Descriptive set theory in the setting of generalized Baire spaces

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Outline

This is an introduction to basic results about Borel and analytic sets and equivalence relations.

We compare results in descriptive set theory with their counterparts in generalized descriptive set theory.

Borel sets

A metric space (X, d) is *complete* if every Cauchy sequence converges.

A *Polish metric space* is a complete metric space (X, d) with a countable base.

Equivalently, (X, d) has a countable dense subset; a set $D \subseteq X$ is *dense* if $D \cap U \neq \emptyset$ for every nonempty open set $U \subseteq X$.

Examples: $\mathbb{R}^n$ with the standard metric

Borel sets

The Baire space ${}^{\omega}\omega = \{x \mid x \colon \omega \to \omega\}$ with the metric $d(x, y) = \frac{1}{2^n}$, where $n \in \omega$ is least with $x(n) \neq y(n)$ for $x \neq y$.

The Cantor space ${}^{\omega}2 = \{x \mid x \colon \omega \to 2\}$.

Note that the Cantor space is homeomorphic to Cantor's middle third subset of $[0, 1]$.

Borel sets

The *κ-Baire space* is the space ${}^{\kappa}\kappa = \{x \mid x\colon \kappa \to \kappa\}$ with the bounded topology.

The basic open sets are $N_t = \{x \in {}^{\kappa}\kappa \mid t \subseteq x\}$ for $t \in {}^{<\kappa}\kappa$.

The *κ-Cantor space* is the subspace ${}^{\kappa}2$ with the relative topology.

Both are 0-*dimensional*, since the sets N_t are clopen.

Borel sets

The collection of κ-*Borel subsets* of ${}^\kappa\kappa$ is generated from the open sets by forming complements and unions of length at most κ.

A subset of ${}^\kappa\kappa$ is called Σ^0_1 if it is open in the bounded topology.

A subset of ${}^\kappa\kappa$ is called Π^0_γ if its complement is Σ^0_γ.

A subset of ${}^\kappa\kappa$ is called Σ^0_γ for $\gamma > 0$ if it is of the form $A = \bigcup_{\alpha<\kappa} A_\alpha$, where each A_α is Π^0_β for some $\beta < \gamma$.

It is easy to see that the collection of κ-Borel sets is equal to the union of Σ^0_γ for all $\gamma < \kappa^+$.

Borel sets

Formulas $\varphi(i_0, \ldots, i_m, x_0, \ldots, x_n)$ have two types of variables:

- for ordinals $i < \kappa$ and
- for elements $x \in {}^\kappa\kappa$.

Atomic formulas are of the form $i = j$, $i < j$ and $x(i) = j$.

A formula $\varphi(i, x)$ is Σ^0_n if it is of the form

$$\exists i_0 < \kappa \; \forall i_1 < \kappa \; \ldots \forall/\exists i_{n-1} < \kappa \; \psi(i_0, \ldots, i_{n-1}, i, x),$$

where ψ is quantifier-free.

Note that a subset A of ${}^\kappa\kappa$ is Σ^0_n if and only if it is definable by a Σ^0_n formula with a parameter in ${}^\kappa\kappa$.

Borel sets

A Σ^0_α subset B of ${}^\kappa\kappa$ is called Σ^0_α-*complete* if for every Σ^0_α set A, there is a continuous function $f\colon {}^\kappa\kappa \to {}^\kappa\kappa$ with $A = f^{-1}(B)$.

$\exists^{ub}\alpha\ \varphi(\alpha)$ means that there are unboundedly many such $\alpha < \kappa$.

$B_1 = \{x \in {}^\kappa 2 \mid \exists\alpha\ x(\alpha) = 1\}$ is Σ^0_1-complete.

$B_2 = \{x \in {}^\kappa 2 \mid \exists^{ub}\alpha\ x(\alpha) = 1\}$ is Π^0_2-complete.

$B_3 = \{(x_\alpha) \in ({}^\kappa 2)^\kappa \mid \exists\alpha\ \exists^{ub}\beta\ x_\alpha(\beta) = 1\}$ is Σ^0_3-complete.

