Aspects of relation algebras

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Outline

1. Introduction to relation algebras.
   Representations.
   Atom structures.
   Examples.
2. Using games to build representations.
   Axiomatising the finite representable relation algebras.
3. 'Rainbow' construction (Hirsch).
   Monk's theorem that the representable relation algebras are not finitely axiomatisable.
4. Relativised representations. Weakly associative algebras (WA, Maddux).
   Finite base property for WA by Herwig's theorem.

Why relation algebras?

- They have a long history—De Morgan Morgan 1860, Peirce Hartshorne and Weiss 1933, Schröder Schröder 1895, Tarski Tarski 1941, Lyndon Lyndon 1950, Monk Monk 1964, Andréka Andréka 1997, Németi Németi 1986, Maddux Maddux 1978, ...
- They provide a simple introduction to parts of algebraic logic.
- Algebraic logic has strong connections to modal and dynamic logics. Recent Dutch–Hungarian work: e.g., **?**,van Benthem and Németi. 1998. Techniques cross over, both sides gain. *Arrow logic* Marx 1996 is the modal analogue of relation algebras.
- Relation algebras have practical applications in computing: eg. databases, artificial planning, specification theory (eg. fork algebras), concurrency Maddux 1996. RelMiCS group.
- Can develop all of mathematics by relation algebras. See Tarski and Givant 1987.

General references

Good introductions to relation algebras and more can be found in Maddux , Németi 1991., Monk and Tarski 1971, Monk and Tarski 1985.

0.1 Boolean algebras and relation algebras

Relation algebras are based on boolean algebras, and we review these first.

0.1.1 Boolean algebras

Definition 0.1.1 A boolean algebra is an algebra $\mathcal{B} = \langle\{\rangle B, +, \cdot, -, 0, 1\}$ satisfying, for all $x, y, z \in B$:

- $+$ is associative and commutative: $^{(}x + y) + z = x + (y + z)$ and $x + y = y + x$
- complement: $-(-x) = x$
- form of distributivity: $x = x \cdot y + x \cdot -y$
- connections: $^{x} \cdot y = -(-x + -y)$, $x + (-x) = 1$, $-1 = 0$.

Abbreviation: $x \leq y$ means $x \cdot y = x$ or equivalently $x + y = y$.

[1]The material in §0.2–0.3 is joint work with Robin Hirsch. Research partially supported by UK EPSRC grants GR/K54946, GR/L85978.

Representations of boolean algebras

The motivation for this definition comes from *fields of sets.* If U is a set, $\wp U$ denotes the power set (set of all subsets) of U. Suppose that $B \subseteq \wp U$ contains $\emptyset, U$ and is closed under union, intersection, complement. For example, $B = \wp U$ itself. Then $\langle B, \cup, \cap, U \setminus -, \emptyset, U\rangle$ is a boolean algebra, called a *field of sets.*

In fact, every boolean algebra is isomorphic to a field of sets (Stone's theorem, Stone 1936).

Unary and binary relations

A *unary relation* on a set U is just a subset of U. So boolean algebras are to do with unary relations. In fact, they embody all truths about unary relations (by Stone's theorem).

Can we do the same for *binary relations* (subsets of $U \times U$)?

0.1.2 Binary relations

We can consider *fields of binary relations on U.* These are subsets A of $\wp(U \times U)$. We want A to be closed under the boolean operations, as before. But there are natural operations on *binary relations* that we want to consider too.

Operations on binary relations

Which operations would we like?

Consider the binary relations *son, daughter.*

son(x,y) = y is a son of x, etc.

$$\text{x} \xrightarrow{\text{son}} \text{y}$$

From these, we know we can derive many other relations, such as:

- child
- parent
- father, mother, brother, sister
- grandchild, grandson
- grandfather, grandmother, grandparent
- aunt, uncle, niece, nephew.

Can we think of basic operations that will let us form these complex relations from *son* and *daughter*?

The relation algebra operations

Clearly, child = son + daughter. This is still boolean; we use + rather than $\cup$.

If we can form the *converse* of a binary relation:

$$R^{\smile}(x, y) \iff R(y, x)$$

then we can do parent:

- parent = child$^{\smile}$, or
- parent = (son + daughter)$^{\smile}$, or
- parent = son$^{\smile}$ + daughter$^{\smile}$.

If we can form the *composition* of two relations:

$$R \,;S(x, y) \iff \exists z(R(x, z) \wedge S(z, y))$$

then we can do more:

- grandchild = child ; child
- grandparent = parent ; parent

Need for equality

Can we say 'brother'?

How about parent ; son ?

Trouble is, parent(me,mum) $\wedge$ son(mum,me). So $\exists z$(parent(me,z) $\wedge$ son(z,me)). So by definition, [parent ; son](me,me). But I am not a brother of me.

But suppose we can express equality, by adding a constant 1' for it. Then

- brother = (parent ; son) $\cdot -$ 1' ($-$ is boolean complement)

With 1' we can express 'mother' too:

- mother = ((parent ; daughter) $\cdot$ 1') ; parent
- father = parent $\cdot -$mother.

The non-boolean operations we usually take are indeed ; , $^\smile$, and 1'.

(1) Can you express *sister* with these operations? How about *nephew*? And *aunt*?

0.1.3 Proper relation algebras

If we use these operations on binary relations, we get a new kind of algebra.

Definition 0.1.2 A *proper relation algebra* is an algebra

$$\mathcal{A} = \langle A, \cup, \cap, E \setminus -, \emptyset, E, =, ^\smile, ; \rangle$$

where $A \subseteq \wp(U \times U)$ for some set U (the 'base'), E is an equivalence relation on U, and the operations are as explained already. (A must be closed under the operations—so, e.g., $\emptyset, E, = \in A$). Usually E will be $U \times U$ itself. But for technical reasons we don't insist on this.

Eg: $\mathcal{A} = \langle \wp(U \times U), \cup, \cap, \setminus, \emptyset, U \times U, =, ^\smile, ; \rangle$, called the *full power set algebra on U*.
Question: Can we axiomatise these algebras, like we did for boolean algebras?

0.1.4 Abstract relation algebras, representations

Definition 0.1.3 (Tarski, 1940s)
A *relation algebra* is an algebra $\mathcal{A} = \langle A, +, \cdot, -, 0, 1, 1', ^\smile, ; \rangle$ where the following hold for all $x, y, z \in A$:

(R0) the axioms for boolean algebras in definition 0.1.1
(R1) $(x \,; y) \,; z = x \,;(y \,; z)$ (associativity)
(R2) $(x + y) \,; z = x \,; z + y \,; z$
(R3) $x \,; 1' = x$
(R4) $x^{\smile\smile} = x$
(R5) $(x + y)^\smile = x^\smile + y^\smile$
(R6) $(x \,; y)^\smile = y^\smile \,; x^\smile$
(R7) $x^\smile \,; -(x \,; y) \leq -y$.

These axioms are an attempted analogue, for binary relations, of the BA axioms. Do they work?

Definition 0.1.4 A *representation* of a relation-type algebra $\mathcal{A}$ is an isomorphism from $\mathcal{A}$ to a proper relation algebra. $\mathcal{A}$ is *representable* if it has a representation.

