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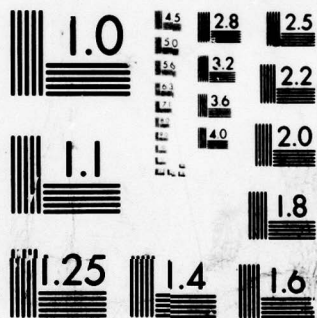
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T-MINIMAX AND MINIMAX DECISION RULES
FOR COMPARISON OF TREATMENTS WITH A CONTROL*

by
⑩ Shanti S. Gupta and Woo-Chul Kim
Purdue University

Department of Statistics
Division of Mathematical Sciences
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LEVEL (R)
Γ-MINIMAX AND MINIMAX DECISION RULES
FOR COMPARISON OF TREATMENTS WITH A CONTROL*

by

Shanti S. Gupta and Woo-Chul Kim
Purdue University

1. Introduction.

In many fields of research one is faced with the problem of comparing k experimental categories with reference to a 'standard' or a 'control'. Following the initial investigation by Paulson (1952), this problem has been studied in several different formulations by Dunnett (1955), Gupta and Sobel (1958) and Lehmann (1961) among others.

Let π_1, \dots, π_k denote the k experimental categories or 'treatment' populations and let π_0 denote the 'control' population, where the quality of each population π_i is characterized by a real-valued parameter θ_i ($i = 0, 1, \dots, k$). Each treatment population π_i is said to be 'superior', 'equivalent' or 'inferior' to the control population π_0 if $\theta_i - \theta_0 \geq \Delta$, $-\Delta < \theta_i - \theta_0 < \Delta$, $\theta_i - \theta_0 \leq -\Delta$, respectively, where Δ is a given positive constant. We consider a problem in which the treatment populations are to be classified as one of the above three cases based on the observations from the populations. Bhattacharyya (1956, 1958) studied this problem for the normal populations with unknown means when the control population is assumed known. A similar problem has been considered by Seeger (1972). We apply the Γ -minimax principle to this problem.

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Γ -minimax principle is known as one of the techniques for the use of incomplete prior information. Such an idea was first used by Robbins (1951) and independently by Hodges and Lehmann (1952) and Menges (1966). The name Γ -minimax was first used by Blum and Rosenblatt (1967). Randles and Hollander (1971) applied such a principle to a problem of selecting the treatments 'better' than the control. It has been applied to various problems, and recently to selection problems by Gupta and Huang (1975, 1977), Berger (1977) and Miescke (1979).

In Section 2, necessary notations, definitions, a loss function and the incomplete prior are introduced. A lemma is given to help find Γ -minimax rules. Section 3 treats the case of known control population, and a Γ -minimax rule and a minimax rule are derived. In Section 4, the case in which the control population parameter θ_0 is unknown is treated. Rules are derived which are Γ -minimax among rules for which the decision about the i -th population depends only on the observations from π_i and π_0 . A minimax rule is also derived. A normal means problem and a normal variances problem are given as specific examples. Section 5 consists of comparisons of Γ -minimax rules with Bayes rules for independent normal priors for the normal means problem.

2. Formulation of the problem.

Let X_0, X_1, \dots, X_k be $k+1$ independent random variables representing the control population π_0 and the k treatment populations π_1, \dots, π_k , respectively, with X_j having pdf $f_j(x-\theta_j)$ with respect to the Lebesgue measure on the real line R where $\theta_j \in \Theta = R$, $i = 0, 1, \dots, k$. The random variables X_0, \dots, X_k may be sufficient statistics or other statistics based on which we wish to make

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statistical decisions. We assume that each $f_i(\cdot)$ ($i = 0, 1, \dots, k$) is symmetric about the origin and strongly unimodal, i.e., $f_i(\cdot)$ is log-concave on the real line. Hence $f_i(x - \theta_i)$ has the monotone likelihood ratio (MLR) property. Obviously, we do not need any observations from π_0 when θ_0 is assumed known; therefore, it will be understood that, in such a case, the random variable X_0 is deleted from our consideration.

The action space G can be written as $G = G_1 \times \dots \times G_k$ where $G_i = \{1, 2, 3\}$ for $i = 1, \dots, k$. The action $a = (a_1, \dots, a_k) \in G$ is to be interpreted in such a way that, for $i = 1, \dots, k$, the treatment population π_i is classified as 'inferior', 'equivalent' and 'superior' to π_0 for $a_i = 1, 2, 3$, respectively. The loss $L(\underline{\theta}, a)$ incurred by the action $a \in G$ for $\underline{\theta} = (\theta_0, \dots, \theta_k)$ is assumed to be of the following form.

$$L(\underline{\theta}, a) = \sum_{i=1}^k L_i(\underline{\theta}, a_i) \quad (2.1)$$

where $L_i(\underline{\theta}, a_i)$ is defined as in the following table;

Table of loss $L_i(\underline{\theta}, a_i)$

State of nature \ a _i	1	2	3
$\theta_i - \theta_0 \leq -\Delta_2$	0	l_1	$l_1 + l_3$
$-\Delta_2 < \theta_i - \theta_0 \leq -\Delta_1$	0	0	l_4
$ \theta_i - \theta_0 < \Delta_1$	l_2	0	l_2
$\Delta_1 \leq \theta_i - \theta_0 < \Delta_2$	l_4	0	0
$\theta_i - \theta_0 \geq \Delta_2$	$l_1 + l_3$	l_1	0

$(l_i \geq 0, i=1, \dots, 4)$

Here, $\Delta_1 = \Delta - \epsilon$, $\Delta_2 = \Delta + \epsilon$ for a given constant $\epsilon: 0 \leq \epsilon < \Delta$ and it will be

understood that the second row and the fourth row will disappear when $\epsilon = 0$. Bhattacharyya (1956) derived a minimax rule assuming the above loss function with $\lambda_1 = \lambda_2 = \lambda_1 + \lambda_3 = 1$ and $\epsilon = 0$ when θ_0 is assumed known and $\theta_1, \dots, \theta_k$ are the unknown means of normal distributions. However, the irregularity of such a loss function has been pointed out in the sense that the minimax risk does not tend to zero even if the sample sizes increase indefinitely, and the same problem has been studied afresh by Bhattacharyya (1958) assuming the above loss function with $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_1 + \lambda_3 = 1$ and $\epsilon > 0$. Note that the above loss function with $\epsilon > 0$ assumes the indifference zones.

For given $\underline{x} = (x_0, x_1, \dots, x_k)$ consider decision rules of the form

$$\delta(\underline{x}) = (\delta_1(\underline{x}), \dots, \delta_k(\underline{x})) \quad (2.2)$$

where $\delta_j(\underline{x}) = (\delta_j(1|\underline{x}), \delta_j(2|\underline{x}), \delta_j(3|\underline{x}))$ and, for $j = 1, 2, 3$, $\delta_j(j|\underline{x})$ denotes the conditional probability of taking action j in the i -th component decision problem. Note that there is no loss of generality in considering decision rules of the form given in (2.3). The risk function of a rule δ for fixed $\underline{\theta}$ is then $R(\underline{\theta}, \delta) = \sum_{i=1}^k R_i(\underline{\theta}, \delta_i)$ where $R_i(\underline{\theta}, \delta_i) = E_{\underline{\theta}}[L_i(\underline{\theta}, \delta_i(\underline{X}))]$. For a prior distribution $\tau(\underline{\theta})$ of $\underline{\theta}$, the overall risk of a rule δ wrt τ is denoted by $r(\tau, \delta) = \sum_{i=1}^k r_i(\tau, \delta_i)$ where $r_i(\tau, \delta_i) = \int R_i(\underline{\theta}, \delta_i) d\tau(\underline{\theta})$.

