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RESEARCH

MATHEMATICS FOR MANAGERS MADE EASY

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MASTER OF SCIENCE
IN
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ABSTRACT

The increasing emphasis on improved management, both in government and in private industry, has created a requirement that persons other than pure mathematicians be knowledgeable of mathematical procedures more sophisticated than the elementary ones of adding, subtracting, multiplying, and dividing. The acquisition of this knowledge can be particularly painful to those who have been away from college for a long period and those who have never been exposed to higher mathematics. Close scrutiny of available texts indicated that there were none which incorporated sufficient uncomplicated explanations of the various mathematical concepts which a management student must comprehend, if he is to be successful in his pursuit of an advanced degree in Management. The authors of this research paper are of the opinion that down to earth, easy to understand explanations of the basic fundamentals of higher mathematics would assist students embarking upon a course of study in the management field. The areas covered in this research paper are: algebra, functions, graphs, equations, logarithms, exponents, progressions, and elementary calculus.

ACKNOWLEDGEMENT

The authors of this research paper were able to consider numerous ideas on topic material and presentation as a result of reviewing all of the books listed in the bibliography. Since this research paper represents a mingling of ideas gleaned from all of the books, with no one idea being specifically attributable to one book, there is a conspicuous absence of footnotes. As a result, the authors desire to thank all of the authors listed in the bibliography for their unwitting assistance.

Likewise, the authors want to thank Associate Professors Cowie and Krauthamer and Lieutenant Commander Commer for the support they gave the authors in writing this research paper.

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CHAPTER 1

INTRODUCTION

The purpose of this research paper is to supplement a formal course in the basic fundamentals of higher mathematics. It may also be used as a self-instruction book for students who are considering embarking on a course in management for which there is no formal mathematics course.

This syllabus is primarily for

(1) The average student who may never have been exposed to the mathematical concepts contained herein.

(2) The student who has been away from school for a number of years.

The titles of the chapters describe the content of the various chapters. An attempt has been made to relate commonly understood and known occurrences in daily life to the concepts explored. Liberal use of solved examples has been made. The student is advised to carefully follow through these solved problems and to satisfy himself as to the reason for the action taken in each step. Obviously, some problems have been over simplified for purposes of easy explanation. As the student becomes more skilled in the mechanical processes, he will probably perform many of the operations mentally rather than with pencil and paper. There are also numerous problems, with answers, for the student to work.

An understanding of the concepts demonstrated in this research paper will greatly ease the rigors of trying to understand the mathematics found in Economics, Statistics, Industrial Management, Data Processing and many other courses.

CHAPTER 2

ALGEBRAIC DEFINITIONS AND SYMBOLS

2.1 Introduction.

The key to the study of a foreign language is a basic knowledge of the vocabulary of that language. Similarly, mathematics has a small basic vocabulary that each student must first grasp prior to studying the specifics of the subject. The paragraphs that follow contain the definitions of terms that are basic to any course in algebra or calculus. Read them; understand them; memorize them . . . they are most important.

2.2 Definitions.

1. Integer - There is nothing mysterious about an integer. It is simply a whole number, the same numbers we all were introduced to in the first grade. Here we are defining it in postgraduate school. It is frequently called a "natural" number such as 1, 2, 3, 87, -15. It may be either positive or negative.

2. Fraction.- Now that we have the integer definition committed to memory we can use it in defining the fraction. A fraction is the ratio of two integers, or one integer divided by another; e.g. 2/3, 17/5, -7/32, 28/-9. The fraction can either be positive or negative.

3. A rational number is any positive or negative integer, or fraction made up of integers. Rational numbers are those which can be represented by the ratio of two integers, e.g. 28/1, -7/3, 3/-2. This concept perhaps will be simplified by learning what is an irrational number. Irrational numbers are those positive and negative numbers which cannot be represented as the ratio of two integers, e.g. $\sqrt{2}$, $\sqrt[3]{7}$, π (since π is not exactly 22/7).

4. The real numbers system is that system of numbers which includes all positive and negative rational and irrational numbers and the number 0. The real system of numbers can be graphically represented as follows:

$- \infty \; \ldots \; -4, -3, -2, -1, 0, 1, 2, 3, 4 \; \ldots \; + \infty$

5. Explicit numbers are just what you would expect them to be. They are numerical symbols which represent specific numbers, e.g. 1, 6, -3/5.

6. Literal numbers - This is perhaps the most important concept for the beginning student of algebra to grasp. Literal numbers are letters which represent or stand for explicit numbers. A literal number can be any letter of the alphabet. Since this letter may often represent different numbers, it is sometimes called a general number. The use of the literal or general number is the principle feature of algebra that distinguishes it from arithmetic, e.g. a, b, c, x, y, z, 3a, 2x.

7. Variable is defined as a symbol which is used to represent different numbers throughout a particular discussion. In practice, letters from the latter portion of the alphabet are utilized as variables, e.g. x, y, z. Variables are literal numbers that may have several values.

8. A constant is a symbol which represents the same number during a discussion; an absolute constant never changes its value, and it usually is an explicit number such as π, 2, 1/3, etc. An arbitrary constant will have the same value throughout any one discussion or exercise, but may be assigned different values in different exercises, and is normally represented by a literal number. The first few letters of the alphabet are conventionally used as arbitrary constants. In the expression $ax + by + cz$, a, b, and c are arbitrary constants.

9. Absolute value - The length of a line on a number scale which represents a certain number, without regard to the direction (negative or positive) of the number from zero. The absolute value of a negative number is that number with its sign changed. Of a positive number or zero, it is the number itself (sometimes called "numerical value"). The expression "the absolute value of" is represented by two vertical lines straddling a number as follows:

$|x|$ means the absolute value of x and equals x.

$|3|$ means the absolute value of 3 and equals 3.

$|-4|$ means the absolute value of 4 and equals 4.

$-|4|$ means minus the absolute value of 4 and equals -4.

10. The algebraic value of a number is equal to its distance and direction, either plus or minus, from a zero point origin. By convention, plus is to the right of the origin and minus is to the left of the origin. The minus sign is always required but the plus sign may be left off. Any number without a minus sign is automatically considered to be a plus, or positive, number.

As an example, in the diagram below the literal numbers have the following values:

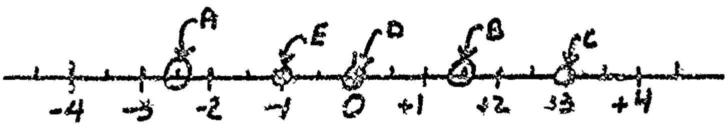
A = -2.5

B = +1.5

C = +3.0

D = 0.0

E = -1.0



11. An algebraic expression is a combination of explicit and literal numbers linked by the four symbols of fundamental mathematical operations; which are addition, subtraction, multiplication, and division. An example

of an algebraic expression follows:

$$A + B - 5D + \frac{6}{V}$$

12. An <u>algebraic term</u> is a distinct part of an algebraic expression which is separated from the rest of the expression by a plus or a minus sign, which is itself a part of the term. As an example, the terms in the algebraic expression above are: A(or +A), +B, -5D, and $+\frac{6}{V}$.

13. A <u>monomial</u> is an algebraic expression with only one term in it.

14. A <u>binomial</u> is an algebraic expression with two terms in it.

15. A <u>trinomial</u> is an algebraic expression with three terms in it.

16. A <u>polynomial</u> is any algebraic expression containing more than one term, although often the word is restricted to mean an expression which is composed of a term or terms that contain a literal number raised to a positive, integral number or zero power. Excluded from the meaning of polynomial are expressions with terms containing literal numbers raised to a negative or a fractional power. Examples are as follows:

Broad definition: $A + B - 5D - 6$

Restricted definition: $3x^2 + 5x + 5$

17. An <u>algebraic factor</u> is a part of an algebraic term which is multiplied by another part or parts of the same term. It can be made up of only one number which cannot be further broken down into integral, rational numbers other than a combination of itself and the number 1 or its negative and the number -1, in which case it is called a prime factor, or it can be made up of several factors multiplied together. As an example, in 7bTV, some of the factors are 7 (which would be a prime factor), b, T, V, or combinations of the individual components such as 7b, bT, 7bT, etc.

18. The <u>coefficient of a term</u> is the factor by which the other factors of a term are multiplied. The explicit number part of the factor

is called a numerical coefficient. The factor normally considered as the coefficient is the factor listed first in the term. As an example, in 7bTV, any of the factors (i.e., 7, b, T, V, 7b, 7bT, etc.) could be considered a coefficient of the other factors making up the term, although 7 would usually be considered the coefficient. The numerical coefficient would be 7 in all cases where the 7 was a part of the factor considered the coefficient.

19. The commutative law for addition indicates that the sum of two numbers is the same regardless of the order in which we add the numbers. That is, a + b is exactly the same as b + a.

20. The associative law for addition indicates that the sum of three or more numbers is the same regardless of the order they are considered in, or how they are grouped. That is, a + b + c is the same as b + c + a or c + a + b.

21. The commutative law for multiplication indicates that the product of two numbers is the same regardless of the order in which they are multiplied. That is, a times b is exactly the same as b times a.

22. The associative law for multiplication indicates that the product of three or more numbers is the same regardless of the order of multiplying the numbers together.

23. Parenthesis ( ), brackets [ ], braces { }, and viniculum -- are all symbols that indicate a grouping of terms or factors within the symbol. Where the symbol is preceded by a literal or explicit number or combination thereof, the entire grouping is multiplied by that coefficient. As an example:

$$6b(a+b+5) = 6ba + 6b^2 + 30b$$

24. The distributive law for multiplication means that the product of a number and a sum of numbers is the same as the sum of the products obtained by multiplying each of the other numbers by the first number. As an example:

$$a(b+c+d) = ab + ac + ad$$

25. The denominator is the number below the line in a fraction, while the numerator is the number above the line. As an example, in the fraction $\frac{235}{678}$, the denominator is 678 and the numerator is 235.

26. In multiplication, the number which is multiplied by another number is known as the multiplicand, while the number that we multiply by is called the multiplier.

27. The product is the result of multiplying the multiplicand by the multiplier. As an example, if we have a multiplicand of 56 and a multiplier of 2, the product is 112.

28. In division, the number which is divided into another number is known as the divisor, while the number divided is known as the dividend. The result of the division is known as the quotient.

2.3 Symbols.

$=$	is equal to
$\equiv$	is identically equal to.
$\neq$	is not equal to.
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to.
$\geq$	is greater than or equal to.
$n!$	n factorial, or, factorial n.
$\propto$ (or $\sim$)	varies as.
$\sum a_n =$	sum of variables of which a_n is the representative.
a^n	a to the nth power, or, a exponent n.
$\sqrt{a}$	square root of a.
$\sqrt[n]{a}$	nth root of a.

$\approx$ is approximately equal to.

$|a|$ absolute value of a.

a_n a subscript n, or, a sub n.

$f(x)$ f-function of x, or, f of x.

(x,y) point whose coordinates are x and y.

x' and x'' x prime and x second respectively

$n \to \infty$ as n increases without bound.

CHAPTER 3

FUNDAMENTAL OPERATIONS OF ALGEBRA

3.1 Introduction.

The basic operations of algebra are the same as arithmetic; that is, addition, subtraction, multiplication, and division. This fact is basic to the study of algebra. The student must constantly remind himself that the a's, b's, x's and y's are literal numbers representing real numbers. Too often, the introduction of a literal number as a variable into a discussion causes the beginning student to develop an unnecessary mental block. Perhaps a simple example may assist the reader in clearing this hurdle. If a student is asked, "What number when added to 4 is equal to 6?", in arithmetic he would say $4 + ? = 6$. By rapid arithmetic the third grade student would supply the answer 2. Now, let us replace the question mark by x. We have $4 + x = 6$. What is x? 2, of course. Too often students block at this point. Similarly 4 divided by what equals 2? In arithmetic, we say $\frac{4}{?} = 2$. Algebra simply approaches this problem by again replacing the question mark by x. The following chapter is perhaps the most important chapter in the book. The student should work and understand all of the exercises and examples.

This knowledge will pay great dividends as the course progresses. Remember always that the operations are really the same as those used in arithmetic. As you work through the exercises you will see that algebra really is not as hard as you expected and that, in fact, the solution of problems can be quite enjoyable.

3.2 Division by Zero.

If we have the expression $\frac{a}{o} = b$, where a is some number other than o, then it would follow that a = bo (by multiplying both sides of the equation by o). Thus, we are now saying that a = o, because o times anything is o. This is illogical because we said that a was some number other than o. For this reason, we say that any number divided by o is undefined.

3.3 Positive and Negative Numbers.

One essential difference between arithmetic and algebra is that in algebra we utilize negative numbers. This should seem logical to the student since we have such things as profit and loss and positive and negative temperatures. The symbols + and - are used to signify whether a number is positive or negative. The same signs are also used to denote addition or subtraction.

Examples:

$(+6) + (-6) = 0$

$(-5) + (-3) = -8$ Parenthesis used for clarity

$(+6) - (+4) = 2$

A number which appears without a sign is by convention considered to be positive.

3.4 Operations with Signed Numbers.

The following rules concerning operations with signed numbers are presented without proof:

Rule 1. To add two numbers of like sign, add their absolute values and prefix the answer by the "like" sign.

Rule 2. To add two numbers of unlike sign, subtract the smaller absolute value number from the larger and prefix the result with the sign of the larger absolute value number.

Addition Examples:

$$\begin{array}{cccc} +8 & +6 & -7 & -15 \\ \underline{+2} & \underline{-4} & \underline{-3} & \underline{+2} \\ +10 & +2 & -10 & -13 \end{array}$$

Rule 3. To subtract one number from another, change the sign of the number to be subtracted and then add the numbers following rules 1 and 2 above.

Subtraction Examples:

$$\begin{array}{cccc} +3 & +15 & -10 & -4 \\ \underline{-(+3)} & \underline{-(-10)} & \underline{-(+5)} & \underline{-(-3)} \\ 0 & +25 & -15 & -1 \end{array}$$

Rule 4. To multiply two numbers, multiply their absolute values. If the signs of the numbers are the same, the answer will be positive. If the signs of the numbers are not the same, the sign of the answer is negative. In short, a plus times a plus or a minus times a minus = a plus answer, while a plus times a minus gives a minus answer.

Multiplication Examples:

$(+2) \times (+4) = +8$

$(-3) \times (-6) = +18$

$(+3) \times (-2) = -6$

$(-1) \times (+8) = -8$

Rule 5. To divide two numbers, divide their absolute values. If the signs of the numbers are alike, the answer will be positive. If the signs of the numbers are unlike, then the answer will be negative.

Division Examples:

$$(+5) \div (+1) = +5$$

$$(+10) \div (-2) = -5$$

$$(-20) \div (-5) = +4$$

$$(-8) \div (+4) = -2$$

As you can see, the multiplication and division rules are practically the same.

3.5 Exponents.

$a \cdot a = a^2$ (the dot is the symbol for times). We say that a is raised to the power 2. a is called the base. Similarly, $b \cdot b \cdot b = b^3$. b is the base of the exponential expression. 3 is the power to which the base is raised. If a letter or number appears without an exponent, then the exponent is taken to be 1. In the expression $6x^2yz^3$, the power of y is 1. A letter or number raised to the zero power is equal to one.

3.6 Addition and Subtraction of Like Terms.

This is a simple topic and can be explained by a simple example. If you have 2 apples and you are given 3 more, you then have 5 apples. Similarly,

$$3x + 2x = 5x$$

$$4xy + 3xy = 7xy$$

$$10\,xyz - 2xyz = 8xyz$$

$$8abc - 7abc = abc$$

3.7 Multiplication of terms.

When we multiply x by x, we arrive at the answer x^2. Similarly:

$$x \cdot x \cdot x = x^3$$

$$x^2 \cdot x^3 = x^5$$

$$x^{12} \cdot x^6 = x^{18}$$

When we multiply like terms, we simply add the exponents of each of the terms and place it as the exponent of the original term. If you think a rule is necessary, then $x^a \cdot x^b = x^{a+b}$

This is another one of those fundamental operations that you ought to know as well as your own name. It should be noted at this point that we cannot multiply unlike terms.

$a^5 \cdot b^4 = a^5b^4$ or b^4a^5, take your choice

Similarly, $x^3 \cdot y^2 = x^3y^2$ or y^2x^3

Another basic point we ought to cover is what happens then our terms to be multiplied contain numerical coefficients. We solve this problem simply by multiplying the numerical coefficients together and going through the same drill that we went through above with the letters and their exponents.

Thus: $6a^2 \cdot 3a^3 = 18\ a^5$

$2b \cdot -3b^4 = -6b^5$

Similarly:

$$2a^2 \cdot 4a \cdot -3a^3 = -24a^6$$

$$a \cdot 6a^3 \cdot 4a^5 = 24a^9$$

Let us now try a few on our own and see if we get the answers in the answer column.

Exercises:

Find the products:

1. $a^2 \cdot a^6$ Ans. a^8
2. $x^2 \cdot x^3 \cdot (-x^4)$ Ans. $-x^9$
3. $15x \cdot 10x^5$ Ans. $150x^6$
4. $a^2 \cdot b^{10} \cdot c^3$ Ans. $a^2b^{10}c^3$
5. $3x^2 \cdot 4y^2$ Ans. $12x^2y^2$
6. $4x^2 \cdot 2y^3 \cdot (-10z^5)$ Ans. $-80x^2y^3z^5$
7. $2a^2b^3c^4 \cdot (-3ab^{-2}c^2)$ Ans. $-6a^3bc^6$
8. $(-x^2) \cdot 2x^5 \cdot 6x^3 \cdot (-x)$ Ans. $12x^{11}$
9. $5x^2y^3z^4 \cdot (-3a)$ Ans. $-15ax^2y^3z^4$
10. $3x^a \cdot 4x^b \cdot (-2x)$ Ans. $-24x^{1+a+b}$

3.8 <u>Addition and Subtraction of Polynomials.</u>

This certainly sounds like a mysterious topic. It really isn't. If I have three horses and 2 cows and buy two more horses and sell one cow, I am left with five horses and one cow. Well, this is the same way that we handle a^2, b^2, ax, my and so forth. <u>We simply add and subtract terms that are the same.</u> Remember, a sign before a parenthesis affects everything within the parenthesis.

Examples:

1. $13a + 4b - a - 2b = 12a + 2b$
2. $20x - 10x + 20y - 15y - 5x - 5y = 5x$
3. $-5xy + 10 - 5yx - 5 + 15xy = 5xy + 5$
4. $10 - 3a^2b^3c - 8 + 4b^3a^2c = a^2b^3c + 2$
5. $10a - (5a + 2b) + 3b - (2a - b) = 3a + 2b$

Exercises:

Carry out the indicated operations.

1. $10a^2 + b^2 - 3b^2 - 5a^2$ Ans. $5a^2 - 2b^2$
2. $a - b + c - (3c - 2b - 2c)$ Ans. $a + b$
3. $3xyz + 2xyz - 4xyz$ Ans. xyz
4. $10a^3b^2 + 5b^2a^3 - a^3$ Ans. $15a^3b^2 - a^3$
5. $3(xy)^2 + x^2y^2$ Ans. $4x^2y^2$
6. $x^{3x} \cdot x^{2y}$ Ans. x^{3x+2y}
7. $10xyz - 12 + 2yzx$ Ans. $12xyz - 12$

3.9 Powers of Terms.

Now that we have mastered the multiplication of terms, it is appropriate that we move forward and develop a means of handling the term $(x^2)^3$. This is the same as $x^2 \cdot x^2 \cdot x^2$ which from our previous discussion equals x^6. It then follows that a rule for handling powers of powers is

$$(x^m)^n = x^{mn}$$

and in the general case where a is a constant coefficient

$$(ax^m)^n = a^n x^{mn}$$

We assume that m and n are positive whole numbers because there are no signs. If there are mixed signs, the rules for multiplying mixed signs apply. For the present, we will use only positive numbers.

Examples:

1. $(a^6)^2 = a^{12}$
2. $(3x^2)^3 = 27x^6$
3. $(-2y^2)^3 = -8y^6$
4. $[(3y^2)^2]^3 = (9y^4)^3 = 729y^{12}$

Exercises:

Carry out the indicated operations:

1. $(a^3)^4 \cdot (b^2)^4$ Ans. $a^{12}b^8$
2. $(x^2)^3 \cdot (x^3)^2$ Ans. x^{12}
3. $(2y^2)^2 \cdot (4x^2)^2$ Ans. $64x^4y^4$
4. $(-3a^2)^4$ Ans. $81a^8$
5. $(-2x^2y^3)^2 \cdot (3x^2y^3)^4$ Ans. $324x^{12}y^{18}$
6. $\left[(-2x^2y)^2\right]^3$ Ans. $64x^{12}y^6$
7. $(a^2)^4 \cdot 3b^2 \cdot (3c^2)^3$ Ans. $81a^8b^2c^6$
8. $(x^a)^b \cdot (x^b)^a$ Ans. x^{2ab}
9. $(2a)^y \cdot (3a)^x$ Ans. $2^y3^xa^{x+y}$
10. $\left[(-2x^2)^y\right]^z$ Ans. $-2^{yz}x^{2yz}$
11. $3ax + 4xa - (2ax - 3xa)$ Ans. $8ax$
12. $14x^2 - 2x + 10 - 5x^2 - 3$ Ans. $9x^2 - 2x + 7$

3.10 Division of Terms.

In one of our earlier paragraphs, we developed the technique for handling the multiplication of like terms. We now want to develop a technique for handling the division of like terms, such as $\frac{x^3}{x^2}$. This expression may be rewritten $\frac{x \cdot x \cdot x}{x \cdot x}$. By simple cancellation, the two x's in the denominator cancel out with two of the three x's in the numerator leaving the answer x. This is the same operation that we carried out in elementary arithmetic as: $\frac{2 \cdot 2 \cdot 2}{2 \cdot 2} = 2$. Now, however, we are using literal numbers. It therefore follows that when we divide powers of like terms we simply subtract the power of the denominator from the power of the numerator and use this as the power of the answer. In symbols, this rule is as follows:

$$\frac{x^m}{x^n} = x^{m-n}$$

Really, it is just a fancy way of dividing. As we move along you will realize that it will be a great aid in simplifying what appear to be very complex expressions.

Examples:

1. $\frac{x^3}{x^2} = x^{3-2} = x$ or another approach which might help you grasp the concept $\frac{x^3}{x^2} = \frac{x^2 \cdot x}{x^2} = x$

2. $\frac{a^b}{a^c} = a^{b-c}$

3. $\frac{(x-y)^{15}}{(x-y)^{14}} = (x-y)^{15-14} = x-y$

4. $\frac{(b^2)^x}{(b)^x} = \frac{b^{2x}}{b^x} = b^{2x-x} = b^x$

5. $\frac{(x^2 - y^2 + 3x^2)^{15}}{(2x^2 + 3y^2 + 2x^2 - 4y^2)^{13}} = \frac{(4x^2 - y^2)^{15}}{(4x^2 - y^2)^{13}} = (4x^2 - y^2)^{15-13}$

$= (4x^2 - y^2)^2$

There we took what looked like a very impossible situation and turned it into something that is easy to handle. Now let us go ahead and try a few problems. Remember that as we move along it will be necessary that you utilize all of the little techniques that you have learned thus far. The exercises may contain review problems that have nothing to do with the immediate paragraphs. The message here is, "Do not develop a set approach". Stay loose. Here in the early stages of the course it is recommended that you quickly review the examples in the book and your own solved problems prior to tackling a new lesson. We all need the reinforcement. With this little advice in mind it is time for a drill.

Exercises:

Carry out the indicated operations.

1.	$\dfrac{x^{35}}{x^{15}}$	Ans.	x^{20}
2.	$\dfrac{x^{y-b}}{x^{-b}}$	Ans.	x^{y}
3.	$\dfrac{(x^{35} \cdot x^{5})^{2}}{x^{2}}$	Ans.	x^{78}
4.	$\dfrac{35a^{3}}{7a^{2}}$	Ans.	$5a$
5.	$\dfrac{(4b^{2})^{3}}{(4b)^{2}}$	Ans.	$4b^{4}$
6.	$\dfrac{(30a^{2} \cdot b^{3} \cdot 4c^{4})^{3}}{(40b^{3} \cdot 3a^{2} \cdot c^{4})^{2}}$	Ans.	$120a^{2}b^{3}c^{4}$
7.	$3x^{2} \cdot 4x^{3} \cdot 2x \cdot x^{15}$	Ans.	$24x^{21}$
8.	$\dfrac{3x^{y} \cdot 4x^{z}}{12x^{n}}$	Ans.	x^{y+z-n}
9.	$\dfrac{(2t^{2})^{2n}}{(2t)^{n}}$	Ans.	$2^{n}t^{3n}$
10.	$2xyz + 3yxz - 2yzx$	Ans.	$3xyz$
11.	$\dfrac{(3a^{3}b^{2}c + 17b^{2}ca^{3})^{10}}{(20cb^{2}a^{3})^{8}}$	Ans.	$400a^{6}b^{4}c^{2}$
12.	$\dfrac{(2x^{2} \cdot 3y^{4} \cdot z)^{2}}{36z^{2}x^{4}y^{8}}$	Ans.	1
13.	$\dfrac{(x-y)^{a+b}}{(x-y)^{a}}$	Ans.	$(x-y)^{b}$
14.	$\dfrac{(a+b+c)^{3/2}}{(c+b+a)^{\frac{1}{2}}}$	Ans.	$a+b+c$

15. $\dfrac{(c^{\frac{1}{4}} \cdot c^{\frac{3}{4}} \cdot c)^2}{c^2}$ Ans. c^2

16. $5xyz - 3xyz + 10$ Ans. $2xyz + 10$

3.11 Fractional Exponents.

Earlier in this chapter, we learned that when we multiply two like quantities we add the exponents to get the product $x^2 \cdot x^3 = x^5$. Following that reasoning, if we multiply $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$ we obtain x^1 or x. Another rule that we used in arithmetic was that the square of the square root of a number equals the number or

$$\sqrt{4} \cdot \sqrt{4} = 4$$
$$\sqrt{16} \cdot \sqrt{16} = 16$$

In algebra the same rule applies.

$$\sqrt{x^4} \cdot \sqrt{x^4} = x^4$$

and

$$\sqrt{y^6} \cdot \sqrt{y^6} = y^6$$

but

$$(y^6)^{\frac{1}{2}} \cdot (y^6)^{\frac{1}{2}} = (y^6)^1 = y^6$$

We then can conclude that if we raise any quantity to the $\frac{1}{2}$ power this is the same operation as taking the square root of the quantity.

Therefore $x^{\frac{1}{2}} = \sqrt{x}$

$$\sqrt{x} \cdot \sqrt{x} = x = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$$

Remembering also in one of the previous paragraphs that when we raise a power to a power we multiply the two powers as:

$$(x^2)^3 = x^{2 \cdot 3} = x^6$$

Similarly $(x^2)^{\frac{1}{2}} = x^{2 \cdot \frac{1}{2}} = \sqrt[2]{x^2} = x$

then $(x^3)^{\frac{1}{3}} = x^{3 \cdot \frac{1}{3}} = \sqrt[3]{x^3}$

but $\sqrt[3]{x^3}$ also $= x$ likewise $(x^4)^{\frac{1}{4}} = \sqrt[4]{x^4} = x$

In the general case then $(x^a)^{\frac{1}{b}} = \sqrt[b]{x^a} = x^{\frac{a}{b}}$

Examples:

1. $a^{\frac{2}{3}} = \sqrt[3]{a^2}$
2. $y^{\frac{1}{2}} = \sqrt[2]{y}$ or $\sqrt{y}$
3. $a^{\frac{9}{5}} = \sqrt[5]{a^9}$
4. $(y^5)^{\frac{1}{3}} = \sqrt[3]{y^5}$
5. $(x^3 \cdot x^3)^{\frac{1}{3}} = \sqrt[3]{x^6} = x^2$

3.12 Negative Exponents.

At this stage of the game, we ought to have the operation $\frac{x^m}{x^n} = x^{m-n}$ down cold. For a moment, you should recall our rule for the multiplication of like terms which was $x^m \cdot x^n = x^{m+n}$. By the same token $x^m \cdot x^{-n} = x^{m-n}$. Well lo and behold this is the same result that we arrived at after carrying out the operation of $\frac{x^m}{x^n}$. We then can conclude that $x^{-n} = \frac{1}{x^n}$. In simple terms, we say that an expression which has a negative exponent is equivalent to the reciprocal $\left(\frac{1}{\text{expression}}\right)$ of the expression to the same absolute value of the exponent with a positive sign. The above few sentences might sound like a little gibberish, however, after you work through the following problems you will have acquired another addition to your bag of tricks. After studying some examples, you will probably agree that there is nothing mysterious about what we are doing.

1. $\frac{1}{x^{-22}} = x^{22}$
2. $\left(a^{-2} + \frac{1}{(a)^2}\right)\left(\frac{2}{a^{-2}}\right) = \left(\frac{2}{a^2}\right)(2a^2) = 4$
3. $\frac{1}{a^{3x}} \cdot \frac{1}{a^{-3x}} = \frac{1}{a^{3x}} \cdot a^{3x} = 1$

4. $\frac{1}{(a+b)^{-2}} \quad \frac{1}{2(a+b)} = (a+b)^2 \times \frac{1}{2(a+b)} = \frac{a+b}{2}$

5. $\frac{1}{a^{-3x}} \cdot \frac{1}{a^{-2y}} \cdot a^{+2x} = a^{3x} \cdot a^{2y} \cdot a^{+2x} = a^{5x+2y}$

Exercises:

1. $3xy + 2yx + \frac{1}{(yx)^{-1}}$ Ans. $6yx$

2. $a + \frac{1}{a^{-2}} + \frac{2}{a^{-3}}$ Ans. $2a^3 + a^2 + a$

3. $(b^2)^2 + \frac{1}{(b^2)^{-2}}$ Ans. $2b^4$

4. $(x^m)^n \cdot \left(\frac{4}{(x^{-m})^2}\right)$ Ans. $4x^{mn+2m}$

5. $(a+b)^{-2} \cdot (a+b)^6 \cdot (a+b)^{-2}$ Ans. $(a+b)^2$

6. $x^{-\frac{1}{2}} \div x^{-\frac{3}{2}}$ Ans. x

7. $3a^2 + 2b^2 + 3ab - a^2 - 3b^2 - ab$ Ans. $2a^2 - b^2 + 2ab$

8. $(a+b)^y \quad (a+b)^{-y} \cdot \left(\frac{1}{(a+b)^{-2y}}\right)$ Ans. $(a+b)^{2y}$

9. $(3a^2)^2 \cdot (a^{-4})$ Ans. 9

10. $a+b - (3a - 4b)$ Ans. $5b - 2a$

11. $\frac{1}{a^{-2}} + \frac{3}{a^{-2}} - \frac{2}{a^{-2}} + \frac{10}{a^{-2}}$ Ans. $12a^2$

12. $3a^2 - 2b^2 - \frac{1}{(3a^2 - 2b^2)^{-1}}$ Ans. 0

13. $(3y)^x \quad (3y)^y \quad \frac{1}{(3y)^{-z}}$ Ans. $3y^{x+y+z}$

14. $-(a+b) + 3b - 2a$ Ans. $2b - 3a$

15. $\frac{1}{x^{-m}} \cdot x^{m+2} \cdot \frac{1}{x^{+2m}}$ Ans. x^2

16. $x^{t} \cdot \frac{1}{x^{-u}} \cdot x^{2t-u}$ Ans. x^{t}

17. $6a + 2b - \frac{1}{(5a - b)^{-1}}$ Ans. $a + 3b$

18. $(a - b)^{2} \cdot \frac{1}{(a-b)^{3}}$ Ans. $\frac{1}{a - b}$

19. $-x + y + 3y + 4x - \frac{1}{x^{-1}}$ Ans. $2x + 4y$

20. $[(3x)^{2}]^{2}$ Ans. $81x^{4}$

21. $(y^{3} \cdot y^{3})^{\frac{1}{3}}$ Ans. y^{2}

22. $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{6}}$ Ans. x

23. $\sqrt[3]{x^{2}} \cdot (x^{\frac{2}{3}})^{-1}$ Ans. 1

Express as a fractional exponent

24. $\sqrt[3]{x^{50}}$ Ans. $x^{\frac{50}{3}}$

25. $\sqrt[4]{x^{8}}$ Ans. $x^{\frac{8}{4}}$

3.13 Multiplying Polynomials.

In arithmetic, we had very little trouble multiplying one number by another. If we had an expression such as 2 (4 + 3), we simply added the 4 to the 3 and then multiplied the sum of 7 by the 2 to get an answer of 14. You should note that you will also get the same answer of 14 if you multiply the 5 by the 2 and then the 3 by the 2 and then add the individual sums which are 8 and 6. By rapid mathematics we arrive at 14. In short:

$$2\ (3 + 4) = (2 \cdot 3) + (2 \cdot 4) = 14$$

Now let's take the algebraic expression a + b. Due to its literal makeup we cannot simplify it any further as we could have in the expression (4 + 3) above. Suppose we wanted to multiply the a + b by c.

$$c\ (a + b)$$

We approach this in the same manner in which we arrived at 14 in the second method above. We simply multiply each part of the polynomial $a + b$ by the monomial c and add up each of the individual products (called partial products) to get our answer. Thus:

$$c\,(a + b) = ca + cb$$

By the same token:

$$a\,(b - c + d) = ab - ac + ad$$

and

$$x\,(y - z + x^2) = xy - xz + x^3$$

Now that we have digested that, let's go on to the next tidbit of knowledge. Here again, the arithmetic analogy is also most appropriate. In arithmetic, if we were asked to find the following product $(2 + 1)\,(3 + 4)$ we could get our answer by two approaches as above.

$$2 + 1 = 3 \quad \text{and} \quad 3 + 4 = 7$$

then

$$3 \times 7 = 21$$

Notice, also, we can multiply

$$2 \cdot 3 \quad \text{and} \quad 2 \cdot 4$$

and

$$1 \cdot 3 \quad \text{and} \quad 1 \cdot 4$$

Then we add up the partial products of 6, 8, 3 and 4 and also obtain the answer 21.

