof of (ii) is similar. □

Note that the proof of Theorem 2.4 depends only on the fact that for any $\mathbf{m} \in X$, $\{p \colon \mathbf{m} * (p) \in X\}$ is finite. Hence, for example, ${}^{\omega}q$ is a compact subspace of ${}^{\omega}\omega$ for any $q \in \omega$.

The original aim of Descriptive Set Theory was the study and classification of sets of real numbers and their properties which are of interest for mathematical analysis. It was early discovered that little is lost and much is gained in simplicity and elegance if one studies sets of irrational numbers. Indeed, for most properties of interest to analysis — measurability, having the power of the continuum, being meager, etc. — the exclusion of a countable set of points (the rationals) has no effect. On the other hand, there are important topological differences between the reals and the irrationals which simplify the theory of sets of irrationals: the irrationals are of topological dimension 0, there is a base for the topology on the irrationals which consists of closed-open sets, and the irrationals are homeomorphic to their own Cartesian powers. Further simplification was obtained by the discovery that the space of irrationals is homeomorphic to ${}^{\omega}\omega$ with the topology described above. Thus many results concerning ${}^{\omega}\omega$ and the product spaces ${}^{k,l}\omega$ have immediate consequences for the spaces of irrational and real numbers (cf. end of § IV.3).

Temporarily, let ${}^{\omega}\omega$ denote (ambiguously) the topological space described above (as well as its underlying set). Let Ir denote similarly the set of irrational numbers x such that $0 < x < 1$ together with the topology induced by the standard topology on the set of real numbers: $Y \subseteq \text{Ir}$ is *open* iff $Y = \text{Ir} \cap Z$ for some open subset Z of the real interval $(0, 1)$. Then the fact we mentioned is: ${}^{\omega}\omega$ and Ir are homeomorphic. We leave the proof of this to Exercise 2.8 and construct here instead a homeomorphism of ${}^{\omega}\omega$ with another subspace of (0, 1), the space BIr of binary irrationals. This correspondence will serve just as well in transferring results from ${}^{\omega}\omega$ to (0, 1) and is somewhat more natural.

A *finite binary decimal* is a representation of a real number in the form:

$$.r_1r_2\ldots r_l = r_1(2^{-1}) + r_2(2^{-2}) + \cdots + r_l(2^{-l})$$

where each $r_i = 0$ or 1. The real numbers which have finite binary representations are exactly those which can be written as a quotient p/q of natural numbers such that q is a power of 2. Clearly such numbers are dense in (0, 1). An *infinite binary decimal* is a representation

$$.r_1r_2\ldots r_l\ldots = \sum_{i=1}^{\infty} r_i(2^{-i})$$

where each $r_i = 0$ or 1. Any such series converges to a real number between 0 and 1 and every such real number has an infinite binary representation. Two infinite binary decimals represent the same real number iff they are of the forms

$$.r_1r_2\ldots r_l\,1\,0\,0\ldots0\ldots,$$

and

$$.r_1r_2\ldots r_l\,0\,1\,1\ldots1\ldots.$$

A *binary irrational* is a real number between 0 and 1 that does not have a finite binary representation. BIr is the topological space consisting of the binary irrationals with the topology induced from (0, 1).

2.5 Theorem. *$^{\omega}\omega$ and* BIr *are homeomorphic.*

Proof. For any $\alpha \in {}^{\omega}\omega$, let $\theta(\alpha)$ be the infinite binary decimal:

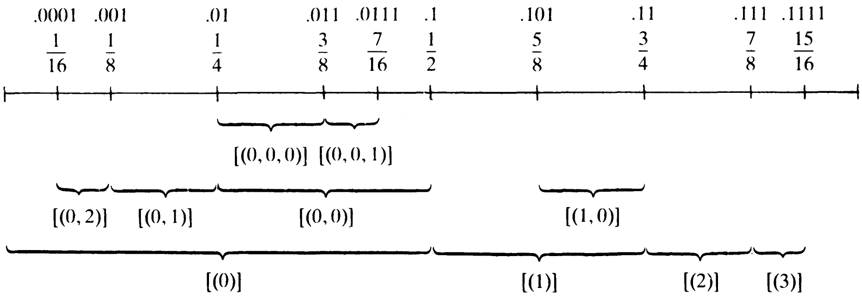
$$\theta(\alpha) = .\underbrace{11\ldots10}_{\alpha(0)+1}\,\underbrace{00\ldots01}_{\alpha(1)+1}\,\underbrace{11\ldots10}_{\alpha(2)+1}\,\underbrace{00\ldots01}_{\alpha(3)+1}\ldots.$$

From the preceding remarks it is obvious that θ maps $^{\omega}\omega$ one-one onto BIr. θ is continuous because if $\alpha\restriction k = \beta\restriction k$, then $|\theta(\alpha) - \theta(\beta)| < 2^{-k}$. For the continuity of θ^{-1}, suppose that $x \in$ BIr and k are given. To insure that $\theta^{-1}(x)\restriction k = \theta^{-1}(y)\restriction k$ it suffices to take $|x-y| < 2^{n}$, where $n = \theta^{-1}(x)(0) + \cdots + \theta^{-1}(x)(k-1) + k$. Hence θ is a homeomorphism. □

To compute the image of a given interval [**m**] in $^{\omega}\omega$, note that $\theta(\alpha)$ has the following representation:

$$\theta(\alpha) = .\underbrace{1\,1\ldots1\,1}_{\alpha(0)+1}$$
$$-(.\underbrace{0\,0\ldots0\,0}_{\alpha(0)+1}\underbrace{1\,1\ldots1\,1}_{\alpha(1)+1})$$
$$+(.\underbrace{0\,0\ldots0\,0}_{\alpha(0)+1}\underbrace{0\,0\ldots0\,0}_{\alpha(1)+1}\underbrace{1\,1\ldots1\,1}_{\alpha(2)+1})$$
$$-\cdots.$$

Thus θ induces the following correspondence between intervals of ${}^{\omega}\omega$ and of Blr:



To extend these results to the spaces ${}^{k,l}\omega$, it suffices to show that for each k and l, ${}^{k,l}\omega$ is homeomorphic to ${}^{\omega}\omega$. For this we need sequence coding functions which are *onto* ω. Set

$$\langle\langle m, n\rangle\rangle^2 = \tfrac{1}{2}(m^2 + 2mn + n^2 + 3m + n)$$

and, recursively, for $l > 1$,

$$\langle\langle m_0, \ldots, m_l\rangle\rangle^{l+1} = \langle\langle\, \langle\langle m_0, \ldots, m_{l-1}\rangle\rangle^l, m_l\rangle\rangle.$$

We leave it as an exercise (2.9) to check that the map $\theta^{k,l}$ defined by:

$$\theta^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = (\mathbf{m}) * \lambda p\,.\langle\langle \alpha_0(p), \ldots, \alpha_{l-1}(p)\rangle\rangle^l$$

is the desired homeomorphism.