It is assumed that partial prior information is available to a decision maker such that, for each i , he can specify $\gamma_i = P[|\theta_i - \theta_0| \geq \Delta_2]$ and $\gamma_i' = P[|\theta_i - \theta_0| < \Delta_1]$ where $\gamma_i + \gamma_i' \leq 1$ for $i = 1, \dots, k$. Let Γ denote the class of all such prior distributions, i.e.,

$$\Gamma = \{\tau(\underline{\theta}) : \int_{|\theta_i - \theta_0| \geq \Delta_2} d\tau(\underline{\theta}) = \gamma_i, \int_{|\theta_i - \theta_0| < \Delta_1} d\tau(\underline{\theta}) = \gamma_i' \text{ for } i=1, \dots, k\}. \quad (2.3)$$

Note that when $\epsilon = 0$, i.e., $\Delta_1 = \Delta_2$, $\gamma_i + \gamma_i' = 1$.

A rule δ^Γ is called a Γ -minimax rule if $\sup_{\tau \in \Gamma} r(\tau, \delta^\Gamma) = \inf_{\delta} \sup_{\tau \in \Gamma} r(\tau, \delta)$, and $\sup_{\tau \in \Gamma} r(\tau, \delta^\Gamma)$ is called the Γ -minimax value. The next result is useful to find the Γ -minimax rule.

Lemma 2.1. Suppose $\{\tau_n, n = 1, 2, \dots\}$ is a sequence of priors in Γ .

If $\overline{\lim}_n \inf_{\delta} r(\tau_n, \delta) \geq c$ and if $\sup_{\tau \in \Gamma} r(\tau, \delta^\Gamma) \leq c$, then δ^Γ is a Γ -minimax rule and c is the Γ -minimax value.

Proof. The result follows from the following inequalities.

$$\begin{aligned} \sup_{\tau \in \Gamma} \inf_{\delta} r(\tau, \delta) &\geq \overline{\lim}_n \inf_{\delta} r(\tau_n, \delta) \\ &\geq c \\ &\geq \sup_{\tau \in \Gamma} r(\tau, \delta^\Gamma) \\ &\geq \inf_{\delta} \sup_{\tau \in \Gamma} r(\tau, \delta) \\ &\geq \sup_{\tau \in \Gamma} \inf_{\delta} r(\tau, \delta). \end{aligned}$$

3. Known control population

In this section θ_0 is assumed known and thus we may assume $\theta_0 = 0$ without loss of generality. Hence \underline{x} and $\underline{\theta}$ in this section denote (x_1, \dots, x_k) and $(\theta_1, \dots, \theta_k)$, respectively. Let us consider a rule $\delta(\underline{x})$ of the form in (2.2) where $\delta_j(j|\underline{x})$ ($j = 1, 2, 3$) is given by

$$\begin{cases} \delta_i(1|\underline{x}) = I_{(-\infty, -d_i]}(x_i), \\ \delta_i(2|\underline{x}) = I_{(-d_i, d_i)}(x_i), \\ \delta_i(3|\underline{x}) = I_{[d_i, \infty)}(x_i), \end{cases} \quad (3.1)$$

for $0 < d_i \leq \infty$ and $i = 1, \dots, k$.

Lemma 3.1. Suppose that a decision rule $\delta(x)$ is given by (2.2) and (3.1).

Then, for $i = 1, \dots, k$,

$$\sup_{\tau \in \Gamma} r_i(\tau, \delta_i) \leq v_i$$

$$\text{where } v_i = \int_{d_i}^{\infty} [\ell_3 \gamma_i f_i(x + \Delta_2) + \ell_2 \gamma_i' (f_i(x - \Delta_1) + f_i(x + \Delta_1)) + \ell_4 (1 - \gamma_i - \gamma_i') f_i(x + \Lambda_1)] dx \\ + \int_{-\infty}^{d_i} \ell_1 \gamma_i f_i(x - \Delta_2) dx.$$

Proof. It follows from the definition of $L_i(\underline{\theta}, a_i)$ and the symmetry of $f_i(\cdot)$ that, for $|\theta_i| < \Delta_1$,

$$R_i'(\underline{\theta}, \delta_i) = \ell_2 [f_i(d_i - \theta_i) - f_i(-d_i - \theta_i)] \\ = \ell_2 f_i(\theta_i + d_i) \left[\frac{f_i(\theta_i - d_i)}{f_i(\theta_i + d_i)} - 1 \right],$$

where R_i' denotes the derivative of R_i wrt θ_i .

It follows from the MLR property of $f_i(x - \theta_i)$ that $R_i'(\underline{\theta}, \delta_i)$ has at most one change of sign, from negative to positive if there is any sign change at all; therefore, $R_i(\underline{\theta}, \delta_i)$ attains the supremum over $\theta_i \in (-\Delta_1, \Delta_1)$ at either $\theta_i = -\Delta_1$ or $\theta_i = \Delta_1$. Hence, for $|\theta_i| < \Delta_1$,

$$R_i(\underline{\theta}, \delta_i) \leq \ell_2 \int_{d_i}^{\infty} [f_i(x - \Delta_1) + f_i(x + \Delta_1)] dx.$$

It can be easily shown that

$$R_i(\underline{\theta}, \delta_i) \begin{cases} \leq \ell_1 \int_{-\infty}^{d_i} f_i(x - \Delta_2) dx + \ell_3 \int_{d_i}^{\infty} f_i(x + \Lambda_2) dx & \text{for } |\theta_i| > \Lambda_2, \\ \leq \ell_4 \int_{d_i}^{\infty} f_i(x + \Delta_1) dx & \text{for } \Lambda_1 < |\theta_i| < \Lambda_2. \end{cases}$$

Therefore, it follows from (2.3) that $\sup_{\tau \in \Gamma} r_i(\tau, \delta_i) \leq v_i$ which completes the proof.

Now we derive a Γ -minimax rule for the case where θ_0 is known.