In algebra again because of its literal formation we often must use the latter method to multiply one polynomial by another. For example:

$$(a + b)\,(c + d) = ac + ad + bc + bd.$$

It follows then that

$$(a + b)\,(c + d + e) = ac + ad + ae + bc + bd + be$$

Then the rule that we follow to multiply one polynomial by another is to multiply each of the terms of either <u>factor</u> by each of the terms of the other factor, and then add up all of the partial products to obtain the answer.

Examples:

1. $(a+b)\ (a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$

2. $(a+b)\ (a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$

3. $(a+b)\ (a^2 + 2ab + b^2)$

for simplicity let's rewrite this as follows:

$$
\begin{array}{lr}
 & a^2 + 2ab + b^2 \\
 & \underline{a \quad + b \qquad\quad} \\
 & a^3 + 2a^2b + b^2a \qquad\quad \\
 & \underline{\quad + \ a^2b + 2b^2a + b^3} \\
\text{Then adding} & a^3 + 3a^2b + 3b^2a + b^3
\end{array}
$$

4. Also

$$
\begin{array}{l}
a^2 - b^2 \\
\underline{a \quad - b} \\
a^3 - b^2a - ba^2 + b^3
\end{array}
$$

Time and time again in the days ahead you will be called upon to multiply one binomial by another binomial. A quick approach is as follows:

$$
\begin{array}{c}
a + b \\
a - b
\end{array}
$$

First multiply the columns $a \cdot a$ and $-b \cdot +b$ getting a^2 and $-b^2$ respectively; then cross multiply the corners. Remember, always your sign conventions; $+ \cdot + = +$, $+ \cdot - = -$, and $- \cdot - = +$.
In the above case we got $+ab$ and $-ab$ which when added together gives zero. Our answer then is $a^2 - b^2$. Let's try a few more for a drill.

Examples:

1. $$\begin{array}{l} 2x - 3y \\ \underline{x + y} \\ 2x^2 - 3xy - 3y^2 \\ \underline{\quad + 2xy \qquad\qquad} \\ 2x^2 - xy - 3y^2 \end{array}$$

(We have arranged like terms vertically for ease of addition and subtraction.)

2. $$\begin{array}{l} 2x + 3 \\ \underline{3y - 2 \qquad\qquad} \\ 6xy - 4x + 9y - 6 \end{array}$$

3. $$\begin{array}{l} x - y \\ \underline{2x - 2y \qquad} \\ 2x^2 - 2xy + 2y^2 \\ \underline{\quad - 2xy \qquad\quad} \\ 2x^2 - 4xy + 2y^2 \end{array}$$

Now I think we are ready to try a few exercises.

Exercises:

Carry out the indicated operations.

1. $b\,(ax + by)$ Ans. $abx + b^2$
2. $x\,(x^3 + x^2 + x + 2)$ Ans. $x^4 + x^3 + x^2 + 2x$
3. $(y^2 + 2)\,y^5$ Ans. $y^7 + 2y^5$
4. $(6x + 2)\,3y$ Ans. $18xy + 6y$
5. $(a - b)\,(a + b)$ Ans. $a^2 - b^2$
6. $(x + y)\,(x + y)$ Ans. $x^2 + 2xy + y^2$
7. $(3y + x)\,(2y - x)$ Ans. $6y^2 - xy - x^2$
8. $(2a + b)\,(a + 2b)$ Ans. $2a^2 + 5ab + 2b^2$
9. $\dfrac{1}{x^{-2}}\quad(x - y)$ Ans. $x^3 - x^2y$
10. $\left(\dfrac{1}{a^{-3}}\right)\ \left(\dfrac{1}{a^2}\right)$ Ans. a

11. $(x - y)\ (x + y)$ Ans. $x^2 - y^2$

12. $\left(\frac{1}{(y - x)^{-1}}\right)\left(\frac{1}{(y + x)^{-1}}\right)$ Ans. $y^2 - x^2$

13. $y^3x^2\ (a + b)$ Ans. $ay^3x^2 + by^3x^2$

14. $(x^2)^3\ (a + b)$ Ans. $ax^6 + bx^6$

15. $(a^2 - b^2)\ (a + b)$ Ans. $a^3 - ab^2 + ba^2 - b^3$

16. $(x + y)\ (x^2 + 2xy + y^2)$ Ans. $x^3 + 3x^2y + 3y^2x + y^3$

17. $(a - b)\ (a + 3 + b)$ Ans. $a^2 + 3a - 3b - b^2$

18. $(3x + 2y)\ (2x + 3y)$ Ans. $6x^2 + 13xy + 6y^2$

19. $(a + b)\ (-(a - b))$ Ans. $\cancel{b^2 - a^2}$

20. $(x^3 + 3x^2 - 2)\ \frac{1}{(3y)^{-1}}$ Ans. $3x^3y + 9x^2y - 6y$

21. $\frac{3x}{y^{-3}}\ (x^2 + 2)$ Ans. $3x^3y^3 + 6xy^3$

22. $(3y^2)^4\ \left(\frac{1}{y^{-4}}\right)$ Ans. $81\ y^{12}$

23. $(3x + y)\ (x^2 + 2xy + y^2)$ Ans. $3x^2 + 7x^2y + 5y^2x + y^3$

24. $a^{100}\ (a^{50} + a^{25})$ Ans. $a^{150} + a^{125}$

25. $(b^{35})\ (b^5 + b)\ (b^5)$ Ans. $b^{45} + b^{41}$

26. $(a-b)\ (a-b)\ (-(-a+b))$ Ans. $a^3 - 3a^2b + 3b^2a - b^3$

27. $2\ (a+b)\ (a+b)^{-1}$ Ans. 2

28. $(3x)\ (x^{30})\ (x^{-25})^{-1}$ Ans. $3x^{56}$

29. $ax^3y^2\ (a + x + y)$ Ans. $a^2x^3y^2 + ax^4y^2 + ax^3y^3$

30. $(4x^{-2}x^4)\ \left(\frac{1}{ax^2}\right)$ Ans. $\frac{4}{a}$

3.14 Dividing Polynomials.

Since division is the mathematical opposite to multiplication, the rules of algebra for division are the reverse of the rules for multiplication. If we desire to divide a polynomial by a monomial, i.e.

$$(x^2 + 2x) \div x$$

we simply divide each term of the polynomial by the monomial to get our answer. This is the same as the following arithmetic operations.

$$(10 + 25) \div 5 = 7$$

or we could attack it this way

$$\frac{10 + 25}{5} = \frac{10}{5} + \frac{25}{5} = 2 + 5 = 7$$

This latter operation is analagous to what we do in algebra. In the above problem we would write $\frac{x^2 + 2x}{x} = \frac{x^2}{x} + \frac{2x}{x} = x + 2$

Examples:

1. $(4x^4 + 2x^2) \div 2x$

   $\frac{4x^4}{2x} + \frac{2x^2}{2x} = 2x^3 + x$

2. $(10a^6 + 5a^3 - 3a^2) \div 5a^2$

   $\frac{10a^6}{5a^2} + \frac{5a^3}{5a^2} - \frac{3a^2}{5a^2}$

   $= 2a^4 + a - \frac{3}{5}$

We will have little need in this cource to divide a polynomial by another polynomial. This technique is analgous to the operation we call long division in arithmetic. We will go ahead and solve a couple of examples however. Most of the problems that we will encounter of this form we will treat as fractions. Through a process of factoring, which we have yet to cover, we will attempt to break down complex fractional expressions into simplified factors which we will endeavor to cancel out. This may confuse you for the moment. Do not worry about it for now.

Examples:

$$(x^2 + 2xy + y^2) \div (x + y)$$

Then

$$\begin{array}{r|l} & \quad x + y \\ \hline x + y & x^2 + 2xy + y^2 \\ & \underline{x^2 + xy} \\ & \qquad xy + y^2 \\ & \qquad \underline{xy + y^2} \\ & \qquad\quad 0 \end{array}$$

This is the long division approach. We divide the first number of the divisor which is x into the first term of the dividend. We get as a partial quotient +x. This we multiply by the divisor to get $x^2 + xy$. We then subtract this from the original dividend and we obtain $xy + y^2$. We then go through the operation again and get the partial quotient +y. In this case there is no remainder. Thus, $x + y$ goes into $x^2 + 2xy + y^2$ exactly $x + y$ times. We see then that $(x + y)\ (x + y)$ equals $x^2 + 2xy + y^2$. When we take up factoring you will learn and then recognize that $x^2 + 2xy + y^2 = (x + y)^2 = (x + y)\ (x + y)$

When asked to divide two polynomials, we would put them into fraction form as follows:

$$\frac{x^2 + 2xy + y^2}{x + y} = \frac{(x + y)\ \cancel{(x + y)}}{\cancel{(x + y)}} = x + y$$

Let's do a long division problem now which does not divide out evenly.

$$
\begin{array}{r|l}
 & a^2 - 4ab + 6b^2 \\ \hline
a + b & a^3 - 3a^2b + 2b^2a - b^3 \\
 & \underline{a^3 + a^2b} \\
 & \quad - 4a^2b + 2b^2a \\
 & \quad \underline{- 4a^2b - 4b^2a} \\
 & \qquad\qquad + 6b^2a - b^3 \\
 & \qquad\qquad \underline{+ 6b^2a + 6b^3} \\
 & \qquad\qquad\qquad\quad 7b^3 \quad \text{Remainder}
\end{array}
$$

The answer then is $a^2 - 4ab + 6b^2 + \frac{7b^3}{a+b}$

Let's try a few of these on our own.

Exercises:

Carry out the indicated operations.

1. $(a^3 + 2a^2b - 2b^2a - b^3) \div (a-b)$ Ans. $a^2 + 3ab + b^2$
2. $(x^2 - y^2) \div (x + y)$ Ans. $x - y$
3. $(K^2 - L^2) \div (K - L)$ Ans. $K + L$
4. $(16a^2 - 9b^2) \div (4a + 3b)$ Ans. $4a - 3b$
5. $(3x^2 - 4x - 4) \div (x - 2)$ Ans. $3x + 2$
6. $(12x^2 + 4x - 8) \div (2x + 2)$ Ans. $6x - 4$
7. $(6x^3 + 6x^2 - 2x - 2) \div (3x^2 - 1)$ Ans. $2x + 2$
8. $(3x^2 - 4x - 2) \div (x + 2)$ Ans. $3x - 10 + \frac{18}{x+2}$
9. $\frac{3}{(3x - 4)^{-1}} \cdot (3x + 4)$ Ans. $9x^2 - 16$
10. $K^{15} \cdot K^{15} \cdot K^{10} \cdot K^{-5}$ Ans. K^{35}
11. $x^2y\,(x^3 + 2)$ Ans. $x^5y + 2x^2y$
12. $(x^2 - y^2) \cdot \frac{1}{(x+y)}$ Ans. $x - y$
13. $(a^2 + 2ab + b^2) \cdot (a + b)^{-1}$ Ans. $a + b$

14. $(4x^4 - 1) \div (2x^2 + 1)$ Ans. $2x^2 - 1$

15. $\left(\dfrac{1}{x + 2}\right) \cdot (2x^2 + 3x - 2)$ Ans. $2x - 1$

16. $(3x^4 - 2x^3 + x^2) \div (x^2)$ Ans. $3x^2 - 2x + 1$

17. $(10y^{10} + 5y^5) \div (5y^5)$ Ans. $2y^5 + 1$

18. $(5x^3 - x^2 + 2) \div (x + 1)$ Ans. $5x^2 - 6x + 6 - \dfrac{4}{x+1}$

CHAPTER 4.

FACTORING AND SIMPLIFICATION OF FACTORS

4.1 Factoring.

In the days ahead you will see how algebra can be used to solve many many problems which we could not solve with arithmetic. Oftentimes in the solution of algebraic problems we develop complicated fractions, polynomials and equations that can be simplified by the process that is called factoring. Factoring is the opposite operation to the one we have learned in the previous chapter, the multiplication of polynomials. As we move along into the subject, we will learn several clever techniques which through drill the student will commit to memory. Remember always that factoring is a technique of breaking down complex expressions into factors or multipliers which when and if multiplied together would become the original expression.

The factors of 10 are: 10 and 1, or 5 and 2. The factors of 100 are: 100 and 1, or 20 and 5, or 10 and 10, or 4 and 25, or 2 and 50.

In algebra, the factors of x^2 are x and x, of x^3 are x^2 and x. Remember that factoring is taking a product and finding out what multiplicand and multiplier were multiplied together to arrive at the given expression. If we were asked to divide $\frac{x^3y}{x^2}$, one way of approaching this problem would be to break the numerator into $x^2 \cdot x \cdot y$, divide out the x^2 in the numerator and denominator and you would be left with an answer of xy. Our approach was to "factor" the x^3 into x^2 and x and then simplify the expression. Keep this simple problem in mind as we learn a few more useful tools of the trade.

The first rule of factoring is to factor out a common term, i.e., $ax + ay$ can be factored into $a(x + y)$. As you see, we have worked just opposite to our multiplication process.

Similarly,

$$x^2a + x^4b = x^2(a + x^2b)$$

$$ax^3 + bx^6 = x^3(a + bx^3)$$

and

$$15x^3 - 10x^2 + 5x = 5x(3x^2 - 2x + 1)$$

In the days ahead you will use this technique over and over again. Remember, the first thing to look for is a common term.

Examples:

1. $\dfrac{ax + bx}{a + b} = \dfrac{x\cancel{(a + b)}}{\cancel{(a + b)}} = x$

2. $\dfrac{x^3 - x}{x^3 - 1} = \dfrac{x\cancel{(x^3 - 1)}}{\cancel{(x^3 - 1)}} = x$

The second rule that we will use over and over again is the "difference of two squares rule". The rule is stated symbolically as follows.

$$x^2 - y^2 = (x - y)(x + y)$$

That is, the difference of two squares can be factored into the sum and the difference of the square roots of the absolute values of the original components.

Examples:

1. $a^2 - b^2 = (a - b)(a + b)$
2. $a^4 - b^4 = (a^2 - b^2)(a^2 + b^2) = (a - b)(a + b)(a^2 + b^2)$
3. $4x^2 - 1 = (2x - 1)(2x + 1)$
4. $16x^2 - 4y^2 = (4x - 2y)(4x + 2y)$

The student will perhaps grasp the concept if he will multiply the above factors to obtain the original expression.

Simplify:

5. $\frac{4a^2 - 1}{2a + 1} = \frac{(2a - 1)\cancel{(2a + 1)}}{\cancel{(2a + 1)}} = 2a - 1$

6. $\frac{16x^4 - 1}{(4x^2+1)(2x-1)} = \frac{(4x^2 - 1)\cancel{(4x^2 + 1)}}{\cancel{(4x^2 + 1)}(2x - 1)} = \frac{4x^2 - 1}{2x - 1} = \frac{\cancel{(2x-1)}(2x+1)}{\cancel{2x - 1}} = 2x + 1$

The next factor rule that we ought to know and learn to recognize to further simplify more complex expressions is:

$$a^2 + 2ab + b^2 = (a + b)^2 = (a + b)\ (a + b)$$

Similarly, $4x^2 + 8xy + 4y^2 = (2x + 2y)\ (2x + 2y)$ (In this case we have said that $a^2 = 4x^2$ or $a = 2x$ and $b^2 = 4y^2$ or $b = 2y$)

and $9x^2 + 6x + 1 = (3x + 1)\ (3x + 1)$

The next factor rule that we must also learn is similar to the one we just talked about. It is:

$$a^2 - 2ab + b^2 = (a - b)^2 = (a - b)\ (a - b)$$

and again similarly,

$$4x^2 - 8xy + 4y^2 = (2x - 2y)\ (2x - 2y)$$

and $9x^2 - 6x + 1 = (3x - 1)\ (3x - 1)$

There are several other factor rules that we could develop now, but, we will take them up in the next section after we have drilled on the above newly acquired knowledge.

Exercises:

Factor the following expressions.

1. $axb + bkl$ Ans. $b(ax + kl)$
2. $pb + ap$ Ans. $p(b+a)$
3. $15x^3 + 10x^2$ Ans. $5x^2\ (3x+2)$

4. $64x^2 - 1$ Ans. $(8x+1)\ (8x-1)$

5. $4x^3 - 4xy^2$ Ans. $4x(x-y)\ (x+y)$

6. $a^3 - 2a^2b + ab^2$ Ans. $a\ (a-b)\ (a-b)$

7. $9x^4 + 6x^3 + x^2$ Ans. $x^2\ (3x+1)\ (3x + 1)$

8. $2x^{13} - 2y^{12}x$ Ans. $2x(x^3-y^3)(x^3+y^3)(x^6+y^6)$

4.2 Some Other Common Factors.

There are two other factors that we ought to learn to recognize. These are the sum of two cubes and the difference of 2 cubes. The sum of two cubes can be factored as follows:

$$x^3 + y^3 = (x+y)\ (x^2-xy+y^2)$$

This can be proved by multiplying the two factors.

$$\begin{array}{l} x^2 - xy + y^2 \\ \underline{x \ + y \qquad\qquad} \\ x^3 - x^2y + xy^2 \\ \underline{\quad + x^2y - xy^2 + y^3} \\ x^3 \qquad\qquad\quad + y^3 \end{array} \quad = x^3 + y^3$$

Similarly,

$$x^3 - y^3 = (x-y)\ (x^2 + xy + y^2)$$

and

$$\begin{array}{l} x^2 + xy + y^2 \\ \underline{x \ - y \qquad\qquad} \\ x^3 + x^2y + y^2x \\ \underline{\quad - x^2y - y^2x - y^3} \\ x^3 \qquad\qquad\quad - y^3 \end{array} \quad = x^3 - y^3$$

4.3 Factoring Trinomials of the Form $ax^2 + bxy + cy^2$.

You will recall from the previous chapter when we discussed the multiplication of two binomials, say $(2a - 4b)\ (3a + 2b)$, our method was to multiply each term of either by the terms of the other. A technique which the student may find helpful to carry out this operation is to write the terms one over the other as follows.

$$\begin{array}{l} 2a - 4b \\ 3a + 4b \end{array}$$

Multiply the left column to get $6a^2$ then multiply the right column to obtain $-16b^2$. The next step is to multiply diagonally $\left(\times\right)$ and then algebraically add the result. In the case we are discussing, we would multiply (3a) by (-4b) and obtain -12ab. Then, we would multiply (+4b) by (2a) and obtain 8ab. In summary, our situation would look like

$$\begin{array}{lr} & 2a - 4b \\ & \underline{3a + 4b} \\ & 6a^2 - 12ab - 16b^2 \\ & \underline{+ 8\,ab} \\ \text{Ans.} & 6a^2 - 4\,ab - 16b^2 \end{array}$$

The student should confirm that this operation conforms to the rule of multiplying binomials which we discussed above.

Remember that factoring is a process of determining a possible set of numbers which were multiplied together to get a certain number. We [illegible] with a trinomial of the above form, $6a^2 - 4ab - 16b^2$, and we may want to simplify the expression. Our approach would be, in a sense, working in a reversed direction to the process above. We would first say to ourselves that the two factors will each be made up of an a and b term. We would then jot down

$$\begin{array}{cc} a & b \\ a & b \end{array}$$

We would then see what were the factors of the $6a^2$ and of the $-16b^2$.

$$+ 6a^2 = (3a)(2a) = (6a)(a) = (-3a)(-2a) = (-6a)(-a)$$

$$- 16b^2 = (16b)(-b) = (8b)(-2b) = (4b)(-4b)\text{, etc.}$$

We then match the different combinations of the factors together until we get a combination, which when multiplied together, will produce the middle term (ab) we are looking for. This, I know, may sound a little hard to understand at first, however, after you work a few of these problems, you will realize that it is really quite simple.

Examples:

Factor the following:

1. $x^2 + 5xy + 4y^2$

$(x + 4y)\ (x + y)$

Proof:

$$\begin{array}{r} x + 4y \\ \underline{x + y} \\ x^2 + 4xy \phantom{{}+4y^2} \\ \underline{+ xy + 4y^2} \\ x^2 + 5xy + 4y^2 \end{array}$$

2. $9x^2 - 3xy - 2y^2$

$(3x - 2y)\ (3x + y)$

Proof:

$$\begin{array}{r} 3x - 2y \\ \underline{3x + y} \\ 9x^2 - 6xy - 2y^2 \\ \underline{+ 3xy \phantom{{}-2y^2}} \\ 9x^2 - 3xy - 2y^2 \end{array}$$

It should be noted at this point that not every expression of the form $ax^2 + b\,x\,y + cy^2$ can be factored. Many such expressions cannot be further simplified. Both $x^2 - 4xy + y^2$ and $3x^2 - 10xy + 2y^2$ are examples of expressions that cannot be factored. Once all of the combinations of factors above have been checked and the student is not able to arrive at the middle term, then he should conclude that the expression cannot be factored.

Further on in the book, when we take up the study of quadratic equations, we will see that one approach to the solution of quadratic equations is the utilization of the techniques of factoring.

Exercises:

Factor the following:

1. $x^2 - 6x + 8$ Ans. $(x - 4)(x - 2)$

2. $12 + x - x^2$ Ans. $(3 + x)(4 - x)$

3. $6a^2 + ab - b^2$ Ans. $(3a-b)(2a+b)$

4. $8x^3 + 16x^2y + 6xy^2$ Ans. $(2x)(2x+3y)(2x+y)$

5. $3x^3 - 12x$ Ans. $(3x)(x-2)(x+2)$

6. $3x^3 - 6x^2 + 21x$ Ans. $(3x)(x^2-2x+7)$

7. $8x^3 + y^3$ Ans. $(2x+y)(4x^2-2xy+y^2)$

8. $a^3 - 8$ Ans. $(a-2)(a^2+2a+4)$

9. $3a^2 + ab - 4b^2$ Ans. $(3a+4b)(a-b)$

10. $24x^2 - 24y^2$ Ans. $24(x-y)(x+y)$

4.4 Simplification of Fractions.

In arithmetic, the general definition of fractions is one number divided by another number. The two numbers are separated by a line which symbolizes the division sign. $\frac{4}{2}$ means 4 divided by 2. $\frac{a}{b}$ in algebra symbolizes a divided by b. In arithmetic, we cancel out like terms to simplify the expressions. As an example:

$$\frac{10 \times \cancel{64} \times \cancel{35} \times 50}{\cancel{8} \times \cancel{8} \times \cancel{7} \times \cancel{5}} = 500$$

In this expression, the 8x8 cancels out with 64 and the 7x5 cancels out with the 35, since like terms cancel. The expression is thus equivalent to 500. However, you must agree that the 500 form is the simpler of the two. Since literal numbers in algebra simply represent the numbers which we deal with in arithmetic, the operation is identical. We break the numerator and denominators down into their simplest factors and then cancel out like terms.

$$\frac{x^3 - y^2x}{x^2 + xy} = \frac{\cancel{x}(x - y)\cancel{(x + y)}}{\cancel{x}\cancel{(x + y)}} = x - y$$

and

$$\frac{a^2 + 2ab + b^2}{a^2 + ab} = \frac{(a + b)\cancel{(a + b)}}{a\cancel{(a + b)}} = \frac{a + b}{a}$$

Since in arithmetic we did not deal with negative numbers, it is considered appropriate that we take the time to drill in the handling of negative numbers in fractions. The student should look at a fraction as having three categories of signs:

(1) The signs of the numerator terms

(2) The signs of the denominator terms

(3) The sign of the fraction itself.

We can change any two of the above categories of signs (from + to - or - to +) without changing the value of the expression. This follows from the rule that:

+ divided by + and - divided by -	gives + answer
+ divided by - - divided by +	gives - answer

A plus sign before a fraction can be considered as a multiplier (having the value +1) of the fraction. Likewise, a negative sign before a fraction can be considered as a multiplier (having the value -1) of the fraction.

Let's drill with a couple of simple arithmetic fractions to see if the above rule is valid.

Take, for example, the fraction $+\frac{1}{2}$

The signs are as follows $+\frac{+1}{+2}$

If we change the numerator and denominator signs from + to - we are left with $+\frac{-1}{-2}$.

We know that -1 divided by -2 gives $+\frac{1}{2}$ for an answer. Therefore our value of our fraction has not changed. Also, starting again, this time with the fraction $-\frac{1}{2}$ or $-\frac{+1}{+2}$, we can change the sign of the fraction from - to + and the sign of the numerator from +1 to -1. We have changed 2 categories of signs, that of the fraction and that of the numerator. We now have $+\frac{-1}{+2}$ or -1 divided by +2, which gives a $-\frac{1}{2}$ when multiplied by the +1 (understood). We are back at our original value of $-\frac{1}{2}$ or $-\frac{+1}{+2}$ which is the same as $+\frac{-1}{+2}$. It follows then, if we take the fraction $+\frac{-1}{+2}$ and change the sign of the fraction from - to + and the sign of the denominator from +2 to -2 the value of the fraction still is $-\frac{1}{2}$ or

$$-\frac{+1}{+2} = +\frac{+1}{-2} = -\frac{+1}{+2}$$

The same rule applies to the literal numbers of algebra, since the letters simply represent real values.

Therefore $\frac{-a}{b} = -\frac{+a}{+b} = +\frac{+a}{-b} = -\frac{+a}{+b}$

Remember the three categories of signs:

1. The sign of the fraction
2. The signs of the terms in the numerator
3. The signs of the terms in the denominator

In the expression $\frac{y - x}{x\,(x - y)}$, we can change the sign of the fraction and the signs of the numerator and obtain $-\frac{x - y}{x\,(x-y)}$ which equals $-\frac{1}{x}$, since by rearranging the signs we are able to cancel out the x - y in both the numerator and denominator.

Examples:

1. $-\frac{a^2-2ab+b^2}{b-a} = -\frac{(a-b)(a-b)}{b-a} = \frac{(a-b)\cancel{(a-b)}}{\cancel{(a-b)}} = a-b$

2. $\frac{4x^3-9xy^2}{3yx-2x^2} = \frac{x(2x^2-9y^2)}{x(3y-2x)} = \frac{(2x-3y)(2x+3y)}{2x-3y} = -(2x+3y) = 3y-2x$

Perhaps we should review for a moment the operation of division of two fractions. In arithmetic, if we were to divide $\frac{3}{2}$ by $\frac{6}{12}$, we would have written $\frac{3}{2} \div \frac{6}{12}$ or $\frac{3/2}{6/12}$. The basic rule we followed was to invert the denominator or divisor and then multiply.

Therefore $\frac{\frac{3}{2}}{\frac{6}{12}} = \frac{3}{2} \cdot \frac{12}{6} = 3$

Similarly $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

Another less important operation is taking a square root of a fraction. If we square $\frac{3}{5}$, we would write

$$\frac{3}{5}^2 = \frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}$$

In other words, we square both the numerator and denominator. Remembering that the square root is the opposite operation to squaring, we would naturally take the square root of both the denominator and the numerator. Thus, the square root of $\frac{9}{25}$ is written as follows: $\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$

Examples:

1. $\frac{\frac{x-a}{b}}{\frac{x}{b}} = \frac{x-a}{b} \cdot \frac{b}{x} = \frac{x-9}{x}$

2. $\sqrt{\frac{a^2}{x^4}} = \frac{a}{x^2}$

3. $\sqrt{\frac{\frac{x^4}{y^2}}{\frac{x}{y}}} = \frac{x^2}{y} \cdot \frac{y}{x} = x$

Another technique we should review from our arithmetic days is the operation of finding the least common denominator. The topic is sometimes called the amalgamation of fractions. In arithmetic, if we had the sum of $\frac{1}{3} + \frac{3}{6} + \frac{3}{2}$ we would find the smallest number that all three denominators would divide into evenly; we called this the least common denominator. In this case, the number would be 6. Then we would divide each of the denominators in turn into 6 and then multiply the numerator of each fraction by the corresponding quotient and simplify. As an example: With $\frac{1}{3} + \frac{3}{6} + \frac{3}{2}$, we obtain $\frac{2+3+9}{6} = \frac{14}{6} = \frac{7}{3}$.

Again the method in algebra is the same. Let's take the expression

$$\frac{2}{x+3} - \frac{(4x-1)}{x^2-9}$$

In this case the least common denominator is $x^2 - 9$ or in its factored form $(x-3)(x+3)$. Then since $x + 3$ goes into $x^2 - 9$ exactly $(x - 3)$ times and $x^2 - 9$ goes into itself once, the original expression then simplifies as follows.

$$\frac{2}{x+3} - \frac{4x-1}{x^2-9} = \frac{2(x-3) - (4x-1)(1)}{(x-3)(x+3)} = \frac{2x-6-4x+1}{(x-3)(x+3)} = \frac{-2x-5}{x^2-9} = -\frac{2x+5}{x^2-9}$$

Examples:

1. $\frac{2x}{3x-2} + \frac{5}{9x^2-4} = \frac{2x(3x+2)+5}{(3x-2)(3x+2)} = \frac{6x^2+4+4x+5}{9x^2-4}$

2. $\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd}$

3. $\dfrac{a^2 - \frac{1}{a}}{a + \frac{1}{a} + 1} = \dfrac{\frac{a^3-1}{a}}{\frac{a^2+1+a}{a}}$

Then, since we are dividing one fraction by another, we invert the denominator fraction and multiply one fraction by the other. The a cancels out of both fractions as follows:

$$\frac{\frac{a^3 - 1}{a}}{\frac{a^2 + 1 + a}{a}} = \frac{a^3 - 1}{\cancel{a}} \times \frac{\cancel{a}}{a^2 + 1 + a}$$

and we are left with $\quad \dfrac{a^3 - 1}{a^2 + 1 + a}$

Exercises:

Factor the following.

1. $8by - 12xy^2$ Ans. $4y\,(2b - 3xy)$
2. $a^2b^2 - 25c^2$ Ans. $(ab - 5c)\,(ab + 5c)$
3. $a^2 + 8a + 16$ Ans. $(a + 4)\,(a + 4)$
4. $9 - 6x + x^2$ Ans. $(x-3)\,(x-3)$ or $(3-x)\,(3-x)$
5. $a^4 + 5a^2 - 14$ Ans. $(a^2 + 7)\,(a^2 - 2)$
6. $2a^2 - a - 10$ Ans. $(2a - 5)\,(a + 2)$
7. $8a^2b^2 - 18b$ Ans. $2b^2\,(2a+3)\,(2a-3)$
8. $x^4 - y^4$ Ans. $(x-y)\,(x+y)\,(x^2+y^2)$
9. $x^2y - y^2 - x^2z + yz$ Ans. $(y-z)\,(x^2 - y)$
10. $x^2 - y^2 + 2yz - z^2$ Ans. $(x-y+z)\,(x+y-z)$
11. $\dfrac{x^2 - y}{x^3 + y^3}$ Ans. $\dfrac{x - y}{x^2 - xy + y^2}$
12. $\dfrac{(x^2 + x - 6)\,(x - 3)}{(x - 2)\,(x^2 - 9)}$ Ans. 1

Combine into a single fraction.

13. $\dfrac{(x - 1)}{(2x^2-18)} + \dfrac{(x + 2)}{(9x - 3x^2)}$ Ans. $\dfrac{x^2 - 13x - 12}{6x\,(x + 3)\,(x - 3)}$

14. $\dfrac{1}{(a-b)\ (a-c)} + \dfrac{1}{(b-c)\ (c-b)} - \dfrac{1}{(a-c)\ (b-c)}$ Ans. $\dfrac{2}{(a-c)\ (a-b)}$

Simply:

15. $\dfrac{x^3}{y^2} \cdot \dfrac{(xy-y^2)}{x^2+xy} \div \dfrac{x^3-y^3}{x^3+y^3}$ Ans. $\dfrac{x^2}{y} \cdot \dfrac{(x^2 - xy + y^2)}{(x^2 + xy + y^2)}$

16. $\dfrac{\dfrac{x-y}{x+y} + \dfrac{y}{y-x}}{1 - y\left(\dfrac{3}{x-y} - \dfrac{2}{x+y}\right)}$ Ans. $\dfrac{x}{x + 2y}$

17. $(-3)^2$ Ans. 9

18. $5^7 \div 5^4$ Ans. 125

19. $\dfrac{3^5}{(-3)^6}$ Ans. $\dfrac{1}{3}$

Perform the indicated operations and simplify when possible.

20. x^3x^5 Ans. x^8

21. $(x^a)^b$ Ans. x^{ab}

22. $(3x^{2n})^4$ Ans. $81x^{8n}$

23. $x^{a(a-b)}\ x^{b(a+b)}\ x^{a^2-b^2}$ Ans. x^{2a^2}

24. $a^{2n-3}b \div \left[a^{n-4}(b^{5-n})^2\right]$ Ans. $a^{n+1}b^{2n-9}$

Simplify:

25. $5b^0$ Ans. 5

26. $\dfrac{6}{(x - y)^0}$ Ans. 6

27. $(3a^{-1}b^2)^{-2}$ Ans. $\dfrac{a^2}{9b^4}$

28. $(-x^{-1})^{-1}$ Ans. $-x$

Simplify:

29. $a\left(b\left[x-c\right]^2-xy\right)$ Ans. $abx^2-2abcx+abc^2-axy$

30. $3x\left(4y+2\left[a+b\left\{-4c\right\}+d\right]+12\right)$ Ans. $12xy+6ax-24bcx+6dx+36x$

31. $\dfrac{a^3b-a^2b^2}{3a^2b^2-3ab^3}$ Ans. $\dfrac{a}{3b}$

32. $\dfrac{x^2+2xy+y^2-z^2}{x+y+z}$ Ans. $x+y-z$

33. $\dfrac{b-2d}{12}+\dfrac{3b+d}{18}$ Ans. $\dfrac{9b-4d}{36}$

34. $\dfrac{45}{x+2y}-\dfrac{36}{x-2y}$ $\dfrac{81x-18y}{x^2-4y^2}$

CHAPTER 5

FUNCTIONS AND GRAPHS

Since functions and graphs are so closely related to one another, the discussion of both has been incorporated into this chapter. Do not worry about the relationship at this time; it will become clearer as you approach the end of the chapter. Since a function must exist before a graph is drawn, we will start off with a discussion of functions.

