Non-Euclidean Symmetry and Indra's Pearls

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Abstract

Escher's well known picture of devils and angels is an example of a symmetrical tiling of two dimensional hyperbolic space. We discuss similar symmetries of three dimensional hyperbolic space, modelled as the inside of a solid ball. The 'shadows' of the solid tiles on the boundary of the ball themselves form patterns governed by a new kind of symmetry, that of Möbius maps on the complex plane. All aspects of such pictures, together with instructions for making them, are explored in the authors' book *Indra's Pearls*. We give examples of beautiful fractal patterns created in this way.

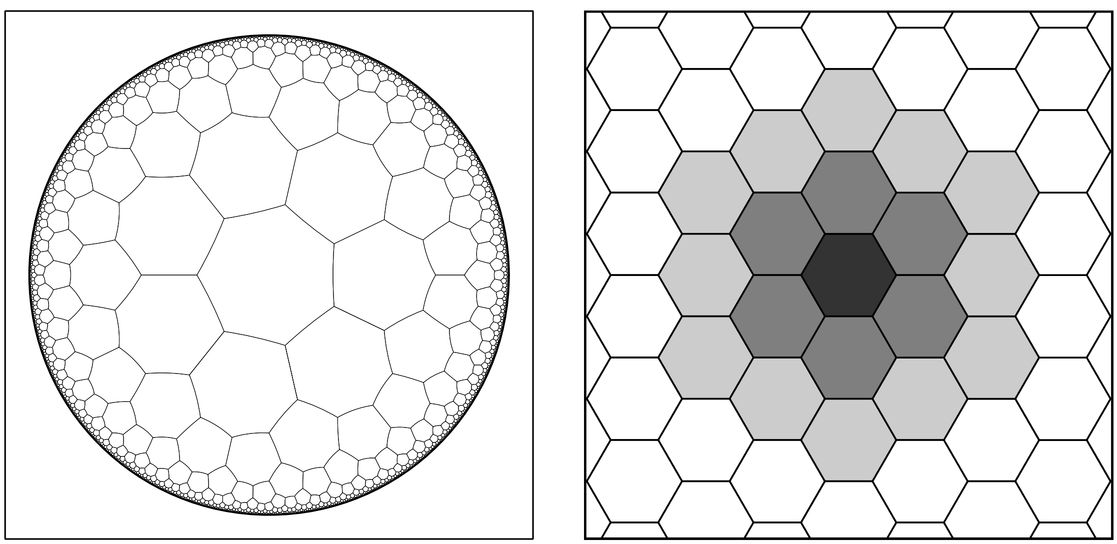

Figure 1: (a) Left: A non-Euclidean tiling of the disk by 'regular' heptagons. (b) Right: A Euclidean tiling of the plane by regular hexagons.

Many people will be familiar with Escher's famous picture of devils and angels. An image with similar symmetry is shown in Figure 1(a). The symmetries are those of hyperbolic, or non-Euclidean geometry. In this geometry, things behave in unexpected ways. For example, the circumference of a circle is proportional not to its radius, but to e^{radius}. This means that to fit into ordinary Euclidean space, a big hyperbolic disk has to crinkle up round its edges like a kale leaf. Once you start looking for it, you see this type of growth throughout the natural world.

The same exponential growth law is manifested in Figure 1(a), in which the tiles are arranged in 'layers' around the centre. If you count carefully, you will find that the n^{th} layer contains roughly 3.1×2.6^n tiles (or exactly 7 times the $2n$-th Fibonacci number). This is in marked contrast with

Euclidean tilings, where the growth law is linear. For example, the honeycomb tiling in Figure 1(b) has exactly $6n$ hexagons in the n^{th} layer.

Despite appearances, in the world of hyperbolic geometry the tiles in Figure 1(a) all have the same size and shape. To fit them into a Euclidean picture, we have to shrink their apparent size as we move away from the centre, so that to our Euclidean glasses the tiles look smaller and smaller as they pile up near the edge of the disk. Since you can fit infinitely many layers of congruent tiles between the centre of the disk and its boundary, the boundary must be infinitely far away from the centre. In this strange geometry, the diameter of the disk is infinite. For this reason, the boundary circle is called the 'circle at infinity'. All the points in the boundary circle are infinitely far away from the centre.

