AST242 LECTURE NOTES PART 5

Contents

1. Waves and instabilities

In this section we will carry out a series of calculations that are similar in that they involve considering small perturbations and how they evolve. We will often derive a dispersion relation and use it to determine if waves are likely to be able to travel or if small perturbations can become unstable and grow. Our first example illustrates the procedure with sounds waves. In this case there is no instability but we do find a dispersion relation relating wavelengths to frequencies for wave solutions.

1.1. Sound waves – compressive waves in 1D.

Consider an isentropic ideal fluid with equation of state $p = K\rho^\gamma$. Recall Euler's equation.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p \tag{1}$$

Consider small perturbations in one dimension about an equilibrium configuration with zero velocity, $u(x,t) = u_1(x,t)$ and $\rho(x,t) = \rho_0 + \rho_1(x,t)$ where u_1 and ρ_1 are both small. We assume that the zeroth order velocity $u_0 = 0$. We have restricted the possible waves to 1 dimension. We also assume that pressure can be described as $p(x,t) = p_0 + p_1(x,t)$ where p_0 is independent of time and position. We insert these expressions in to Euler's equation

$$\frac{\partial}{\partial t}(u_0 + u_1) + (u_0 + u_1)\frac{\partial}{\partial x}(u_0 + u_1) = -\frac{1}{\rho_0 + \rho_1}\frac{\partial}{\partial x}(p_0 + p_1) \tag{2}$$

We can approximate $\frac{1}{\rho_0+\rho_1} \sim \frac{1}{\rho_0}(1 - \rho_1/\rho_0)$

Any term that is a produce of two first order terms is second order so we can drop it. We can also drop zero-th order terms as they should satisfy Euler's equation on their own. Euler's equation becomes

$$\frac{\partial u_1}{\partial t} + u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} + \frac{\rho_1}{\rho_0^2}\frac{\partial p_0}{\partial x} \tag{3}$$

As $u_0 = 0$ we can drop the second term on the left hand side. As u_0 doesn't depend on position we can drop the third term on the left hand side. As p_0 does not depend on position we can drop the second term on the right hand side. Euler equation now becomes

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial x} \tag{4}$$

We note that

$$dp_1 = \frac{\partial p_0}{\partial \rho}d\rho \quad \rightarrow \quad \frac{\partial p_1}{\partial x} = \left(\frac{\partial p}{\partial \rho}\right)_0 \frac{\partial \rho_1}{\partial x}. \tag{5}$$

and Euler's equation can be written in terms of density and velocity only

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_0}\left(\frac{\partial p}{\partial \rho}\right)_0 \frac{\partial \rho_1}{\partial x} \tag{6}$$

To first order in our perturbations Euler's equation has become

$$u_{1,t} = -\frac{1}{\rho_0}\left(\frac{\partial p}{\partial \rho}\right)_0 \rho_{1,x} \tag{7}$$

where $_{,x}$ refers to a derivative with respect to x. The $\mathbf{u}\cdot\nabla\mathbf{u}$ term is second order in our perturbation strength and so has been dropped. Let's take the time derivative of this equation

$$u_{1,tt} = -\frac{1}{\rho_0}\left(\frac{\partial p}{\partial \rho}\right)_0 \rho_{1,xt} \tag{8}$$

We now use the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{9}$$

Following the same procedure of using first order perturbations, conservation of mass becomes

$$\rho_{1,t} = -\rho_0 u_{1,x} \tag{10}$$

Let's take the x derivative of this equation

$$\rho_{1,tx} = -\rho_0 u_{1,xx} \tag{11}$$

Putting these together we find

$$u_{1,tt} = \left(\frac{\partial p}{\partial \rho}\right)_0 u_{1,xx} \tag{12}$$

This is a wave equation with solutions

$$u_1 = A\exp(i(kx - \omega t)) \tag{13}$$

$$\rho_1 = B\exp(i(kx - \omega t)) \tag{14}$$

for amplitudes A, B. Inserting these wavelike solutions into our equation for conservation of mass and that derived from Euler's equation we find a dispersion relation

$$\omega^2 = \left(\frac{\partial p}{\partial \rho}\right)_0 k^2 \tag{15}$$

The sound speed is

$$c_s = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_0} = \sqrt{\gamma\frac{p}{\rho}} \tag{16}$$

where the derivative is often describe in terms of keeping the entropy fixed. Furthermore Equation (7) relates the amplitude in velocity to that in density with $A = c_s B/\rho_0$.

The above procedure of using a first order perturbation approximation and looking for a dispersion relation is seen again and again in fluid dynamics. It is used for stability analysis, for example in derivation of the Kelvin Helmholtz instability. This procedure is also used to find the dispersion relation for spiral density waves in a fluid disk.

Consider how we would have done a similar derivation but not restricting the system to 1 dimension. Equation 7 (derived from Euler's equation) becomes

$$\mathbf{u}_{1,t} = -\frac{1}{\rho_0}\left(\frac{\partial p}{\partial \rho}\right)_0 \boldsymbol{\nabla}\rho_1 \tag{17}$$

and the equation of continuity becomes

$$\rho_{1,t} = -\rho_0 \boldsymbol{\nabla}\cdot\mathbf{u}_1 \tag{18}$$

Taking the divergence of equation 17 and time derivative of equation 18

$$\nabla \cdot \mathbf{u}_{1,t} = -\frac{1}{\rho_0}\left(\frac{\partial p}{\partial \rho}\right)_0 \nabla^2 \rho_1 \tag{19}$$

$$\rho_{1,tt} = -\rho_0 \nabla \cdot \mathbf{u}_{1,t} \tag{20}$$

These can be combined to give

$$\rho_{1,tt} = \left(\frac{\partial p}{\partial \rho}\right)_0 \nabla^2 \rho_1. \tag{21}$$

This is a wave equation. Solutions are

$$\propto e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}.$$

Note that equation 20 implies that there is no density perturbation if the velocity perturbation is incompressible.

