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COMPLEMENTS, MOLLIFIERS AND THE PROPENSITY TO DISRUPT. (U)
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6 COMPLEMENTS, MOLLIFIERS AND THE PROPENSITY TO DISRUPT.

by

10 A. Charnes,
J. Rousseau
L. Seiford

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CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director
Business-Economics Building, 203E
The University of Texas
Austin, Texas 78712
(512) 471-1821

406 197

ABSTRACT

Gately (1974) and Littlechild and Vaidya (1976) defined and studied ratio measures of "disruption propensity" of coalitions in an n-person game. We define and study new incremental measures giving rise to a wide variety of "disruption solution" concepts free of various ratio defects and affording advantages of analysis and acceptability in terms of solution specifications. Various "mollifier" and "homomollifier" solution concepts are characterized which appear to be of promising utility.

1. INTRODUCTION

Gately [1974] introduced the concept of an individual player's "propensity to disrupt" for the case of a three-person characteristic function game. Littlechild and Vaidya [1976] have recently extended this concept to n-person games. For a given payoff vector, they define a coalition's "propensity to disrupt" as the ratio of what the complementary coalition stands to lose if that payoff vector is abandoned to what the coalition itself stands to lose. They then define a disruption nucleolus using the vector of propensities to disrupt rather than the vector of coalitional excesses.

The choice of the ratio of the two quantities as the measure of their disparity, however, is not without its attendant difficulties. To avoid infinite disruption propensities, one is restricted to games with a strict core. The calculation of solution concepts involving the normalized propensities also becomes much more involved. Thus Littlechild and Vaidya do not employ the normalized, or "per person", disruption ratio which they indicate they would prefer.

In this paper we propose alternate measures of the disparity in terms of the difference rather than the ratio. In particular, we normalize the two quantities (to obtain the per person loss) and use their difference rather than their ratio. These incremental forms of the propensity to disrupt have a number of interesting and suggestive properties. Thus the complement of a game is defined and the propensity to disrupt is characterized as a weighted coalitional excess for a particular "mixture", e.g. componentwise convex combination of a game function and its complement function. We call games arising from such mixtures "mollifiers" and the per person forms, "homomollifiers." A large variety of "disruption" solution concepts is now immediately available using the extremal ("convex nucleus") characterizations of Charnes and Kortanek [1967] and Charnes and Keane [1969].

2. THE PROPENSITY TO DISRUPT

Let (N, v) be a characteristic function game where $N = \{1, 2, \ldots, n\}$ is the set of players and v is a characteristic function, i.e. a non-negative function defined on the subsets of N with $v(\emptyset) = 0$. A payoff vector is an n-tuple $x=(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} x_i = v(N)$ and $x_i \geq 0$, $i = 1, \ldots, n$.

Littlechild and Vaidya define, for a given payoff vector x, the "propensity to disrupt" of a coalition $S \subset N$ $(S \neq \emptyset, N)$ by

$$\hat{d}(x, S) = \frac{x(N-S) - v(N-S)}{x(S) - v(S)} \tag{2.1}$$

where $x(S) = \sum_{i \in S} x_i$.

To avoid the difficulties of infinite, negative or artificially positive propensities, they limit themselves to games where the strict core is non-empty and require the solution vectors to be in this set.

To avoid this severe restriction and other difficulties we suggest that the difference of the quantities $x(N-S) - v(N-S)$ and $x(S) - v(S)$ be employed as a measure of their variation. More generally, one could consider a weighted difference of the form

$$\alpha(N-S)[x(N-S) - v(N-S)] - \beta(S)[x(S) - v(S)]$$

where $\alpha(N-S)$ and $\beta(S)$ are suitably chosen weights or "normalization" factors.

A measure of the desired "average" or "per-person" disruption quality is then had by taking the weights to be the reciprocals of the coalitional size. As we shall shortly see, it turns out also to yield a class of games, "homomollifiers", with particularly convenient analytical properties reflecting those of v(S). Thus we define the incremental form of the "average" propensity to disrupt of coalition S as

$$d(x,S) = \frac{x(N-S) - v(N-S)}{|N-S|} - \frac{x(S) - v(S)}{|S|} \qquad (2.2)$$

where $|S|$ is the cardinality of the set S. Hence the average propensity to disrupt is the difference of the normalized coalitional excesses for S and N-S.

