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A NEW METHOD OF ESTIMATING THE WEIBULL SHAPE PARAMETER

V. K. MURTHY
SYSTEM DEVELOPMENT CORPORATION
SANTA MONICA, CALIFORNIA

Contract No. F33-615-67-C-1865
Project No. 7071

OFFICE OF AEROSPACE RESEARCH

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UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This report was prepared for the Applied Mathematics Research Laboratory, Aerospace Research Laboratories, Wright-Patterson Air Force Base, by Dr. V. K. Murthy, System Development Corporation, under Contract F33-615-67-C-1865. In this report the author develops the method of random hazard functions and applies it to estimating the Weibull shape parameter.

The author wishes to thank Dr. H. Leon Harter of the Applied Mathematics Laboratory, Wright-Patterson Air Force Base, for many helpful suggestions which improved the organization of this paper.

ABSTRACT

The method of random hazard functions is applied to the case of the Weibull distribution and the following results are obtained: It is well known that, in the case of a Weibull distribution with two parameters α and β, there is no way of estimating and testing for the shape parameter β without knowledge of the scale parameter α using the usual methods based on maximum likelihood. The method of random hazard functions is now used to obtain a consistent and asymptotically normal class of estimates for β independent of any specification whatsoever on α. This result enables one to test for the randomness of the underlying failure phenomena under the Weibull setup. An illustrative example is given.

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1. INTRODUCTION*

The main object in reliability analysis is to estimate for each specified time, $t > 0$, the probability that the given item (where item denotes a component, or a subsystem of components, or a system of subsystems) survives time instant t. This function of t is usually denoted by R(t). The complement of R(t), which denotes the probability that the item failed by time t, is denoted by $F(t) = 1-R(t)$. If the random variable T denotes the observed time to failure of an item, F(t) is its distribution function, which is also referred to in the literature as "the underlying law of failures." Assuming that F(t) is absolutely continuous,

$$dF(t) = f(t)dt \tag{1}$$

where f(t)dt is the unconditional probability that the item fails during the interval (t, t + dt). The function f(t) is called the probability density of the underlying law of failures.

Z(t)dt denotes the conditional probability that the item having survived time t fails during the next interval (t, t + dt). If one uses this notation, the definition of conditional probability yields

$$Z(t)dt = \frac{f(t)dt}{1-F(t)} \ . \tag{2}$$

Solving Eq. (2) for F(t) gives

$$F(t) = 1-e^{-\int_0^t Z(t)dt} \ . \tag{3}$$

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The function $Z(t)$ is called the hazard rate. The quantities $R(t)$, $Z(t)$, and the expected value of the random variable T (representing the mean time to failure [m.t.t.f.] of the item) are called the life quality or item effectiveness parameters. It should be said at this stage that the ultimate end in reliability analysis is to estimate or establish the life quality or item effectiveness parameters.

Where the experimenter has sufficient experience with the item concerned to specify the mathematical form of the underlying law of failures, except for the knowledge of certain parameters involved, the statistical estimation procedures are called parametric. Where the form of the underlying law of failures cannot be assumed with reasonable accuracy, the applicable statistical procedures are called nonparametric.

However, there are many significant gaps in both the parametric and the nonparametric areas of reliability estimation. In the parametric area, exact methods of statistical inference are possible only under the exponential law of failures. With respect to other plausible specifications of the law of failures, such as the Weibull distribution, methods based on maximum likelihood and order statistics provide asymptotic (large-sample) statistical procedures for any given sampling plan (for example, random, truncated, and censored). These methods are usually complicated from the point of view of evaluating the estimates and nuisance parameters. The nonparametric approach in reliability is relatively new and largely unexplored.

If in Eq. (2) one regards $Z(t)$ not as a function of the real variable t but as a function of the time to failure T, then one has what is called a random hazard function. At this moment we do not go into detailed properties of the random hazard function as such. However, we will use the random hazard function to obtain a consistent and asymptotically normal class of estimators for the Weibull shape parameter β and we discover that the sampling properties of our estimate for β are asymptotically free from α. This is a significant property which the class of estimators for β given by the method of maximum likelihood does not possess even for large samples.