The homeomorphism θ induces a natural measure on ${}^{\omega}\omega$. Let mes_{Lb} denote Lebesgue measure restricted to Blr. For $\mathsf{A} \subseteq {}^{\omega}\omega$, we set

$$\text{mes}(\mathsf{A}) = \text{mes}_{\text{Lb}}\{\theta(\alpha) \colon \alpha \in \mathsf{A}\}$$

and say A is *measurable* just in case its θ-image is Lebesgue measurable. Because θ is a homeomorphism, all open and closed sets are measurable. The measure is clearly countably additive and has the property that all subsets of a set of measure 0 are measurable (completeness). This measure may also be described as the *product measure* on ${}^{\omega}\omega$ generated by the measure on ω which assigns $\{n\}$ the measure $2^{-(n+1)}$. Thus for any sequence $\mathbf{m} = (m_0, \ldots, m_{k-1})$,

$$\mathrm{mes}([\mathbf{m}]) = 2^{-(m_0+1)} \cdot \cdots \cdot 2^{-(m_{k-1}+1)} = 2^{-(m_0+\cdots+m_{k-1}+k)}.$$

Similarly, we may define a measure on ${}^{k,l}\omega$ either via the homeomorphism $\theta^{k,l}$ or directly by setting

$$\mathrm{mes}(\{m_0\} \times \cdots \times \{m_{k-1}\} \times [p^0] \times \cdots \times [p^{l-1}])$$
$$= 2^{-(m_0+1)} \cdot \cdots \cdot 2^{-(m_{k-1}+1)} \cdot \mathrm{mes}([p^0]) \cdot \cdots \cdot \mathrm{mes}([p^{l-1}]).$$

2.6–2.13 Exercises

2.6. Let α^* range over ${}^{\omega}2$. Show that for any set X of finite sequences of 0's and 1's,

$$\forall \alpha^* \exists p . \alpha^* \restriction p \in X \leftrightarrow \exists n \forall \alpha^* (\exists p < n) . \alpha^* \restriction p \in X.$$

2.7. For any $\mathsf{A} \subseteq {}^{\omega}\omega$, let $s * \mathsf{A} = \{s * \alpha : \alpha \in \mathsf{A}\}$ and $\mathsf{A}^{(s)} = \{\alpha : s * \alpha \in \mathsf{A}\}$. The class Ka of *Kalmar sets* is the smallest class X of subsets of ${}^{\omega}\omega$ such that $\varnothing, {}^{\omega}\omega \in X$ and if for all $n, \mathsf{A}_n \in X$, then $\bigcup \{\langle n \rangle * \mathsf{A}_n : n \in \omega\} \in X$. Show for all A and s,

(i) $\mathsf{A} \in \mathrm{Ka} \to \mathsf{A}^{(s)} \in \mathrm{Ka}$;
(ii) $\mathsf{A} \in \mathrm{Ka} \leftrightarrow s * \mathsf{A} \in \mathrm{Ka}$;
(iii) $\forall n . \mathsf{A}^{(\langle n \rangle)} \in \mathrm{Ka} \to \mathsf{A} \in \mathrm{Ka}$;
(iv) $\forall \beta \exists n . \mathsf{A}^{(\bar{\beta}(n))} \in \mathrm{Ka} \to \mathsf{A} \in \mathrm{Ka}$;
(v) $\mathsf{A} \in \mathrm{Ka} \leftrightarrow \mathsf{A}$ is closed-open.

2.8. Prove that the topological spaces ${}^{\omega}\omega$ and Ir are homeomorphic. (Since the sets of rationals and binary rationals are each countable and dense in (0, 1), there is a one-one order-preserving correspondence between them. This may be extended in a unique way to a homeomorphism of (0, 1) with itself. The restriction of this homeomorphism to Ir is a homeomorphism of Ir with BIr.)

2.9. Show that $\theta^{k,l}$ is a homeomorphism of ${}^{k,l}\omega$ onto ${}^{\omega}\omega$.

2.10. (The Zero-One Law). Show that for any measurable set $\mathsf{A} \subseteq {}^{\omega}\omega$, if for all s,

$$\mathrm{mes}(\mathsf{A} \cap [s]) = \mathrm{mes}(\mathsf{A}) \cdot \mathrm{mes}([s]),$$

then mes(A) is either 0 or 1. (Show that this equation holds with $[s]$ replaced by any measurable set.)

2.11. Show that in the usual topology on the real interval (0, 1), Ir and BIr are G_δ sets (countable intersection of open sets) but not F_σ sets (countable union of closed sets).

2.12. Show that a relation $\mathsf{R} \subseteq {}^{k,l}\omega$ is open iff for some $S \subseteq {}^{k+1}\omega$,

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \exists p\, S(\mathbf{m}, \bar{\alpha}_0(p), \ldots, \bar{\alpha}_{l-1}(p)).$$

2.13. Show that a homeomorphism between ${}^\omega\omega$ and Ir may be constructed directly. (To each finite sequence $\mathbf{m}$ assign a rational number $\theta(\mathbf{m})$ recursively by the rules:

$$\theta(\emptyset) = 0 \quad \text{and} \quad \theta((p, \mathbf{m})) = \frac{1}{p + 1 + \theta(\mathbf{m})}.$$

For all α, $\theta(\alpha \restriction k)$ converges to a limit $\theta^*(\alpha)$.)

3. *Inductive Definitions*

Let X be any fixed set. A function Γ from the power set of X into itself is called an *operator over* X. Γ is said to be *inclusive* iff for all $Y \subseteq X$, $Y \subseteq \Gamma(Y)$, *monotone* iff for all $Y \subseteq Z \subseteq X$, $\Gamma(Y) \subseteq \Gamma(Z)$, and *inductive* iff Γ is either inclusive or monotone. An operator Γ *defines inductively* a subset $\bar{\Gamma}$ of X as follows. We define by transfinite recursion the sequence Γ^σ by $\Gamma^\sigma = \Gamma(\bigcup\{\Gamma^\tau : \tau < \sigma\})$ and set $\bar{\Gamma} = \bigcup\{\Gamma^\sigma : \sigma \in \mathrm{Or}\}$. We write $\Gamma^{(\sigma)}$ for $\bigcup\{\Gamma^\tau : \tau < \sigma\}$ so that $\Gamma^\sigma = \Gamma(\Gamma^{(\sigma)})$.

We think of the set $\bar{\Gamma}$ as being "built up" in stages. Starting from the empty set we get successively $\Gamma(\emptyset)$, $\Gamma(\Gamma(\emptyset))$, ... Γ^σ is called the σ-th *stage* or *level.*

3.1. Lemma. *For any inductive operator* Γ *and any ordinal* σ,

(i) $\Gamma^{(\sigma)} \subseteq \Gamma^\sigma$;

(ii) $\Gamma^{\sigma+1} = \Gamma(\Gamma^\sigma)$;

(iii) $\Gamma^{(\sigma)} = \Gamma^\sigma \rightarrow \Gamma^\tau = \Gamma^\sigma = \bar{\Gamma}$ *for all* $\tau \geqslant \sigma$;

(iv) $\Gamma^{(\sigma)} = \Gamma^\sigma$ *for some* σ *such that* $\mathrm{Card}(\sigma) \leqslant \mathrm{Card}(X)$.

Proof. For inclusive Γ, (i) is immediate from the definitions; for monotone Γ it follows from the obvious fact that for $\tau \leqslant \sigma$, $\Gamma^{(\tau)} \subseteq \Gamma^{(\sigma)}$. (ii) is immediate from the observation that by (i), $\Gamma^{(\sigma+1)} = \Gamma^\sigma$. (iii) is proved by induction on τ: for $\tau = \sigma$, clearly $\Gamma^\tau = \Gamma^\sigma$; for $\tau > \sigma$, the induction hypothesis yields $\Gamma^{(\tau)} = \Gamma^\sigma$ and we have $\Gamma^\tau = \Gamma(\Gamma^{(\tau)}) = \Gamma(\Gamma^\sigma) = \Gamma(\Gamma^{(\sigma)}) = \Gamma^\sigma$. For (iv), suppose that for each σ with $\mathrm{Card}(\sigma) \leqslant \mathrm{Card}\, X$, $\Gamma^{(\sigma)} \subsetneq \Gamma^\sigma$, and let x_σ be an element of $\Gamma^\sigma \sim \Gamma^{(\sigma)}$. If $\tau \neq \sigma$, also

$x_\tau \neq x_\sigma$, and this defines an injection of $\{\sigma\colon \mathrm{Card}(\sigma) \leq \mathrm{Card}(X)\}$ into X. But this set is exactly the least cardinal *larger* than Card X, so this is impossible. □

We denote by $|\Gamma|$ the least σ such that $\Gamma^{(\sigma)} = \Gamma^\sigma$, the *closure ordinal* of Γ. Then

3.2 Corollary. *For any inductive operator Γ over X,* $\mathrm{Card}(|\Gamma|) \leq \mathrm{Card}(X)$ *and* $\bar{\Gamma} = \Gamma^{(|\Gamma|)}$. □

Thus we need not think of the sequence Γ^σ as extended over all ordinals but only over those less than $|\Gamma|$. In particular, if $X = \omega$ we need only consider countable ordinals.