Definition 0.1.5 RA is the class of all relation algebras. RRA is the class of representable relation-type algebras.

It's easily seen that the RA axioms are sound:

(2) RRA $\subseteq$ RA.

Equivalently, a proper relation algebra is a relation algebra.
Can we prove an analogue of Stone's theorem, showing completeness? That is, is *any* relation algebra representable?

Sadly, no [Lyndon, 1950]:

$$\mathsf{RRA} \subset \mathsf{RA}.$$

We prove this in example 0.1.12. Moreover, RRA is not finitely axiomatisable in first-order logic [Monk, 1964]: see section 0.3. In fact, RRA is not finitely axiomatisable in 2nd-order logic, 3rd-order logic, ... It can't be axiomatised by equations using a finite set of variables, nor by Sahlqvist equations. See Hirsch and Hodkinson 1999: To appear, Hodkinson 1988, Andréka 1997, Venema 1997.

At least it is recursively axiomatisable (see section 0.2).

0.1.5 Atoms and atom structures

We will mostly consider finite algebras. For these, it's easier to work with their atoms.

Definition 0.1.6 An *atom* of a (boolean algebra or) relation algebra is a $\leq$-minimal non-zero element. A relation algebra is *atomic* if every non-zero element is $\geq$ an atom.

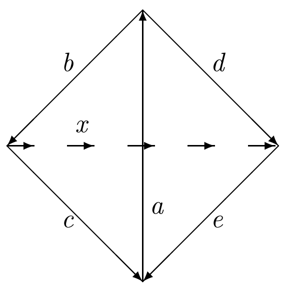
Any finite relation algebra is atomic. It can be shown that in an atomic relation algebra $\mathcal{A}$:
• 1' is determined by {atoms a : a $\leq$ 1'}.
• if a is an atom then so is $a^{\smile}$. And $^{\smile}$ is determined by its values on atoms.
• ; in $\mathcal{A}$ is determined by the set of triples of atoms (x, y, z) such that $x\,;y \geq z^{\smile}$. We call these *consistent triangles.*

Atom structures

So if we specify
• a finite set X of atoms,
• the set Id of atoms under 1',
• how the function $^{\smile}$ behaves on the atoms,
• the set C of consistent triangles,
is there a relation algebra with these atoms?

Not quite:

Definition 0.1.7 A *'relation algebra atom structure'* is a structure $(X, Id, {}^{\smile}, C)$ satisfying:
• $x^{\smile\smile} = x$ (for all $x \in X$)
• $x = y^{\smile}$ iff $\exists z \in Id((x, y, z)$ consistent)
• if (x, y, z) is consistent then so are (y, z, x) and $(z^{\smile}, y^{\smile}, x^{\smile})$ ('Peircean transforms')
• diamond completion (for associativity):



If $(a, b, c), (a, d, e)$ are consistent then there is x with $(b, x, d^{\smile}), (c^{\smile}, x, e)$ consistent.

(3) Show that if $a \in Id$ then $a^{\smile} = a$.

Atom structures and relation algebras

Definition 0.1.8 For an atomic relation algebra $\mathcal{A}$, write $At\mathcal{A}$ for its *atom structure*:

$$At\mathcal{A} = (\{\text{atoms of } \mathcal{A}\}, Id, {}^{\smile}, C),$$

where $Id = \{a : a \leq 1'\}$, $^{\smile}$ is as in $\mathcal{A}$ (restricted to atoms), and $C = \{(a, b, c) : c^{\smile} \leq a\,; b\}$.

Theorem 0.1.9 (Lyndon, 1950)

1. *If $\mathcal{A}$ is an atomic relation algebra then $At\mathcal{A}$ is a relation algebra atom structure as in definition 0.1.7.*
2. *For any relation algebra atom structure $\mathcal{S} = (X, Id, \smile, C)$, there is an atomic relation algebra $\mathcal{A}$ with atom structure $\mathcal{S}$. If X is finite, $\mathcal{A}$ is unique.*

Proof. 1. Exercise!
2. $\mathcal{A} = \langle \wp X, \cup, \cap, \setminus, \emptyset, \wp X, Id, {}^\smile, ; \rangle$ is a relation algebra with atom structure $\mathcal{S}$, where for $a, b \subseteq X$:

$$\begin{aligned} a^\smile &= \{x^\smile : x \in a\}, \\ a\,; b &= \{z : \exists x \in a\ \exists y \in b\ (x, y, z^\smile) \in C\}. \end{aligned}$$

If $\mathcal{S}$ is finite and $At\mathcal{B} = \mathcal{S}$, then $b \mapsto \{x \in X : x \leq b\}$ is an isomorphism : $\mathcal{B} \to \mathcal{A}$. ■

So we can specify a finite relation algebra by specifying its atom structure.

0.1.6 Examples of relation algebras

Example 0.1.10 The smallest non-trivial relation algebra, $\mathcal{I}$, has atoms 1' and $\sharp$, both self-converse. The consistent triangles are the Peircean transforms of (1', 1', 1') and ($\sharp, \sharp$, 1'). It is representable: take $U = \{0, 1\}$ and interpret 1' as $=$ and $\sharp$ as $\neq$.

Example 0.1.11 The *point algebra* $\mathcal{P}$ has 3 atoms, 1'$, a, a^\smile$. The consistent triangles are all Peircean transforms of $(x, x^\smile, 1')$ for all atoms x (of course), and of $(a, a, a^\smile)$. The point algebra is representable: take U to be the rational numbers, interpret 1' as $=$, and a as $<$.

McKenzie's algebra (1970)

Example 0.1.12 This is the smallest non-representable relation algebra. We call it $\mathcal{K}$. It has 4 atoms:

The consistent triangles are all Peircean transforms of:
- $(x, x^\smile, 1')$ for all x
- $(a, a, a^\smile)$ (like point algebra so far)
- $(a, a^\smile, \sharp)$, $(a^\smile, a, \sharp)$, and $(a, \sharp, \sharp)$.

This defines a relation algebra (exercise). Exercise 11 shows that it's not representable.

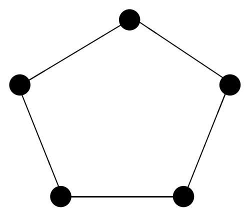
Another example

Example 0.1.13 This algebra (say $\mathcal{J}$) has 4 atoms: 1', r, b, g. They are all self-converse.
The *inconsistent* triangles are the Peircean transforms of
- identity ones (x,y,1') if $x \neq y$, as usual
- (b,b,b)
- (b,b,g)
- (b,g,g)

r is called a *flexible atom,* as all triangles with an r and no 1' are consistent, and it causes $\mathcal{J}$ to be representable (see exercise 10). As of 1996, it's not known whether it has a finite representation.

(4) If in $\mathcal{I}$ we say that $(\sharp, \sharp, \sharp)$ is consistent too, do we get a (representable) relation algebra?

(5) In the pentagon,



let $e(x, y)$ hold iff (x, y) is an edge, and $n(x, y)$ iff (x, y) is a non-edge (and $x \neq y$). Show that $\{1', e, n\}$ form the atoms of a proper relation algebra. What are the consistent triangles?