Borel sets

Any Σ^0_α-complete set is Σ^0_α but not Π^0_α.

Hence the κ-Borel hierarchy has length κ^+.

Borel sets

Suppose that $X \subseteq Y$. A *retraction* $f\colon Y \to X$ is a continuous function with $f{\upharpoonright}X = \mathrm{id}_X$.

Proposition

Every closed subset C of ${}^{\omega}\omega$ is a retract of ${}^{\omega}\omega$.

Proof.

Let $T = \{t \in {}^{<\omega}\omega \mid \exists x \in C\ t \subseteq x\}$.

Let $f{\upharpoonright}C = \mathrm{id}_C$.

Borel sets

For $x \notin C$, let t_x be the longest initial segment of x in T.

Let $f(x)$ be an arbitrary element y of C with $t_x \subseteq y$.

Then $f\colon {}^{\omega}\omega \to C$ is a retraction. □

Borel sets

Proposition (Lücke-S.)

There is a closed subset of ${}^{\kappa}\kappa$ that is not a retract of ${}^{\kappa}\kappa$.

Proof.

Let T denote the subtree of ${}^{<\kappa}\kappa$ consisting of all $t \in {}^{<\kappa}\kappa$ such that $t(\alpha) = 0$ for only finitely many $\alpha \in dom(t)$.

Let $C = [T]$.

Borel sets

Suppose that $f\colon {}^{\kappa}\kappa \to C$ is a continuous retraction onto C.

We inductively construct sequences $\langle x_n \in C \mid n < \omega \rangle$ and $\langle \alpha < \kappa \mid n < \omega \rangle$ such that

- $x_n(\alpha_n) = 0$,
- $\gamma_n < \alpha_{n+1}$, and
- $x_n \upharpoonright \alpha_{n+1} = x_{n+1} \upharpoonright \alpha_{n+1}$ for all $n < \omega$.

Let $\alpha_0 = 0$ and x_0 be an arbitrary element of C with $x_0(0) = 0$.

Borel sets

Now assume that x_n and α_n are already constructed.

Since $f(x_n) = x_n$ and f is continuous, we can find some $\alpha_{n+1} > \alpha_n$ with $f[N_{x_n \restriction \alpha_{n+1}}] \subseteq N_{x_n \restriction (\alpha_n + 1)}$.

Pick $x_{n+1} \in C \cap N_{x_n \restriction \alpha_{n+1}}$ with $x_{n+1}(\alpha_{n+1}) = 0$.

Now pick $x \in {}^{\kappa}\kappa$ with $x_n \restriction \alpha_{n+1} \subseteq x$ for all $n < \omega$.

Then $f(x)(\alpha_n) = x_n(\alpha_n) = 0$ for all $n \in \omega$.

Hence $f(x) \notin C$, contradicting our assumption on f. □

Borel sets

Lemma

Suppose that X is a Polish metric space, F is an Σ^0_2 subset of X and $\epsilon > 0$.

Then there is a partition $\langle F_n \mid n \in \omega \rangle$ of F into Σ^0_2 sets of diameter at most ϵ whose closures are contained in F.

Proof.

Let $\langle C_n \mid n \in \omega \rangle$ be a sequence of closed sets with diameter at most ϵ with $F = \bigcup_{n \in \omega} C_n$.

Let $F_n = C_n \setminus \bigcup_{i<n} C_i$. Its closure is contained in C_n.

Since F_n is the intersection of two Σ^0_2 sets, it is also Σ^0_2.

Borel sets

Proposition

Every nonempty Polish space X is a continuous image of ${}^{\omega}\omega$.

Proof.

We construct a sequence $\langle C_t \mid t \in {}^{<\omega}\omega \rangle$ of Σ^0_2 subsets of X by induction with $C_\emptyset = X$.

For all $t \in {}^{<\omega}2$, the sets $C_{t^\frown \langle i \rangle}$ partition C_t into sets of diameter at most $\frac{1}{2^{|t|}}$ whose closures are contained in C_t.