Using the relation

$$x(N-S) = v(N) - x(S)$$

(2.2) can be rewritten as

$$d(x,S) = \left[\frac{1}{|N-S|} + \frac{1}{|S|}\right]\left[\frac{|S|(v(N) - v(N-S)) + |N-S|v(S)}{|S| + |N-S|} - x(S)\right] \qquad (2.3)$$

We notice that d(x, S) is a weighted coalitional excess for the "game"

$$w(S) = \frac{|S|(v(N) - v(N-S)) + |N-S|v(S)}{|S| + |N-S|} \qquad (2.4)$$

Evidently w(S) is a convex combination of v(S) and v(N) - v(N-S). This observation motivates the next section.

3. COMPLEMENTS AND MOLLIFIERS

The complement[2/] of a game v, denoted by $\overline{v}$, is defined by

$$\overline{v}(S) = v(N) - v(N-S).$$

Clearly, the complement transformation is involutory (i.e. $\overline{\overline{v}} = v$) with $\overline{v}(\emptyset) = 0$ and $\overline{v}(N) = v(N)$.

It will be helpful in what follows to keep in mind the 3-person game with $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 1$, $v(23) = 0$, $v(123) = 1$. For it, $\overline{v}(1) = 1$, $\overline{v}(2) = \overline{v}(3) = 0$, $\overline{v}(12) = \overline{v}(13) = \overline{v}(23) = 1$, $\overline{v}(123) = 1$.

While thereby $\overline{v}$ will not necessarily be superadditive, even if v is superadditive, $\overline{v}$ does inherit some of the structure of v.

Theorem (3.1)

(i) If v is monotone, i.e. $A \subseteq B \Rightarrow v(A) \leq v(B)$, then $\overline{v}$ is monotone.

(ii) If g is strategically equivalent to v, then $\overline{g}$ is strategically equivalent to $\overline{v}$.

Proof: (i) $A \subseteq B \Rightarrow N - A \supseteq N - B$

$$\Rightarrow v(N-A) \geq v(N-B)$$

$$\Rightarrow \overline{v}(A) = v(N) - v(N-A) \leq v(N) - v(N-B) = \overline{v}(B)$$

(ii) Suppose $g(S) = r \cdot v(S) + \sum_{i \in S} \alpha_i$ with $r > 0$.

Then
$$\begin{aligned}\overline{g}(S) &= g(N) - g(N-S)\\ &= \Big(r \cdot v(N) + \sum_{i \in N} \alpha_i\Big) - \Big(r \cdot v(N-S) + \sum_{i \in N-S} \alpha_i\Big)\\ &= r(v(N) - v(N-S)) + \sum_{i \in S} \alpha_i\\ &= r\,\overline{v}(S) + \sum_{i \in S} \alpha_i\end{aligned}$$

Q.E.D.

If we assume that v is superadditive, the structure of $\overline{v}$ becomes more fixed.

2/ As noted by the referee, the name "dual" has been used for this concept. We prefer "complement" since as he states (and we later prove) this "dual" of a superadditive game cannot be supperadditive if not equal to the "primal".

Theorem 3.2 If v is superadditive, then

(i) $\bar{v}(S) \geq v(S) \quad (\forall\ S \subseteq N)$,

(ii) $\bar{v}(S) + \bar{v}(N-S) \geq \bar{v}(N) \qquad (\forall\ S \subseteq N)$.

(iii) $\bar{v}(S \cup T) \geq \bar{v}(S) + v(T)$, whenever $S \cap T = \emptyset$.

(iv) $\bar{v}$ is superadditive iff $\bar{v} = v$.

Proof: (i) $\bar{v}(S) = v(N) - v(N-S) \geq v(S)$, where the last inequality is due to superadditivity.

(ii) $\bar{v}(S) + \bar{v}(N-S) = v(N) - v(N-S) + v(N) - v(S)$

$= v(N) + v(N) - v(N-S) - v(S)$.

Since $v(N) = \bar{v}(N)$ and $v(N) \geq v(N-S) + v(S)$, the result follows.

(iii) $v(N-S) \geq v(N-S \cup T) + v(T)$, so $\bar{v}(S) \leq \bar{v}(S \cup T) - v(T)$.