2. ESTIMATION OF THE WEIBULL SHAPE PARAMETER β BASED ON RANDOM HAZARD FUNCTION

Consider the Weibull distribution

$$F(t) = 1-e^{-\alpha t^{\beta}}, \alpha > 0, \beta > 0. \quad (4)$$

The cumulative hazard rate is given by

$$Y(t) = \int_0^t Z(x)dx = \alpha t^{\beta} . \quad (5)$$

Differentiating (5) with respect to t we have for the hazard rate Z(t) of the Weibull distribution

$$Z(t) = \alpha\beta t^{\beta-1} . \quad (6)$$

Now let N items be put to a life testing experiment and let $T_1, T_2, \ldots, T_N$ denote the observed times to failure of these N items. Let us assume that the underlying law of failures is Weibull given by (4). In other words $\{T_i\}$, $i=1, 2, \ldots, N$ are independently identically distributed random variables with common distribution given by (4).

Now if T is distributed according to (4)

$$E(T^K) = \alpha\beta \int_0^\infty t^K t^{\beta-1} e^{-\alpha t^\beta} dt$$

$$= \alpha^{-K/\beta} \int_0^\infty x^{K/\beta} e^{-x} dx$$

$$= \alpha^{-K/\beta} \Gamma(1+K/\beta) \quad , \qquad (7)$$

where $\Gamma(x)$ denotes the gamma function. Consider now the following random hazard function:

$$g(T) = TZ(T) \quad , \qquad (8)$$

where T is the observed time to failure and Z(T) is the random hazard function of the Weibull distribution given by (6).

Let us now consider the following random variable and investigate its properties. Define

$$\hat{\beta} = \frac{1}{N} \sum_{i=1}^{N} g(T_i) \quad . \qquad (9)$$

Before we go any further let us note that $\hat{\beta}$ is not a statistic as we cannot compute its value as soon as the observations $T_1, T_2, \ldots, T_N$ are available. We have

$$E(\hat{\beta}) = \frac{1}{N} \sum_{i=1}^{N} E(g(T_i)) = E(g(T)) \quad , \qquad (10)$$

where T is distributed according to (4). Now

$$E(g(T)) = E(TZ(T))$$
$$= E(T\alpha\beta T^{\beta-1}) \tag{11}$$
$$= \alpha\beta E(T^{\beta}) \quad .$$

Substituting for $E(T^{\beta})$ from (7) we obtain

$$E(g(T)) = \alpha\beta\ \alpha^{-1} = \beta \quad . \tag{12}$$

Thus the random variable $\hat{\beta}$ is unbiased for β.

We will now compute the variance of the random variable $\hat{\beta}$. From (9) we have

$$\text{Var}(\hat{\beta}) = \frac{1}{N}\,\text{Var}(g\ (T)) \quad . \tag{13}$$

Now

$$E(g^2(T)) = E(\alpha^2\beta^2T^{2\beta}), \tag{14}$$

$$E(\alpha^2\beta^2T^{2\beta}) = \alpha^2\beta^2\alpha^{-2}\Gamma(3)$$
$$= 2\beta^2 \quad . \tag{15}$$

Combining (12), (13) and (15) we obtain

$$\text{Var}(\hat{\beta}) = \beta^2/N \tag{16}$$

At this stage we have discovered that the random variable $\hat{\beta}$ is exactly unbiased for β and its variance β^2/N does not depend on the scale parameter α of the Weibull distribution. We will now construct a statistic β^* with the following property, viz;

$$\underset{N\to\infty}{\text{Plim}}(\hat{\beta}-\beta^*) = 0 \quad , \tag{17}$$

in other words the statistic β^* converges in probability to the random variable $\hat{\beta}$. If we prove (17), we have proved that the statistic β^* has the same asymptotic properties, in fact, has the same asymptotic distribution as the random variable $\hat{\beta}$. In fact, in view of the Lindeberg and Levy version of the central limit theorem (see Cramér [1], p. 215)

$$\sqrt{N}\,\frac{\hat{\beta}-\beta}{\beta} \xrightarrow{D} N(0,1) \quad , \tag{18}$$

in other words $\hat{\beta}$ is asymptotically normally distributed with mean β and variance β^2/N independent of α. Thus if we can define a statistic β^* suitably and show (17) for that β^* then we have produced a statistic β^* with all the desired properties (asymptotically or for large samples) for estimating the shape parameter β of the Weibull distribution.