(6) Does $\mathcal{P}$ have a finite representation?

(7) Show that the atom structure for $\mathcal{K}$ in example 0.1.12 is a relation algebra atom structure.

0.2 Games

How can we tell whether a (finite) relation algebra is representable?

We use networks and games. This view developed in Hirsch and Hodkinson. 1997c based on Lyndon Lyndon 1950.

We make some assumptions, for simplicity:

- Most relation algebras here will be finite. So we can work with (relation algebra) atom structures instead of algebras.
- We will always assume 1' is an atom of relation algebras here. So $Id = \{1'\}$ in atom structures.

It follows that in a representation of a relation algebra, we can always assume that 1 is represented as $U \times U$ (not just an equivalence relation on U):

(8) Let $\mathcal{A}$ be a representable relation algebra and suppose that 1' is an atom of $\mathcal{A}$. Show that $\mathcal{A}$ has a representation h such that $h(1)$ is of the form $U \times U$.

0.2.1 Networks

A network approximates a representation.

Definition 0.2.1 (network) Let $\mathcal{S} = (A, Id, \smile, C)$ be an atom structure. An *(atomic) pre-network* over $\mathcal{S}$ is a pair $N = (N_1, N_2)$ where N_1 is a non-empty set and $N_2 : N_1 \times N_1 \to A$ is a labelling function.

N is a *network* if:

- $N_2(x, x) = 1'$ for all $x \in N_1$
- $(N_2(x, y), N_2(y, z), N_2(z, x))$ is consistent for all $x, y, z \in N_1$.

Some networks satisfy $N(x, y) = 1' \Rightarrow x = y$. We call these *strict networks.*

Some basic facts about pre-networks

Let $N = (N_1, N_2)$ and $N' = (N_1', N_2')$ be pre-networks (over some atom structure).

Definition 0.2.2 N, N' are *isomorphic*, written $N \cong N'$, if there is a bijection $\theta : N_1 \to N_1'$ with $N_2(x, y) = N_2'(\theta(x), \theta(y))$ for $x, y \in N_1$.

Definition 0.2.3 We write $\boldsymbol{N \subseteq N'}$ if $N_1 \subseteq N_1'$ and $N_2' \restriction (N_1 \times N_1) = N_2$ (that is, $N_2 \subseteq N_2'$).

Definition 0.2.4 If $N_0 \subseteq N_1 \subseteq \cdots$ are pre-networks, define $\bigcup_{t<\omega} N_t$ to be the pre-network

$$(\bigcup_{t<\omega} (N_t)_1, \bigcup_{t<\omega} (N_t)_2).$$

Clearly, if the N_t are all networks then so is $\bigcup N_t$.

Notation 0.2.5 We generally write N for any of N, N_1, N_2 (sometimes write N_1 as $dom(N)$, but usually distinguish by context). We let $|N|$ denote $|N_1|$ and say N is finite if N_1 is.

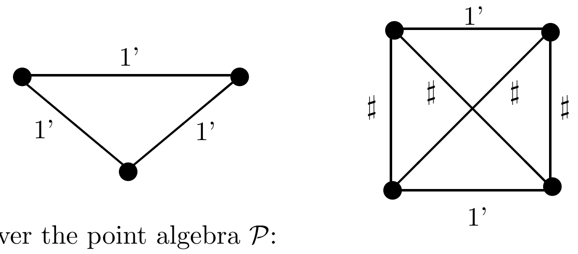
Lemma 0.2.6 *In a network N we have $N(x, y) = N(y, x)^\smile$ for all $x, y \in N$.*

Proof. $(N(x, y), N(y, x), N(x, x))$ is consistent. But $N(x, x) = 1'$. So $N(x, y) = N(y, x)^\smile$. ∎

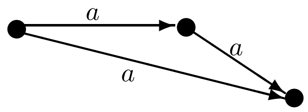
So in diagrams, we only need label arcs one way. And if atoms are self-converse, we don't need arrows. Labels on reflexive arcs ($N(x, x)$) are 1', so we don't need to write this.

Examples of networks

Here are 2 networks over (the atom structure of) the small algebra $\mathcal{I}$ (example 0.1.10):



And a network over the point algebra $\mathcal{P}$:

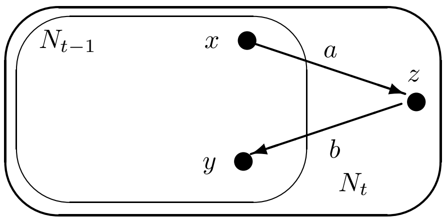


0.2.2 Game on atomic networks

Definition 0.2.7 (game) Fix $n \leq \omega$, an atom structure $\mathcal{S}$, and a pre-network N (over $\mathcal{S}$). Players $\forall$ and $\exists$ play a game $\boldsymbol{G_n(N, \mathcal{S})}$ with n rounds: $0, 1, \ldots, t, \ldots$, for $t < n$, in which pre-networks $N_0, N_1, \ldots$ over $\mathcal{S}$ are built. G_0 is a trivial game with no rounds, which $\exists$ wins by default. For $n > 0$, $G_n(N, \mathcal{S})$ is played as follows.

- In round 0, $\exists$ places a copy of N on the board (so $N_0 \cong N$).
- In round $t > 0$ (with $t < n$), if N_{t-1} has been built so far, $\forall$ chooses $x, y \in N_{t-1}$ and atoms a, b of $\mathcal{S}$ such that $(a, b, N_{t-1}(y, x))$ is consistent.
  $\exists$ must respond with a pre-network $N_t \supseteq N_{t-1}$ with a node $z \in N_t$ such that $N_t(x, z) = a$ and $N_t(z, y) = b$.

We can assume $z \notin N_{t-1}$ if we want.



Winning: After n rounds, the game is over; if all N_t played are *networks* then $\exists$ wins. Otherwise, $\forall$ wins.

0.2.3 Winning strategies

Definition 0.2.8 A *winning strategy* for a player in $G_n(N, \mathcal{S})$ is a set of rules that, if followed, always lead to a win for that player.

We need not formalise the notion of strategy. We note the following straightforward lemma:

Lemma 0.2.9 *Fix $n < \omega$ and a pre-network N over an atom structure $\mathcal{S}$. If $n = 1$ then $\exists$ has a winning strategy in $G_n(N, \mathcal{S})$ iff N is a network. If $n > 0$, the following are equivalent:*

1. *$\exists$ has a winning strategy in $G_{n+1}(N, \mathcal{S})$,*
2. *for any move $\forall$ makes in round 0 of $G_{n+1}(N, \mathcal{S})$, $\exists$ has a response leading to a pre-network N_1 such that she has a winning strategy in $G_n(N_1, \mathcal{S})$.*

0.2.4 Games and representations

Theorem 0.2.10 *Let $\mathcal{A}$ be a countable atomic relation algebra. Let I denote the unique (up to isomorphism) 1-node network over $At\mathcal{A}$.*

1. *If $\exists$ has a winning strategy in $G_\omega(I, At\mathcal{A})$, then $\mathcal{A}$ is representable.*
2. *The converse holds if $\mathcal{A}$ is finite.*