1.1.1. *Plane waves.* We now redo the above but do not restrict the analysis to one spatial dimension. We assume that pressure, velocity and density

$$\propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{22}$$

so that

$$p = p_0 + p_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{23}$$

$$\rho = \rho_0 + \rho_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{24}$$

$$\mathbf{u} = \mathbf{u}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{25}$$

The continuity equation to first order becomes

$$-i\omega\rho_1 + i\rho_0 \mathbf{k}\cdot\mathbf{u}_1 = 0 \tag{26}$$

Euler's equation becomes

$$-i\omega\mathbf{u}_1 = -c_s^2 i\mathbf{k}\frac{\rho_1}{\rho_0} \tag{27}$$

We take $\mathbf{k}\cdot$ Euler's equation

$$\omega\mathbf{k}\cdot\mathbf{u}_1 = c_s^2 k^2 \frac{\rho_1}{\rho_0} \tag{28}$$

and insert the above into equation (26), finding a dispersion relation

$$\omega^2 = c_s^2 k^2 \tag{29}$$

as long as $\mathbf{k}\cdot\mathbf{u}_1 \neq 0$. This condition implies that $\frac{\partial \rho_1}{\partial t} \neq 0$ and so the waves are compressive in nature.

2. Jeans Instability

We consider a constant density medium at rest with density ρ and sound speed c_s. Consider perturbations to density, velocity and gravitational potential

$$\begin{aligned}\rho(x,t) &= \rho_0 + \rho_1(x,t) \\ u(x,t) &= 0 + u_1(x,t) \\ \Phi(x,t) &= \text{something} + \Phi_1(x,t) \qquad (30)\end{aligned}$$

with forms for the perturbations

$$\begin{aligned}\rho_1(x,t) &= \rho_a e^{i(kx-\omega t)} \\ u_1(x,t) &= u_a e^{i(kx-\omega t)} \qquad (31) \\ \Phi_1(x,t) &= \Phi_a e^{i(kx-\omega t)} \qquad (32)\end{aligned}$$

The procedure of completely ignoring the zeroth order term in the gravitational potential is known as the *Jeans swindle*. Usually one assumes that the zero-th order variables satisfy the hydrodynamic equations.

The perturbation to the gravitational potential can be found from Poisson's equation

$$\nabla^2 \Phi_1 = 4\pi G \rho_1 \qquad (33)$$

we take only the derivative in the x direction finding

$$\Phi_a = -\frac{4\pi G \rho_a}{k^2} \qquad (34)$$

Conservation of mass to first order

$$\rho_{1,t} + \rho_0 u_{1,x} = 0 \qquad (35)$$

Inserting our perturbations

$$-\omega \rho_a + \rho_0 k u_a = 0 \qquad (36)$$

or

$$u_a = \frac{\rho_a}{\rho_0} \frac{\omega}{k} \qquad (37)$$

Euler's equation to first order

$$u_{1,t} + c_s^2 \frac{\rho_{1,x}}{\rho_0} = -\Phi_{1,x} \qquad (38)$$

Inserting our Fourier components

$$-i\omega u_a + ikc_s^2 \frac{\rho_a}{\rho_0} = ik \frac{4\pi G \rho_a}{k^2} \qquad (39)$$

Inserting our relation between u_a and ρ_a we find a dispersion relation

$$\omega^2 = c_s^2 k^2 - 4\pi G\rho_0 \tag{40}$$

Let's look at the dispersion relation. In the limit of larger k we recover our sound waves. In the limit of small k our frequency becomes complex and modes are unstable. We set $\omega^2 = 0$ and solve for k finding a Jeans wave-vector

$$k_J = \sqrt{\frac{4\pi G\rho_0}{c_s^2}} \tag{41}$$

or an associated Jeans wavelength $\lambda_J = 2\pi/k_J$. The maximum stable wavelength

$$\lambda_J = \sqrt{\frac{\pi c_s^2}{G\rho_0}} \tag{42}$$

For $k < k_J$ or $\lambda > \lambda_J$ modes are unstable and there can be gravitational collapse. We can also consider a mass contained within this wavelength $\rho_0\lambda_J^3$ and call it the Jeans mass

$$M_J \sim \frac{\pi^{3/2} c_s^3}{G^{3/2}\rho_0^{1/2}} \tag{43}$$

As $k \to 0$ we find that $\omega^2 \to -4\pi G\rho_0$ becomes increasingly negative and reaches a maximum at $k = 0$ which corresponds to an infinite wavelength. In this limit the growth inverse timescale $\gamma = i\omega = \sqrt{4\pi G\rho_0}$. Given a particular cloud the fastest growing wavelength will be the largest one of order the size of the cloud itself.

It is useful to estimate a timescale for gravitational collapse to happen. In this case $\sqrt{G\rho}$ is an inverse collapse timescale or a free-fall timescale.