Now define

$$\beta^* = \frac{1}{N} \sum_{i=1}^{N} T_i \, Z_N(T_i) \quad , \tag{19}$$

where

$$Z_N(T_i) = \frac{f_N(T_i)}{R_N(T_i)} \tag{20}$$

and

$$R_N(T_i) = \frac{1}{N} \;[\text{number of observations among } T_1, T_2, \ldots, T_N > T_i] \quad , \tag{21}$$

$$f_N(T_i) = \frac{B_N}{N} \sum_{j=1}^{N} K(B_N(T_j - T_i)) \quad , \tag{22}$$

and B_N is a sequence of nonnegative constants depending on the sample size N such that $B_N \to \infty$ as $N \to \infty$ and $\frac{B_N}{N} \to 0$ as $N \to \infty$, and finally $K(x)$ is a window function (see Murthy [2], [3]) satisfying

$$\begin{cases} K(x) \geq 0 \\ K(x) = K(-x) \\ \lim_{|x| \to \infty} xK(x) = 0 \\ \int_{-\infty}^{\infty} K(x)dx = 1 \quad . \end{cases} \tag{23}$$

Clearly as soon as one chooses the sequence $\{B_N\}$ and a window function $K(x)$ satisfying the above properties one can compute the statistic β^* given by (19) based on observed values of time to failure $T_1, \ldots, T_N$ of N identical items put to a life testing experiment.

Now

$$\hat{\beta}-\beta^* = \frac{1}{N} \sum_{i=1}^{N} T_i \left[Z(T_i) - Z_N(T_i)\right] \quad . \tag{24}$$

It is evident that if we can show

$$\operatorname*{Plim}_{N\to\infty} (Z(T_i) - Z_N(T_i)) = 0 \quad , \tag{25}$$

it follows from (24) that

$$\operatorname*{Plim}_{N\to\infty} (\hat{\beta}-\beta^*) = 0 \quad . \tag{26}$$

Now

$$Z_N(T_i) = \frac{f_N(T_i)}{R_N(T_i)} \quad , \tag{27}$$

where $f_N(T_i)$ and $R_N(T_i)$ are respectively given by (22) and (21) and

$$Z(T_i) = \frac{f(T_i)}{R(T_i)} \quad , \tag{28}$$

where $f(t)$ and $R(t)$ are respectively the density and reliability functions. We will establish that $Z_N(T_i)$ converges in probability to $Z(T_i)$ as the sample size $N \to \infty$ by establishing that the denominator $R_N(T_i)$ of $Z_N(T_i)$ converges in probability to $R(T_i)$ and the numerator $f_N(T_i)$ converges in probability to $f(T_i)$.

We will prove that

$$\underset{N\to\infty}{\text{Plim}} \left[F_N(T_i) - F(T_i) \right] = 0 \quad . \tag{29}$$

Now

$$F_N(T_i) = \frac{1}{N} \sum_{i=1}^{N} U(T_i - T_j) \quad , \tag{30}$$

where $U(x)$ is the Heaviside unit function

$$\begin{aligned} U(x) &= 1 \qquad x \geq 0 \\ &= 0 \qquad \text{otherwise} . \end{aligned} \tag{31}$$

We have

$$F_N(T_i) - F(T_i) = \frac{1}{N} \left\{ 1 + U(T_i - T_1) + \ldots + U(T_i - T_{i-1}) + U(T_i - T_{i+1}) + \ldots + U(T_i - T_N) - NF(T_i) \right\} . \tag{32}$$

Since $F(T_i)$ is uniformly distributed in (0,1)

$$E(F(T_i)) = \frac{1}{2} \quad . \tag{33}$$

Also since T_i and, say, T_1 are independently identically distributed

$$E(U(T_i - T_1)) = P(T_i \geq T_1) = \frac{1}{2} \quad . \tag{34}$$

Using (33) and (34) and taking expectation on both sides of (32) we discover that

$$E\left[F_N(T_i) - F(T_i) \right] = \frac{1 + \frac{1}{2}(N-1-N)}{N} = \frac{1}{2N} \tag{35}$$

and hence

$$\lim_{N \to \infty} E\left[F_N(T_i) - F(T_i) \right] = 0 \quad . \tag{36}$$

Now

$$F_N(T_i) - F(T_i) = \frac{1}{N} \left\{ \left[U(T_i - T_1) - F(T_i) \right] + \ldots + \left[U(T_i - T_{i-1}) - F(T_i) \right] + \left[U(T_i - T_{i+1}) - F(T_i) \right] + \ldots + \left[U(T_i - T_N) - F(T_i) \right] + 1 - F(T_i) \right\} . \tag{37}$$