Let T denote the tree of all $t \in {}^{<\omega}2$ with $C_t \neq \emptyset$.

Note that $\bigcap_{n\in\omega} C_{x \upharpoonright n}$ contains a unique element $f(x)$ for each $x \in [T]$.

Borel sets

Then $f\colon [T] \to X$ is a continuous bijection.

Since $[T]$ is a retract of ${}^{\omega}\omega$, X is a continuous image of X.

Analytic sets

A subset of ${}^\kappa\kappa$ is *(κ)-analytic* if it satisfies one of the following equivalent properties:

- There is a closed subset C of ${}^\kappa\kappa$ and a continuous function $f\colon {}^\kappa\kappa \to {}^\kappa\kappa$ with $f(C) = A$.
- There is a subtree of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ with $p[T] = \{x \in {}^\kappa\kappa \mid \exists y \in {}^\kappa\kappa\ (x,y) \in [T]\} = A$.
- Is is definable by some Σ^1_1 formula φ over ${}^\kappa\kappa$: φ is of the form $\exists y \in {}^\kappa\kappa\ \psi(x,y)$, where $\psi(x,y)$ contains only quantifiers ranging over κ.

Analytic sets

Proposition

Every nonempty analytic subset A of a Polish space Y is a continuous image of ${}^{\omega}\omega$.

Proof.

Let $f\colon X \to Y$ be continuous with $\mathrm{ran}(f) = A$.

Let $g\colon {}^{\omega}\omega \to X$ be a continuous surjection.

Then $f \circ g\colon {}^{\omega}\omega \to A$ is a continuous surjection.

Analytic sets

Proposition (Lücke-S.)

There is a closed subset of ${}^\kappa\kappa$ that is not a continuous image of ${}^\kappa\kappa$.

Proof.

Given $\gamma \leq \kappa$ closed under Gödel pairing and $s\colon \gamma \to 2$, we define
$R_s \;=\; \{(\alpha,\beta) \in \gamma \times \gamma \mid s(\langle \alpha,\beta \rangle) = 1\}$.

Let $W \;=\; \{x \in {}^\kappa 2 \mid (\kappa, R_x) \textit{ is a well-order}\}$.

Let $S = \{x \restriction \alpha \mid x \in W,\ \alpha < \kappa\}$.

Given $x \in W$ and $\alpha < \kappa$, let $\mathrm{rank}_\alpha(x)$ denote the rank of α in (κ, R_x).

We will write $\alpha <_x \beta$ instead of $(\alpha,\beta) \in R_x$.

Analytic sets

It is sufficient to show the next lemma.

Lemma

There is no ω-closed tree subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ with cofinal branches through all its nodes and $W = p[T]$.

Analytic sets

Proof.

Assume, towards a contradiction, that T is such a tree.

Given $(s,t) \in T$ and $\alpha < \kappa^+$, we define

$$T_{s,t} \ = \ \{(u,v) \in T \mid (s \subseteq u \vee u \subseteq s) \wedge (t \subseteq v \vee v \subseteq t)\}$$

to be subtree of T induced by the node (s,t).

Let $r(s,t,\alpha) \ = \ \sup\{\mathrm{rank}_\alpha(x) \mid x \in p[T_{s,t}]\} \ \leq \ \kappa^+$ to be the supremum of the ranks of α in (κ, R_x) with $x \in p[T_{s,t}]$.

Then we have $r(\emptyset,\emptyset,\alpha) = \kappa^+$ for all $\alpha < \kappa$.

Analytic sets

Claim

Let $(s,t) \in T$ *and* $\alpha < \kappa$ *with* $r(s,t,\alpha) = \kappa^+$. *If* $\gamma < \kappa^+$, *then there is* $(u,v) \in T$ *extending* (s,t) *and* $\beta < \kappa$ *such that*

- $\mathrm{dom}(u)$ *is closed under Gödel pairing,*
- $\alpha < \beta < \mathrm{length}(u)$,
- $\beta <_u \alpha$, *and*
- $r(u,v,\beta) \geq \gamma$.