Proof. We first prove (1). Let the game commence. Let $\exists$ use her winning strategy, and let $\forall$ make every possible move at some stage of play (he can do so because $At\mathcal{A}$ is countable). Play builds a chain of networks

$$I = N_0 \subseteq N_1 \subseteq \cdots$$

Let $N = \bigcup_{t<\omega} N_t$. Define an equivalence relation $\sim$ on $dom(N)$ by

$$x \sim y \iff N(x,y) = 1\text{'}.$$

For $r \in \mathcal{A}$, let $\widehat{r} = \{(x/{\sim}, y/{\sim}) : x, y \in N,\ N(x,y) \leq r\}$. Then

$$\widehat{\mathcal{A}} = \langle \{\widehat{r} : r \in \mathcal{A}\}, \cup, \cap, \setminus, \emptyset, N/{\sim} \times N/{\sim}, =, ^{\smile}, ;\rangle$$

is a proper relation algebra, and $r \mapsto \widehat{r}$ is an isomorphism from $\mathcal{A}$ to $\widehat{\mathcal{A}}$. So $\mathcal{A}$ is representable. For more details see Hirsch and Hodkinson. 1997c, Hirsch and Hodkinson. 1997a.

Before proving (2), we need a lemma. Assume $\mathcal{A}$ is finite, and (without loss of generality) a proper relation algebra of the form $\mathcal{A} = \langle A, \cup, \cap, \setminus, \emptyset, U \times U, =, ^{\smile}, ;\rangle$ on a set U (so $A \subseteq \wp(U \times U)$).

Lemma 0.2.11
a) If $x, y \in U$ then there is a unique atom $\alpha(x,y) \in \mathcal{A}$ with $(x,y) \in \alpha(x,y)$.
b) Let $X \subseteq U$. Then $Nwk(X) \stackrel{\text{def}}{=} (X, \alpha \restriction X \times X)$ is a network over $At\mathcal{A}$.

Proof.
a) Let $a \in A$ be $\leq$-minimal with $(x,y) \in a$: a exists as $\mathcal{A}$ is finite and $(x,y) \in U \times U = 1^{\mathcal{A}}$. Check that a is an atom of $\mathcal{A}$.

If a, b are distinct atoms containing (x,y), then $(x,y) \in a \cdot b = \emptyset$, contradiction.
b) Easy. ■

Now we prove (2). $\exists$'s strategy in $G_\omega(I, At\mathcal{A})$ is to ensure that each N_t played satisfies:

- $dom(N_t) \subseteq U$
- $(x,y) \in N_t(x,y)$ for all $x, y \in U$.

That is, $N_t = Nwk(dom(N_t))$. This is a network, by lemma 0.2.11. So if she can do it, it's a winning strategy.

It's easy to do in round 0: because 1' is an atom, $\exists$ can take $N_0 = Nwk(x) \cong I$ for any $x \in U$. If she has done it as far as N_{t-1}, let $\forall$ pick $x, y \in N_{t-1}$ and atoms a, b with $(a, b, N_{t-1}(y,x))$ consistent. So in $\mathcal{A}$ we have $N_{t-1}(x,y) \leq a \,;b$. By assumption, $(x,y) \in N_{t-1}(x,y)$, so $(x,y) \in a\,;b$. So there is $z \in U$ with $(x,z) \in a$ and $(z,y) \in b$. $\exists$ defines $N_t \stackrel{\text{def}}{=} Nwk(dom(N_{t-1}) \cup \{z\})$. ■

Finite vs. infinite games

Theorem 0.2.12 *Let $\mathcal{S}$ be a finite atom structure and N a finite pre-network over $\mathcal{S}$. The following are equivalent:*
1. $\exists$ has a winning strategy in $G_\omega(N, \mathcal{S})$.
2. $\exists$ has a winning strategy in $G_n(N, \mathcal{S})$ for all finite n.

Proof. $1 \Rightarrow 2$: trivial.
$2 \Rightarrow 1$: $\exists$'s strategy in $G_\omega(N, \mathcal{S})$ is:
$(*)$ 'in each round, t, ensure I have a winning strategy in $G_n(N_t, \mathcal{S})$ for infinitely many n'.

$(*)$ is true in round 0 of $G_\omega(N, \mathcal{S})$ by assumption. Inductively assume that $(*)$ holds for N_{t-1}, and let $\forall$ make his move in round t of $G_\omega(N, \mathcal{S})$. We can evidently regard this move as $\forall$'s move in round 0 of a play of $G_{n+1}(N_{t-1}, \mathcal{S})$ for any $n > 0$. By $(*)$, $\exists$ has a winning strategy in this game for each n in some infinite set $X \subseteq \omega \setminus \{0\}$. Let the strategy's response to this move of $\forall$ in round 0 be N_t^n, say. Clearly (cf. lemma 0.2.9), $\exists$ has a winning strategy in $G_n(N_t^n, \mathcal{S})$ for each $n \in X$. As $\mathcal{S}$ and the N_t^n are finite, there is an infinite set $Y \subseteq X$ such that the N_t^n $(n \in Y)$ are all isomorphic. $\exists$ lets $N_t = N_t^n$ (any $n \in Y$). This keeps $(*)$.

$(*)$ implies each N_t is a network. So this strategy is winning for $\exists$. ■

Expressing winning strategy in logic

Theorem 0.2.13 *Let $n < \omega$. There is a first-order sentence σ_n such that for any atom structure $\mathcal{S}$, $\exists$ has a winning strategy in $G_n(I, \mathcal{S})$ iff $\mathcal{S} \models \sigma_n$. And $\{\sigma_n : n < \omega\}$ is recursive.*

Proof. For each finite set X, we write a formula φ_n^X with free variables in $\{v_{xy} : x, y \in X\}$, such that for any pre-network N with domain X,

$$\exists \text{ has a winning strategy in } G_n(N, \mathcal{S}) \iff \mathcal{S} \models \varphi_n^X(N).$$

The notation $\mathcal{S} \models \varphi_n^X(N)$ means that we evaluate φ_n^X in $\mathcal{S}$ with the variable v_{xy} assigned to the atom $N(x, y) \in \mathcal{S}$, for each $x, y \in X$.

We define φ_n^X by induction on n. We let $\varphi_0^X = \top$, and

$$\varphi_1^X \stackrel{\text{def}}{=} \bigwedge_{x \in X} (v_{xx} = 1\text{'}) \wedge \bigwedge_{x,y,z \in X} C(v_{xy}, v_{yz}, v_{zx})).$$

Clearly, $\mathcal{S} \models \varphi_1^X(N)$ iff N is a network, which by lemma 0.2.9 is iff $\exists$ has a winning strategy in $G_1(N, \mathcal{S})$.