Write

$$Z_1 = U(T_i - T_1) - F(T_i) \quad ,$$

$$Z_2 = U(T_i - T_2) - F(T_i) \quad ,$$

$$\cdots$$

$$Z_{i-1} = U(T_i - T_{i-1}) - F(T_i) \quad , \tag{38}$$

$$Z_i = U(T_i - T_{i+1}) - F(T_i) \quad ,$$

$$\cdots$$

$$Z_{N-1} = U(T_i - T_N) - F(T_i) \quad .$$

Then

$$\begin{aligned} \mathrm{Var}\left[F_N(T_i) - F(T_i)\right] &= \frac{1}{N^2}\left[\mathrm{Var}(Z_1 + Z_2 + \ldots + Z_{N-1} + 1 - F(T_i))\right] \\ &= \frac{1}{N^2}\left[\frac{1}{12} + \mathrm{Var}(Z_1 + \ldots + Z_{N-1}) \right. \\ &\qquad \left. + \mathrm{Cov}\left((1-F(T_i), Z_1 + Z_2 + \ldots + Z_{N-1})\right)\right] . \end{aligned} \tag{39}$$

Now

$$\begin{aligned} \mathrm{Var}(Z_1) &= \mathrm{Var}(U(T_i - T_1)) + \frac{1}{12} - 2\mathrm{Cov}\left[F(T_i), U(T_i - T_1)\right] \\ &= \frac{1}{4} + \frac{1}{12} - \frac{1}{6} = \frac{1}{6} \quad . \end{aligned} \tag{40}$$

$$\begin{aligned} \mathrm{Cov}(Z_1, Z_2) = \int_0^\infty \int_0^\infty \int_0^\infty &\left[U(T_i - T_1) - F(T_i)\right]\left[U(T_i - T_2) - F(T_i)\right] \\ &\times dF(T_1)\,dF(T_2)\,dF(T_i) \quad , \end{aligned} \tag{41}$$

since $E(Z_i) = 0$, $i = 1, 2, \ldots, N-1$.

Expanding (41) we obtain

$$\mathrm{Cov}(Z_1,Z_2) = \int_0^\infty \int_0^\infty \int_0^\infty U(T_i-T_1)U(T_i-T_2)dF(T_1)dF(T_2)dF(T_i) \quad (42)$$

$$-\int_0^\infty \int_0^\infty \int_0^\infty F(T_i)U(T_i-T_2)dF(T_1)dF(T_2)dF(T_i)$$

$$-\int_0^\infty \int_0^\infty \int_0^\infty F(T_i)U(T_i-T_1)dF(T_1)dF(T_2)dF(T_i)$$

$$+\int_0^\infty \int_0^\infty \int_0^\infty F^2(T_i)dF(T_1)dF(T_2)dF(T_i)$$

$$= 1-E(F(\mathrm{Max}(T_1,T_2))) - \frac{1}{3} - \frac{1}{3} + \frac{1}{3} = \frac{2}{3} - E(F(\max(T_1,T_2))) \quad .$$

Now

$$E(F(\max(T_1,T_2))) = \int_0^\infty F(x)2F(x)f(x)dx = 2\int_0^\infty F^2(x)f(x)dx = \frac{2}{3} \quad . \quad (43)$$

Combining (42) and (43) we discover that

$$\mathrm{Cov}(Z_1, Z_2) = 0 \quad . \tag{44}$$

Similarly

$$\left.\begin{aligned} &\mathrm{Var}(Z_i) = \frac{1}{6} \ , \qquad i = 1,2,\ldots,N-1, \\ &\mathrm{Cov}(Z_i, Z_j) = 0 \qquad i \neq j \quad . \end{aligned}\right\} \tag{45}$$

Finally

$$\begin{aligned} \mathrm{Cov}(Z_1,(1-F(T_i)) &= E(Z_1(1-F(T_i)) \qquad \left[\text{since } E(Z_1) = 0\right] \\ &= E(\left[1-F(T_i)\right]\left[U(T_i-T_1)-F(T_i)\right]) \\ &= E(U(T_i-T_1)-F(T_i)U(T_i-T_1)-F(T_i)+F^2(T_i)) \\ &= \frac{1}{2} - \frac{1}{3} - \frac{1}{2} + \frac{1}{3} = 0 \quad . \end{aligned} \tag{46}$$