Analytic sets

Proof of the Claim.

The assumption $r(s, t, \alpha) = \kappa^+$ allows us to find $(x, y) \in [T]$ with $s \subseteq x$, $t \subseteq y$ and $\mathrm{rank}_\alpha(x) \geq \gamma + \kappa$.

This implies that there is a β with $\alpha < \beta < \kappa$ and $\gamma \leq \mathrm{rank}_\beta(x) < \gamma + \kappa$.

Pick $\delta > \max\{\alpha, \beta, \mathrm{length}(s)\}$ closed under Gödel pairing and define (u, v) to be the node $(x{\restriction}\delta, y{\restriction}\delta)$ extending (s, t).

Since R_u is a well-ordering of $\mathrm{length}(u)$, we have $\beta <_u \alpha$.

Finally, (x, y) witnesses that $r(u, v, \beta) \geq \gamma$. □

Analytic sets

Claim

If $(s,t) \in T$ *and* $\alpha < \kappa$ *with* $r(s,t,\alpha) = \kappa^+$, *then there is a node* (u,v) *in* T *extending* (s,t) *and* $\alpha < \beta < \mathrm{length}(u)$ *such that*

- $\mathrm{length}(u)$ *is closed under Gödel pairing,*
- $\beta <_u \alpha$ *and*
- $r(u,v,\beta) = \kappa^+$.

Proof of the claim.

We obtain $\gamma < \kappa^+$, let $(u_\gamma, v_\gamma) \in T$ and $\beta_\gamma < \kappa$ by the previous claim.

Then we can find $(u,v) \in T$, $\beta < \kappa$ and $X \in [\kappa^+]^{\kappa^+}$ such that $(u,v) = (u_\gamma, v_\gamma)$ and $\beta = \beta_\gamma$ for every $\gamma \in X$.

This implies that $r(u,v,\beta) = \kappa^+$. □

Analytic sets

Hence there are strictly increasing sequences $\langle (s_n, t_n) \mid n < \omega \rangle$ of nodes in T and $\langle \beta_n \mid n < \omega \rangle$ of elements of κ such that

- $\mathrm{length}(s_n)$ is closed under Gödel pairing and
- $\beta_{n+1} <_{s_{n+1}} \beta_n$ for all $n < \omega$.

Let $s = \bigcup_{n<\omega} s_n$ and $t = \bigcup_{n<\omega} t_n$.

Then $\mathrm{length}(s)$ is closed under Gödel pairing and R_s is ill-founded, hence $s \notin S$.

But $(s, t) \in T$, since T is ω-closed.

By our assumptions on T, there is a cofinal branch (x, y) in $[T]$ through (s, t) and this implies that $s = x \restriction \mathrm{length}(s) \in S$, a contradiction. □

Analytic sets

A *perfect subset* of ${}^{\omega}\omega$ is one that is closed and homeomorphic to ${}^{\omega}2$.

A subset of ${}^{\omega}\omega$ has the *perfect set property* if it either

- is countable, or
- has a perfect subset.

Analytic sets

Proposition

Every analytic subset of ${}^{\omega}\omega$ has the perfect set property.

Proof.

Let $f\colon {}^{\omega}\omega \to A$ be a continuous surjection.

We construct $\langle t_s \mid s \in {}^{<\omega}2 \rangle$ such that

- $t_{\emptyset} = \emptyset$
- If $s \subsetneq s'$, then $t_s \subsetneq t_{s'}$.
- $f(N_{t_s})$ is uncountable for all $s \in {}^{<\omega}2$.
- $f(N_{t_{s^\frown 0}}) \cap f(N_{t_{s^\frown 1}}) = \emptyset$ for all $s \in {}^{<\omega}2$.

Analytic sets

Suppose that t_s is constructed such that $f(N_{t_s})$ is uncountable.

Then there are extensions $u \perp v$ of t_s with both $f(N_{t_u})$ and $f(N_{t_v})$ uncountable.

Otherwise all $u \supseteq t_s$ with $f(N_u)$ uncountable lie on a single branch.

