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ANALYSIS OF TRANSMISSION-LINE
ACCELERATOR CONCEPTS

J. K. Temperley
D. Eccleshall

May 1978

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both a high accelerating voltage per stage and nominal unit efficiency can be achieved. A recirculating accelerator is described, in which advantage is taken of a repetitive voltage waveform present in the transmission-line cavities to repeatedly accelerate a current pulse which is recirculated through the accelerator. Expressions for the open-circuit output voltage, accelerating voltage per stage, and efficiency of energy transfer to the beam are derived for this case also. It is shown that, with proper choice of parameters, this type of design again affords the possibility of nominal unit efficiency for energy transfer to the beam.

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1. INTRODUCTION

During the last few years considerable interest has developed in the feasibility of designing accelerators capable of accelerating pulsed high-current electron beams to tens of MeV. Possible applications of such devices include electron-beam fusion, collective ion accelerator injection, and deep flash radiography. While single-stage diode accelerators exist which are capable of handling current pulses of up to 10^6 amperes, the energy limitation of these single-stage machines is on the order of 10 MeV. The AURORA facility at the Harry Diamond Laboratories, for example, was designed to produce a 120-ns, 1.6-MA pulse of 15-MeV electrons.¹ To get to higher energy the beam must be subjected to a succession of properly-phased accelerating fields. Multi-gap concepts for achieving this include the magnetic linear induction accelerator,²⁻⁵ the autoaccelerator,^{6,7} and accelerators based on iron-free inductors^{8,9}

¹B. Bernstein and I. Smith, "AURORA, An Electron Accelerator," *IEEE Trans. Nucl. Sci.* NS-20, No. 3, 294-300, 1973.

²R. Avery, G. Behrsing, W.W. Chupp, A. Faltens, E.C. Hartwig, H.P. Hernandez, C. MacDonald, J.R. Meneghetti, R.G. Nemetz, W. Popenuck, W. Salsig, and D. Vanecsek, "The ERA 4 MeV Injector," *IEEE Trans. Nucl. Sci.* NS-18, No. 3, 479-483, 1971.

³N.C. Christofilos, R.E. Hester, W.A.S. Lamb, D.D. Reagen, W.A. Sherwood, and R.E. Wright, "High Current Linear Induction Accelerator for Electrons," *Rev. Sci. Inst.* 35, 886-890, 1964.

⁴J. Leiss, NBS, private communication.

⁵A.I. Anatskii, O.S. Bogdanev, P.V. Buckaev, Yu P. Vakhrushin, I.F. Malyshev, G.A. Nalivaika, A.I. Pavlov, V.A. Suslov, and E.P. Khalchitskii, "Linear Induction Accelerator," *Sov. At. En.* 21, 1134-1140, 1966.

⁶M. Friedman, "Autoacceleration of an Intense Relativistic Electron Beam," *Phys. Rev. Lett.* 31, 1107-1110, 1973.

⁷J.K. Burton, D. Conte, M. Friedman, T. O'Connell, and J. Shipman, "The Autoaccelerator - A Generator for Producing a High Energy Intense Relativistic Electron Beam by the Autoaccelerator Process," *IEEE Trans. Nucl. Sci.* NS-24, No. 3, 1628-1630, 1977.

⁸A.I. Pavlovskii, A.I. Gerasimov, D.I. Zenkov, V.S. Bosamykin, A.P. Klementev, and V.A. Tananakin, "An Iron-Free Linear Induction Accelerator," *Sov. At. En.* 28, 549-550, 1970.

⁹A.I. Pavlovskii, and V.S. Bosamykin, "Linear Inductive Accelerators Without Iron," *Sov. At. En.* 37, 942-947, 1974.

or pulse transmission lines.¹⁰⁻¹³ Pulse-line accelerators which use internal switches to generate the accelerating voltage have received little attention, at least in this country, probably because of the number of high-current, high-voltage switches that must be operated synchronously. Recently, however, Pavlovskii et al.¹¹ have described a radial pulse-line accelerator of this type for high current (100 kA), in which they claim to have solved the switching problems. Furthermore, very significant advances are being made in this country in high-speed simultaneous switching for application to electron-beam fusion.¹⁴ It is therefore appropriate to reconsider the possibilities for accelerator design based on energy storage in pulse transmission lines. In this report we present the mathematical details of an analysis of internally-switched pulse-line accelerators based on constant-impedance lines. It is assumed throughout that the lines and switches are ideal and lossless.¹³ The results of this analysis have been submitted to the open literature.¹³ Also included is a discussion of how a repetitive voltage waveform present in the transmission-line cavities might be used to repeatedly accelerate a current pulse which is recirculated through the accelerator. Techniques for recirculating high-current beams are presently being investigated at NBS with a magnetic induction linear accelerator.¹⁵

2. BASIC PULSE LINE CONCEPTS

2.1 The Blumlein Pulse Line

The workhorse of most single-stage diode accelerators is the Blumlein pulse line. A simplified diagram of a strip-line version of such a line is shown in Figure 1. Other geometries are also employed - in

¹⁰E.C. Hartwig, "Pulsed Line Acceleration of Electron Rings," *Proceedings of Symposium on Electron Ring Accelerators*, Feb. 1968, pp 44-64, UCRL 18103, 1968.

¹¹A.I. Pavlovskii, V.S. Bosamykin, G.D. Kuleshov, A.I. Gerasimov, V.A. Tananakin, and A.P. Klementev, "Multielement Accelerators Based on Radial Lines," *Sov. Phys. Dokl.* 20, 441-443, 1975.

¹²V.I. Kazacha and I.V. Kozhukhov, "Use of Radial Transmission Lines in Pulsed Accelerators," *Sov. Phys. Tech. Phys.* 21, 841-844, 1976.

¹³D. Eccleshall and J.K. Temperley, "Transfer of Energy from Charged Transmission Lines with Applications to Pulsed High-Current Accelerators," submitted to *J. Appl. Phys.*

¹⁴K.R. Prestwich and T.H. Martin, Sandia Laboratories, private communication.

¹⁵J. Leiss and M. Wilson, National Bureau of Standards, private communication.

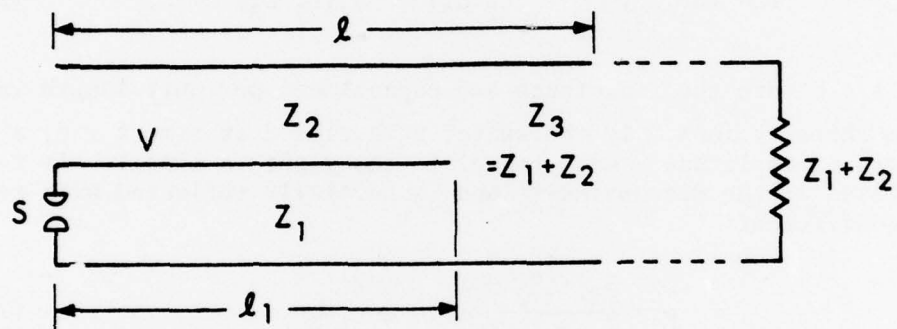


Figure 1. Blumlein pulse line.

AURORA, for example, coaxial Blumleins are used.¹ The main feature of the Blumlein line is that the full charging voltage is applied to a matched load, whereas with a simple coaxial line, for example, only half the charging voltage appears across a matched load. The principle is easily understood. The center strip in the diagram of Figure 1 is originally charged to some voltage V , and no voltage appears across the output end. We note that if the same dielectric material is used in all regions of the line, then the speed of an electromagnetic wave is the same in all three regions. The time of travel per unit length, T , is just

$$T = (L_1 C_1)^{1/2} = (L_2 C_2)^{1/2} = (L_3 C_3)^{1/2}, \quad (2.1)$$

where L_i, C_i are the inductance and capacitance per unit length in each of the three regions. If the switch S is closed at time $t = 0$, a voltage step of amplitude V will travel to the right in line 1. At $t = T\ell_1$ it arrives at the discontinuity and is partially reflected with reflection coefficient

$$\rho = \frac{Z_3 + Z_2 - Z_1}{Z_3 + Z_2 + Z_1} = \frac{Z_2}{Z_2 + Z_1}. \quad (2.2)$$

The transmission coefficient is

$$\tau = 1 + \rho = \frac{2Z_2 + Z_1}{Z_2 + Z_1}. \quad (2.3)$$

The transmitted voltage divides across lines 2 and 3 according to the ratio of their impedances. Hence at time $t = T\ell_1 + T(\ell - \ell_1)$ a voltage step

$$V_{\text{out}} = \frac{Z_3}{Z_3 + Z_2} \frac{2Z_2 + Z_1}{Z_2 + Z_1} V = V \quad (2.4)$$

arrives at the output of line 3. If a matched load of impedance $Z_1 + Z_2$ is connected across the output this voltage will be developed across the load and no reflections occur at this end. The voltage step reflected back into line 1 at $t = T\ell_1$ has magnitude

$$V_1 = \frac{Z_2}{Z_2 + Z_1} V, \quad (2.5)$$

and is reflected out-of-phase from the shorted end of line 1 at $t = 2T\ell_1$. The voltage step transmitted into line 2 at $t = T\ell_1$ is

$$V_2 = \frac{Z_2}{Z_2 + Z_1} V, \quad (2.6)$$

and is reflected in-phase from the open end of line 2 at $t = 2T\ell_1$. At time $t = 3T\ell_1$ both these steps arrive back at the junction with line 3, and each is partially reflected and partially transmitted. The voltage step generated at $t = 3T\ell_1$ and traveling to the left in line 1 is

$$\begin{aligned} V_1 &= \frac{Z_2}{Z_2 + Z_1} \left(\frac{-Z_2 V}{Z_2 + Z_1} \right) + \frac{Z_1}{Z_1 + Z_3} \left(1 + \frac{Z_1 + Z_3 - Z_2}{Z_1 + Z_3 + Z_2} \right) \frac{Z_2}{Z_2 + Z_1} V \\ &= \frac{Z_2(Z_1 - Z_2)}{(Z_1 + Z_2)^2} V. \end{aligned} \quad (2.7)$$

In line 2 we have

$$\begin{aligned} V_2 &= \frac{Z_1}{Z_1 + Z_2} \frac{Z_2 V}{Z_1 + Z_2} + \frac{Z_2}{Z_2 + Z_3} \left(1 + \frac{Z_2 + Z_3 - Z_1}{Z_2 + Z_3 + Z_1} \right) \left(-\frac{Z_2}{Z_2 + Z_1} V \right) \\ &= \frac{Z_2(Z_1 - Z_2)}{(Z_1 + Z_2)^2} V. \end{aligned} \quad (2.8)$$

The voltage step in the output line generated at $3T\ell_1$ is

$$\begin{aligned} V_3 &= \frac{Z_3}{Z_3 + Z_2} \frac{2Z_2 + Z_1}{Z_2 + Z_1} \left(\frac{-Z_2}{Z_2 + Z_1} V \right) - \frac{Z_3}{Z_3 + Z_1} \frac{2Z_1 + Z_2}{Z_2 + Z_1} \frac{Z_2}{Z_2 + Z_1} V \\ &= \frac{-2Z_2}{Z_2 + Z_1} V. \end{aligned} \quad (2.9)$$

This step arrives at the load at time $t = 3T\ell_1 + T(\ell - \ell_1)$. Hence the load voltage is constant during the period $T\ell < t < T\ell + 2T\ell_1$, so that for constant voltage output the useful pulse length is $2T\ell_1$. In particular, we see by adding all steps in the lines, that if $Z_1 = Z_2$, the voltage goes to zero in lines 1 and 2 after $3T\ell_1$, and drops to zero

across the load at $t = T\ell + 2T\ell_1$. To see that in this situation the total energy stored in the line is delivered to the load, we note that the stored energy is

$$E_{st} = \frac{1}{2} \ell_1 C_1 V^2 + \frac{1}{2} \ell_1 C_2 V^2, \quad (2.10)$$

and the energy delivered to the load is

$$E_{load} = \frac{2T\ell_1 V^2}{Z_1 + Z_2}. \quad (2.11)$$

Now, $C_1 = T/Z_1$ and $C_2 = T/Z_2$. Defining the efficiency ϵ to be the ratio of the energy delivered to the energy stored, we have

$$\begin{aligned} \epsilon &= \frac{2T\ell_1 V^2 / (Z_1 + Z_2)}{\frac{1}{2} \ell_1 V^2 \left(\frac{T}{Z_1} + \frac{T}{Z_2} \right)} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} \\ &= 1 \text{ when } Z_1 = Z_2. \end{aligned} \quad (2.12)$$

We also note that ϵ is close to unity even for quite dissimilar lines (e.g., $\epsilon = 89\%$ for $Z_2/Z_1 = 2$). While multi-gap transmission-line accelerators based on various Blumlein configurations have been proposed,^{10,16} the geometry of these designs is extremely cumbersome. We turn now to a different type of transmission-line accelerator.

2.2 The Radial Pulse-Line Accelerator

Consider a series of radial pulse lines arranged as in Figure 2a. With the switches open, the center electrodes are charged to a voltage V . An electron beam sent down the accelerator tube will be alternately accelerated and decelerated with no net energy gain, as shown in Figure 2b. Suppose now that each set of switches, which are situated around the inner circumference of the cavity structure, is closed just before the electron beam reaches that gap. A voltage pulse will be generated within the cavity, but until this pulse has traveled to the open gap the electron beam will see only accelerating voltages and hence will gain energy as indicated in Figure 2c. There is unfortunately a basic objection to this conceptually simple design. The impedance of the radial pulse line varies as a function of the radial distance from the center. Hence any waves traveling in the line see a time-varying

¹⁶G. Yonas, T.H. Martin, and K.R. Prestwich, Sandia Laboratories, private communication.