(iv) By (i), $\bar{v} \neq v$ implies $\bar{v}(S) > v(S)$ for some S.

Thus $v(N) - v(N-S) - v(S) > 0$.

For $\bar{v}$ superadditive,

$$\begin{aligned}\bar{v}(N) &\geq \bar{v}(S) + \bar{v}(N-S)\\ &= v(N) - v(N-S) - v(S) + v(N)\\ &> v(N)\text{ , a contradiction, since } \bar{v}(N) = v(N).\end{aligned}$$

Hence $\bar{v} = v$.

Q.E.D.

Corollary 3.2 v is constant sum iff $\bar{v}(S) = v(S)$ for all $S \subseteq N$.

Proof: If $\bar{v}(S) = v(S)$ then $v(S) + v(N-S) = v(N)$.
Conversely, $v(S) + v(N-S) = \text{constant} = v(N)$, taking $S = \emptyset$.

Q.E.D.

As will next be shown, the core of a game may be characterized alternatively by means of the complement in an "upper bound" form.

<u>Theorem 3.3</u> Let $C(v)$ denote the core of a superadditive game v. The following three conditions are equivalent for x an imputation of v and $x(S) = \sum_{i \in S} x_i$:

(i) $x \in C(v)$.

(ii) $v(S) \leq x(S)$, $\forall S \subseteq N$.

(iii) $x(S) \leq \overline{v}(S)$, $\forall S \subseteq N$

<u>Proof:</u> (i) iff (ii) is well known. For (ii) iff (iii) notice that

$$v(S) \leq x(S) \text{ iff } v(N) - v(S) \geq v(N) - x(S)$$
$$\text{iff } \overline{v}(N-S) \geq x(N-S).$$

<u>Corollary 3.3:</u> $x \in C(v)$ iff $v(S) \leq x(S) \leq \overline{v}(S)$, $\forall S \subseteq N$, where x is an imputation of v.

The value of $\overline{v}(S)$ can be considered as a maximum feasible "goal" of coalition S. It is the largest amount that they can reasonably "expect" to get, just as $v(S)$ is the least they would "accept". For coalition N-S, however, these bounds are reversed. We thus define a <u>mollifier</u> of a game v as any componentwise convex combination of the function v and its complement function $\overline{v}$. In particular, w_μ , a "constant" mollifier of v is defined for $0 \leq \mu \leq 1$ by

$$w_\mu(S) = \mu\overline{v}(S) + (1-\mu)v(S).$$

It is again immediate that $w_\mu(\emptyset) = 0$ and $w_\mu(N) = v(N)$. As a convex combination $w_\mu(S)$ lies between $v(S)$ of the game and $\overline{v}(S)$ of its complement. From this we conclude that the core of such a mollifier is contained in the core of v. Thus

<u>Theorem 3.4</u> $C(w_\mu) \subseteq C(v)$ for a superadditive game v.

Q.E.D.

As was the case for $\overline{v}$, w_μ is not necessarily superadditive. If v is constant sum, however, $w_\mu(S) = v(S)$ for all $\mu \in [0,1]$.

Some further properties of constant mollifiers are given in the following theorem.

Theorem 3.5 If v is a superadditive game then:

(i) $w_\mu(S)$ is linear and monotone non-decreasing in μ.

(ii) if w_{μ_1} is superadditive, then w_{μ_2} is superadditive for all $\mu_2 \leq \mu_1$.

Proof: (i) Let $\mu_1 < \mu_2$. Then

$$w_{\mu_1}(S) - w_{\mu_2}(S) = (\mu_1 - \mu_2)\overline{v}(S) + (1-\mu_1)v(S) - (1-\mu_2)v(S)$$

$$= (\mu_1 - \mu_2)\overline{v}(S) - (\mu_1 - \mu_2)v(S)$$

$$= (\mu_1 - \mu_2)[\overline{v}(S) - v(S)]$$

$$\leq 0$$

since $(\mu_1 - \mu_2) < 0$ and $\overline{v}(S) \geq v(S)$.