5.1 Functions.

Functions are merely a symbolic way of indicating that a relationship exists between two or more variables. We can easily understand functions if we say to ourselves that the amount of money we have in our pockets is a function of, or is somehow related to, the number of quarters in our pockets (if we assume that we have quarters), or the number of quarters, dimes, and nickels (assuming that we have only quarters, dimes, and nickels in our pockets), etc. If, in the first case, we let m stand for the money, expressed in dollars, and q stand for the number of quarters in our pockets we could write:

m depends in some manner upon or is related somehow to q.

Now, since we are saying that m is dependent upon the value of q, we call m the dependent variable and q the independent variable. Then, since mathematicians are, in general, lazy when it comes to writing things out, we substitute a literal symbol (usually f, probably because it is the first letter of the word function, but any symbol may be used) followed by a parenthesis in place of the words "depends in some manner upon or is related somehow to". Next, we place the symbol for the independent variable within the parenthesis. If there are two or more independent variables, we put the symbol for each within the parenthesis, separating the symbols with commas. Then in order to separate the symbol used to indicate "depends in some manner upon or is related somehow to," from the dependent variable which goes on the left of the equation, we place an equal sign (=) between the two. We should be cautioned that the equal sign does not mean that the dependent variable is equal to the literal symbol used to indicate function times the contents of the parenthesis in the same manner that 3(5) means 3 times 5. The symbology =f(x) should all be considered at one time in determining its meaning; which is, something is a function of whatever is indicated within the parenthesis. Putting the above information together, we should be able to see that the fast way of expressing the fact that the money is a function of the quarters in our pocket is:

$m = f(q)$ - which simply means that the amount of money in our pockets is a function of the number of quarters that we have in our pockets.

In the second case, where we had quarters, dimes, and nickels, we could use the symbols m for money in dollars, q for the number of quarters, d for the number of dimes, and n for the number of nickels and easily, I hope, see that we can write the relationship in the following manner:

$m = f(q,d,n)$ - meaning that the money in our pockets is a function of the number of quarters, dimes, and nickels that we have.

Notice that so far we have only said that =f( ) indicates only the fact that a dependent variable is somehow related to an independent variable or to independent variables. We have not shown the exact relationship because we must first of all get used to looking at the symbology and realizing that it means that something, which we will call the dependent variable, is a function of or is somehow related to one or more other variables, which we will call the independent variable or variables. Remember, now, that the symbol indicating function can be any literal symbol that we want to use. Thus the symbols f( ), g( ), F( ), G( ), h( ), etc., all mean the same thing. Namely, that something is a function of the independent variable indicated by the symbol within the parenthesis. Likewise, $f(x,y)$, $g(x,y)$, $F(x,y)$, etc., all mean the same thing; namely, that something is dependent upon the values of the variables indicated by the symbols x and y.

Now that we know that $f(x)$, $g(x)$, etc., only means that something is dependent upon the value of the independent variable x, we can begin to wonder what the exact relationship is. It should be intuitively obvious that we must know the exact relationship before we can determine any values of the dependent variable regardless of how much we know about the independent variable. For example, all we know so far in the money case is the fact that the amount of money is somehow related to the number of quarters. From experience, we know that four quarters are the same as one dollar so we can write the following:

$m = .25q$ (That is, for every quarter we have in our pockets we have .25 dollars).

Then, if we had 10 quarters, we would substitute the number 10 for the literal number q and we would find that money, in dollars, equals 2.50. Going back and putting the data into functional language we would have:

$m = f(q)$ where $f(q) = .25q$ – That is, money is a function of the number of quarters and the exact relationship is .25 times the number of quarters.

Once given the relationship of m to q, we would place within the parenthesis the value of q that we wanted to solve m for, and would substitute this value for the independent variable symbol wherever it appeared in the equation which gives the exact relationship. Then, if we assume that we have 10 quarters, we proceed as follows:

$m = f(q) = .25q$ find the value of m for f(10), in other words solve for f(10). Then, substituting in the number 10 wherever a q appeared we would get:

$$m = f(q) = .25(10) = 2.50$$

Now getting along a little bit, we will go to the quarters, dimes, and nickels case. We can write this function statement as:

$m = f(q,d,n) = .25q + .10d + .05n$ (This means that the amount of money that we have in dollars is equal to $\frac{1}{4}$ the number of quarters, plus 1/10 the number of dimes, plus 1/20 the number of nickels.)

Thus, if we wanted to find the value of four quarters, ten dimes and ten nickels, we would write:

$m = f(q,d,n) = .25q + .10d + .05n$ Solve for $f(4,10,10)$

Since, by convention, the numerical values within the parenthesis are interpreted as being applicable to the literal number occupying the same relative position in the parenthesis, we substitute 4 for q, 10 for d, and 10 for n in the function equation. We can now solve the problem as follows:

$$m = .25q + .10d + .05n = .25(4) + .10\,(10) + .05(10)$$
$$= 1.00 + 1.00 + .50 = \underline{2.50} \text{ Ans.}$$

At this time, we should review the fact that we can use any literal symbol that we wish in order to represent the dependent variable, the independent variable or variables, or to indicate a function. The only requirement is that we be consistent. In other words, we would not want the symbol x to stand for one variable one place in a problem, and for a different variable in a different place in the same problem. As an example, we could have just as easily written:

$y = f(x,z,w) = .25x + .10z + .05w$, (where y stood for money in dollars, x stood for the number of quarters, z stood for the number of dimes, and w stood for the number of nickels.)

In order to ensure that an understanding of functions has been gained, you should study the following problems step by step.

1. $n = f(y) = y+2$ solve for $f(2)$

   therefore: $n = 2 + 2 = 4$ Ans.

2. $n = g(y) = y^2 + 4$ solve for $g(2)$

   therefore: $n = (2)^2 + 4 = 4 + 4 = \underline{8}$ Ans.

3. $y = f(x) = x - 3$ solve for $f(3)$

   therefore: $y = 3-3 = \underline{0}$ Ans.

4. $t = f(x,w) = 2x + 4w + 1$ solve for $f(3,4)$

   therefore: $t = 2(3) + 4(4) + 1 = 6 + 16 + 1 = \underline{23}$ Ans.

5. $r = f(x,w) = 3x^2 + 2x + w$ solve for $f(3,4)$

   therefore: $r = 3(3) \;\; + 2(3) + (4) = 27 + 6 + 4 = \underline{37}$ Ans.

6. $f = f(m,a) = ma$ solve for $f(3,16)$

   therefore: $f = (3)\ (16) = \underline{48}$ Ans.

7. $y = f(x) = x^2 + x + 1$ solve for $f(x + 1)$

   therefore: $y = (x+1)^2 + (x+1) + 1 = x^2 + 2x + 1 + x+1 + 1 = \underline{x^2+3x+3}$ Ans.

5.2 Functions of Functions.

It is now time to extend our knowledge of functions to include finding the value of the dependent variable when the independent variable is, itself, dependent upon a third variable. To illustrate this condition, which may be called a "function of a function," we will go back to our money problem and assume that we have only nickels but, for some unknown reason, we want to express our money as being a function of quarters alone and not quarters, dimes, and nickels. We will assume, therefore, that we can consider 5 nickels as one quarter. Now, in compliance with the restriction that we placed on the problem, we can say that:

$$m = f(q) = .25q$$

Remember, though, that the value of q depends upon the number of nickels that we have. Knowing this, we can write a function statement as follows:

$q = g(n)$ (We could have just as well used $f(n)$ but $g(n)$ was used to avoid confusion that could have resulted later on.)

Since five nickels are equivalent to a quarter, we can indicate the exact relationship as:

$$q = g(n) = .20n$$

Continuing to assume that we only have nickels, it follows that what we really want to do is solve the equation:

$m = f(q) = .25q$ for the value of $q = g(n) = .20n$

Therefore, we may substitute $g(n)$ in the equation wherever q appears and we find that:

$m = .25\,(g(n))$, but in this case $g(n)$ is equal to $.20n$, so we get,

$$m = .25\,(.20n) = .05n$$

Assuming that we had 15 nickels, we can state that the dollar value of our money would be .05 times 15 or .75, which is also intuitively obvious.

Now, if we want to expand this example still further to include quarters and dimes as well as nickels, with the same stipulation with regards to expressing m in terms of q, we get the following equations:

$$m = f(q) = .25q$$

$q = h(q,d,n) = q + .40d + .20n$ (Note that we have said that q is a function of itself and that we set the relationship as one to one. This makes sence, doesn't it?)

As a problem this would probably be stated as follows:

$m = f(q) = .25q$ and $q = h(q,d,n)$ solve for $f(q)$ given that we have 4 quarters, 10 dimes, and 10 nickels

In this case we proceed as follows:

1. Substitute $h(q,d,n)$ in place of q in the exact relationship and get $m = .25(h(q,d,n))$
2. Replace $h(q,d,n)$ with the exact relationship $q + .40d + .20\,n$ and get $m = .25(q + .40d + .20n)$
3. Put in the values of q, d, and n and we get $m = .25(4 + .40(10) + .20\,(10)) = .25(4 + 4 + 2) = .25(10) = \underline{2.50}$ Ans.

We also have a function of a function when we determine (or attempt to determine) the value of the dependent variable for a value of the independent variable, where the value used is dependent upon the same independent variable as the original dependent variable. In order to simplify this apparent talking in circles, I'll indicate what is meant by an example.

Assume:

$y = f(x) = \frac{x + 1}{x}$ solve for $f(w)$ where $w = f(x) = x + 1$

Remember, now, that $f(x)$ indicates only that y and w are both functions of x. It does not indicate that they are equal. In fact, they are not equal. Solving the problem, we get:

$$y = \frac{w + 1}{w} \text{ but } w = x + 1 \text{ so we get } y = \frac{(x+1) + 1}{x + 1} = \frac{x + 2}{x + 1}$$

5.3 Domain.

Having seen how functions work, we must think about the range over which the exact relationship holds true. This range is called the domain. A rather simple example would be the case of the salesman who receives a commission of 10% on all sales under \$1,000. and a commission of \$100.

plus 15% of the price in excess of $1,000. on all sales over $1,000. We could write his commission schedule then as:

$C = f(S) = .10\ S$ where C stands for commission and S for sales

It is true for the range or domain of 0-$1,000.

$C = f(S) = \$100 + .15\ (S-1,000)$ True in the domain of over $1,000.

We can see that the relationship between the same variables was different in two areas of sales. The area where each holds true is that function's domain.

Before progressing to the next half of the chapter, which is on graphs, the student should work through the following examples and problems.

Examples:

1. $y = F(x) = x^2 + 5x + 3$ solve for a. $F(3)$ b. $F(5)$

   a. $y = (3)^2 + 5(3) + 3 = 9 + 15 + 3 = 27$

   b. $y = (5)^2 + 5(5) + 3 = 25 + 25 + 3 = 53$

2. $z = G(a) = 5a^2 + ba + 3$ solve for a. $G(2)$, b. $G(3)$

   a. $z = 5(2)^2 + b(2) + 3 = 20 + 2b + 3 = 23 + 2b$

   b. $z = 5(3)^2 + b(3) + 3 = 45 + 3b + 3 = 48 + 3b$

3. $y = f(x,w) = x + w$ find a. $f(2,3)$ b. $f(3,4)$

   a. $y = 2 + 3 = 5$

   b. $y = 3 + 4 = 7$

4. $t = f(x,w) = x^2 + x + w^2 - 1$ find a. $f(2,3)$ b. $f(3,4)$

   a. $t = (2)^2 + (2) + (3)^2 - 1 = 4 + 2 + 9 - 1 = 14$

   b. $t = (3)^2 + (3) + (4)^2 - 1 = 9 + 3 + 16 - 1 = 27$

5. $n = f(x) = 2x-1$ Find a. $f(x+1)-f(x)$ b. $f(x+3)+f(x-1)$

a. $n = 2(x+1)-1 - 2(x)-1 = 2x+2-1-2x-1 = 0$

b. $n = 2(x+3)-1 + 2(x-1)-1 = 2x+6-1+2x-2-1 = 4x+2$

As you can see this is merely a problem of solving for one value of the independent variable and then another value and then performing the algebraic operations called for.

6. $y = g(x) = x+5$ and $t = h(x) = x^2 - 3$ Find a. $\dfrac{g(3)}{h(5)}$ b. $\dfrac{g(4)}{h(6)}$

a. $\dfrac{y}{t} = \dfrac{3+5}{(5)^2-3} = \dfrac{8}{25-3} = \dfrac{8}{22} = \dfrac{4}{11}$

b. $\dfrac{y}{t} = \dfrac{4+5}{(6)^2-3} = \dfrac{9}{36-3} = \dfrac{9}{33} = \dfrac{3}{11}$

7. $y = g(x) = x^2+2$ and $t = h(x) = x^3+4x-1$ Find a. $g(3) + h(4)$

b. $g(5) - h(2)$

a. $y + t = (3)^2 + 2 + (4)^3 + 4(4) - 1 = 6 + 8 + 12 + 16 - 1 = 41$

b. $y - t = (5)^2 + 2 - \left[(2)^3 + 4(2) - 1\right] = 25+2 - \left[8+8-1\right] = 27-15 = 12$

8. $g(x) = x^2 + 4$ and $h(x) = 3 + x$ Find a. $g\left[h(x)\right]$ b. $h\left[g(x)\right]$

a. $\left[h(x)\right]^2 + 4 = \left[3+x\right]^2 + 4 = 9+6x + x^2 + 4 = 13 + 6x + x^2$

b. $3 + \left[g(x)\right]^2 = 3 + (x^2+4)^2 = 3+x^4+8x^2+16 = x^4+8x^2+19$

9. $g(x) = x^2 + 13$ and $h(x) = 6x^3 + 5x + 3$ Prove $g(7) = h(2)$

$g(7) = (7)^2 + 12 = 49 + 12 = 61$

$h(2) = 6(2)^3 + 5(2) + 3 = 48 + 10 + 3 = 61$

10. If $z = f(y) = \dfrac{3y^2 - 12}{y + 2}$ and $y = g(x) = x + 1$, express z in terms of x.

$$z = \frac{3(x+1)^2 - 12}{(x+1) + 2} = \frac{3(x^2 + 2x + 1) - 12}{x + 3}$$

$$= \frac{3x^2+6x+3-12}{x+3} = \frac{3x^2+6x-9}{x+3} = \frac{(3x-3)\,(x+3)}{x+3} = 3x-3$$

Problems.

1. If $y = f(x) = x^2+5x-3$, solve for $f(3)$ Ans. 21

2. If $y = f(x) = \frac{x+4}{x^2}$, solve for $f(4)$ Ans. 1/2

3. If $t = g(y) = y^2+y+2$, solve for $g(2y)$ Ans. $4y^2+2y+2$

4. If $t = g(y) = \frac{y^2+1}{y^2-1}$, solve for $g(y)$ Ans. $\frac{y^2+1}{y^2-1}$

5. If $m = f(n) = n^2+n$, find $f(n-1) - f(n)$ Ans. 1

6. If $z = f(y) = \frac{y+1}{y}$ and $y = f(x) = x+3$, express z in terms of x Ans. $z = \frac{x+4}{x+3}$

7. If $s = f(y) = y+3y^2$ and $w = g(y) = y+1$, find $f(w)$ in terms of y Ans. $s = 3y^2+7y+4$

8. If $z = f(y) = \frac{y^2-1}{y+1}$, solve for $f(y+1)$ Ans. y

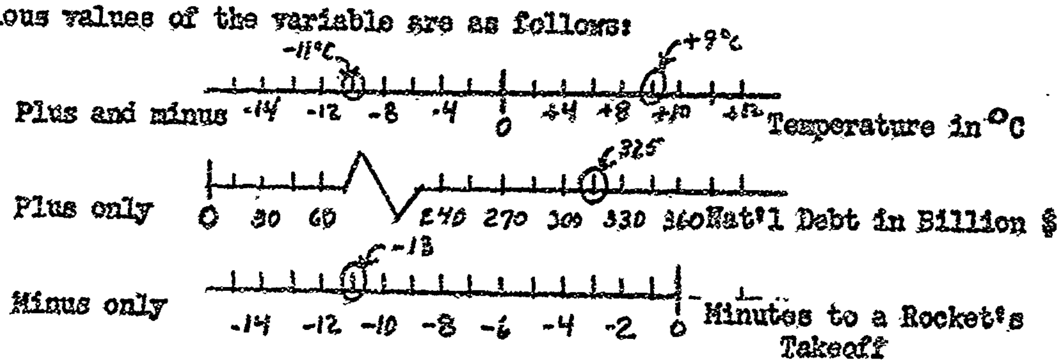
9. If $F(x) = x^2+x-18$ and $G(x) = x-3$, does $F(4) = G(5)$? Ans. Yes

10. If $F(x) = x^2+18$ and $G(x) = -x^3+18$, does $F(-3) = G(3)$? Ans. Yes

5.4 Graphs.

Graphs are nothing but a pictorial means of showing the value of a single variable, the relationship between two variables, or the relationships among three variables. Graphs of more than three variables are not drawn because we cannot easily depict more than three dimensions and in graphing one dimension is given to each variable.

To start the discussion, we will assume we want to show only the various possible values of a single variable. All we need for this is a straight line which has been divided into equal length elements by dividing lines which have been numbered in an appropriate, logical manner. By appropriate is meant marking the divisions in units, tens, hundreds, etc., as best fits the situation and having minus and/or plus values, as necessary. By logical is meant starting at one place, called the origin, and marking the divisions as they fall in sequence in distance away from the origin. Thus, we don't have 1 next to the origin followed by 3 and then 2. We must have 1, 2, 3, etc. We normally label the straight line or axis, as it is commonly called, so that we will know what we are measuring. Some examples of single line graphs and the indication of various values of the variable are as follows:



In order to show the relationship between two variables, commonly called plotting the function, we must form what is called a system of coordinates. A system of coordinates merely consists of two straight lines in a plane which cross each other at right angles and which have been divided into appropriate divisions, which are called coordinates. There is no need for the measurements to be the same along both axes. By convention, we make one line horizontal and the other line vertical. Then we call the point of intersection the origin and assign positive or plus number values to the coordinates or divisions along the horizontal axis to the right of the origin. We do likewise to the divisions along the vertical axis above the origin. Negative or minus values are assigned to the divisions along the horizontal axis to the left of the origin and to the divisions along the vertical axis beneath the origin. Repeating, these values assigned to the divisions are called coordinates. The next consideration is the four areas or quadrants that the two axes divide the graph into. These we number from I to IV in the following manner. The upper right quadrant is number I, the upper left quadrant is number II, the lower left quadrant is quadrant III, and the lower right quadrant is number IV. Looking at Figure 5-1, we can see that in quadrant I, both coordinates are positive, while in quadrant II, the horizontal coordinate is negative while the vertical coordinate is positive, etc. Along with understanding the numbering system there are a few conventions that we should know. First, the horizontal axis is called the abscissa, while the vertical axis is called the ordinate. Second, the abscissa is usually used for plotting the value of the independent variable, while the ordinate is used for plotting the value of the dependent variable. Third, that for a given abscissa value a vertical line is drawn through that abscissa value to establish one line of constant abscissa value. Fourth, for a given ordinate value we draw a horizontal line through that value to establish one line with constant ordinate value. The last two points are illustrated in Figure 5-1. Normally, we do not actually draw the lines but instead just imagine them to be there.

Again by convention, when giving the coordinates of points we give the abscissa value first and then the ordinate value. If we are given

FIGURE 5-1

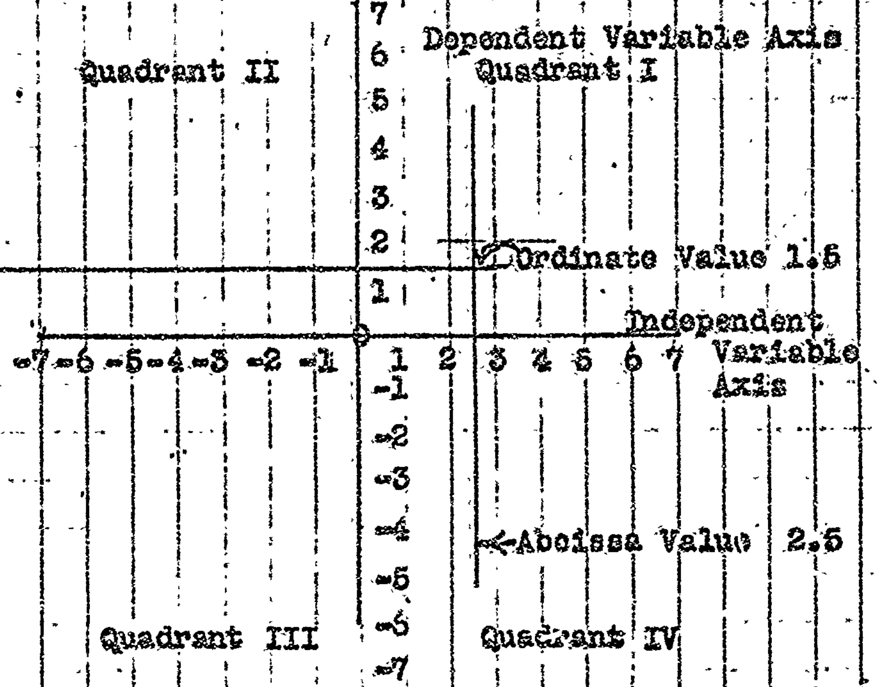
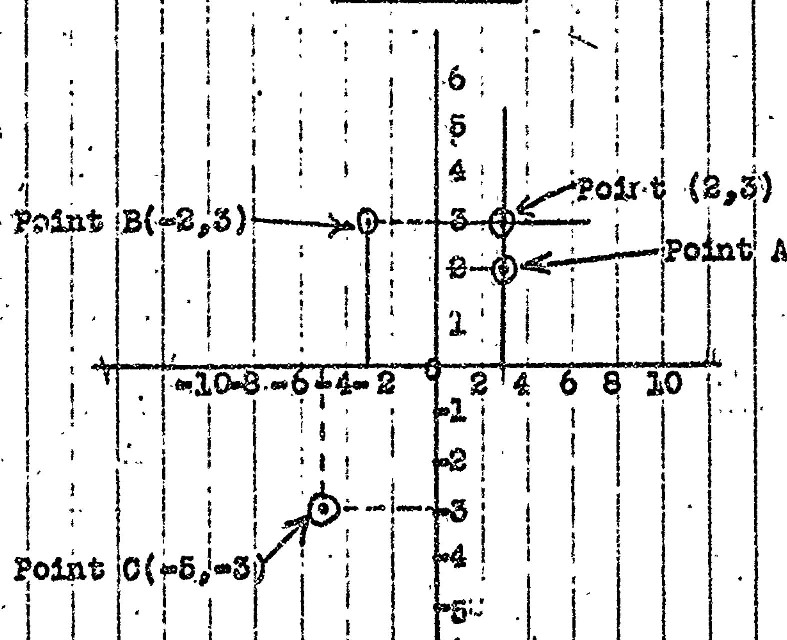


Figure 5-2



an abscissa value and an ordinate value for a point, we have what is called an ordered pair. Given an ordered pair, such as (2,3), we plot the point by drawing a vertical line through the abscissa coordinate 2 and a horizontal line through the ordinate coordinate 3, and where they intersect is the point that we want. This little maneuver is also illustrated in Figure 5-2. Similarly, if we want to find the coordinates of a point on a graph, we run lines through the points that are perpendicular to the two axes. Where the vertical line crosses the horizontal axis is the abscissa coordinate and where the horizontal line crosses the vertical axis is the ordinate coordinate. The ordered pair coordinates for several points have been shown in Figure 2. We would describe point A as the ordered pair (+3, +2), point B as the ordered pair (-2, +3), etc.

If we wanted to show the relationships among three variables, we simply add another axis to the graph, the third axis being perpendicular to the plane of the first two axes. Now that we are in three dimensions, a specific value for any axis could lie anywhere on a plane through the axis and perpendicular to the axis at that value. If we are given three coordinates, and asked to locate the point represented by them, we can draw the three planes. The intersection of the three planes is the point that we want.

Now, knowing how graphs are drawn and developed, we can get into the plotting of functions. To plot one relationship between the dependent variable and the independent variable, we need to solve the function equation with a selected value of the independent variable in order to get the dependent variable value and thus have an ordered pair. Obviously, the coordinate for the independent variable will be the value we substitute into the equation, while the coordinate of the dependent variable will be the result of substituting the independent value into the formula. Knowing the coordinates, all we have to do is draw our horizontal line through the ordinate coordinate and our vertical line through the abscissa coordinate. For example: If we know the independent variable is 4 and the dependent variable is 6, we draw a line perpendicular to the independent variable axis at 4 and a line perpendicular to the dependent variable axis at 6. The point at which they cross is the point we are looking for. Now, if we continue to determine ordered pairs and to plot them, we will get a series of points that could be connected together somehow. If we assume that the function is continuous, that is, that the independent variable can have any value with the function's domain, we can connect the points with a solid line. Then we can determine the values of the independent and dependent variables that make ordered pairs by dropping perpendicular lines to the axis. The shape of the line will depend upon the function plotted; the reasons for the differences will be taken up in later chapters. If the variables can only assume certain values within the domain of a function, as an example, assume that only whole number values can be assigned to the independent variable, we have what is called a discrete function. We cannot "legally" connect the points with a straight line but we could plot several points and connect them with a broken line and then read off values of the dependent variable for values of the independent variable which can be assumed.

To plot a graph, the first thing to do is draw up a table showing the values of the independent variable to be considered and then determine the corresponding dependent variable values. Then we would plot the ordered pairs. As an example, let's draw for the function $y = f(x) = x + 3$ and another function where $y = f(x) = x^2 + 3$. The first thing to do is to draw up tables similar to the ones below. Then plot the points as shown in Figure 5-3.

$$y = f(x) = x + 3$$

if x =	-9	-3	0	+3	+6	+9
y =	- 6	0	+3	+6	+9	+12

$$y = f(x) = x^2 + 3$$

if x =	-3	-2	-1	0	+1	+2	+3	+4
y =	12	7	4	3	4	7	12	19

Oftentimes as we are plotting a graph of a function, we notice that the curve seems to approach a certain value of one or the other of the variables but just never seems to reach that value. As an example, consider the graph of $y = f(x) = \frac{1}{x}$ as has been done in Figure 5-4. Notice that the curve approaches the y and x axes but that it never quite makes contact. Lines such as these are referred to as asymptotes. In our function $y = \frac{1}{x}$, the place of contact would be at $x = \infty$ when $y = 0$ and $y = \infty$ when $x = 0$.

Since the drawing of three dimensional graphs is a lengthy process and results in a perspective problem, no illustrations of three dimensional graphs will be given. However, all we really have to remember is that we would be working with planes rather than lines and that it takes three coordinate values to obtain a point.

The student should now plot graphs for the following functions. By using ordered pairs other than the ones in the answer column and

FIGURE 5-3.

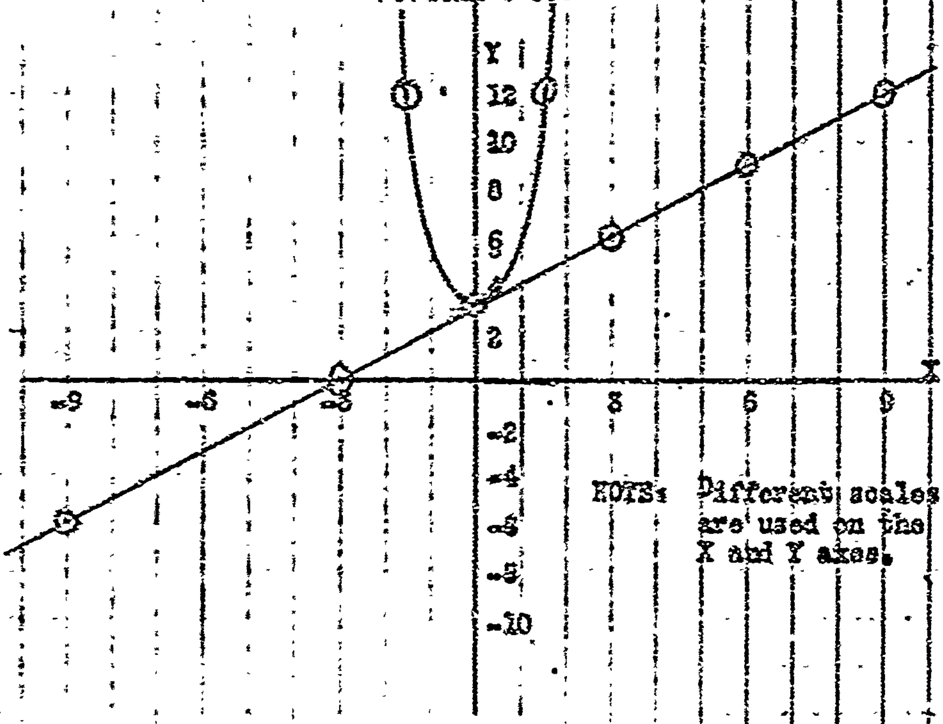
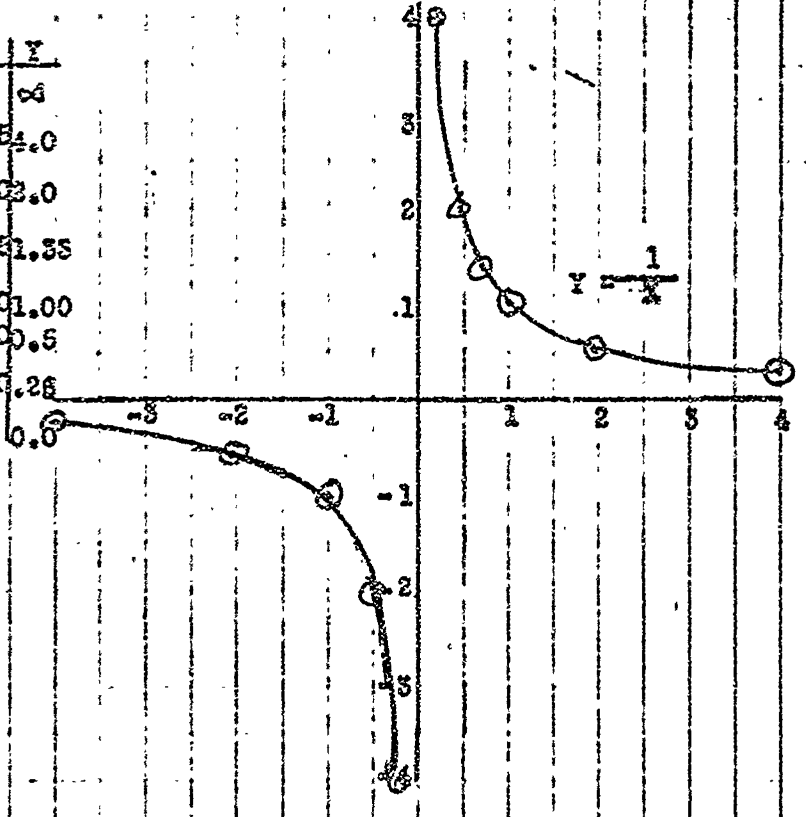


FIGURE 5-4

X	Y
0	∞
.25	4.0
.50	2.0
.75	1.33
1.0	1.00
2.0	0.5
4.0	0.25
∞	0.0



X	Y
0	-∞
-.25	-4.0
-.50	-2.0
-.75	-1.33
-1.0	-1.0
-2.0	-0.5
-4.0	-0.25
-∞	0

drawing lines between the points, he can check his work by seeing if the ordered pairs in the answer column fall on the line.

Problem:

1.	$y^2 + 4x = 4$	Ans. $(x=0, y = 2)$ $(x=1, y=0)$
2.	$x^2 - 4 = 0$	Ans. (Vertical lines at $x = \pm 2$
3.	$y = 4$	Ans. (Horizontal line; at $y = 4$)
4.	$y^2 + 4y + 3x = 0$	Ans. $(x=1, y=-1)(x = 4, y = 2)$
5.	$x^2 + 2x + 3y^2 = 0$	Ans. $(x=1, y=1)$ $(x=6, y=4)$
6.	$16x^2 + 9y^2 = 144$	Ans. $(x=3, y=0)$ $(x=0, y=4)$

CHAPTER 6

EQUATIONS OF THE FIRST DEGREE

6.1 Introduction.

An algebraic equation is a mathematical statement that two algebraic expressions are equal. The symbol " = ", is used to express this equality. $a + b = c + d$ is an equation which could literally represent the equality $4 + 3 = 5 + 2$.

Equations are classified by degree, first degree, second degree, and so on. The degree corresponds to the term of the highest power in the equation.

$x + 6 = 10$ is an equation of the first degree,

$x^2 + 4x + 2 = 7$ is an equation of the second degree,

$a^2b^3 + 2a^2 = 42$ is an equation of the fifth degree, since a^2b^3 is a fifth power term.

Algebraic equations are further generally divided into two types, the identical equation, or identity, and the conditional equation. The identical equation is true for all permissible values of the letters. "Permissible", in this case, means values of the variables, which when substituted in the equation, result in both sides of the equation being defined. The conditional equation is only true for particular values of the variables. These particular values we call roots or solutions of the equations.[1] The following are illustrations of identities:

1. $\sqrt{x^4} = x^2$

   $x^2 = x^2$

[1]Ray Dubisch, Vernon E. Howes and Steven J. Bryant, Intermediate Algebra (New York, London: John Wiley & Sons, Inc., 1960), pp. 79-81.

The real proof that an algebraic equation is an identity is that by simplification each side of the equation can be reduced to the same expression as shown both above and below.

2. $$\frac{8x^3 + 16x^2y + 6xy^2}{2x + y} = 4x^2 + 6xy$$

$$\frac{2x\,(2x + 3y)\,\cancel{(2x + y)}}{\cancel{(2x + y)}} = 4x^2 + 6xy$$

$$2x\,(2x + 3y) = 4x^2 + 6xy$$

$$4x^2 + 6xy = 4x^2 + 6xy$$

The student should verify for himself that the above definition of an identity holds true, that is, that the equation is valid for all possible values of the variable.

6.2 Equations of the First Degree.

In this chapter we will take up the study of the solution of conditional equations of the first degree. After we become proficient in this operation we will learn to graph equations of the first degree on coordinate axes. For reasons of simplification, we will refer to identical equations as identities and to conditional equations simply as equations.