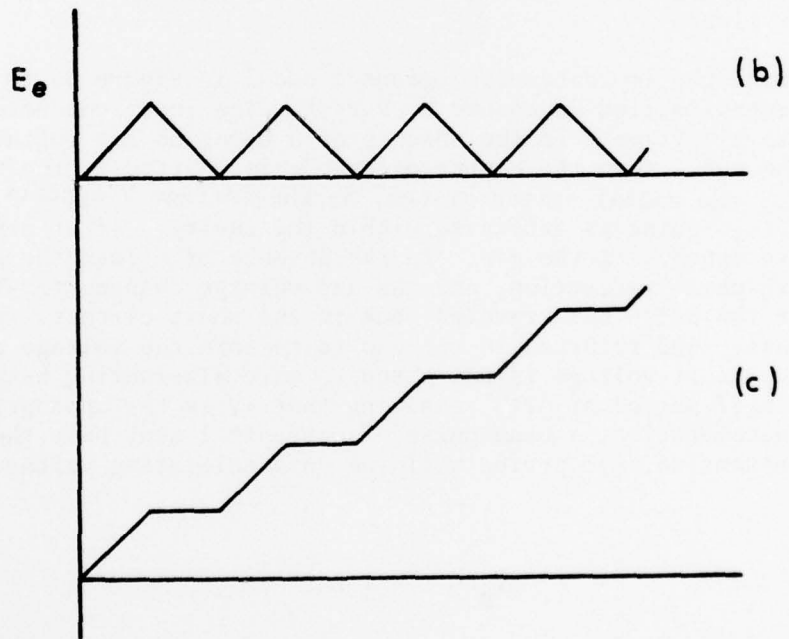
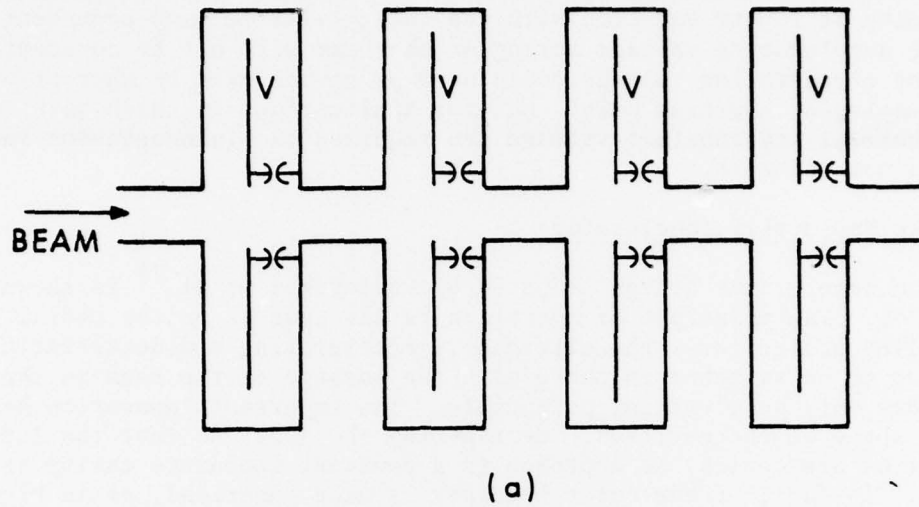


Figure 2. (a) Non-constant impedance radial pulse line accelerator. (b) Electron energy with switches open. (c) Electron energy with sequential closing of switches.

impedance and, in particular, the interaction of a uniform high-current beam pulse of finite duration with the cavity will be time-dependent and the net accelerating voltage acting on the beam will not be constant. A constant accelerating voltage could perhaps be achieved by appropriate time-shaping of the beam pulse, but for applications in which both constant current and constant voltage are required the inhomogeneous radial line is not suitable.

2.3 The Pavlovskii Accelerator

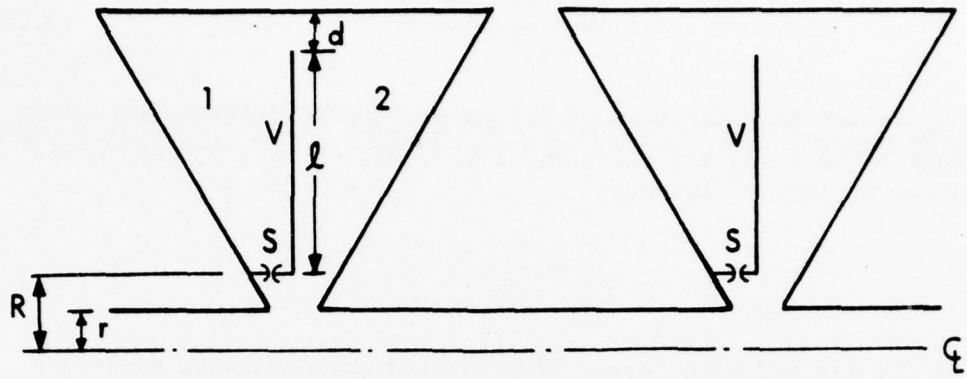
The accelerator design proposed by Pavlovskii et al.¹¹ is shown in Figure 3a. The principle of operation is the same as in the radial pulse-line accelerator - the alternately accelerating and decelerating gaps are to be switched in phase with the passage of the beam so that the beam sees only accelerating potentials. The important innovation here is the shape of the cavities. By tapering the sides so that the lateral boundaries are conical an approach to a constant-impedance cavity is obtained. In fact, if the outer boundary is made spherical, as in Figure 3b, rather than cylindrical, one arrives at a constant-impedance spherical reentrant cavity with a disc in the middle. In the following we treat the configuration of Figure 3a as being made up of constant-impedance lines.

We take the impedance of regions 1 and 2 in Figure 3a to be Z . For a first approximation we assume $d, R-r \ll \ell$. The inner conductor is charged to a voltage V in the absence of a beam; no net voltage appears across the gap. When the center electrode is shorted to the cavity side by closing the radial switch system, S , the voltage V appears at the gap and a voltage pulse is generated within the cavity. After a period $2T\ell$ this pulse appears at the gap. In the absence of a load the pulse undergoes an in-phase reflection, and the gap voltage changes to $-V$. A period $4T\ell$ later the pulse has traveled back to the short circuit, reflected out-of-phase, and returned to the gap to restore the voltage again to V . The open-circuit voltage is now a square wave alternating between $-V$ and V with a half period of $4T\ell$. Assuming that $+V$ is the appropriate polarity for acceleration, a beam pulse of current* I sent past the gap during a constant-voltage period will see an accelerating voltage V_g given by

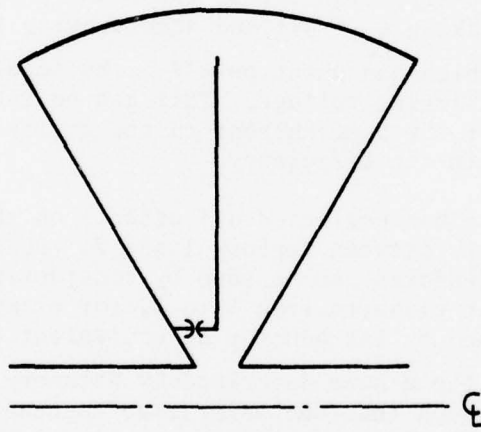
$$V_g = V - I Z , \quad (2.13)$$

where the polarity of the voltage step $I Z$ generated by the beam is always such as to oppose acceleration. Defining the efficiency as in Section 2.1, we obtain

*We assume throughout that the beam pulse consists of highly relativistic electrons, so that the velocity and therefore the current are independent of particle energy.



(a)



(b)

Figure 3. (a) Pavlovskii accelerator design.
 (b) Spherical-reentrant-cavity-like modification.

$$\epsilon = \frac{L_p I Z (V - IZ)}{T \ell V^2}, \quad (2.14)$$

where L_p is the time duration of the pulse. If we differentiate this with respect to I and set the derivative equal to zero we obtain the condition for maximum efficiency:

$$I = \frac{V}{2Z}. \quad (2.15)$$

This is the matched-load case. If a matched-current pulse of length $L_p = 2T\ell$ is sent past the gap during the period $0 \leq t \leq 2T\ell$, which corresponds to the first output voltage pulse, the accelerating voltage is

$$V_g = \frac{V}{2}, \quad (2.16)$$

and the efficiency of energy transfer to the beam is 50%. This can be increased to 100% by taking $L_p = 4T\ell$ and accelerating the beam during a later voltage pulse, which has duration $4T\ell$. The accelerating voltage is again half of the charging voltage. This can be increased only by a deliberate mis-match of the beam current to the cavity impedance, with a consequent decrease in the efficiency.

The above analysis has neglected all effects on the voltage pulse due to the discontinuity between regions 1 and 2. Possible complications to the accelerating waveforms can be seen by considering what happens to a step voltage as it transits from 1 to 2, for example. The discontinuity can be described by introducing an equivalent circuit in parallel with the lines in the manner described by Whinnery and others.¹⁷⁻¹⁹ However, for a design such that the major time constant of the equivalent circuit is less than $2Td$ an approximate analysis based on principal modes can be used. In this approximation a voltage step V in line 1 traveling toward the outer discontinuity will see line 2 in series with a line of impedance approximately $2Z$ which is shorted after an electrical length Td . We will call this line 3. When the voltage step arrives

¹⁷J.R. Whinnery and H.W. Jamieson, "Equivalent Circuits for Discontinuities in Transmission Lines," *Proc. IRE* 32, 98-115, 1944.

¹⁸J.R. Whinnery, H.W. Jamieson, and T.E. Robbins, "Coaxial-Line Discontinuities," *Proc. IRE* 32, 695-709, 1944.

¹⁹J.R. Whinnery and D.C. Stinson, "Radial Line Discontinuities," *Proc. IRE* 43, 46-51, 1955.

at the discontinuity it is partially reflected with $\rho = \frac{1}{2}$. A step of amplitude $\frac{V}{2}$ is transmitted into line 2, and a step of amplitude V is transmitted into line 3. This latter step is reflected out-of-phase at the shorted end and arrives back at the junction a time $2T_d$ later, where it is totally transmitted into lines 1 and 2 with amplitude $\frac{V}{2}$ in each line. The steps in lines 1 and 2 then appear as shown in Figure 4a, if the initial voltage on the center electrode is taken to be negative. Thus the effect of the discontinuity on the step transmitted into line 2 is essentially a deterioration of the risetime. These steps can now be followed through their subsequent history in an analogous manner, taking account of reflections at the ends of lines 1 and 2 and of partial reflections at the junction with line 3. The resulting open-circuit waveform at the output end of line 2 is shown in Figure 4b.

We see from the above analysis that periods of length $4T_l$ during which the open-circuit output voltage is constant at $\pm V$ still exist, at least in this approximate, idealized analysis, even when the discontinuity at the outer end of the line is taken into account. However, if we now try to achieve 100% efficiency by accelerating a pulse with $I = \frac{V}{2Z}$ during one of these periods, the accelerating voltage will be further affected by the discontinuity. To see this, we note that the arrival of the beam front at the gap causes a voltage step V_B to be sent up line 2. Assume for purposes of argument that the beam front arrives at the gap at $t = 2T_l + 2T_d$ which, according to Figure 4b, is the beginning of a period of $4T_l$ during which the open-circuit voltage is constant at V . For the matched-current case the beam-generated step is

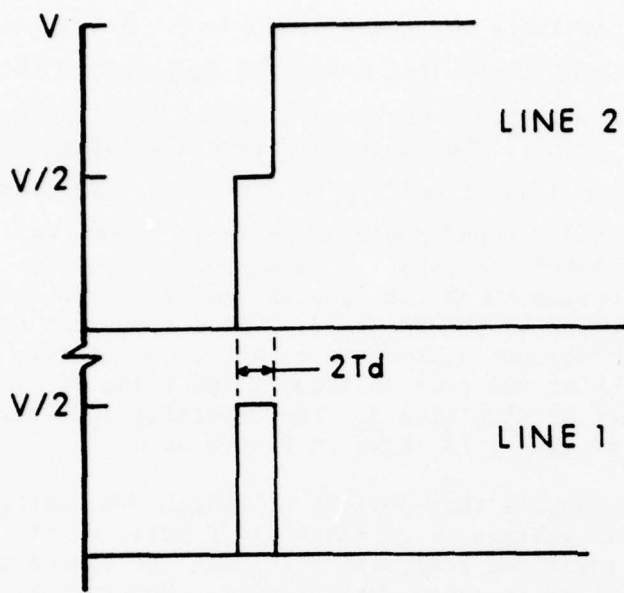
$$V_B = -\frac{V}{2} \quad (2.17)$$

Hence the accelerating voltage is

$$V_g = V - \frac{V}{2} = \frac{V}{2} \quad (2.18)$$

The step V_B is partially reflected at the junction with line 3 and returns down line 2 with

$$V_B' = -\frac{V}{4} \quad (2.19)$$



(a)

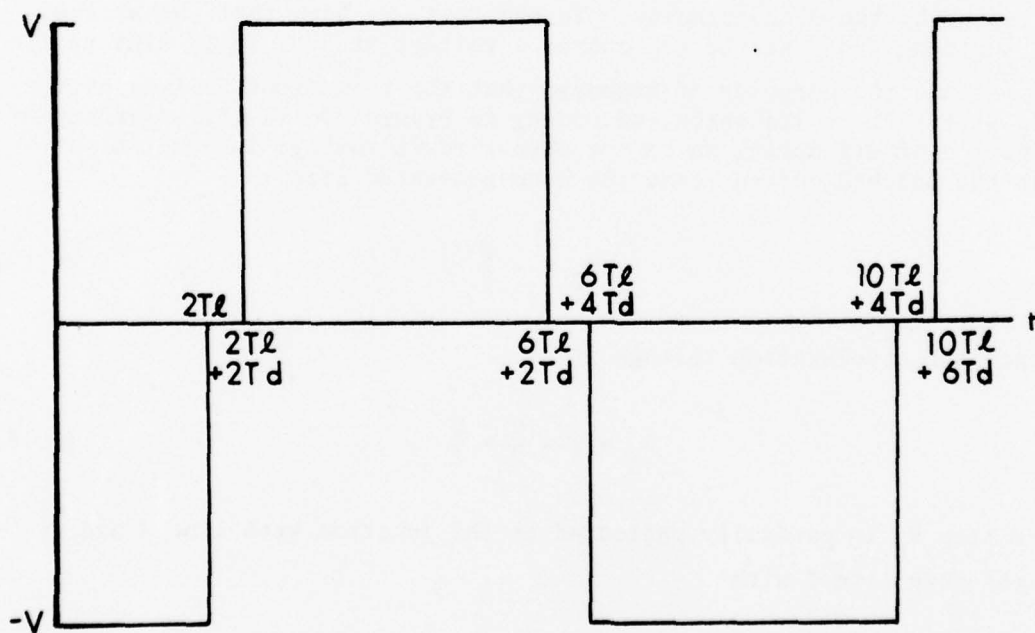


Figure 4. (a) Effect of coupling region on steps in lines 1 and 2. (b) Resulting open-circuit output voltage.

