Math Logic: Model Theory & Computability
Lecture 14

Nonstandard models of arithmetic.

Def. Call $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$, as well as any other σ_{arithm}-structure isomorphic to $\underline{N}$, the *standard model* of PA or $Th(\underline{N})$. Call a model $\underline{M}$ of PA *nonstandard* otherwise, i.e. if $\underline{M}$ is not isomorphic to $\underline{N}$.

The weak upward Löwenheim-Skolem theorem implies that PA and even $Th(\underline{N})$, where $\underline{N} := (\mathbb{N}, 0, S, +, \cdot)$, has unctbl models. In particular, these would be nonstandard.

Def. For a model $\underline{M} \models PA$, denote by $\mathbb{N}^{\underline{M}} := \{\dot{n}^{\underline{M}} : n \in \mathbb{N}\}$ where for each $n \in \mathbb{N}$, $\dot{n} := \underbrace{S(S(S(\dots S(0))))}_{n}$. We call the elements of $\mathbb{N}^{\underline{M}}$ *standard* and those in $M \setminus \mathbb{N}^{\underline{M}}$ *nonstandard*. We call the set $\mathbb{N}^{\underline{M}}$ the *standard part of* $\underline{M}$.

Obs. A model $\underline{M} \models PA$ is standard if and only if $M = \mathbb{N}^{\underline{M}}$.

Proof. $\Leftarrow$. If $M = \mathbb{N}^{\underline{M}}$ then $h : \mathbb{N} \to M$, $n \mapsto \dot{n}^{\underline{M}}$ is an isomorphism, hence $\underline{M}$ is standard.

$\Rightarrow$. Trivial. $\square$

Prop. There are countable nonstandard models of $Th(\underline{N})$ (hence of PA).

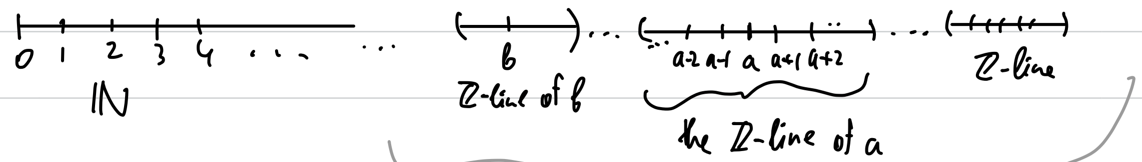
Proof. Let $\sigma := \sigma_{arithm} \cup \{c\}$, where c is a constant symbol. Then the theory

$$T := Th(\underline{N}) \cup \{c \neq \dot{n} : n \in \mathbb{N}\}$$

is finitely satisfiable: indeed, if $T_0 \subseteq T$ is finite, then $T_0 \subseteq Th(\underline{N}) \cup \{c \neq \dot{0}, c \neq \dot{1}, \dots, c \neq \dot{n}\}$ for some $n \in \mathbb{N}$, hence, $\underline{N}' := (\mathbb{N}, 0, S, +, \cdot, c^{\underline{N}'})$ satisfies

T_0, where $c^{\underline{N}'} := n+1$. By compactness, T is satisfiable, hence has a countable model $\underline{\tilde{M}}$ by the weak downward L-S. The reduct $\underline{M}$ of $\underline{\tilde{M}}$ to a σ_{arith}-structure is then nonstandard because $c^{\tilde{M}} \neq n^{M}$ for all $n \in \mathbb{N}$, i.e. $c^{\tilde{M}} \in M \setminus \mathbb{N}^{M}$. $\square$

What do ctbl nonstandard models of arithmetic look like? For a model $\underline{M} \models PA$, define $\leq$ on M by setting $a \leq b$:<=> there is $m \in M$ such that $a+m=b$. We will prove in homework that when $\underline{M}$ is nonstandard, this order is not a well-order, moreover, $\underline{M}$ looks like this:



the order on the $\mathbb{Z}$-lines is isomorphic to $\mathbb{Q}$.

the order on the $\mathbb{N}$-line and $\mathbb{Z}$-lines together is isomorphic to $\mathbb{Q}_{\geq 0}$.