(ii) Suppose $w_{\mu_1}(S \cup T) \geq w_{\mu_1}(S) + w_{\mu_1}(T)$. Then

$$\mu_1\overline{v}(S\cup T) + (1-\mu_1)v(S\cup T) \geq \mu_1\overline{v}(S) + (1-\mu_1)v(S) + \mu_1\overline{v}(T) + (1-\mu_1)v(T).$$

Regrouping, we obtain

$$v(S\cup T) - v(S) - v(T) \geq \mu_1[\overline{v}(S) - v(S) + \overline{v}(T) - v(T) + v(S\cup T) - \overline{v}(S\cup T)].$$

Multiplying by $\dfrac{\mu_2}{\mu_1} > 0$ yields

$$\frac{\mu_2}{\mu_1}[v(S\cup T) - v(S) - v(T)] \geq \mu_2[\overline{v}(S) - v(S) + \overline{v}(T) - v(T) + v(S\cup T) - \overline{v}(S\cup T)].$$

Since $\dfrac{\mu_2}{\mu_1} \leq 1$,

$$v(S\cup T) - v(S) - v(T) \geq \frac{\mu_2}{\mu_1}[v(S\cup T) - v(S) - v(T)].$$

Hence, $v(S\cup T) - v(S) - v(T) \geq \mu_2[\overline{v}(S) - v(S) + \overline{v}(T) - v(T) + v(S\cup T) - \overline{v}(S\cup T)]$.

Upon regrouping, the result follows.

Q.E.D.

Recalling the "game", w(S), which motivated this section, we see that in (2.4) the convex combination weights are a function of the coalition. We therefore define a coalitional mollifier by

$$w(S) = \mu_S \overline{v}(S) + (1 - \mu_S) v(S)$$

where $\mu_S \in [0, 1]$, $\forall S$. This allows us to "mollify" different coalitional values to a greater or lesser degree than others. In particular, if $\mu_S = \frac{|S|}{|N|}$, we call the associated w(S) a "homomollifier".

Coalitional mollifiers have several attractive properties. Specifically, w(S) is superadditive if the weights are additive for disjoint coalitions. This is formalized in the following theorem.

Theorem 3.6 Let v be a superadditive game and $\mu_S \in [0, 1]$ the weight associated with coalition S. If the weights satisfy $\mu_S + \mu_T = \mu_{S \cup T}$ whenever $S \cap T = \emptyset$, then $w(S) = \mu_S \overline{v}(S) + (1-\mu_S) v(S)$ is a superadditive game.

Proof: Let $S \cap T = \emptyset$

$$\begin{aligned} w(S \cup T) &= \mu_{S \cup T}\overline{v}(S \cup T) + (1-\mu_{S \cup T}) v(S \cup T) \\ &= \mu_S\overline{v}(S \cup T) + \mu_T\overline{v}(S \cup T) + (1-\mu_S - \mu_T) v(S \cup T) \\ &\geq \mu_S\overline{v}(S \cup T) + \mu_T\overline{v}(S \cup T) + (1-\mu_S - \mu_T) [v(S) + v(T)] \\ &\geq \mu_S[\overline{v}(S) + v(T)] + \mu_T[\overline{v}(T) + v(S)] + (1-\mu_S-\mu_T)[v(S) + v(T)] \\ &\quad \text{(by Theorem 3.2 (iii))} \\ &= \mu_S\overline{v}(S) + \mu_T\overline{v}(T) + (1-\mu_S)v(S) + (1-\mu_T)v(T) \\ &= w(S) + w(T). \end{aligned}$$

Q.E.D.

Even more interesting is the fact that w(S) is a constant sum game, if, in addition to disjunctive additivity, the weights are normalized, i.e. $\mu_N = 1$.

Theorem 3.7 If in addition to the assumptions of Theorem 3.6, $\mu_N = 1$, then w(S) is constant sum.

Proof:

$$\begin{aligned} w(S) + w(N-S) &= \mu_S \bar{v}(S) + (1-\mu_S)v(S) + \mu_{N-S}\bar{v}(N-S) + (1-\mu_{N-S})v(N-S) \\ &= (\mu_S + \mu_{N-S})v(N) + (1-\mu_{N-S}-\mu_S)v(N-S) + (1-\mu_S-\mu_{N-S})v(S) \\ &= \mu_N v(N) + (1-\mu_N)v(N-S) + (1-\mu_N)v(S) \\ &= v(N) = w(N) \end{aligned}$$

Q.E.D.