What does this instability mean physically? The Jeans wavelength is the wavelength at which the sound crossing timescale is equivalent to the free fall or gravitational collapse timescale. The Jeans instability implies that all density distributions are unstable, from large scale cosmological structure to molecular clouds. Clumps will form with a minimum mass given by the Jeans mass.

3. Stratified Fluid Flows – Waves or Instabilities on a Fluid Boundary

Consider two incompressible inviscid fluids one lying above the other in a uniform gravitational field. The fluid on the top has density ρ' and that on the bottom ρ. The fluid on the top has mean velocity U' and that on the bottom U. We orient our coordinate system so that $-z$ is the direction of gravitational acceleration, the interface between the two fluids lies at $z = 0$ and x increases along the interface.

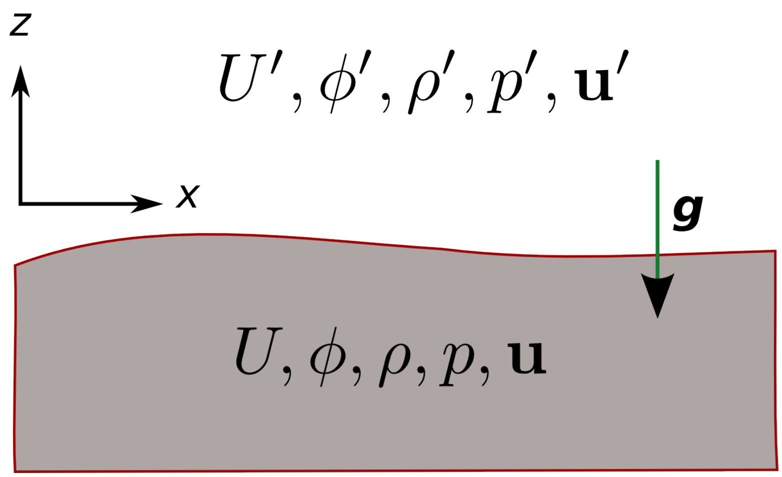

FIGURE 1. Two fluids, one lying above the other in a gravitational field. The top fluid has horizontal velocity U' and the bottom one has velocity U. When the top fluid is denser than the bottom fluid $\rho' > \rho$ then the boundary is unstable to the Rayleigh-Taylor instability. When $\rho' < \rho$ gravity waves can propagate on the boundary. When $U' \neq U$ then the boundary is unstable to the Kelvin-Helmholtz instability.

This problem is somewhat more complicated than our previous example of sound waves as we must consider an interface.

Recall Euler's equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla \mathbf{u})\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla \Phi \tag{44}$$

We use the vector identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \left(\frac{u^2}{2}\right) - u \times \nabla \times u \tag{45}$$

If the flow is irrotational we can drop the second term on the right. For an irrotational flow Euler's equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{u^2}{2}\right) = -\frac{1}{\rho}\nabla p - \nabla \Phi \tag{46}$$

If the flow is irrotational then we can use a potential function for the velocity $\mathbf{u}$ such that

$$\nabla \psi = -\mathbf{u} \tag{47}$$

Euler's equation becomes

$$\nabla\left(-\frac{\partial\psi}{\partial t}+\frac{u^2}{2}+\frac{p}{\rho}+\Phi\right)=0 \tag{48}$$

Here moving the density inside the gradient is equivalent to assuming that the flow is incompressible. We can integrate the previous equation finding

$$-\frac{\partial\psi}{\partial t}+\frac{u^2}{2}+\frac{p}{\rho}+\Phi=F(t) \tag{49}$$

where $F(t)$ does not depend on position.

We now consider an interface between the two fluids and describe the position of the interface (in z) as $\xi(x,t)$. We assume that

$$\xi = Ae^{i(kx-\omega t)} \tag{50}$$

and based on a derived dispersion relation determine if small perturbations grow, decay or oscillate with time.

Since our velocity is U' and U above and below the interface our velocity potential can be written

$$\psi(x,z,t)=\begin{cases} -U'x+\phi'(x,z,t) & \text{for} \quad z>0 \\ -Ux+\phi(x,z,t) & \quad\quad\;\; z<0 \end{cases} \tag{51}$$

We regard the functions ϕ,ϕ' as perturbations. The above is equivalent to describing the velocity as the sum of zero-th and first order solutions. We assume that $\phi',\phi \propto e^{i(kx-\omega t)}$. However ϕ',ϕ also depend on z.

For an incompressible fluid $\nabla\cdot\mathbf{u}=0$ so

$$\nabla^2\psi=0. \tag{52}$$

For a solution with perturbation exponentially decaying with z (and decaying with increasing z for $z>0$ and decaying with decreasing z for $z<0$), and depending on $e^{i(kx-\omega t)}$, Laplaces's equation allows us to relate the exponential decay rate with z to the wavenumber in x. In other words

$$\begin{aligned} \phi' &= \phi_1' e^{i(kx-\omega t)-kz} \\ \phi &= \phi_1 e^{i(kx-\omega t)+kz} \end{aligned} \tag{53}$$

satisfy Laplace's equation.