Now imagine a similar geometry in 3-dimensions. Solid tiles or crystals of the same hyperbolic size will be fitted together to fill up 3-dimensional hyperbolic space. This hyperbolic universe can be enclosed in a Euclidean sphere whose boundary is infinitely far, in hyperbolic terms, away from its centre.

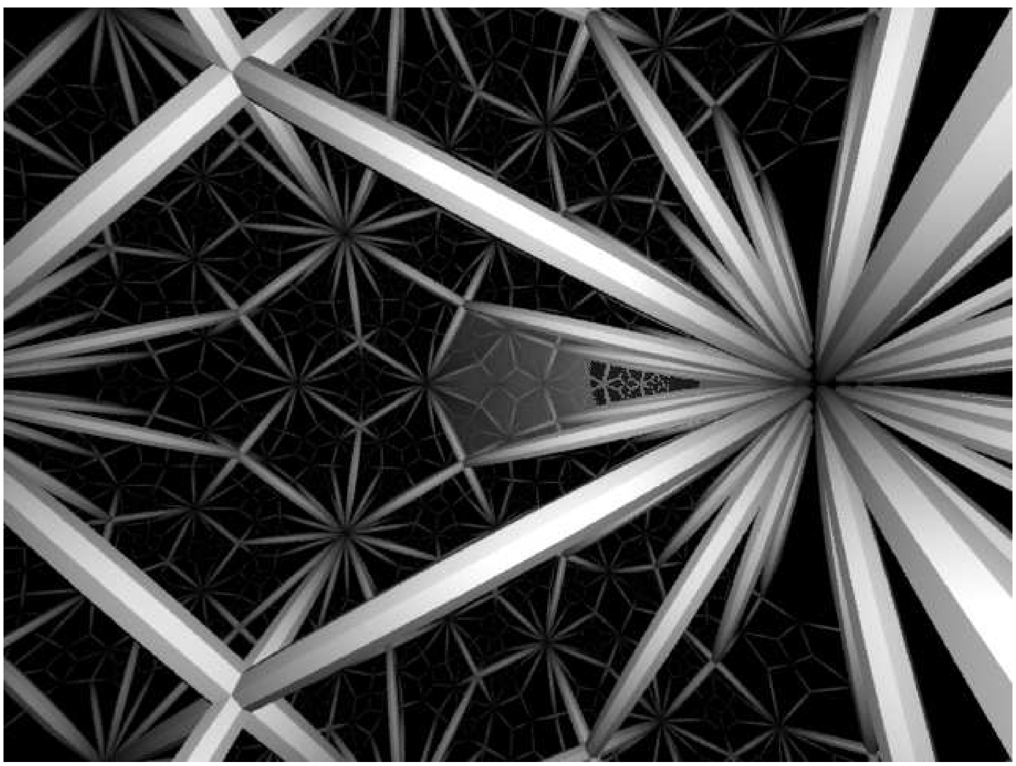

Figure 2: Non-Euclidean tiling of three-dimensional space. Still from *Not Knot!*

Tilings of hyperbolic 3-space are rather hard to draw, although some remarkable pictures have been made by Charles Gunn at the Geometry Center of the University of Minnesota, for the film *Not Knot!* published by A K Peters, Ltd. As an easier substitute, mathematicians usually study what they see on the boundary of the sphere. Patterns seen here, being two dimensional, can be flattened out by projecting onto a plane.

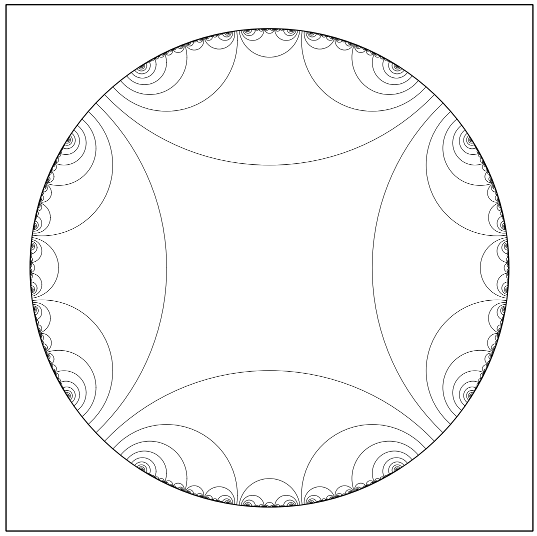

Figure 3: Non-Euclidean tiling of the disk by a polygon with some sides at infinity.