The student probably recalls learning the arithmetic multiplication or "times" table in elementary school. Perhaps this was our first brush with an arithmetic identity.

$$2 \times 4 = 8$$

$$4 \times 3 = 12$$

In algebra we simply substitute letters for some or all of the numbers and we have an equation. Take for example the equations

$$2x = 8$$

$$3y = 12$$

To solve the equations above for x and y, we would simply divide both sides of the first equation by 2.

$$\frac{2x}{2} = \frac{8}{2}$$

$$x = 4$$

and both sides of the second equation by 3.

$$3y = 12$$

$$\frac{3y}{3} = \frac{12}{3}$$

$$y = 4$$

In arithmetic we would have set up this problem by saying respectively: what number when multiplied by 2 equals 8 and what number when multiplied by 3 equals 12? In algebra we simply substitute a letter for the unknown and then express the relationship as an equation.

<u>Problem.</u>

If I multiply a number by 2, then add 4, my answer is 10. What is the number?

If we let x stand for the unknown, the problem can be expressed as follows:

$$2x + 4 = 10$$

Once you have an equation expressed, the solution is found by manipulating the equation until the unknown, in the above case x, is on one side of the equation alone. We then have solved the equation. The manipulations I refer to are the fundamental processes of multiplying, dividing, adding, and subtracting. The fundamental rule which governs the application of these processes is that since each side of the equation is equal to the other, then anything that we do to one side of the equation we must also do to the other. In short summary then, if we are given a first degree equation to solve, we work towards getting our unknown alone on one side of the equation by carrying out identical processes to both sides of the equation. When working with equations of any kind, the student must always remember that "anything we do to one side of the equation we must do to the other side" . . . because the sides are equal.

Now let's solve the above equation.

$$2x + 4 = 10$$

We will first subtract 4 from both sides of the equation. We then get

$$2x + 4 - 4 = 10 - 4$$

$$2x = 6$$

Then we divide both sides by 2.

Thus $$\frac{2x}{2} = \frac{6}{2}$$

$$x = 3 \text{ Ans.}$$

Because of the great importance of this topic many solved problems and exercises are provided.

Examples:

Solve the following for the unknown variable.

1. $3x + 36 = 2x$
   $3x - 2x = -36$
   $x = -36$

2. $2x = 5x - 100$
   $2x - 5x = -100$
   $-3x = -100$

Then multiplying both sides by -1 and dividing both sides by 3

$$3x = 100$$

$$x = \frac{100}{3} = 33\ 1/3$$

3. In the equation $F = MA$ solve the equation (a) for M (b) for A

(a) $F = MA$

dividing both sides by A

$$\frac{F}{A} = M$$

$$M = \frac{F}{A}$$

(b) $F = MA$

dividing both sides by M

$$\frac{F}{M} = A$$

$$A = \frac{F}{M}$$

4. $x + 2 = 10$

then by squaring both sides

$$\sqrt{x + 2} = 100$$

$$x = 100 - 2 = 98$$

$$x = 98$$

There are a couple of short cuts that the student should quickly master. In the equation

$$3x - 9 = 6,$$

the first step in the solution is to add +9 to both sides. Since the student is working to get the numbers on one side and x on the other, the above operation can be looked upon as moving the -9 from one side of the equation to the other and making it a +9.

$$3x - 9 = 6$$

$$3x = 6 + 9$$

Similarly,

$$3x + 9 = 24$$
$$3x = 24 - 9$$

In short, if we move a number from one side of the equation to the other, we change its sign. The operation is identical to adding or subtracting the same number from both sides.

Similarly,

$$3x - 9 = 6$$
$$3x = 6 + 9 = 15$$
$$3x = 15$$

the next step would be to divide both sides by 3

$$\frac{3x}{3} = \frac{15}{3}$$

This is identical to moving the 3, which is a multiplier on the left side, and making it a divider on the right side.

$$3x = 15$$
$$x = \frac{15}{3} = 5$$

Similarly,

$$3x + 9 = 24$$
$$3x = 24 - 9 = 15$$
$$3x = 15$$
$$x = \frac{15}{3} = 5$$

If this is the student's first acquaintance with this process, take as much time as is necessary to practice the above operations until they are known so well that they will never be forgotten.

5. $36x - 42 = 24x + 6$
$36x - 24x = 42 + 6$
$12x = 48$
$x = \frac{48}{12} = 4$

6. $6 = \dfrac{1}{(\sqrt{x})^{-1}}$

$\sqrt{x} = 6$

$x = 36$

7. Solve for a.

$\dfrac{bc}{d} = \dfrac{1}{a}$

$a = \dfrac{d}{bc}$

Solve for c in the above equation.

$a = \dfrac{d}{bc}$

$c = \dfrac{d}{ba}$

8. $6(x + 2) = 3(x + 8)$

$6x + 12 = 3x + 24$

$6x - 3x = 24 - 12$

$3x = 12$

$x = 4$

9. $4(a - 12) = -2(a - 3)$

$4a - 48 = -2a + 6$

$6a = 48 + 6 = 54$

$6a = 54$

$a = 9$

10. Solve for x in terms of a, b and c.

$\dfrac{4x + a - b}{c} = 1$

$4x + a - b = c$

$4x = c + b - a$

$x = \dfrac{c + b - a}{4}$

11. Solve for n.

$$1 = \frac{2}{3} + \frac{3}{\sqrt{n}}$$

$$1 - \frac{2}{3} = \frac{3}{\sqrt{n}}$$

$$\frac{1}{3} = \frac{3}{\sqrt{n}}$$

$$\sqrt{n} = 9$$

$$n = 81$$

12. Find x in terms of the other variables.

$$\frac{y+z}{2} + 2xz = x + y + z$$

$$x - 2xz = \frac{y+z}{2} - y - z$$

Then factoring out the common term x

$$x\ (1 - 2z) = \frac{y+z}{2} - y - z$$

$$x\ (1 - 2z) = \frac{y + z - 2y - 2z}{2} = \frac{-y - z}{2}$$

$$x = \frac{-y - z}{2\ (1 - 2z)} = \frac{y + z}{2\ (2z - 1)}$$

Exercises:

1. Six subtracted from two times a number is equal to the number plus six. What is the number?

Ans. $x = 12$

Assume that x may represent any real number for which both members are defined and classify each of the following as either equations or identities.

2. $2\ (x-3) + 5 = 3\ (x-2) + 5$ Ans. Equation

3. $6x + 4 \div 2 = (6x + 4) \div 2$ Ans. Equation

4. $(5x-55)\ (3x^2 + 7x) = 5x\ (3x+7)\ (x-11)$ Ans. Identity

5. $(9x^2-36y^2) \div (3x-6y) = 3x + 6$ Ans. Equation

Solve for x in terms of the other variables.

6. $\frac{y + z}{\sqrt{x}} = 5$ Ans. $x = \frac{y^2 + 2yz + z^2}{25}$

7. $\frac{1}{y} - \frac{1}{z} = \sqrt{\frac{z}{1 - x}}$ Ans. $x = 1 - \frac{y^2z^3}{(z-y)^2}$

8. Solve for y

$\frac{y - 7}{y^2-2y} = \frac{y}{y-2} - \frac{y + 4}{y}$ Ans. $y = 5$

Solve each of the following equations, if possible, and then check the results.

9. $y + 2 = 7$ Ans. $y = 5$

10. $2r = \frac{1}{2}$ Ans. $r = \frac{1}{4}$

11. $\frac{2t}{3} = \frac{-4}{5}$ Ans. $t = -6/5$

12. $3a - 5 = 2a + 5$ Ans. $a = 10$

13. $3z + 2 = 3z$ Ans. Impossible

14. $h\sqrt{5} - 3 = h + 1$ Ans. $h = 4/(\sqrt{5} - 1)$

15. $\frac{2x + 3}{2} + \frac{3x}{x - 1} = x$ Ans. $x = 1/3$

16. $\frac{y - 7}{y^2-2y} = \frac{y}{y - 2} - \frac{y + 4}{y}$ Ans. $y = 5$

17. $\frac{5}{x-1} + \frac{1}{4-3x} = \frac{3}{6x-8}$ Ans. $x = 7/5$

Set up equations which express each of the following conditions. Solve them if you can.

18. Twice a number n is equal to the number increased by 5.

Ans. $n = 5$

19. A number x is 5 more than a number y.

Ans. $x = y + 5$

20. True or False

$$(a + b) \div c = \frac{a}{c} + \frac{b}{c}$$ Ans. True

21. True or False

$$c \div (a + b) = \frac{c}{a} + \frac{c}{b}$$ Ans. False

22. Solve for g in

$$S = \frac{1}{2} g T^2$$ Ans. $g = \frac{2S}{T^2}$

23. Solve for e and then solve for g in

$$T = 2\pi\sqrt{\frac{e}{g}}$$ Ans. $e = \frac{gT^2}{4\pi^2}$

$$g = \frac{4\pi^2 e}{T^2}$$

24. Solve for P in Ans. $P = \frac{FQ}{Q-F}$

$$\frac{1}{F} = \frac{1}{P} + \frac{1}{Q}$$

25. Solve for T in

$$z + \frac{2PQR}{T} = \sqrt{\frac{Q^2 - Z^2}{3PS}} + \sqrt{\frac{Z^3}{4PQT^2}}$$ Ans. $$T = \frac{2PQR - \sqrt{\frac{z^3}{4PQ}}}{\sqrt{\frac{Q^2-Z^2}{3PS}} - Z}$$

6.3 Word Problems.

Now that we have learned the basic techniques of solving linear equations, we are then able to use them in the solution of practical problems. Our basic approach will be to select a certain variable that is being talked about and then express this in literal terms as x, y, z, t and so on. Then we should analyze the problem to determine what is equal in the problem. The next step is to write an equation in terms of the literal variable for the equivalent expressions. The next step is to solve the equation for the variable. We then will have obtained the answer desired. There remains then only the task of substituting the answer back into the equation to check the solution. What we have just discussed will become much clearer after we solve a few of these problems.

Examples:

1. A number when multiplied by 2 and then added to 15 is equivalent to six times the number subtracted from 55.

Let x = the number

Then the equation that we can write which expresses the word problem in algebraic terms is:

$$2x + 15 = 55 - 6x$$

Then solving for x

$$8x = 40$$

$$x = 5 = \text{the number.}$$

The solution is checked by substituting the value of x into the original equation. In this case

$$2\,(5) + 15 = 55 - 6\,(5)$$

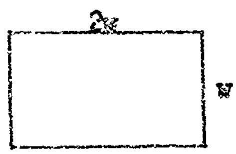
$$25 = 25 \qquad \text{Solution is correct.}$$

2. What are the dimensions of a cornfield whose length is twice its width and whose perimeter is equal to 600 feet?

Let w = the width of the field

Then $2w$ = the length of the field

Since the perimeter of a rectangle is equal to two times the length added to two times the width, we get



$$2w + 2\,(2w) = 600$$

$$6w = 600$$

$$w = 100' = \text{width}$$

$$2w = 200' = \text{length}$$

Checking:

$$2\,(100) + 2\,(200) = 600$$

$$600 = 600 \qquad \text{Solution is correct.}$$

3. Two cities A and B are located 100 miles apart. A car leaves A headed for B traveling at 20 miles per hour. Another car leaves B headed for A at the same time that the first car left. This car is traveling at 30 miles per hour. Where will the cars pass one another?

It is often helpful to draw a little picture to describe the problem.

$$A \; \overbrace{\qquad\qquad\qquad}^{D} \; \overbrace{\qquad\qquad\qquad}^{100-D} \; B$$

Remember: Distance = Rate x Time.

Let's let the distance the car which left A travels = D. Then the distance the other car travels before they pass one another is equal to 100 - D. Using the simple formula $D = R \times T$

Car from A to B $\quad D = 20 \times T$

Car from B to A $\quad 100-D = 30 \times T$

We cannot solve either of the above equations separately because each equation has two unknowns T and D, however, we know that the time which each car travels prior to passing one another is the same.

The time of the first car $= \frac{D}{20}$

The time of the second car $= \frac{100 - D}{30}$

Then $\quad \frac{D}{20} = \frac{100 - D}{30}$

$$30\,D = 20\,(100 - D)$$
$$30\,D = 2000 - 20\,D$$
$$50\,D = 2000$$
$$D = 40 \text{ miles}$$
$$100 - D = 60 \text{ miles}$$

Checking: $\quad \frac{D}{20} = \frac{100 - D}{30}$

$$\frac{40}{20} = \frac{100 - 40}{30}$$

$$2 = 2$$

The point is 40 miles from A and 60 miles from B.

We should always remember to let a letter equal an unknown, then try to determine what is equal in the problem. This is the key. Once we have determined what is equal, the equation and its solution are elementary.

Exercises:

Solve each of the following exercises by introducing only one unknown.

1. Paul Jones won \$64,000 in a TV contest, put aside \$28,000 for taxes on the income and split the balance with his consultant by giving the consultant \$10,000 less than he kept for himself. How much did he give the consultant?

   Ans. Paul Jones kept \$23,000 and gave his consultant \$13,000.

2. The sum of three consecutive integers is 105. Find the smallest of these integers.

   Ans. 34.

3. One number is 12 more than another. The smaller number is 25 per cent of the larger. Find the numbers.

   Ans. 4 and 16.

4. Bob has twice as much cash as Bill. If he lent Bill a dollar, they would have the same amount. How much did Bill have?

   Ans. \$2.00.

5. Don usually drives from his home to the college in 12 minutes. When rushed, he increases his average speed by 5 miles per hour and makes the trip in 10 minutes. How far does Don travel?

   Ans. 5 miles.

6. In a given time, iron mine A produces 500 more tons of ore than mine B. The ore from the former contains 25 per cent pure iron

as compared with 60 per cent pure iron from B, and the latter produces 400 more tons of pure iron than does A. What is the output of each in tons of ore?

Ans. A = 2000 tons B = 1500 tons

7. Forty gallons of milk whose butterfat content is 5% is mixed with thirty-two gallons of milk containing 4% butterfat. Find how many gallons of skim milk with ¼% of butterfat must be (a) added to give milk with 4½% butterfat (b) removed to give cream with 16% butterfat.

Ans. (a) 1 gal. (b) 53 5/31 gal.

8. A motor radiator contains 24 quarts of a solution which is 20% alcohol and 80% water. How much of the solution must be drained off and replaced by pure alcohol to give a 30% solution?

Ans. 3 quarts.

6.4 The Graph of a First Degree Equation.

If we have an equation which expresses a relationship between two variables such as y and x and the power of the variables is 1 for both, as in $y = 2x + 2$ or the power of one of the variables is 1 and the other is 0 understood, i.e. $y = 10$ or $x = 4$, we are able to graph the equation on coordinate axes such as we discussed in the chapter on functions and graphs.

Let's take the equation $y = 2x + 2$ and make up a table of coordinates, graph them and then discuss what we have done. By substituting values of x into the equation we can solve for corresponding values of y in the following manner:

When $x = -3$, $y = 2(-3) + 2 = -4$

When $x = -2$, $y = 2(-2) + 2 = -2$

When $x = -1$, $y = 2(-1) + 2 = 0$

When $x = 0$, $y = 2(0) + 2 = +2$

When $x = +1$, $y = 2(1) + 2 = +4$

When $x = +2$, $y = 2(2) + 2 = +6$

When $x = +3$, $y = 2(3) + 2 = +8$

These values are then plotted in Figure 6-1 and the result is a straight line. We could have originally been given $-4x + 2y = 4$. When we solve this equation for y we get our original equation $y = 2x + 2$. This form of the first degree equation where we have a dependent variable (y) in terms of an independent variable (x) is called the slope-intercept form of the equation. In general terms this equation is expressed as

$$y = mx + b$$

x	y
-3	-4
-2	-2
-1	0
0	2
1	4
2	6
3	8

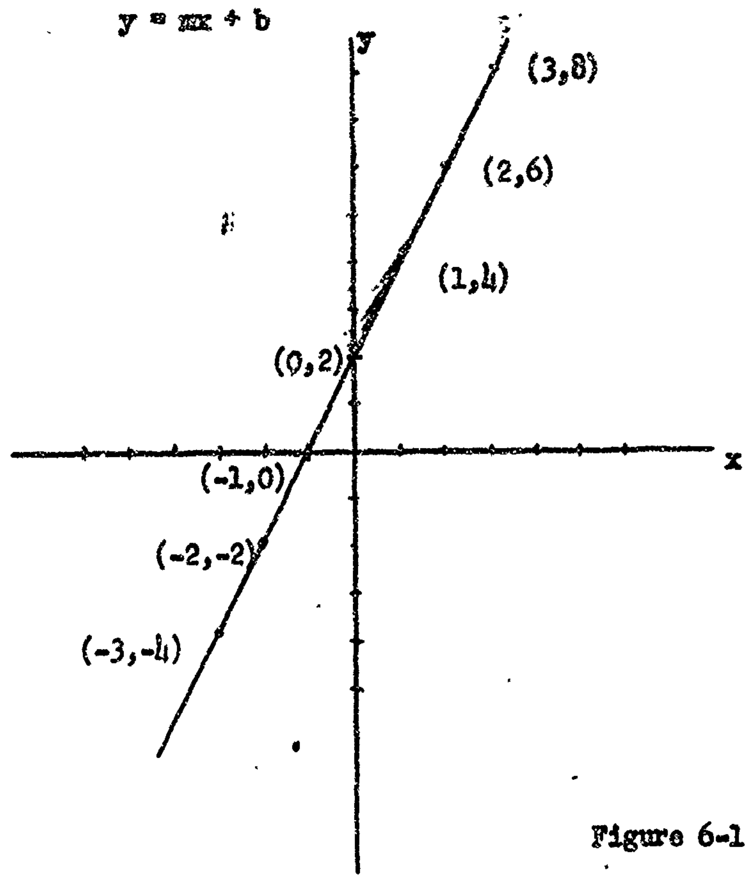


Figure 6-1

Where: y is the dependent variable (vertical axis)

x is the independent variable (horizontal axis)

m is the slope of the ~~line~~, or the change in y divided by the change in x

b is the y intercept, or where the graph crosses the y axis.

6.5 Slope-Intercept Form.

In the equation that we graphed in Figure 6-1, $y = 2x + 2$, the coefficient of x is 2 and corresponds to m in the general form $y = mx + b$. By our above definition of slope, change in y divided by a change in x, we can see that when we move a distance of a +1 in the x direction the corresponding change in y for the graph of the line is a +2 or $\frac{+2}{+1} = 2 =$ slope. When we have the equation in the slope-intercept form, if the coefficient of x is positive the graph of the line will slope up to the right; if the coefficient of x is negative the line slopes down to the right.

The b portion of the slope-intercept form tells us the point where the graph crosses the y axis. This is proven in the following manner: At any point on the y axis the value of x is 0. In the general form then

$$y = mx + b$$

When $\quad x = 0$

then $\quad y = m(0) + b$

$$y = +b$$

Examples:

1. What is the slope and the y intercept of the equation $2y - 4x + 10 = 0$

$$2y - 4x + 10 = 0$$
$$2y = 4x - 10$$
$$y = 2x - 5$$
$$\text{Slope} = +2$$
$$y \text{ intercept} = -5$$

The graph of this equation is a line which slopes up to the right with a +2 increase in y for every +1 increase in x. It crosses the y axis at $y = -5$.

2. What is the slope and the y intercept of $3y + 12x = 12$?

$$3y + 12x = 12$$
$$3y = -12x + 12$$
$$y = -4x + 4$$
$$\text{Slope} = -4$$
$$\text{y intercept} = +4$$

The graph of this equation is a line which slopes down to the right with a -4 decrease in y for each +1 increase in x. The graph crosses the y axis at $y = +4$.

It should be noted at this time that the equations $y = +4$, $y = -3$, etc., are horizontal lines parallel to the x axis with a slope which is 0. $y = 0$ is the x axis. Similarly $x = -4$, $x = 3$, etc., are vertical lines parallel to the y axis which all have an infinite slope. The equation $x = 0$ is the y axis.

If we were asked to determine the x intercept (where the line crosses the x axis) for an equation such as $y = 3x - 9$, we would substitute 0 for y since the value of y at any point on the x axis is 0.

Example:

Given $y = 3x - 9$ Find the x intercept.

$$y = 3x - 9$$
$$0 = 3x - 9$$
$$x = 3 \quad \text{Ans.}$$

Exercises:

Graph the following equations and determine the slope and the y intercept of each.

1. $10x + 2y = 8$ — Ans. Slope $= -5$, y intercept $= +4$

2. $3x - 6 = 3y$ — Ans. Slope $= +1$, y intercept $= -2$

3. $8x - 4y - 16 = 0$ — Ans. Slope $= +2$, y intercept $= -4$

4. $x + y - 4 = 0$ — Ans. Slope $= -1$, y intercept $= +4$

5. $-50x - 25y = 75$ — Ans. Slope $= -2$, y intercept $= -3$

6. Explain why the slope of the equation $y = 4$ is 0.

7. Explain why the slope of the equation $x = 2$ is infinite.

8. What can we say about the graphs of two lines parallel to one another?

9. What is the simplest means of obtaining the x intercept?

6.6 Obtaining Equation of a Line Given Two Points.

Now that we have learned how to graph an equation of the first degree, we will derive a formula which we can use to obtain the equation of a line if we are given any two sets of coordinates on the line.

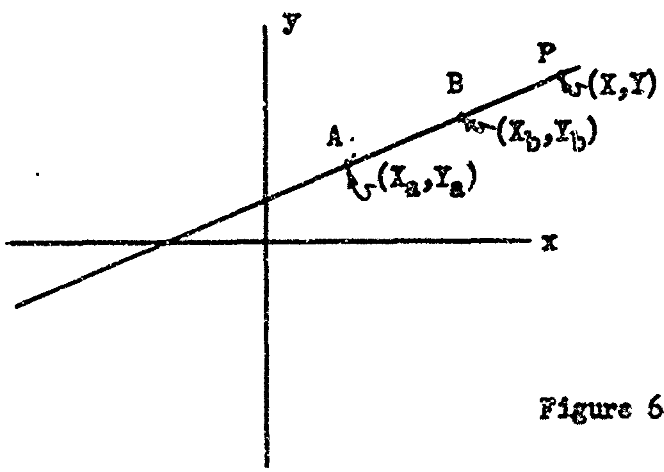


Figure 6-2

If we have some line drawn on coordinate axes as shown in Figure 6-2 and we know the coordinates of points A and B, which are respectively (X_a, Y_a) and (X_b, Y_b), we can determine the slope of the line. Since the slope is defined as the change in y divided by the change in x, the slope of the line through A and B is equal to $\frac{Y_b - Y_a}{X_b - X_a}$. Now let us take any other point on the line, call it point P with coordinates (X, Y). If we then determine the slope between point P and point B we will obtain $\frac{Y - Y_b}{X - X_b}$. If we determine the slope between point P and point A we will obtain $\frac{Y - Y_a}{X - X_a}$. Since each of these three points, P, A, and B are on the same line the slopes we obtained must be equal to one another.

Therefore

$$\frac{Y - Y_a}{X - X_a} = \frac{Y - Y_b}{X - X_b} = \frac{Y_b - Y_a}{X_b - X_a} = \text{slope}$$

With the above formula we now can write the equation of a line, if we are given the coordinates of any two points on the line. We can also use the formula to write the equation of the line if we are given one set of coordinates and the slope of the line.

Examples:

1. What is the equation of the line through (1, 2) and (4, 3)?

$$\frac{Y - Y_1}{X - X_1} = \frac{Y_2 - Y_1}{X_2 - X_1} = m$$

$$m = \text{slope}$$

$$\frac{Y - 2}{X - 1} = \frac{3 - 2}{4 - 1}$$

$$\frac{Y - 2}{X - 1} = \frac{1}{3}$$

Cross multiplying $\times$

$$3(Y - 2) = X - 1$$

$$3Y - 6 = X - 1 \quad \text{Ans.}$$

This is "OK" but for practice, let's put it into slope-intercept form.

$$3Y = X + 5$$

$$Y = \frac{X}{3} + \frac{5}{3} \qquad \text{Ans.}$$

Looking back at our solution, we should realize that when we substitute the coordinates of the two points in the right side of our equation, we would obtain the slope of the line $\frac{3 - 2}{4 - 1} = \frac{1}{3}$. We proved this when we put the equation in the slope-intercept form and found that the coefficient of x was $\frac{1}{3}$.

It follows then that if we were given a point and the slope of a line through that point, we could obtain the equation of the line. If we were given the point (2,3) and asked to find the equation of a line through that point with a slope of 3 we would proceed as follows:

$$\frac{Y - Y_1}{X - X_1} = m = \text{slope}$$

$$\frac{Y - 3}{X - 2} = 3$$

Then

$$Y - 3 = 3x - 6$$

$$Y = 3x - 3$$

$$\text{Slope} = 3$$

$$Y \text{ intercept} = -3$$

Exercises:

Find the equation of the line through the following sets of coordinates. Put the equation in the slope-intercept form. What is the slope the y intercept, and the x intercept?

1. $(1,-3)$ $(2,0)$ Ans. $y = 3x - 6$
$m = 3$
y intercept $= -6$
x intercept $= +2$

2. $(-2,-6)$ $(2,10)$ Ans. $y = 4x + 2$
$m = 4$
y intercept $= +2$
x intercept $= -\frac{1}{2}$

3. $(-2,-2)$ $(2,-6)$ Ans. $y = -x-4$
$m = -1$
y intercept $= -4$
x intercept $= -4$

4. $(2,3)$ $(4,4)$ Ans. $y = \frac{x}{2} + 2$
$m = \frac{1}{2}$
y intercept $= +2$
x intercept $= -4$

5. $(-2,-16)$ $(3,4)$ Ans. $y = 4x - 8$
$m = 4$
y intercept $= -8$
x intercept $= +2$

Determine the equation of the line through the indicated points having the given slopes. Then graph the line and determine the y and x intercepts.

6. $(2,6)$, $m = 2$ Ans. $y = 2x + 2$
y intercept $= +2$
x intercept $= -1$

7. $(-2,-3)$, $m = 3$ Ans. $y = 3x + 3$
y intercept $= +3$
x intercept $= -1$

8. $(1,-1)$, $m = -2$ Ans. $y = -2x + 3$
y intercept $= +3$
x intercept $= 3/2$

9. $(5,-1)$, $m = -1$ Ans. $y = -x + 4$

y intercept $= +4$

x intercept $= +4$

Find the equation of the line which has the following x and y intercepts.

10. x intercept $= -2$
y intercept $= +4$ Ans. $y = 2x + 4$

11. x intercept $= +6$
y intercept $= +3$ Ans. $y = \frac{x}{2} + 3$

12. x intercept $= +4$
y intercept $= -8$ Ans. $y = 2x - 8$

6.7 Systems of Linear Equations.

We have just become acquainted with linear equations of two variables. Let's suppose we were asked to find two numbers which when added together equalled 4. An algebraic statement of this problem using the variables x and y would be $x + y = 4$. There are an infinite number of combinations of values of x and y which satisfy this equation, i.e., 2 and 2, 3 and 1, +12 and -8 and so on. There is no way for us to select one set of values over any other and be sure that this is the one we are looking for. However, if we are given additional information such as $y = 8$, we can then substitute the value of $y = 8$ in the first equation as follows:

$$x + y = 4$$

Then substituting $y = 8$ for y

$$x + 8 = 4$$

Then we can solve for x

$$x = 4 - 8 = -4$$

It is not possible to solve an equation of two variables for specific values. It is necessary to have some other relationship expressed as an equation. If we have two linear equations and two unknowns we can solve the "system" of equations for specific values. In this section, we will learn two methods of arriving at solutions of systems of two equations and two unknowns. The first method we will call the substitution method. We just employed this method in solving the system of

$$x + y = 4$$
$$y = 8$$

The second method we will learn we will call the arithmetic method.

6.8 Substitution Method - Two Equations and Two Unknowns.

In this method we solve for one of the variables in one equation in terms of the other variable, and then substitute this value in the other equation and solve for the other variable. Then we substitute the value we have obtained in either of the equations and solve for the variable which is still unknown. Then check your results in both equations. The above may sound quite complicated, however, after we work through a couple of example problems you will see that they are really quite simple.

Examples:

1. Solve the following system of equations for x and y.

(1) $x + y = 7$

(2) $2x - 3y = -6$

The equations have been numbered to facilitate the explanation.

First solve equation (1) for x in terms of y $\qquad x = 7 - y.$

Then substitute this value of x in terms of y into equation (2)

$$2(7 - y) - 3y = -6$$

Since we now have one equation and one unknown, we can solve for that unknown.

$$2\,(7-y) - 3y = -6$$
$$14 - 2y - 3y = -6$$
$$5y = 20$$
$$y = 4$$

Then taking the value of $y = 4$ and substituting it into the simplest equation, in this case equation (1), and solve for x.

$$x + y = 7$$
$$x + 4 = 7$$
$$x = 3$$

We could have substituted the $y = 4$ in equation (2) in the following manner.

$$2x - 3y = -6$$
$$2x - 3(4) = -6$$
$$2x = -6 + 12 = 6$$
$$x = 3$$

Our next step is to check the values of $x = 3$ and $y = 4$ in both equations to prove our work to ourselves.

(1) $x + y = 7$

$3 + 4 = 7$

$7 = 7$ Correct

(2) $2x - 3y = -6$

$2(3) - 3(4) = -6$

$6 - 12 = -6$

$-6 = -6$ Correct

2. Solve the following system of equations for x and y and check your results.

(1) $x - 2y = 0$

(2) $3x + 2y = 8$

Solving (1) for x

$$x = 2y$$

Substituting this in equation (2)

(2) $3x + 2y = 8$

$$3\,(2y) + 2y = 8$$
$$6y + 2y = 8$$
$$8y = 8$$
$$y = 1$$

Substituting $y = 1$ in equation (1)

(1) $\quad x - 2y = 0$

$$x - 2(1) = 0$$
$$x = 2$$

Then checking the values of x and y in both equations

(1) $\quad x - 2y = 0$ $\qquad$ and $\qquad$ (2) $\quad 3x + 2y = 8$

$$2 - 2(1) = 0 \qquad\qquad 3\,(2) + 2(1) = 8$$
$$0 = 0 \qquad\qquad 8 = 8$$

Values check.

3. Solve the following system of equations.

(1) $\quad 2x - 3y = 4$

(2) $\quad x + 6y = 2$

Solving (2) for x

$$x = 2 - 6y$$

Substituting in (1)

$$2\,(2 - 2y) - 3y = 4$$
$$4 - 4y - 3y = 4$$
$$7y = 0$$
$$y = 0$$

Substituting in equation (2)

$$x + 2y = 2$$
$$x + 0 = 2$$
$$x = 2$$

Checking

(1) $\quad 2x - 3y = 4$ $\qquad$ (2) $\quad x + 6y = 2$

$$2(2) - 3(0) = 4 \qquad\qquad 2 + 6(0) = 2$$
$$4 = 4 \qquad\qquad 2 = 2$$

The student should now realize that each of the equations in a system can be plotted as a straight line. If the graphs of the two equations cross (if they are not parallel) then there is a value of x and y which "satisfies" both equations. It is left as a drill for the student to graph each of the systems of equations to prove to himself that this is in fact true. The common point should coincide with the values obtained by the substitution method.

6.9 Arithmetic Method - Two Equations and Two Unknowns.

When we solve a system of two equations and two unknowns by the Arithmetic method we multiply, divide, add and subtract in an effort to eliminate one of the variables, remembering always that anything we do to one side of an equation, we must do to the other. The other principle upon which this method is based is that if we add one equation to another equation we obtain a third equation, since we are adding equals to both sides of an equation. This may sound confusing, however, after we work through a couple of solved problems you should have no trouble in applying this method.

Example:

1. Solve the following system of equations by the arithmetic method.

(1) $6x + 2y = 10$

(2) $3x - 4y = -5$

We should aim at eliminating one variable from the system. We can eliminate the y's by multiplying equation (1) by 2 and then adding equation (1) and (2). We could just as easily eliminate the x's by multiplying equation (2) by -2 and then adding equations (1) and (2). Let's eliminate the y's. Equation (1) becomes:

(1) $12x + 4y = 20$

(2) $3x - 4y = -5$

Adding (1) and (2)

$$15x = 15$$
$$x = 1$$

At this point, our method is the same as the one we used in the substitution method above, after we had solved for one of the variables. We simply substitute the value known in either of the original equations and solve for the other and then check both values.

Substituting $x = 1$ in equation (2)

$$3(1) - 4y = -5$$
$$4y = 8$$
$$y = 2$$

Then

Checking in (1) and checking in (2)

$12(1) + 4(2) = 20$ $\qquad$ $3(1) - 4(2) = -5$

$20 = 20$ $\qquad$ $-5 = -5$

2. Solve the following system of equations by the arithmetic method.

(1) $3x + 2y = 1$

(2) $4x - y = 16$

We can eliminate the y's by multiplying equation (2) by 2 and then adding (1) to (2) or we could eliminate the x's by multiplying (1) by 4 and (2) by -3 and then adding. Let's eliminate the y's since this involves only one multiplication.

(1) $3x + 2y = 1$

(2) $8x - 2y = 32$

$$11x = 33$$
$$x = 3$$

Substituting $x = 3$ in original equation (2)

$$(2) \qquad 4(3) - y = 16$$
$$y = -4$$

It is left to the student to check these values in the original equations.

6.10 Simultaneous Equations with Three Unknowns.

In order to solve equations of the first degree with two unknowns, we found that we needed two equations. Similarly, to solve a system of equations with three unknowns we need to have three equations which express relationships between or among some or all of the variables.

Our approach is very similar to the substitution and arithmetic methods used in the previous paragraphs. Suppose we are given the system of equations

$$(1) \qquad x + y + z = 6$$
$$(2) \qquad 2x - 3y + 2z = 2$$
$$(3) \qquad z = 3$$

Our approach would be to use the value of z which we are given in equation (3) and substitute it in equations (1) and (2).

We obtain:

$$(1) \qquad x + y + 3 = 6$$
$$(2) \qquad 2x - 3y + 6 = 2$$

transposing the integers

$$(1) \qquad x + y = 3$$
$$(2) \qquad 2x - 3y = -4$$

We now have two equations and two unknowns which we can solve. Using the arithmetic method, we would multiply equation (1) by -2 and then add the two equations and eliminate the x's and then solve for y.