We write out the velocity $\mathbf{u}=-\nabla\psi$

$$\mathbf{u}=\begin{cases} U'\hat{\mathbf{x}}+k\phi_1'(-i\hat{\mathbf{x}}+\hat{\mathbf{z}})e^{i(kx-\omega t)-kz} & \text{for} \quad z>0 \\ U\hat{\mathbf{x}}+k\phi_1(-i\hat{\mathbf{x}}-\hat{\mathbf{z}})e^{i(kx-\omega t)+kz} & \quad\quad\;\; z<0 \end{cases} \tag{54}$$

We require that fluid elements not cross the interface, $\xi(x,t)$. This means that

$$u_z = -\frac{d\psi'}{dz} = \frac{\partial \xi}{\partial t} + U'\frac{\partial \xi}{\partial x} \qquad \text{for} \qquad z > 0 \tag{55}$$

and

$$u_z = -\frac{d\psi}{dz} = \frac{\partial \xi}{\partial t} + U\frac{\partial \xi}{\partial x} \qquad \text{for} \qquad z < 0. \tag{56}$$

Equation 50 gives partial derivatives

$$\frac{\partial \xi}{\partial t} = -i\omega A e^{i(kx-\omega t)} \qquad \frac{\partial \xi}{\partial x} = ikAe^{i(kx-\omega t)} \tag{57}$$

Inserting these into equations 55, 56 we find

$$\begin{aligned} i(kU' - \omega)A &= k\phi_1' \\ i(kU - \omega)A &= -k\phi_1. \end{aligned} \tag{58}$$

and each is satisfied. So far the above are two equations for three unknowns A, ϕ_1', ϕ_1. However we have another equation derived from Euler's equation (or Bernoulli's equation).

We can invert equation 49 to solve for pressure

$$p = -\rho\left(-\frac{\partial \psi}{\partial t} + \frac{u^2}{2} + \Phi\right) + \rho F(t) \tag{59}$$

At the interface we require pressure balance. The gravitational potential at the interface $\Phi = gz = g\xi$ so

$$-\rho'\left(-\frac{\partial \psi'}{\partial t} + \frac{u'^2}{2} + g\xi\right) + \rho' F(t) = -\rho\left(-\frac{\partial \psi}{\partial t} + \frac{u^2}{2} + g\xi\right) + \rho F(t) \tag{60}$$

For large and small z above and below the boundary we assume that perturbations in the fluid will decay. Because $F(t)$ does not depend on position we can set it to zero.

Zeroth order terms should balance, so we need only consider terms that are first order in our perturbation quantities. To first order

$$u'^2 \to -2U'\frac{\partial \phi'}{\partial x} \qquad u^2 \to -2U\frac{\partial \phi}{\partial x}. \tag{61}$$

Equation 54 implies that $\frac{\partial \psi'}{\partial t} = \frac{\partial \phi'}{\partial t}$ and likewise for ψ and ϕ. Inserting these expressions into our pressure relation (equation 60) and evaluating it at $z = 0$

$$\rho'\left[-\left(\frac{\partial}{\partial t} + U'\frac{\partial}{\partial x}\right)\phi' + g\xi\right] = \rho\left[-\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi + g\xi\right]. \tag{62}$$

Evaluating at $z = 0$ is equivalent to assuming that the amplitude of perturbation A is very small (specifically $A < k^{-1}$). Using our exponential forms for ϕ, ϕ', ξ this becomes

$$\rho' \left(-i(kU' - \omega)\phi_1' + gA\right) = \rho \left(-i(kU - \omega)\phi_1 + gA\right). \tag{63}$$

Using equations 58 to eliminate ϕ_1', ϕ_1 we find an equation where each term is proportional to A. Dividing this by A we find a dispersion relation that can be put in the following form

$$\rho(kU - \omega)^2 + \rho'(kU' - \omega)^2 = kg(\rho - \rho'). \tag{64}$$

This is our **dispersion relation**.

3.1. **Surface gravity waves.** For two fluids at rest ($U = U' = 0$) our dispersion relation is

$$\omega = \pm\sqrt{gk}\sqrt{\frac{\rho - \rho'}{\rho + \rho'}} \tag{65}$$

If $\rho > \rho'$ then ω is real and we have wave-like solutions. In the limit of $\rho' \ll \rho$ (air over water) $\omega = \sqrt{gk}$ and we have surface gravity waves.

3.2. **Rayleigh-Taylor instability.** Consider fluids at rest with $\rho' > \rho$. In this case ω has to be complex which implies that perturbations can grow exponentially and so are unstable. The solution is

$$\omega = \pm i\sqrt{gk}\sqrt{\frac{\rho' - \rho}{\rho + \rho'}} \tag{66}$$

with solutions $\propto e^{\pm i\omega t}$. We can construct a growth timescale

$$t_{grow} = \frac{i}{|\omega|} = (gk)^{-1/2}\left|\frac{\rho' - \rho}{\rho + \rho'}\right|^{-1/2} \tag{67}$$

so we can write solutions as

$$\propto e^{\pm t/t_{grow}} \tag{68}$$

As we have assumed uniform acceleration by gravity, this instability could also take place in other settings where there is uniform acceleration such as in the shell of a supernova remnant. Outward acceleration is equivalent to inwardly directed gravity.