Note that for any inductive Γ, $\Gamma(\bar{\Gamma}) = \bar{\Gamma}$; but for $\sigma < |\Gamma|$, $\Gamma(\Gamma^{(\sigma)}) \neq \Gamma^{(\sigma)}$. In other words, $\bar{\Gamma}$ is the first *fixed point* of Γ in the sequence Γ^σ.

3.3 Theorem. *For any monotone operator Γ over a set X, $\bar{\Gamma}$ is the smallest fixed point of Γ — that is,*

$$\bar{\Gamma} = \bigcap \{Z\colon Z \subseteq X \wedge \Gamma(Z) = Z\}.$$

Proof. By the preceding remark, $\bar{\Gamma} \in \{Z\colon Z \subseteq X \wedge \Gamma(Z) = Z\}$ so that the intersection of this set is included in $\bar{\Gamma}$. Conversely, let Z be any subset of X such that $\Gamma(Z) = Z$; we prove by induction on σ that for all σ, $\Gamma^\sigma \subseteq Z$. Assume as induction hypothesis that this holds for all $\tau < \sigma$ so $\Gamma^{(\sigma)} \subseteq Z$. Then by monotonicity, $\Gamma^\sigma = \Gamma(\Gamma^{(\sigma)}) \subseteq \Gamma(Z) = Z$. □

Note that the proof yields also that for monotone Γ,

$$\bar{\Gamma} = \bigcap \{Z\colon Z \subseteq X \wedge \Gamma(Z) \subseteq Z\}.$$

These results have two distinct aspects. First, they give a characterization of $\bar{\Gamma}$ which does not involve ordinals. Second, they provide a very convenient way of proving that all $x \in \bar{\Gamma}$ have some property: one shows that the set Z of all $x \in X$ which have the property satisfies $\Gamma(Z) \subseteq Z$. In applying this method we say that the proof is by *Γ-induction* or by *induction over $\bar{\Gamma}$*.

In many contexts where we are defining inductively a particular set Y it will be convenient to avoid direct reference to the inductive operator involved. Thus if Y is defined as $\bar{\Gamma}$, we may write Y^σ and $Y^{(\sigma)}$ instead of Γ^σ and $\Gamma^{(\sigma)}$ and describe proofs by Γ-induction as proofs by *induction over Y*.

In the remainder of this section we consider the properties of two special classes of inductive definitions. Let Y be a subset of X and $\mathscr{F}$ a family of finitary functions on X — that is, for each $\varphi \in \mathscr{F}$, there is a natural number $k(\varphi)$ such that $\mathrm{Dm}\,\varphi = {}^{k(\varphi)}X$ and $\mathrm{Im}\,\varphi \subseteq X$. For each such pair $(Y, \mathscr{F})$, we define an inductive operator $\Gamma_{Y,\mathscr{F}}$ by:

$$\Gamma_{Y,\mathscr{F}}(Z) = Y \cup \{\varphi(\mathbf{z}) \colon \varphi \in \mathscr{F} \wedge \mathbf{z} \in {}^{k(\varphi)}Z\}.$$

The resulting set $\bar{\Gamma}_{Y,\mathscr{F}}$ is called the *closure of Y under $\mathscr{F}$*. Since $\Gamma_{Y,\mathscr{F}}$ is clearly monotone, $\bar{\Gamma}_{Y,\mathscr{F}}$ is also the *smallest set including Y and closed under $\mathscr{F}$*.

3.4 Lemma. *For any Y and $\mathscr{F}$ as above, $|\Gamma_{Y,\mathscr{F}}| \leq \omega$.*

Proof. Let Y and $\mathscr{F}$ be fixed and write Γ for $\Gamma_{Y,\mathscr{F}}$. By 3.1 (i) it suffices to show $\Gamma^{\omega} \subseteq \Gamma^{(\omega)}$. Let x be any element of $\Gamma^{\omega} = \Gamma(\Gamma^{(\omega)})$. If $x \in Y$, then $x \in \Gamma^0 \subseteq \Gamma^{(\omega)}$. Otherwise, for some $\varphi \in \mathscr{F}$ and some $\mathbf{z} \in {}^{k(\varphi)}(\Gamma^{(\omega)})$, $x = \varphi(\mathbf{z})$. For each $i < k(\varphi)$, let r_i be the least natural number such that $z_i \in \Gamma^{r_i}$, and set $r = \max\{r_i \colon i < k(\varphi)\}$. Then $\mathbf{z} \in {}^{k(\varphi)}\Gamma^r$ so $x \in \Gamma(\Gamma^r) = \Gamma^{r+1} \subseteq \Gamma^{(\omega)}$. □

The method of inductive definition is a generalization of the definition of the set ω of natural numbers in set theory: ω is the smallest set including $\{0\}$ and closed under the successor function, $\mathrm{Sc}(x) = x \cup \{x\}$. Many of the fundamental notions of elementary logic are most naturally defined inductively, often by operators of the form $\Gamma_{Y,\mathscr{F}}$. For example, the set of formulas of a (finitary) first-order formal language is the closure of the set of atomic formulas under functions corresponding to the propositional connectives and quantifiers (cf. § III.5). The set of formal theorems of an axiomatic theory is the closure of the set of axioms under functions corresponding to the rules of inference. An example which is not a closure under finitary functions is the class of formulas of the infinitary language $L_{\omega_1\omega}$ (cf. Keisler [1971]).

We shall also need a generalization of the method of definition by recursion. Roughly speaking, for any set X^* we may define a function $\theta \colon \omega \to X^*$ by specifying a value $\theta(0)$ and a method for calculating $\theta(m+1)$ from $\theta(m)$. The corresponding generalization will allow us to define a function $\theta \colon \bar{\Gamma}_{Y,\mathscr{F}} \to X^*$ by specifying the values $\theta(y)$ for $y \in Y$ and methods for calculating $\theta(\varphi(\mathbf{x}))$ from $\theta(x_0), \ldots, \theta(x_{k(\varphi)-1})$ for all $\varphi \in \mathscr{F}$ (the x_i should be thought of as the immediate predecessors of $\varphi(\mathbf{x})$). In the case of ω, m is uniquely determined by $m+1$, but for arbitrary Y and $\mathscr{F}$ it may happen that $\varphi(\mathbf{x}) = \varphi'(\mathbf{x}')$ or $\varphi(\mathbf{x}) \in Y$ so that the rules would not determine a unique value for $\theta(\varphi(\mathbf{x}))$.

We call the pair $(Y, \mathscr{F})$ *monomorphic* iff all $\varphi \in \mathscr{F}$ are one-one and the sets Y and $\{\mathrm{Im}\,\varphi \colon \varphi \in \mathscr{F}\}$ are pairwise disjoint. The inductive definitions of ω and the class of formulas of a first-order language are monomorphic whereas that of the class of formal theorems is not.