This step arrives back at the output at $t = 4T\ell + 2Td$, is reflected in-phase, and cancels out V_g . In the meantime the part of V_B which was transmitted into line 3 has reflected out-of-phase at the shorted end and been partially transmitted back down line 2. It arrives at the output at $t = 4T\ell + 4Td$ and restores the accelerating voltage to $\frac{V}{2}$. Hence the perturbation on the accelerating voltage due to the discontinuity at the outer circumference of the cavity is of short duration, but very severe.

While the above analysis is approximate and ideal, it strongly suggests that if one requires a constant accelerating voltage for a constant beam current the beam pulse length should be restricted to $2T\ell$. As has already been pointed out, the maximum efficiency possible in the Pavlovskii accelerator for this pulse length is 50%. We turn now to a detailed analysis of a design which overcomes this drawback and also permits accelerating voltages greater than one-half the charging voltage.

3. THE ASYMMETRIC-LINE ACCELERATOR

3.1 The Equivalent Circuit

Figure 5 shows examples of general line configurations in which regions 1 and 2 can have different but constant characteristic impedance. In this chapter, in which we treat the open-circuit behavior of such lines, we will ignore complications at the line junction. Arguments similar to those in Section 2.3 for the Pavlovskii accelerator indicate that the effect of the coupling region on the open-circuit output voltage will be a deterioration in the pulse risetime. We will return to this later. The basic accelerator operation is the same as before: the center electrode is originally charged to V_0 , and at $t = 0$ the switch system S shorts this electrode to the side of the cavity. In the approximation that $d, R-r, c \ll \ell$, the equivalent circuit is as shown in Figure 5c. For the line of Figure 2a the characteristic impedance is²⁰

$$Z_1 = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln \left(\frac{\tan \frac{\alpha}{2}}{\tan \frac{\beta}{2}} \right), \quad (3.1)$$

and similarly for Z_2 . For the coaxial line of Figure 5b we have²⁰

$$Z_1 = \frac{1}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \ln \frac{b}{a}, \quad (3.2)$$

²⁰I.A.D. Lewis and F.H. Wells, "Millimicrosecond Pulse Techniques," Pergamon Press, New York, 1959, p 367.

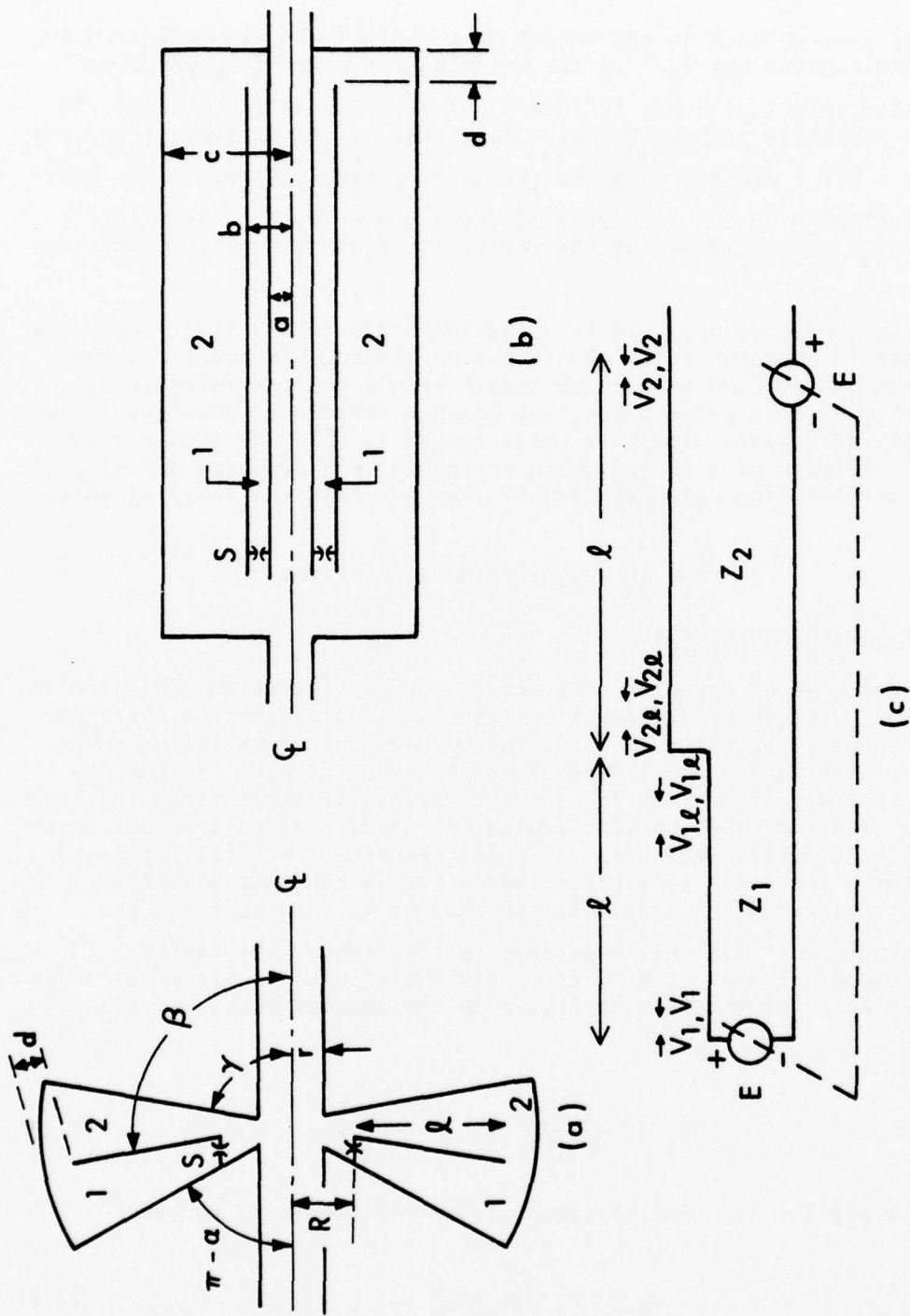


Figure 5. (a) Asymmetric radial line. (b) Asymmetric coaxial line. (c) Equivalent circuit for asymmetric-line cavities.

and similarly for Z_2 , where μ , ϵ are the magnetic permeability and permittivity of the cavity medium. Throughout we assume that all regions of the cavity contain the same dielectric. The special case considered in section 2.3 corresponds to Figure 5a with $\beta = 90^\circ$ and $\gamma = \pi - \alpha$.

We will analyze the equivalent circuit with the aid of Laplace transforms. The voltage waveforms at various points in the equivalent circuit are designated as in Figure 5c, where the arrows denote the direction of travel of the waves. We let the Laplace transform of V be \bar{V} and let the Laplace transform variable be s . We take ρ to be the reflection coefficient at the discontinuity for a wave traveling from 1 to 2:

$$\rho = (Z_2 - Z_1)/(Z_2 + Z_1) \quad . \quad (3.3)$$

The equivalent circuit generators set up the initial conditions, i.e., the generator in line 1 generates a voltage step $-V_0$ at $t = 0$ and the generator in line 2 applies a voltage V_0 across the (open-circuit) output at $t = 0$. We assume ideal switches so that the voltage waveforms are step functions and

$$\bar{E} = \frac{1}{s} V_0 \quad . \quad (3.4)$$

The transformed output voltage is given by

$$\bar{V} = \bar{E} + \bar{V}_2 \quad , \quad (3.5)$$

where

$$\bar{V}_2 = \overset{\rightarrow}{\bar{V}}_2 + \overset{\leftarrow}{\bar{V}}_2 \quad . \quad (3.6)$$

We have

$$\frac{\overset{\rightarrow}{\bar{V}}_1}{\overset{\leftarrow}{\bar{V}}_1} = -1 \quad , \quad \frac{\overset{\rightarrow}{\bar{V}}_2}{\overset{\leftarrow}{\bar{V}}_2} = 1 \quad , \quad \bar{V}_2 = \overset{\rightarrow}{2\bar{V}}_2 = \overset{\leftarrow}{2\bar{V}}_2 \quad . \quad (3.7)$$

Noting that the reflection coefficient for a wave traveling from 2 to 1 is just $-\rho$, and that the transmission coefficient is one plus the reflection coefficient, we can write

$$\overleftarrow{V}_{1\ell} = \rho \overrightarrow{V}_{1\ell} + (1 - \rho) \overleftarrow{V}_{2\ell} \quad (3.8)$$

and

$$\overrightarrow{V}_{2\ell} = -\rho \overleftarrow{V}_{2\ell} + (1 + \rho) \overrightarrow{V}_{1\ell} . \quad (3.9)$$

Furthermore, since the time for a wave to travel a line of length ℓ is $T\ell$, we have

$$\overrightarrow{V}_{1\ell} = e^{-sT\ell} (\overrightarrow{V}_1 - \overline{E}) , \quad (3.10)$$

$$\overleftarrow{V}_{1\ell} = \overleftarrow{V}_1 e^{sT\ell} , \quad (3.11)$$

$$\overrightarrow{V}_2 = \overrightarrow{V}_{2\ell} e^{-sT\ell} , \quad (3.12)$$

$$\overleftarrow{V}_2 = \overleftarrow{V}_{2\ell} e^{sT\ell} . \quad (3.13)$$

From (3.7), (3.10), and (3.11) we get

$$\overrightarrow{V}_{1\ell} + \overleftarrow{V}_{1\ell} e^{-2sT\ell} = -\overline{E} e^{-sT\ell} . \quad (3.14)$$

From (3.8), (3.9) we obtain

$$\overrightarrow{V}_{1\ell} = \frac{1}{1+\rho} [\rho \overleftarrow{V}_{2\ell} + \overrightarrow{V}_{2\ell}] , \quad (3.15)$$

$$\overleftarrow{V}_{1\ell} = \frac{1}{1+\rho} [\rho \overrightarrow{V}_{2\ell} + \overleftarrow{V}_{2\ell}] . \quad (3.16)$$

Substituting (3.15) and (3.16) into (3.14) gives

$$\overleftarrow{V}_{2\ell} (\rho + e^{-2sT\ell}) + \overrightarrow{V}_{2\ell} (1 + \rho e^{-2sT\ell}) = - (1 + \rho) \overline{E} e^{-sT\ell} . \quad (3.17)$$

Using (3.12) and (3.13) in (3.17) we obtain

$$\overline{V}_2 e^{-sT\ell} (\rho + e^{-2sT\ell}) + \overline{V}_2 (1 + \rho e^{-2sT\ell}) = -(1 + \rho) \overline{E} e^{-sT\ell}. \quad (3.18)$$

From (3.7) and (3.18), then

$$\overline{V}_2 = \frac{-2(1 + \rho) \overline{E} e^{-2sT\ell}}{1 + e^{-4sT\ell} + 2\rho e^{-2sT\ell}}. \quad (3.19)$$

Using (3.4), (3.5), and (3.19) we obtain for the transformed output voltage

$$\overline{V} = \frac{V_0}{s} - 2V_0(1 + \rho) \frac{e^{-2sT\ell}}{s} \frac{1}{1 + e^{-4sT\ell} + 2\rho e^{-2sT\ell}}. \quad (3.20)$$

3.2 The Open-Circuit Output Voltage

To obtain the inverse transform we rewrite (3.20) in the form

$$\overline{V} = \frac{V_0}{s} - V_0(1 + \rho) e^{-2sT\ell} \left[\frac{1}{s e^{-2sT\ell} (\cosh 2sT\ell + \rho)} \right], \quad (3.21)$$

The poles of the second term in (3.21) are

$$s_n = 0, \pm \frac{i}{2T\ell} (2n\pi + \theta), \quad n = \dots, -2, -1, 0, 1, 2, \dots, \quad (3.22)$$

where* $\theta = \arccos(-\rho)$. These poles are all distinct. Setting $D(s) = s e^{-2sT\ell} (\cosh 2sT\ell + \rho)$, we obtain for the derivative

$$\begin{aligned} D'(s) &= e^{-2sT\ell} (\cosh 2sT\ell + \rho) - 2sT\ell e^{-2sT\ell} (\cosh 2sT\ell + \rho) \\ &\quad + 2sT\ell e^{-2sT\ell} \sinh 2sT\ell. \end{aligned} \quad (3.23)$$

Evaluating (3.23) at the poles gives

$$D'(0) = 1 + \rho$$

$$D' \left[\pm \frac{i}{2T\ell} (2n\pi + \theta) \right] = - (2n\pi + \theta) \sin \theta [\cos \theta \mp i \sin \theta].$$

* We use the notation $\arccos x$ to mean the principal value of $\cos^{-1} x$.