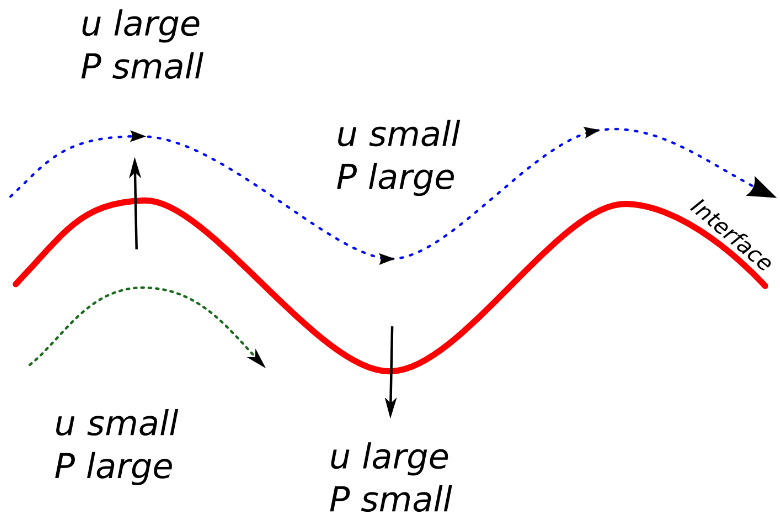

FIGURE 2. Kelvin Helmholtz instability grows because of pressure differences caused by the velocity perturbations. It's not obvious here, but there must be a small lag between pressure and velocity differences for water waves to be excited by wind. If the velocity and pressure perturbations are exactly in phase then there is no energy transfer between media.

3.3. **Kelvin-Helmholtz Instability.** We consider the case when fluids are stable to the Rayleigh-Taylor instability but moving with respect to one another. Our dispersion relation (equation 64) is a quadratic equation for ω. Grouping terms we can write the dispersion relation as a polynomial of ω,

$$\omega^2(\rho+\rho') - \omega 2k(\rho U + \rho' U') + k^2(\rho U^2 + \rho' U'^2) - kg(\rho - \rho') = 0 \tag{69}$$

The quadratic equation gives

$$\omega = \frac{1}{2(\rho+\rho')}\Big(2k(\rho U + \rho' U') \pm \sqrt{4k^2(\rho U + \rho' U')^2 - 4(\rho+\rho')(k^2(\rho U^2 + \rho' U'^2) - kg(\rho-\rho'))}\Big). \tag{70}$$

There is no real solution and instability occurs when

$$k > \frac{(\rho^2 - \rho'^2)g}{\rho\rho'(U-U')^2}. \tag{71}$$

Note ω is not necessarily small at the transition point. When gravity is unimportant, we can take $g \to 0$, and then find that all wavelengths are unstable as long as $U' \neq U$. When $\rho > \rho'$, then gravity stabilizes short k or long wavelengths. The larger k

are more unstable, consequently the smallest wavelengths grow the fastest, however surface tension and other small scale processes can stabilize the smallest wavelengths.

4. Thermal Instability

We consider the case of the interstellar medium, gas with different temperatures but with each phase approximately in pressure equilibrium. There are different heating and cooling mechanisms. We start with an equation for energy conservation. We would like to work with variables density, temperature and pressure.

$$TdS = de - p\frac{d\rho}{\rho^2} \tag{72}$$

For an ideal gas with $e = \frac{p}{\rho}\frac{1}{\gamma-1}$

$$\rho T dS = \frac{dp}{\gamma - 1} - \frac{\gamma}{\gamma-1}\frac{p}{\rho}d\rho \tag{73}$$

Using the Lagrangian derivative and equating heating rate with that due to thermal conductivity and heating and cooling

$$\rho T\frac{DS}{Dt} = \frac{1}{\gamma-1}\frac{Dp}{Dt} - \frac{\gamma}{\gamma-1}\frac{p}{\rho}\frac{D\rho}{Dt} = \nabla\cdot(\lambda\nabla T) + Q^+(\rho,T) - Q^-(\rho,T) \tag{74}$$

where λ is the thermal conductivity, Q^+ is the heating rate per unit volume and Q^- is the cooling rate. The form of the cooling rate, Q^-, is expected to be non-trivial in shape. It contains a peak at about 100K, due to molecular line emission (vibrational and rotational transitions), and one at 10^5K due to optical and UV emission lines from ions (e.g., bound-free, bound-bound, and during recombination).

We include three more equations, the continuity equation (conservation of mass) and Euler's equation (conservation of momentum) and an equation of state

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{75}$$

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p \tag{76}$$

$$p = \frac{\rho k_B T}{\bar{m}} \tag{77}$$

We consider perturbations of velocity, density, temperature and pressure in the form

$$\begin{aligned}
\rho(\mathbf{x},t) &= \rho_0 + \rho_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
p(\mathbf{x},t) &= p_0 + p_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
T(\mathbf{x},t) &= T_0 + T_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\
\mathbf{u}(\mathbf{x},t) &= \mathbf{u}_1 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}
\end{aligned} \tag{78}$$

We assume the gas without perturbations is uniform and at rest ($u_0 = 0$, and p_0, ρ_0, T_0 independent of $\mathbf{x}$ and time).