In the two dimensional analogue Figure 1(a) this is not very interesting, since tiles pile up all the way round the circle at infinity. But suppose that the original tile, still a polygon with a finite number of sides, stretched all the way out to the boundary, as in Figure 3. Each copy also meets the circle in 4 circular arcs. Although the tiles fill up all of hyperbolic 2-space (the interior of the disk), the totality of the arcs do not fill up the whole circle. The set of omitted points has interesting properties, for example it has a 'fractal dimension'. It is an example of what is called mathematically a *Cantor set*, more colloquially, a 'fractal dust'.

Returning to 3-dimensions, the analogue is a polyhedron with a finite number of faces, some of which reach all the way out to the sphere at infinity. Outside observers will see these faces as 'shadows' where the polyhedron meets the sphere. Figure 4 shows the shadows of the tiles (in this case triangular) piling up in remarkable patterns on the boundary of the sphere, like noses pressed against a window pane.

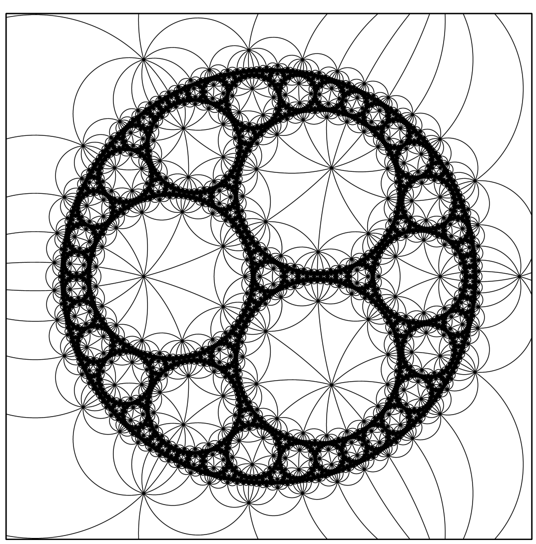

Figure 4: Faces of polyhedra in a 3-dimensional hyperbolic tiling pressed against the sphere at infinity.

Patterns like this were among the novel symmetries studied by the German mathematician Felix Klein (1849 – 1925). In the 1980's, David Mumford realised they were a natural target for computer exploration. With David Wright, he embarked on a systematic study which eventually resulted, not only in inspiring new mathematics, but also in our book *Indra's Pearls* [1].

The book shows off some of the remarkable pictures which resulted. We wanted to explain them with the minimum of mathematical baggage, but with enough detail for the mathematically inclined to follow the reasoning and for the computationally inclined to make their own pictures. By writing down mathematical formulae which describe the symmetries, we can discover what happens to a basic tile as the symmetries move it around.

All rigid motions of Euclidean space can be obtained by repeated reflections in lines or planes. The same is true in hyperbolic space, as long as we remember to replace 'reflection' by 'hyperbolic reflection'. Without thinking about that too hard, remember we are interested in what happens on the window pane at infinity. In Figure 1, two dimensional hyperbolic lines appear as circular arcs. Similarly in 3-dimensions, hyperbolic planes look like pieces of spheres. Such a spherical shell inside hyperbolic space meets infinity in a circle. So the motion we want to pin down could be described as 'reflecting in circles'. There is a nice bit of elementary mathematics which implements exactly this: 'reflecting' in a plane inside hyperbolic space translates to 'inverting' in the circle in which the plane hits infinity. Figure 5 shows a straight-laced stick figure inverted in a circle into slightly curvier figure.

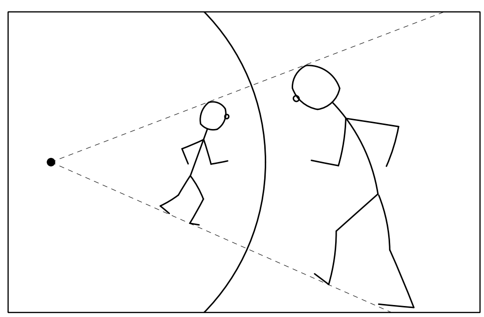

Figure 5: Inversion of a stick figure in a circle.

There is an important difference with the familiar symmetries of Euclidean space, which preserve Euclidean distance. Inside hyperbolic space, the symmetries preserve *hyperbolic* distance. On the window pane at infinity, however, it is impossible to find a way to measure distance which is preserved. All is not lost: it turns out that the transformations we need are exactly those which transform circles into circles, with changes of radius being allowed. Such transformations are called *Möbius maps*, after the German mathematician August Möbius (1790 – 1868).