Theorem 3.1. Assume that independent random variables X_1, \dots, X_k have $f_1(x_1 - \theta_1), \dots, f_k(x_k - \theta_k)$, respectively, with $f_i(\cdot)$ being symmetric and strongly unimodal, and that the loss function is given by (2.1). Then the Γ -minimax rule δ^Γ is given by (2.2) and (3.1) where each $d_i = d_i^\Gamma$ in (3.1) is defined by $d_i^\Gamma = \max(c_i, 0)$ with c_i being determined by

$$\begin{aligned} & \ell_3 \gamma_i f_i(x + \Delta_2) + \ell_2 \gamma_i' [f_i(x - \Delta_1) + f_i(x + \Delta_1)] + \ell_4 (1 - \gamma_i - \gamma_i') f_i(x + \Delta_1) \\ & \leq, > \ell_1 \gamma_i f_i(x - \Delta_2) \quad \text{as } x \geq, < c_i. \end{aligned} \quad (3.2)$$

Proof. The existence of a c_i satisfying (3.2) follows from the MLR property of $f_i(x - \theta_i)$. Therefore, the decision rule δ^Γ is well defined. First, we will consider the case when $\epsilon > 0$, i.e., $\Delta_2 > \Delta_1$. For $n > \Delta_1^{-1}$, let τ_n be a prior distribution in Γ under which $\theta_1, \dots, \theta_k$ are independent, $P(\theta_i = \Delta_2) = P(\theta_i = -\Delta_2) = \gamma_i/2$, $P(\theta_i = \Delta_1) = P(\theta_i = -\Delta_1) = (1 - \gamma_i - \gamma_i')/2$ and $P(\theta_i = \Delta_1 - n^{-1}) = P(\theta_i = -\Delta_1 + n^{-1}) = \gamma_i'/2$ for $i = 1, \dots, k$. Then it can be easily verified that $\inf_{\delta} r(\tau_n, \delta) = \sum_{i=1}^k \inf_{\delta_i} r_i(\tau_n, \delta_i)$ and, for $i = 1, \dots, k$,

$$\inf_{\delta_i} r_i(\tau_n, \delta_i) = \int_{-\infty}^{\infty} p_n(x) dx / 2,$$

where $p_n(x) = \min\{p_n(1, x), p_n(2, x), p_n(1, -x)\}$ with $p_n(2, x) = \ell_1 \gamma_i [f_i(x + \Delta_2) + f_i(x - \Delta_2)]$ and $p_n(1, x) = \ell_2 \gamma_i' [f_i(x - \Delta_1 + n^{-1}) + f_i(x + \Delta_1 - n^{-1})] + \ell_4 (1 - \gamma_i - \gamma_i') f_i(x - \Delta_1) + (\ell_1 + \ell_3) \gamma_i f_i(x - \Delta_2)$. Since $f_i(\cdot)$ is strongly unimodal on the real line, $f_i(\cdot)$ is

continuous and thus $p_n(x)$ converges, as $n \rightarrow \infty$, to $p(x) = \min\{p(1,x), p(2,x), p(1,-x)\}$ where $p(1,x) = \lim_{n \rightarrow \infty} p_n(1,x)$. Note that $p(1,x) \geq p(1,-x)$ if and only if $x \geq 0$. This follows from the fact that, for any $t > 0$ $f_i(x-t) \geq f_i(x+t)$ if and only if $x \geq 0$. Since $p_n(x)$ is bounded above by $p(2,x)$ which is integrable, it follows from the Lebesgue convergence theorem that

$$\begin{aligned} \liminf_n \inf_{\delta_i} r_i(\tau_n, \delta_i) &= \int_{-\infty}^{\infty} p(x) dx / 2 \\ &= \int_0^{\infty} \min\{p(2,x), p(1,-x)\} dx. \end{aligned} \quad (3.3)$$

Note that $\int_0^{\infty} \min\{p(2,x), p(1,-x)\} dx$ can be written as

$$\begin{aligned} &\int_0^{\infty} \min\{p(2,x), p(1,-x)\} dx \\ &= \int_0^{\infty} \min\{\ell_3 \gamma_i f_i(x+\Delta_2) + \ell_4 (1-\gamma_i - \gamma_i') f_i(x+\Delta_1) + \ell_2 \gamma_i' [f_i(x+\Delta_1) + f_i(x-\Delta_1)], \\ &\quad \ell_i \gamma_i f_i(x-\Delta_2)\} dx + \int_{-\infty}^0 \ell_1 \gamma_i f_i(x-\Delta_2) dx \\ &= \int_{d_i}^{\infty} [\ell_3 \gamma_i f_i(x+\Delta_2) + \ell_4 (1-\gamma_i - \gamma_i') f_i(x+\Delta_1) + \ell_2 \gamma_i' (f_i(x-\Delta_1) + f_i(x+\Delta_1))] dx \\ &\quad + \int_{-\infty}^{d_i} \ell_1 \gamma_i f_i(x-\Delta_2) dx, \end{aligned}$$

where $d_i = \max(0, c_i)$ with c_i defined as in (3.2).

It follows from Lemma 3.1 that $\liminf_n \inf_{\delta_i} r_i(\tau_n, \delta_i) \geq \sup_{\tau \in \Gamma} r_i(\tau, \delta_i^{\Gamma})$. Therefore,

$$\begin{aligned} \liminf_n \inf_{\delta} r(\tau_n, \delta) &= \lim_n \sum_{i=1}^k \inf_{\delta_i} r_i(\tau_n, \delta_i) \\ &\geq \sum_{i=1}^k \sup_{\tau \in \Gamma} r_i(\tau, \delta_i^{\Gamma}) \\ &\geq \sup_{\tau \in \Gamma} r(\tau, \delta^{\Gamma}). \end{aligned}$$

Hence Lemma 2.1 yields that δ^I is a Γ -minimax rule. This completes the proof of the case when $\epsilon > 0$. Note that $\Delta_1 = \Delta_2 = \Delta$ and $\gamma_i + \gamma_i' = 1$ for $i = 1, \dots, k$ if $\epsilon = 0$. When $\epsilon = 0$, let us consider a sequence of prior distributions, $\{\pi_n, n > \Delta^{-1}\}$, in Γ under which $\theta_1, \dots, \theta_k$ are independent, $P(\theta_i = \Delta) = P(\theta_i = -\Delta) = \gamma_i/2$ and $P(\theta_i = \Delta - n^{-1}) = P(\theta_i = -\Delta + n^{-1}) = \gamma_i'/2$ for $i = 1, \dots, k$. Then we can prove in the exactly same manner as the above that δ^Γ is a Γ -minimax rule.

Now we discuss the derivation of the minimax rule for some special cases.

A minimax rule can be derived from the arguments in the proof of Theorem 2.1. For this purpose, assume that $\lambda_1 = \lambda_2$, $\lambda_4 \leq 2\lambda_1$ and $\lambda_3 \leq \lambda_1$. We may assume that $\lambda_1 = \lambda_2 = 1$ without loss of generality. Let us consider a rule δ^* of the type given by (2.2) and (3.1) where each $d_i = d_i^*$ in (3.1) is determined so that, for $F_i(x) = \int_{-\infty}^{\infty} f_i(t) dt$,

$$F_i(d_i - \Delta_2) + \lambda_3 F_i(-d_i - \Delta_2) = F_i(-d_i - \Delta_1) + F_i(-d_i + \Delta_1). \quad (3.3)$$

Note that the existence of such a non-negative d_i^* follows from the strong unimodality and the symmetry of $f_i(\cdot)$. Let us define γ_i and $\gamma_i' = 1 - \gamma_i$ for $i = 1, \dots, k$ by

$$\gamma_i = [f_i(d_i^* - \Delta_1) + f_i(d_i^* + \Delta_1)] / [f_i(d_i^* - \Delta_2) - \lambda_3 f_i(d_i^* + \Delta_2) + f_i(d_i^* - \Delta_1) + f_i(d_i^* + \Delta_1)].$$