The incremental propensity to disrupt of coalition S was given by (2.3) as

$$d_v(x,S) = \left(\frac{1}{|N-S|} + \frac{1}{|S|}\right)\left[\frac{|S|(v(N) - v(N-S)) + |N-S|\, v(S)}{|S| + |N-S|} - x(S)\right].$$

Since $\mu_S = \frac{|S|}{|N|}$ satisfies the conditions of Theorem 3.7, we observe that this propensity to disrupt is a weighted coalitional excess, i.e.

$$d_v(x,S) = \left(\frac{1}{|N-S|} + \frac{1}{|S|}\right)[w(S) - x(S)] \tag{3.1}$$

where w(S) (the homomollifier of v) is a constant sum game.

We conclude the present section by observing that our incremental propensity to disrupt is additive due to the additivity of the underlying components. This is made explicit by the following theorem whose proof is straightforward.

Theorem 3.8 Suppose u, v and g are n-person games with g strategically equivalent to v. ($g(S) = r \cdot v(S) + \sum_{i \in S} \alpha_i$ with $r > 0$.)

Then

(i) $\overline{u+v} = \overline{u} + \overline{v}$ (i.e. the complement of a sum is the sum of the complements)

$$\overline{g}(S) = r\overline{v}(S) + \sum_{i \in S} \alpha_i$$

(ii) $w^{u+v} = w^u + w^v$, where w^z denotes the "w" mollifier of z, i.e. w designates a particular weighting of the z(S) and $\overline{z}(S)$.

$$w^g(S) = r \cdot w^v(S) + \sum_{i \in S} \alpha_i$$

(iii) $d_u(x, S) + d_v(y, S) = d_{u+v}(x+y, S)$

$$d_g(rx+\alpha, S) = rd_v(x, S).$$

4. DISRUPTIVE SOLUTIONS.

We first observe (from 3.1) that any solution concept defined in terms of the incremental disruption propensities can be equivalently expressed using the weighted coalitional excesses for w. Thus solution concepts for w give rise to corresponding "disruption" solution concepts for v. For example, the disruption nucleolus could be expressed using the vector of disruption propensities as in Littlechild and Vaidya or it could equivalently be expressed as a weighted nucleolus of w.

Charnes and Kortanek generalized the nucleolus and introduced the class of "convex nucleus" solutions. Charnes and Keane have shown that this class includes, among others, the core and the Shapley value. Their mathematical programming characterizations are:

(i) The core $C(g) \neq \emptyset$ consists of the solutions to

$$\min \sum_{S \subset N} |g(S) - x(S)| \qquad (4.1)$$

subject to

$$x(N) = g(N).$$

(ii) The Shapley value $\varphi(g)$ is the solution x^* to

$$\min_x \sum_{S \subset N} [g(S) - x(S)]^2 m(S) \tag{4.2}$$

subject to

$$x(N) = g(N),$$

where the weights $m(S) = m(|S|) = \binom{n-2}{|S|-1}^{-1}$.

In view of the above, we could define a disruption core, a disruption Shapley value and a host of other disruption solution concepts, each characterized a la Charnes and Kortanek as the *solution(s) to*

$$\min_x \sum_{S \subset N} f_S(d(x,S))$$

subject to

$$x(N) = v(N)$$

or equivalently

$$\min_x \sum_{S \subset N} f_S\left((w(S) - x(S))\frac{n}{(n-s)s}\right)$$

subject to

$$x(N) = w(N)$$

for suitably chosen convex functions f_S.

In particular, we define the <u>disruption value</u> with power α for the game v as the solution to

$$\min_x \sum_{S \subset N} \left(w_v(S) - x(S)\right)^2 \left[\frac{n}{(n-s)s}\right]^{\alpha(s)} \tag{4.3}$$

subject to

$$x(N) = w(N) .$$

We remark that as $\alpha \geq 0$ increases, α independent of S, the weights of the single player coalitions and their complements increase while the weights of the others decrease. Another extreme case of interest is

$$\min_x \sum_{i=1}^{n} [w(i) - x(i)]^2 \tag{4.4}$$

subject to

$$x(N) = w(N) .$$

The solution of (4.4) is easily shown to be

$$x_j = w(j) + \frac{w(N) - \sum_{i=1}^{N} w(i)}{n}$$

which equalizes the propensity to disrupt for coalitions of size 1.