Similarly

$$\mathrm{Cov}\left[Z_j,(1-F(T_i))\right] = 0 \ , \qquad j = 1,2,\ldots,N-1 \quad . \tag{47}$$

Combining (39), (45) and (47) we discover that

$$\begin{aligned} \mathrm{Var}\left[F_N(T_i)-F(T_i)\right] &= \frac{1}{N^2}\left[\frac{N-1}{6} + \frac{1}{12}\right] \\ &= \frac{2N-1}{12N^2} \quad . \end{aligned} \tag{48}$$

Hence

$$\lim_{N\to\infty} \mathrm{Var}\left[F_N(T_i)-F(T_i)\right] = 0 \quad . \qquad (49)$$

Combining (36) and (49) we finally discover

$$\operatorname{Plim}_{N\to\infty} (F_N(T_i)-F(T_i)) = 0 \quad ,$$

which implies

$$\operatorname{Plim}_{N\to\infty} (R_N(T_i)-R(T_i)) = 0 \quad . \qquad (50)$$

To complete the proof of (25) we now have only to show that $f_N(T_i)$ converges in probability to $f(T_i)$.

In other words we have to show that

$$\operatorname{Plim}_{N\to\infty} (f_N(T_i)-f(T_i)) = 0 \qquad (51)$$

where

$$f_N(T_i) = \frac{B_N}{N}\sum_{j=1}^{N} K(B_N(T_j-T_i)) \quad . \qquad (52)$$

Let

$$X_1 = K(B_N(T_1-T_i)) \quad ,$$

$$\cdots$$

$$X_{i-1} = K(B_N(T_{i-1}-T_i)) \quad , \qquad (53)$$

$$X_i = K(B_N(T_{i+1}-T_i)) \quad ,$$

$$\cdots$$

$$X_{N-1} = K(B_N(T_N-T_i)) \quad .$$

Then $f_N(T_i)$ can be written as

$$f_N(T_i) = \frac{B_N K(0)}{N} + \frac{B_N}{N} \sum_{j=1}^{N-1} X_j \quad . \tag{54}$$

Consider now

$$\begin{aligned} E(f_N(T_i)-f(T_i)) &= E\left[\frac{B_N K(0)-f(T_i)}{N} + \frac{B_N}{N} \sum_{j=1}^{N-1} (X_j - f(T_i))\right] \\ &= \frac{B_N}{N} K(0) - \frac{E(f(T_i))}{N} + \frac{B_N}{N} \sum_{j=1}^{N-1} E(X_j - f(T_i)) \\ &= \frac{B_N}{N} K(0) - \frac{E(f(T_i))}{N} + \frac{N-1}{N} B_N E(X_1 - f(T_i)) \quad . \end{aligned} \tag{55}$$

Since by assumption

$$\frac{B_N}{N} \to 0 \text{ as } N \to \infty \text{ and } K(0) \text{ is a constant,}$$

$$\lim_{N \to \infty} \frac{K(0)B_N}{N} = 0 \quad . \tag{56}$$

Now

$$\begin{aligned} \frac{1}{N} E(f(T_i)) &= \frac{1}{N} \int_0^\infty f(t)f(t)dt \\ &= \frac{1}{N} \int_0^\infty f^2(t)dt \quad . \end{aligned}$$

Assuming that

$$\int_0^\infty f^2(t)dt < \infty \quad , \tag{57}$$

which is certainly true in the case of the Weibull distribution, we obtain

$$\lim_{N \to \infty} \frac{1}{N} E(f(T_i)) = 0 \quad . \tag{58}$$

Now

$$E(B_N(X_1-f(T_i))) = \int_0^\infty \int_0^\infty B_N[K(B_N(T_1-T_i))-f(T_i)]dF(T_1)dF(T_i)$$

$$= \int_0^\infty dF(T_i) \int_0^\infty B_N\Big[K(B_N(T_1-T_i))-f(T_1)\Big]f(T_1)dT_1 \quad . \tag{59}$$

It now follows from a lemma (see Murthy [3], p. 1028) that

$$\lim_{N \to \infty} \int_0^\infty B_N[K(B_N(T_1-T_i))-f(T_i)] f(T_1)dT_1 = 0 \quad . \tag{60}$$