Since $\gamma_i \in [0, 1]$, we can consider a family of prior distributions, Γ , given by (2.3). Then it follows from Theorem 2.1 that the corresponding Γ -minimax rule is of the same type as δ^* except that now $d_i^\Gamma = \max(c_i, 0)$ where c_i is determined so that

$$H(c_i) = \gamma_i [\lambda_3 f_i(c_i + \Delta_2) - f_i(c_i - \Delta_2)] + \gamma_i' [f_i(c_i - \Delta_1) + f_i(c_i + \Delta_1)] = 0.$$

Since $H(d_i^*) = 0$ and $d_i^* \geq 0$, $d_i^\Gamma = d_i^*$, i.e., the rule δ^* is the Γ -minimax

rule; therefore it follows from the arguments in the proof of Theorem 2.1 that

$$\begin{aligned}
 \liminf_n \inf_{\delta} r(\tau_n, \delta) &\geq \sum_{i=1}^k \gamma_i [F_i(d_i^* - \Delta_2) + \ell_3 F_i(-d_i^* - \Delta_2)] + \gamma_i' [F_i(-d_i^* - \Lambda_1) + F_i(-d_i^* + \Lambda_1)] \\
 &= \sum_{i=1}^k [F_i(-d_i^* - \Delta_1) + F_i(-d_i^* + \Lambda_1)] \\
 &= \sum_{i=1}^k \sup_{\underline{\theta}} R_i(\underline{\theta}, \delta^*) \\
 &\geq \sup_{\underline{\theta}} R(\underline{\theta}, \delta^*)
 \end{aligned}$$

Therefore, we have the next result which includes the results in Bhattacharyya (1956, 1958) as special cases.

Corollary 3.1. Under the assumptions in Theorem 3.1, if $\ell_1 = \ell_2 = 1$, $\ell_3 \leq 1$ and $\ell_4 \leq 2$, then a rule δ^M of the type given by (2.2) and (3.1) with $d_j = d_j^M$ in (3.1) being determined by (3.3) is minimax.

4. Unknown control population.

In this section we will consider the case when θ_0 is unknown and will derive a Γ -minimax decision rule δ^Γ in the class \mathfrak{D}_0 of decision rules for which $\delta_i(\underline{x})$ in (2.2) depends only on x_0 and x_i for $i = 1, \dots, k$. Let us consider rules $\delta(\underline{x})$ in \mathfrak{D}_0 where $\delta_i(j|\underline{x})$ ($j = 1, 2, 3$) are given by

$$\begin{cases}
 \delta_i(1|\underline{x}) = I_{(-\infty, -d_i]}(x_i - x_0), \\
 \delta_i(2|\underline{x}) = I_{(-d_i, d_i)}(x_i - x_0), \\
 \delta_i(3|\underline{x}) = I_{[d_i, \infty)}(x_i - x_0),
 \end{cases} \quad (4.1)$$

for $0 \leq d_i \leq \infty$ and $i = 1, \dots, k$.

Note that the pdf of $Y_i = X_i - X_0$ is given by

$$g_i(y - (\theta_i - \theta_0)) = \int_{-\infty}^{\infty} f_i(t + y - \theta_i) f_0(t - \theta_0) dt, \quad (4.2)$$

and that $g_i(\cdot)$ is strongly unimodal by the result of Ibragimov (1956) and symmetric about the origin. Therefore, the next follows from this fact and Lemma 3.1.

Lemma 4.1. Suppose that a rule $\delta(\underline{x})$ in \mathfrak{D}_0 is given by (2.2) and (4.1).

Then, for $i = 1, \dots, k$,

$$\sup_{\tau \in \Gamma} r_i(\tau, \delta_i) \leq \omega_i,$$

where, for $i = 1, \dots, k$,

$$\begin{aligned} \omega_i = & \int_{d_i}^{\infty} [\lambda_3 \gamma_i g_i(y + \Delta_2) + \lambda_2 \gamma_i' (g_i(y - \Delta_1) + g_i(y + \Delta_1)) + \lambda_4 (1 - \gamma_i - \gamma_i') g_i(y + \Delta_1)] dy \\ & + \int_{-\infty}^{d_i} \lambda_1 \gamma_i g_i(y - \Delta_2) dy. \end{aligned}$$

We now proceed as in Theorem 3.1 by considering the following sequence $\{\tau_n, n > \Delta^{-1}\}$ of prior distributions in Γ for the case when $\epsilon > 0$. Under τ_n ,

(i) $\theta_1 - \theta_0, \dots, \theta_k - \theta_0$ are independent,

(ii) $P[\theta_i - \theta_0 = \Delta_2] = P[\theta_i - \theta_0 = -\Delta_2] = \gamma_i/2$,

$$P[\theta_i - \theta_0 = \Delta_1] = P[\theta_i - \theta_0 = -\Delta_1] = (1 - \gamma_i - \gamma_i')/2,$$

$$P[\theta_i - \theta_0 = \Delta_1 - n^{-1}] = P[\theta_i - \theta_0 = -\Delta_1 + n^{-1}] = \gamma_i'/2 \text{ and}$$

(iii) θ_0 has uniform distribution over $[-n, n]$ and is independent of $\theta_1 - \theta_0, \dots, \theta_k - \theta_0$.

It can be easily shown that the overall risk of the Bayes rule is given by

$$\inf_{\delta \in \mathcal{D}_0} r(\tau_n, \delta) = \frac{1}{4n} \sum_{i=1}^k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(i, x, y) dx dy \quad (4.3)$$

where $p_n(i, x, y) = \min\{s_n(i, x, y), t_n(i, x, y), s_n(i, -x, -y)\}$ with

$$\begin{aligned} s_n(i, x, y) &= \ell_2 \gamma_i \int_{-n}^n [f_i(x-u-\Delta_1+n^{-1}) + f_i(x-u+\Delta_1-n^{-1})] f_0(y-u) du + \\ &+ \ell_4 (1-\gamma_i - \gamma_i') \int_{-n}^n f_i(x-u-\Delta_1) f_0(y-u) du + \\ &+ (\ell_1 + \ell_3) \gamma_i \int_{-n}^n f_i(x-u-\Delta_2) f_0(y-u) du \text{ and} \end{aligned}$$

$$t_n(i, x, y) = \ell_1 \gamma_i \int_{-n}^n [f_i(x-u+\Delta_2) + f_i(x-u-\Delta_2)] f_0(y-u) du.$$

From change of variables $x = nv-w$ and $y = nv+w$, it follows that

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(i, x, y) dx dy / 4n &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(i, nv-w, nv+w) dv dw / 2 \\ &\geq \int_{-\infty}^{\infty} [\int_{-1}^1 p_n(i, nv-w, nv+w) dv] dw / 2. \end{aligned} \quad (4.4)$$