Let $f \colon {}^{\omega}2 \to A$ with $f(x) = \bigcup_{n \in \omega} t_{x \restriction n}$.

One can extend u and v so that $f(N_{t_u})$ and $f(N_{t_v})$ are disjoint.

Then f is a continuous injection into A. By compactness, $f(A)$ is perfect. $\square$

Analytic sets

A *κ-perfect* subset of ${}^{\kappa}\kappa$ is one that is closed and homeomorphic to ${}^{\kappa}2$.

A subset of ${}^{\kappa}\kappa$ has the *κ-perfect set property* if it either

- has either size at most κ, or
- has a κ-perfect subset.

A tree of height ω_1 is a *Kurepa tree* if

- it has at least ω_2 many cofinal branches, and
- all levels are countable.

A tree is called *κ-perfect* if it is $<\kappa$-closed and every node extends to a splitting node.

Analytic sets

Proposition

If $\kappa = \omega_1$ and T is a Kurepa subtree of ${}^{<\kappa}2$, then $[T]$ is a counterexample to the κ-perfect set property.

Proof.

Suppose that $[T]$ has a κ-perfect subset.

Then T has a κ-perfect subtree S.

Let $\mathrm{Split}_{<\omega}(S)$ denote the union of all finite splitting levels of S.

Let $\alpha = \sup_{s \in \mathrm{Split}_{<\omega}(S)} \mathrm{ht}(s) < \omega_1$.

Then the α-th level $\mathrm{Lev}_\alpha(T)$ of T is uncountable. □

Analytic sets

It is consistent that the κ-perfect set property holds for all analytic subsets of ${}^{\kappa}\kappa$, and in fact much more complex sets.

In a model of set theory where the κ-perfect set property holds for all closed subsets of ${}^{\kappa}\kappa$ for $\kappa = \omega_1$, there are no Kurepa trees.

The Baire property

A set $Y \subseteq X$ has the *Baire property* if there is an open set $U \subseteq X$ with the property that $U \triangle Y$ is meager.

A function $f : X \to Y$ is called *Baire measurable* if pre-images of open sets have the Baire property.

The Baire property

A set $Y \subseteq {}^{\kappa}\kappa$ is *κ-meager* if it is the union of κ many nowhere dense subsets of ${}^{\kappa}\kappa$.

A set $Y \subseteq {}^{\kappa}\kappa$ has the *κ-Baire property* if there is an open set $U \subseteq {}^{\kappa}\kappa$ with the property that $U \triangle Y$ is κ-meager.

A function $f : {}^{\kappa}\kappa \to {}^{\kappa}\kappa$ is called *κ-Baire measurable* if pre-images of open sets have the κ-Baire property.

The Baire property

Lemma

Suppose that $A \subseteq {}^{\kappa}\kappa$ has the Baire property. Then the following properties are equivalent:

- *A is not meager.*
- *A is comeager in some non-empty open set.*

Proof.

By Baire's category theorem. □

The Baire property

Proposition

Every analytic subset of a Polish space X has the Baire property.

Proposition (Halko)

There is an analytic subset of ${}^{\kappa}\kappa$ without the κ-Baire property.

We will show something a bit stronger.

The Baire property

Suppose that A, B are disjoint subsets of ${}^{\kappa}\kappa$. We say that C *separates* A and B if $A \subseteq C$ and $B \subseteq \neg C$.

The *Club filter* C_κ on κ is the set of $x \in {}^{\kappa}2$ such that $\{\alpha < \kappa \mid x(\alpha) = 1\}$ contains a club.

The *non-stationary ideal* NS_κ on κ is the set of $x \in {}^{\kappa}2$ such that $\{\alpha < \kappa \mid x(\alpha) = 0\}$ contains a club.

Both are analytic subsets of ${}^{\kappa}2$.

The Baire property

Proposition (Väänänen)

The club filter and the non-stationary ideal on κ cannot be separated by a set with the κ-Baire property.