Let consider perturbations to the heating and cooling functions

$$Q(\rho, T) = Q^+(\rho_0, T_0) - Q_-(\rho_0, T_0) + Q_\rho \rho_1 + Q_T T_1 \tag{79}$$

so that

$$\begin{aligned}
Q_\rho &= \left.\frac{\partial(Q^+ - Q^-)}{\partial \rho}\right|_{\rho_0, T_0} \\
Q_T &= \left.\frac{\partial(Q^+ - Q^-)}{\partial T}\right|_{\rho_0, T_0}
\end{aligned} \tag{80}$$

To first order in our perturbations, the continuity equation, Euler's equation, the equation of state and our equation for conservation of energy become

$$-i\omega\rho_1 + i\mathbf{k}\cdot\mathbf{u}_1\rho_0 = 0 \tag{81}$$

$$-i\omega\mathbf{u}_1 = -i\mathbf{k}\frac{p_1}{\rho_0} \tag{82}$$

$$\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} + \frac{T_1}{T_0} \tag{83}$$

$$-\frac{i\omega}{\gamma - 1}\left(p_1 - \frac{\gamma p_0}{\rho_0}\rho_1\right) = -\lambda k^2 T_1 + Q_T T_1 + Q_\rho \rho_1 \tag{84}$$

Note $\frac{\gamma p_0}{\rho_0} = c_s^2$ where c_s is the sound speed. We can dot $\mathbf{k}$ with equation (82) to find

$$\omega \mathbf{k}\cdot\mathbf{u}_1 = k^2 \frac{p_1}{\rho_0} \tag{85}$$

and this can be used to eliminate the velocity from the continuity equation

$$\omega^2 \rho_1 = k^2 p_1 \tag{86}$$

Inserting this into the last two perturbation equations

$$\left(\frac{\omega^2}{k^2}\frac{\rho_0}{p_0} - 1\right)\frac{\rho_1}{\rho_0} = \frac{T_1}{T_0} \tag{87}$$

$$-\frac{i\omega}{\gamma-1}\left(\frac{\omega^2}{k^2} - c_s^2\right)\rho_1 = Q_T T_1 - \lambda k^2 T_1 + Q_\rho \rho_1 \tag{88}$$

These can be manipulated to give a dispersion relation

$$-\frac{i\omega}{\gamma-1}\left(\frac{\omega^2}{k^2} - c_s^2\right) - \left(Q_T - \lambda k^2\right)\frac{T_0}{p_0}\left(\frac{\omega^2}{k^2} - \frac{c_s^2}{\gamma}\right) = Q_\rho \tag{89}$$

where we have used our definition for sound speed. This differs only in the sign of ω from the expression given by Pringle & King (equation 8.10) because of the different sign in our assumed form for the perturbation. The dispersion relation is a cubic equation in ω. The three roots (real or/and complex) can be studied as a function of k.

Note: We have assumed an ideal gas described with an adiabatic index even though we are considering heating and cooling.

4.1. **Acoustic waves damped by thermal conductivity.** If there is no thermal conductivity, $\lambda = 0$, and heating and cooling are negligible ($Q_\rho = Q_T = 0$) then the dispersion relation (equation 89) simplifies to

$$\omega^2 = k^2 c_s^2 \tag{90}$$

which is the relation for acoustic or sound waves.

Let's consider the dispersion relation when heating balances cooling and we are at an equilibrium point $Q_\rho = Q_T = 0$. In this case the dispersion relation becomes

$$i\omega\left(\omega^2 - k^2c_s^2\right) = \lambda k^2 \frac{T_0}{\rho_0}(\gamma - 1)\left(\omega^2 - k^2 c_s^2 \gamma^{-1}\right) \tag{91}$$

If we assume that λ is small we can try a solution $\omega = kc_s + x$ where x is small. To first order in λ and x

$$i2k^2c_s^2 x = \lambda k^2 \frac{T_0}{\rho_0}(\gamma-1)^2\gamma^{-1}k^2c_s^2 \tag{92}$$

$$x = -\frac{i\lambda}{2}\frac{T_0}{\rho_0}(\gamma-1)^2\gamma^{-1} \tag{93}$$

Thus

$$\omega = c_s k - i\delta \qquad \delta = \frac{\lambda}{2}\frac{T_0}{\rho_0}(\gamma-1)^2\gamma^{-1} \tag{94}$$

with $\delta > 0$. With our form for the perturbations $\propto \exp(i(\mathbf{k}\cdot\mathbf{x} - \omega t)$ we find that our perturbations exponentially damp due to thermal conductivity.

Note that δ is not small if k is sufficiently large. Hence our assumption that δ is small is only satisfied for moderate values of k. For k sufficiently large (or over sufficiently small distances) perturbations are rapidly damped due to conduction.

The parameter δ has units of inverse time. We can estimate how much the amplitude is reduced per wavelength traveled with the factor $e^{-\delta P}$ where P is the period, $P = \frac{2\pi}{\omega} = \frac{2\pi}{c_s k}$.

4.2. **Field Stability Criterion.** We consider slow changes or small ω. In this limit $\omega \ll kc_s$ and our dispersion relation (equation 89) reduces to

$$\frac{i\omega}{\gamma - 1} c_s^2 + \left(Q_T - \lambda k^2\right) \frac{T_0}{p_0} \frac{c_s^2}{\gamma} = Q_\rho \tag{95}$$

We can rewrite this as

$$i\omega = \frac{\gamma - 1}{\gamma p_0} \left(Q_\rho \rho_0 - Q_T T_0 + \lambda k^2 T_0\right). \tag{96}$$

The ISM is nearly in pressure equilibrium. Since $p \propto \rho T$ we can use $\left.\frac{\partial \rho}{\partial T}\right|_p = -\frac{\rho}{T}$ and

$$\left.\frac{\partial Q}{\partial \ln T}\right|_p = Q_T T + Q_\rho T \left.\frac{\partial \rho}{\partial T}\right|_p = Q_T T - Q_\rho \rho \tag{97}$$

Thus our dispersion relation (equation 95) becomes

$$-i\omega = \frac{\gamma - 1}{\gamma p_0} \left[\left.\frac{\partial Q}{\partial \ln T}\right|_p - \lambda k^2 T_0 \right] \tag{98}$$

In the case of no conductivity ($\lambda = 0$) there is instability when

$$\text{instability} \iff \left.\frac{\partial Q}{\partial \ln T}\right|_p > 0 \tag{99}$$

This is called the Field stability criterion.