Note that

$$\begin{aligned} s_n(i, nv-w, nv+w) &= \ell_2 \gamma_i \int_{n(v-1)}^{n(v+1)} [f_i(z-w-\Delta_1+n^{-1}) + f_i(z-w+\Delta_1-n^{-1})] f_0(z+w) dz + \\ &+ \ell_4 (1-\gamma_i - \gamma_i') \int_{n(v-1)}^{n(v+1)} f_i(z-w-\Delta_1) f_0(z+w) dz + \\ &+ (\ell_1 + \ell_3) \gamma_i \int_{n(v-1)}^{n(v+1)} f_i(z-w-\Delta_2) f_0(z+w) dz \text{ and} \\ t_n(i, nv-w, nv+w) &= \ell_1 \gamma_i \int_{n(v-1)}^{n(v+1)} [f_i(z-w+\Delta_2) + f_i(z-w-\Delta_2)] f_0(z+w) dz. \end{aligned}$$

Therefore, for any $(v, w) \in (-1, 1) \times \mathbb{R}$, $p_n(i, nv-w, nv+w)$ converges, as $n \rightarrow \infty$,

to $p(i,w) = \min\{s(i,w), t(i,w), s(i,-w)\}$ where

$$\begin{aligned} s(i,w) &= \ell_2 \gamma_i \int_{-\infty}^{\infty} [f_i(z-w-\Delta_1) + f_i(z-w+\Delta_1)] f_0(z+w) dz + \\ &+ \ell_4 (1-\gamma_i-\gamma_i') \int_{-\infty}^{\infty} f_i(z-w-\Delta_1) f_0(z+w) dz + \\ &+ (\ell_1 + \ell_3) \gamma_i \int_{-\infty}^{\infty} f_i(z-w-\Delta_2) f_0(z+w) dz \text{ and} \end{aligned}$$

$$t(i,w) = \ell_1 \gamma_i \int_{-\infty}^{\infty} [f_i(z-w+\Delta_2) + f_i(z-w-\Delta_2)] f_0(z+w) dz.$$

It follows from (4.4) that, for $i = 1, \dots, k$,

$$\begin{aligned} \frac{\lim}{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_n(i,x,y) dx dy / 4n &\geq \int_{-\infty}^{\infty} p(i,w) dw \\ &= \int_0^{\infty} h(i,y) dy, \end{aligned}$$

$$\begin{aligned} \text{where } h(i,y) &= \min\{\ell_2 \gamma_i' [g_i(y-\Delta_1) + g_i(y+\Delta_1)] + \ell_4 (1-\gamma_i-\gamma_i') g_i(y+\Delta_1) + \\ &+ (\ell_1 + \ell_3) \gamma_i g_i(y+\Delta_2), \ell_1 \gamma_i [g_i(y+\Delta_2) + g_i(y-\Delta_2)]\}. \end{aligned}$$

Then from (4.3), we have

$$\frac{\lim}{n} \inf_{\delta \in \mathcal{D}_0} r(\tau_n, \delta) \geq \sum_{i=1}^k \int_0^{\infty} h(i,y) dy. \quad (4.5)$$

Note that $\int_0^{\infty} h(i,y) dy$ can be written as

$$\begin{aligned} \int_0^{\infty} h(i,y) dy &= \int_0^{\infty} \min\{\ell_3 \gamma_i g_i(y+\Delta_2) + \ell_4 (1-\gamma_i-\gamma_i') g_i(y+\Delta_1) + \ell_2 \gamma_i' (g_i(y+\Delta_1) + g_i(y-\Delta_1)), \\ &\ell_1 \gamma_i g_i(y-\Delta_2)\} dy + \int_{-\infty}^0 \ell_1 \gamma_i g_i(y-\Delta_2) dy \end{aligned}$$

$$\begin{aligned}
&= \int_{d_i}^{\infty} [\ell_3 \gamma_i g_i(y+\Delta_2) + \ell_2 \gamma_i' (g_i(y-\Delta_1) + g_i(y+\Delta_1)) + \ell_4 (1-\gamma_i - \gamma_i') g_i(y+\Delta_1)] dy \\
&\quad + \int_{-\infty}^{d_i} \ell_1 \gamma_i g_i(y-\Delta_2) dy,
\end{aligned}$$

where $d_i = \max(c_i, 0)$ with c_i being determined so that

$$\ell_3 \gamma_i g_i(y+\Delta_2) + \ell_2 \gamma_i' (g_i(y-\Delta_1) + g_i(y+\Delta_1)) + \ell_4 (1-\gamma_i - \gamma_i') g_i(y+\Delta_1) \tag{4.6}$$

$$\leq, \geq \ell_1 \gamma_i g_i(y-\Delta_2) \quad \text{as } y \geq, \leq c_i.$$

Let δ^Γ be the rule given by (2.2) and (4.1) where $d_i = d_i^\Gamma$ in (4.1) is defined by $d_i = \max(c_i, 0)$ with c_i determined by (4.6). Then it follows from (4.5) and Lemma 4.1 that

$$\begin{aligned}
\frac{\liminf}{n} \inf_{\delta \in \mathcal{D}_0} r(\tau_n, \delta) &\geq \sum_{i=1}^k \sup_{\tau \in \Gamma} r_i(\tau, \delta_i^\Gamma) \\
&\geq \sup_{\tau \in \Gamma} r(\tau, \delta^\Gamma).
\end{aligned}$$

Therefore, Lemma 2.1 yields the next result.

Theorem 4.1. Assume that independent random variables X_0, \dots, X_k have pdf's $f_0(x_0 - \theta_0), \dots, f_k(x_k - \theta_k)$, respectively, with $f_i(\cdot)$ being strongly unimodal and symmetric, and that the loss function is given by (2.1). Then the Γ -minimax rule δ^Γ in \mathcal{D}_0 is given by (2.2) and (4.1) where $d_i = d_i^\Gamma$ in (4.1) is defined by $d_i^\Gamma = \max(c_i, 0)$ with c_i being determined by (4.6), for all $i = 1, 2, \dots, k$.

Remark 4.1. It can be easily shown that the symmetry of $f_i(\cdot)$ in Theorem 4.1 can be replaced by that of $g_i(\cdot)$. It should be noted that the symmetry of $g_i(\cdot)$ follows when $f_0(\cdot), \dots, f_k(\cdot)$ are identical.

The next result follows in exactly the same manner as Corollary 3.1 was proved.

Corollary 4.1. Under the assumptions in Theorem 4.1, if $\ell_1 = \ell_2 = 1$, $\ell_3 \leq 1$ and $\ell_4 \leq 2$, then a minimax rule δ^M in \mathcal{D}_0 is given by (2.2) and (4.1) where $d_i = d_i^M$ in (4.1) is determined so that, for $G_i(x) = \int_{-\infty}^x g_i(t)dt$,

$$G_i(d_i - \Delta_2) + \ell_3 G_i(-d_i - \Delta_2) = G_i(-d_i - \Delta_1) + G_i(-d_i + \Delta_1).$$

Now we provide some examples to illustrate the application of the above results.

Example 4.1. Suppose π_i represents a normal population $N(\theta_i, \sigma_i^2)$ for $i = 0, \dots, k$ with σ_i^2 ($i = 0, \dots, k$) known. We assume that a random sample of size n_i is taken from each of the $k+1$ populations π_0, \dots, π_k . By sufficiency we can restrict our attention to the decision rules depending only on the sample means X_0, \dots, X_k where X_i has normal distribution with mean θ_i and variance $\eta_i^2 = \sigma_i^2/n_i$ for $i = 0, 1, \dots, k$.

