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A finite basis theorem for product varieties of groups

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It is shown that, if $\underline{\underline{U}}$ is a subvariety of the join of a nilpotent variety and a metabelian variety and if $\underline{\underline{V}}$ is a variety with a finite basis for its laws, then $\underline{\underline{UV}}$ also has a finite basis for its laws. The special cases $\underline{\underline{U}}$ nilpotent and $\underline{\underline{U}}$ metabelian have been established by Higman (1959) and Ivanjuta (1969) respectively. The proof here, which is independent of Ivanjuta's, depends on a rather general sufficient condition for a product variety to have a finite basis for its laws.

All varieties considered in this note are varieties of groups. For notation, terminology and basic results see Hanna Neumann's book [4], but note that German letters are here represented by double-underlined Roman letters. A variety $\underline{\underline{V}}$ will be called *finitely based* if the laws of $\underline{\underline{V}}$ have a finite basis.

Graham Higman has shown [2] that if $\underline{\underline{U}}$ is a nilpotent variety, then the product variety $\underline{\underline{UV}}$ is finitely based whenever the variety $\underline{\underline{V}}$ is finitely based. In a recent paper [3], Ivanjuta proves a similar result with "nilpotent" replaced by "metabelian". He, of course, relies on D.E. Cohen's theorem that every metabelian variety is finitely based [1]. The purpose of this note is to point out that the stronger results of Cohen's paper, together with an adaptation of Higman's method, easily (and independently of Ivanjuta) yield the following:

THEOREM. *If* $\underline{\underline{U}}$ *is any subvariety of* $\underline{\underline{A}}^2 \vee \underline{\underline{N}}_c$ *(where* c *is*

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arbitrary), then $\underline{\underline{UV}}$ *is finitely based whenever* $\underline{\underline{V}}$ *is finitely based.*

Let F be an absolutely free group of countably infinite rank, and let $E(F)$ be the set of all endomorphisms of F. For any subsets $\underline{\underline{a}} \subseteq F$ and $\Sigma \subseteq E(F)$ denote by $\langle \underline{\underline{a}} \rangle_\Sigma$ the "Σ-closure" of $\underline{\underline{a}}$, i.e., the least subgroup of F which contains $\underline{\underline{a}}$ and admits Σ. Two subsets of F will be called *Σ-equivalent* if their Σ-closures coincide. If $\underline{\underline{f}}$ is a free generating set for F, then an endomorphism $\beta \in E(F)$ will be called *brief* with respect to $\underline{\underline{f}}$ if $\underline{\underline{f}}\beta \subseteq \{1, f^a \mid f \in \underline{\underline{f}}, a \in F\}$. The set of all brief endomorphisms with respect to $\underline{\underline{f}}$ will be denoted by $B(\underline{\underline{f}})$. Now a variety $\underline{\underline{U}}$ is finitely based if and only if there exists a finite subset $\underline{\underline{u}} \subseteq F$ such that $U(F) = \langle \underline{\underline{u}} \rangle_{E(F)}$. By analogy, $\underline{\underline{U}}$ will be called *strongly finitely based* if and only if there exists a finite subset $\underline{\underline{u}} \subseteq F$ such that $U(F) = \langle \underline{\underline{u}} \rangle_{B(\underline{\underline{f}})}$. Contrary to appearances, the choice of the free generating set $\underline{\underline{f}}$ in this definition is immaterial. For if $\underline{\underline{f}}'$ is another free generating set, there exists an automorphism α of F such that $\underline{\underline{f}}' = \underline{\underline{f}}\alpha$, and it is straightforward to check that for any subset $\underline{\underline{a}} \subseteq F$

$$\left(\langle \underline{\underline{a}} \rangle_{B(\underline{\underline{f}})}\right)\alpha = \langle \underline{\underline{a}}\alpha \rangle_{B(\underline{\underline{f}}\alpha)} ;$$