Let us consider the Field stability criterion physically. Pressure equilibrium is expected otherwise there will be large velocities generated and mixing. If increasing the temperature increases the heating rate then the temperature will continue to rise and we will have instability. If increasing the temperature decreases the heating rate then we will have convergence to a stable situation. For the second case, giving stability, we have *negative feedback*. For the first case, giving instability, we have *positive feedback*. I note that there can be instability even in the negative feedback case if the feedback (heating or cooling) is delayed, a situation that we have not taken into account here but does happen for example when star formation heats or perturbs the medium affecting future generations of stars.

When there is conductivity and k is large there is stability even when the stability criterion shown in equation (99) is violated. In this case the thermal conductivity smoothes or damps temperature fluctuations faster than they can grow. We can define a critical wavevector by setting $\omega = 0$ in equation (99) and solving for k

$$k_F^2 \equiv \frac{1}{\lambda T_0} \left.\frac{\partial Q}{\partial \ln T}\right|_p \tag{100}$$

The Field length $\lambda_F \equiv 2\pi/k_F$. For wavelengths larger than the critical one perturbations will grow. Small perturbations are smoothed out by the conductivity and so stabilized.

4.3. **An aside on positive and negative feedback.** The very simplest system

$$\frac{d\Sigma}{dt} = f(\Sigma) \tag{101}$$

can achieve equilibrium near a fixed point Σ_* where $f(\Sigma_*) = 0$. Expanding about the fixed point and letting $x = \Sigma - \Sigma_*$

$$\frac{dx}{dt} = f'(\Sigma_*)x \tag{102}$$

This has an exponential solution that is exponentially decaying to the fixed point if $f'(\Sigma_*) < 0$. If the derivative is negative we say the system has negative feedback and we expect a stable system. Otherwise we say the system has positive feedback and we expect it to be unstable. Note that $f'(\Sigma)$ is in units of inverse time and so defines a decay or growth timescale;

$$t_{decay} = -\frac{1}{f'(\Sigma)}. \tag{103}$$

A system with delayed feedback might look like this

$$\frac{d\Sigma(t)}{dt} = f(\Sigma(t-\tau)) \tag{104}$$

so that its rate of change depends on the density Σ at a time τ in the past. We could imagine a complicated function f that depends on Σ at various times. There is a class of differential equations called delayed differential equations that can be employed. Instability can occur even when there is negative feedback in systems with delayed feedback. Instability or oscillating solutions can occur when the feedback delay timescale τ exceeds the timescale for decay to equilibrium, t_{decay}. For example

$$\dot{\Sigma} = -\frac{\pi}{2\tau}\Sigma(t-\tau) \tag{105}$$

has solution

$$\Sigma \propto \sin\left(\frac{\pi t}{2\tau}\right). \tag{106}$$

. Oscillating solutions can have frequencies that period related to the delay; here $P \sim 4\tau$.

5. Convective Instability

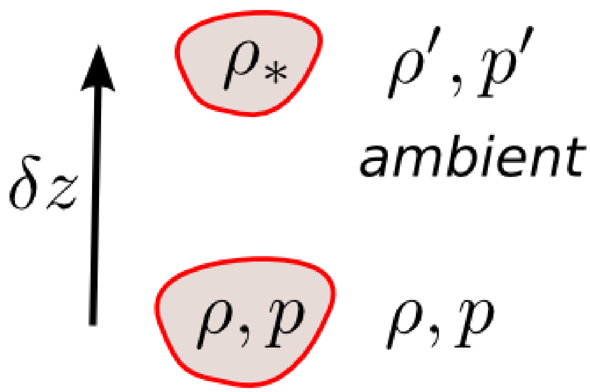

FIGURE 3. A small fluid element is displaced by δz. Originally it had pressure and density p, ρ. At its new location the ambient pressure and density are p', ρ'. We let the fluid element expand or contract to reach pressure equilibrium at its new location without exchanging heat. Its new density is ρ_*. If $\rho_* > \rho'$ and the fluid element is denser than the ambient medium then it will sink back to its original location. Otherwise it will keep rising and the system is convectively unstable.

Consider a small fluid element that is located in fluid that is in hydrostatic equilibrium in a gravitational field. Our fluid element starts in equilibrium with pressure and density p, ρ. We then displace it by a vertical amount δz. At this new position the ambient pressure and density are slightly different with pressure and density p', ρ'. When we displaced our fluid element it may have contracted or expanded. If its new density ρ_* is larger than the new ambient value ρ' then the fluid element will sink back to its original position. In this case the system is stable. If the new fluid element density ρ_* is lower than the new ambient value then the fluid element is buoyant and we say the system is convectively unstable.