Combining (59) and (60) we obtain

$$\lim_{N \to \infty} E(B_N(X_1-f(T_i)) = 0 \quad . \tag{61}$$

Finally combining (55), (56), (58) and (61) we discover that

$$\lim_{N \to \infty} E(f_N(T_i)-f(T_i)) = 0 \quad . \tag{62}$$

Appealing to the lemma (see Murthy [3], p. 1028) repeatedly and carrying out a straightforward but laborious computation, one discovers that

$$\lim_{N \to \infty} \mathrm{Var}(f_N(T_i)-f(T_i)) = 0 \quad . \tag{63}$$

Combining (62) and (63) we obtain

$$\operatorname*{Plim}_{N \to \infty} (f_N(T_i)-f(T_i)) = 0 \quad . \tag{64}$$

In view of (50), (64) and (25) we finally prove that

$$\operatorname*{Plim}_{N \to \infty} (\hat{\beta}-\beta^*) = 0 \quad . \tag{65}$$

Hence

THEOREM
The sequence of statistics $\sqrt{N}\,\frac{\beta^*-\beta}{\beta}$ converges in distribution to a normal distribution with zero mean and unit variance as the sample size N tends to infinity.

3. TEST FOR EXPONENTIALITY IN THE WEIBULL SETUP

The Weibull distribution given by (4) reduces to the exponential distribution if the shape parameter $\beta = 1$.

To test

$$H_0 \colon \beta = 1 \quad , \tag{66}$$

we use the theorem just established as follows. Suppose that the sample size N is large enough for the normal approximation proved in the theorem to be valid. Choose a sequence $\{B_N\}$ and a window $K(x)$. Compute β^* and $(\beta^*-1)\sqrt{N}$. If the latter value exceeds the $\alpha\%$ value of a normal distribution with zero mean and unit variance reject the hypotheses H_0.

4. AN EXAMPLE

Consider the estimate β^* of β given by (19), namely:

$$\beta^* = \frac{1}{N} \sum_{i=1}^{N} T_i Z_N(T_i) \quad , \tag{67}$$

where $T_1, T_2, \ldots, T_N$ are the observed times to failure and

$$Z_N(T_i) = \frac{f_N(T_i)}{R_N(T_i)} \quad , \tag{68}$$

$$R_N(T_i) = \frac{1}{N} \left[\text{number of observations among } T_1, T_2, \ldots, T_N > T_i \right] \quad , \tag{69}$$

$$f_N(T_i) = \frac{B_N}{N} \sum_{j=1}^{N} K(B_N(T_j - T_i)) \quad , \tag{70}$$

and B_N is a sequence of nonnegative constants depending on the sample size N such that

$$\lim_{N \to \infty} \frac{B_N}{N} = 0 \quad , \tag{71}$$

and finally the window K(x) satisfies (23).

To illustrate the use of the statistic to test the hypothesis H_0 given by (66), let us consider

$$B_N = \log N \tag{72}$$

$$K(x) = \frac{1}{2}, \; |x| \le 1 \quad ,$$

$$= 0 \text{ otherwise;}$$

clearly B_N given by (72) satisfied (71) and K(x) defined by (73) satisfies the window condition (23).

If we substitute (72) and (73) in (67) the statistic β^* in this illustration reduces to

$$\beta^* = \frac{\log N}{2N} \sum_{i=1}^{N} T_i \left\{ \frac{1 + \left[\text{number of observations } T_j (j \neq i), \text{ such that } |T_j - T_i| < \frac{1}{\log N}\right]}{\left[\text{number of observations } T_j \text{ such that } T_j > T_i\right]} \right\}$$

As soon as the observations T_1, T_2,...,T_N are available the statistic (74) can at once be computed and the procedure for rejection or acceptance of the hypothesis H_0 can be carried out.

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13. ABSTRACT

The method of random hazard functions is applied to the case of the Weibull distribution and the following results are obtained: It is well known that, in the case of a Weibull distribution with two parameters α and β, there is no way of estimating and testing for the shape parameter β without knowledge of the scale parameter α using the usual methods based on maximum likelihood. The method of random hazard functions is now used to obtain a consistent and asymptotically normal class of estimates for β independent of any specification whatsoever on α. This result enables one to test for the randomness of the underlying failure phenomena under the Weibull set up. An illustrative example is given.

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