Hence the inverse transform of (3.21) is²¹

$$\begin{aligned}
 V &= V_0 u(t) - V_0 (1+\rho) u(t-2T\ell) \left\{ \frac{1}{1+\rho} + \sum_{n=-\infty}^{\infty} \frac{e^{i(2n\pi+\theta)\left(\frac{t-2T\ell}{2T\ell}\right)}}{-(2n\pi+\theta)\sin\theta[\cos\theta - i\sin\theta]} \right. \\
 &\quad \left. + \sum_{n=-\infty}^{\infty} \frac{e^{-i(2n\pi+\theta)\left(\frac{t-2T\ell}{2T\ell}\right)}}{-(2n\pi+\theta)\sin\theta[\cos\theta + i\sin\theta]} \right\} \\
 &= V_0 u(t) - V_0 u(t-2T\ell) \times \\
 &\quad \times \left\{ 1 + (1+\rho) \left[2 \sum_{n=-\infty}^{\infty} \frac{\cos\theta \cos\left[\frac{n\pi(t-2T\ell)}{2T\ell} + \frac{\theta(t-2T\ell)}{2T\ell}\right]}{-(2n\pi+\theta)\sin\theta} \right. \right. \\
 &\quad \left. \left. + 2 \sum_{n=-\infty}^{\infty} \frac{-\sin\theta \sin\left[\frac{n\pi(t-2T\ell)}{2T\ell} + \frac{\theta(t-2T\ell)}{2T\ell}\right]}{-(2n\pi+\theta)\sin\theta} \right] \right\}, \quad (3.24)
 \end{aligned}$$

where the unit step function is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}. \quad (3.25)$$

Noting that $\cos\theta = -\rho$, $\sin\theta = \sqrt{1-\rho^2}$, we can rewrite (3.24) in the form

$$\begin{aligned}
 V &= V_0 u(t) - V_0 u(t-2T\ell) \times \\
 &\quad \times \left\{ 1 + 2\sqrt{\frac{1+\rho}{1-\rho}} \left[\rho \frac{(t-2T\ell)}{2T\ell} \sum_{n=-\infty}^{\infty} \frac{\cos\left[\frac{n\pi(t-2T\ell)}{T\ell} + \frac{\theta(t-2T\ell)}{2T\ell}\right]}{\frac{n\pi(t-2T\ell)}{T\ell} + \frac{\theta(t-2T\ell)}{2T\ell}} \right. \right. \\
 &\quad \left. \left. + \sqrt{1-\rho^2} \frac{(t-2T\ell)}{2T\ell} \sum_{n=-\infty}^{\infty} \frac{\sin\left[\frac{n\pi(t-2T\ell)}{T\ell} + \frac{\theta(t-2T\ell)}{2T\ell}\right]}{\frac{n\pi(t-2T\ell)}{T\ell} + \frac{\theta(t-2T\ell)}{2T\ell}} \right] \right\}. \quad (3.26)
 \end{aligned}$$

²¹R.V. Churchill, "Modern Operational Mathematics in Engineering," McGraw-Hill Book Co., Inc., New York and London, 1944, p. 170.

The summations in (3.26) can be evaluated explicitly²² to give

$$V = V_0 u(t) - V_0 u(t - 2T\ell) \times \left\{ 1 + \sqrt{\frac{1+\rho}{1-\rho}} \left[\rho \frac{\cos(2k+1)\frac{\theta}{2}}{\sin \frac{\theta}{2}} + \sqrt{1-\rho^2} \frac{\sin(2k+1)\frac{\theta}{2}}{\sin \frac{\theta}{2}} \right] \right\}, \quad 2kT\ell < t - 2T\ell < 2(k+1)T\ell. \quad (3.27)$$

We now define $x = \arccos \rho$, so that $x = \pi - \theta$, and substitute into (3.27) to obtain

$$\begin{aligned} V &= V_0 u(t) - V_0 u(t - 2T\ell) \\ &\times \left\{ 1 + \sqrt{\frac{1+\rho}{1-\rho}} \left[\cos x \frac{(-1)^k \sin(2k+1)\frac{x}{2}}{\cos \frac{x}{2}} + \sin x \frac{(-1)^k \cos(2k+1)\frac{x}{2}}{\cos \frac{x}{2}} \right] \right\} \\ &= V_0 u(t) - V_0 u(t - 2T\ell) \\ &\times \left\{ 1 + (-1)^k \left[\frac{\sin(2k+3)\frac{x}{2}}{\sin \frac{x}{2}} \right] \right\}, \quad 2(k+1)T\ell < t < 2(k+2)T\ell. \quad (3.28) \end{aligned}$$

We now define

$$V^{(m)} = V(t), \quad 2mT\ell < t < 2(m+1)T\ell$$

and obtain from (3.28)

$$\begin{aligned} V^{(0)} &= V_0 \\ V^{(m)} &= -V_0 (-1)^{m-1} \frac{\sin(2m+1)\frac{x}{2}}{\sin \frac{x}{2}}, \quad m \geq 1, \end{aligned}$$

which is the same as

²²V. Mangulis, "Handbook of Series for Scientists and Engineers," Academic Press Inc., New York 1965, p. 102.

$$V^{(m)} = (-1)^m V_0 \frac{\sin(2m+1)\frac{x}{2}}{\sin \frac{x}{2}}, \quad m \geq 0. \quad (3.29)$$

Expression (3.29) is our final expression for the open-circuit output voltage. An alternate derivation, which is somewhat longer but perhaps retains more physical insight, is given in the Appendix.

We see from the form of (3.29) that, if the impedances in the asymmetric line are chosen so that x is a rational multiple of π , then the open-circuit output voltage will be repetitive. To determine the period of this repetition, we note that a given sequence of voltage pulses will repeat after some number k of intervals of $2T\ell$ if there exists a value of k such that $V^{(m+k)} = V^{(m)}$ for all m , that is, if

$$(-1)^{m+k} \sin\left(\frac{2m+2k+1}{2}x\right) = (-1)^m \sin\left(\frac{2m+1}{2}x\right)$$

Put $x = \frac{p}{q}\pi$, where p and q are integers with no common factors. Then we have

$$\begin{aligned} & (-1)^{m+k} \sin\left(\frac{2m+2k+1}{2}\frac{p}{q}\pi\right) \\ &= (-1)^{m+k} \left[\sin\frac{2m+1}{2}\frac{p}{q}\pi \cos\frac{kp}{q}\pi + \cos\frac{2m+1}{2}\frac{p}{q}\pi \sin\frac{kp}{q}\pi \right]. \end{aligned}$$

If $\frac{kp}{q}$ is an integer, the second term in the above expression vanishes. The smallest value of k for which this occurs is $k = q$. If p and $q = k$ are both odd, the first term is $(-1)^m \sin\frac{2m+1}{2}\frac{p}{q}\pi$, and $V^{(m+q)} = V^{(m)}$. If either p or $q = k$ is even, the first term is $-(-1)^m \sin\frac{2m+1}{2}\frac{p}{q}\pi$. In this latter case the smallest value of k for which the pattern repeats is $k = 2q$. Therefore the voltage patterns repeat after q intervals of $2T\ell$ if $x = \frac{p}{q}\pi$ where p and q are both odd, and after $2q$ intervals if $x = \frac{p}{q}\pi$ and either p or q is even.

It is also easy to show that these repetitive voltage patterns contain intervals of $4T\ell$ for which the open-circuit output voltage remains constant. For this to be true we must have

$$V^{(k-1)} = V^{(k)},$$

or

$$(-1)^{k-1} \sin \left(\frac{2k-1}{2} \right) x = (-1)^k \sin \left(\frac{2k+1}{2} \right) x$$

for some value of $k > 0$. Taking $x = \frac{p}{q} \pi$ and letting n be any positive integer, we observe that

$$(-1)^{nq-1} \sin \left(\frac{2nq-1}{2} \right) \frac{p}{q} \pi = (-1)^{nq} \cos nq\pi \sin \frac{p}{q} \frac{\pi}{2}$$

and

$$(-1)^{nq} \sin \left(\frac{2nq+1}{2} \right) \frac{p}{q} \pi = (-1)^{nq} \cos nq\pi \sin \frac{p}{q} \frac{\pi}{2} .$$

Therefore the open-circuit voltage remains constant for all intervals of $4T\ell$ given by $(nq-1)2T\ell < t < (nq+1)2T\ell$. Furthermore

$$\begin{aligned} v^{(nq)} &= (-1)^{nq} V_0 \frac{\sin \left(\frac{2nq+1}{2} \right) \frac{p}{q} \pi}{\sin \frac{p}{q} \frac{\pi}{2}} \\ &= (-1)^{nq} V_0 \cos nq\pi = (-1)^{n(p+q)} V_0 . \end{aligned}$$

That is, the open-circuit voltage during the double-length intervals is $\pm V_0$. If p and q are both odd the voltage is $+V_0$. If either p or q is even the voltage alternates between $+V_0$ and $-V_0$ in successive double-length intervals. Indeed, in this latter case we have in general

$$\begin{aligned} v^{(m+q)} &= (-1)^{m+q} V_0 \frac{\sin \left(\frac{2m+1}{2} + q \right) \frac{p}{q} \pi}{\sin \frac{p}{q} \frac{\pi}{2}} \\ &= (-1)^{m+q} V_0 \frac{\cos p\pi \sin \frac{2m+1}{2} \frac{p}{q} \pi}{\sin \frac{p}{q} \frac{\pi}{2}} \\ &= -v^{(m)} . \end{aligned}$$

The idealized Pavlovskii line treated in Section 2.3 can serve as a simple illustration of these general results. For this case we have $\rho = 0$, $x = \frac{1}{2} \pi = \frac{p}{q} \pi$. The voltage pattern repeats every $2q=4$ intervals of $2T\ell$. During the last 2 intervals of each half-period (which in this case constitutes the total half-period) the voltage is constant, alternating between $+V_0$ and $-V_0$ in successive half-periods. The open-circuit voltage patterns for this case and for the simple asymmetric line cases $\rho = \frac{1}{2}$, $x = \frac{\pi}{3}$ and $\rho = -\frac{1}{2}$, $x = \frac{2\pi}{3}$ are shown in Figure 6.

To summarize the results on the open-circuit line pair, then, we have shown that the output voltage is

$$V^{(m)} = (-1)^m V_0 \frac{\sin \left(\frac{2m+1}{2} x \right)}{\sin \frac{x}{2}}, \quad x = \arccos \rho. \quad (3.29)$$

If $x = \frac{p}{q} \pi$, p and q integers with no common factors, the voltage is a repetitive pattern with period $2qT\ell$ if p and q are both odd and period $4qT\ell$ if either p or q is even. For the repeating patterns the voltage remains constant for intervals of $4T\ell$ given by $(nq - 1)2T\ell < t < (nq + 1)2T\ell$, and is equal to $+V_0$ if p and q are both odd. If either p or q is even it alternates between $+V_0$ and $-V_0$ in successive half-periods. In this latter case we have $V^{(m+q)} = -V^{(m)}$.

4. ENERGY TRANSFER TO THE LOAD

4.1 Maximum Efficiency

In the principal mode approximation considered in Section 3 an axially symmetric constant-current beam of short duration ($< 2T\ell$) will couple resistively to the line of impedance Z_2 through image currents in the conducting walls of the structure.¹³ If the beam pulse is timed so that it passes the line opening during the interval $2mT\ell < t < 2(m+1)T\ell$, the net accelerating voltage at the gap, V_g , is

$$V_g = V^{(m)} - I Z_2, \quad (4.1)$$

where I is the beam current, equal in magnitude to the current induced in the walls. We have implicitly assumed the convention that $V^{(m)}$ and

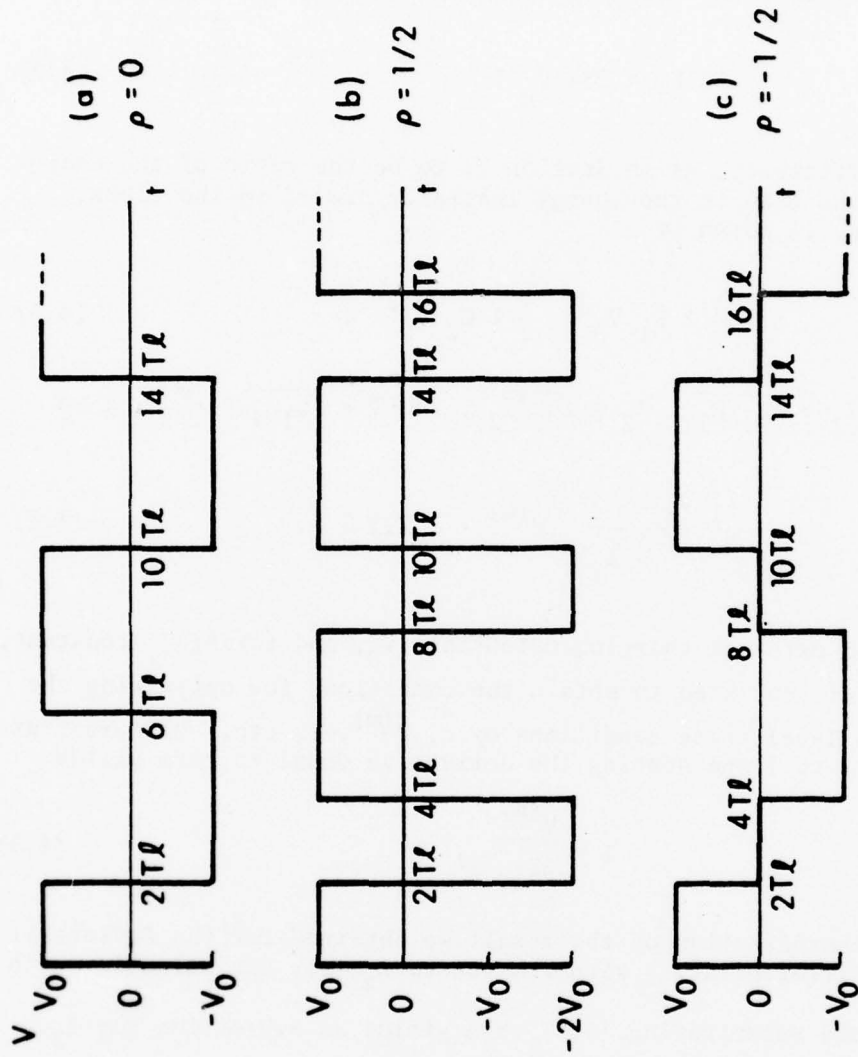


Figure 6. Open-circuit output voltage for several line-impedance pairs. (a) $Z_2 = Z_1$. (b) $Z_2 = 3Z_1$. (c) $Z_2 = Z_1/3$.