A subset A of ${}^{\kappa}2$ is called *super-dense* if $A \cap (\bigcap_{\alpha<\kappa} U_\alpha) \neq \emptyset$, whenever $\langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of dense open subsets of some non-empty open subset of ${}^{\kappa}2$.

Lemma

The club filter and the non-stationary ideal are super-dense.

Proof.

It is sufficient to prove this for the club filter C_κ.

The Baire property

Suppose that $\langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of dense open subsets of N_t.

We construct a strictly increasing sequence $\langle t_\alpha \mid \alpha < \kappa \rangle$ in ${}^{<\kappa}\kappa$.

Let $t_0 = t$.

Let $t_{\alpha+1}$ be chosen such that $U_\alpha \cap N_{t_{\alpha+1}} \neq \emptyset$.

For limits $\gamma < \kappa$, let $u = \bigcup_{\alpha<\gamma} t_\alpha$ and $t_\gamma = u \cup \{(|u|, 1)\}$.

Then $x = \bigcup_{\alpha<\kappa} t_\alpha \in \mathrm{C}_\kappa \cap \bigcap_{\alpha<\kappa} U_\alpha$. $\square$

The Baire property

Lemma

If A and B are disjoint super-dense sets, then they cannot be separated by a set with the κ-Baire property.

Proof.

Suppose that C is a set with the κ-Baire property that separates A and B.

Let $U \subseteq {}^{\kappa}2$ be open such that $C \triangle U$ is κ-meager.

Suppose that $U \neq \emptyset$.

The Baire property

Since $C \triangle U$ is κ-meager, there is a sequence $\langle U_\alpha \mid \alpha < \kappa \rangle$ of dense open subsets of U such that $C \triangle U$ is disjoint from $\bigcap_{\alpha<\kappa} U_\alpha$.

Since NS_κ is super-dense, there is some $x \in B \cap U \cap \bigcap_{\alpha<\kappa} U_\alpha$.

Then $x \in C$. But $x \in B$ and B is disjoint from C.

If $U = \emptyset$, a similar argument works by exchanging the roles of A and B.
$\square$

Equivalence relations

We now connect the perfect set property and the Baire property with the study of equivalence relations.

Silver showed that every coanalytic equivalence relation either

- has countably many equivalence classes, or
- there is a perfect set of pairwise inequivalent reals.

Friedman showed that the analogue to Silver's theorem for κ-Borel equivalence relations on ${}^\kappa\kappa$ is consistent relative to $0^\#$.

Equivalence relations

Lemma

If Silver's theorem is valid for all κ-Borel equivalence relations, then all κ-Borel sets have the perfect set property.

Proof.

Suppose that $A \subseteq {}^{\kappa}\kappa$ is a κ-Borel set of size at least κ^+.

Then

$$E = \{(x, y) \in ({}^{\kappa}\kappa)^2 \mid x = y \vee x, y \notin A\}$$

is a κ-Borel equivalence relation with at least κ^+ many equivalence classes.

The assumption yields a κ-perfect subset of A. $\square$

Equivalence relations

Suppose that E and F are equivalence relations on Borel subsets X, Y of Polish spaces.

A *Borel reduction* from E to F is a Borel measurable function $f\colon X \to Y$ with

$$(x, y) \in E \iff (f(x), f(y)) \in F.$$

We write $E \leq_B F$ if such a reduction exists.

Moreover, E is called *smooth* if $E \leq_B \mathrm{id}_Z$ for some Polish space Z.

Equivalence relations

Suppose that E and F are equivalence relations on κ-Borel subsets of ${}^{\kappa}\kappa$.

A *κ-Borel reduction* from E to F is a κ-Borel measurable function $f\colon {}^{\kappa}\kappa \to {}^{\kappa}\kappa$ with

$$(x, y) \in E \iff (f(x), f(y)) \in F.$$

We write $E \leq_B F$ if such a reduction exists.

Moreover, E is called *smooth* if $E \leq_B \mathrm{id}_{{}^{\kappa}\kappa}$.

Equivalence relations

E_0 is the equivalence relation on ${}^{\omega}2$ defined by $(x, y) \in E_0$ if the set $\{n \in \omega \mid x(n) \neq y(n)\}$ is bounded.