For mathematicians, what we do is flatten out the sphere by stereographic projection and view it as the Riemann sphere or extended complex plane. Möbius maps are exactly the maps which send the complex number z (possibly including ∞) to the new complex number

$$\frac{az+b}{cz+b},$$

where a, b, c, d are fixed complex numbers. (The shape of the formula gives Möbius maps their other common name *linear fractional transformations*.) This enables us to study Klein's new symmetries using the algebra of two-by-two matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Indra's Pearls begins with a review of the language of symmetry and complex numbers, before going into the detailed effects of maps like these. What we really want to study is this. Take a pair of matrices such as for example

$$\begin{pmatrix} 1 & 0 \\ -2i & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1-i & 1 \\ 1 & 1+i \end{pmatrix}.$$

Then take a basic figure, say a stick man, and apply both of these transformations to the figure over and over again. As you can see in Figure 6, the images get small and are distorted as the level of the repetition increases. If you choose the starting matrices cleverly, the patterns which emerge when the images pile up are can be amazingly beautiful.

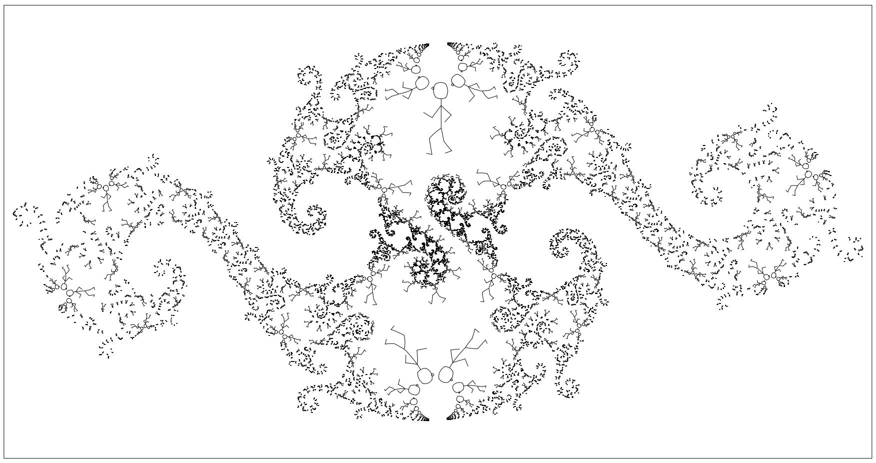

Figure 6: Images of a man piling up on the limit set.

To see more clearly what is going on, we often drop the original figure altogether and just look at the region where its smaller and smaller images pile up. This is called the *limit set* or *chaotic set* of the iteration, because in this part of the pictures, the symmetry group acts in a chaotic way. (Though to a mathematician, the chaos is very controlled.)

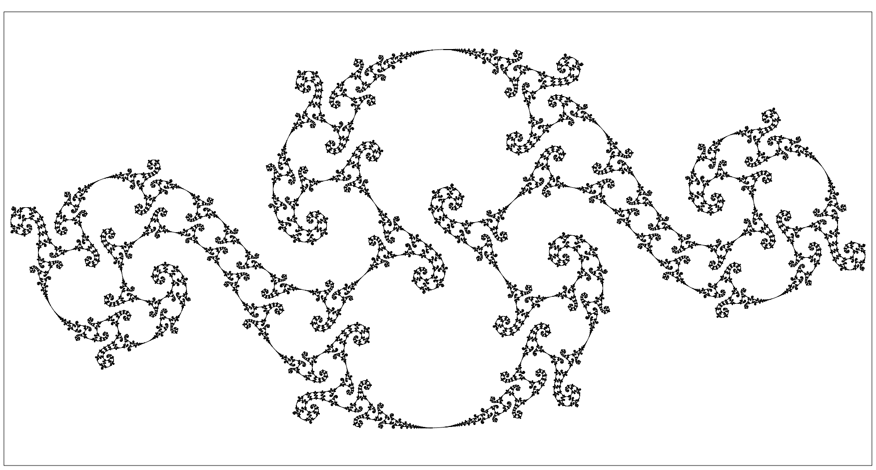

Figure 7: The same limit set with the man taken away to get a better view.