I have the same sign for the case of acceleration. If the length of the beam pulse is $2T\ell$, the energy gained by the beam at the gap is

$$E_g = 2T\ell V_g I . \quad (4.2)$$

We define the efficiency, as in Section 2, to be the ratio of the energy transferred to the beam to the energy initially stored in the lines. The stored energy is given by

$$E = \frac{1}{2} \ell C_1 V_0^2 + \frac{1}{2} \ell C_2 V_0^2 . \quad (4.3)$$

Recalling that $Z_1 = \sqrt{L_1/C_1}$, $Z_2 = \sqrt{L_2/C_2}$, and $T = \sqrt{L_1 C_1} = \sqrt{L_2 C_2}$, we obtain

$$\epsilon = \frac{4 Z_1 Z_2}{V_0^2 (Z_1 + Z_2)} (V^{(m)} - I Z_2) I . \quad (4.4)$$

For a fixed external charging potential, V_0 , and taking Z_1 constant, (4.4) can be differentiated to obtain the conditions for optimizing the efficiency. We label these conditions by $\hat{\epsilon}$, $\hat{V}^{(m)}$, \hat{I} , etc. Differentiating with respect to I and setting the derivative equal to zero yields

$$\hat{I} = \frac{\hat{V}^{(m)}}{2Z_2} , \quad (4.5)$$

which is the generalization of the result we obtained for the Pavlovskii accelerator. Differentiating with respect to Z_2 (or, equivalently, with

respect to ρ) and substituting (4.5) then yields an expression for \hat{Z}_2 .

It is somewhat less laborious, however, simply to note that the largest value the efficiency can possibly have is 1. For $m = 0$ this requires $Z_2 = 0$, a physically unrealizable case. In the following we take $m > 0$.

If we substitute (4.5) into (4.4) and set $\epsilon = 1$, the solutions (if any) will clearly yield the conditions for maximum efficiency. Making the substitutions gives

$$\begin{aligned}\hat{\epsilon} &= \frac{4 Z_1 \hat{Z}_2}{V_0^2 (Z_1 + \hat{Z}_2)} \left(\hat{V}^{(m)} - \frac{\hat{V}^{(m)}}{2\hat{Z}_2} \hat{Z}_2 \right) \frac{\hat{V}^{(m)}}{2\hat{Z}_2} = \\ &= \frac{\hat{V}^{(m)^2}}{V_0^2} \frac{Z_1}{Z_1 + \hat{Z}_2} = \frac{\hat{V}^{(m)^2}}{2V_0^2} (1 - \hat{\rho}) .\end{aligned}\quad (4.6)$$

Putting $\hat{\epsilon} = 1$ then yields

$$\hat{V}^{(m)} = \pm V_0 \left(\frac{2}{1-\hat{\rho}} \right)^{1/2} .\quad (4.7)$$

We now use (3.29) in (4.7) to get

$$\begin{aligned}(-1)^m V_0 \frac{\sin \left(\frac{2m+1}{2} \hat{x} \right)}{\sin \frac{\hat{x}}{2}} &= \pm V_0 \left(\frac{2}{1 - \cos \hat{x}} \right)^{1/2} = \pm \frac{V_0}{\sin \frac{\hat{x}}{2}} , \\ (-1)^m \sin \left(\frac{2m+1}{2} \hat{x} \right) &= \pm 1 .\end{aligned}$$

Therefore

$$\hat{x} = \frac{2j+1}{2m+1} \pi , \quad j = 0, 1, \dots, m-1 ; m > 0 .\quad (4.8)$$

Hence we see that for $m > 0$ it is always possible to choose an impedance ratio such that all the energy stored in the lines will be transferred to a matched-current pulse of length $2T\ell$ during the period $2mT\ell < t < 2(m+1)T\ell$. In fact, there are m different ways in which this impedance ratio may be chosen. We note in passing that the values of \hat{x} are the same as those which result in an open-circuit output voltage pattern which repeats with period $(2m+1)2T\ell$. The beam may therefore in principle be timed to pass the accelerating gap during any period of $2T\ell$ beginning at $[m+n(2m+1)]2T\ell$, where n is zero or a positive integer. We further note that using (4.7) in (4.1) gives

$$|\hat{V}_g| = \frac{|V_0|}{2} \left(\frac{2}{1-\hat{\rho}} \right)^{1/2} > \frac{|V_0|}{2} .\quad (4.9)$$

That is, the accelerating voltage in the maximum-efficiency case is always greater than the accelerating voltage in the Pavlovskii accelerator.

The conditions which maximize the efficiency are shown in Table I for $m = 1, 2,$ and 3 . We see that for the $m = 1$ case, with $\hat{\rho} = \frac{1}{2}$, the system operation is similar to a symmetric Blumlein line in transferring all stored energy to a matched load at a voltage across the load of V_0 . For other values of m and $\hat{\rho}$, accelerating voltages in excess of V_0 can be achieved while still maintaining $\epsilon = 1$. The impedance ratios required in these cases are large, however, and such configurations might be very difficult to achieve in practice.

We remind the reader at this point that the effect on the output voltage of the coupling region between the two lines has been neglected in our analysis. The effects of the coupling region can best be determined experimentally or through computer analysis of the equivalent circuit using lumped circuit elements¹⁷⁻¹⁹ to simulate the coupling region. To gain a qualitative understanding, however, we can approximate the coupling region as a shorted transmission line of impedance $Z_1 + Z_2$ and electrical length Td , and trace a voltage step through the system as we did in Section 2.3 for the symmetric lines. The results of this analysis for $x = \frac{\pi}{3}$ and $x = \frac{5}{7}\pi$ are shown in Figure 7, where we compare the open-circuit output voltage obtained for $d = 0.05\ell$ with that obtained when the coupling region is completely neglected ($d=0$). We see that additional short pulses arising from reflections in the coupling region are superimposed on the waveform at intervals of $2T\ell$. Except for the double-length pulses, where small pulses occur at the mid-point, the net effect is that the rise- or fall-time of the waveform at $t = 2mT\ell$ is $2mTd$ instead of zero. We therefore expect that in a real cavity the effect of the coupling region will be to cause rise-time deterioration which will become increasingly severe in successive periods of $2T\ell$. The constant-voltage pulses will therefore be shorter than $2T\ell$, and the length of the beam pulse to be accelerated must be shortened accordingly. The efficiency will be correspondingly reduced.

4.2 The Double-Length Pulse

In our initial discussion of the Pavlovskii accelerator in Section 2.3 we showed that, neglecting perturbations caused by the discontinuity at the outer circumference of the line, 100% efficiency for energy transfer from the cavity to the beam can be achieved by accelerating a pulse of length $4T\ell$ during an interval of that length for which the open-circuit voltage is constant. In Section 3.2 we showed that there are also asymmetric line configurations in which the voltage remains constant for an interval of $4T\ell$. In Section 4.1 we showed that some of these configurations (namely, those in which the open-circuit voltage

TABLE I. Conditions for maximum efficiency ($\hat{\rho}$, etc.) and maximum energy transfer ($\tilde{\rho}$, etc.) to a beam load.

	Reflection Coefficient	Impedance Ratio	Gap Voltage	Efficiency	Beam Current	Energy Transfer
	$\hat{\rho}$	$\frac{\hat{Z}_2}{Z_1}$	$\frac{\hat{V}_g}{V_0}$	$\hat{\epsilon}$	$\frac{\hat{I}}{V_0/Z_1}$	$\frac{\hat{V}_g \hat{I}}{V_0^2/Z_1}$
	$\tilde{\rho}$	$\frac{\tilde{Z}_2}{Z_1}$	$\frac{\tilde{V}_g}{V_0}$	$\tilde{\epsilon}$	$\frac{\tilde{I}}{V_0/Z_1}$	$\frac{\tilde{V}_g \tilde{I}}{V_0^2/Z_1}$
m=1	0.5	3	-1.0	100%	-0.333	0.333
	0.366	2.16	-0.866	95%	-0.402	0.348
m=2	0.809	9.47	1.62	100%	0.171	0.276
(a)	0.794	8.65	1.55	~100%	0.180	0.280
(b)	-0.309	0.528	-0.618	100%	-1.17	-1.40
	-0.423	0.406	-0.565	91%	-1.40	0.724
m=3	0.901	19.2	-2.25	100%	-0.117	-0.120
(a)	0.897	18.4	-2.21	~100%	-0.117	0.264
(b)	0.222	1.57	0.806	100%	0.504	0.406
(c)	-0.623	0.232	-0.570	100%	-2.46	-2.81
	-0.693	0.181	-0.510	88%	-2.46	1.40
						1.43

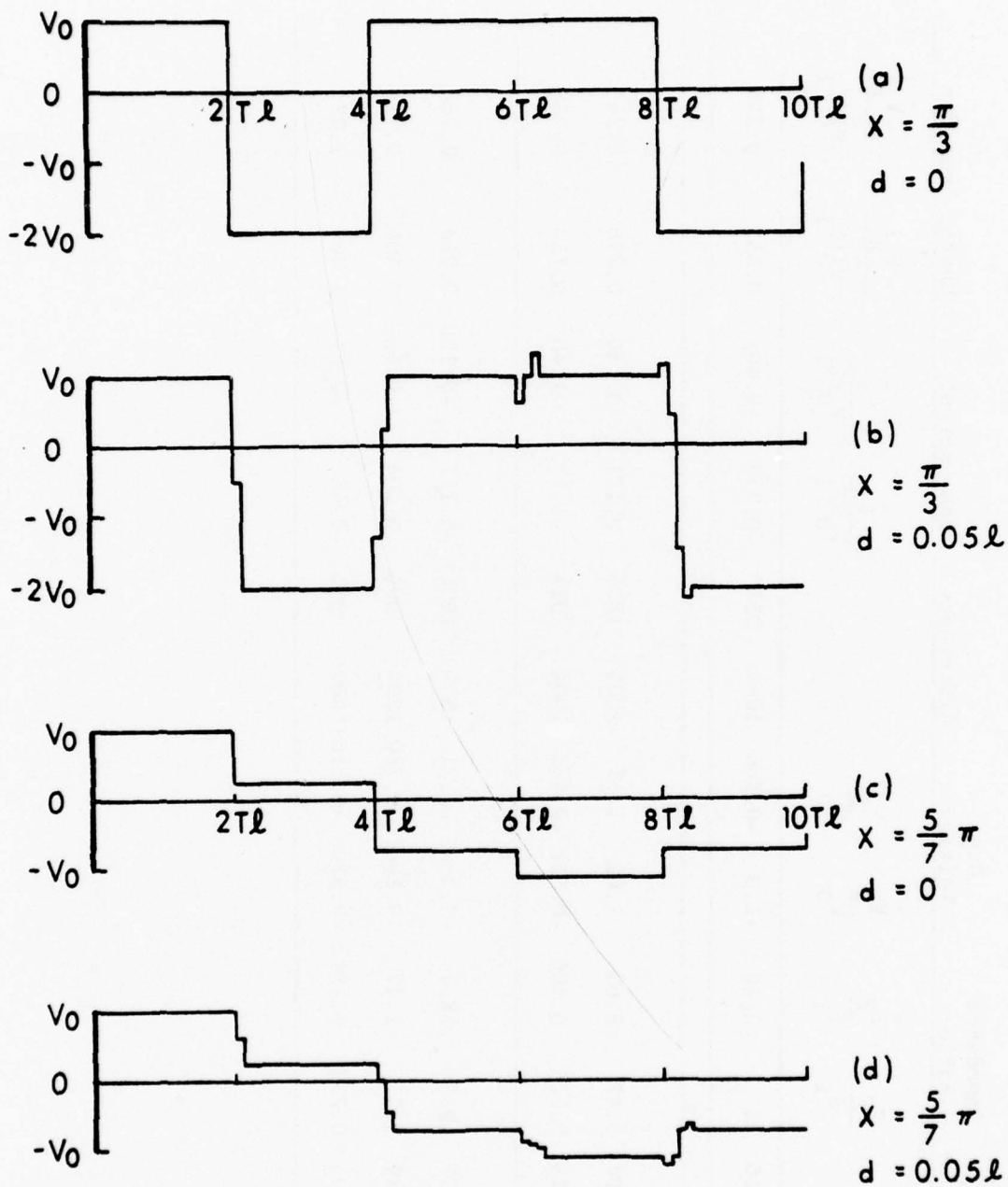


Figure 7. Effect of coupling region on open-circuit output voltage. (a) $x = \pi/3$, coupling region neglected. (b) $x = \pi/3$, coupling region approximated by transmission line of length $d = 0.05 l$. (c) $x = 5\pi/7$, coupling region neglected. (d) $x = 5\pi/7$, coupling region approximated by transmission line of length $d = 0.05 l$.

is always $+V_0$ during the interval $4T\ell$ for which the voltage is constant) correspond to 100% efficiency for energy transfer to a properly-matched, appropriately-phased beam pulse of length $2T\ell$. It is of some interest to ask what happens if an accelerator designed for $\epsilon = 1$ in the case of a pulse of length $2T\ell$ is used to accelerate a constant-current pulse of length $4T\ell$ during one of the double-length voltage intervals.