It is left to the student to finish the solution. The answers are $x = 1$, $y = 2$ and $z = 3$.

We might be given a system such as:

$$(1) \quad x + y + z = 4$$

$$(2) \quad 2x - 3y + z = -1$$

$$(3) \quad 2x - 4y = -2$$

We should notice right away that equation (3) contains only 2 variables, x and y. Our approach in this case would be to apply the arithmetic approach to equations (1) and (2) and eliminate the variable z. This we can do by multiplying equation (1) by -1 and then adding (1) and (2) to get a new equation (4) with unknowns x and y.

$$\begin{array}{lrcl} (1) & -x - y - z & = & -4 \\ (2) & \underline{2x - 3y + z} & \underline{=} & \underline{-1} \\ (4) & x - 4y & = & -5 \end{array}$$

We then can take the combination of equations (3) and (4), multiply (4) by -1 and eliminate the y's, and then solve for x.

$$\begin{array}{lrcl} (3) & 2x - 4y & = & -2 \\ (4) & \underline{-x + 4y} & \underline{=} & \underline{+5} \\ & x & = & 3 \end{array}$$

We can then substitute $x = 3$ in one of the equations which has only 2 variables, in this case either (3) or (4), and solve for the other variable. Substituting in (3) we get

$$\begin{array}{lrcl} (3) & 2(3) - 4y & = & -2 \\ & 4y & = & 6 + 2 \\ & y & = & 2 \end{array}$$

Now we are almost "home". We simply have to take an equation which contains three variables, substitute the values of x and y which we have already determined and solve for z. Let's use equation (1) for this substitution. We, however, could also have used equation (2).

$$(1) \quad 3 + 2 + z = 4$$

$$z = -1$$

A check is then made by substituting all the values we have determined in the three original equations.

In the general case, we would be given three equations each containing the same three variables. Our approach would then be to take any two equations and eliminate one variable, say x. Our next step would be to take another combination of two equations and eliminate the same variable x. We would then have obtained two equations which contain two variables, say y and z, which we can solve. Once we have obtained a value of one variable, we then substitute in an equation which contains only the known variable and one other. When we have solved for 2 variables, we substitute these values in one of the equations containing the three variables and find the third and last. We then check all three values in all original equations.

Example:

Solve the following system of equations:

$$(1) \quad 3x - 2y + z - 1 = 0$$

$$(2) \quad x + 2y - 3z - 13 = 0$$

$$(3) \quad x + y + 2z = -3$$

Taking equations (1) and (2) and putting the constants on the right side of the equation and eliminating the y's by adding

$$(1) \quad 3x - 2y + z = 1$$

$$(2) \quad x + 2y - 3z = 13$$

we get

$$(4) \quad 4x - 2z = 14$$

Taking equations (2) and (3), we can multiply (3) by -2, then add it to equation (2) to obtain equation (5).

$$
\begin{array}{lrcl}
(2) & x + 2y - 3z & = & 13 \\
(3) & \underline{-2x - 2y - 4z} & \underline{=} & \underline{+6} \\
(5) & -x \qquad -7z & = & 19
\end{array}
$$

We then take equations (4) and (5) and eliminate x by multiplying equation (5) by +4 and adding the result from equation (4).

$$
\begin{array}{lrcl}
(4) & 4x - 2z & = & 14 \\
(5) & \underline{-4x - 28z} & \underline{=} & \underline{76} \\
 & -30z & = & 90 \\
 & z & = & -3
\end{array}
$$

Substituting $z = -3$ in equation (4)

$$
\begin{array}{rcl}
4x - 2(-3) & = & 14 \\
4x & = & 8 \\
x & = & 2
\end{array}
$$

Then substituting $x = 2$ and $z = -3$ in equation (3)

$$
\begin{array}{lrcl}
(3) & 2 + y + 2(-3) & = & -3 \\
 & y & = & -3 + 6 - 2 \\
 & y & = & 1
\end{array}
$$

It would be a good drill for the student to check the values in the original equations.

Exercises:

Solve the following systems of equations by both the substitution and arithmetic methods and check the values obtained in the original equations.

1. $2x - 3y = 1$ Ans. $x = 5$
   $x - 2y = -1$ $\quad$ $y = 3$

2. $x - 2y = 28$ Ans. $x = 8$
$2x + y = 6$ $y = -10$

3. $3x + 3y = -15$ Ans. $x = -2$
$4x - 2y = 2$ $y = -3$

4. $6x - 5y = 16$ Ans. $x = 6$
$4x - 3y = 12$ $y = 4$

5. $2x - 3y = 14$ Ans. $x = 4$
$3x + 6y = 0$ $y = -2$

Solve the following systems of equations for x, y and z and then check the results in the original equations.

6. $3x - 4y + 2z = 8$ Ans. $x = 2$
$4x - 8y - 2z = -6$ $y = 1$
$x - 7y + 2z = 1$ $z = 3$

7. $6x - 3y + 2z = -14$ Ans. $x = -2$
$x + y - z = -3$ $y = 4$
$-2x - 2y + 3z - 11 = 0$ $z = 5$

8. $z + 7 = 0$ Ans. $x = 5$
$6x - 3y = 6$ $y = 8$
$4x - y + 2z = -2$ $z = -7$

9. $y + 3 = 0$ Ans. $x = -1$
$4x + 4 = 0$ $y = -3$
$x + y - z - 2 = 0$ $z = -6$

10. $3x + 4y = 60$ Ans. $x = 8$
$2y + z - 8 = 0$ $y = 9$
$2x + 3y + 4z - 3 = 0$ $z = -10$

6.11 Inequalities.

There are occasions in a management curriculum, particularly in the quantitative disciplines, when we wish to symbolically state that one number is "greater than" or "less than" another number. Conventionally we write

$$a > b \quad \text{which means a is greater than b}$$

or

$$c < d \quad \text{which means c is less than d}$$

A good way to remember these relationships is to remember that the point always "points at the smaller number".

Similarly:

$$e \geqq f \quad \text{means e equal to or greater than f}$$

and

$$g \leqq h \quad \text{means g is equal to or less than h}$$

We can apply most of the techniques that we have learned in the solution of equalities to the handling of inequalities. Let's take the simple inequality $8 > 4$ and see what we can do with it.

1. We can add like quantities to both sides of an inequality and the inequality will still have the same "sense".

$$8 + 2 > 4 + 2$$

$$10 > 6$$

(Same sense meaning, in this case, is still greater than.)

Similarly:

2. We can subtract like quantities from both sides of an inequality and the inequality will still have the same sense.

$$8 - 3 > 4 - 3$$

$$5 > 1$$

3. We can multiply or divide both sides of an inequality by the same "positive" number and again still maintain the validity or sense of the inequality.

$$8 > 4$$

$$8 \cdot 2 > 4 \cdot 2$$

$$16 > 8$$

and

$$8 > 4$$
$$\frac{8}{2} > \frac{4}{2}$$
$$4 > 2$$

4. Let's examine what happens if we multiply both sides of our inequality by a minus number. Starting again with

$$8 > 4$$

If we multiply both sides of this inequality by -2

$$8 \cdot (-2) > 4(-2)$$
$$-16 > -8$$

Notice that if we keep the same inequality symbol $>$ (same sense) as we did in the previous examples, we get a nonsense answer, since we all know that -16 is less than -8 or using our newly acquired knowledge

$$-16 < -8$$

Let's take another inequality such as

$$3 < 6$$

Multiplying both sides by -3 we obtain

$$3(-3) < 6(-3)$$
$$-9 < -18$$

which again is a nonsense answer since $-9 > -18$.

Similarly, if we divide both sides of the inequality $3 < 6$ by -3,

$$\frac{3}{-3} < \frac{6}{-3}$$

$-1 < -2$ which also is nonsense since $-1 > -2$.

Therefore, we can conclude that if we multiply or divide both sides of an inequality by a negative number, we still have an inequality, however, the <u>sense</u> of the new inequality is reversed.

We simply change the inequality from $>$ to $<$ or from $<$ to $>$, whichever is appropriate.

5. We can also raise both sides of inequalities to the same power or we can take the same root (square, cube, etc.) of both sides. However, if we take negative roots we must change the sense of the inequality. For example, if we take the inequality

$$6 > 4$$

and square both sides, we get

$$36 > 16$$

Taking another inequality, such as

$$64 < 100$$

we can take square roots of both sides, and get

$$+8 < +10$$

or $$-8 > -10$$

We can also "solve" inequalities . Suppose we were told to solve the following inequality for x.

$$3x + 4 \geq 28$$

Using the techniques discussed above, we subtract 4 from each side and get

$$\begin{aligned} 3x + 4 &\geq 28 \\ 3x &\geq 28 - 4 \\ 3x &\geq 24 \\ x &\geq 8 \end{aligned}$$

Similarly, we can solve for y in the following inequality:

$$\begin{aligned} 2y - 6 &< 36 \\ 2y &< 36 + 6 \\ 2y &< 42 \\ y &< 21 \end{aligned}$$

As we can see, solving inequalities is just like solving equalities.

6.12 Graphing Inequalities.

Inequalities, when graphically presented, are areas in the coordinate axes, rather than lines. For example, $x > 0$ is all of the area to the right of the y axis and $y < 1$ is all of the area below the line $y = 1$. Likewise, $y \geqq 0$ is the area on and above the x axis.

We might be asked to graph on a set of coordinate axes the region or area described by the following three relationships:

(1) $x > 0$

(2) $y > 0$

(3) $y \leqq -2x + 8$

Referring to Figure 6-3, the first inequality, $x > 0$ excludes the y axis and the area to its left. The next inequality, $y > 0$, excludes the x axis and the area which is below the x axis. At this point, we have limited the area we are looking for to the area in the first quadrant. The third relationship $y \leqq -2x + 8$ would exclude the area above the line $y = -2x + 8$. The area which fulfills the "specifications" of all three relationships is the area of the cross-hatched triangle including the portion of the line $y = -2x + 8$, but excluding the portions of the x and y axes which border the triangle.

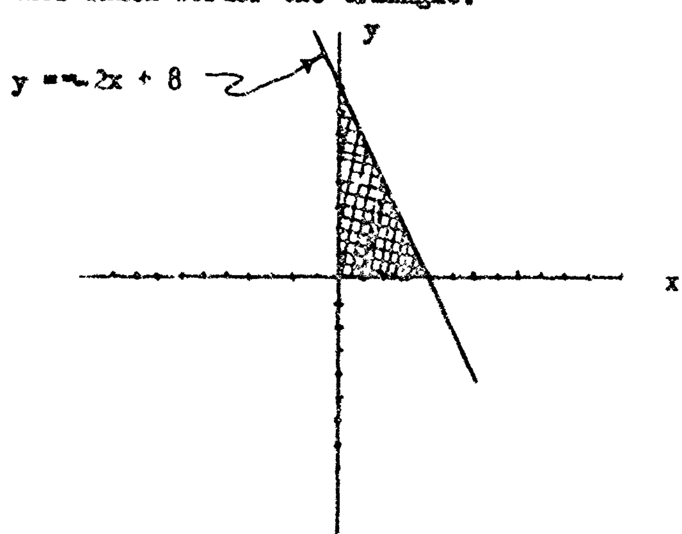


Figure 6-3

Exercises:

Solve the following inequalities.

1. $x + 6 > 15$ Ans. $x > 9$

2. $2x - 10 < 30 - 6x$ Ans. $x < 5$

Graph the following groups of inequalities.

3. $x \geq 0$

$y \geq 0$

$y \geq -x + 4$

4. $-4 < x < +4$

$-2 < y < +2$

CHAPTER 7

EQUATIONS OF THE SECOND DEGREE AND HIGHER

7.1 Introduction.

A step above linear equations, in degree of difficulty of solving, are those equations and expressions which (1) contain at least one term in which the sum of the exponents of the variables in that term, or (2) in the case of a term with only one variable where that variable's exponent, is equal to or greater than two. The sums of the exponents of the variables in a term is what determines the degree of a term. The degree of an equation or expression is considered to be the same as the degree of the highest degree term in that equation. Thus, in the equation $4x^2y^3 + 3x^2 + 2 = 0$, $4x^2y^3$ is a fifth degree term (exponent of $x = 2$ plus exponent of $y = 3$, with sum $= 5$), $3x^2$ is a second degree term, the constant 2 is a zero degree term because there is no variable, and the equation as a whole is of the fifth degree. Note carefully that it is the sum of the exponents that determines the degree of the term. Just as first degree equations are commonly referred to as linear equations, second degree equations are commonly referred to as _quadratic equations_. The solving of second degree, or quadratic, equations is usually not too difficult and will be demonstrated in the following paragraphs. However, the solving of equations of a degree higher than the second degree is more often than not a matter of approximating the solution, and then checking its accuracy by substituting the approximation back into the original equation. Due to the problems of approximating solutions, let's just confine our discussion to the solving of quadratic equations. The solutions which we obtain are called _roots_ of the equation and must in all cases be substituted back into the original equation to ensure that they are correct and accurate because, as we will see, it is possible to get some incorrect answers. To start with, let's first consider quadratic equations with one unknown, and then go into solving equations with two unknowns and finally take up the graphing of quadratics.

7.2 Quadratics with One Unknown.

When we talk of quadratics with one unknown, what we really mean is that all the unknown parts to a problem can be expressed in terms of a single variable. For example, we may know the area of a plot of land but not know its length or width. However, if we know the relationship between the length and width we can express one in terms of the other and we would thus consider that we had a problem with only one unknown. Knowing what we mean by one unknown, let's now develop a general equation which expresses this condition. In our development, let's assign the unknown variable the literal symbol x. Since it is to be a quadratic equation, there must be one term containing the variable raised to the second power. Knowing that this term will be x^2, but not knowing how many x^2's we have, we put a symbol for an unknown constant before the x^2. Since a is the first letter of the alphabet, we'll use a and get the first term of our general quadratic equation to be ax^2. Now, we have filled the minimum requirements for a quadratic equation. It is, of course, possible, but not necessary, to have the variable raised to the first power and to have a constant. Assuming we have some first degree power of x present and a constant, but that we do not know how many x's there are or what the constant is, we substitute the literal symbols b and c for these unknowns, respectively. Thus we can come up with the general quadratic equation

$$ax^2 + bx + c = 0.$$

Remember, b and/or c can be zero but that a cannot be zero; if it were, we would not have a quadratic.

There are several methods of solving the quadratic equation, that is finding values of the variable that will satisfy the equation. If we consider our land parcel to be 15,000 square feet and to have sides of equal length, we can easily determine the length of the sides. First, let x stand for the length of a side and set up the equation for the area; that is

$$x \text{ times } x = 15{,}000$$

Multiplying the x's, we get

$$x^2 = 15{,}000$$

If we take the square root of both sides of the equation, the solution to our land problem would thus be plus or minus $\sqrt{15{,}000}$ feet. Be sure to remember that when we take square roots, we get a plus and minus value. Substituting back into the original equation,we find that both roots satisfy the equation. However, now we must use a little practical common sense and realise that while the $-\sqrt{15{,}000}$ satisfies the equation, that there may be some question as to its being meaningful in a practical sense. Obviously, it does not make sense because we cannot have negative dimensions of land, so we disregard this root and use only the root $+\sqrt{15{,}000}$ and say the length of a side is $\sqrt{15{,}000}$ feet.

For those quadratics with one unknown where b is not zero, we have a little more complicated problem. If we can manipulate the equation to get the right side of the equation equal to zero, and still have on the left side of the equation an expression that is factorable, we have an easy task. All we do is factor the left side, set each factor equal to zero and solve for the value of the variable that will make the individual factor equal to zero. This is a perfectly legal maneuver because, if one factor is zero, the whole expression will be zero, because zero times anything is zero. As an example, let us assume our land plot is now 15,000 square feet and rectangular, with the length being 50 feet greater than the width. Letting x stand for the width in feet, we can write the formula for the area as

$$x (x + 50) = 15,000.$$

Multiplying the left side and then shifting the constant to the left side, we get $x^2 + 50x - 15,000 = 0$

which we factor into $(x + 150) (x - 100) = 0$

setting each factor $= 0$, we get

$x + 150 = 0$, or $x = -150$

$x - 100 = 0$, or $x = 100$

Substituting both roots back into the original equation, we find both roots satisfy it. But think, the -150 is impossible so we only consider the +100 as a solution to our problem. This tells us that the width is 100 feet. Knowing that the length is equal to the width plus 50 feet, we find the length to be 150 feet. By rapid mathematics, 100 times 150 equals 15,000 and we see that we have a valid solution. In addition to the problem of sometimes getting solutions that are not valid from a practical point of view, we can also get solutions that won't even satisfy the original equation. These we call extraneous roots. For example, if we multiply both sides of the equation by a common denominator we may get an extraneous root. To facilitate understanding this concept, let's try solving the following equation.

$$\frac{x^2 + 3x}{3 + x} = 0$$

Multiplying both sides by $3 + x$, we got

$$x^2 + 3x = 0$$

which can be factored into

$$x(x + 3) = 0$$

Then setting the factors = 0, we get

$$x = 0 \quad \text{and} \quad x = 3$$

Now substituting back into the original equation, we find $x = 3$ is not possible, because we would be dividing by zero. Thus, the only satisfactory root is $x = 0$. The moral of the story is to check all roots by substituting them back into the original equation and by ascertaining whether or not they are practical.

Now let's go through a few examples to ensure that we know what we have been talking about.

1. Solve $x^2 + 5x + 6 = 0$

   Solution: factor into $(x + 2)(x + 3) = 0$

   setting each factor = 0, then $x = -2$ and $x = -3$

   checking $x = -2$, $(-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0$ checks

   checking $x = -3$, $(-3)^2 + 5(-3) + 6 = 9 - 15 + 6 = 0$ checks

2. Solve $200x^2 - 750x + 625 = 0$

   Solution: divide by 25, $8x^2 - 30x + 25 = 0$

   factoring into $(2x - 5)(4x - 5) = 0$

   setting each factor = 0, $2x - 5 = 0$, $x = 2.5$

   $4x - 5 = 0$, $x = 1.25$

   Checking $x = 2.5$, $200(2.5)^2 - 750(2.5) + 625 = 1250 - 1875 + 625 = 0$ checks

   Checking $x = 1.25$, $200(1.25)^2 - 750(1.25) + 625 = 312.5 - 937.5 + 625 = 0$ checks

3. A car dealer bought some cars for a total price of \$36,000. Two cars were destroyed on his lot by fire before he could sell them. By selling the remaining cars at \$950 above his average cost he was able to make a profit of \$400. How many cars did he originally buy?

Solution: Let x stand for the number of cars the dealer bought. Then, his average cost was 36,000 divided by x and his total sales were 36,000 divided by x plus 950 which is all multiplied by x minus 2. The difference between sales and costs was his profit. Symbolically we can write

Sales	- Costs	= Profits
$\left(\frac{36,000}{x} + 950\right)(x - 2)$	$-36,000$	$= 400$

Multiplying both sides by x, we get

$$(36,000 + 950x)\ (x - 2) \qquad -36,000\,x \qquad = 400\,x$$

$$36,000\,x - 72,000 + 950\,x^2 - 1900\,x - 36,000\,x = 400\,x$$

Collecting terms, we get

$$950\,x^2 - 2300\,x - 72,000 = 0$$

dividing by 50, we get

$$19x^2 - 46x + 1440 = 0$$

which we can factor into

$$(x - 10)\ (19x + 144) = 0$$

setting factors = 0, we get

$$x = +10 \qquad \text{and} \qquad x = -\frac{144}{19}$$

In this case, it is easiest to just ignore the $-\frac{144}{19}$ because it is not possible for the dealer to buy negative quantities of cars, and then check to see if 10 cars would satisfy the conditions of the problem. It does, so the solution is 10 cars.

7.3 Quadratic Formula.

Sometimes when we collect terms on the left hand side of the equation, we find we have an unfactorable expression. In this case, we can utilize the solution technique known as completing the square. In this method, by mathematical manipulations, we make the left side of the equation factorable into two factors which are exactly the same without regard to what is on the right side of the equation. To start off with, we shift the constant term to the right side of the equation and divide through by the constant coefficient of the second degree term. As an example, in $3x^2 - 2x - 4 = 0$, we would get $x^2 - \frac{2x}{3} = \frac{4}{3}$. Studying $(a + 2)(a + 2) = a^2 + 4a + 4$, we can see that if an expression with the constant coefficient of the second degree term equal to one is to be factorable into two identical factors, the constant term must be the square of one half the constant coefficient of the first degree term. We can see that we must add the term $\left(\frac{2}{3} x \frac{1}{2}\right)^2$ to both sides of the equation. Thus we get

$$x^2 - \frac{2}{3}x + \left(\frac{2}{3} \times \frac{1}{2}\right)^2 = \frac{4}{3} + \left(\frac{2}{3} \times \frac{1}{2}\right)^2$$

or

$$x^2 - 2/3x + \left(\frac{1}{3}\right)^2 = \frac{4}{3} + \left(\frac{1}{3}\right)^2$$

factoring, we get

$$\left(x - \frac{1}{3}\right)^2 = \frac{4}{3} + \left(\frac{1}{3}\right)^2$$

taking the square root of both sides, we get

$$x - \frac{1}{3} = \pm\sqrt{\frac{4}{3} + \frac{1}{9}} = \pm\sqrt{\frac{12 + 1}{9}} = \pm\sqrt{\frac{13}{9}}$$

or

$$x = \frac{1}{3} \pm\sqrt{\frac{13}{9}}$$

It should be apparent to us that if we could develop a general formula for this method we could save a lot of time. To develop this formula let's go back to the general form of the quadratic equation and do the same thing with symbols rather than numbers. Taking the general form of the quadratic equation, $ax^2 + bx + c = 0$, let's subtract c from both sides and divide by a and get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now if we add $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ to both sides

we get $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$

which can be factored into

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

now taking square roots of both sides

$$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

subtracting $\frac{b}{2a}$ from both sides, we get

$$x = -\frac{b}{2a} \pm\sqrt{\frac{b^2-4ac}{4a^2}} = -\frac{b}{2a} \pm\frac{\sqrt{b^2-4ac}}{2a}$$

combining, we get

$$x = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$$

which is the general formula for solving any quadratic equations with one unknown. Knowing this formula, all we have to do to solve any quadratic equation with one unknown is substitute in the values of a, b, and c.

The $\sqrt{b^2 - 4ac}$ part of the formula is called the <u>determinant</u> because, knowing its value, we can foretell what form the roots of the equation will take.

If,

1. $b^2 - 4ac > 0$ we get two unequal roots

2. $b^2 - 4ac = 0$ we get two equal roots

3. $b^2 - 4ac < 0$ we get two unreal, or imaginary, roots. (Not included in this course)

Before beginning the topic of quadratics with two unknowns, let's try solving two equations using the formula.

1. Solve $x^2 - 2x - 3 = 0$

Solution: $a = 1$, $b = -2$, $c = -3$ in the general equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{+2 \pm \sqrt{4 + 12}}{2} \qquad \frac{+2 \pm \sqrt{16}}{2}$$

therefore $x = \frac{2 + 4}{2} = \underline{3}$ or $x = \frac{2 - 4}{2} = \underline{-1}$

Both answers check and the solution is a good one.

2. Solve $3x^2 + 2x - 4 = 0$

Solution: $a = 3$, $b = 2$, $c = -4$ in the general equation

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-4)}}{2(3)}$$

$$= \frac{-2 \pm \sqrt{4 + 48}}{6} = \frac{-2 \pm \sqrt{52}}{6}$$

$$= \frac{-2 \pm 2\sqrt{13}}{6} = \frac{-1 \pm \sqrt{13}}{3}$$

therefore $x = \frac{-1 + \sqrt{13}}{3}$ and $x = \frac{-1 - \sqrt{13}}{3}$

Substituting the roots back into the equation, we find they check and we have valid roots.

Now try to solve the problems listed below and see if you get the answers in the answer column.

Exercises:

1. Solve $x^2 + x - 20 = 0$	Ans. $x = 4$ and $x = -5$
2. Solve $x^2 - 16 = 0$	Ans. $x = +4$ and $x = -4$
3. Solve $x^2 + 2\sqrt{2}\,x - 6 = 0$	Ans. $x = +\sqrt{2}$ and $x = -3\sqrt{2}$
4. Solve $5x^2 - 2x - 2 = 0$	Ans. $x = \frac{1+\sqrt{11}}{5}$ and $x = \frac{1-\sqrt{11}}{5}$
5. Solve $t^2 + \sqrt{3}\,t - 1 = 0$	Ans. $t = \frac{-3+\sqrt{7}}{2}$ and $t = \frac{-3-\sqrt{7}}{2}$
6. Solve $\frac{4}{3}x^2 + \frac{1}{2}x - 5 = 0$	Ans. $x = \frac{-3 + 3\sqrt{\frac{323}{3}}}{16}$ and $x = \frac{-3 - 3\sqrt{\frac{323}{3}}}{16}$
7. Solve $x^2 + 2x = 8$	Ans. $x = 2$ and $x = -4$
8. Solve $x^2 + 6x + 5 = 0$	Ans. $x = -1$ and $x = -5$
9. Solve $12x^2 + 7x = 12$	Ans. $x = \frac{3}{4}$ and $x = -\frac{4}{3}$
10. Solve $9x^2 + 12x + 4 = 0$	Ans. $x = -\frac{2}{3}$ and $x = -\frac{2}{3}$

7.4 Quadratics with Two Unknowns.

Just as with linear equations, we must have as many equations as we have unknowns in order to get a solution to a problem. Therefore, with two unknowns we must have two equations. With only one equation and two unknowns, we will be reduced to determining one variable in terms of another and will not be able to do anything except develop a series of ordered pairs. We can then use the ordered pairs to plot the graph of the equation. Having two equations with two unknowns, however, will change this situation, and we will be able to get a solution to the problem mathematically; that is, we will know what values of the variables will satisfy both equations. Of course, we could also solve the problem graphically just as can be done with two unknowns in linear equations.

The method we use to solve our problem will depend upon whether both equations are quadratics or one is linear while the other is a quadratic. In the latter case, the problem is relatively simple because we just use the linear equation as a means of expressing one variable in terms of the other, and then substitute this relationship into the quadratic equation so that we get a quadratic with one unknown. Then, we can obtain the roots for one variable fairly easily by utilizing the general formula for solving quadratic equations with one unknown or by factoring. As an example, suppose we want to find the roots that satisfy the following equations.

$$3x - 2y = 5$$

and

$$x^2 - xy + 2y = 7$$

Solution: Manipulate the linear equation and express y in terms of x, that is $y = \frac{3x - 5}{2}$

Substitute this into the second equation and get

$$x^2 - x\left(\frac{3x - 5}{2}\right) + 2\left(\frac{3x - 5}{2}\right) = 7$$

$$x^2 - \frac{3x^2 - 5x}{2} + \frac{6x - 10}{2} = 7$$

We can multiply by 2 and get

$$2x^2 - 3x^2 + 5x + 6x - 10 = 14$$

Collecting terms, we get

$$-x^2 + 11x - 24 = 0$$

Now, if we multiply by -1, we get

$$x^2 - 11x + 24 = 0$$

which we can factor into

$$(x - 3)\,(x - 8) = 0$$

and determine that

$$x = 3$$

$$x = 8$$

Both answers check, so we substitute the roots into the linear equation and get two corresponding values of y.

In this problem for $x = 3$, $y = \frac{3(3) - 5}{2} = 2$

and $x = 8$, $y = \frac{3(8) - 5}{2} = \frac{19}{2}$.

Thus we see that we have two possible solutions to the problem. We arrange the roots in the form of ordered pairs because the x and y values are dependent upon one another; that is, we write the solution as

$$(x = 3,\ y = 2), \quad (x = 8,\ y = \tfrac{19}{2}).$$

Another situation that we could encounter would be one where we had only the squares of the unknowns. Thus if our unknowns were x and y, we would only have x^2 and y^2 terms. This type of problem is very easy to solve since it is similar to solving linear equations with two unknowns. That is, we mathematically manipulate the equations until the constant coefficients of one of the squared unknowns are alike in both equations. Then we add the two equations or subtract one from the other so that we end up with one squared unknown and a constant. Obviously, the next thing we would do is take the square root of the constant. This gives us a plus and a minus root for one of the unknowns. Then what we have to do is substitute these two values into one of the two equations and get the value of the other squared unknown. By mathematical manipulations, we get a new constant which is equal to the squared unknown. Taking square roots of both sides, we get a plus and a minus root for the second unknown. Remember, however, that we now have four pairs of roots. That is, we have the two values of the first unknown and a plus and a minus root of the second unknown to go along with each of the roots of the first unknown. To clarify our thinking, let's find the solution to the following equations.

$$8x^2 + 5y^2 = 65$$
$$2x^2 + 3y^2 = 25$$

Solution: Multiply second equation by 4 and get

$$8x^2 + 12y^2 = 100$$

Subtract this from the first equation in the following manner:

$$\begin{aligned} 8x^2 + 5y^2 &= 65 \\ \underline{8x^2 + 12y^2} &\underline{= 100} \\ -7y^2 &= -35 \\ y^2 &= 5 \end{aligned}$$

taking square roots $y = \pm\sqrt{5}$

Then we substitute both roots of y in the first equation and we get

$$\text{for } y = +\sqrt{5}, \quad \begin{aligned} 8x^2 + 5(+\sqrt{5})^2 &= 65 \\ 8x^2 + 5(5) = 65 &= 65 \\ 8x^2 &= 65 - 25 \\ x^2 &= 5 \end{aligned}$$

taking square roots, $x \pm 5$

$$\text{for } y = -\sqrt{5}, \quad \begin{aligned} 8x^2 + 5(-\sqrt{5})^2 &= 65 \\ 8x^2 + 5(5) &= 65 \\ 8x^2 &= 65 - 25 \\ x^2 &= 5 \end{aligned}$$

taking square roots, $x = \pm\sqrt{5}$

In this particular problem the roots for each unknown are similar, but this occurs rather infrequently. One thing that is always true though when we have only squared unknowns is that there will always be four pairs of values that satisfy the equations, and that there will be only one absolute value of the root for each unknown. This is pointed out by the paired roots of the equations in the above example, which are $(x = +\sqrt{5},\ y = +\sqrt{5})$, $(x = +\sqrt{5},\ y = -\sqrt{5})$, $(x = -\sqrt{5},\ y = +\sqrt{5})$ and $(x = -\sqrt{5},\ y = -\sqrt{5})$. Note how the absolute value of the x root is always $\sqrt{5}$ and the absolute value of the y root is always $\sqrt{5}$. Since another example where the roots are not the same would probably solidify our thinking on this type problem, let's try another problem.

Solve: $3x^2 + 2y^2 = 29$

$$4x^2 - 3y^2 = 33$$

Solution: Multiply the first equation by 3 and second equation by 2 and get

$$9x^2 + 6y^2 = 87$$

$$8x^2 - 6y^2 = 66$$

Adding the second equation to the first, we get

$$17x^2 = 153$$

$$x^2 = 9$$

taking square roots we get, $x = \pm\sqrt{9} = \pm 3$

Substituting $x = +3$ in the first equation, we get

$$3(+3)^2 + 2y^2 = 29$$

$$27 + 2y^2 = 29$$

$$2y^2 = 2$$

$$y^2 = 1$$

Taking square roots, we get, $y = \pm 1$

to go with $x = +3$

Substituting $x = -3$ in the first equation, we get

$$3(-3)^2 + 2y^2 = 29$$
$$27 + 2y^2 = 29$$
$$2y^2 = 2$$
$$y^2 = 1$$

This means we have the roots $y = \pm 1$ to go with $x = -3$. Thus we can see that our statement holds true even when the roots of the variables are different.

Our approach when we are faced with all the terms being of the second degree, such as in the equation $x^2 + 3xy + 4y^2 = 6$ is a little different. In this case, we manipulate the equations so that we can eliminate the constant and then we solve one unknown in terms of the other unknown and substitute this value into one of the equations. Next, we work until we can get a number value for the second unknown. Since this is probably a little confusing, let's work an example and see how it's done.

Let's solve

$$2x^2 - 4xy + 6y^2 = 6$$
$$2x^2 + 5xy - 10y^2 = 8$$

Solution: First, multiply the first equation by 4 and the second equation by 3 to get the constants equal. This gives us

$$8x^2 - 16xy + 24y^2 = 24$$
$$6x^2 + 15xy - 30y^2 = 24$$

Subtracting the second from the first, we get

$$2x^2 - 31\ xy + 54\ y^2 = 0$$

We can now factor what we have and solve x in terms of y or y in terms of x. That is: $(x - 2y)\ (2x - 27y) = 0$

Setting factors equal to 0

$$x = 2y \quad \text{and } x = \frac{27}{2}\ y$$

We could also solve by the formula for solving a quadratic of one unknown if we consider $a = 2$, $b = -31y$, and $c = 54y^2$ in which case we get

$$x = \frac{-(-31y) \pm \sqrt{(-31y)^2 - 4(2)\ (54y^2)}}{2\ (2)}$$

$$x = \frac{31y \pm \sqrt{961y^2 - 432\ y^2}}{4}$$

$$x = \frac{31y \pm \sqrt{529y^2}}{4} = \frac{31y \pm 23y}{4}$$

$$x = \frac{8y}{4} = 2y \quad \text{and} \quad x = \frac{54y}{4} = \frac{27}{2}\,y$$

The next thing we do after getting one unknown in terms of the second unknown is substitute this value into one of the equations. Let's substitute $x = 2y$ back into the second equation and we'll get

$$2(2y)^2 + 5(2y)\ (y) - 10y^2 = 8$$
$$8y^2 + 10y^2 - 10y^2 = 8$$
$$8y^2 = 8$$
$$y = \pm 1$$

Knowing that when $x = 2y$ that $y = \pm 1$, we can turn our thinking around and see that when $y = +1$ that $x = +2$ and when $y = -1$ that $x = -2$. We now have two ordered pairs, namely, $(x = 2,\ y = 1)$ and $(x = -2,\ y = -1)$.