3.5 Theorem (Definition by Recursion). *For any monomorphic pair $(Y, \mathscr{F})$ and any set X^*, suppose that $\psi \colon X^* \to X^*$ and for each $\varphi \in \mathscr{F}$, $\varphi^* \colon {}^{k(\varphi)}X^* \to X^*$. Then there exists a unique function $\theta \colon \bar{\Gamma}_{Y,\mathscr{F}} \to X^*$ such that*

(i) *for all $y \in Y$, $\theta(y) = \psi(y)$;*

(ii) *for all $\varphi \in \mathscr{F}$ and all $\mathbf{x} \in {}^{k(\varphi)}\bar{\Gamma}_{Y,\mathscr{F}}$,*

$$\theta(\varphi(\mathbf{x})) = \varphi^*(\theta(x_0), \ldots, \theta(x_{k(\varphi)-1})).$$

Proof. Let $\Gamma = \Gamma_{Y,\mathscr{F}}$, X^*, ψ, and φ^* be given as in the hypothesis. We define functions $\theta_r \colon \Gamma^r \to X^*$ by ordinary recursion as follows. $\theta_0 = \psi$. Suppose θ_r is defined and $x \in \Gamma^{r+1}$. If $x \in \Gamma^r$ we set $\theta_{r+1}(x) = \theta_r(x)$. If $x \in \Gamma^{r+1} \sim \Gamma^r$, then by the assumption that $(Y, \mathscr{F})$ is monomorphic there exist unique $\varphi \in \mathscr{F}$ and $\mathbf{z} \in {}^{k(\varphi)}\Gamma^r$ such that $x = \varphi(\mathbf{z})$. We then set $\theta_{r+1}(x) = \varphi^*(\theta_r(z_0), \ldots, \theta_r(z_{k(\varphi)-1}))$. Finally, $\theta = \bigcup\{\theta_r \colon r \in \omega\}$. We leave to the reader the easy verification that θ satisfies conditions (i) and (ii) (Exercise 3.11). □

Our second special class consists of operators over a product space $X \times Y$. Operators of this type will be used in defining subsets of ${}^{k,l}\omega$.

3.6 Definition. An operator Γ over $X \times Y$ is *decomposable* iff there exists a family of operators Γ_y over X, indexed by $y \in Y$, such that for any $Z \subseteq X \times Y$,

$$\Gamma(Z) = \{(x, y) \colon x \in \Gamma_y(\{z \colon (z, y) \in Z\})\}.$$

3.7 Lemma. *For any decomposable operator Γ over $X \times Y$,*

(i) $\bar{\Gamma} = \{(x, y) \colon y \in Y \wedge x \in \bar{\Gamma}_y\}$;

(ii) $|\Gamma| = \sup\{|\Gamma_y| \colon y \in Y\}$.

Proof. Both parts follows easily from the assertion that for all σ

$$\Gamma^\sigma = \{(x, y) \colon y \in Y \wedge x \in \Gamma_y^\sigma\}.$$

To establish this by induction, suppose that it holds for all $\tau < \sigma$. Then

$$\Gamma^{(\sigma)} = \bigcup_{\tau<\sigma}\{(x, y) \colon y \in Y \wedge x \in \Gamma_y^\tau\} = \{(x, y) \colon y \in Y \wedge x \in \Gamma_y^{(\sigma)}\},$$

and

$$\Gamma^\sigma = \Gamma(\Gamma^{(\sigma)}) = \{(x, y) \colon y \in Y \wedge x \in \Gamma_y(\Gamma_y^{(\sigma)})\} = \{(x, y) \colon y \in Y \wedge x \in \Gamma_y^\sigma\}. \quad \square$$

Decomposable inductive operators over ${}^{k,l}\omega$ are given by families of operators Γ_α over ${}^k\omega$. By Corollary 3.2 each $|\Gamma_\alpha|$ is countable and thus $|\Gamma| \leq \aleph_1$, whereas the closure ordinal of an arbitrary operator over ${}^{k,l}\omega$ is bounded only by the least cardinal greater than $2^{\aleph_0}$. This fact will play an important role in § III.3.

3.8–3.13 Exercises

3.8. Show that any monotone operator over a set X has a largest fixed point $\bar{\Gamma}$. In fact, $\bar{\Gamma} = \sim\Gamma^\circ$, where Γ° is a monotone operator defined by $\Gamma^\circ(Y) = \sim\Gamma(\sim Y)$.

3.9. Construct an example of a non-monotone inductive operator which has no smallest fixed point.

3.10. An operator Γ is called *κ-compact* (for any cardinal κ) iff whenever $x \in \Gamma(Y)$, also $x \in \Gamma(Z)$ for some $Z \subseteq Y$ with $\text{Card}(Z) < \kappa$. Note that any $\Gamma_{Y,\mathcal{F}}$ is ω-compact. What can be said in general about the closure ordinal of a κ-compact inductive operator?

3.11. Complete the proof of Theorem 3.5. Sketch an alternative proof in which θ is defined inductively as the smallest set of pairs (x, x^*) such that ... .

3.12. Suppose $Y \subseteq X$ and $\mathcal{F}$ is a family of finitary functions on X such that $(Y, \mathcal{F})$ is monomorphic. For each $x \in \bar{\Gamma}_{Y,\mathcal{F}}$, define Sp($x$), the *support* of x, recursively by:

$$\text{Sp}(y) = \varnothing, \quad \text{for} \quad y \in Y;$$

$$\text{Sp}(\varphi(\mathbf{x})) = \bigcup \{\text{Sp}(x_i)\colon i < k(\varphi)\} \cup \{x_i\colon i < k(\varphi)\}.$$

Establish the following *principle of proof by course-of-values induction* over $\bar{\Gamma}_{Y,\mathcal{F}}$: for any $Z \subseteq \bar{\Gamma}_{Y,\mathcal{F}}$, if $(\forall x \in \bar{\Gamma}_{Y,\mathcal{F}})[\text{Sp}(x) \subseteq Z \rightarrow x \in Z]$, then $Z = \bar{\Gamma}_{Y,\mathcal{F}}$.

3.13. There is also a natural notion of definition by course-of-values recursion that says roughly that we may define a function $\theta\colon \bar{\Gamma}_{Y,\mathcal{F}} \rightarrow X^*$ by specifying the values $\theta(y)$ for $y \in Y$ and methods for calculating $\theta(x)$ from values $\theta(z)$ for $z \in \text{Sp}(x)$. Formulate precisely a principle of this kind as general as possible and prove that it is valid.

3.14 Notes. Inductive definitions have long played a fundamental role in many areas of mathematics but have been studied as objects only much more recently. Definitions in Algebra of the subgroup, subring, etc. generated by certain elements are all inductive. The class of Borel sets of a topological space is inductively defined as is (the complement of) the perfect kernel of a set of reals (cf. Exercise 3.8). The principal objects of study of Logic and Recursion Theory are all inductively defined. The general study of inductive definability begins explicitly with Spector [1961] but it is close to the surface in many earlier papers of Kleene, especially [1955] and [1955a]. Moschovakis [1974, pp. 3–4] gives a more extended history of the subject.

Chapter II

Ordinary Recursion Theory

The notion of a recursive function resulted from an attempt in the 1930's to provide a precise mathematical characterization of the concept of a mechanically or algorithmically calculable function from ${}^k\omega$ into ω. One way to understand this concept is to imagine an idealized digital computer not subject to error or limitations of memory or storage space. Then a partial function F is mechanically calculable just in case there is a finite program (or algorithm) for this computer which directs it to accept inputs of the form $\mathbf{m}$ and carry out a computation with two possible results: if $\mathbf{m} \in \mathrm{Dm}\, F$, the computation terminates after finitely many steps with the correct value $F(\mathbf{m})$ as output; if $\mathbf{m} \notin \mathrm{Dm}\, F$, the computation does not terminate.

As this is an intuitive concept, however, it cannot be described completely except by convention. Not only is any attempt subject to legitimate disagreement on the basis of current knowledge, but also the possibility remains open that in the future a new means of calculation will be discovered which will be agreed by mathematicians to be mechanical but will not fall under the proposed description. Still, from a practical point of view, the notion seems to be a viable one: most people with a thorough understanding of the concepts involved will agree on the question of whether or not a given method of calculation is mechanical.