Since we are interested in a total time of length $4T\ell$, it is easiest to analyze the situation by following the reflections and transmissions of voltage steps in the lines. As in our initial analysis of the Pavlovskii accelerator, we will neglect the complications caused by the region where the lines are coupled together. We denote by V_{g1} the accelerating voltage during the first half of the beam pulse and by V_{g2} the accelerating voltage during the second half. We know that the open-circuit voltage is V_0 . When the front of the beam pulse passes the gap, a voltage step $-IZ_2$ is sent up line 2. As before, we use a caret to designate parameters corresponding to a design which gives unit efficiency for a pulse of length $2T\ell$. We therefore have

$$V_{g1} = V_0 - I \hat{Z}_2 . \quad (4.10)$$

A time $T\ell$ later this step reaches the line junction and is reflected with reflection coefficient

$$\rho' = \frac{Z_1 - \hat{Z}_2}{Z_1 + \hat{Z}_2} = -\hat{\rho} . \quad (4.11)$$

At time $2T\ell$ the reflected step arrives back at the output end and is reflected in-phase, changing the accelerating voltage to

$$V_{g2} = V_0 - I\hat{Z}_2 + 2\hat{\rho}I\hat{Z}_2 . \quad (4.12)$$

We see that unless I is negligibly small, which is not a case of interest, the accelerating voltage is not constant for the duration of the pulse, but rather changes abruptly at the midpoint.

While for most applications this is not acceptable, we will for the sake of completeness note some efficiency formulas for this case. We have

$$\epsilon = \frac{4 Z_1 \hat{Z}_2}{V_0^2 (Z_1 + \hat{Z}_2)} I (V_{g1} + V_{g2}) . \quad (4.13)$$

If we take

$$I = \frac{V_0}{2\hat{Z}_2} , \quad (4.14)$$

which is optimum for the case of pulse length $2Tl$, we obtain

$$\epsilon = \frac{2\hat{Z}_1}{\hat{Z}_1 + \hat{Z}_2} (1 + \hat{\rho}) = (1 - \hat{\rho})(1 + \hat{\rho}) = 1 - \hat{\rho}^2 . \quad (4.15)$$

For this situation the efficiency is less than 1. It is possible, however, to choose an I such that $\epsilon = 1$. We demand

$$\frac{4\hat{Z}_1\hat{Z}_2}{V_0^2(\hat{Z}_1 + \hat{Z}_2)} I [2(V_0 - I\hat{Z}_2) + 2\hat{\rho}I\hat{Z}_2] = 1 . \quad (4.16)$$

Solving (4.16) for I we obtain

$$I = \frac{V_0}{2\hat{Z}_2(1 - \hat{\rho})} . \quad (4.17)$$

The accelerating voltages for this case are

$$V_{g1} = \frac{V_0}{2} \frac{1 - 2\hat{\rho}}{1 - \hat{\rho}} ,$$

$$V_{g2} = \frac{V_0}{2} \frac{1}{1 - \hat{\rho}} . \quad (4.18)$$

4.3 Maximum Energy Transfer

Although it seems paradoxical at first, it is possible to transfer more total energy to a pulse of length $2Tl$ than is transferred in an accelerator designed for $\epsilon = 1$, while still keeping the same charging voltage V_0 . The penalty for doing this is that even more energy must be stored than is transferred to the pulse, so that $\epsilon < 1$. To determine the conditions for maximum energy transfer to the beam, which we denote \tilde{I} , $\tilde{V}^{(m)}$, etc., we use the expression (4.2):

$$E_g = 2Tl V_g I . \quad (4.2)$$

Writing this in terms of Z_1 and ρ , we obtain

$$E_g = 2T\lambda I \left[V^{(m)}(\rho) - IZ_1 \frac{1+\rho}{1-\rho} \right] \quad (4.19)$$

Taking the derivative with respect to I and setting equal to zero gives

$$\tilde{I} = \frac{\tilde{V}^{(m)}}{2Z_2}, \quad (4.20)$$

in analogy to the case for maximum efficiency. Unlike the maximum efficiency treatment, however, there is no way to derive general expressions which completely determine the optimum conditions without taking the derivative with respect to ρ (or Z_2). Setting $\frac{\partial E_g}{\partial \rho} = 0$ yields

$$\frac{\partial V^{(m)}(\rho)}{\partial \rho} - \frac{2I Z_1}{(1-\rho)^2} = 0. \quad (4.21)$$

Substituting (4.18) into (4.19) we obtain

$$(1 + \tilde{\rho})(1 - \tilde{\rho}) \left. \frac{\partial V^{(m)}(\rho)}{\partial \rho} \right|_{\rho = \tilde{\rho}} = V^{(m)}(\tilde{\rho}) \quad (4.22)$$

as the equation which specifies $\tilde{\rho}$. Referring to (3.29), we see that (4.22) is a trigonometric equation in $x = \arccos \rho$, which must in general be solved numerically. For $m=1$, however, we have

$$\begin{aligned} V^{(1)} &= -V_0 \frac{\sin \frac{3x}{2}}{\sin \frac{x}{2}} \\ &= -V_0 \frac{\sin x \cos \frac{x}{2} + \cos x \sin \frac{x}{2}}{\sin \frac{x}{2}} \\ &= -V_0 \left[2 \left(\frac{1+\cos x}{2} \right) + \cos x \right] \\ &= -V_0 (1+2\rho) \end{aligned}$$

The solutions of (4.22) in this case are

$$\tilde{\rho} = \frac{-1 \pm \sqrt{3}}{2},$$

and only the plus sign is physically meaningful.

It is shown in the Appendix that $V^{(m)}$ can always be written as a polynomial of degree m in ρ . Hence (4.22) is a polynomial of degree $m+1$ in ρ . The complete set of conditions for maximum energy transfer for $m=1, 2,$ and 3 are shown in Table I. Note that in each case there are only m physically meaningful roots of (4.22). We see that in each case

$|\hat{I}| < |\tilde{I}|$ and $|\frac{\hat{V}}{V_0} g| > |\frac{\tilde{V}}{V_0} g|$. The maximum energy transfer is therefore

achieved by accelerating a larger current and not by increasing the energy gain per particle per gap.

5. THE RECIRCULATING ACCELERATOR

5.1 Time-Dependent Voltage Generated By a Beam Pulse

The recirculating accelerator is based on the idea that a beam pulse can be accelerated to a higher energy by repeatedly circulating the beam pulse through an accelerator. For example, in a single-gap accelerator, if the output voltage exists across the gap for a period several times as long as the time required for one recirculation, the pulse will be accelerated on each passage. For the transmission line accelerators we have been discussing the desired accelerating voltage exists only for a period $2T\ell$, which might typically be on the order of 10 nanoseconds for reasonably-sized cavities. To circulate the pulse several times through the accelerator in this time frame is not possible, and it would be difficult to recharge and switch the cavities between beam passes. The repeating voltage patterns derived in Section 3.5, however, offer the interesting possibility that the cavities could be charged up once, and the beam recirculation timed so that the pulse arrives back at the cavity when the open-circuit output voltage repeats. The derivation of the repeating waveforms assumed lossless lines and there will in practice be an attenuation of the voltage amplitude as well as progressively shorter periods of constant acceleration voltage (Section 4.1) available for each successive pass through the gap. Experiments* on line-pair systems using high-quality coaxial lines, however, have yielded results which encourage us to pursue the concept further analytically. Of course, the current must be less than \hat{I} , or else all the stored energy will be transferred to the beam on the first pass. We shall see that with proper choice of parameters it is theoretically possible to achieve $\epsilon = 1$.

*Experiments performed by C.E. Hollandsworth, BRL.

To analyze this case we require a general expression for the time-dependent voltage generated in the cavity structure by the passage of a beam pulse. We have encountered this problem before, in Section 2.3 and 4.2, but there we were interested in times so short that it was possible simply to follow the reflected voltage steps and determine their influence on the output voltage. To derive the general expression applicable to a recirculating pulse we will proceed with a Laplace transform analysis.

The idealized transmission line configuration is shown in Figure 8a, where we are now concerned with the time behavior of a voltage step generated at the output end of line 2, with line 1 shorted as shown. As in our earlier analysis we neglect the effect of the intermediate region which in the real configuration serves to couple the lines together. The input impedance looking back into the lines is²³

$$Z_{in} = Z_2 \frac{1 + \rho_\ell e^{-2sT\ell_2}}{1 - \rho_\ell e^{-2sT\ell_2}}, \quad (5.1)$$

where ρ_ℓ is the reflection coefficient at the line junction looking back from the open end,

$$\rho_\ell = \frac{Z_\ell - Z_2}{Z_\ell + Z_2}, \quad (5.2)$$

and Z_ℓ is the input impedance looking into line 1 from the line junction:

$$Z_\ell = Z_1 \frac{1 - e^{-2sT\ell_1}}{1 + e^{-2sT\ell_1}} = Z_1 \tanh sT\ell_1. \quad (5.3)$$

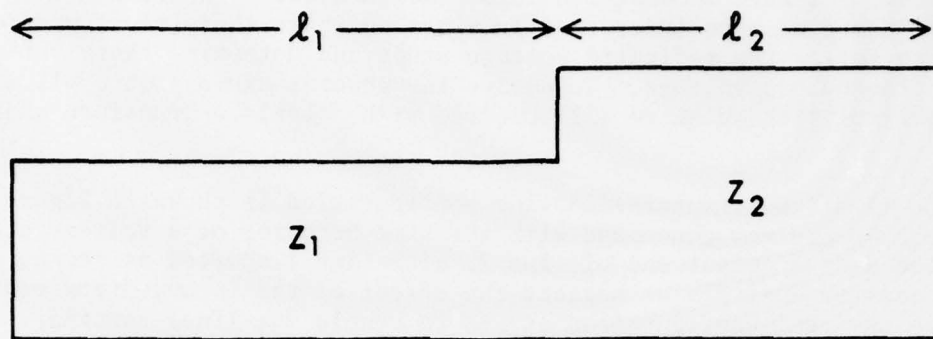
Using (5.2) and (5.3) in (5.1) yields, after some algebra,

$$Z_{in} = Z_2 \frac{Z_1 \tanh sT\ell_1 + Z_2 \tanh sT\ell_2}{Z_1 \tanh sT\ell_1 \tanh sT\ell_2 + Z_2}. \quad (5.4)$$

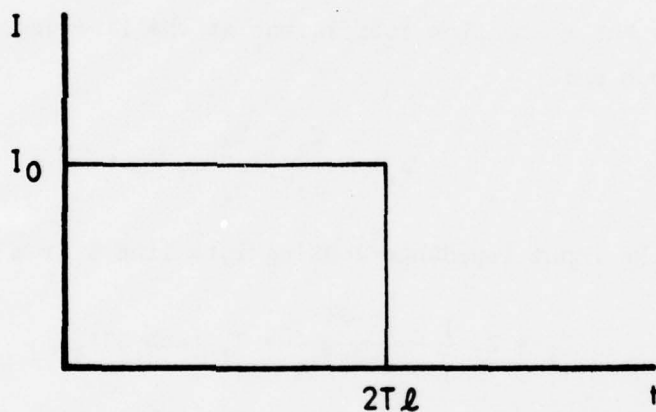
Since in the cases of interest the two lines are of the same length, we now put $\ell_1 = \ell_2 = \ell$. Then in terms of the reflection coefficient

$\rho = \frac{Z_2 - Z_1}{Z_2 + Z_1}$ used in our earlier analysis we can obtain (5.4) in the form

²³I.A.D. Lewis and F.H. Wells, "Millimicrosecond Pulse Techniques," Pergamon Press, New York, 1959, p. 39.