Now let's substitute $x = \frac{27}{2}\,y$ back into the second equation and see what we get.

$$2\left(\frac{27}{2}y\right)^2 + 5\left(\frac{27y}{2}\right)(y) - 10y^2 = 8$$

$$\left(\frac{729}{2}\right)y^2 + \frac{135}{2}\,y^2 - 10y^2 = 8$$

Multiplying through by the least common denominator, 2, we get

$$729y^2 + 135y^2 - 20y^2 = 8$$
$$844\, y^2 = 16$$
$$y^2 = \frac{16}{844} = \frac{4}{211}$$

Taking square roots we get

$$y = \pm\sqrt{\frac{2}{211}}$$

Again turning our thinking around, we can determine that when $y = +\frac{2}{\sqrt{211}}$, x must equal $\frac{27}{2} \times \frac{2}{\sqrt{211}}$ or $\frac{27}{211}$

and when $y = -\frac{2}{\sqrt{211}}$, $x = -\frac{27}{\sqrt{211}}$. This gives us our second pair of roots, namely:

$$\left(x = \frac{27}{\sqrt{211}},\ y = \frac{2}{\sqrt{211}}\right) \quad \text{and} \left(x = \frac{-27}{\sqrt{211}},\ y = \frac{-2}{\sqrt{211}}\right)$$

Now that we know how to find the roots of two quadratic equations, we might say "so what". Well, the value of this knowledge is that we can solve problems where we have two unknowns that affect two or more phases of a problem. Using these unknowns, we express each phase of the problem in the form of an equation and solve the equations. As an example, consider supply and demand problems. If the functions for supply and demand were quadratic equations, each with the same unknowns, we could find the point where the curves crossed (equilibrium point) and could thus tell at what prices supply would meet demand.

Now try the problems given below and see if you can get the answers given.

Exercises:

1. Solve: $3x - 2y = 5$ Ans. $(x = 3,\ y = 2)\ (x = 8,\ y = \frac{19}{2})$

$x^2 - xy + 2y = 7$

2. Solve: $2xy - 3y^2 + 1 = 0$ Ans. $(x=1, y=1)$ (only one solution)

$$2x - 3y + 1 = 0$$

3. Solve: $3x^2 - 5y^2 = -5$ Ans. $(x = 5, y = 4)$ $(x = 5, y = -4)$

$$4x^2 + y^2 = 116$$

$(x = -5, y = 4)$ $(x = -5, y = -4)$

4. Solve: $x^2 + 4y^2 = 4$ Ans. $(x = -1.4, y = -0.7)$

$$xy = 1$$

$(x = 1.4, y = 0.7)$

7.5 Graphs of Quadratics.

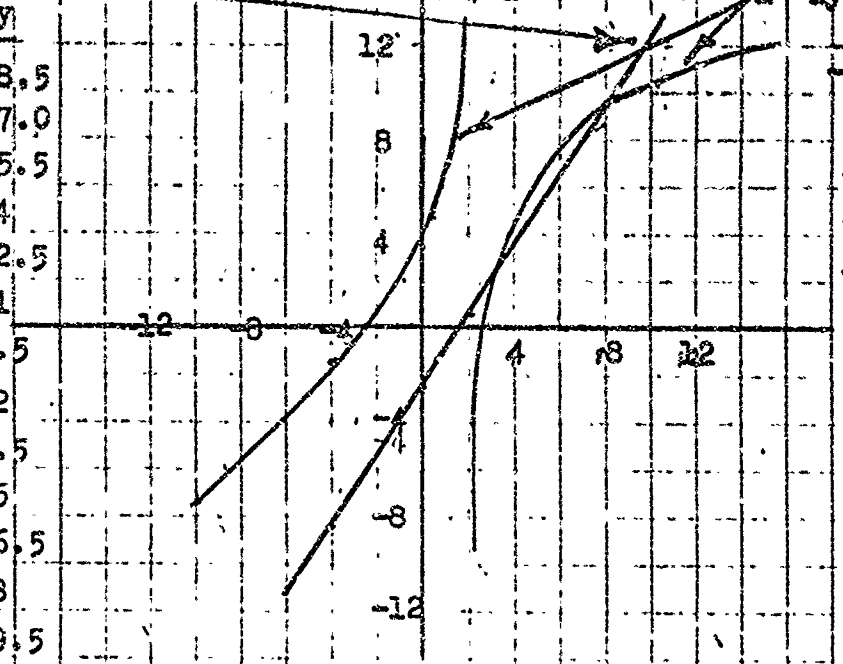
The use of the mathematical procedures outlined above are fine, but they can be somewhat tedious, so whenever we can, we solve quadratic equation problems with two unknowns with graphs. The method is really simple and all we do is lay out our graph and for each equation determine for various values of one variable the corresponding values of the other variables. Then we plot the ordered pair for each equation and connect the points. Examples of this procedure are shown in Figures 7-1 through 7-4. The points where the curves cross are solutions. In plotting the graph of quadratic equations, we must be particularly careful to remember that where we have square roots there are two answers, plus and minus, for every value. In addition, we should instantly dismiss any plots of negative quantities, when such cannot practically exist. We should recall from an earlier chapter that if we have only one unknown, the solution lies on a straight line perpendicular to that unknown's axis at the value or values of the unknown that satisfies or satisfy the equation.

Now we'll take the same problems as in section 7.4 and solve them by graphical methods. The solutions are shown in Figures 7-1 through 7-4, respectively.

FIGURE 7-1

$3x-2y=5$

x	y
-4	-8.5
-3	-7.0
-2	-5.5
-1	-4
0	-2.5
1	-1
2	.5
3	2
4	3.5
5	55
6	6.5
7	8
8	9.5



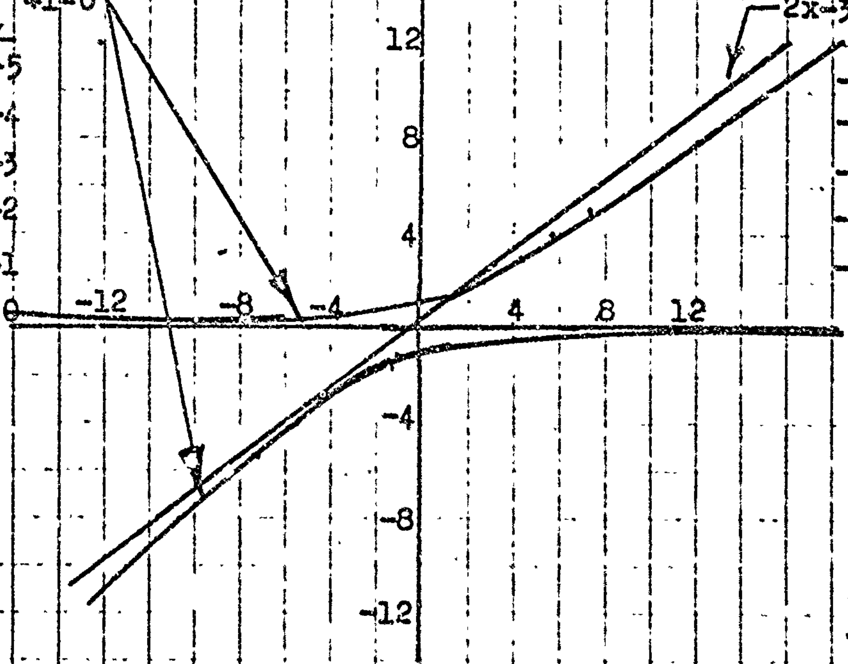
$x^2-xy+2y=7$

x	y
-10	-7.6
-4	-1.5
-3	-0.4
-4	.75
-1	2
0	3.3
1	6
2	∞
3	2
4	4.5
5	6
6	7.3

FIGURE 7-2

$2xy-3y^2+1=0$

x	y
-7.4	-5
-5.9	-4
-4.3	-3
-2.8	-2
-1	-1
∞	0
1	1
2.3	2
4.3	3
5.9	4
7.4	5



$2x-3y+1=0$

x	y
-5	-3
-4	-2.3
-3	-1.7
-2	-1
-1	-.3
0	.3
1	1
2	1.7
3	2.3
4	3
5	3.6
6	4.3
7	5

FIGURE 7-3

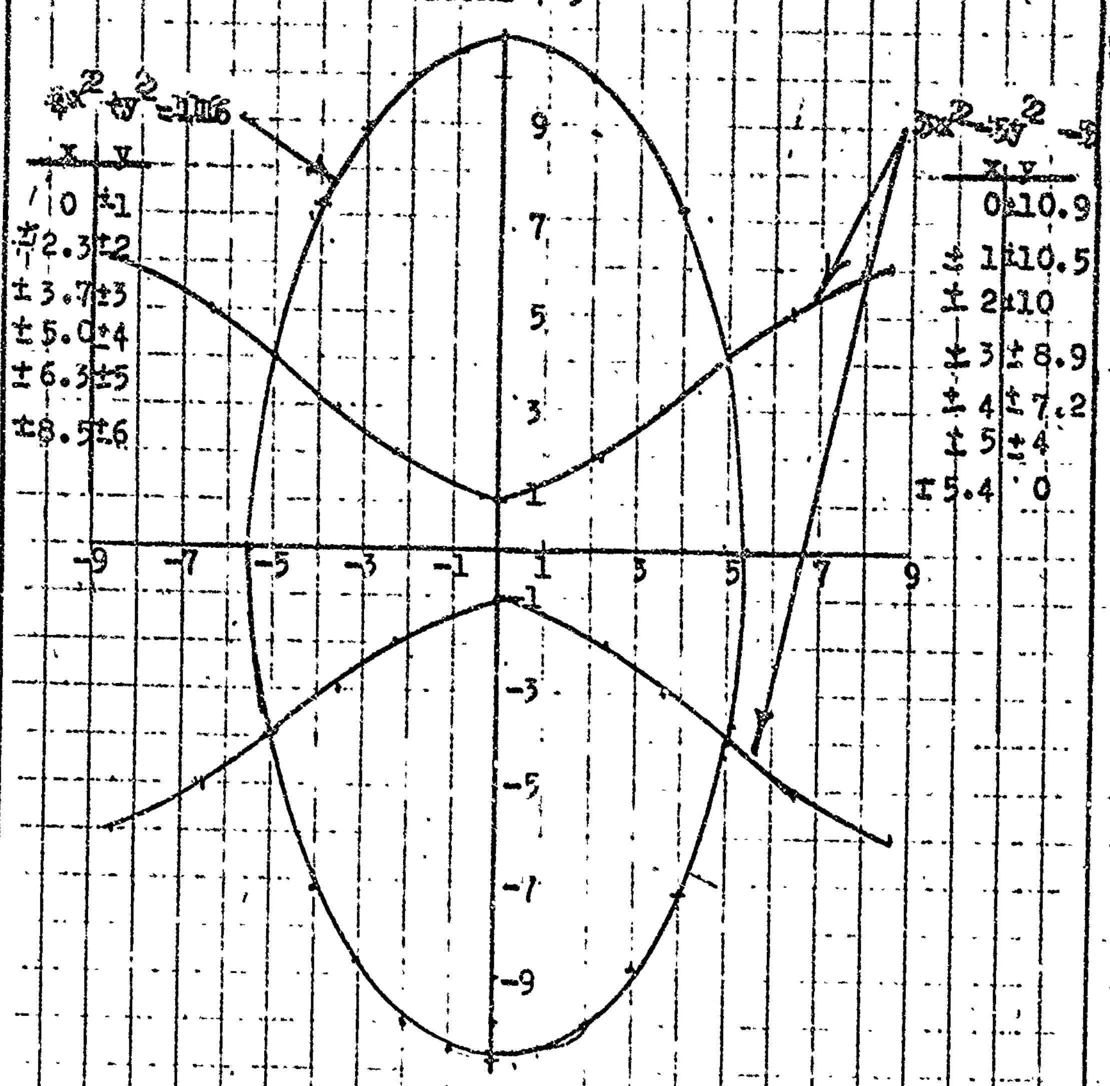
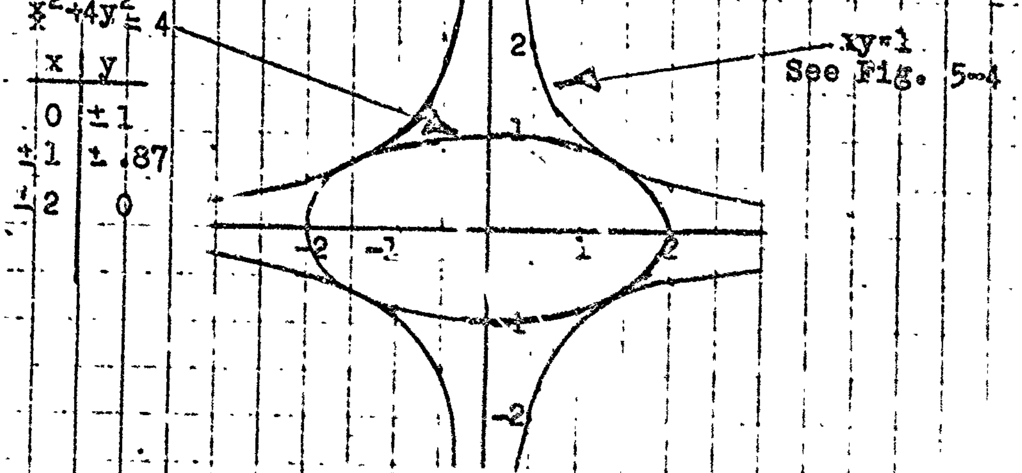


FIGURE 7-4



If you understand what we have covered so far, the solution of the following problems should be no great effort. It is suggested that you solve the problems mathematically and then solve them graphically.

1. Solve: $x - y = 1$, $x^2 + y^2 = 13$ Ans. $(x = 3, y = 2)$ $(x = -2, y = -3)$

2. Solve: $x + 4y = 9$, $x^2 + 4y = 9$ Ans. $(x = 0, y = \frac{9}{4})$ $(x = 1, y = 2)$

3. Solve: $3x + 4y = 7$, $x^2 + x = 2y^2$ Ans. $(x = 1, y = 1)$ $(x = 49, y = -35)$

4. Solve: $xy - 7x + 4y = 12$, $3y - 2x = 1$ Ans. $(x = 8, y = \frac{17}{3})$ $(x = -2, y = -1)$

5. Solve: $x^2 + y^2 = 13$, $3x^2 - 4y^2 = 1$ Ans. $(x = 3, y = 2)$ $(x = 3, y = -2)$ $(x = -3, y = 2)$ $(x = -3, y = -2)$

6. Solve: $4x^2 - 2xy + y^2 = 3$, $8x^2 + 6xy - 3y^2 = -4$ Ans. $(x = \frac{1}{2}, y = 2)$ $(x = -\frac{1}{2}, y = -2)$ $(x = \frac{1}{2}, y = 1)$ $(x = -\frac{1}{2}, y = 1)$

CHAPTER 8

PROGRESSIONS

8.1 Introduction.

If we were to walk down an avenue in a large city and observed that first we crossed First Street, then Second Street, then Third Street, etc., we would soon realize that we were crossing streets which were numbered in some sequence. Unless we were real dunces, we could heuristically determine that the next street we would cross would probably have a number one unit greater than the number of the last street we crossed. If we try to visualize what has occurred, we can see that we have followed some rule or formula which indicates what the next street number will be. In this case, we have simply added one to the previous street number. An orderly sequencing of numbers such as this allows us to determine what the next, or the second following,or the nth (where n is the literal symbol for the number we want) following number will be. Such a sequence is what is known in the mathematic's world as a progression. Due to the mathematical manipulations involved,we classify progressions as being either arithmetic or geometric progressions.

8.2 Arithmetic Progressions.

Arithmetic progressions are those progressions in which the next number is determined by adding, or subtracting, a constant amount from the previous number. In our street crossing example, we added one to the previous number to determine the number of the next street we would cross. If when we got to 30th street we decided to retrace our steps, we would assume the next street we crossed would be numbered one less than the most recent street we had crossed, 29th street. In this case, we have subtracted one. In other arithmetic progressions, it is possible that the constant difference will be something other than 1. An example of a difference other than 1 would be successive Leap Years in which the constant difference is 4 years, (i.e., 1932, 1936, 1940, 1944, etc.).

Knowing what arithmetic progressions are and how they are formed, we can proceed to determine any term in a progression once we know one term, its relative location in the progression and the common difference between terms. In order to reduce our confusion to a minimum, we will start off with those cases in which we know the first term. Now let's develop a system for determining the value of the nth (where n stands for the number of the term we are looking for) term of an arithmetic progression. First, we let the literal symbol a stand for the value of the first term and the literal symbol d stand for the constant difference between terms. Now we can express the terms of a five term progression as follows:

First term	=	a		
Second term	=	a + d	=	a + d
Third term	=	second term + d	=	a + 2d
Fourth term	=	third term + d	=	a + 3d
Fifth term	=	fourth term + d	=	a + 4d

By reflecting for a moment on the value of the multiplier of d, we can see that the value of a term is equal to the value of the first term plus the number of the term we want, less 1, times the common difference. Symbolically, if we let the literal symbol t_n stand for the value of the term we are interested in, we get:

$$t_n = a + (n-1)\,d$$

Proof that this formula will work can be shown mathematically. For example, let $a = 4$ and $d = 3$ and assume that we want to find the value of the seventh term. By substitution, we get:

$$t_7 = 4 + (7-1) \times 3 = 4 + (6) \times 3 = 22$$

We can check this by doing it the long way and getting: 4, 7, 10, 13, 16, 19, 22.

In some cases, we may desire to know not only the value of the nth term but also the cumulative total of the terms in the arithmetic progression up to and including the nth term. One reason might be to determine total earnings over a period of time if we were to start with a stated salary and would receive yearly salary increases. An example would be trying to compare whether you would receive more money over a ten year period of time from a plan that started you at $1,000 a year with $100 yearly increases, or from a plan that started you at $1300 a year with $50 yearly increases. Obviously, we could figure out the value for each term and then add all of the values, however, this could involve considerable work and would be a more complex problem. Therefore, let's develop a way to figure this easily. First, if we were to sum all the values and divide by the number of terms we would have the average value of the terms in the progression. It should follow, then, that if we know the average value of the terms and number of terms in a progression that we can get the sum of the terms in the progression just by multiplying the average value by the number of terms. Fortunately for us, the average is easy to find when there is a constant difference between consecutive terms because all we have to do is add the first term and the last term together and divide by 2. This can be proved mathematically but a few examples will suffice for this text. Assume the progression is 1, 3, 5, 7.

Average $= \frac{1 + 3 + 5 + 7}{4} = \frac{16}{4} = 4$. Also $1 + 7 = 8$ and $\frac{8}{2} = 4$.

For the progression 7, 11, 15, 19, 23, the average is $\frac{7 + 11 + 15 + 19 + 23}{5}$ $= \frac{75}{5} = 15$. Also $\frac{7 + 23}{2} = \frac{30}{2} = 15$. If we let S_n stand for the sum of the value of the first n terms, t_n, the nth term, and n for the number of terms, we get the following equation:

$$S_n = n \frac{(a + t_n)}{2}$$

but

$$t_n = a + (n-1) d$$

so

$$S_n = n \left[\frac{a + a + (n-1) d}{2}\right] = n \left[\frac{2a + (n-1) d}{2}\right]$$

In our example where $a = 4$ and $d = 3$, we would find the sum of the first 7 terms in the following manner:

$$\begin{aligned} S_7 &= 7 \left[\frac{2 (4) + (7 - 1) (3)}{2}\right] \\ &= 7 \left[\frac{8 + 18}{2}\right] \\ &= (7)\ (13) \\ &= 91 \end{aligned}$$

It is left to the student to prove this fact by adding the values of the 7 terms of the progression.

An infrequently seen variation of the arithmetic progression is the harmonic progression. In a harmonic progression, we have a series of fractions, the reciprocals of which form an arithmetic progression. An example would be: $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, etc. For ease of computation, it would be best to always manipulate the fractions so that the numerators were equal to 1. Then, knowing the reciprocals form an arithmetic progression, we can easily see that what we have is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}$, etc.

Obviously, we determine the values of the various denominators in the same way we determine the value of the nth term of an arithmetic progression. Examples are given below. There is no known easy method of getting the sum of the n terms of a harmonic progression.

Before going on to determining the values of missing terms, it would be wise to go through the following examples until you are convinced of the correctness of the answers.

1. Find the eighth term of an arithmetic progression where $a = 8$ and $d = 5$.

   Solution: $t_8 = 8 + (8-1)(5) = 8 + 35 = \underline{43}$ Ans.

2. Find the sixth term of an arithmetic progression where $a = 5$ and $d = -6$.

   Solution: $t_6 = 5 + (6-1)(-6) = 5 - 30 = \underline{-25}$ Ans.

3. Find the seventh term of an arithmetic progression where $a = \frac{1}{4}$ and $d = \frac{1}{4}$.

   Solution: $t_7 = \frac{1}{4} + (7-1)(\frac{1}{4}) = \frac{1}{4} + \frac{6}{4} = \frac{7}{4}$ Ans.

4. Find the fourth term of the harmonic progression where the first term is $\frac{1}{4}$ and $d = 2$.

   Solution: Find the reciprocal of $\frac{1}{4}$. It is 4.

   Then $t_4 = 4 + (4-1)(2) = 4 + 6 = 10$.

   Fourth term = reciprocal of 10 or $\frac{1}{10}$ Ans.

5. Find the fourth term of the harmonic progression where the first term is 2/3 and $d = 3$.

Solution: Change fraction so that the numerator is equal to 1. To do this we divide numerator and denominator by 2 in the following manner:

$$2/3 = \frac{2/2}{3/2} \qquad \frac{1}{3/2}$$

Then we find the reciprocal of the fraction. It is $3/2$.

Thus $t_4 = 3/2 + (4-1)(3) = 3/2 + 9 = \frac{21}{2}$

The fourth term of the harmonic progression equals the reciprocal of $\frac{21}{2}$ or $\frac{1}{\frac{21}{2}} = \frac{2}{21}$ Ans.

Now that we know how to find the nth term of an arithmetic progression, let's find a way to determine the values of terms in between two known terms. The values of these unknown terms are called arithmetic means. If we think of a straight line of fence posts as forming an arithmetic progression,we can work up to our method for finding arithmetic means. Assume between the first and last post that we want to put in 6 posts, giving us a total of eight posts, and that we want all posts equally spaced apart. Then the first post will serve as the beginning of the first space. The second post will serve as the end of the first space and as the beginning of the second space. The third post will serve as the end of the second space as the beginning of the third space. Skipping along a bit, we find the seventh post serves the end of the sixth space and the beginning of the seventh space, while the eighth post serves merely as the end post of the seventh space. Thus, we see that there is one more post than the number of spaces. Now if we attach consecutive term numbers of a progression one to a post, we can see that the difference of values of consecutive terms will be equal to the difference between the values of the first term and the last term divided by a number equal to the number of terms less one because there is one less number of spaces than there are posts. Symbolically, we could write:

$$d = \frac{t_n - a}{n-1}$$

The value of d can also be obtained by solving the equation $t_n = a + (n-1)\,d$ for d. Having found the value of d, we can then proceed to find the values of the missing terms.

Examples:

1. Find the 4 arithmetic means in the six term arithmetic progression where $a = 4$ and $t_n = 14$.

   Solution: $d = \frac{14-4}{6-1} = \frac{10}{5} = 2$

   Arithmetic progression = 4, 6, 8, 10, 12, 14. Ans.

2. Find the 4 arithmetic means in the six term arithmetic progression where $a = 4$ and $t_n = 16$.

   Solution: $d = \frac{16-4}{5-1} = \frac{12}{5} = 2.4$

   Arithmetic progression = 4, 6.4, 8.8, 11.2, 13.6, 16 Ans.

3. Find the 3 harmonic means in the five term harmonic progression where the first term is $\frac{1}{2}$ and the last term is $\frac{1}{6}$.

   Solution: The reciprocals of the fractions are 2 and 6, respectively. then $d = \frac{6-2}{4} = \frac{4}{4} = 1$. Therefore, the values of the reciprocals of the terms are 2, 3, 4, 5, 6 and the harmonic series = $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ Ans.

8.3 Geometric Progressions.

Geometric progressions are those progressions in which the successive terms differ from each other by some constant multiplier. The multiplier can be either a fraction or a whole number. If we let a = the value of the first term, n = the number of the terms we are interested in, and r = the value of the multiplier, we would have the following values for the first five terms of a geometric progression:

First term	=	a	$= a\ r^0$
Second term	=	first term x r	$= a\ r^1$
Third term	=	second term x r	$= a\ r^2$
Fourth term	=	third term x r	$= a\ r^3$
Fifth term	=	fourth term x r	$= a\ r^4$

By inspection, we can see that the exponent of the constant multiplier is n-1. Thus we can set up the general rule that:

$$t_n = a\ r^{(n-1)}$$

As proof, let us determine the seventh term of a geometric progression in which a = 6 and r = 2. Substituting in the formula, we find:

$$t_n = 6 \times 2^{(7-1)} = 6 \times 2^6 = 6 \times 64 = 384$$

Checking by multiplying each term by the multiplier gives us the following terms: 6, 12, 24, 48, 96, 192, 384. It checks.

The most common application of geometric progressions is the determination of the amount of money we will have at the end of a certain period, if we invest it at a compound interest rate. In computing the amount, we must consider the initial amount invested as being the first post in our line of fence posts. Then at the end of the first period, we are at the second post, where the value equals the amount invested plus the interest rate times the amount invested. Symbolically, if we let the literal symbol A_k stand for the amount at the end of period K, the literal symbol P stands for the amount initially invested, and the literal symbol i stand for the period interest rate, we could write:

For the end of first period $A_1 = P + i P$

For the end of second period $A_2 = A_1 + i A_1$

but $A_1 = P + iP$, so $A_2 = P + i P + i (P + i P)$

$A_2 = P + i P + i P + i^2 P = P (1 + 2 i + i^2)$

Factoring we get $A_2 = P (1 + i)^2$

We should note that the exponent of the multiplier $(1 + i)$ is the same as the subscript K for A. Since it can be proved, but won't be here, that this relationship always holds true, we generalize and write the equation as follows:

$$A_k = P (1 + i)^k$$

That is, the amount of money at the end of k periods equals the initial investment multiplied by the factor 1 plus the interest rate raised to the k power.

As an example, suppose we invest $1,000 at a compound interest rate of 6% per annum, compounded annually, and want to know how much money we will have at the end of ten years.

Solution: $A_{10} = P (1 + i)^{10}$

$= 1000 (1 + .06)^{10}$

$= 1000 (1.06)^{10}$

$= \$1,790.80$

Note that this is the amount of money we will have at the end of 10 years. It is not the amount of interest earned. To determine the amount of interest earned, we must subtract out the $1,000 which we initially invested. As another example, suppose we want to find out how much interest an investment of $1,000 will earn over 10 years if the interest rate is 3% per six month period and compounded semi-annually.

Solution: Notice that there will be 20 periods so we are looking for the amount earned over 20 periods. Then:

$$A_{20} = 1000\ (1 + .03)^{20}$$

$$A_{20} = 1000\ (1.03)^{20}$$

$$A_{20} = \$1,806.10$$

But the amount of interest earned equals

$$A_{20} - A_0 = \$1,806.10 - 1000 = \underline{\$806.10} \text{ Ans.}$$

Geometric progression procedures can also be applied in determining the amount of some characteristic remaining of a material, if we know the rate of deterioration of the characteristic and the amount of the characteristic we are starting with. In this case, the multiplier would be something less than one. It must be remembered here also that the initial amount corresponds to the first term so that at the end of the first period we are looking for the second term in the progression. As an example, let's assume we have a policy of spending half of the money we start a year with during that year, that we start with $1000, and that we want to know how much we will have spent during the third year.

Solution: What we really want is the difference between what we have at the end of the second year and the end of the third year. Since zero years is at the first term, we are trying to find the difference between the third and fourth terms. To do so, let's find the third and fourth terms and subtract the fourth term from the third term.

$$t_3 = 1000\ (\tfrac{1}{2})^2 \qquad\qquad t_4 = 1000\ (\tfrac{1}{2})^3$$

$$t_3 = 1000\ (\tfrac{1}{4}) \qquad\qquad t_4 = 1000\ (\tfrac{1}{8})$$

$t_3 = 250$ $\qquad\qquad$ $t_4 = 125$

$t_3 - t_4 = \$250 - \$125 = \$125$ spent during the third year.

As with arithmetic progressions, we may desire to obtain the sum of all the terms in a geometric progression. For example, if we were doubling our bets at the dice table without winning, we might like to know how much we have lost, so we will develop a formula for the sum. If we let the literal symbol S_n stand for the sum of n terms, we have

$S_n = a + ar^1 + ar^2 + \text{-----} + ar^{(n-1)}$ (where + -----+ indicates that terms have been left out.)

Then if we multiply both sides of the equation by r, we get a second equation where

$$r\,S_n = ar + ar^2 + ar^3 + \text{-----} + ar^n$$

Now let's subtract the second equation from the first and we will get

$$S_n - r\,S_n = a + ar + ar^2 + \text{-----} ar^{(n-1)} - ar + ar^2 + ar^3 + \text{-----} + ar^n$$

which simplifies into $S_n - r\,S_n = a - ar^n$

which can be further simplified into

$$S_n\,(1-r) = a\,(1-r^n) \quad \text{or}$$

$$S_n = \frac{a\,(1-r^n)}{(1-r)} \quad \text{or} \quad \frac{a\,(r^n - 1)}{(r-1)}$$

As an example, if we want the sum of the first five terms of a geometric progression in which a = 10 and r = 2, we would get:

$$S_5 = \frac{10\,(2^5 - 1)}{(2-1)}$$

$$S_5 = \frac{10\,(32 - 1)}{1} = \frac{10\,(31)}{1}$$

$$S_5 = 310$$

Where the number of terms increases without limit, we have what is called an <u>infinite geometric series</u>. Finding the sum of the terms in this case is impossible if our multiplier is greater than 1 because a number greater than 1 raised to infinity is infinity. Where the multiplier is 1, the sum is also infinity because we would be dividing by zero. However, when the multiplier is less than one, we can determine a definite maximum <u>sum</u> or <u>limit</u> for the progression because r^n approaches zero and we wind up with

$$S_n = \frac{a}{1-r} \text{ or } \frac{-a}{r-1} \text{ for } n = \text{infinity.}$$

As an example, take the infinite geometric progression 2, 2/3, 2/9 and determine the limit.

Solution: $a = 2 \quad r = \frac{1}{3}$

$$S = \frac{2}{1 - 1/3} = \frac{2}{\frac{2}{3}}$$

$$S = \left(\frac{2}{\frac{2}{3}}\right) = 2 \times \frac{3}{2}$$

$$S = 3$$

In order to help convince ourselves that there is no limit to those progressions where the multiple is greater than 1, let us see what sum of money we would have spent after suffering 30 straight losses at the dice table, assuming our first bet was one dollar and that everytime we lost, we doubled the amount lost on the last bet.

$$S_{30} = \frac{1\,[(2)^{30} - 1]}{(2 - 1)}$$

$$S_{30} = \frac{1073761824 - 1}{1}$$

$$S_{30} = 1,073,761,823$$

It is easy to see that if we lost a few more times, then our losses would exceed the "national debt". It is pretty easy to see now why if we increase without limit, we'll soon approach infinity when the multiple is greater than 1.

Terms between two given terms of a geometric progression are known as <u>geometric means</u>. Thus if we are given the first and last terms of a five term geometric progression we would have three geometric means. Here the problem is not as easy as considering a line of posts as we did with arithmetic progressions so we must go back to our formula for deriving the value of the nth term and work backwards. Remembering that

$$t_n = ar^{n-1}$$

we can see that knowing t_n, n, and a, we can solve for r. Doing so we get

$$r^{n-1} = \frac{t_n}{a}$$

or

$$r = \sqrt[n-1]{\frac{t_n}{a}}$$

Having found r, we can now determine the values of the missing terms starting with t_2, then t_3, etc.

As an example, suppose we want to find the geometric means in the geometric progression where $a = 16$, and $t_5 = 81$. (Obviously $n = 5$, and we have 3 geometric means).

Solution: Solve $r = \sqrt[(n-1)]{\frac{t_5}{a}}$

$$r = \sqrt[(5-1)]{\frac{81}{16}}$$

$$r = \sqrt[4]{81/16}$$

$$r = 3/2$$

then $t_2 = 16 \times 3/2 = 24$

$t_3 = 24 \times 3/2 = 36$

$t_4 = 36 \times 3/2 = 54$

$t_5 = 54 \times 3/2 = 81$ (checks)

Exercises:

1. Find the nth term of the arithmetic progression where:

a) $a = 3$	b) $a = -2$	c) $a = 20$
$d = 1$	$d = -1$	$d = 5$
$n = 8$	$n = 7$	$n = 4$
Ans. 10	Ans. −8	Ans. 35

2. Find the sum of the first n terms of the arithmetic progressions where:

a) $a = 4$	b) $a = 5$	c) $a = 20$
$d = 1$	$d = -2$	$d = -2$
$n = 8$	$n = 6$	$n = 4$
Ans. 60	Ans. 0	Ans. 68

3. Find the values of arithmetic means for the arithmetic progression where:

a) $a = 6$	b) $a = 12$	c) $a = 5$
$t_6 = 16$	$t_4 = -6$	$t_5 = 15$
Ans. 8,10,12,14	Ans. 60	Ans. 1,−3, −7, −11

4. Find the nth term in the geometric progressions where:

a) $a = 6$	b) $a = 4$	c) $a = -20$
$r = 3$	$r = 3/4$	$r = 1/2$
$n = 5$	$n = 4$	$n = 5$
Ans. 486	Ans. 27/16	Ans. $-20/16$

In problem 4.c., you should note that the progression alternates between a positive value and a negative value. This will occur whenever r is equal to a minus number.

5. Find the sum of the first n terms of the geometric progression where:

a) $a = 2$	b) $a = -2$	c) $a = 6$
$r = 3$	$r = -3$	$r = 1/3$
$n = 4$	$n = 4$	$n = 3$
Ans. 80	Ans. +40	Ans. 26/3

6. Find the values of the geometric means for the geometric progression where:

a) $a = 1$	b) $a = 16$	c) $a = 1/4$
$t_5 = 16$	$t_6 = 1/2$	$t_4 = 2$
Ans. 2,4,8	Ans. 8,4,2,1	Ans. 1/2, 1

CHAPTER 9

LOGARITHMS

9.1 Introduction.

Many mathematics books begin the development of the topic of logarithms with a complicated definition which has the tendency to scare the student or, in some way, cause the student to feel that the subject is incomprehensible. On the contrary, logarithms are easy; they are simply a mathematical technique which is principally used to facilitate extremely laborious arithmetic calculations. Suppose the student of arithmetic were asked to evaluate the expression 15^{250}. This is an arithmetic calculation which requires the student to multiply 15 by itself 250 times. This operation would be tedious and would require several hours of diligent work. After we have learned the mathematical technique of logarithms, we will be able to carry out this computation and obtain an approximate answer in just a few minutes. Keep this thought in mind as you read and work through the development of the theory of logarithms. Remember, it is principally used to simplify arithmetic calculations.