The general case (4.3) can also be solved explicitly. (See the appendix for the details.) If we let, even more generally,

$$m(|S|) = \left[\frac{n}{(n-s)s}\right]^{\alpha(s)} p(s) \qquad \text{where } s = |S|,\ p(s) \geq 0,$$

then the solution to (4.3) is given by

$$x_i(v) = \frac{1}{\beta}\left[\mu_i + \frac{\beta w(N) - M}{n}\right] \tag{4.5}$$

where

$$\mu_i = \sum_{S \supseteq i} w(S)\, m(|S|)$$

$$M = \sum_{i=1}^{n} \mu_i$$

$$\beta = \sum_{s=1}^{n-1} m(s) \binom{n-2}{s-1} .$$

It is readily verified that the disruption value solution given by (4.5) is additive over games with the same number of players. Thus if v and u are two such characteristic functions, then

$$x_i(v) + x_i(u) = x_i(v+u) .$$

This property and the intuitive appeal of a solution which minimizes the propensity to disrupt makes it an attractive alternative distribution scheme to, say, the ordinary Shapley value. Thus, for instance, when the two are close in value the disruption concept affords another interpretation of the properties of the Shapley value in achieving equitable solutions.

Let us now illustrate these various matters by reference to our little example which we have chosen because of its importance in elucidation of discrepancies between standard solution presentations and acceptable ones. For it $w(1) = 1/3$, $w(2) = w(3) = 0$, $w(12) = w(13) = 1$, $w(23) = 2/3$, $w(123) = 1$. Note that the core of w is empty whereas the core of v is the unique solution $x_1 = 1$, $x_2 = x_3 = 0$. This is not an acceptable prescription, for why should players 2 or 3 join 1 to assure his $x_1 = 1$, while they receive zero? Re the Gately-Littlechild-Vaidya formulation, v has <u>no</u> strict core. Therefore it is outside the class which is covered by their results.

Suppose, further, responding to the referee's query as to what might the G-L-V Theory say for games in the neighborhood of v, we consider our example as the limit (as $\epsilon \to 0$) of games having a strict core:

$$\begin{aligned} v(i) &= 0 \\ v(12) = v(13) &= 1-2\epsilon \\ v(23) = \epsilon\ , \quad v(123) &= 1 \end{aligned} \tag{4.6}$$

Then the G-L-V solution (which equalizes the ratio propensities to disrupt) is:

$$\begin{aligned} x_1 &= (1-\epsilon)/(1+3\epsilon) \\ x_2 = x_3 &= 2\epsilon/(1+3\epsilon) \end{aligned} \tag{4.7}$$

Thus the limit of the ratio disruption solutions as $\epsilon \to 0$ is the (unacceptable) $x_1 = 1$, $x_2 = x_3 = 0$!

We calculate the Shapley value to be $x_1 = 2/3$, $x_2 = x_3 = 1/6$. In comparison with (4.4) our "individual" disruption value yields $x_1 = 5/9$, $x_2 = x_3 = 2/9$. The two differ in that the Shapley value gives players 2 and 3 each twenty-five percent less (1/18) than does the disruption value.

Thus even this extreme incremental disruption value checks the Shapley value well in this simple, but critical case. In further work underway we expect to make deeper and broader studies of the properties of our disruption values (such as constant sum) in relation to those of other possible solution notions including extensions to meta-game notions such as core-stem solutions [5] and unions [6]. Toward this program we record the following (easily proved) result:

Theorem 4.1: *The Shapley Value of a game v and its complement $\overline{v}$ are identical, e.g.* $\Phi_i(v) = \Phi_i(\overline{v})$, $\forall i$.

APPENDIX

For the explicit solution of the disruption value convex programming problem, let

$$m(s) = \left[\frac{n}{(n-s)s}\right]^{\alpha(s)} p(s)$$

where $s = |S|$, $p(s) \geq 0$.

Then (4.3) becomes

$$\min_x \sum_{S \subset N} [w(S) - x(S)]^2 m(s)$$

subject to

$$x(N) = w(N)$$

Thus we shall follow the arguments of Charnes & Keane [1].