(a)



(b)

Figure 8. (a) Idealized transmission line configuration for calculation of voltage pulse by beam. (b) Time dependence of beam current.

$$Z_{in} = \frac{Z_2 (1 - e^{-4sT\ell})}{1 + 2\rho e^{-2sT\ell} + e^{-4sT\ell}} \quad (5.5)$$

We assume that the current is a square pulse, as shown in Figure 8b:

$$I = I_0 [u(t) - u(t - 2T\ell)] , \quad (5.6)$$

so that the transform is

$$I = I_0 \left(\frac{1}{s} - \frac{e^{-2sT\ell}}{s} \right) . \quad (5.7)$$

Hence for the transformed voltage we obtain

$$\begin{aligned} \frac{\bar{V}}{Z_2 I_0} &= \frac{1}{s} \frac{(1 - e^{-4sT\ell})}{1 + 2\rho e^{-2sT\ell} + e^{-4sT\ell}} - \frac{e^{-2sT\ell}}{s} \frac{(1 - e^{-4sT\ell})}{1 + 2\rho e^{-2sT\ell} + e^{-4sT\ell}} \\ &= \frac{\sinh 2sT\ell}{s(\cosh 2sT\ell + \rho)} - e^{-2sT\ell} \frac{\sinh 2sT\ell}{s(\cosh 2sT\ell + \rho)} . \end{aligned} \quad (5.8)$$

5.2 The Inverse Transform

We introduce the notation

$$\bar{V}_1 = \frac{\sinh 2sT\ell}{s(\cosh 2sT\ell + \rho)} . \quad (5.9)$$

Then from (5.8) we have

$$\begin{aligned} \frac{V}{Z_2 I_0} &= \mathcal{L}^{-1} \{ \bar{V}_1 - e^{-2sT\ell} \bar{V}_1 \} \\ &= V_1(t) - V_1(t - 2T\ell) . \end{aligned} \quad (5.10)$$

The poles of \bar{V}_1 are

$$s_n = \pm \frac{j}{2T\ell} (2n\pi + \theta) , \quad n = \dots -2 , -1 , 0 , 1 , 2 , \dots$$

where $\theta = \arccos(-\rho)$, and these are all distinct. Evaluating the derivative of the denominator of \bar{V}_1 at the poles gives

$$D'(s_n) = -(2n\pi + \theta) \sqrt{1-\rho^2} ,$$

so that the inverse transform is²¹

$$\begin{aligned} V_1(t) &= \sum_{n=-\infty}^{\infty} \frac{i \sqrt{1-\rho^2} e^{(2in\pi + i\theta) \frac{t}{2T\ell}}}{-(2n\pi + \theta) \sqrt{1-\rho^2}} \\ &+ \sum_{n=-\infty}^{\infty} \frac{-i \sqrt{1-\rho^2} e^{-(2in\pi + i\theta) \frac{t}{2T\ell}}}{-(2n\pi + \theta) \sqrt{1-\rho^2}} \\ &= \frac{2t}{2T\ell} \sum_{n=-\infty}^{\infty} \frac{\sin \left[\frac{n\pi t}{T\ell} + \frac{\theta t}{2T\ell} \right]}{\frac{n\pi t}{T\ell} + \frac{\theta t}{2T\ell}} . \end{aligned} \quad (5.11)$$

From Reference 22 we then have

$$V_1(t) = \frac{\sin(2k+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}} , \quad 2kT\ell < t < 2(k+1)T\ell . \quad (5.12)$$

Similarly, for $V_1(t - 2T\ell)$ we obtain

$$\begin{aligned} V_1(t - 2T\ell) &= 2 u(t-2T\ell) \sum_{n=-\infty}^{\infty} \frac{\sin(2n\pi + \theta) \frac{(t-2T\ell)}{2T\ell}}{2n\pi + \theta} \\ &= u(t-2T\ell) \frac{\sin(2k+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}} , \quad 2kT\ell < t-2T\ell < 2(k+1)T\ell . \end{aligned} \quad (5.13)$$

We again define

$$V^{(m)} = V(t) , \quad 2mT\ell < t < 2(m+1)T\ell . \quad (5.14)$$

From (5.12) and (5.13) we then have

$$\begin{aligned}
 V^{(0)} &= Z_2 I_0 , \\
 \frac{V^{(m)}}{Z_2 I_0} &= \frac{\sin(2m+1)\frac{\theta}{2}}{\sin \frac{\theta}{2}} - \frac{\sin[2(m-1)+1]\frac{\theta}{2}}{\sin \frac{\theta}{2}} \\
 &= \frac{2 \cos \frac{1}{2} [(2m+1)\frac{\theta}{2} + (2m-1)\frac{\theta}{2}] \sin \frac{1}{2} [(2m+1)\frac{\theta}{2} - (2m-1)\frac{\theta}{2}]}{\sin \frac{\theta}{2}} \\
 &= 2 \cos m \theta , \quad m \geq 1 .
 \end{aligned} \tag{5.15}$$

To conform with our previous notation, we again substitute $x = \arccos \rho = \pi - \theta$ to obtain from (5.15) our final expression:

$$\begin{aligned}
 V^{(0)} &= Z_2 I_0 , \\
 V^{(m)} &= 2 (-1)^m Z_2 I_0 \cos mx , \quad m \geq 1 .
 \end{aligned} \tag{5.16}$$

5.3 Acceleration of a Recirculating Beam

We now analyze the situation in which a cavity is charged externally to a voltage V_0 , and the device is used to accelerate a recirculating beam pulse of length $2T\ell$ and current I . For the voltage at the gap due to the switched applied voltage we have

$$V_A^{(m)} = (-1)^m V_0 \frac{\sin(\frac{2m+1}{2}x)}{\sin \frac{x}{2}} , \tag{3.29}$$

where m counts the intervals of $2T\ell$ which have elapsed since the closing of the switches. For the voltage at the gap due to a single passage of the beam pulse, which opposes the impressed voltage, we have

$$V_B^{(k)} = (-1)^k 2Z_2 I \cos kx , \quad k \geq 1 , \tag{5.16}$$

where k counts the intervals of $2T\ell$ which have elapsed since the beam arrived at the gap. We wish to choose an impedance ratio which results in a repeating open-circuit output pattern, so that according to the results of Section 3.2 we must have $x = \frac{p}{q} \pi$, p and q integers. If p and q have no common factors, the pattern repeats every $2qT\ell$ if p and q are both odd, and every $4qT\ell$ otherwise.

Assume that the beam pulse passes the gap for the first time during the interval $2nT\ell < t < 2(n+1)T\ell$, where t is measured from the time the internal cavity switches are closed, and every $2jqT\ell$ thereafter, where j is an integer and must be even if p and q are not both odd. Then the accelerating voltage during the first pass is

$$V_g^{(n)} = (-1)^n \frac{\sin\left(\frac{2n+1}{2} \frac{p}{q} \pi\right)}{\sin \frac{p}{q} \frac{\pi}{2}} - Z_2 I = V_A^{(n)} - Z_2 I \quad (5.17)$$

Between the first and second pass, $2(n+1)T\ell < t < 2(n+jq)T\ell$, we have

$$V_g^{(n+k)} = V_A^{(n+k)} - (-1)^k 2Z_2 I \cos k \frac{p}{q} \pi, \quad 1 \leq k \leq jq - 1 \quad (5.18)$$

The second time the beam is sent through we get, since $V_A^{(n)} = V_A^{(n+jq)}$,

$$V_g^{(n+jq)} = V_A^{(n)} - (-1)^{jq} 2Z_2 I \cos jp\pi - Z_2 I \quad (5.19)$$

Similarly, for the third time through the accelerating voltage is

$$\begin{aligned} V_g^{(n+2jq)} &= V_A^{(n)} - (-1)^{2jq} 2Z_2 I \cos 2jp\pi \\ &\quad - (-1)^{jq} 2Z_2 I \cos jp\pi - Z_2 I \quad (5.20) \end{aligned}$$

The s^{th} time through we get

$$V_g^{(n+(s-1)jq)} = V_A^{(n)} - Z_2 I - \sum_{r=1}^{s-1} (-1)^{rjq} 2Z_2 I \cos rjp\pi, \quad s > 1 \quad (5.21)$$

Now, if j is even, $(-1)^{rjq} = +1$, $\cos rjp\pi = +1$, and we have

$$\sum_{r=1}^{s-1} (-1)^{rjq} 2Z_2 I \cos rjp\pi = 2(s-1) Z_2 I \quad (5.22)$$

If j is odd, then so are p and q . Then $(-1)^{rjq} = (-1)^r$ and $\cos rjp\pi = (-1)^r$ so that the sum again has the value (5.22). Hence we have for (5.21)

$$V_g^{(n+(s-1)jq)} = V_A^{(n)} - Z_2 I (2s-1) \quad (5.23)$$

Suppose the beam pulse is to be circulated through the accelerator a total of κ times. The energy gained will be

$$\begin{aligned} E_g &= \sum_{s=1}^{\kappa} \{V_A^{(n)} - Z_2 I (2s-1)\} 2Tl I = \\ &= 2Tl I \kappa (V_A^{(n)} - \kappa Z_2 I) \end{aligned} \quad (5.24)$$

The energy stored initially is

$$E_s = \frac{1}{2} Tl V_0^2 \frac{Z_1 + Z_2}{Z_1 Z_2} \quad (5.25)$$

Taking the ratio of (5.24) to (5.25) we have

$$\epsilon = \frac{4 I \kappa \{V_A^{(n)} - \kappa Z_2 I\}}{V_0^2 \frac{(Z_1 + Z_2)}{Z_1 Z_2}} \quad (5.26)$$

To maximize the efficiency we first set $\frac{\partial \epsilon}{\partial I} = 0$ and find

$$\hat{I} = \frac{V_A^{(n)}}{2\kappa Z_2} \quad (5.27)$$

That is, the matched current in the recirculating case is $\frac{1}{\kappa}$ times the matched current in the single-pass case, where κ is the number of times the beam circulates through the accelerator. To determine the conditions for unit efficiency we substitute (5.27) into (5.26) and set $\epsilon = 1$, giving

$$4 \frac{\hat{V}_A^{(n)}}{2\kappa \hat{Z}_2} \kappa \left\{ \hat{V}_A^{(n)} - \frac{\hat{V}_A^{(n)}}{2} \right\} = V_0^2 \frac{Z_1 + \hat{Z}_2}{Z_1 \hat{Z}_2}$$

Recognizing that $\frac{Z_1}{Z_1 + \hat{Z}_2} = \frac{1 - \hat{\rho}}{2}$ we have

$$\hat{V}_A^{(n)} = \pm V_0 \left(\frac{2}{1 - \hat{\rho}} \right)^{1/2} = \pm \frac{V_0}{\sin \frac{x}{2}} . \quad (5.28)$$

Putting in the explicit form of $\hat{V}_A^{(n)}$ which we required to get a repeating voltage pattern gives

$$(-1)^n V_0 \frac{\sin \left(\frac{2n+1}{2} \right) \frac{p}{q} \pi}{\sin \frac{p}{q} \frac{\pi}{2}} = \pm \frac{V_0}{\sin \frac{p}{q} \frac{\pi}{2}}$$

which implies

$$(2n + 1) p = q (2\ell + 1) ,$$

where ℓ is any integer. Hence we have

$$\hat{x} = \frac{2\ell+1}{2n+1} \pi , \quad \ell = 0, 1, 2, \dots, n-1 , \quad (5.29)$$

where n designates the interval during which the beam is first circulated past the gap. In the unit efficiency case the total voltage through which the beam is accelerated is obtained by summing (5.23), which gives

$$\hat{V}_{\text{tot}} = \frac{\kappa \hat{V}_A^{(n)}}{2} . \quad (5.30)$$

The beam pulse gains a decreasing amount of energy each time it passes the gap, the gap voltage decreasing $\frac{\hat{V}_A^{(n)}}{\kappa}$ each time and going to zero after κ passes. If the beam were to continue to be recirculated it would of course lose energy to the cavity.

These results are surprisingly simple. The particles in the matched low-current beam which circulates through the accelerator κ times gain κ times as much energy as would the particles in a matched high-current beam which passes the gap only once. There are, of course,

$1/\kappa$ times as many particles in the low-current beam as in the high-current beam. The conditions for unit efficiency are the same in the two cases: make $\rho = \cos \frac{2\ell+1}{2n+1} \pi$, and accelerate the beam pulse during those intervals of length $2T\ell$ starting at $t = 2nT\ell$ and/or any multiple of $2(2n+1)2T\ell$ thereafter, the "and" referring to the recirculating case, the "or" to the single-pass case. We note additionally that the gap referred to could be a multi-gap accelerator through which the beam circulates. As was pointed out earlier, these results were obtained under the assumption of ideal, lossless lines and switches, we have treated only the principal mode in the lines, and we have ignored complications arising from the cavity region which couples the two lines together.

As a simple numerical example, suppose that a cavity is designed with $\rho = \frac{1}{2}$, $2T\ell = 10$ ns, $V_0 = 1$ MV, and $Z_2 = 50 \Omega$. Then the open-circuit output voltage pattern will be as shown in Figure 6b. A 20-kA, 10-ns beam pulse will be accelerated through 1 MV by passing it through the cavity during the 10-ns period beginning at 10 ns (or 40 ns, or 70 ns, ...) after the closing of the internal cavity switches, and all the stored energy will be transferred to the beam. In a recirculating mode we can circulate a 2-kA, 10-ns beam pulse through the cavity 10 times at intervals of 30 ns (or 60 ns, or 90 ns, ...) starting at $t = 10$ ns (or 40 ns, or 70 ns, ...) resulting in a total acceleration through 10 MV, and again all the stored energy will be transferred to the beam. Since the accelerating voltage decreases with each pass, 75% of the total acceleration is achieved after 5 passes.