(A) Γ -minimax rule: The Γ -minimax rule δ^Γ in \mathcal{D}_0 in Theorem 4.1 is determined by $d_i^\Gamma = (\eta_i^2 + \eta_0^2)^{\frac{1}{2}} \max(c_i, 0)$ where c_i is defined so that

$$\begin{aligned} & \ell_3 \gamma_i e^{-2(\lambda_i + \epsilon_i)x} + \ell_2 \gamma_i' [e^{-2\epsilon_i(x - \lambda_i)} + e^{-2\lambda_i(x - \epsilon_i)}] + \\ & + \ell_4 (1 - \gamma_i - \gamma_i') e^{-2\lambda_i(x - \epsilon_i)} - \ell_1 \gamma_i \geq, \leq 0 \text{ as } x \leq, \geq c_i, \end{aligned} \tag{4.7}$$

where $\lambda_i = \Lambda(\eta_i^2 + \eta_0^2)^{-\frac{1}{2}}$ and $\epsilon_i = \epsilon(\eta_i^2 + \eta_0^2)^{-\frac{1}{2}}$.

(B) Minimax rule: Assume $\ell_1 = \ell_2 = 1$, $\ell_3 \leq 1$ and $\ell_4 \leq 2$. Then the minimax rule δ^M in \mathcal{D}_0 in Corollary 4.1 is determined by $d_i^\Gamma = (\eta_i^2 + \eta_0^2)^{\frac{1}{2}} t_i$ where t_i is defined so that

$$\phi(t_i - \lambda_i - \epsilon_i) + \epsilon_3 \phi(-t_i - \lambda_i - \epsilon_i) = \phi(-t_i - \lambda_i + \epsilon_i) + \phi(-t_i + \lambda_i - \epsilon_i) \quad (4.8)$$

with λ_i and ϵ_i defined as in (A) and ϕ denoting the cdf of the standard normal distribution.

Example 4.2. Assume that π_i represents a normal population $N(0, \sigma_i^2)$ for $i = 0, 1, \dots, k$ with σ_i^2 unknown, and that we have a random sample of size n taken from each population π_i . Consider a problem of partitioning the treatment populations in terms of variances with a loss structure analogous to that given by (2.1), i.e., a loss function obtained from the latter by substituting $\log \sigma_i^2$, $\log \Delta$ and $\log \epsilon$ for θ_i , Δ and ϵ , respectively. Thus Δ and ϵ are assumed such that $1 \leq \epsilon < \Delta$. By sufficiency we need to consider only the decision rules depending on s_0^2, \dots, s_k^2 where s_i^2 denotes the sample variance corresponding to π_i . Since ns_i^2/σ_i^2 ($i = 0, 1, \dots, k$) are independently distributed chi-square random variables with degrees of freedom n , it can be easily seen that the associated location parameter problem satisfies the assumptions in Theorem 4.1 except the symmetry which is not necessary in this problem because of Remark 4.1. Therefore, with obvious modifications we have the following results. Let \mathcal{D}_0 denote the class of decision rules $\delta = (\delta_1, \dots, \delta_k)$ for which δ_i depends only on s_0^2 and s_i^2 and let x_i denote s_i^2/s_0^2 for $i = 1, \dots, k$.

(A) Γ -minimax rule: A Γ -minimax rule δ^Γ in \mathcal{D}_0 is given by

$$\delta_i^\Gamma(1|x_i) = I_{(0, d_i^{-1}]}(x_i), \quad \delta_i^\Gamma(2|x_i) = I_{(d_i^{-1}, d_i)}(x_i) \quad \text{and} \quad \delta_i^\Gamma(3|x_i) = I_{[d_i, \dots)}(x_i)$$

for $i = 1, \dots, k$ where $d_i = \max(c_i, 1)$ with c_i being determined so that

$$\begin{aligned}
& \ell_3 \gamma_i \left(\frac{\Delta_2 + y}{1 + \Delta_2 y} \right)^n + \ell_2 \gamma_i \left[\left(\frac{\Delta_2 + y}{\Delta_1 + y} \right)^n + \left(\frac{\Delta_2 + y}{1 + \Delta_1 y} \right)^n \right] \left(\frac{\Delta_1}{\Delta_2} \right)^{n/2} + \\
& + \ell_4 (1 - \gamma_i - \gamma_i') \left(\frac{\Delta_2 + y}{1 + \Delta_1 y} \right)^n \left(\frac{\Delta_1}{\Delta_2} \right)^{n/2} \leq, \geq \ell_1 \gamma_i \text{ as } y \geq, \leq c_i.
\end{aligned} \tag{4.9}$$

Here $\Delta_1 = \Delta \epsilon^{-1}$, $\Delta_2 = \Delta \epsilon$.

(B) Minimax rule: Assume $\ell_1 = \ell_2 = 1$, $\ell_3 \leq 1$ and $\ell_4 \leq 2$. The minimax rule δ^M in \mathcal{D}_0 is the same as δ^Γ in (A) except that $d_i = d$ is determined so that

$$G_n(d/\Delta_2) + \ell_3 [1 - G_n(d\Delta_2)] = G_n(\Delta_1/d) + 1 - G_n(d\Delta_1) \tag{4.10}$$

where G_n denotes the cdf of F-distribution with degrees of freedom n and n .

We note that if π_i represents $N(\mu_i, \sigma_i^2)$ with both μ_i and σ_i^2 unknown, then the above results still hold with $n-1$ replacing n .

5. Comparison of Γ -minimax rules with Bayes rules.

When we represent our a priori information about the parameters by prior distributions over the parameter space, one method for the use of such information is to find a rule which is Γ -minimax with respect to the class, Γ , of such prior distributions.

Another way is to select one such prior distribution and use the corresponding Bayes rule. Thus Bayes rules wrt prior distributions in Γ are natural competitors of a Γ -minimax rule.

In this section we consider $k+1$ normal populations $N(\theta_i, \sigma^2)$ with σ^2 known, and derive Bayes rules wrt normal priors and then compare them with the corresponding Γ -minimax rules from both points of view.

For this purpose, assume that $(\theta_0, \dots, \theta_k)$ have prior distribution τ_0 under which $\theta_0, \dots, \theta_k$ are independent and each θ_i has a normal distribution with mean μ_i and variance v_i^2 . Let x_0, \dots, x_k denote the observed sample means based on samples of size n_i ($i = 0, 1, \dots, k$). To simplify forthcoming formulas, let us introduce the following notations;

$$\begin{aligned} \sigma_i^2 &= \sigma^2/n_i, \quad b_i = [(\sigma_i^{-2} + v_i^{-2})^{-1} + (\sigma_0^{-2} + v_0^{-2})^{-1}]^{-1}, \\ m_i &= (\sigma_i^{-2}x_i + v_i^{-2}\mu_i)(\sigma_i^{-2} + v_i^{-2})^{-1}, \quad y_i = (m_i - m_0)/b_i. \end{aligned} \quad (5.1)$$

The following theorem describes the Bayes rule.