In particular, although we cannot give a rigorous proof that every recursive function is mechanically calculable, our justification of this assertion in § 2 below should be convincing to almost everyone. The converse proposition, known as Church's Thesis, that all mechanically calculable functions are recursive, is somewhat more problematic. Without a precise delineation of the class of mechanically calculable functions, we are in no position to *prove* that all of its members are recursive. We are forced, therefore, to rely on what might be called circumstantial evidence. Most importantly, no one has exhibited a function which is agreed to be mechanically calculable but is not recursive. In a similar vein, every known procedure which produces from calculable functions another calculable function also produces a recursive function from recursive functions.

Another kind of evidence is given by the variety of ways that the class of recursive functions can be characterized. Although these characterizations have quite different intuitive content (based on different conceptions of mechanical calculability) they all describe exactly the class of recursive functions. This shows that this is a very natural class and is at least intimately connected with the notion of mechanical calculability.

A discussion of these diverse characterizations and a more detailed examination of the evidence for Church's Thesis may be found in Kleene [1952].

As we shall be discussing functionals as well as functions, we shall want, for comparison, also a notion of mechanical calculability for partial functions from ${}^{k,l}\omega$ into ω. At first glance there seems to be no way for our idealized computer to accept inputs of the form $(\mathbf{m}, \boldsymbol{\alpha})$. Even if we allow the computer to have infinite memory facilities sufficient to store all the values of an argument α, it would seemingly take infinitely long just to "read in" these values. Hence to preserve the finiteness of computations we say that the computer receives an input $(\mathbf{m}, \boldsymbol{\alpha})$ when it is connected to an infinite memory device in which have previously been stored $m_0, \ldots, m_{k-1}$ and the complete graphs of $\alpha_0, \ldots, \alpha_{l-1}$. The computer may then refer to this device at any point in the computation to transfer to its working "registers" either an m_i or a value $\alpha_j(p)$. Since the computation of a value $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha})$ must be finite, only finitely many values of each argument are actually used. Thus mechanically calculable functionals are continuous.

1. Primitive Recursion

We examine first the class of primitive recursive functionals. We shall show that this class includes many familiar functionals but fails to exhaust the class of mechanically calculable functionals. Although in this section we are concerned only with total functionals, we state some of the definitions with '$\simeq$' rather than '$=$' for future application to partial functionals.

1.1 Definition. For any k, l, and n, any $i < k$ and $j < l$, and any $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$,

(i) (*the initial functionals*)

$$\mathsf{Cs}_n^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = n, \qquad \mathsf{Pr}_i^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = m_i,$$

$$\mathsf{Sc}_i^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = m_i + 1, \quad \text{and} \quad \mathsf{Ap}_{i,j}^{k,l}(\mathbf{m}, \boldsymbol{\alpha}) = \alpha_j(m_i);$$

(ii) (*functional composition*) for any k' and any functionals $\mathsf{G}, \mathsf{H}_0, \ldots, \mathsf{H}_{k'-1}$, $\mathsf{FCmp}_{k'}^{k,l}(\mathsf{G}, \mathsf{H}_0, \ldots, \mathsf{H}_{k'-1})$ is the functional F of rank (k, l) such that

(a) if G is of rank (k', l) and $\mathsf{H}_0, \ldots, \mathsf{H}_{k'-1}$ are all of rank (k, l), then

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{G}(\mathsf{H}_0(\mathbf{m}, \boldsymbol{\alpha}), \ldots, \mathsf{H}_{k'-1}(\mathbf{m}, \boldsymbol{\alpha}), \boldsymbol{\alpha});$$

(b) otherwise, $\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq 0$;

(iii) (*primitive recursion*) for any functionals G and H, $\mathrm{Rec}^{k+1,l}(\mathsf{G},\mathsf{H})$ is the functional F of rank $(k+1, l)$ such that

(a) if G is of rank (k, l) and H is of rank $(k+2, l)$, then $\mathsf{F}(0, \mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha})$, and for all p,

$$\mathsf{F}(p+1, \mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{H}(\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}), p, \mathbf{m}, \boldsymbol{\alpha});$$

(b) otherwise, $\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq 0$.

1.2 Definition. The class Prf of *primitive recursive functionals* is the smallest class of total functionals which contains the initial functionals and is closed under functional composition and primitive recursion.

Note that Prf is inductively defined by closure under finitary functions. By induction over Prf it follows that every primitive recursive functional is total and mechanically calculable: clearly this is true for the initial functionals and these properties are preserved by functional composition and primitive recursion.

1.3 Examples. The *addition* function $(+)$ is defined by the equations $0 + m = n$ and $(p+1) + m = (p+m) + 1$. A simple calculation shows that

$$+ = \mathrm{Rec}^{2,0}(\mathsf{Pr}_0^{1,0}, \mathsf{Sc}_0^{3,0})$$

and is thus primitive recursive. *Multiplication* satisfies $0 \cdot m = 0$ and $(p+1) \cdot m = p \cdot m + m$, so that

$$\cdot = \mathrm{Rec}^{2,0}(\mathrm{Cs}_0^{1,0}, \mathrm{FCmp}_2^{3,0}(+, \mathsf{Pr}_0^{3,0}, \mathsf{Pr}_2^{3,0}))$$

and is thus primitive recursive. The *exponential* function $\exp(p, m) = m^p$ satisfies $\exp(0, m) = 1$ and $\exp(p+1, m) = \exp(p, m) \cdot m$ and is similarly shown to be primitive recursive. The *factorial* function (!) satisfies $0! = 1$ and $(p+1)! = p!(p+1)$ and is primitive recursive. Let

$$\mathrm{sg}^+(p) = \begin{cases} 0, & \text{if } p = 0; \\ 1, & \text{if } p > 0; \end{cases} \quad \text{and} \quad \mathrm{sg}^-(p) = \begin{cases} 1, & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

Then

$$\mathrm{sg}^+ = \mathrm{Rec}^{1,0}(\mathrm{Cs}_0^{0,0}, \mathrm{Cs}_1^{2,0}) \quad \text{and} \quad \mathrm{sg}^- = \mathrm{Rec}^{1,0}(\mathrm{Cs}_1^{0,0}, \mathrm{Cs}_0^{2,0}),$$

so both are primitive recursive. Let f be the primitive recursive function $\mathrm{Rec}^{1,0}(\mathrm{Cs}_0^{0,0}, \mathsf{Pr}_1^{2,0})$ so that $f(0) = 0$ and $f(p+1) = p$ (the *predecessor* function). Then if we set

$$g = \mathrm{Rec}^{2,0}(\mathsf{Pr}_1^{1,0}, \mathrm{FCmp}^{3,0}(f, \mathsf{Pr}_0^{3,0})),$$

it is straightforward to check that

$$g(p, m) = \begin{cases} m - p, & \text{if} \quad m \geqslant p; \\ 0, & \text{otherwise.} \end{cases}$$

$g(p, m)$ is usually written $m \dot{-} p$.

We call a relation R *primitive recursive* just in case its characteristic functional K_R is primitive recursive. Then $\mathsf{K}_\leqslant(m, p) = \mathrm{sg}^+(m \dot{-} p)$, $\mathsf{K}_\geqslant(m, p) = \mathrm{sg}^+(p \dot{-} m)$, $\mathsf{K}_<(m, p) = \mathrm{sg}^-(p \dot{-} m)$, $\mathsf{K}_>(m, p) = \mathrm{sg}^-(m \dot{-} p)$, and $\mathsf{K}_=(m, p) = \mathrm{sg}^+(\mathsf{K}_\leqslant(m, p) + \mathsf{K}_\geqslant(m, p))$ so these relations are all primitive recursive. Furthermore, if R and S are primitive recursive relations of the same rank, then

$$\mathsf{K}_{\mathsf{R}\cup\mathsf{S}}(\mathbf{m}, \boldsymbol{\alpha}) = \mathsf{K}_\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \cdot \mathsf{K}_\mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}),$$

$$\mathsf{K}_{\mathsf{R}\cap\mathsf{S}}(\mathbf{m}, \boldsymbol{\alpha}) = \mathrm{sg}^+(\mathsf{K}_\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) + \mathsf{K}_\mathsf{S}(\mathbf{m}, \boldsymbol{\alpha})), \quad \text{and}$$

$$\mathsf{K}_{\sim\mathsf{R}}(\mathbf{m}, \boldsymbol{\alpha}) = 1 \dot{-} \mathsf{K}_\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha})$$

so that the class of primitive recursive relations of a given rank forms a Boolean algebra.