We now record the behavior of the output voltage between beam passes, since this could be important for identifying voltage excursions which cause breakdown. As before, we have $x = \frac{p}{q} \pi$, where p and q are integers with no common factors. Then during the period between the s^{th} and the $(s+1)^{\text{th}}$ passes at time t between $2T\ell(n+(s-1)jq+k)$ and $2T\ell(n+(s-1)jq+k+1)$, where $1 \leq k \leq jq-1$, we find

$$\begin{aligned}
 v^{(n+(s-1)jq+k)} &= v_A^{(n+k)} - 2Z_2 I \sum_{m=0}^{s-1} (-1)^{mjq+k} \cos(mjq+k) \frac{p}{q} \pi \\
 &= v_A^{(n+k)} - 2Z_2 I \sum_{m=0}^{s-1} (-1)^{mjq+k} \cos mjp\pi \cos k \frac{p}{q} \pi \\
 &= v_A^{(n+k)} - (-1)^k 2Z_2 I s \cos k \frac{p}{q} \pi . \quad (5.31)
 \end{aligned}$$

In particular, for the unit efficiency case we have $q = 2n+1$ and p odd, say $p = 2r+1$. Then (5.31) becomes

$$\begin{aligned} \hat{V}^{(n+(s-1)j(2n+1)+k)} &= \\ &= \hat{V}_A^{(n+k)} - (-1)^k 2sZ_2 \hat{I} \cos k \frac{2r+1}{2n+1} \pi . \end{aligned} \quad (5.32)$$

But

$$\begin{aligned} \hat{V}^{(n+k)} &= (-1)^{n+k} V_0 \frac{\sin \left(\frac{2n+1}{2} + k \right) \frac{2r+1}{2n+1} \pi}{\sin \frac{2r+1}{2n+1} \frac{\pi}{2}} = \\ &= (-1)^{n+k} \frac{V_0 (-1)^r \cos k \frac{2r+1}{2n+1} \pi}{\sin \frac{2r+1}{2n+1} \frac{\pi}{2}} . \end{aligned} \quad (5.33)$$

Using (5.33) and (5.27) in (5.32) results in

$$\hat{V}^{(n+(s-1)j(2n+1)+k)} = \frac{(-1)^{k+n+r} V_0 \cos k \frac{2r+1}{2n+1} \pi}{\sin \frac{2r+1}{2n+1} \frac{\pi}{2}} \left[1 - \frac{s}{\kappa} \right] . \quad (5.34)$$

Clearly the voltage given by (5.34) is identically zero after κ passes, when all the stored energy has been delivered to the beam.

It is of some interest to note that the recirculating accelerator described here affords the possibility of a quite high accelerating gradient for relatively low-current beams. It is possible to show that a single-pass accelerator designed to accelerate a high-current beam with given efficiency will always exhibit a higher accelerating gradient than a single-pass accelerator designed for the same efficiency at a lower current. Furthermore, as shown above, if a beam of current I_0 is circulated κ times through a machine designed to accelerate a current $I = \kappa I_0$ through a voltage V , the lower-current beam will be accelerated through a voltage κV . In the recirculating case one thus increases the accelerating gradient through two effects: the recirculation of the beam and the use of design parameters appropriate to a higher current than that to be accelerated. The accelerating gradient in a recirculating accelerator is therefore more than a factor of κ greater than in a single-pass machine designed for the same current and efficiency; for some designs, in fact, this gain in accelerating gradient approaches a factor of κ^2 . The details of the analysis leading to these conclusions will be published in a separate report.

6. SUMMARY

We have shown that, under the assumptions of ideal, lossless lines and switches, and within the approximations of a principal-mode analysis, a transmission-line accelerator using asymmetric line pairs can be designed which will accelerate a constant-current beam pulse of length $2T\ell$ through a constant voltage with a theoretical efficiency of unity. Furthermore, we have demonstrated that the accelerating voltage in this case is always greater than half the charging voltage applied to the lines. We have demonstrated that, while periods of time of length $4T\ell$ exist for which the magnitude of the open-circuit output voltage remains constant at V_0 , a constant-current beam pulse of length $4T\ell$ will not experience a constant accelerating voltage for its full duration. In addition, we have shown that a lower-current beam pulse can be circulated through the accelerator several times resulting in a total accelerating voltage several times that of the single-pass case, and that in this case also total transfer of the stored energy to the beam pulse can be achieved.

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APPENDIX

We here present for the open-circuit case an alternative derivation of the time-dependent output voltage. While somewhat lengthier than the derivation given in Section 3.2, it has the heuristic advantage of explicitly demonstrating that the voltage periods of length $2T\ell$ are due to the step-function character caused by repeated reflections in the transmission lines. We start with (3.19):

$$\bar{V}_2 = \frac{-2(1+\rho)\bar{E}e^{-2sT\ell}}{1 + e^{-4sT\ell} + 2\rho e^{-2sT\ell}} \quad (A.1)$$

From the binomial theorem for negative integral exponents, we obtain for (A.1)

$$\bar{V}_2 = -2(1+\rho)\bar{E}e^{-2sT\ell} \sum_{n=0}^{\infty} (-1)^n (e^{-4sT\ell} + 2\rho e^{-2sT\ell})^n \quad (A.2)$$

We further expand the summand in (A.2):

$$\begin{aligned} \bar{V}_2 &= -2(1+\rho)\bar{E}e^{-2sT\ell} \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^n \binom{n}{k} e^{-4(n-k)sT\ell} (2\rho)^k e^{-2ksT\ell} \\ &= -2(1+\rho)\bar{E}e^{-2sT\ell} \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^n (2\rho)^k \binom{n}{k} e^{-(4n-2k)sT\ell} \end{aligned} \quad (A.3)$$

Using (3.4), (3.5), and (A.3) we obtain for the transformed output voltage

$$\bar{V} = \frac{1}{s}V_0 - 2(1+\rho)V_0 \frac{e^{-2sT\ell}}{s} \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^n (2\rho)^k \binom{n}{k} e^{-(4n-2k)sT\ell} \quad (A.4)$$

Taking the inverse transform of (A.4) we obtain the output voltage as a function of time:

$$V(t) = V_0 u(t) - 2(1+\rho)V_0 \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^n \binom{n}{k} (2\rho)^k u[t - 2(2n - k + 1)T\ell] \quad (A.5)$$

We note that the output voltage is a series of step functions, and that the value changes every $2T\ell$. In particular, we see that for any t , the summand is identically zero for values of n and k such that $2(2n - k + 1) T\ell > t$. We therefore define

$$v^{(m)} = V(t) , \quad 2mT\ell < t < 2(m + 1)T\ell . \quad (\text{A.6})$$

We now make the substitution $i = 2n - k$ and replace the sum on n with a sum on i . We see from the above that $v^{(0)} = V_0$, and for any value of $m > 0$, i is summed from 0 to $m - 1$. To obtain the new limits on k we note the following correspondence:

n	k	i
0	0	0
1	0	2
	1	1
2	0	4
	1	3
	2	2
3	0	6
	1	5
	2	4
	3	3
4	0	8
	1	7
	2	6
	3	5
	4	4

We see that for i even, k is summed from zero to i in steps of 2, and for i odd, k is summed from 1 to i in steps of 2. We therefore substitute $k = 2j$ when i is even and $k = 2j + 1$ when i is odd. Making these substitutions and using the definition of $u(t)$, we obtain

$$v^{(0)} = V_0 ,$$

$$\begin{aligned}
V^{(m)} = V_0 - 2(1 + \rho)V_0 & \left\{ \sum_{\substack{i=0 \\ i \text{ even}}}^{m-1} \sum_{j=0}^{\frac{i}{2}} (-1)^{\frac{i+2j}{2}} \binom{\frac{i+2j}{2}}{2j} (2\rho)^{2j} \right. \\
& \left. + \sum_{\substack{i=0 \\ i \text{ odd}}}^{m-1} \sum_{j=0}^{\frac{i-1}{2}} (-1)^{\frac{i+2j+1}{2}} \binom{\frac{i+2j+1}{2}}{2j+1} (2\rho)^{2j+1} \right\}, \quad m > 0. \quad (\text{A.7})
\end{aligned}$$

Putting $i = 2n$ in the first sum and $i = 2n + 1$ in the second sum gives

$$\begin{aligned}
V^{(m)} = V_0 - 2(1 + \rho)V_0 & \left\{ \sum_{n=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{j=0}^n (-1)^{n+j} \binom{n+j}{2j} (2\rho)^{2j} \right. \\
& \left. + \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{j=0}^n (-1)^{n+j+1} \binom{n+j+1}{2j+1} (2\rho)^{2j+1} \right\}, \quad m > 0, \quad (\text{A.8})
\end{aligned}$$

where the square brackets denote "largest integer contained in." Interchanging the order of summation results in

$$\begin{aligned}
V^{(m)} = V_0 - 2(1 + \rho)V_0 & \left\{ \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{n=j}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^{n+j} \binom{n+j}{2j} (2\rho)^{2j} \right. \\
& \left. + \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{n=j}^{\lfloor \frac{m}{2} \rfloor - 1} (-1)^{n+j+1} \binom{n+j+1}{2j+1} (2\rho)^{2j+1} \right\}, \quad m > 0. \quad (\text{A.9})
\end{aligned}$$

Making the substitution $i = n - j$ yields

$$\begin{aligned}
V^{(m)} = V_0 - 2(1 + \rho)V_0 & \left\{ \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor - j} (-1)^{2j+i} \binom{i+2j}{2j} (2\rho)^{2j} \right. \\
& \left. + \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1 - j} (-1)^{2j+1+i} \binom{i+2j+1}{2j+1} (2\rho)^{2j+1} \right\}, \quad m > 0. \quad (\text{A.10})
\end{aligned}$$

To recombine these into a single sum we put $2j = k$ in the first sum, $2j+1 = k$ in the second sum:

$$V^{(m)} = V_0 - 2(1 + \rho)V_0 \left\{ \sum_{\substack{k=0 \\ k \text{ even}}}^{2\left[\frac{m-1}{2}\right]} \sum_{i=0}^{\left[\frac{m-1}{2}\right] - \frac{k}{2}} (-1)^{k+i} \binom{i+k}{k} (2\rho)^k \right. \\ \left. + \sum_{\substack{k=0 \\ k \text{ odd}}}^{2\left[\frac{m}{2}\right]} \sum_{i=0}^{\left[\frac{m}{2}\right] - \frac{k+1}{2}} (-1)^{k+i} \binom{i+k}{i} (2\rho)^k \right\}, \quad m > 0 \quad . \quad (\text{A.11})$$

We note that if m is odd (even) the first sum goes to $m-1$ ($m-2$) and the second sum to $m-2$ ($m-1$). We can therefore combine the sums, and we obtain a power series in 2ρ for the open-circuit output voltage:

$$V^{(0)} = V_0 \\ V^{(m)} = V_0 - 2(1 + \rho)V_0 \sum_{k=0}^{m-1} (-1)^k (2\rho)^k \sum_{i=0}^{\left[\frac{m-k-1}{2}\right]} (-1)^i \binom{i+k}{k}, \quad m > 0 \quad . \quad (\text{A.12})$$

Values of $V^{(m)}$ for $m = 0, 1, \dots, 6$ are given in Table AI.

TABLE AI. EXPRESSIONS FOR OPEN CIRCUIT VOLTAGE ON AN ASSYMETRIC LINE

$$\begin{aligned} V^{(0)} &= V_0 \\ V^{(1)} &= -V_0 (1 + 2\rho) \\ V^{(2)} &= -V_0 (1 - 2\rho - 4\rho^2) \\ V^{(3)} &= V_0 (1 + 4\rho - 4\rho^2 - 8\rho^3) \\ V^{(4)} &= V_0 (1 - 4\rho - 12\rho^2 + 8\rho^3 + 16\rho^4) \\ V^{(5)} &= -V_0 (1 + 6\rho - 12\rho^2 - 32\rho^3 + 16\rho^4 + 32\rho^5) \\ V^{(6)} &= -V_0 (1 - 6\rho - 24\rho^2 + 32\rho^3 + 80\rho^4 - 32\rho^5 - 64\rho^6) \end{aligned}$$

To reduce (A.12) to the closed-form expression (3.29), we first interchange the order of summation to obtain

$$V^{(m)} = V_0 - 2V_0 (1+\rho) \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{k=0}^{m-(2i+1)} (-1)^{k+i} \binom{i+k}{k} (2\rho)^k. \quad (\text{A.13})$$

We now put $n = k+2i+1$, and replace the sum over k with a sum over n :

$$V^{(m)} = V_0 - 2V_0 (1+\rho) \sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{n=2i+1}^m (-1)^{n-i-1} \binom{n-i-1}{i} (2\rho)^{n-2i-1}. \quad (\text{A.14})$$

Once more reversing the order of summation we get

$$V^{(m)} = V_0 - 2V_0 (1+\rho) \sum_{n=1}^m \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-i-1} \binom{n-i-1}{i} (2\rho)^{n-2i-1}. \quad (\text{A.15})$$

We now define $x = \arccos \rho$, as in the main text. From Reference A-1 we find

$$\sin nx = \sin x \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^i \binom{n-i-1}{i} (2 \cos x)^{n-2i-1}. \quad (\text{A.16})$$

Using (A.16) in (A.15) yields

$$V^{(m)} = V_0 + 2V_0 \frac{(1+\cos x)}{\sin x} \sum_{n=1}^m (-1)^n \sin nx. \quad (\text{A.17})$$

This sum can be explicitly evaluated as^{A-2}

$$\sum_{n=1}^m (-1)^n \sin nx = \frac{-\sin x + (-1)^m [\sin mx + \sin (m+1)x]}{2(1 + \cos x)}. \quad (\text{A.18})$$

We therefore have

^{A-1}V. Mangulis, "Handbook of Series for Scientists and Engineers," Academic Press, Inc., New York, 1965, p. 16.

^{A-2}V. Mangulis, "Handbook of Series for Scientists and Engineers," Academic Press, Inc., New York, 1965, p. 104.

$$\begin{aligned}
V^{(m)} &= V_0 - V_0 + V_0 (-1)^m \frac{\sin m x + \sin (m+1) x}{\sin x} \\
&= (-1)^m V_0 \frac{2 \sin \left(\frac{2m+1}{2}\right) x \cos \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \\
&= (-1)^m V_0 \frac{\sin \left(\frac{2m+1}{2}\right) x}{\sin \frac{x}{2}}, \quad m > 0, \quad (A.19)
\end{aligned}$$

and

$$V^{(0)} = V_0 .$$

This reproduces (3.29), as required.

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