Theorem 5.1. Assume the loss function is given by (2.1). Then the Bayes rule δ^B wrt τ_0 is given by $\delta_i^B(1|\underline{y}) = I_{(-\infty, -d_i]}(y_i)$, $\delta_i^B(2|\underline{y}) = I_{(-d_i, d_i)}(y_i)$ and $\delta_i^B(3|\underline{y}) = I_{[d_i, \infty)}(y_i)$ for $i = 1, \dots, k$ where $d_i = \max(c_i, 0)$ with c_i being determined so that

$$\begin{aligned} & \lambda_3 \phi(-\Delta_2 b_i^{-1} - y) + \lambda_4 [\phi(-\Delta_1 b_i^{-1} - y) - \phi(-\Delta_2 b_i^{-1} - y)] \\ & + \lambda_2 [\phi(\Delta_1 b_i^{-1} - y) - \phi(-\Delta_1 b_i^{-1} - y)] - \lambda_1 \phi(-\Delta_2 b_i^{-1} + y) \\ & \geq, \leq 0 \quad \text{as } y \leq, \geq c_i. \end{aligned}$$

Proof. It suffices to find the Bayes rule for each of the k component decision problems. This reduces to the comparison of posterior risks of three possible actions. We will do this for the first component decision problem without loss of generality. Let $p_1(y_1)$, $p_2(y_1)$ and $p_3(y_1)$ denote the posterior risks of the actions 1, 2 and 3, respectively, in the first component problem. Then it can be shown that

$$p_1(y) = (\ell_1 + \ell_3)\phi(-\Delta_2 b_1^{-1} + y) + \ell_4[\phi(\Delta_2 b_1^{-1} - y) - \phi(\Delta_1 b_1^{-1} - y)] + \\ + \ell_2[\phi(\Delta_1 b_1^{-1} - y) - \phi(-\Delta_1 b_1^{-1} - y)],$$

$$p_2(y) = \ell_1[\phi(-\Delta_2 b_1^{-1} - y) + \phi(-\Delta_2 b_1^{-1} + y)] \text{ and}$$

$$p_3(y) = p_1(-y).$$

Note that $p_1(y) - p_3(y)$ can be written as $E_y H(Z)$ where Z has a normal distribution with mean y and variance 1 and $H(\cdot)$ is given by

$$H(z) = \begin{cases} \ell_1 + \ell_3 & \text{if } z \geq \Delta_2 b_1^{-1} \\ \ell_4 & \text{if } \Delta_1 b_1^{-1} < z < \Delta_2 b_1^{-1} \\ 0 & \text{if } -\Delta_1 b_1^{-1} \leq z \leq \Delta_1 b_1^{-1} \\ -\ell_4 & \text{if } -\Delta_2 b_1^{-1} < z < -\Delta_1 b_1^{-1} \\ -(\ell_1 + \ell_3) & \text{if } z \leq -\Delta_2 b_1^{-1}. \end{cases}$$

Since the density of the normal distribution $N(y, 1)$ has the MLR property, it follows that $p_1(y) - p_3(y)$ has at most one sign change.

Furthermore, it can be shown that $p_1(y) - p_3(y)$ is strictly increasing on $(-\Delta b_1^{-1}, \Delta b_1^{-1})$ and $p_1(0) - p_3(0) = 0$. Thus $p_1(y) - p_3(y) \geq, \leq 0$ as $y \geq, \leq 0$. Similarly, we can show that $p_3(y) - p_2(y) \geq, \leq 0$ as $y \leq, \geq c_1$ for some real number c_1 unless $p_3(y) - p_2(y) \leq 0$ for all y . Therefore the result follows.

Now we compare the Γ -minimax rule δ^Γ given in Example 4.1 and the Bayes rule δ^B given in Theorem 5.1 under the assumption that $\ell_1 = \ell_2 = \ell_4 = 1$, $v_3 = v$, $n_i = n$ and $v_i^2 = v^2$ for $i = 0, \dots, k$. Note that we compare these rules under the relations $\gamma_i = \phi[(-\Delta_2 + \mu_i - \mu_0)(2v^2)^{-\frac{1}{2}}] + \phi[(-\Delta_2 - \mu_i + \mu_0)(2v^2)^{-\frac{1}{2}}]$ and $\gamma_i' = \phi[(\Delta_1 - \mu_i + \mu_0)(2v^2)^{-\frac{1}{2}}] - \phi[(-\Delta_1 - \mu_i + \mu_0)(2v^2)^{-\frac{1}{2}}]$ for $i = 1, \dots, k$. Each of them is the best in its own merit. Therefore there are two ways of any

meaningful comparison of these rules. One way is to examine the increase in the overall risk wrt τ_0 resulting from the use of δ^I . Another way is to compare them in terms of $\sup_{\tau \in \Gamma} r(\tau, \delta)$. When $n_i = n$ and $v_j^2 = v^2$ for $i = 0, 1, \dots, k$, the Bayes rule depends on \underline{x} only through $x_1 - x_0, \dots, x_k - x_0$ and it can be shown that $\sup_{\tau \in \Gamma} r(\tau, \delta^B) = \sum_{j=1}^k \sup_{\tau \in \Gamma} r_j(\tau, \delta_j^B)$. Thus it suffices to compare these rules wrt classification of one population. We choose μ_1 for this purpose without loss of generality.

Now we introduce the parameters used in the comparison as follows.

$$\beta_1 = \frac{nv^2}{\sigma^2}, \quad \beta_2 = \frac{\Delta}{\sqrt{2}v^2}, \quad \beta_3 = \frac{\epsilon}{\sqrt{2}v^2} \text{ and } \beta_4 = \frac{\mu_1 - \mu_0}{\sqrt{2}v^2}.$$

It can be verified that the overall risk wrt τ_0 of these rules can be written as

$$\begin{aligned} & \ell \phi(-A-B-C) + \phi(A-B-C) + \phi(D-E) + \ell \phi(-D-E) \\ & - \phi_0(-A-B-C, -D-E; \rho) + (1-\ell) \phi_0(-A-B-C, D-E; \rho) \\ & - \phi_0(-A+B-C, D-E; \rho) + \phi_0(A-B-C, -D-E; \rho) \\ & - \phi_0(-A+B-C, -D-E; \rho) - \phi_0(A-B-C, D-E; \rho) \\ & + \phi_0(-A+B-C, D-E; \rho) - \phi_0(A-B-C, -D-E; \rho) \\ & - \phi_0(A+B-C, D-E; \rho) - (1-\ell) \phi_0(A+B-C, -D-E; \rho) \end{aligned}$$

where $\phi_0(\cdot, \cdot; \rho)$ is the cdf of a bivariate normal distribution with zero means, unit variances and correlation coefficient ρ , and where $A = \beta_2$, $B = \beta_3$, $C = \beta_4$, $\rho = \beta_1^{-\frac{1}{2}}(1+\beta_1)^{-\frac{1}{2}}$, $D = d_1 \beta_1^{-\frac{1}{2}}$ for δ^B , $D = \max(c_1, 0)(1+\beta_1)^{-\frac{1}{2}}$ for δ^I , $E = \rho^{-1} \beta_4$ for δ^B and $E = \rho \beta_4$ for δ^I with d_1 and c_1 being those in Theorem 5.1 and Example 4.1, respectively. Also $\sup_{\tau \in \Gamma} r_1(\tau, \delta_1)$ for both rules