Inductively, given φ_n^X for $n > 0$, we pick $z \notin X$, let $Z = X \cup \{z\}$, and let φ_{n+1}^X be

$$\bigwedge_{x,y \in X} \forall ab \Big(C(a, b, v_{yx}) \to \exists_{w \in X} v_{wz}, v_{zz}, v_{zw} (v_{xz} = a \wedge v_{zy} = b \wedge \varphi_n^Z) \Big).$$

($\exists_{w \in X} v_{wz}$ denotes a string of quantifiers $\exists v_{wz}$ for all $w \in X$.) By lemma 0.2.9 and the inductive hypothesis, $\exists$ has a winning strategy in $G_{n+1}(N, \mathcal{S})$ iff whatever move $\forall$ makes in round 0, $\exists$ has a response that leaves a network N', which we can assume has domain Z, such that $\mathcal{S} \models \varphi_n^Z(N')$. By examining the rules governing $\forall$'s moves and $\exists$'s responses, we see that this holds iff $\mathcal{S} \models \varphi_{n+1}^X(N)$, as required. This completes the induction; the theorem follows by letting $\sigma_n = \varphi_n^{\{x\}}(1\text{'}/v_{xx})$ (any x). ■

Axioms for representability

We can now axiomatise the finite representable relation algebras (in which 1' is an atom).

Definition 0.2.14 For $n < \omega$, λ_n is the sentence obtained by translating σ_n into the language of relation algebras, by:

- relativising quantifiers to atoms ('x is an atom' is definable by $\forall y(y < x \leftrightarrow y = 0)$).
- replacing $C(x, y, z)$ by $z^{\smile} \leq x ; y$.

The λ_n are essentially the 'Lyndon conditions' of Lyndon 1950. The following is immediate:

Lemma 0.2.15 *For any atomic relation algebra $\mathcal{A}$, $\mathcal{A} \models \lambda_n$ iff $At\mathcal{A} \models \sigma_n$.*

Theorem 0.2.16 (ess. Lyndon, Lyndon 1950) *Let $\mathcal{A}$ be a finite relation algebra (with 1' $\in At\mathcal{A}$). Then $\mathcal{A}$ is representable iff $\mathcal{A} \models \lambda_n$ for all finite n.*

Proof. By theorem 0.2.10, $\mathcal{A}$ is representable iff $\exists$ has a winning strategy in $G_\omega(I, At\mathcal{A})$.
By theorem 0.2.12, this is iff $\exists$ has a winning strategy in $G_n(I, At\mathcal{A})$ for all finite n.
By theorem 0.2.13, this is iff $At\mathcal{A} \models \sigma_n$ for all n.
By lemma 0.2.15, this is iff $\mathcal{A} \models \lambda_n$ for all n. ■

So the equations defining relation algebras together with the λ_n (all $n < \omega$) axiomatise the finite algebras in RRA in which 1' is an atom. (For when it isn't, see Hirsch and Hodkinson. 1997b.)

0.2.5 What about infinite relation algebras?

Answer: One direction of theorem 0.2.16 still holds.

Fact 0.2.17 RRA is elementary (first-order axiomatisable). In fact, it is a variety—equationally axiomatised Tarski 1955.

Theorem 0.2.18 *If $\mathcal{A}$ is any atomic relation algebra of which 1' is an atom, and $\mathcal{A} \models \lambda_n$ for all finite n, then $\mathcal{A}$ is representable.*

Proof (sketch). By lemma 0.2.15, $At\mathcal{A} \models \sigma_n$ for all n. By theorem 0.2.13, $\exists$ has a winning strategy in $G_n(I, At\mathcal{A})$ for all $n < \omega$. By (eg) saturation (see Chang and Keisler. 1990, Hodkins 1993), there is countable $\mathcal{B} \equiv \mathcal{A}$ such that $\exists$ has a winning strategy in $G_\omega(I, \mathcal{B})$; for details, see Hirsch and Hodkinson. 1997b. By theorem 0.2.10, $\mathcal{B} \in$ RRA. By fact 0.2.17, $\mathcal{A} \in$ RRA, too. ■

The converse fails, even for atomic relation algebras (have a look at Lemma 0.2.11). But arbitrary representable relation algebras (RRA) can be axiomatised by games in a similar way. See Lyndon 1956, Hirsch and Hodkinson. 1997c, Hirsch and Hodkinson. 1997a.

(9) Show that $\exists$ has a winning strategy in $G_\omega(I, At\mathcal{P})$ where $\mathcal{P}$ is the point algebra of example 0.1.11. [Hint: if stuck, use the known representation of $\mathcal{P}$ and the proof of theorem 0.2.10(2).]

(10) Show that $\exists$ has a winning strategy in $G_\omega(I, At\mathcal{J})$ where $\mathcal{J}$ is the 4-atom algebra of example 0.1.13. [Hint: use atom r to fill in the edges.]

(11) Show that $\forall$ has a winning strategy in $G_\omega(I, At\mathcal{K})$ where $\mathcal{K}$ is McKenzie's 4-atom algebra (example 0.1.12).

0.3 Rainbow construction, Monk's theorem

Here, we are going to prove what is often called the most important negative result in algebraic logic:

Theorem 0.3.1 (Monk, Monk 1964) RRA *is not finitely axiomatisable in first-order logic.*

The idea is to construct finite relation algebras $\mathcal{A}_n$ $(2 \leq n < \omega)$ such that for all n:

- $\forall$ has a winning strategy in $G_\omega(I, At\mathcal{A}_n)$
- $\exists$ has a winning strategy in $G_n(I, At\mathcal{A}_n)$.

Lemma 0.3.2 *Monk's theorem follows.*

Proof. Assume RRA is finitely axiomatised, by σ, say. Let

$$T = \{\text{RA axioms}\} \cup \{\lambda_n : n < \omega\} \cup \{\alpha, \neg\sigma\},$$

where α expresses atomicity: $\forall x(x > 0 \rightarrow \exists y(y \leq x \wedge \forall z(z < y \leftrightarrow z = 0)))$. Then by theorem 0.2.13 and lemma 0.2.15, for any finite $\Sigma \subseteq T$ there is $n < \omega$ such that $\mathcal{A}_n \models \Sigma$. By first-order compactness, T has a model, say $\mathcal{B}$. By theorem 0.2.18, $\mathcal{B} \in$ RRA, contradicting $\mathcal{B} \models \neg\sigma$. ■

0.3.1 Rainbow algebras

We build the $\mathcal{A}_n$ using a simplification of a recent idea due to R. Hirsch Hirsch 1995, called the *rainbow construction.* The atom structure $At\mathcal{A}_n$ $(n \geq 2)$ is given by:

Atoms:
- 1'
- g_i for $i \leq n$ (greens)
- w (white)
- y, b (yellow, black)
- r_{ij} for distinct $i, j < n$ (red).

All atoms are self-converse, except reds, where $\mathsf{r}_{ij}^{\smile} = \mathsf{r}_{ji}$. The *inconsistent* triangles are all Peircean transforms of:
• $(x, y, 1')$ if $x \neq y^{\smile}$.
• green ones: $(\mathsf{g}_i, \mathsf{g}_j, \mathsf{g}_k)$ for any $i, j, k \leq n$
• $(\mathsf{g}_i, \mathsf{g}_j, \mathsf{w})$ (any $i, j \leq n$), (y,y,y), (y,y,b)
• any red triangle except those of the form $(\mathsf{r}_{ij}, \mathsf{r}_{jk}, \mathsf{r}_{ki})$.