The Lagrangean is

$$L = \sum_{S \subset N} [w(S) - x(S)]^2 m(s) + 2\lambda [x(N) - w(N)]$$

Differentiating with respect to each of the variables, we obtain

$$0 = \frac{\partial L}{\partial x_i} = -2 \sum_{\substack{S \subset N \\ S \supseteq i}} [w(S) - x(S)] m(s) + 2\lambda \qquad i=1,\ldots,n$$

$$0 = \frac{\partial L}{\partial \lambda} = x(N) - w(N)$$

or equivalently

$$\lambda = \sum_{S \supseteq i} [w(S) - x(S)] m(S) \tag{1}$$

$$x(N) = w(N) \tag{2}$$

Let $\mu_i = \sum_{S \supseteq i} w(S) m(s)$

Then (1) can be written as

$$\sum_{S \supseteq i} x(S) m(s) = \mu_i - \lambda \tag{3}$$

But $$\sum_{S \supseteq i} x(S) m(s) = \sum_{r=1}^{n-1} \sum_{\substack{|S|=r \\ S \supseteq i}} x(S) m(s)$$

$$= \sum_{r=1}^{n-1} m(r) \cdot \sum_{\substack{|S|=r \\ S \supseteq i}} x(S)$$

$$= x_i m(i) + \sum_{r=2}^{n-1} m(r) \left[x_i \binom{n-1}{r-1} + \sum_{j \neq i} x_j \binom{n-2}{r-2} \right]$$

$$= \sum_{r=1}^{n-1} m(r) \binom{n-1}{r-1} x_i + \sum_{r=2}^{n-1} m(r) \binom{n-2}{r-2} \sum_{j \neq i} x_j$$

$$= \sum_{r=1}^{n-1} m(r) \binom{n-1}{r-1} x_i - \sum_{r=2}^{n-1} m(r) \binom{n-2}{r-2} x_i$$

$$+ \sum_{r=2}^{n-1} m(r) \binom{n-2}{r-2} w(N)$$

$$= \sum_{r=1}^{n-1} m(r) \binom{n-2}{r-1} x_i + \sum_{r=2}^{n-1} m(r) \binom{n-2}{r-2} w(N)$$

$$= \beta x_i + \eta w(N)$$

where $\beta = \sum_{r=1}^{n-1} m(r) \binom{n-2}{r-1}$ and $\eta = \sum_{r=2}^{n-1} m(r) \binom{n-2}{r-2}$.

Substitution into (3) yields

$$\beta x_i + \eta w(N) = \mu_i - \lambda \tag{4}$$

or

$$x_i = [-\eta w(N) + \mu_i - \lambda] \frac{1}{\beta} \tag{5}$$

To obtain the expression for λ, we sum (4) over the set of players

$$\sum_{i=1}^{n} \left[\beta x_i + \eta w(N)\right] = \sum_{i=1}^{n} \mu_i - n\lambda$$

or

$$\beta w(N) + \eta n w(N) = \sum_{i=1}^{n} \mu_i - n\lambda$$

Solving for λ we obtain

$$\lambda = \frac{1}{n} [M - \beta w(N) - \eta n w(N)] \qquad \text{where } M = \sum_{i=1}^{n} \mu_i$$

Replacing λ in (5) and simplifying we finally arrive at

$$x_i = \frac{1}{\beta} \left[\mu_i + \frac{\beta w(N) - M}{n} \right]$$

where $\beta = \sum_{r=1}^{n-1} m(r) \binom{n-2}{r-1}$ and $\binom{k}{0} = 1$

$\mu_i = \sum_{S \supseteq i} w(S)m(s)$, $S \neq N$

$M = \sum_{i=1}^{n} \mu_i$

Q.E.D.

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13. ABSTRACT

Gately (1974) and Littlechild and Vaidya (1976) defined and studied ratio measures of "disruption propensity" of coalitions in an n-person game. ~~We~~ The authors define and study new incremental measures giving rise to a wide variety of "disruption solution" concepts free of various ratio defects and affording advantages of analysis and acceptability in terms of solution specifications. Various "mollifier" and "homomollifier" solution concepts are characterized which appear to be of promising utility.

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