E_0^κ is the equivalence relation on ${}^{\kappa}2$ defined by $(x, y) \in E_0^\kappa$ if the set $\{\alpha < \kappa \mid x(\alpha) \neq y(\alpha)\}$ is bounded.

Equivalence relations

Lemma

Suppose that $B \subseteq {}^{\kappa}2$ is an E_0^{κ}-invariant set with the κ-Baire property. Then B is either κ-meager or κ-comeager.

Proof.

Suppose that B is not κ-meager.

Fix $s \in 2^{\alpha}$ such that B is κ-comeager in N_s.

Equivalence relations

It is sufficient to show that B is κ-comeager in N_t for all $t \in 2^n$.

Let $f\colon {}^\kappa 2 \to {}^\kappa 2$ with $f(s^\frown x) = t^\frown x$, $f(t^\frown x) = s^\frown x$ and $f(u^\frown x) = u^\frown x$ for all $u \neq s, t$ in $2^{|s|}$.

Then $(x, f(x)) \in E_0^\kappa$ for all $x \in {}^\kappa 2$. Hence $f(B) \subseteq B$.

So B is κ-comeager in N_t. □

Equivalence relations

Proposition

E_0^κ is not smooth.

Proof.

Suppose that $f\colon {}^\kappa\kappa \to {}^\kappa\kappa$ is a κ-Baire measurable reduction from E_0^κ to $\mathrm{id}_{{}^\kappa\kappa}$.

It is sufficient to find some $x \in {}^\kappa\kappa$ such that $f^{-1}(\{x\})$ is κ-comeager.

Then $f^{-1}(\{x\})$ is not contained in one equivalence class.

Equivalence relations

For each $t \in {}^{<\kappa}2$, $f^{-1}(N_t)$ is either κ-meager or κ-comeager.

Thus for each $\alpha < \kappa$, there is a unique $t_\alpha \in 2^\alpha$ such that $f^{-1}(N_{t_\alpha})$ is κ-comeager.

Let $x = \bigcup_{\alpha<\kappa} t_\alpha$. Then $f^{-1}(\{x\}) = \bigcap_{\alpha<\kappa} f^{-1}(N_{t_\alpha})$ is κ-comeager. □

Equivalence relations

Harrington, Kechris and Louveau proved the E_0-dichotomy: E_0 is Borel reducible to every non-smooth Borel equivalence relation E on ${}^\kappa\kappa$.

Hyttinen proved that the analogue for ${}^\kappa\kappa$ fails: E_0^κ is not κ-Borel reducible to all non-smooth κ-Borel equivalence relations on ${}^\kappa\kappa$.

Equivalence relations

Suppose that $\mathcal{L} = \{R_i \mid i < \lambda\}$ is a relational language of size $|\mathcal{L}| = \lambda \leq \kappa$. Assume that R_i has arity n_i.

The *space of $\mathcal{L}$-structures*

$$\mathrm{Mod}_{\mathcal{L}} = \prod_{i<\lambda} 2^{\kappa^{n_i}}$$

with the $<\kappa$-support topology.

Its basic open sets are of the form $\prod_{\alpha<\lambda} N_{t_\alpha}$, where $t_\alpha \neq \emptyset$ for strictly less than κ many $i < \lambda$.

$\mathrm{Mod}_{\mathcal{L}}$ is homeomorphic to 2^κ.

Equivalence relations

Let $\mathcal{C}$ denote κ-Borel classes of $\mathcal{L}$-structures closed under isomorphism.

Let $\cong_{\mathcal{C}}$ denote the isomorphism relation on $\mathcal{C}$.

A class $\mathcal{C}$ is called *Borel complete* if for every other κ-Borel class $\mathcal{B}$

$$\cong_{\mathcal{B}} \leq_B \cong_{\mathcal{C}} .$$

Equivalence relations

Proposition

The class of (undirected) graphs is Borel complete.

Proof.

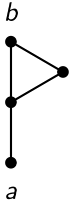
Suppose that M is a class of $\mathcal{L}$-structures.