Now with that brief introduction we are ready to digest the somewhat sticky definition referred to above. The logarithm of a positive number to a given base, other than 1, is the exponent of the power to which the base must be raised to equal the number.

This definition can be expressed as an equation $N = b^x$, where N is the positive number, b is the base which is greater than 0 and not equal to 1. The exponent is x and it is the exponent to which a base b is raised to produce a number N. The equation, as expressed above in the form $N = b^x$, we shall call the exponential form.

This same equation can be expressed in logarithmic form as follows:

$\log_b N = x$ which reads as follows: the logarithm of the number N to the base b is x.

To drill on the definition of a logarithm, we have included exercises which require the student to write the logarithmic forms of exponential equations and vice versa. The student will find it most helpful to continually remind himself that a logarithm is an exponent to which a base is raised to produce a number. Let's try a few.

Examples:

Write the logarithmic form of the following exponential equations.

1. $2^3 = 8$

When we look at this problem we should immediately determine what is the exponent or logarithm. We then can write

$$\log_{\_} \_\_\_ = 3$$

Then we find the base, or in other words, the number which is being raised to the power. In this case, the base is 2. We then can fill in the base blank on the log side of the equation.

$$\log_2 \_\_\_ = 3$$

We complete the transformation by inserting the number that we obtain by raising 2 to the 3 power in its proper place in the logarithmic form.

$$\log_2 8 = 3$$

2. $5^2 = 25$

In logarithmic form this expression becomes

$$\log_5 25 = 2$$

3. $4^3 = 64$

Similarly, $\log_4 64 = 3$

4. $3^4 = 81$

Then $\log_3 81 = 4$

5. $15^2 = 225$

$\log_{15} 225 = 2$

The transformation from logarithmic to exponential form is just the reverse of the above operation.

Examples:

Transform the following logarithmic equations into their exponential forms.

1. $\log_a B = C$

   $a^c = B$

2. $\log_x Y = 2$

   $x^2 = Y$

3. $\log_{10} 100 = 2$

   $10^2 = 100$

4. $\log_e A = B$

   $e^b = A$

5. $\log_2 16 = 4$

   $2^4 = 16$

The student should re-read and study the preceding part of this chapter until the basic principles contained therein are well understood.

Remember always that a logarithm is an exponent. . . and that logarithms are a mathematical technique used to facilitate arithmetic computations.

9.2 Laws of Logarithms.

With the preceding basic theory known, we are now ready to discuss the basic laws of logarithms which we will use extensively in computation. In developing these relationships, we will use the rules of exponents which we have learned earlier which are:

Law A. $a^x \cdot a^y = a^{x+y}$

Law B. $\dfrac{a^x}{a^y} = a^{x-y}$

Law C. $(a^x)^y = a^{xy}$

From these three exponential laws we are able to derive useful laws of logarithms.

First Law - Logarithm of a Product

Rewriting the multiplication law of exponents

$$a^x \cdot a^y = a^{x+y}$$

Transferring this exponential equation into logarithmic form we obtain:

$$\log_a a^x a^y = x + y$$

but

$$\log_a a^x = x$$

and

$$\log_a a^y = y$$

therefore,

$$\log_a a^x a^y = \log_a a^x + \log_a a^y$$

therefore, in general terms

$$\log_a KL = \log_a K + \log_a L$$

or in words: The log of a product to a certain base is equal to the sum of the logs of the factors of the product to the same base.

Proof: $\log_2 8 \cdot 4 = \log_2 8 + \log_2 4$

$$\log_2 32 = 3 + 2$$

$$5 = 5$$

Similarly: $\log_x ABC = \log_x A + \log_x B + \log_x C$

Second Law - Logarithm of a Quotient

Law B above for exponents is

$$\frac{a^x}{a^y} = a^{x-y}$$

Expressing this exponential equation in logarithmic form

$$\log_a \frac{a^x}{a^y} = x - y$$

but

$$\log_a a^x = x$$

and

$$\log_a a^y = y$$

therefore:

$$\log_a \frac{a^x}{a^y} = \log_a a^x - \log_a a^y$$

or in general terms:

$$\log_a \frac{K}{L} = \log_a K - \log_a L$$

or in words: the log of a quotient expressed as a fraction, to a certain base, is equal to the log of the numerator to that base minus the log of the denominator to that same base.

Proof: $\log_2 \frac{32}{8} = \log_2 32 - \log_2 8$

$$2 = 5 - 3$$

$$2 = 2$$

Third Law - Logarithm of a Number to a Power

Rewriting law C for exponents

$$(a^x)^y = a^{xy}$$

Transforming this exponential equation into logarithmic form

$$\log_a (a^x)^y = xy$$

but

$$\log_a a^x = x$$

Substituting

$$\log_a (a^x)^y = y \log_a a^x$$

or in general terms

$$\log_a (K)^L = L \log_a K$$

or in words. . . The log of a number to a power to a certain base is equal to the power times the log of the number to that same base.

Proof: $\log_2 (4)^3 = 3 \log_2 4$

$$\log_2 64 = 3 \log_2 4$$

$$6 = 3 \cdot 2$$

$$6 = 6$$

Examples:

Transform using the laws of logarithms.

1. $\log_b AB$

$$\log_b AB = \log_b A + \log_b B$$

2. $\log_b X^y$

$$\log_b X^y = y \log_b X$$

3. $\log_b \frac{X}{Y}$

$$\log_b \frac{X}{Y} = \log_b X - \log_b Y$$

4. $\log_b \sqrt{X}Y$

$$\log_b \sqrt{X}Y = \log_b \sqrt{X} + \log_b Y$$

$$= \log_b X^{\frac{1}{2}} + \log_b Y$$

$$= \tfrac{1}{2} \log_b X + \log_b Y$$

5. $\log_b \frac{2^3}{\sqrt{5}}$

$$\log_b \frac{2^3}{\sqrt{5}} = \log_b 2^3 - \log_b \sqrt{5}$$

$$= 3 \log_b 2 - \tfrac{1}{2} \log_b 5$$

9.3 Bases of Logarithms.

Logarithmic tables have been computed for a base of 10 and a base of e (which is equal to 2.718). Logarithms to the base e are called natural logarithms and are written using the symbol ln, i.e., ln 6 means the logarithm of 6 to the base e ($e = 2.718$). The e base is understood when ln is used.

Logarithms to the base 10 are written using the symbol log without a base written in. Log 100 means the logarithm of 100 to the base 10. Again the 10 base is understood. Logarithms to the base 10 are called common logarithms. The choice of 10 for a base was made because our numbering system is based on 10 and multiples of 10. At 10 our numbering system jumps from one to two integers, at 100 it jumps from two to three and so on. The reason for the selection of base 10 will become more apparent as we learn more about common logarithms.

The exponent to which 10 must be raised to produce a given number is a common logarithm.

$\log 10 = 1$ which means	$10^1 = 10$
$\log 100 = 2$ which means	$10^2 = 100$
$\log 1000 = 3$ which means	$10^3 = 1000$
$\log 10{,}000 = 4$ which means	$10^4 = 10{,}000$

The student will probably say to himself, "Well this is neat but what about numbers between 10 and 100 and between 100 and 1000, etc.?"

The answer to this question is simple and gets to the heart of the subject of common logarithms. A common logarithm has two parts, one called a characteristic and the other called a mantissa. The characteristic is the whole number part of a logarithm (exponent of 10) which can be obtained by inspection. For whole numbers, it relates to the number of numbers before the decimal. The mantissa is the decimal portion of the logarithm and it is obtained from a logarithmic table similar to Table XI pp. 598-599 of Rider's "First Year Mathematics for Colleges" second edition published by the Macmillan Company. Let's go ahead and look up the logarithm of a couple of numbers and see how easy the process is. The student is encouraged to have a table of logarithmic mantissas available while reading the rest of this chapter.

Another law of logarithms which we will call the fourth law is offered here without proof. It concerns transforming a logarithm from one base to another. It is written symbolically as follows:

$$\log_a M = \frac{\log_b M}{\log_b a}$$

This rule will be found useful in transferring logarithms from a natural base to a common base and vice versa. We will not have a great need for this rule in a management environment.

Example:

1. Find the log 57.2

Remembering that the log 10 = 1 and log 100 = 2, we know that the log 57.2 is somewhere between 1.000 and 2.000. This is what is meant by saying that we can obtain the characteristic of the logarithm by inspection. In the case of log 57.2, the characteristic of the logarithm is 1. The mantissa or decimal part is obtained from the log table (Rider). Going to 57 on the vertical scale and across to 2 on the horizontal scale we obtain 7574.

$$\log 57.2 = 1.7574$$

Again, exponentially this means

$$10^{1.7574} = 57.2$$

Similarly, $\log 572 = 2.7574$

and $\log 5720 = 3.7574$

In general terms, the characteristic of a logarithm of a positive number ≥ 1 is one less than the number of integers in the number. The mantissa is obtained from the log table using the sequence (order) of integers in the number for which the log is required.

Examples:

1. $\log 35 = 1.5441$
2. $\log 350 = 2.5441$
3. $\log 16.3 = 1.2122$
4. $\log 163 = 2.2122$
5. $\log 8.3 = 0.9191$

Since $\log 1 = 0$ because $10^0 = 1$ and remembering the $\log 10 = 1$, it follows that the characteristic of a positive number from 1 to 10 but not including 10 is 0 and the logarithm is simply the mantissa for that particular sequence.

6. $\log 9.2 = 0.9638$
7. $\log 92 = 1.9638$
8. $\log 920 = 2.9638$

9.4 Finding the Antilogarithms of Positive Logarithms.

If we are given the expression log A = N and then given the value (antilog) of N and asked to find A, we say that we are asked to find the antilogarithm of N. Using example 3 above as an example, we could be told that the logarithm of some number is 1.2122 and be asked to determine the number. The question might be written as follows:

Find N if log N = 1.2122

Another way to state this problem would be to rewrite the equation in its exponential form $10^{1.2122} = N$. In finding the "antilog", we are finding N.

The operation of finding the antilog is the reverse of finding the logarithm. We should realize right away that since the characteristic is 1, there are two integers to the left of the decimal. The actual sequence of numbers is dependent on the mantissa. To find the antilog we go into the table to find 2122. This mantissa corresponds to the sequence 163. Our characteristic, to repeat, tells us that there are two digits to the left of the decimal. So . . .

$$\log N = 1.2122$$
$$N = 16.3$$

It follows that if $\log N = 2.2122$

$$N = 163$$

Examples:

1. $\log N = 3.4997$

   $N = 3160$

2. $\log X = 1.7396$

   $X = 54.9$

3. $\log Y = 2.6561$

   $Y = 453$

4. $\log M = .4031$

$M = 2.53$

5. $\log M = .1847$

$M = 1.53$

At this point, the student should stop and go back over the material covered so far in this chapter again paying particular attention to the points which are not yet completely understood. The basic theory of logarithms has now been covered. It is only necessary now to cover the case of finding the logarithm of decimal numbers (which are negative) and then the case of finding the antilogarithm given a negative logarithm. The payoff of logarithms will come at the end of the chapter when we apply the technique of logarithms to facilitate complex arithmetic computations.

9.5 Logarithms of Decimal Numbers.

You will recall from our previous discussions that the characteristic of a logarithm of a number from 1 to 10 was 0. Now let's find the logarithm of a decimal number such as .1.

or $\log .1 = X$

If we put this equation in exponential form it becomes

$$10^x = .1$$

x must equal -1 since $\frac{1}{10} = 10^{-1} = .1$

It follows then that

$$\log .1 = -1$$

Similarly:

$$\log .01 = -2$$

$$\log .001 = -3$$

$$\log .0001 = -4$$

Now we are ready to find the logarithm of a number such as .324

or $\log .324 = ?$

The logarithm of a decimal number also has two parts, a characteristic and a mantissa, just as its whole number counterpart. The logarithm of a decimal number will always be negative and its characteristic will always be one more than the number of zeros immediately following the decimal. The mantissa is obtained in the same manner that it was obtained in the whole number case, that is, by entering the log tables with the sequence of numbers and obtaining the correct mantissa.

Find: log .324

The characteristic is negative and is one more than the number of zeros immediately following the decimal; in this case it it -1. The mantissa is obtained from the table by entering the vertical column at 32 and moving across horizontally to the 4 column. We read 5105.

Then $\log .324 = -1.5105$

To facilitate the manipulation of logarithms, a logarithm such as -1.5105 is conventionally expressed as 9.5105 - 10.

Examples:

1. $\log .00257 = -3.4099 = 7.4099 - 10$
2. $\log .521 = -1.7168 = 9.7168 - 10$
3. $\log .0414 = -2.6170 = 8.6170 - 10$

9.6 Finding the Antilogarithm Given a Negative Logarithm.

To find the antilogarithm given a negative logarithm, our procedure again is the reverse operation to finding a logarithm given a decimal number. The negative sign of a logarithm is the indicator or "clue" that the antilog is a decimal number. The mantissa determines the sequence of numbers while the characteristic determines where the decimal is placed. The number of zeros immediately following the decimal is one less than the absolute numerical value of the characteristic.

Examples:

Find the antilogarithms, given the following logarithms.

1. $\log N = 9.6444 - 10 = -1.6444$

   Since the logarithm is negative, the antilog is a decimal. The mantissa yields a sequence of numbers from the table of 441. Since the number of zeros immediately following the decimal is one less than the absolute numerical value of the characteristic, one less than one is 0, therefore,

   since, $\log N = 9.6444 - 10$

   then, $N = .441$

2. $\log M = 7.2201 - 10$

   $M = .00166$

3. $\log X = 8.3502 - 10$

   $X = .0224$

9.7 Logarithmic Computation.

In order to grasp the technique of logarithmic computation quickly, we will work through a simple arithmetic problem using logarithms. The basic method for each problem will be the same.

Suppose we were asked to carry out the following calculation.

$$\frac{3^2 \cdot 6}{9}$$

Our first step is to set X = to the computation required. X is the answer we are trying to obtain.

Then,
$$X = \frac{3^2 \cdot 6}{9}$$

Then take the logarithm of both sides of the equation
$$\log X = \log \frac{3^2 \cdot 6}{9}$$

and $\log X = \log 3^2 \cdot 6 - \log 9$ (By the second law)

$\log X = \log 3^2 + \log 6 - \log 9$ (By the first law)

$\log X = 2 \log 3 + \log 6 - \log 9$ (By the third law)

Now we go to the log table and look up the necessary logarithms.

$\log 3 = .4771$

$\log 6 = .7782$

$\log 9 = .9542$

Then,
$$\begin{array}{rlr} 2 \log 3 & = 2x \cdot .4771 = & .9542 \\ + \log 6 & = & +.7782 \\ & & 1.7324 \\ - \log 9 & = & -.9542 \\ & & .7782 \end{array}$$

Therefore $\log X = .7782$

The next step is to take the antilog to obtain X

$$X = 6.0$$

This certainly looks like quite a bit of work to obtain an answer which we could have obtained simply by carrying out the indicated operations to obtain 6.

$$\frac{3^2 \cdot 6}{9} = \frac{9 \times 6}{9} = 6$$

Now let's do the computation

$$36^5.$$

This computation could also be done quite easily by arithmetic. Our main concern at this point is to learn the method of logarithmic computation. Again, set X equal to the computation $X = 36^5$.

Then, $\log X = \log 36^5$

and $\log X = 5 \log 36$ (by the third law)

Using the log tables

$\log 36 = 1.5563$

Then, $5 \times 1.5563 = 7.7815$

Then, $\log X = 7.7815$

Our next step is to take the antilogarithm. Going into the table, we find that there is no exact mantissa listed for 7815. The mantissas in the table which straddle this value are:

7818 which corresponds to the sequence 605 and

7810 which corresponds to the sequence 604

We are looking for the sequence which corresponds with the mantissa 7815. We approximate this sequence by going through a little bit of mathematical gymnastics called interpolation. The sequence corresponding to 7815 is somewhere between 6040 and 6050. The sequence is approximated by taking $\frac{5}{8} \times 10 = 6.2$ since 7815 is $\frac{5}{8}$ of the distance between 7810 and 7818, or referring to the log table the 5, 8 and 10 are obtained as follows:

$$\begin{array}{r} 7815 \\ -7810 \\ \hline 5 \end{array} \quad \text{and} \quad \begin{array}{r} 7818 \\ -7810 \\ \hline 8 \end{array} \quad \text{and} \quad \begin{array}{r} 6050 \\ -6040 \\ \hline 10 \end{array}$$

Therefore, the sequence corresponding to 7815 is 6046. We have interpolated in three place tables to obtain a fourth place, which is the best we can expect from this approximate approach. Therefore, the .2 is dropped. If the interpolation had produced a value of 6.6, we would have made the figure 6047. With a characteristic of 7, we know that there are 8 digits to the left of the decimal. Our approximate answer to the computation 36^5 then is

$$X = 60,460,000$$

A more exact answer would require a more complete set of logarithmic mantissas.

9.8 Interpolation.

In the example above, the student has been exposed to the technique of interpolation. The student would be wise to view this as a means of obtaining one more significant figure from a table of x significant figures. Using perhaps a coarser vernacular, we are approximating an answer using 25¢ tables rather than buying the 50¢ higher priced variety. We are "approximating" because we are assuming a straight line or uniform change between any two numbers in the table, when in fact it is not. It is an exponential relationship. This is another reason why we dropped the .2 in the previous example.

Let's now use our three place tables to obtain a logarithm of a four place number.

Find log 2733.

We proceed as follows. We place our number between the two numbers which "straddle" it in the table. Since it is between 2730 and 2740, we can arrange our problem as follows:

$$10\left(\begin{array}{l} \log\ 2740 \\ \left.\begin{array}{l}\log\ 2733 \\ \log\ 2730\end{array}\right\} 3 \end{array}\right) \qquad \left(\begin{array}{l} 4378 \\ x\left\{\begin{array}{l} \\ 4362\end{array}\right. \end{array}\right) 16$$

We can find the mantissas which correspond to the sequences 2740 and 2730. They are respectively, 4378 and 4362. The mantissa that we are looking for then is approximately $\frac{3}{10}$ of the distance between 4362 and 4378. $(4378 - 4362 = 16)$

Therefore,

$$\frac{3}{10} \times 16 = 4.8 \quad \text{say } 5$$

The mantissa then is 4362 + 0005 = 4367 and log 2733 = 3.4367.

We could also solve for the mantissa by setting up a proportion where

$$\frac{x}{16} = \frac{3}{10}$$

$$x = \frac{3}{10} \cdot 16 = 4.8 \quad \text{say } 5$$

Interpolating to obtain the antilog is the process we used in the last example, under logarithmic computation above, to obtain the fourth significant figure. It is considered worthwhile to do one more example problem in this section. In this case, we will be finding the antilog of a negative logarithm (which we all know must be a decimal).

Find N if log N = 8.4688 - 10

Remember that the characteristic 8 _____ -10 or -2 merely tells us that N has one 0 to the right of the decimal point. The mantissa falls between 4683 and 4698. A recommended way of setting up the problem is as follows:

$$15\left(\begin{array}{lll} .4698 & \text{corresponds to sequence} & 2950 \\ .4688 & & \\ .4683 \;\; 5 & \text{corresponds to sequence} & x \;\; 2940 \end{array}\right)10$$

Therefore, the sequence we are looking for is about $\frac{5}{15}$ of the distance from 2940 to 2950, which is 10.

$$\frac{5}{15} \cdot 10 = 3.3 \quad \text{Say } 3$$

or by proportion $\frac{x}{10} = \frac{5}{15}$

$$x = \frac{5}{15} \cdot 10 = 3.3 \quad \text{Say } 3$$

Therefore, the sequence is

$$2940 + 3 = 2943$$

Then

$$N = .02943$$

Exercises:

1. Express the following in logarithmic form:

a) $5^4 = 625$ Ans. $\log_5 625 = 4$

b) $32^{1/5} = 2$ Ans. $\log_{32} 2 = 1/5$

c) $7^{-2} = 1/49$ Ans. $\log_7 1/49 = -2$

d) $10^{-4} = 0.0001$ Ans. $\log_{10} .0001 = -4$

e) $a^b = c$ Ans. $\log_a c = b$

2. Express the following in exponential form:

a) $\log_6 36 = 2$ Ans. $6^2 = 36$

b) $\log_8 32 = 5/3$ Ans. $8^{5/3} = 32$

c) $\log_{27} 1/9 = 2/3$ Ans. $27^{-2/3} = 1/9$

d) $\log_{17} 1 = 0$ Ans. $17^0 = 1$

e) $\log_x y = z$ Ans. $x^z = y$

3. Express as a sum, difference, or multiple of logarithm of simpler quantities:

a) $\log_4 uv$ Ans. $\log_4 u + \log_4 v$

b) $\log_3 \sqrt[5]{4}$ Ans. $2/5 \log_3 2$

c) $\log_a (b^c d^f)$ Ans. $c \log_a b + f \log_a d$

d) $\log_{10} (2^5/3^4)$ Ans. $5 \log_{10} 2 - 4 \log_{10} 3$

4. Write the characteristics of the logs (base 10) of the following:

a) 46.8	Ans. 1
b) 27,600	Ans. 4
c) 93,000,000	Ans. 7
d) 0.1	Ans. -1 or 9 - 10
e) 2.674×10^{-3}	Ans. -3 or 7 - 10

5. If log N has a mantissa such that the significant digits of N are 3406, find N for each of the following characteristics:

a) 1	Ans. 34.06
b) 8 - 10	Ans. .03406
c) 11	Ans. 340,600,000,000
d) -3	Ans. .003406
e) 10^2	Ans. 3.406×10^{100}

6. Compute the following using logarithms:

a) $\dfrac{83.40 \times 2.019}{0.000006423 \times 195.8}$	Ans. 133,900
b) $\sqrt[3]{\dfrac{86.37}{6.143}}$	Ans. 2.414
c) $(2.138)^3 \times (42.10)^{-2}$	Ans. .005513
d) $(-0.03420)^{1/3}$	Ans. -0.3249
e) $(-12.36)^{-2/5}$	Ans. 0.3658

7. If a curve in a road is banked to prevent skids or overturning at V miles per hour, proper elevation (h) in feet of the outside edge is given by:

$$\frac{h}{w} = \left(\frac{22V}{15}\right)^2 \left(\frac{1}{gr}\right)$$

Find h for $g = 32.16$, $r = 4000$, $w = 26.0$ and $v = 40$

Ans. 0.697

8. Find the following logarithms using a table of common logarithms.

a) $\ln 0.39$ Ans. $9.0584 - 10$

b) $\log_5 2$ Ans. 0.4306

c) $\log_{100} 31$ Ans. 0.7457

9. Solve for the unknown:

a) $2^{x+6} = 32$ Ans. -1

b) $10^{2x-3} = 43$ Ans. 2.3168

c) $2^{3x} = 3^{2x+1}$ Ans. -9.32

d) $\log_3 (x+1) + \log_3 (x+3) = 1$ Ans. $0, -4$

e) $\dfrac{\log (7x - 12)}{\log x} = 2$ Ans. $3, 4$

CHAPTER 10

CALCULUS

10.1 Introduction.

We have previously covered most of the manipulations that can be performed on algebraic equations. We have seen how we can: (1) factor them; (2) graph them; and (3) solve systems of equations to determine where their graphs intersect. We have also learned how to set up equations to help us solve problems. We still must, however, investigate ways of predicting the affect varying the value of one variable will have on the other variable or variables. For example, if we are driving along a highway between two cities, what affect will changing our speed have on the time it takes us to make the trip?

In order to find the various times for all the various possible speeds, we could make numerous calculations and develop a table, we could determine a few relationships, draw a graph and then read from the graph, or we could determine the relationship for one speed and determine an average rate of change in time for each incremental change in speed. Then we could take our known relationship, and knowing the incremental change in speed, compute the time it takes to complete the trip. It is this latter process that is known as calculus. In other words, calculus is merely the process of finding how much the dependent variable varies for incremental changes in the value of the independent variable. When we want to compute the rate of change for extremely small changes in the value of the independent variable, we get what we call the instantaneous rate of change. The determination of the instantaneous rate of change is commonly called differential calculus.

The principles of calculus are by no means mystical. In fact, they are rather simple. The complexities of calculus only creep in when the relationships between variables are complex. However, the student should not worry because most of the equations encountered in management are not complicated. Now that we know what calculus is, let's proceed and determine how it helps us predict the effects of changing the value of the independent variable.

10.2 First Principles of Calculus.

In order that we may better understand the formulas that we will later use in differential calculus, let's develop our own rate of change for a rather simple problem. For ease of following, let's assume that we have only two variables; namely, x, the independent variable, and y, the dependent variable. Let's also agree that we will represent small changes in the variable x by the symbol Δx and small changes in the variable y by Δy. The student is cautioned that the variables could have any literal symbols assigned to them, but that by convention we usually use x and y. The case in which we have more than two variables will be taken up later.

Let's proceed now and see what happens to the dependent vari ble, when we change the independent variable a slight bit. It should be obvious that we must first know the relationships between the two variables before we could hope to determine the effects of changing the value of the independent variable by a slight bit. For purposes of explanation, let's assume the relationship is expressed by the equation:

$$y = 3x^2 + x + 2$$

Since the method we will use is the basic theory behind calculus, we call it the first principles of calculus.

If we increase the value of the variable x by a small amount, which we agreed we would call Δx, we will obviously change the value of y by some amount, which we agreed to call Δy. Therefore, using our relationship given above, we would get

$$y + \Delta y = 3\,(x + \Delta x)^2 + (\ x + \Delta x) + 2$$

or, $$y + \Delta y = 3x^2 + 6x\,(\Delta x) + 3(\Delta x)^2 + x + \Delta x + 2$$

or, $$\Delta y = 3x^2 + 6\,x(\Delta x) + 3(\Delta x)^2 + x + \Delta x + 2 - y$$

but, $$y = 3x^2 + x + 2$$

so $$\Delta y = 3x^2 + 6x(\Delta x) + 3(\Delta x)^2 + x + \Delta x + 2 - 3x^2 - x - 2$$

or $$\Delta y = 6x\,(\Delta x) + 3\,(\Delta x)^2 + \Delta x$$

Now if we want to get the average rate of change of y with respect to changes in x, we must divide the change in value of our y variable, which is Δy, by the change in value of our x variable, which is Δx. However, since we must perform exactly the same operations to both sides of an equation to retain the validity of an equation, we must divide both sides of the equation by Δx, if we want to obtain the average rate of change of Δy with respect to Δx. This then gives us:

$$\frac{\Delta y}{\Delta x} = \frac{6x(\Delta x)}{(\Delta x)} + \frac{3(\Delta x)^2}{(\Delta x)} + \frac{\Delta x}{\Delta x}$$

cancelling out on the right side, we find

$$\frac{\Delta y}{\Delta x} = 6x + 3\Delta x + 1 = \text{the average rate of change}$$

Thus, we can see that the rate of change of the value of the dependent variable with respect to the independent variable is 6 times the value of the independent variable plus 3 times the change in the independent variable's value plus 1. To complete the example all we need to know is the initial value of the independent variable and its change in value. To convince ourselves of what we have done, let's assume that the initial value of x is 3 and that the change in value of x is +1 and that we want to find the change in y for these conditions. Substituting the values into our last formula we find:

$$\frac{\Delta y}{1} = 6\,(3) + 3\,(1) + 1$$

$$\text{or} \quad \Delta y = 18 + 3 + 1 = 22$$

We can check this by determining the values of y for $x = 3$ and $x = 4$ and subtracting the first from the second. We find $y = 32$ for $x = 3$ and $y = 54$ for $x = 4$ and the difference is 22, the same as we expected it to be.

Since we normally want the instantaneous rate of change, that is when Δx is extremely small, we will work through only two problems of finding $\frac{\Delta y}{\Delta x}$ before proceeding on to the concept of differential calculus.

Exercises:

1. Find $\frac{\Delta y}{\Delta x}$ for $y = x^2 + x + 5$

Solution: $y + \Delta y = (x + \Delta x)^2 + (x + \Delta x) + 5$

$$\Delta y = x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x + 5 - y$$

$$\Delta y = x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x + 5 - x^2 - x - 5$$

$$\Delta y = 2x\,(\Delta x) + (\Delta x)^2 + \Delta x$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x + 1$$

2. Find $\frac{\Delta y}{\Delta x}$ for $y = 6x^2 + 7$

Solution: $y + \Delta y = 6\,(x + \Delta x)^2 + 7$

$$\Delta y = 6x^2 + 12x\,(\Delta x) + (\Delta x)^2 + 7 - y$$

$$\Delta y = 6x^2 + 12\,x\,(\Delta x) + (\Delta x)^2 + 7 - 6x^2 - 7$$

$$\Delta y = 12x\,(\Delta x) + (\Delta x)^2$$

$$\frac{\Delta y}{\Delta x} = 12x + \Delta x$$

10.3 Differentiating.

Knowing how to find the rate of change using the first principles of calculus, we can proceed to find the instantaneous rate of change by letting Δx become extremely small; so small, in fact, that it approaches the value zero. In this case, we change the symbology from Δx to dx and Δy to dy. Symbolically, we write the fact as follows:

$$\frac{dy}{dx} = \underset{\Delta x \to 0}{\text{limit}} \frac{\Delta y}{\Delta x}$$

The meaning of Δx and dx is the same; that is, a small change in the values of the variable. The degree of the change is all that differs. That is, dx means a very very small change. Therefore, we could take the three $\frac{\Delta y}{\Delta x}$'s developed above and change them as follows:

$$\frac{\Delta y}{\Delta x} = 6x + 3\Delta x + 1 \quad \text{to} \quad \frac{dy}{dx} = 6x + 3dx + 1$$

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x + 1 \quad \text{to} \quad \frac{dy}{dx} = 2x + dx + 1$$

and

$$\frac{\Delta y}{\Delta x} = 12x + \Delta x \quad \text{to} \quad \frac{dy}{dx} = 12x + dx$$

Now, if we consider dx to be infinitesimal, we can drop all terms which contain it because something almost zero times anything gives us almost zero. Thus, we would not affect our accuracy very much by dropping the terms with dx. Our equations would now read:

$$\frac{dy}{dx} = 6x + 1$$

$$\frac{dy}{dx} = 2x + 1$$

and $$\frac{dy}{dx} = 12x$$

That is, we have developed much simpler expressions for the instantaneous rate of change.

We all probably agree that it seems like there should be an easier way than the first principles to develop the value of this instantaneous rate of change. There is. We can develop the method by analyzing the results of our three previous efforts using the first principles of calculus. We should note that we have multiplied each term by the exponent of the independent variable in that term and that we have reduced the value of the exponent by 1. If we consider a term containing only a constant, we consider the independent variable to be there but to be invisible because its exponent is zero and we have mentally replaced it with a factor of 1. As an example, $6x^0 = 6 \times 1 = 6$, so rather than write $6x^0$, we write just plain 6. Looking at it in this way, we can see that the constants "drop out" when we differentiate because when we multiply by the exponent of the independent variable, zero, we get zero. Symbolically, if we consider each term independently of the other terms in the equation, and if we let a stand for the constant coefficient and x^n stand for the independent variable raised to the nth power, we can write the general expression:

$$y = ax^n$$

and $$\frac{dy}{dx} = nax^{n-1}$$

Due to the additive law, we can differentiate each term independently and then add them together to get the total derivative for the equation.

Now let's look at a few examples and then try a few on our own.

Exercises:

Differentiate the following equations:

1. $y = 3x^3 + 4x^2 + 5x + 1$

Solution: $\frac{dy}{dx} = (3)\ 3x^2 + (2)\ 4x + (1)\ 5 + 0$

$\frac{dy}{dx} = 9x^2 + 8x + 5$

2. $y = 16x^2 - 6x + 1$

Solution: $\frac{dy}{dx} = (2)\ 16x - (1)\ 6 + 0$

$\frac{dy}{dx} = 32x - 6$

3. $m = n^2 + n - 16 + n^{-2} + 5n^{-1}$

Solution: $\frac{dm}{dn} = 2n + 1 - 0 + (-2)\ n^{-3} + (-1)\ 5n^{-2}$

$\frac{dm}{dn} = 2n + 1 - 2n^{-3} - 5n^{-2}$

4. $t = r^2 + 3$ Ans. $\frac{dt}{dr} = 2r$

5. $R = 16G + 5$ Ans. $\frac{dR}{dG} = 16$

6. $y = 17x^4 + 6x^2 + 5$ Ans. $\frac{dy}{dx} = 68x^3 + 12x$

7. $s = 32t^2$ Ans. $\frac{ds}{dt} = 64t$

10.4 Differentiation of Higher Orders.

Sometimes we need to differentiate more than once to find the rate of change that we want. For example, if we wanted the rate of change of the rate of change, we would simply take the derivative of the derivative of the original equation. The derivative of the original equation is called the first derivative, while its derivative is called the second derivative. The derivative of the second derivative is called the third derivative, and so on. In order to indicate the exact derivative that we are working with, we insert, except for the first derivative, the number of the derivative we want between the d and the symbol of the dependent variable concerned and after the symbol of the independent variable. For example, the second derivative of y with respect to x is written $\frac{d^2y}{dx^2}$, and orally we would say dee-two-why and dee-x-two.

Exercises:

Find the derivative indicated in the following equations.