Our next aim is to show that the sequence coding and decoding functions of § I.1.4 are primitive recursive. To this end we establish some further closure properties of the classes of primitive recursive functionals and relations.

1.4 Definition. For any k, l, k', and l', any functionals $\mathsf{G}, \mathsf{G}_0, \ldots, \mathsf{G}_{k'}$, and any relations $\mathsf{R}_0, \ldots, \mathsf{R}_{k'-1}$, and S,

(i) (*expansion*) if G has rank (k, l), then $\mathrm{Ex}_{k',l'}(\mathsf{G})$ is the functional F of rank $(k + k', l + l')$ such that

$$\mathsf{F}(\mathbf{m}, \mathbf{n}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \simeq \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha});$$

(ii) (*bounded search*) if G has rank $(k + 2, l)$, then $\mathrm{Bs}(\mathsf{G})$ is the functional F of rank $(k + 1, l)$ such that

$$\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \begin{cases} q, & \text{if} \quad q < p,\ \mathsf{G}(q, \mathbf{m}, \boldsymbol{\alpha}) \simeq 0, \quad \text{and} \\ & \qquad (\forall r < q)(\exists n > 0) . \mathsf{G}(r, \mathbf{m}, \boldsymbol{\alpha}) \simeq n; \\ p, & \text{if} \quad (\forall q < p)(\exists n > 0) . \mathsf{G}(q, \mathbf{m}, \boldsymbol{\alpha}) \simeq n; \end{cases}$$

we write

$$\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \text{"least"} \quad q < p . \mathsf{G}(q, \mathbf{m}, \boldsymbol{\alpha}) \simeq 0;$$

(iii) (*definition by cases*) if $\mathsf{G}_0, \ldots, \mathsf{G}_{k'}$, $\mathsf{R}_0, \ldots, \mathsf{R}_{k'-1}$ all have rank (k, l) and for any $(\mathbf{m}, \boldsymbol{\alpha}) \in {}^{k,l}\omega$ there is at most one $i < k'$ such that $\mathsf{R}_i(\mathbf{m}, \boldsymbol{\alpha})$, then $\mathrm{Cases}_{k'}(\mathsf{G}_0, \ldots, \mathsf{G}_{k'}, \mathsf{R}_0, \ldots, \mathsf{R}_{k'-1})$ is the functional F of rank (k, l) such that

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) \simeq \begin{cases} \mathsf{G}_0(\mathbf{m}, \boldsymbol{\alpha}), & \text{if} \quad \mathsf{R}_0(\mathbf{m}, \boldsymbol{\alpha}); \\ \quad \vdots & \\ \mathsf{G}_{k'-1}(\mathbf{m}, \boldsymbol{\alpha}), & \text{if} \quad \mathsf{R}_{k'-1}(\mathbf{m}, \boldsymbol{\alpha}); \\ \mathsf{G}_{k'}(\mathbf{m}, \boldsymbol{\alpha}), & \text{otherwise} \quad (\forall i < k' \sim \mathsf{R}_i(\mathbf{m}, \boldsymbol{\alpha})); \end{cases}$$

(iv) (*relational composition*) if $\mathsf{G}_0, \ldots, \mathsf{G}_{k'-1}$ have rank (k, l) and S has rank (k', l), then $\mathrm{RCmp}_{k'}(\mathsf{S}, \mathsf{G}_0, \ldots, \mathsf{G}_{k'-1})$ is the relation R of rank (k, l) such that

$$\mathsf{R}(\mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow \mathsf{S}(\mathsf{G}_0(\mathbf{m}, \boldsymbol{\alpha}), \ldots, \mathsf{G}_{k'-1}(\mathbf{m}, \boldsymbol{\alpha}), \boldsymbol{\alpha});$$

(v) (*bounded quantification*) if S is of rank $(k+2, l)$, then $\exists^0_<(\mathsf{S})$ and $\forall^0_<(\mathsf{S})$ are the relations P and Q of rank $(k+1, l)$ such that

$$\mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\exists q < p)\mathsf{S}(q, \mathbf{m}, \boldsymbol{\alpha}), \quad \text{and}$$
$$\mathsf{Q}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow (\forall q < p)\mathsf{S}(q, \mathbf{m}, \boldsymbol{\alpha}).$$

1.5 Theorem. *The class of primitive recursive functionals and relations is closed under expansion, bounded search, definition by cases, relational composition, and bounded quantification.*

Proof. (i) Clearly any expansion of an initial functional is still an initial functional. Any expansion of $\mathrm{FCmp}(\mathsf{G}, \mathsf{H}_0, \ldots, \mathsf{H}_{k'-1})$ is $\mathrm{FCmp}(\mathsf{G}', \mathsf{H}'_0, \ldots, \mathsf{H}'_{k'-1})$ for suitable expansions G' and H'_i of G and H_i. Similarly, any expansion of $\mathrm{Rec}(\mathsf{G}, \mathsf{H})$ is $\mathrm{Rec}(\mathsf{G}', \mathsf{H}')$ for suitable G' and H'. Hence by induction the expansion of any primitive recursive functional is primitive recursive.

(ii) If G is primitive recursive, then so is F defined by:

$$\mathsf{F}(0, \mathbf{m}, \boldsymbol{\alpha}) = 0;$$
$$\mathsf{F}(p+1, \mathbf{m}, \boldsymbol{\alpha}) = \mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) + \mathrm{sg}^+(\mathsf{G}(\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}), \mathbf{m}, \boldsymbol{\alpha})).$$

We leave to the reader the amusing verification that F is $\mathrm{Bs}(\mathsf{G})$.

(iii) Suppose that $\mathsf{G}_0, \ldots, \mathsf{G}_{k'}$ and $\mathsf{R}_0, \ldots, \mathsf{R}_{k'-1}$ satisfy the hypothesis for definition by cases. Then the F defined there is also given by

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) = [\mathsf{G}_0(\mathbf{m}, \boldsymbol{\alpha}) \cdot \mathrm{sg}^-(\mathsf{K}_{\mathsf{R}_0}(\mathbf{m}, \boldsymbol{\alpha}))] + \cdots + [\mathsf{G}_{k'-1}(\mathbf{m}, \boldsymbol{\alpha}) \cdot \mathrm{sg}^-(\mathsf{K}_{\mathsf{R}_{k'-1}}(\mathbf{m}, \boldsymbol{\alpha}))]$$
$$+ [\mathsf{G}_{k'}(\mathbf{m}, \boldsymbol{\alpha}) \cdot \mathsf{K}_{\mathsf{R}_0}(\mathbf{m}, \boldsymbol{\alpha}) \cdot \cdots \cdot \mathsf{K}_{\mathsf{R}_{k'-1}}(\mathbf{m}, \boldsymbol{\alpha})],$$

and is thus seen to be primitive recursive.