hence if $U(F) = \langle \underline{\underline{u}} \rangle_{B(\underline{\underline{f}})}$, then, since $U(F)$ is characteristic in F, it follows that

$$U(F) = U(F)\alpha = \left(\langle \underline{\underline{u}} \rangle_{B(\underline{\underline{f}})}\right)\alpha = \langle \underline{\underline{u}}\alpha \rangle_{B(\underline{\underline{f}}\alpha)} = \langle \underline{\underline{u}}\alpha \rangle_{B(\underline{\underline{f}}')} ,$$

and moreover $\underline{\underline{u}}$ is finite if and only if $\underline{\underline{u}}\alpha$ is finite. Thus the property of being strongly finitely based is genuinely varietal. Accordingly, it will be convenient in the sequel to use for F the "word group" $X = X_\infty$, and to define brief endomorphisms with respect to the distinguished free generating set $\underline{\underline{x}} = \{x_i \mid i = 1, 2, \ldots\}$. In addition, the following abbreviated notation will be used: $E = E(X)$, $B = B(\underline{\underline{x}})$ and $V = V(X)$, the latter for any variety $\underline{\underline{V}}$.

The proof of the theorem depends upon the following lemma, which is essentially an adaptation of Hanna Neumann's exposition in [*4*] of Higman's method.

LEMMA. *If* $\underline{\underline{U}}$ *is a strongly finitely based variety, then* $\underline{\underline{UV}}$ *is finitely based whenever* $\underline{\underline{V}}$ *is finitely based.*

Proof. It is well-known that if $\underline{\underline{w}}$ is a finite subset of X , then $\underline{\underline{w}}$ is E-equivalent to a singleton $\{w\}$, where w may be assumed to involve precisely the letters $x_1, \ldots, x_k$ for some k . In fact the endomorphisms used to establish this are all brief, so that "E-equivalent" may be replaced by "B-equivalent". Thus the assumptions about $\underline{\underline{U}}$ and $\underline{\underline{V}}$ mean that

(1) $$U = \langle u \rangle_B = gp(u\beta | \beta \in B)$$

and

(2) $$V = \langle v \rangle_E = gp(v\theta | \theta \in E)$$

for some $u = u(x_1, \ldots, x_m)$ and $v = v(x_1, \ldots, x_n)$. For the proof of the lemma it will now be shown that

$$U(V) = \langle u\nu \rangle_E$$

where $\nu \in E$ is defined by $x_i\nu = v\tau_{(i-1)n}$ for all $x_i \in \underline{\underline{x}}$ (τ_k is the "translation" $\tau_k : x_i \mapsto x_{i+k}$) .

Now by definition

(3) $$U(V) = gp(w\phi | w \in U \ ; \ \phi \in E \text{ such that } X\phi \subseteq V) \ ,$$

so that trivially $\langle u\nu \rangle_E \subseteq U(V)$. For the reverse inclusion, observe first that by using (2) and an argument similar to that used in the proof of 34.22 in [4] it may be readily verified that if an endomorphism $\phi \in E$ satisfies $X\phi \subseteq V$, then there exist $\pi, \theta \in E$ (depending on ϕ) such that $\phi = \pi\nu\theta$. Hence from (3)

$$\begin{aligned} U(V) &\subseteq gp(w\pi\nu\theta | w \in U \ ; \ \pi, \theta \in E) \\ &\subseteq \langle w'\nu | w' \in U \rangle_E \ , \end{aligned}$$

and so, using (1),

(4) $$U(V) \subseteq \langle u\beta\nu | \beta \in B \rangle_E \ .$$

But it is easy to see that for any $\beta \in B$ there exists a $\gamma \in E$ such that $\beta\nu = \nu\gamma$, and therefore, from (4)

$$U(V) \subseteq \langle u\nu\gamma \mid \gamma \in E\rangle_E = \langle u\nu\rangle_E .$$