By choosing the initial symmetries with enough care, we can create limit sets with intricate patterns of tangent circles. The two matrices written down above produce the famous Apollonian Gasket shown in Figure 8.

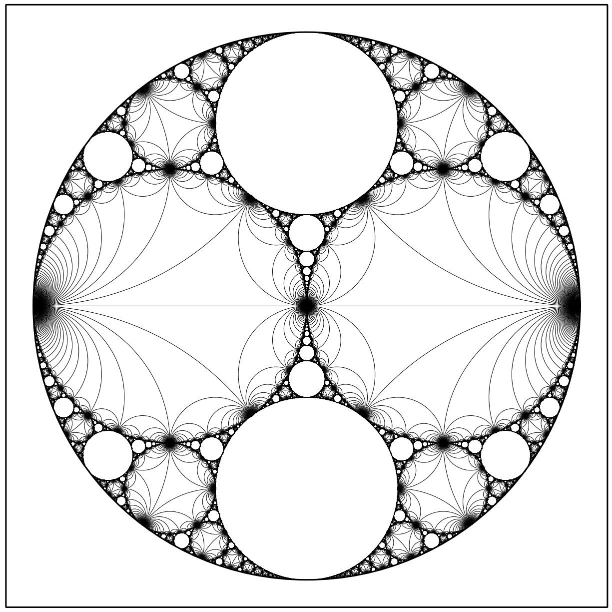

Figure 8: The Apollonian gasket.

In other examples like those in Figures 9 and 10, the tangent circles spiral in beautiful patterns. How and why this happens is explored in great detail in the book, which also contains instructions for making such images.

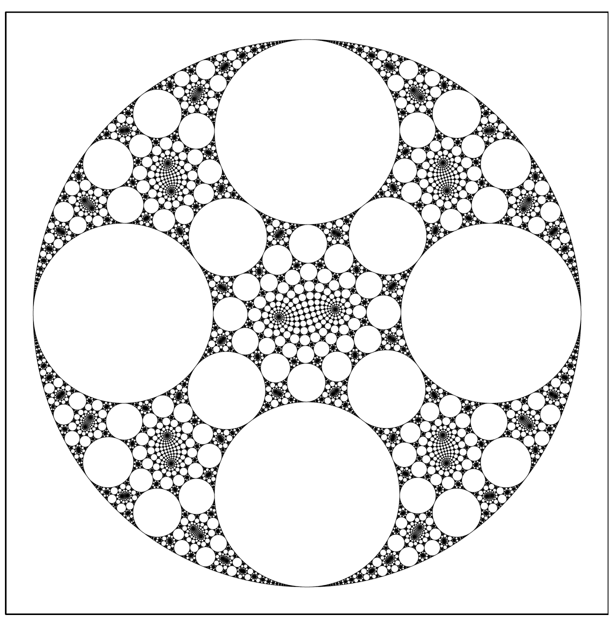

Figure 9: A spiralling cusp group.

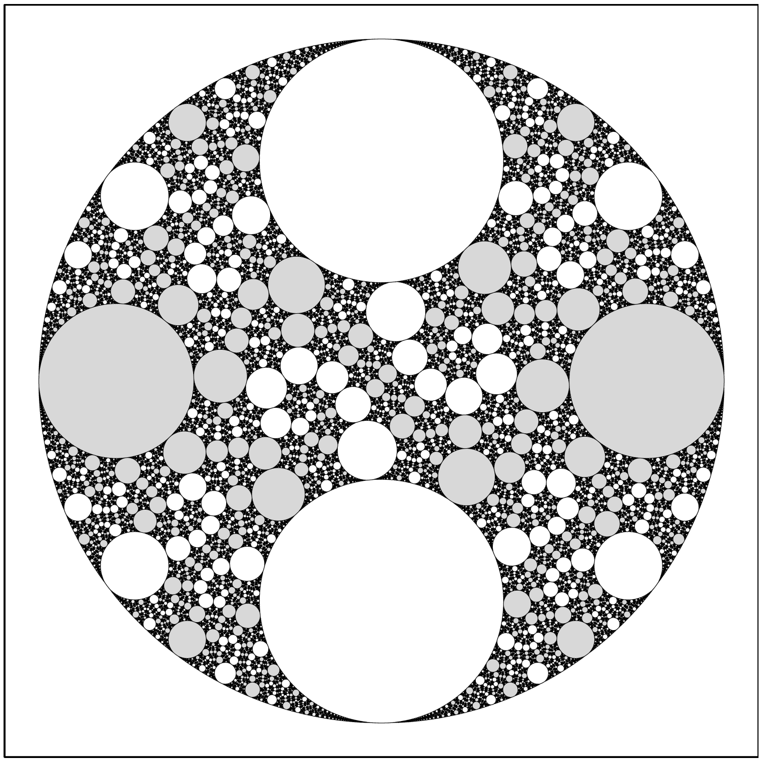

Figure 10: A more intricate cusp group with one side of circles shaded.