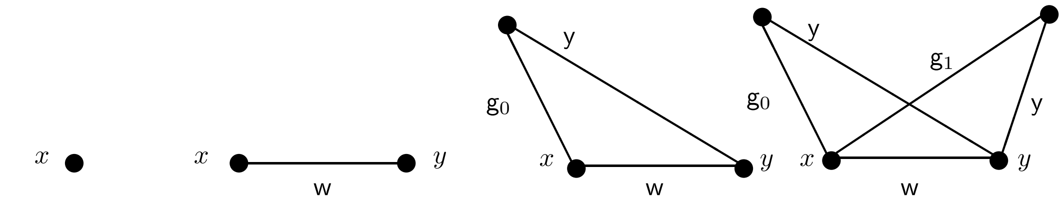
This defines a finite relation algebra There are finitely many atoms, so any relation algebra with this atom structure is finite. We check that it's a relation algebra atom structure (definition 0.1.7):

- $x^{\smile\smile} = x$ for all atoms x: clear.
- $x = y^{\smile}$ iff $(x, y, 1')$ consistent: clear.
- if (x, y, z) is consistent then so are (y, z, x) and $(z^{\smile}, y^{\smile}, x^{\smile})$ ('Peircean transforms'): by definition.
- diamond completion (for associativity): this will follow if we show ∃ has a winning strategy in $G_4(I, \mathcal{A}_n)$. We do this in theorem 0.3.4.

0.3.2 ∀'s winning strategy

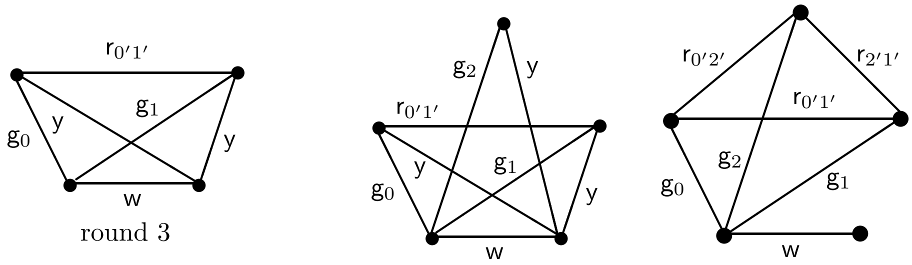
Theorem 0.3.3 *∀ has a winning strategy in $G_\omega(I, At\mathcal{A}_n)$ $(\forall n)$.*

Proof. His first 4 moves are:



round 0 — round 1: ∀ picks $x, x, \mathsf{w}, \mathsf{w}$ — round 2: ∀ picks $x, y, \mathsf{g}_0, \mathsf{y}$ — round 3: ∀ picks $x, y, \mathsf{g}_1, \mathsf{y}$

Now ∃ has to fill in the missing edge. Which atom can she choose? Only a red—say, $\mathsf{r}_{0'1'}$ (below left). ∀ continues by picking $x, y, \mathsf{g}_2, \mathsf{y}$ (below middle), and ∃ has to fill in 2 edges with reds (below right, yellow edges omitted for clarity):



round 3 — round 4

And so on: ∀ plays $x, y, \mathsf{g}_{i-2}, \mathsf{y}$ in round i. ∃ will run out of reds in round $n + 2$. ■

0.3.3 ∃'s winning strategy

Theorem 0.3.4 *∃ has a winning strategy in $G_{n+2}(I, At\mathcal{A}_n)$ for each $n \geq 2$.*

Note: $n + 2 \geq 4$, giving diamond completion.

Proof. At the start of some round t, $0 < t < n+2$, suppose the current pre-network is $N = N_{t-1}$. We suppose inductively that $\exists$ has ensured that N is a network. Note that $|N| \leq t \leq n+1$. Let $\forall$ move in round t by picking nodes x, y and atoms a, b with $(a, b, N(y, x))$ consistent. We can assume $\forall$ never plays so that $\exists$ doesn't need to add a node. So

- N is strict: if $N(v, w) = 1'$ then $v = w$,
- $a, b \neq 1'$.

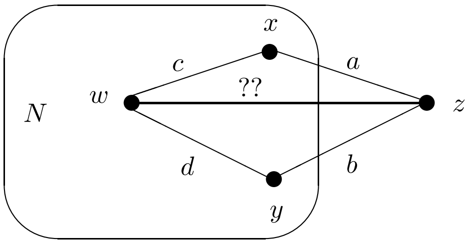
$\exists$ makes N_t by adding a new node z, defining $N_t(x, z) = a$, $N_t(z, y) = b$, $N_t(z, z) = 1'$, and then choosing $N_t(w, z)$ for all $w \in N \setminus \{x, y\}$. Of course, she defines $N_t(z, w) = N_t(w, z)^{\smile}$.

What does $\exists$ choose for the $N_t(w, z)$?

1. Use white if possible.
2. If not, use black if possible.
3. If not, use a suitable red.

Here, 'if possible' means 'if triangles w, x, z and w, y, z are rendered consistent'.

In more detail



Let $N(w, x) = c \ldots$

$\ldots$ and $N(w, y) = d$.

As N is strict, $c, d \neq 1'$.

1. If a, c are not both green, and b, d are not both green, then because $a, b, c, d \neq 1'$ she can pick $?? = N_t(w, z) = \mathsf{w}$ (white).
2. Otherwise, if a, c not both yellow, and b, d not both yellow, then, again bearing in mind that $a, b, c, d \neq 1'$, she lets $?? = \mathrm{b}$ (black).
3. Otherwise, we can assume a, c are green, and b, d, yellow. $\exists$ must choose a red for $N_t(w, z)$ for each w in the set
$$R = \{w \in N : N(w, x) \text{ green}, \ N(w, y) = \mathsf{y}\}.$$
If $v, w \in R$, $v \neq w$, then because N is (inductively) a strict network, $N(v, w)$ is red. Indeed, the rule for consistency of red atoms ensures that there is a function $\rho : R \to n$ with
$$N(v, w) = \mathsf{r}_{\rho(v), \rho(w)} \quad \text{(all distinct } v, w \in R).$$
(If $R = \{w\}$, we let $\rho(w) = 0$.) Note that $|rng(\rho)| \leq |N| - 2 \leq n - 1$, so there is $i \in n \setminus rng(\rho)$. $\exists$ lets $?? = N_t(w, z) = \mathsf{r}_{\rho(w), i}$ for all $w \in R$.

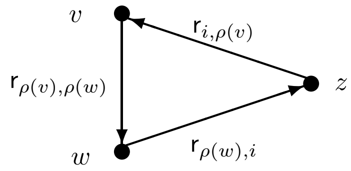
This completes the labelling and she defines N_t to be the result.

Is N_t a network?

It's enough to check that all triangles (v, w, z) with $v, w \in N$, $v \neq w$, are consistently labelled.

Triangles of the form (x, y, z), (v, x, z), (v, y, z) are clearly consistent.