can be written as

$$\begin{aligned} & \gamma_1 [\phi(R+|S|-T-U) - \phi(-R+|S|-T+U) + \lambda \phi(-R+|S|-T-U)] \\ & + \gamma_1' [\phi(-R-S+T-U) + \phi(-R+S-T+U)]^V [\phi(-R+S+T-U) + \phi(-R-S-T+U)] \\ & + (1-\gamma_1') \phi(-R+|S|-T+U) \end{aligned}$$

where $x^V y = \max(x, y)$, $T = \beta_2 \beta_1^{\frac{1}{2}}$, $U = \beta_3 \beta_1^{\frac{1}{2}}$, $S = \beta_4 \beta_1^{-\frac{1}{2}}$ for δ^B , $S = 0$ for δ^Γ , $R = \beta_1^{-\frac{1}{2}} (1 + \beta_1)^{\frac{1}{2}} d_1$ for δ^B , $R = \max(c_1, 0)$ for δ^Γ with d_1 and c_1 being those in Theorem 5.1 and Example 4.1, respectively. For selected values of β_i ($i = 1, \dots, 4$), Table I and Table II give $r_1(\tau_0, \delta_1)$ and $\sup_{\tau \in \Gamma} r_1(\tau, \delta_1)$ for $\delta_1 = \delta_1^\Gamma$, δ_1^B for $\lambda = 0$ and $\lambda = 1$, respectively. It can be observed from these tables that, in many cases, the increase in the overall risk wrt τ_0 from the use of δ_1^Γ is only slight compared to that in $\sup_{\tau \in \Gamma} r_1(\tau, \delta_1)$ from the use of δ_1^B . In this sense, δ^Γ is more robust against other formulation than δ^B . Such properties of δ^Γ become more prominent as the difference between the prior means (β_4) increases and the prior variance (β_1) gets smaller. When we have the same prior means and the prior variance is large, both rules compare favorably with each other. In most cases, we can observe that δ^Γ compares favorably with δ^B in terms of the overall risk.

Table I

Overall risks and the values of $\sup_{\tau \in \Gamma} r(\tau, \delta)$ of δ^B and δ^Γ when $\lambda = 0$.

		δ^Γ	δ^B	δ^Γ	δ^B	δ^Γ	δ^B
β_4	β_1	.25		1.0		4.0	
		$\beta_2 = 0.5$			$\beta_3 = 0.125$		
1.0		.4961	.9696	.4097	.7784	.2816	.5404
		.3930	.2587	.2864	.2097	.2001	.1189
0.5		.5344	.8274	.4577	.5201	.3312	.4359
		.4520	.3957	.3321	.2963	.2121	.1615
0.0		.5164	.5527	.4568	.4626	.3413	.3492
		.4599	.4544	.3341	.3315	.2103	.1786
		$\beta_2 = 0.5$			$\beta_3 = 0.05$		
1.0		.5250	.9731	.4471	.8117	.3261	.6276
		.4140	.2853	.3103	.2395	.2413	.1493
0.5		.5768	.8460	.5088	.6703	.3999	.5306
		.4898	.4330	.3840	.3368	.2899	.2021
0.0		.5680	.5937	.5170	.5190	.4210	.4391
		.5027	.4978	.3828	.3764	.2856	.2232
		$\beta_2 = 0.8$			$\beta_3 = 0.2$		
1.0		.4770	.9598	.3793	.7074	.2574	.4408
		.3909	.2824	.2539	.1931	.1369	.0923
0.5		.3739	.7920	.3450	.5344	.2692	.3449
		.3594	.3163	.2424	.2237	.1152	.1083
0.0		.3172	.5226	.3066	.3794	.2546	.2667
		.3151	.3050	.2404	.2298	.1133	.1131
		$\beta_2 = 0.8$			$\beta_3 = 0.08$		
1.0		.5365	.9699	.4541	.7840	.3573	.5860
		.4380	.3309	.3190	.2411	.2407	.1353
0.5		.4357	.8308	.4278	.6286	.3960	.4890
		.4311	.3742	.3137	.2801	.1632	.1588
0.0		.3789	.5827	.3785	.4658	.3679	.3967
		.3789	.3658	.3509	.2887	.1939	.1659

The numbers on the first (second) row in each box are the values of

$\sup_{\tau \in \Gamma} r(\tau, \delta)$ ($r(\tau_0, \delta)$).

Table II

Overall risks and the value of $\sup_{\tau \in \Gamma} r(\tau, \delta)$ of δ^B and δ^P when $\ell = 1$.

		δ^P	δ^B	δ^P	δ^B	δ^P	δ^B
β_4	β_1						
		.25		1.0		4.0	
		$\beta_2 = 0.5$		$\beta_3 = 0.125$			
1.0		.6655	1.6167	.5435	1.0979	.3258	.5696
		.5218	.2929	.2947	.2195	.1674	.1194
		.5746	1.2157	.5098	.7758	.3496	.4498
0.5		.5280	.4325	.3361	.3082	.1941	.1620
		.5299	.6991	.4865	.5298	.3529	.3560
0.0		.5085	.4817	.3474	.3437	.1985	.1791
		$\beta_2 = 0.5$		$\beta_3 = 0.05$			
1.0		.7117	1.6582	.6084	1.1721	.4027	.6698
		.5688	.3264	.3280	.2523	.2059	.1500
		.6242	1.2762	.5775	.8566	.4372	.5520
0.5		.5872	.4778	.3828	.3522	.2445	.2030
		.5817	.7627	.5545	.6032	.4452	.4502
0.0		.5683	.5316	.4017	.3923	.2510	.2242
		$\beta_2 = 0.8$		$\beta_3 = 0.2$			
1.0		.5021	1.4114	.3982	.8181	.2580	.4413
		.4248	.2883	.2465	.1938	.1355	.0923
		.3745	.9929	.3470	.5702	.2693	.3451
0.5		.3642	.3195	.2448	.2243	.1151	.1083
		.3172	.5774	.3070	.3900	.2546	.2667
0.0		.3157	.3062	.2420	.2303	.1133	.1131
		$\beta_2 = 0.8$		$\beta_3 = 0.08$			
1.0		.5665	1.4765	.4861	.9276	.3617	.5872
		.4891	.3393	.2966	.2423	.2239	.1353
		.4357	1.0748	.4289	.6799	.3960	.4894
0.5		.4337	.3789	.3222	.2811	.1632	.1588
		.3789	.6540	.3785	.4823	.3679	.3969
0.0		.3789	.3675	.3520	.2895	.1939	.1659

The numbers on the first (second) row in each box are the values of

$\sup_{\tau \in \Gamma} r(\tau, \delta)$ ($r(\tau_0, \delta)$).

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optimal selection rules are derived for classifying the treatments into superior, equivalent, or inferior groups. These Γ -minimax rules are compared with Bayes rules for the normal means problem. It is shown that the Γ -minimax rules compare quite favorably with the Bayes rules. Minimax rules are also derived for the same problem.

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