Let $\mathcal{L} = \{R_i \mid i < \lambda\}$, where $\lambda \leq \kappa$ and n_i the arity of R_i.

We need to associate a graph $\Gamma(M)$ on κ to each $M \in \mathcal{C}$.

Equivalence relations

The following graph G is used to mark the direction from a to b.



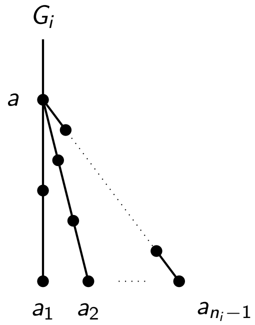
For each ordinal $\gamma < \kappa$, we consider a graph G_γ obtained from the graph $(\gamma, <)$ by replacing all edges between $\alpha < \beta$ with copies of G.

Note that G_γ codes γ.

Equivalence relations

By an *i-tag* we mean the following graph H_i.

It is obtained by connecting the bottom node of G_i with the node a.



Equivalence relations

We now glue an i-tag to every n_i-tuple $(a_0, \ldots, a_{n_i-1}) \in R_i$.

Let $\Gamma(M)$ an isomorphic copy on κ of this graph.

If $\Gamma(M)$ is chosen in a reasonable way, then Γ is continuous.

We can assume there is a trivial unary relation; then $\Gamma(M)$ codes M. $\square$

Equivalence relations

The next example illustrates a way of showing that an equivalence relation is not smooth.

Proposition

E_0^κ is κ-Borel reducible to graph isomorphism GI_κ.

Proof.

For any $x \in {}^\kappa 2$, let $G_x = (\kappa, <, \{\alpha < \kappa \mid x(\alpha) = 1\})$.

Let f send $x \in {}^\kappa 2$ to the disjoint union of all graphs G_y for $y \in [x]_{E_0^\kappa}$.

If f is chosen in a reasonable way, then it is continuous. □

Equivalence relations

Let $E_S = \{(x, y) \in ({}^{\kappa}2)^2 \mid (x^{-1}(\{1\}) \triangle y^{-1}(\{1\})) \cap S \in \mathrm{NS}_\kappa\}$.

Equivalence relations of the form E_S are important in recent work of Hyttinen, Kulikov and Moreno on classification theory.

They connect model-theoretic properties of a class $\mathcal{C}$ with the fact that $E_{S^\kappa_\omega}$ reduces to $\cong_{\mathcal{C}}$, where $S^\kappa_\lambda = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \omega\}$.

Equivalence relations

Suppose that S is a stationary subset of κ.

A $\Diamond_\kappa(S)$-sequence $\langle S_\alpha \mid \alpha \in S \rangle$ is a sequence of subsets of κ such that for every $A \subseteq \kappa$,

$$\{\alpha \in S \mid A \cap \alpha = S_\alpha\}$$

is stationary.

$\Diamond_\kappa(S)$ states that such a sequence exists.

Equivalence relations

Proposition (Kulikov)

Assume that $\diamondsuit_\kappa(S)$ holds. Then E_0^κ is Borel reducible to E_S.

Proof.

We define a reduction $f\colon {}^\kappa 2 \to {}^\kappa\kappa$ as follows.

Let $f(X) = \{\alpha \in S \mid S_\alpha \text{ and } X \cap \alpha \text{ agree on a final segment of } \alpha\}$.

Equivalence relations

If X, Y are E_0-equivalent, then $f(X)$, $f(Y)$ are E_S-equivalent, since they are even E_0-equivalent.

If X, Y are not E_0-equivalent, then there is a club C of α where X, Y differ cofinally below α.

Hence $f(X)$, $f(Y)$ differ on a stationary subset of S: those $\alpha \in C \cap S$ with $S_\alpha = X \cap \alpha$. □

Outlook

Many important results from descriptive set theory are open in the setting of generalized Baire spaces.

For instance, an analogue to the G_0-dichotomy.

An analogue to Borel uniformization for Borel relations with countable sections.

Analogues to various regularity properties.