Consider a triangle (v, w, z) where $v, w \notin \{x, y\}$. $\exists$ labelled (v, z) and (w, z) with w, b, or a red. All triangles with 2 edges like this are consistent, except perhaps all-red triangles. But $\exists$ only uses reds for (v, z) and (w, z) when $v, w \in R$. In this case, we have



and this is consistent. ■

0.3.4 Other results

The full rainbow construction and variations can be used to prove many more negative results, such as:

1. The class of relation algebras with representations satisfying the condition given in lemma 0.2.11(1) is not elementary Hirsch 1995, Hirsch and Hodkinson. 1997b.
2. RRA is not closed under 'completions' Hodkinson 1997.
3. (Hence) RRA is not Sahlqvist-axiomatisable Venema 1997.
4. It is undecidable whether a finite relation algebra is representable Hirsch and Hodkinson 1999: To appear.

Similar results to at least (1–3) for 'cylindric algebras' (higher-arity relations) can be proved too.

5. For all $n \geq 5$ there is a first-order sentence that can be proved with n variables but not with $< n$ Hodkinson and Maddux . It also holds for $n = 4$ Tarski and Givant 1987.
6. For $n \geq 5$, the classes RA_n, SRaCA_n, and (for $m \geq 3$, $m + 2 \leq n$) $\mathsf{SNr}_m\mathsf{CA}_n$ are not finitely axiomatisable Hirsch and Hodkinson. Submitted, 1999b.

0.4 Finite base property

Relation algebras, even non-representable ones, are surprisingly complicated. We mentioned that it is undecidable whether a finite relation algebra is representable, and other negative results (§0.1.4). Tarski proved that the equational theory of RA is undecidable: see Givant and Németi. 1997.

One of the chief 'causes' is associativity. So researchers like Maddux proposed weakening the associative law. The semi-associative law is $(x\,;1)\,;1 = x\,;(1\,;1)$. Even assuming only this, the equational theory of the resulting class, SA, is undecidable.

0.4.1 Weak associativity

Maddux Maddux 1982 proposed an even weaker law,

$$([x \cdot 1']\,;1)\,;1 = [x \cdot 1']\,;(1\,;1)$$

The class of all algebras satisfying this law and (R0), (R2)–(R7) of definition 0.1.3 is called WA ('weakly associative algebras'). The equational theory of WA is decidable. Every weakly associative algebra is representable if we allow special 'relativised representations'. And, as we now prove, finite weakly associative algebras have finite relativised representations Hodkinson and Németi. 1999.

0.4.2 Relativised representations

These provide a weak version of 'proper relation algebra' or 'ordinary' representation: all operations are relativised to (intersected with) the unit.

Definition 0.4.1 A *relativised proper relation algebra* is an algebra $\langle A, \cup, \cap, -, \emptyset, 1, 1', ^\smile, ;\rangle$, where $A \subseteq \wp(U \times U)$ for some set U, and for $r, s \in A$,

$$\begin{array}{rcll} -r &=& 1 \setminus r & \text{(as before)},\\ 1' &=& \{(x,y) \in 1 : x = y\}, & \\ r^\smile &=& \{(x,y) \in 1 : (y,x) \in r\}, & \\ r\,;s &=& \{(x,y) \in 1 : \exists z((x,z) \in r \wedge (z,y) \in s))\}. & \end{array}$$

1 (the 'unit') need not be an equivalence relation on U.

Definition 0.4.2 A *relativised representation* of a relation-type algebra $\mathcal{A}$ is an isomorphism from $\mathcal{A}$ onto a relativised proper relation algebra.

WA and relativised representations

Weakly associative algebras are associated with relativised representations with reflexive symmetric unit.

Theorem 0.4.3 (Maddux, Maddux 1982) *Let $\mathcal{A}$ be a relation-type algebra. $\mathcal{A} \in$ WA iff $\mathcal{A}$ has a relativised representation in which the unit is reflexive and symmetric.*

We will prove a related result:

Theorem 0.4.4 (Hodkinson and Németi. 1999) *Let $\mathcal{A}$ be a* finite *relation-type algebra. $\mathcal{A} \in$ WA iff $\mathcal{A}$ has a relativised representation* with finite base *in which the unit is reflexive and symmetric.*

This is known as the finite base property for WA.

(12) Prove $\Leftarrow$ without using theorem 0.4.3.

0.4.3 Herwig's theorem

Before proving theorem 0.4.4, we need some preliminaries.

Definition 0.4.5 Let M, N be structures for a finite relational language L.

1. A subset $X \subseteq M$ is *live* if $|X| = 1$ or there are $a_1, \dots, a_n \in M$ and n-ary $R \in L$ such that $M \models R(a_1, \dots, a_n)$ and $X \subseteq \{a_1, \dots, a_n\}$.
2. M is *packed* if any two elements of M form a live set.
3. A *partial isomorphism* of M is a one-one partial map $f : M \to M$ such that for any $a_1, \dots, a_n \in dom(f)$ and n-ary $R \in L$, $M \models R(a_1, \dots, a_n) \leftrightarrow R(f(a_1), \dots, f(a_n))$.
   A partial isomorphism is an *automorphism* if it's bijective.
4. A *homomorphism* from M to N is a map $f : M \to N$ such that for any $a_1, \dots, a_n \in M$ and n-ary $R \in L$, if $M \models R(a_1, \dots, a_n)$ then $N \models R(f(a_1), \dots, f(a_n))$.

Theorem 0.4.6 (Herwig, Herwig. 1998.) *Let M be a finite structure in a finite relational language. There is a finite structure $M^+ \supseteq M$ such that*

1. *Any partial isomorphism of M extends to an automorphism of M^+.*
2. *If $X \subseteq M^+$ is live then $g(X) \stackrel{\text{def}}{=} \{g(x) : x \in X\} \subseteq M$ for some automorphism g of M^+.*
3. *If N is a packed structure, and $f : N \to M^+$ is a homomorphism, then there exists a homomorphism $f^- : N \to M$.*

0.4.4 Finite base property for WA

Proof of theorem 0.4.4 $\Rightarrow$. Let $\mathcal{A}$ be a finite weakly associative algebra. We assume for simplicity that 1' is an atom of $\mathcal{A}$. Regarding $At\mathcal{A}$ as a binary relational language, we define a finite $At\mathcal{A}$-structure M as follows. Pick one representative of each isomorphism type of strict network over $At\mathcal{A}$. The copies are assumed pairwise disjoint. Strict networks are defined as for relation algebras. Let the domain of M be the set of all the nodes of these copies. Interpret the relation symbols of

$At\mathcal{A}$ in M by:

$$M \models a(x,y) \quad \text{iff} \quad \exists N{\in}M(x,y \in N \wedge N(x,y)=a).$$

M is thus a finite $At\mathcal{A}$-structure. Let $M^+ \supseteq M$ be as in Herwig's theorem. Let $h : \mathcal{A} \to \wp(M^+ \times M^+)$ be given by

$$h(r) = \{(x,y) : x,y \in M^+, M^+ \models a(x,y) \text{ for some atom } a \leq r\}.$$

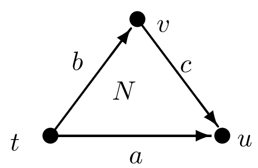
We show that h is a relativised representation of $\mathcal{A}$ (with range a relativised proper relation algebra with base M^+). Let $r,s \in \mathcal{A}$.