This completes the proof of the lemma. //

Observe now that if $\underline{\underline{S}}$ and $\underline{\underline{T}}$ are strongly finitely based varieties then so is $[\underline{\underline{S}}, \underline{\underline{T}}]$. (For the proof, simply make the obvious modifications to the proof of the corresponding statement about finitely based varieties.) By a straightforward induction argument starting from the trivial variety it follows that the variety consisting of all the polynilpotent groups of a given class row, and in particular the variety $[\underline{\underline{N}}_a, \underline{\underline{N}}_b]$ for any a, b , is strongly finitely based. Further, Theorem 17.2 of M.A. Ward's paper [6] (with $m = 2$, $m+n = c+1$) gives

$$\underline{\underline{A}}^2 \vee \underline{\underline{N}}_c = \bigwedge_{d=1}^{c-2} [\underline{\underline{N}}_d, \underline{\underline{N}}_{c-1-d}] ,$$

and since the meet of finitely many strongly finitely based varieties is clearly again strongly finitely based, it follows that $\underline{\underline{A}}^2 \vee \underline{\underline{N}}_c$ is strongly finitely based. For the proof of the theorem it remains to show that the same is true of the subvarieties of $\underline{\underline{A}}^2 \vee \underline{\underline{N}}_c$, and for this it is clearly sufficient to show that the B-closed subgroups of X which contain $\left(X^{(2)} \cap X_{(c+1)}\right)$ satisfy the ascending chain condition. However

(i) Since only brief endomorphisms are used in the proof of 34.12 in [4], the term "equivalent" there can be replaced by "B-equivalent", and so the proof of 34.14 yields that the B-closed subgroups of X containing $X_{(c+1)}$ satisfy the a.c.c. .

(ii) Similarly, since Cohen's "order-preserving maps" on the generating set $\left\{x_i X^{(2)} \mid x_i \in \underline{\underline{x}}\right\}$ of $X/X^{(2)}$ are all induced by brief endomorphisms of X , it follows from Remark 2 in his paper [1] that the B-closed subgroups of X containing $X^{(2)}$ also satisfy the a.c.c. .

The conclusions of (i) and (ii), and a theorem of G. Pickert [5] (cf. R.A. Bryce's proof of 16.25 in [4]) may now be combined to complete the proof of the theorem.

REMARKS. Two questions which arise naturally from the lemma are the following:

(a) Does the property of strongly finitely based varieties expressed in the lemma in fact characterise such varieties?

(b) Does there exist a finitely based variety which is not strongly finitely based?

We have not been able to answer these questions, but in relation to (a) we have noted firstly that a slight modification to the proof of the lemma yields that if $\underline{\underline{U}}$, $\underline{\underline{V}}$ are strongly finitely based then so is $\underline{\underline{UV}}$, and secondly (by a direct argument) that if $\underline{\underline{UV}}$ is strongly finitely based then so is $\underline{\underline{V}}$.

DISCLAIMER (M.F. Newman). This note properly contains the remains of my "proof" of 36.13 in [4]. I withdraw my claim to have a proof of that statement.

Note added on 20 October, 1969. We are indebted to Professor M.R. Vaughan-Lee for sending us a preprint of a paper ["Abelian by nilpotent varieties", to appear in *Quart. J. Math. Oxford*] in which he confirms 36.13 of [4]. His proof in fact yields that the B-closed subgroups of X containing $(\underline{\underline{AN}}_c \wedge \underline{\underline{N}}_c\underline{\underline{A}})(X)$ satisfy the a.c.c. , and this together with our lemma immediately gives that *if* $\underline{\underline{U}} < \underline{\underline{AN}}_c \wedge \underline{\underline{N}}_c\underline{\underline{A}}$ *and* $\underline{\underline{V}}$ *is finitely based, then* $\underline{\underline{UV}}$ *is finitely based*; a result which supersedes our theorem.

References

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