(iv) If $\mathsf{R} = \mathrm{RCmp}(\mathsf{S}, \mathsf{G}_0, \ldots, \mathsf{G}_{k'-1})$, then $\mathsf{K}_\mathsf{R} = \mathrm{FCmp}(\mathsf{K}_\mathsf{S}, \mathsf{G}_0, \ldots, \mathsf{G}_{k'-1})$; so if $\mathsf{S}, \mathsf{G}_0, \ldots,$ and $\mathsf{G}_{k'-1}$ are all primitive recursive, so is R.

(v) Let P, Q, and S be as in the definition. Then

$$\mathsf{P}(p, \mathbf{m}, \boldsymbol{\alpha}) \leftrightarrow [\text{``least''}\ q < p\,.\,\mathsf{S}(q, \mathbf{m}, \boldsymbol{\alpha})] < p,$$

and $\mathsf{Q} = {\sim}\exists^0_<({\sim}\mathsf{S})$. □

These results will be used, usually without reference, to justify the claim that some explicitly defined functional or relation is primitive recursive. For example, if G, H, I, R, S, and T are primitive recursive, and F is defined by:

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) = \begin{cases} m_0 + \alpha_2(m_1), & \text{if} \quad (\exists q < s)\mathsf{R}(q, \mathsf{G}(q, \mathbf{m}, \boldsymbol{\alpha}), \alpha_4); \\ \alpha_1(\text{``least''}\ q < \mathsf{H}(m_3, \alpha_0, \alpha_2) \\ \qquad [q \cdot m_1 \geq \mathsf{I}(m_2, \alpha_3)]), & \text{if} \quad \mathsf{S}(\mathbf{m}, \boldsymbol{\alpha}) \wedge \neg\mathsf{T}(m_0, \boldsymbol{\alpha}); \\ 0, & \text{otherwise}; \end{cases}$$

then a succession of applications of the clauses of Theorem 1.5 together with the remarks preceding it shows that F is primitive recursive.

As a first application, we obtain the primitve recursiveness of the relation "m divides p" and the function $\langle p_m : m \in \omega \rangle$ which enumerates the prime numbers:

$$m \text{ divides } p \leftrightarrow (\exists q < p + 1)(q \cdot m = p);$$

$$p_0 = 2, \quad \text{and}$$

$$p_{m+1} = (\text{``least''}\ q < p_m! + 2)[p_m < q \wedge (\neg\exists r < q)(1 < r \wedge r \text{ divides } q)].$$

1.6 Corollary. *The sequence coding and decoding functions and the set* Sq *of* §I.1.4 *are all primitive recursive.*

Proof. For the functions, this is immediate from their definitions. Also $s \in \mathrm{Sq} \leftrightarrow (\forall i < s)[p_i \text{ divides } s \rightarrow i < \lg(s)]$. □

For any functional F of rank $(k + 1, l)$ we set

$$\bar{\mathsf{F}}(0, \mathbf{m}, \boldsymbol{\alpha}) = \langle\ \rangle \quad \text{and}$$

$$\bar{\mathsf{F}}(p + 1, \mathbf{m}, \boldsymbol{\alpha}) = \bar{\mathsf{F}}(p, \mathbf{m}, \boldsymbol{\alpha}) * \langle \mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) \rangle.$$

Thus $\bar{\mathsf{F}}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \langle \mathsf{F}(0, \mathbf{m}, \boldsymbol{\alpha}), \ldots, \mathsf{F}(p - 1, \mathbf{m}, \boldsymbol{\alpha}) \rangle$. From the definition and the preceding Corollary it is clear that if F is primitive recursive, so is $\bar{\mathsf{F}}$. Furthermore, for any $q > p$, $\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq (\bar{\mathsf{F}}(q, \mathbf{m}, \boldsymbol{\alpha}))_p$ so the primitive recursiveness of $\bar{\mathsf{F}}$ implies that of F. The same argument shows that the functional $\mathsf{H}(p, \alpha) = \bar{\alpha}(p)$ is primitive recursive.

1.7 Definition (*course-of-values recursion*). For any k and l, and any functional G of rank $(k+2, l)$, CvRec(G) is the functional F of rank $(k+1, l)$ such that

$$\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) \simeq \mathsf{G}(\bar{\mathsf{F}}(p, \mathbf{m}, \boldsymbol{\alpha}), p, \mathbf{m}, \boldsymbol{\alpha}).$$

1.8 Theorem. *The class of primitive recursive functionals is closed under course-of-values recursion.*

Proof. If G is of rank $(k+2, l)$ and $\mathsf{F} = \mathrm{CvRec}(\mathsf{G})$, then

$$\bar{\mathsf{F}}(p+1, \mathbf{m}, \boldsymbol{\alpha}) = \bar{\mathsf{F}}(p, \mathbf{m}, \boldsymbol{\alpha}) * \langle \mathsf{G}(\bar{\mathsf{F}}(p, \mathbf{m}, \boldsymbol{\alpha}), p, \mathbf{m}, \boldsymbol{\alpha})\rangle.$$

Hence, if G is primitive recursive, so is $\bar{\mathsf{F}}$. But then by the preceding remarks, also F is primitive recursive. □

In applying these theorems to show that a particular relation R is primitive recursive, we must formally work with K_R and show this to be a primitive recursive functional. Usually, however, it is more perspicuous to describe directly recursive conditions on R. For example, the condition

$$R(p) \leftrightarrow p = 0 \vee (\exists q < p)[q + 7 = p \wedge R(q)]$$

is equivalent to

$$\mathsf{K}_R(p) = \begin{cases} 0, & \text{if} \quad p = 0; \\ F(p, \bar{\mathsf{K}}_R(p)), & \text{if} \quad p > 0; \end{cases}$$

where

$$F(p, s) = \begin{cases} 0, & \text{if} \quad (\exists q < p)[q + 7 = p \wedge (s)_q = 0]; \\ 1, & \text{otherwise.} \end{cases}$$

In such cases we shall leave to the reader the translation of the conditions on R to conditions on K_R.

Another technique we shall use frequently is to give definitions of the form

$$\mathsf{F}(i, j, \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle) = \mathsf{G}(\alpha_j(m_i), \langle \mathbf{m} \rangle, \langle \boldsymbol{\alpha} \rangle).$$

This should be taken as an abbreviation for

$$\mathsf{F}(i, j, s, \gamma) = \mathsf{G}(((\gamma((s)_i))_j, s, \gamma)$$

so that F is defined for all arguments, not only those of the form $\langle \mathbf{m} \rangle$ and $\langle \boldsymbol{\alpha} \rangle$. Since the decoding functions are primitive recursive, if G is primitive recursive, so is F.

We turn now to the assertion that not all mechanically calculable functionals are primitive recursive. Our method derives from the description of mechanical computability in terms of an idealized computer. We shall in effect specify a particular computer and a "programming language" which suffices to write programs for computing all primitive recursive functionals. We can then exhibit a functional which is mechanically calculable but cannot be "programmed" in this language and hence is not primitive recursive.

The "language" is simply a set $\mathrm{Pri} \subseteq \omega$, members of which we call *primitive recursive indices*. To each $a \in \mathrm{Pri}$ is assigned a primitive recursive functional $[a]$ by interpreting a as a program for an idealized computer whose basic operations correspond to the clauses of Definition 1.1.

1.9 Definition. Pri is the smallest subset of ω such that for all k, l, and n, all $i < k$, and all $j < l$,
(0) $\langle 0, k, l, 0, n\rangle$, $\langle 0, k, l, 1, i\rangle$, $\langle 0, k, l, 2, i\rangle$, and $\langle 0, k, l, 3, i, j\rangle$ all belong to Pri;
(1) for any k' and any $b, c_0, \ldots, c_{k'-1} \in \mathrm{Pri}$, $\langle 1, k, l, b, c_0, \ldots, c_{k'-1}\rangle \in \mathrm{Pri}$;
(2) for any $b, c \in \mathrm{Pri}$, $\langle 2, k+1, l, b, c\rangle \in \mathrm{Pri}$.