Nonaxiomatizable classes of structures.

We already saw examples of nonaxiomatizable classes as a consequence of the weak upward L-S theorem. We give more examples of different kinds here.

When is a class $\mathcal{C}$ of σ-structures and its complement axiomatizable? For example, when $\mathcal{C}$ is finitely axiomatizable, hence by a single sentence φ, then the complement of $\mathcal{C}$ is also axiomatizable by $\neg\varphi$. The following says that this is the only possible scenario:

Thm. If a class $\mathcal{C}$ of σ-structures and its complement are both axiomatizable, then they are finitely axiomatizable.

Proof. This is just a rephrasing of Q6 of HW5. Indeed, letting T, T_c be axiomatizations for $\mathcal{C}$ and $\mathcal{C}^c$, we see that T and

$$S := \{\neg\varphi : \varphi \in T_c\}$$

satisfy the hypothesis of Q6 HW5. $\square$

Example. The class of non-bipartite graphs is not axiomatizable because the class of bipartite graphs is not finitely axiomatizable.

Classes defined by infinite disjunctions.

Instead of a general statements, let's consider a couple of examples.

Examples. (a) The class of all torsion groups, i.e. groups in which every element has finite order, i.e. the groups $\underline{G}$ satisfying

$$\forall g \in G \bigvee_{n \in \mathbb{N}} \underbrace{g \cdot g \cdot \ldots \cdot g}_{n} = 1^{\underline{G}}.$$

(b) The class of connected graphs. Indeed a graph $\underline{G} := (V, E)$ connected iff

$$\forall u, v \in V \bigvee_{n \in \mathbb{N}} \underbrace{\left(\begin{array}{l}\text{there is a path in } \underline{G} \\ \text{of length } n \text{ between } u \text{ and } v\end{array}\right)}_{\varphi_n(u,v)},$$

where $\varphi_n(u,v) := \exists x_0 \exists x_1 \ldots \exists x_n \left(x_0 = u \wedge x_n = v \bigwedge_{i=0}^{n-1} (x_i E x_{i+1}) \wedge \bigwedge_{i=0}^{n-1} (x_i \neq x_{i+1})\right).$

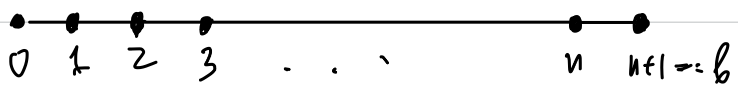
It is left as an exercise to show that (a) is not axiomatizable, and we show (b).

Prop. The class $\mathcal{C}$ of connected graphs is not axiomatizable.

Proof. Suppose the contrary and let T be a σ_{graph}-theory axiomatizing $\mathcal{C}$. We extend the signature to $\sigma := \sigma_{graph} \cup \{a, b\}$ where a, b are constant symbols. Let

$$\tilde{T} := T \cup \{d_{>n}(a,b) : n \in \mathbb{N}\},$$

where $d_{>n}(x,y) := \neg \bigvee_{i=0}^{n} \varphi_i(x,y)$, where $\varphi_i(x,y)$ are as in Example (b) above, so $d_n(x,y)$ holds iff the graph distance between x and y is $>n$. Then $\tilde{T}$ is finitely satisfiable: letting $T_0 \subseteq \tilde{T}$ be a finite subset, we see that $T_0 \subseteq T \cup \{d_0(a,b), d_1(a,b), \dots, d_n(a,b)\}$ for some $n \in \mathbb{N}$, so the graph



satisfies T_0.

By compactness, $\tilde{T}$ is satisfiable by a model $\underline{\tilde{M}}$, whose reduct to the σ_{graph}-structure is disconnected because there is no path between $a^{\tilde{M}}$ and $b^{\tilde{M}}$ in $\underline{M}$. But by Q1 of HW4, $\underline{M} \models T$ because $\underline{\tilde{M}} \models T$, contradicting that a graph satisfies T iff it is connected. $\square$