- h is 1–1: if $r \neq s$, take an atom $a \leq r \oplus s$ (symmetric difference). There are $x,y \in M$ with $M \models a(x,y)$. Then $(x,y) \in h(r) \oplus h(s)$.
- $h(r+s) = h(r) \cup h(s)$, $h(r \cdot s) = h(r) \cap h(s)$: clear.
- $h(-r) = h(1) \setminus h(r)$, $h(1') = \{(x,x) : (x,x) \in h(1)\}$, $h(r^\smile) = \{(x,y) \in 1 : (y,x) \in h(r)\}$, $h(1)$ is reflexive and symmetric — use liveness.

(13) Check the above.

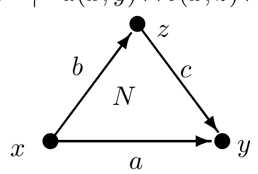
Finally we check $h(r\,;s) = \{(x,y) \in h(1){:}\exists z((x,z) \in h(r) \wedge (z,y) \in h(s))\}$.

$\subseteq$: If $(x,y) \in h(r\,;s)$, there's an atom $a \leq r\,;s$ with $M^+ \models a(x,y)$. So certainly $(x,y) \in h(1)$. Also, $\{x,y\}$ is live. Let g be an automorphism of M^+ with $g(x), g(y) \in M$. Then $M \models a(g(x), g(y))$. Now we may take atoms $b \leq r$, $c \leq s$ with $a \leq b\,;c$. So there's a network N in M with nodes t,u,v, say:



Clearly, $t \mapsto g(x), u \mapsto g(y)$ is a partial isomorphism of M. So it extends to an automorphism f of M^+. Let $z = g^{-1}f(v)$. Then $M^+ \models b(x,z) \wedge c(z,y)$. So $(x,z) \in h(r)$ and $(z,y) \in h(s)$, as required.

$\supseteq$: Let $x,y,z \in M^+$ with $(x,y) \in h(1)$, $(x,z) \in h(r)$, $(z,y) \in h(s)$. We require $(x,y) \in h(r\,;s)$. Let $b \leq r, c \leq s$, and a be atoms with $M^+ \models a(x,y) \wedge b(x,z) \wedge c(z,y)$. Then the substructure



is packed, and the inclusion map is a homomorphism: $N \to M^+$. By Herwig's theorem there's a homomorphism $x \mapsto x', y \mapsto y', z \mapsto z'$ from N into M. So

$$M \models a(x',y') \wedge b(x',z') \wedge c(z',y').$$

So by definition of M, $(b,c,a^\smile)$ is consistent. So $a \leq b\,;c \leq r\,;s$. By definition of h, $(x,y) \in h(r\,;s)$, as required.

■

Remarks

1. The size of the finite representation of $\mathcal{A}$ can be bounded recursively in $|\mathcal{A}|$.
2. The same idea proves that any universal sentence valid in finite WAs is valid in all WAs. This gives the 'finite model property' for *arrow logic* in its relativised interpretation.

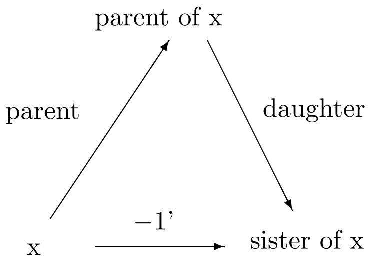
3. The method applies in several other situations. E.g., can find a finite relativised representation (of a certain kind) for any finite relation algebra, and associativity gives it more properties. See Grädel. 1998, Hirsch and Hodkinson. Submitted, 1999a.

(14) Let h be a relativised representation of $\mathcal{A} \in \mathsf{WA}$. Show that the unit, $h(1)$, is a *reflexive and symmetric* binary relation on the base of h.

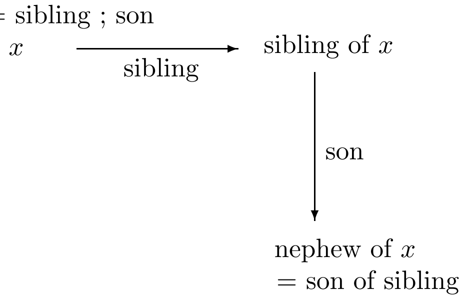
0.5 Answers to selected exercises

Exercise 1

1. sister = (parent ; daughter) $\cdot -$ 1'



2. nephew = sibling ; son



3. aunt = parent ; sister

Exercise 3

If $\mathcal{S} = (X, Id, ^\smile, C)$ is a relation algebra atom structure and $a \in Id$, show $a^\smile = a$.

We know $(a, a^\smile, b) \in C$ for some $b \in Id$. As C is closed under Peircean transforms, $(a^\smile, b, a) \in C$. But $a \in Id$, so $a^\smile = b^\smile$. So $a = a^{\smile\smile} = b^{\smile\smile} = b$. Thus, $(a, a^\smile, a) \in C$. As C is closed under Peircean transforms, $(a^\smile, a^\smile, a) \in C$. But $a \in Id$, so $a^\smile = a^{\smile\smile} = a$.

Exercise 4

The algebra with consistent triangles $(x, x^\smile, 1')$ and $(\sharp, \sharp, \sharp)$ is a representable relation algebra: one representation is on a set with three elements, interpreting 1' as $=$ and $\sharp$ as $\neq$.

Exercise 5

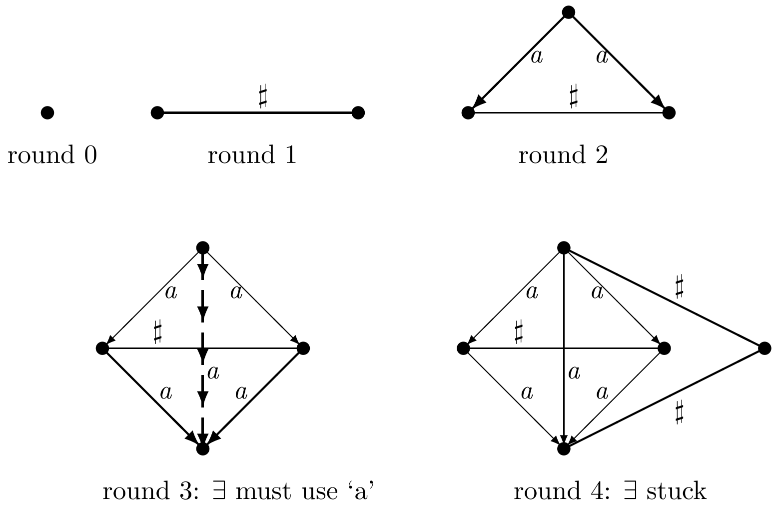
The consistent triangles are $(x, x, 1')$ for $x \in \{1', e, n\}$, (e, e, n), (n, n, e). The algebra with atoms $1', e, n$ is closed under the operations and the pentagon is a representation of it.

Exercise 6

In any representation of the point algebra, a is interpreted as an (irreflexive) dense partial order. ($a \,;\, a = a$ gives transitivity and density, and $a \neq a^\smile$ gives antisymmetry.) The partial order has at least 2 elements, because a is interpreted as a non-empty relation. But there is no such finite order.

Exercise 11

$\forall$ can win $G_5(I, \mathcal{K})$ as follows:



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