This is clearly a monomorphic inductive definition, so by Theorem I.3.5 there exists a unique map $[\cdot]$ from Pri into the class of functionals such that

(0) $[\langle 0, k, l, 0, n\rangle] = \mathsf{Cs}_n^{k,l}$;

$[\langle 0, k, l, 1, i\rangle] = \mathsf{Pr}_i^{k,l}$;

$[\langle 0, k, l, 2, i\rangle] = \mathsf{Sc}_i^{k,l}$;

$[\langle 0, k, l, 3, i, j\rangle] = \mathsf{Ap}_i^{k,l}$;

(1) $[\langle 1, k, l, b, c_0, \ldots, c_{k'-1}\rangle] = \mathrm{FCmp}_{k'}^{k,l}([b], [c_0], \ldots, [c_{k'-1}])$;
(2) $[\langle 2, k+1, l, b, c\rangle] = \mathrm{Rec}^{k,l}([b], [c])$.

1.10 Theorem. $\mathrm{Prf} = \{[a]: a \in \mathrm{Pri}\}$.

Proof. For the inclusion ($\subseteq$) we observe that $\{[a]: a \in \mathrm{Pri}\}$ clearly contains the initial functions and is closed under composition and primitive recursion. For ($\supseteq$) consider $\{a: a \in \mathrm{Pri} \wedge [a]$ is primitive recursive$\}$. This set satisfies clauses (0)–(2) of Definition 1.9 and thus includes Pri. □

For each k and l, set

$$\mathsf{Ev}^{k,l}(a,\mathbf{m},\boldsymbol{\alpha}) = \begin{cases} [a](\mathbf{m},\boldsymbol{\alpha}), & \text{if} \quad a \in \mathrm{Pri} \wedge (a)_1 = k \wedge (a)_2 = l; \\ 0, & \text{otherwise.} \end{cases}$$

$\mathsf{Ev}^{k,l}$ is called an *evaluation function.*

1.11 Theorem. *for all $k > 0$ and all l, $\mathsf{Ev}^{k,l}$ is mechanically calculable but not primitive recursive.*

Proof. Suppose first that $\mathsf{Ev}^{k+1,l}$ were primitive recursive. Then if

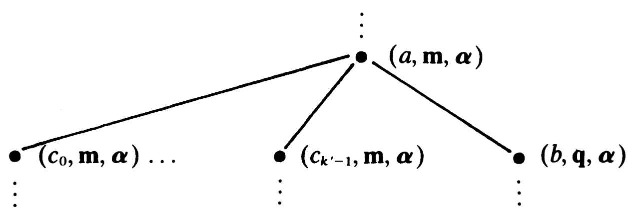
$$\mathsf{F}(a,\mathbf{m},\boldsymbol{\alpha}) = \mathsf{Ev}^{k+1,l}(a,a,\mathbf{m},\boldsymbol{\alpha}) + 1$$

also F is primitive recursive. Hence by Theorem 1.10, $\mathsf{F} = [b]$ for some $b \in \mathrm{Pri}$. But then

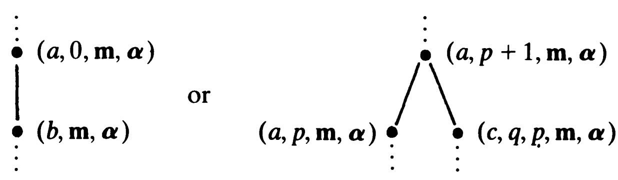
$$\mathsf{F}(b,\mathbf{m},\boldsymbol{\alpha}) = \mathsf{Ev}^{k+1,l}(b,b,\mathbf{m},\boldsymbol{\alpha}) + 1 = [b](b,\mathbf{m},\boldsymbol{\alpha}) + 1 = \mathsf{F}(b,\mathbf{m},\boldsymbol{\alpha}) + 1,$$

a contradiction.

To see that $\mathsf{Ev}^{k,l}$ is mechanically calculable we examine the notion of a *computation tree.* Such a tree may be thought of as a schematic representation of the action of the idealized computer in calculating a value of a given functional. Each computation tree has a top node x_0. Each node x has 0 or more *immediate predecessors* which lie just below x. Each node x is labeled with a triple $(a,\mathbf{m},\boldsymbol{\alpha})$. x is said to be *evaluated* when $[a](\mathbf{m},\boldsymbol{\alpha})$ is computed. If a is an index for one of the initial functions, then x has no immediate predecessors. If $a = \langle 1,k,l,b,c_0,\ldots,c_{k'-1}\rangle$, then x has $k'+1$ immediate predecessors labeled as follows:



where $\mathbf{q} = (q_0,\ldots,q_{k'-1})$ and for all $i < k'$, $[c_i](\mathbf{m},\boldsymbol{\alpha}) = q_i$. If $a = \langle 2, k+1, l, b, c\rangle$, then x has either 1 or 2 immediate predecessors:



where $[a](p, \mathbf{m}, \boldsymbol{\alpha}) = q$. Thus the labels on the immediate predecessors of x correspond to the *subcomputations* necessary to evaluate the label at x.

For any triple $(a, \mathbf{m}, \boldsymbol{\alpha})$ with $a \in \text{Pri}$, $(a)_1 = k$, and $(a)_2 = l$, we generate and label a computation tree as follows. The top node is labeled $(a, \mathbf{m}, \boldsymbol{\alpha})$. Depending on a, the appropriate number of immediate predecessors of x_0 are constructed and all but possibly the right-most one labeled. The number of immediate predecessors of these may then be determined and so on. If node x lies below node y, then the index at x is not greater than the index at y, and if they are equal, then the first argument at x is strictly less than the first argument at y. Hence each branch terminates with a node labeled with an index for one of the initial functions. This is immediately evaluable. If at some stage all nodes at a given level except the right-most one have been evaluated, then this one may be labeled. When all immediate predecessors of a given node x have been evaluated, then x may be evaluated and has the value of the right-most immediate predecessor as its value. In any application of composition, $k' < a$. Hence by the Infinity Lemma (I.2.3) the tree is finite and this process terminates after a finite number of steps with an evaluation of the top node x_0 and hence with the value $\mathsf{Ev}^{k,l}(a, \mathbf{m}, \boldsymbol{\alpha})$.

A mechanical procedure for calculating $\mathsf{Ev}^{k,l}(a, \mathbf{m}, \boldsymbol{\alpha})$ now goes as follows. Determine first whether or not $a \in \text{Pri}$, $(a)_1 = k$, and $(a)_2 = l$. This is possible by Corollary 1.6 and Exercise 1.16. If not, the value is 0. If so, construct the computation tree as described above and read off the value of the top node. □

1.12–1.18 Exercises

1.12. Find explicitly primitive recursive indices for addition and multiplication.

1.13. Show that any primitive recursive function has infinitely many indices.

1.14. Show that if G is primitive recursive and

$$\mathsf{F}(p, \mathbf{m}, \boldsymbol{\alpha}) = \mathsf{G}(p, \mathbf{m}, \boldsymbol{\alpha}, \lambda q \,.\, \mathsf{F}_p(q, \mathbf{m}, \boldsymbol{\alpha}))$$

where

$$\mathsf{F}_p(q, \mathbf{m}, \boldsymbol{\alpha}) = \begin{cases} \mathsf{F}(q, \mathbf{m}, \boldsymbol{\alpha}), & \text{if} \quad q < p; \\ 0, & \text{otherwise}; \end{cases}$$

then also F is primitive recursive.

1.15. Show that if G and H are both primitive recursive and

$$\mathsf{F}(\mathbf{m}, \boldsymbol{\alpha}) = \mathsf{G}(\mathbf{m}, \boldsymbol{\alpha}, \lambda p \,.\, \mathsf{H}(p, \mathbf{m}, \boldsymbol{\alpha})),$$

then F is primitive recursive.