1. $\frac{d^2y}{dx^2}$ for $y = 3x^4 + 4x^3 + 2x + 5$

   Solution: $\frac{dy}{dx} = 12x^3 + 12x^2 + 2$

   derivative of $\frac{dy}{dx} = \frac{d^2y}{dx^2} = 36x^2 + 24x$

2. $\frac{d^4y}{dx^4}$ for $y = x^5 + 6x^2 + 3 - x^{-2}$

   Solution: $\frac{dy}{dx} = 5x^4 + 12x + 2x^{-3}$

   $$\frac{d^2y}{dx^2} = 20x^3 + 12 - 6x^{-4}$$

   $$\frac{d^3y}{dx^3} = 60x^2 + 24x^{-5}$$

   $$\frac{d^4y}{dx^4} = 120x - 120x^{-6}$$

3. $\frac{d^2y}{dx^2}$ for $y = 4x^2 + 6x + 10$ Ans. 8

4. $\frac{d^3y}{dx^3}$ for $y = 16x^5 + x^3 + x$ Ans. $960x^2 + 6$

5. $\frac{dy}{dx}$ for $y = 16x^5 + x^3 + x$ Ans. $80x^4 + 3x^2 + 1$

6. $\frac{d^2y}{dx^2}$ for $y = \frac{1}{x^2} + \frac{3}{x} + 4$ Ans. $6x^{-4} + 6x^{-3}$

7. $\frac{d^2y}{dx^2}$ for $y = x^2 + 100x + 1000$ Ans. 2

10.5 Derivatives with More Than 2 Variables.

In some cases we may find that a variable is dependent upon more than one independent variable. For example, y might be dependent upon m and t as shown in the following equation.

$$y = mt^2$$

In this case we can take the derivative of y with respect to one of the independent variables and consider the other one to be constant. This is what is called partial differentiation. When we take a partial derivative, we take the derivative with respect to one variable and consider everything else to be a constant. If we take $y = mt^2$ and find the first derivative with respect to t, we would get:

$$\frac{dy}{dt} = (2)\ mt = 2mt$$

With respect to m, we get:

$$\frac{dy}{dm} = t^2$$

Remember now, each of these is a partial derivative. If we want to get the total rate of change we would have to manipulate the equations so that only the dy's were on the left side of the equations. In our example, we would have:

(1) $dy = 2mt\ (dt)$ (Differential with respect to t)

and (2) $dy = t^2\ (dm)$ (Differential with respect to m)

We must add the two differentials together to get the total differential, thus,

$$dy = 2mt\ (dt) + t^2\ (dm)$$

This is what is known as the <u>total differential</u>.

Now let's try a few exercises and see if we can get the answers given.

Examples.

1. Find the partial derivative with respect to m for

   $y = m^2 + m + x^2 + x$

   Solution: Consider x to be a constant

   $$\frac{dy}{dm} = 2m + 1$$

2. Find the partial derivative with respect to x for

   $y = m^2 + m + x^2 + x$

   Solution: Consider m to be a constant

   $$\frac{dy}{dx} = 2x + 1$$

3. Find the total derivative of

   $y = m^2 + m + x^2 + x$

   $dy = (2m + 1)\ dm + (2n + 1)\ dx$

Exercises:

1. Find $\frac{dy}{dm}$ for $y = m^2 + xm + x$ Ans. $2m + x$
2. Find $\frac{dy}{dx}$ for $y = m^2 + xm + x$ Ans. $m + 1$
3. Find $\frac{dy}{dx}$ for $y = m^3 + xm^2 + x^2$ Ans. $m^2 + 2x$

4. Find $\frac{dy}{dm}$ for $y = m^3 + xm^2 + x^2$ Ans. $3m^2 + 2xm$

5. Find $\frac{d^2y}{dm^2}$ for $y = m^3 + xm^2 + x^2$ Ans. $6m + 2x$

6. Find $\frac{d^2y}{dx^2}$ for $y = m^3 + xm^2 + x^2$ Ans. 2

7. Find $\frac{dy}{dx}$ for $y = m^2 + n^2 + x^2$ Ans. $2x$

8. Find $\frac{dy}{dn}$ for $y = m^2 + n^2 + x^2$ Ans. $2n$

9. Find $\frac{dy}{dm}$ for $y = m^2 + n^2 + x^2$ Ans. $2m$

10. Find total derivative for

$y = m^2 + n^2 + x^2$ Ans. $2xdx + 2ndn + 2mdm$

10.6 Special Derivatives.

While all equations can be differentiated by the first principles, the work involved sometimes becomes unduly lengthy. For this reason, some derivatives are usually memorized. The more commonly found derivatives are listed below, with an example of each. The formulas used are offered without proof:

1. Sum of terms $y = ax^n + bx^{n-1} + cx^{n-2} + \text{----} + c$

$$\frac{dy}{dx} = nax^{(n-1)} + (n-1)\, bx^{(n-2)} + (n-2)\, cx^{(n-3)} + \text{----} + 0$$

Example: $y = 3x^2 + 2x + 3$

$$\frac{dy}{dx} = 6x + 2$$

2. Products of terms, say $y = (u)(v)$

$$\frac{dy}{dx} = \frac{du}{dx}(v) + \frac{dv}{dx}(u)$$

Example: $y = (x^2 + 3)(4x^4 + 5)$

$$\frac{dy}{dx} = 2x(4x^4 + 5) + 16x^3(x^2 + 3)$$

$$\frac{dy}{dx} = 8x^5 + 10x + 16x^5 + 48x^3 = 24x^5 + 48x^3 + 10x$$

3. Quotients of terms, say $y = \frac{u}{v}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example: $y = \frac{x^2 + 3}{x}$

$$\frac{dy}{dx} = \frac{x(2x) - (x^2 + 3)(1)}{x^2}$$

$$\frac{dy}{dx} = \frac{2x^2 - x^2 - 3}{x^2} = \frac{x^2 - 3}{x^2}$$

4. Derivative of functions of a function, say $y = u^n$ where u is a function of x.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example: $y = 4u$, where $u = x^2 + 2$

$$\frac{dy}{du} = 4 \quad , \quad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = 4 \cdot 2x = 8x$$

Exercises:

1. Differentiate $y = (x^2 + 1)(x^2)$ Ans. $4x^3 + 2x$

2. Differentiate $y = (x^3 + 3x)(x + 1)$ Ans. $4x^3 + 3x^2 + 6x + 3$

3. Differentiate $y = \dfrac{x^3 + 2}{x^2}$ Ans. $\dfrac{2x^4 - 4x}{x^4}$

4. Differentiate $y = \dfrac{x^4}{x^2 + 1}$ Ans. $\dfrac{2x^5 + 4x^3}{x^4 + 2x^2 + 1}$

5. Differentiate $y = \frac{3}{2} u^{\frac{3}{2}}$ Ans. $3x\,(x^2 + 4)^{\frac{1}{2}}$
   where $u = x^2 + 4$

6. Differentiate $y = 6u^{-3}$ Ans. $-36x^{-7}$
   where $u = x^2$

10.7 Maximum and Minimum Points.

The student should recall from our original derivation of the first derivative, that $\frac{dy}{dx}$ is the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero. We should reflect on this statement for a moment and visualise what would happen in Figure 10-1 if we let Δx approach zero.

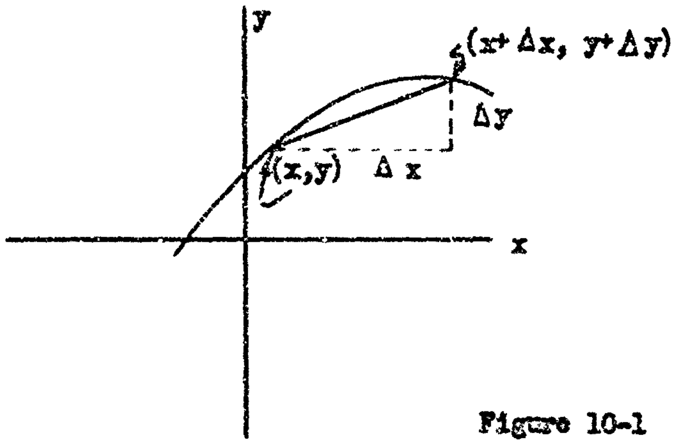


Figure 10-1

The hypotenuse of our triangle would rotate counterclockwise, passing through the various ordered pairs on the curve, until it coincided with the line tangent to the curve at (x,y). When this occurs, Δx has approached zero, and $\frac{\Delta y}{\Delta x}$ becomes the slope of the tangent to the curve at point (x,y). It should now be evident to the student that the slope of the curve at any particular point is equal to the slope of the line tangent to the curve at that point. Therefore, we can conclude that the first derivative, $\frac{dy}{dx}$, is, geometrically speaking, the slope of the curve of the function at any point on the curve.

If we had the function

$$y = 6x^2 - 12x + 2$$

$$\frac{dy}{dx} = 12x - 12$$

The expression, $12x - 12$, is the slope of the curve $y = 6x^2 - 12x + 2$ at any point. If it were necessary to obtain the slope at a particular point, say $x = -1$, we would simply substitute the value $x = -1$ in the

expression $\quad 12x - 12$

Then $\quad 12(-1) - 12$

$$-12 - 12 = -24$$

The slope of $y = 6x^2 - 12x + 2$ at $x = -1$ is the high negative slope, -24.

At $x = 0$, $\quad 12x - 12$

$$12(0) - 12 = -12$$

the slope equals -12.

At $x = +1$, $\quad 12x - 12$

$$12(1) - 12 = 0$$

the slope is zero, therefore the slope of the curve at the point $x = 1$ is parallel to the x axis.

At $x = 2$, $\quad 12x - 12 = 0$

$$12(2) - 12 = +12$$

At $x = 3$, $\quad 12(3) - 12 = +24$

The function $y = 6x^2 - 12x + 2$ is sketched in Figure 10-2. The y values corresponding to the values of x plotted are obtained by substituting the appropriate x values in the original equation.

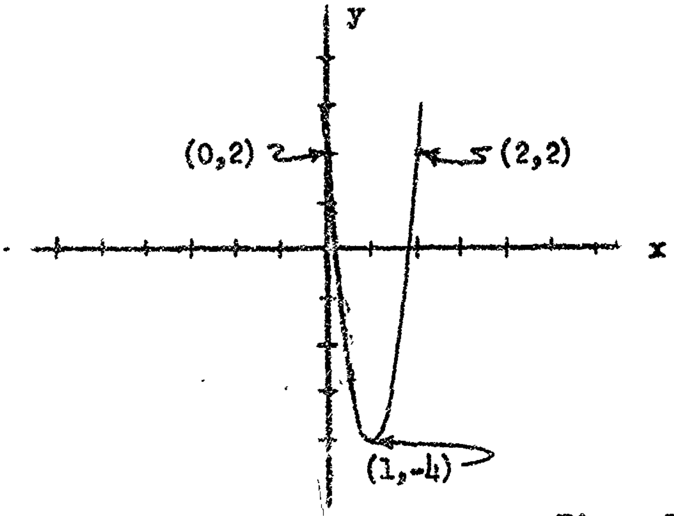


Figure 10-2

At point (0,2) we calculated the slope to be -12; at point (1,-4) it was 0; and at point (2,2) it was +12.

You will recall from our study of linear equations that an equation such as $y = 2x + 4$ is in the slope-intercept form of the equation. The graph of this equation was a line with a slope of +2. Let's see what happens if we take this equation and differentiate it.

$$y = 2x + 4$$

$$\frac{dy}{dx} = +2$$

The first derivative is the coefficient of x which we already know is the slope of the equation of this line. If we understand the simple concept of slope as covered in the chapter on linear equations and as developed above, the subject of determining maximum or minimum points, relative maximum or minimum points or points of inflection of graphs of functions will be very simple.

Now, what do we mean when we say maximum, minimum, relative maximum or relative minimum points? In the graph of the parabola in Figure 10-2, the point (1,-4) is definitely a minimum point on the graph since there is no possible value of x which can be substituted in the equation, which will result in a lower value of y than -4. A parabola such as the one sketched in Figure 10-3 would have a definite maximum at x = 0.

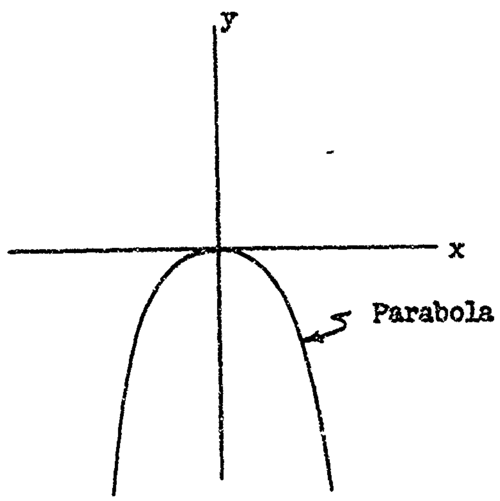


Figure 10-3

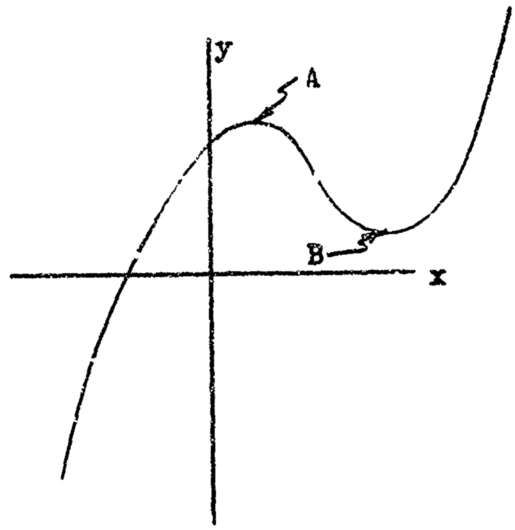


Figure 10-4

At any other value of x, the value of y would be less than zero. Looking now at the function sketched in Figure 10-4, we can see that as x decreases to the left of point A the corresponding value of y also decreases. As X increases to the right of point B the corresponding value of y also increases. There obviously must be higher y values than the y value at point A and lower y values than the y value at point B. Point A, then is referred to as a relative maximum point, and point B is referred to as a relative minimum point. The explanations above are presented without the use of formal definitions. The student is invited to develop in his own words appropriate definitions for these concepts. It should be noted by the student that at the turning points in Figures 10-2, 10-3, and 10-4, the slopes of the curve and of the tangents to the curve are equal to zero.

The slope of a curve can also equal zero at points of inflection. The student should turn ahead to Figures 10-7 and 10-8 to see what points of inflection look like. We will study them in detail in subsection 10.8.

At the coordinates $(1,-4)$ of Figure 10-2, we found that the slope was zero. This point coincided with the minimum point on the graph. The student should also realize that as we moved along the curve from $x = 0$ to $x = 2$ our slope went from $-$ to 0 to $+$ or the "rate of change" of slope in that area was positive.

Rewriting our original function

$$y = 6x^2 - 12x + 2$$

and differentiating

$$\frac{dy}{dx} = 12x - 12 = \text{slope}$$

To obtain a point where the slope is zero, we simply set the expression we have obtained as our first derivative equal to zero and solve for the unknown, in this case x.

$$12x - 12 = 0$$
$$12x = 12$$
$$x = 1$$

Therefore, at $x = 1$ we know the slope is 0. (We then know that we have a maximum or minimum point or a point of inflection as will be shown later). What then is the y coordinate at this point? This is obtained by substituting the value of $x = 1$ in the original equation.

$$y = 6x^2 - 12x + 2$$
$$y = 6(1)^2 - 12(1) + 2$$
$$y = -4$$

Since the graph is sketched in Figure 10-2 already, we know in this case, that the point is a minimum point. Suppose, however, that we did not know whether the point on the curve was a maximum or a minimum at this point. We can determine this without plotting the graph by taking the second derivative, and then substituting the value of x for which we know that the slope is zero. Since the first derivative gives us the slope, or the rate of change of y with respect to x, the second derivative should give us the rate of change of the first derivative or rate of change of slope with respect to x at any value of x. If the slope rate is positive, we should obtain a positive figure when we substitute our value of x where the slope is zero in the second derivative expression. The reverse is true if the slope rate is negative at the value where the slope is zero.

In the parabola example

$$\frac{dy}{dx} = 6x - 12$$

The second derivative, $\frac{d^2y}{dx^2} = +6$, which indicates to us that if we move in the positive x direction on the parabola the rate of change of slope will always be positive. This concept can more easily be visualized by the solution of a problem.

Example:

1. Determine the turning points for the graph of the equation $y = x^3 + x^2 + 1$ and sketch the graph.

$$y = x^3 + x^2 + 1$$

First, take the first derivative to obtain the expression for the slope.

$$\frac{dy}{dx} = 3x^2 + 2x = \text{slope}$$

Find out where we have turning points by equating the slope expression to zero, since the slope will be zero at any turning point (or at a point of inflection as discussed in subsection 10.8).

$$3x^2 + 2x = 0$$

factoring
$$x\,(3x + 2) = 0$$
$$x = 0$$
$$3x + 2 = 0$$
$$3x = -2$$
$$x = -\frac{2}{3}$$

Therefore, we know that at $x = 0$ and $x = -2/3$, the slope is 0 and we have turning points.

Next, we obtain the values of y which correspond to $x = 0$ and $x = -2/3$ by separately substituting these values in the original equation.

$$y = x^3 + x^2 + 1$$

When $x = 0$

$$y = 0 + 0 + 1$$
$$y = +1$$

$$y = x^3 + x^2 + 1$$

When $x = -2/3$

$$y = \left(\frac{-2}{3}\right)^3 + \left(\frac{-2}{3}\right)^2 + 1$$
$$y = \frac{-8}{27} + \frac{4}{9} + 1$$
$$y = 1 \ 4/27$$

Turning points are located at

$$(0,1) \text{ and } \left(\frac{-2}{3}, \ \frac{31}{27}\right)$$

Then to determine if these points are relative maximum or minimum points, we take the second derivative.

$$\text{Since } \frac{dy}{dx} = 3x^2 + 2x$$

$$\frac{d^2y}{dx^2} = 6x + 2$$

Substituting $x = 0$ in $6x + 2$

$$6(0) + 2 = \underline{+2}$$

This tells us that the slope rate is positive at $x = 0$. Therefore, point (0,1) is a minimum point or relative minimum point.

Substituting $x = -2/3$ in $6x + 2$

$$6\left(\frac{-2}{3}\right) + 2 = -2$$

This tells us that the slope rate is negative at $x = -2/3$, therefore, we have either a "maximum" or a relative "maximum." We then can sketch these two points on a coordinate axis as shown by the solid curve in Figure 10-5.

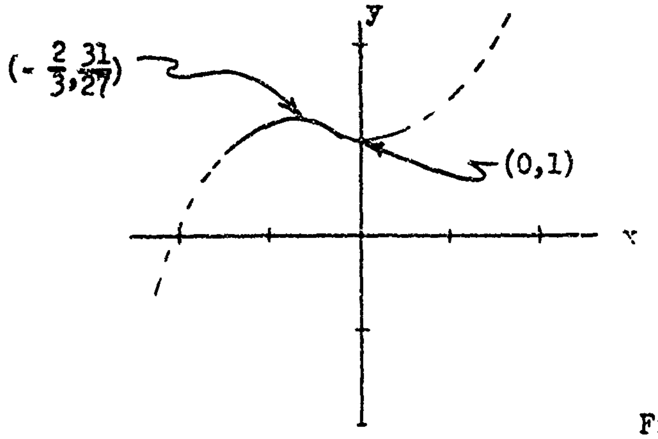


Figure 10-5

We can also see from the original equation $y = x^3 + x^2 + 1$, that as x increases beyond $x = 0$, y will continue to increase toward + infinity; also as x decreases below $-2/3$, y will continue to decrease indefinitely. Therefore, we can clearly see that the turning points we have obtained are relative maximum and minimum points. We can then sketch in the rest of our curve. (Shown in dotted lines in Figure 10-5.)

Example:

2. Determine the relative maximum and minimum points for the graph of the equation $y = \frac{x^3}{3} - 2x^2 + 3x + 1$ and sketch the graph of this equation.

$$y = \frac{x^3}{3} - 2x^2 + 3x + 1$$

then $\qquad \dfrac{dy}{dx} = \dfrac{3x^2}{3} - 4x + 3$

equating $\dfrac{dy}{dx}$ to 0

$$x^2 - 4x + 3 = 0$$

factoring $\qquad (x - 3)\,(x - 1) = 0$

then $\qquad x = 3$ and $x = 1$

and $\qquad \dfrac{d^2y}{dx^2} = 2x - 4$

at $x = 3$, $\qquad 2x - 4$

$2(3) - 4 = \underline{+2}$ $\qquad$ Concave up or relative "minimum"

at $x = 1$, $\qquad 2x - 4$

$2(1) - 4 = \underline{-2}$ $\qquad$ Concave down or relative "maximum"

Finding the corresponding y values from the <u>original</u> equation:

at $x = 3$,	at $x = 1$,
$y = \frac{x^3}{3} - 2x^2 + 3x + 1$	$y = \frac{x^3}{3} - 2x^2 + 3x + 1$
$y = \frac{(3)^3}{3} - 2(3)^2 + 3(3) + 1$	$y = \frac{(1)^3}{3} - 2(1)^2 + 3(1) + 1$
$y = 9 - 18 + 9 + 1$	$y = \frac{1}{3} - 2 + 3 + 1$
$y = \underline{+1}$	$y = \frac{7}{3}$

As in the first problem, for values of $x > 3$, y continues to increase indefinitely, and for values of $x < 1$, y decreases indefinitely. We then confirm that our values are relative maximum and minimum points. Our sketch will look like Figure 10-6.

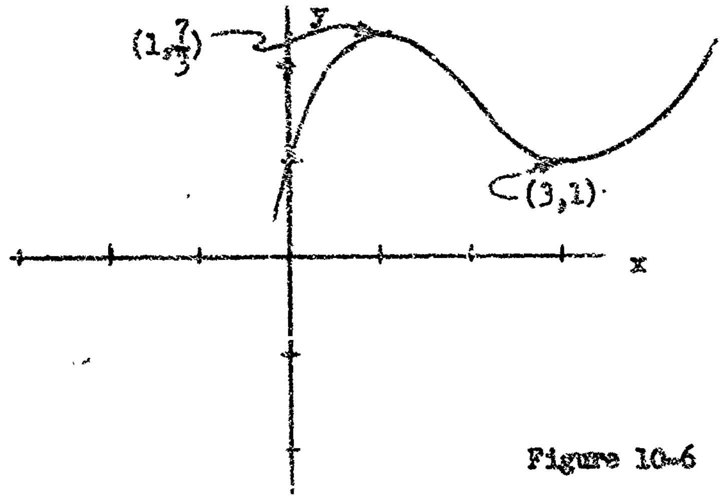


Figure 10-6

10.8 Points of Inflection.

Relative maximum and minimum points are not the only times that a curve may have its slope equal to zero. Referring now to Figure 10-7,

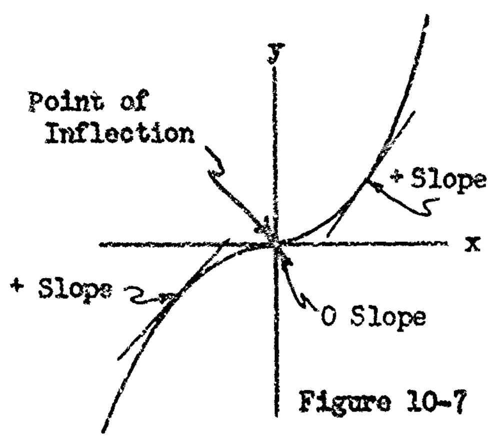


Figure 10-7

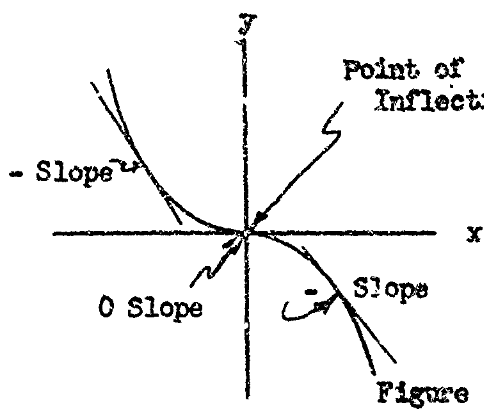


Figure 10-8

we can see that if we had a curve such as that shown, our slope would be positive as we approached $x = 0$, in the positive direction, the slope would be 0 at $x = 0$, and then it would again become and remain positive at $x > 0$. The reverse is true for the curve shown in Figure 10-8. There the slope goes from minus to zero to minus as indicated. The points $(0,0)$ in each case are called points of inflection.

Example:

1. Sketch the graph of the curve $y = x^3$.

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

Equating $3x^2$ to 0

$$3x^2 = 0$$

$$x = 0$$

then $y = 0$, by substitution in original equation

This then tells us that at $x = 0$, $y = 0$, the slope is 0. Then taking the second derivative,

$$\frac{d^2y}{dx^2} = 6x$$

When $x = 0$, $\frac{d^2y}{dx^2} = 6x = 6(0) = \underline{0}$

When we substitute a value of x for which the slope is zero in the second derivative and obtain a zero answer, we have a point of inflection. If we graph the equation $y = x^3$, we will see that it has the general slope of the curve in Figure 10-7. The slope goes from plus to zero to plus as we move from negative values of x to positive values of x and the rate of change of slope, $\frac{d^2y}{dx^2}$, is zero where the slope is zero.

The student is invited to plot the curve $y = -x^3$ and determine if it has an inflection point.

10.9 <u>Summary of Turning Points and Points of Inflection.</u>

When asked to graph a function,

1. Take the first derivative and equate it to zero. The values obtained tell us we have either a turning point (maximum or minimum) or a point of inflection.
2. Take the second derivative and substitute the values obtained in step 1. If we obtain:

(1) a <u>positive</u> sign, we have a minimum or a relative minimum point.

(2) a <u>negative</u> sign, we have a maximum or a relative maximum point.

(3) zero, we have a point of inflection.

3. We obtain the corresponding values of the dependent variable (y) from substituting the values of the independent variable (x) obtained in step 1 in the original equation.

Exercises:

1. Find the coordinates of the minimum point on the graph of the equation $y = x^2 + 2x + 4$. Ans. $x = -1$ $y = 3$

2. Find the coordinates of the relative maximum and minimum points on the graph of the equation $y = \frac{x^3}{3} + 2x^2$. Sketch the graph of this equation. Ans.

   Relative maximum $x = -4$ $y = 10\ 2/3$

   Relative minimum $x = 0$ $y = 0$

3. Sketch the graph of the equation $y = 3x^3$. What are the coordinates of the point of inflection? Ans. $x = 0$ $y = 0$

10.10 <u>Introduction to Integration.</u>

In differentiation we are given a function and asked to find a certain derivative, or we are given some derivative and asked to find a higher order derivative. In integration, we work in the opposite direction. Here we are given a derivative and we are required to "integrate", or obtain either the original function or the next lower derivative. Let's take the simple function

$$y = 3x^2 + 4$$

When we differentiate it, we obtain:

$$\frac{dy}{dx} = 6x$$

Now let's take this derivative and integrate it in an attempt to obtain the original function. To indicate the integration operation, we arrange our derivative in a special manner. First of all, we multiply both sides by dx and obtain:

$$dy = 6x\ dx$$

Then we insert an integral sign, "$\int$", on both sides of the equation as indicated below.

$$\int dy = \int 6x\ dx$$

We now are ready to integrate.

If we raise the power of x to the next power, 2, and divide by the same number 2, we obtain $\frac{6x^2}{2}$ or $3x^2$, the first term of the original function. With the meager amount of information given, it is impossible to obtain the constant, +4, we originally had. The original function could have had any number for a constant, since the derivative of a constant is zero. We conventionally compensate for this inability by indicating that there could be a constant by adding +c to the function developed by integration. Summarizing our integration problem,

$$\int dy = \int 6x\ dx$$

$$y = 3x^2 + c$$

Similarly, $\int dy = \int 1\ dx$

$$y = x + c$$

and $\int dy = \int (x^2 + 2x)\ dx$

$$y = \frac{x^3}{3} + x^2 + c$$

Our general rule for integration then for a derivative, such as

$\frac{dy}{dx} = ax^n$, is arrived at as follows. Changing this derivative to the integration form,

$$\int dy = \int ax^n \, dx$$

which becomes

$$y = \frac{ax^{n+1}}{n+1} + c$$

Examples:

1. Integrate: $\frac{dy}{dx} = x^4 - 3x^3 + 2x^2 + x + 1$

then $$\int dy = \int (x^4 - 3x^3 + 2x^2 + x + 1) \, dx$$

$$y = \frac{x^5}{5} - \frac{3x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x + c$$

2. Perform the indicated operation.

$$\int dy = \int (3x^2 + 2) \, dx$$

$$y = \frac{3x^3}{3} + 2x + c$$

$$y = x^3 + 2x + c$$

10.11 Distance, Velocity and Acceleration.

If we have an equation which expresses some distance traveled expressed as a function of time, the first derivative is an equation for the velocity, since velocity is defined as the rate of change of distance with respect to time. The second derivative of the original equation is equal to the acceleration, since the acceleration is the rate of change of velocity with respect to time.

Therefore, if the distance, S, a certain object travels (in feet) is expressed as a function of time t (in seconds) is $S = 16t^3 + 2t^2 + 3t + 10$,

$$s = 16t^3 + 2t^2 + 3t + t \qquad \text{feet}$$

then $$\frac{ds}{dt} = \text{velocity} = 48t^2 + 4t + 3 \qquad \text{feet/sec}$$

and $$\frac{d^2s}{dt^2} = \text{acceleration} = 96t + 4 \qquad \text{feet/sec/sec}$$

Examples:

1. The velocity (in ft/sec) of a body at any time t (in seconds) is related to time by the following equation: $V = t^2 - t + 1$. Find an equation which expresses the distance traveled as a function of time if the distance traveled at time 0 seconds is 0 feet.

$$V = \frac{ds}{dt} = t^2 - t + 1$$

$$\int ds = \int (t^2 - t + 1)\, dt$$

$$s = \frac{t^3}{3} - \frac{t^2}{2} + t + c$$

Since $s = 0$ when $t = 0$, then $\quad 0 = 0 - 0 + 0 + c$

$$c = 0$$

therefore $$s = \frac{t^3}{3} - \frac{t^2}{2} + t$$

2. Determine an expression for the acceleration of the object at any time t in the above example.

$$V = \frac{ds}{dt} = t^2 - t + 1$$

$$\frac{dv}{dt} = \frac{d^2s}{dt^2} = 2t - 1 \text{ in units of ft/sec/sec}$$

Exercises:

1. $\int (3x^2 + \frac{1}{x^2})\, dx$ — Ans. $x^3 - \frac{1}{x} + c$

2. $\int (-3 + 4x^3)\, dx$ — Ans. $x^4 - 3x + c$

3. $\int (x^{-3} - \frac{1}{x^3})\,dx$ Ans. $0 + c$

4. $\int (x^2 + 2)\,dx$ Ans. $\frac{x^3}{3} + 2x + c$

5. If distance, y in yards, and time, x in seconds, are connected by the following formula, $y = x^2 + x + 1$, what is the velocity at time (a) zero? (b) after 2 seconds? (c) after 10 seconds?

Ans. (a) 1 yard/sec (b) 5 yards/sec (c) 21 yards/sec

6. What is the acceleration in problem 5?

Ans. 2 yards/sec/sec

7. The velocity of an object after 2 seconds is equal to 10 feet/sec. The distance traveled at time 0 seconds is 0 feet. If the acceleration of this object in feet/sec/sec is given by the formula $a = x + 2$, where x is expressed in seconds, what is the equation which expresses the distance, s, traveled in terms of x?

Ans. $s = \frac{x^3}{6} + x^2 + 4x$ feet

8. The velocity of a body in ft/sec at time t seconds is given by $v = t^2 - t + 1$. Find the distance from an observer at time t, if the position is 1 foot at time zero seconds.

Ans. $s = \frac{t^3}{3} - \frac{t^2}{2} + 1$ feet

BIBLIOGRAPHY

1. Allendorfer, C. B. and Oakley, C.O. Fundamentals of Freshman Mathematics. McGraw-Hill, 1955.

2. Andres, P. G., Miser, H. J. and Rheingold, H. Basic Mathematics for Engineers. Wiley, 1947.

3. Ayre, H. G. Basic Mathematical Analysis for Junior and Senior Colleges. McGraw-Hill, 1956.

4. Banks, J. H. Elements of Mathematics. Allyn and Bacon, 1956.

5. Cooley, H. R., Gans, P., Kline, M. and Wahlert, H. E. Introduction to Mathematics. Houghton Mifflin Co., 1937.

6. Davis, H. T. and McCown, M. G. General Mathematics. Principia Press, 1962.

7. Fine, H. B. A College Algebra. Ginn and Company, 1946.

8. Fowler, F. P. and Sandberg, E. W. Basic Mathematics for Administration. Wiley, 1962.

9. Freund, J. E. A Modern Introduction to Mathematics, Prentice-Hall, 1956.

10. Geary, A., Lowry, H. V. and Hayden, H. A. Mathematics for Technical Students. Longmans, 1941.

11. Granville, W. A., Smith, P. F. and Longley, W. R. Elements of the Differential and Integral Calculus. Ginn and Company, 1941.

12. Hamilton, W. T. and Hamilton, J. R. Mathematical Analysis, A Modern Approach. Harper, 1956.

13. Hart, W. L. First Year College Mathematics. Heath, 1943.

14. Hogben, L. T. Mathematics for the Million. W. W. Norton Co., 1937.

15. Kaltenborn, H. S., Anderson, S. A. and Kaltenborn, H. H. Basic Mathematics. Ronald Press, 1958.

16. Kelley, J. L. Introduction to Modern Algebra. D. Van Nostrand, 1960.

17. McNeil, D. B. Modern Mathematics for the Practical Man. D. Van Nostrand, 1963.

18. Meier, R. C. and Archer, S. H. Mathematics for Management. McGraw-Hill, 1960.

19. Rider, P. R. College Algebra. The MacMillan Co., 1943.

20. Rider P. R. First Year Mathematics for Colleges. The MacMillan Co., 1940.

21. Rosenbach, J. B. and Whitman, E. A. College Algebra. Ginn and Company, 1939.

22. Thompson, S. P. Calculus Made Easy. The MacMillan Co., 1921.

23. Wardhill, T. H. Mathematics for the Layman. Philosophical Library, 1958.

24. Wilczynski, E. J. College Algebra with Applications. Allyn and Bacon, 1942.