Why did we call our book Indra's Pearls? In western thought, the infinite is conceived as counting without end, a flock of sheep going through the gate forever: one, two, three, .. one hundred and one, one hundred and two, ... But there are other ways of getting to infinity. Remember the man with seven wives. Kits, cats, sacks and wives, quite a lot of traffic enroute to St Ives! In fact the traffic increases exponentially with the number of levels (kits, cats,...).

Such exponential growth reminds us of hyperbolic tilings, typically formed by a similar repetitive process. In Figure 11, we start with six disjoint circles. Suppose we reflect in one of these circles C. The five other initial circles which are outside C get reflected into five smaller circles inside C. The repeat operation produces five more small circles inside each of these second level circles: $5^2 = 25$ circles in all. At the next level, we will have $5^3 = 125$ tiny circles. And so on.

In many eastern philosophies, especially Buddhist, this idea of the infinite appearing from copies within copies is pervasive: *"In a single atom, great and small lands, as many as atoms."* This concept was so exactly reflected in the mathematics of our pictures that it inspired our title, taken from the ancient Buddhist myth of Indra's Web:

In the heaven of the great god Indra is said to be a vast and shimmering net, finer than a spider's web, stretching to the outermost reaches of space. Strung at the each intersection of its diaphanous threads is a reflecting pearl. Since the net is infinite in extent, the pearls are infinite in number. In the glistening surface of each pearl are reflected all the other pearls, even those in the furthest corners of the heavens. In each reflection, again are reflected all the infinitely many other pearls, so that by this process, reflections of reflections continue without end.

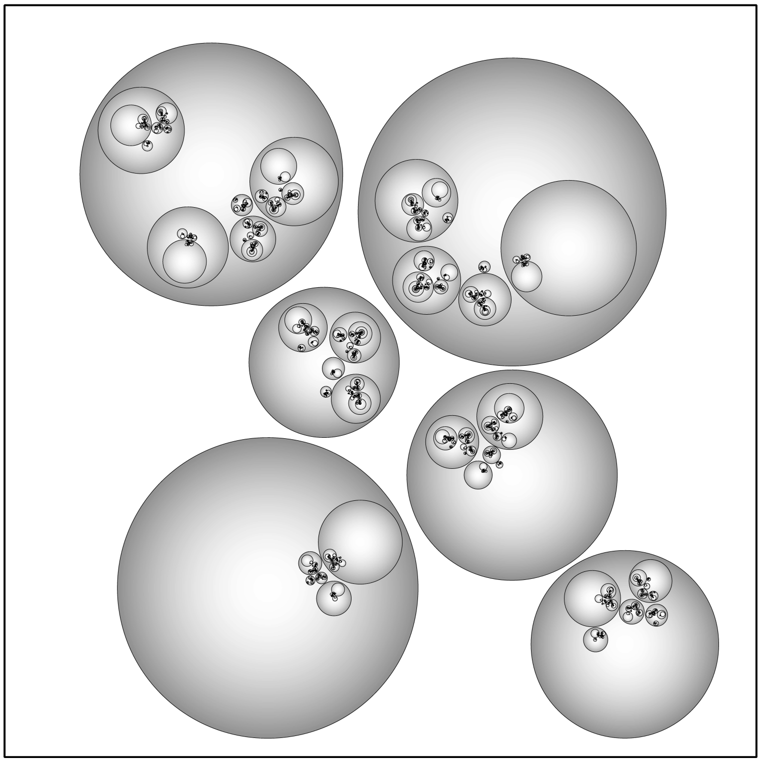

Figure 11: Worlds within worlds.

References

[1] *Indra's Pearls, The Vision of Felix Klein,* D. Mumford, C. Series and D. Wright, Cambridge University Press, 2002.