How are we going to let the fluid element change? We let it remain in pressure equilibrium with its surroundings. This means its new pressure is p'. We don't allow it to heat or cool while we displace it. This is equivalent to assuming that heat exchange is slow. This is equivalent to moving the fluid element adiabatically. Because we consider adiabatic changes $p \propto \rho^\gamma$ and

$$\frac{p_*}{p} = \frac{p'}{p} = \left(\frac{\rho_*}{\rho}\right)^\gamma \qquad \text{or} \qquad \rho_* = \rho\left(\frac{p'}{p}\right)^{1/\gamma} \tag{107}$$

The pressure gradient for our atmosphere is $\frac{dp}{dz}$. If we go up by δz from a particular location, then the ambient pressure and density are

$$p' = p + \frac{dp}{dz}\delta z \qquad \rho' = \rho + \frac{d\rho}{dz}\delta z. \tag{108}$$

We insert p' into our previous equation finding

$$\begin{aligned} \rho_* &= \rho\left(\frac{p'}{p}\right)^{1/\gamma} \\ &= \rho\left(\frac{p + dp/dz\delta z}{p}\right)^{1/\gamma} \\ &= \rho\left(1 + \frac{1}{\gamma p}\frac{dp}{dz}\delta z\right) \\ &= \rho + \frac{\rho}{\gamma p}\frac{dp}{dz}\delta z. \end{aligned} \tag{109}$$

The difference in densities

$$\rho_* - \rho' = \left(\frac{\rho}{\gamma p}\frac{dp}{dz} - \frac{d\rho}{dz}\right)\delta z \tag{110}$$

The system is stable if $\rho_* > \rho'$ or

$$\frac{\rho}{\gamma p}\frac{dp}{dz} > \frac{d\rho}{dz} \tag{111}$$

The stability criterion is often called the Schwarzschild criterion but it is usually expressed in terms of temperature.

5.1. **Schwarzschild criterion and the Brunt-Väisälä frequency.** Using an ideal gas $p \propto \rho T$ or

$$\rho \propto \frac{p}{T}.$$

Expanding the derivative $\frac{d\rho}{dz}$,

$$\rho' = \rho + \frac{d\rho}{dz}\delta z = \rho + \left[\frac{\rho}{p}\frac{dp}{dz} - \frac{\rho}{T}\frac{dT}{dz}\right]\delta z. \tag{112}$$

We can modify equation 110

$$\rho_* - \rho' = \left[-(1-\gamma^{-1}) \frac{\rho}{p} \frac{dp}{dz} + \frac{\rho}{T} \frac{dT}{dz} \right] \delta z \tag{113}$$

must be positive for stability. Both $\frac{dT}{dz}$ and $\frac{dp}{dz}$ are expected to be negative in an atmosphere. The first term dominates and the medium is stable if

$$\left| \frac{dT}{dz} \right| < (1-\gamma^{-1}) \frac{T}{\rho} \left| \frac{dp}{dz} \right| . \tag{114}$$

This is the Schwarzschild criterion.

The force per unit volume acting on the displaced fluid element is $g(\rho_* - \rho)$ where the difference in densities is given by equation (113). An equation of motion

$$\rho \frac{d^2(\delta z)}{dt^2} = -g(\rho - \rho_*) \tag{115}$$

or

$$\frac{d^2(\delta z)}{dt^2} = -\frac{g}{\rho} \left[-(1-\gamma^{-1}) \frac{\rho}{p} \frac{dp}{dz} + \frac{\rho}{T} \frac{dT}{dz} \right] \delta z \tag{116}$$

The Brunt-Väisälä frequency, N,

$$N^2 = \frac{g}{T} \left[\frac{dT}{dz} - (1-\gamma^{-1}) \frac{T}{p} \frac{dp}{dz} \right] \tag{117}$$

gives the frequencies of small oscillations when the system is stable. These are called internal gravity waves to differentiate them from surface waves. The frequency is that for buoyant oscillations. The Brunt-Väisälä frequency can also be written (see equation 110)

$$N^2 = \frac{g}{\rho} \left(\frac{\rho}{\gamma p} \frac{dp}{dz} - \frac{d\rho}{dz} \right) \tag{118}$$

For an isothermal gas $\gamma = 1$, the pressure is proportional to the density, $p \propto \rho$, and the temperature gradient is zero, $dT/dz = 0$. In an isothermal atmosphere the Brunt-Väisälä frequency vanishes; $N = 0$. An atmosphere with a steep temperature gradient is likely to be convective (with negative N^2 and not satisfying the Schwarzchild criterion for stability) whereas one with a shallow temperature gradient should be stable (and with positive N^2).

6. Waves traveling in a plane parallel atmosphere

The background setting is an atmosphere in hydrostatic equilibrium. We encounter two restoring forces for small perturbations: pressure and buoyancy. We suspect that waves can travel at the sound speed, c_s if they are primarily acoustic in nature, and with a frequency that is related to the Brunt-Väisälä frequency, N if they are buoyant in nature. We expect a dispersion relation that has limits of $\omega = c_s k$ for acoustic waves and $\omega = N$ for internal gravity waves.

Our zero-th order or equilibrium solution has density and pressure $\rho_0(z)$ and $p_0(z)$ related via hydrostatic equilibrium

$$\frac{dp_0}{dz} = -\rho_0 g \tag{119}$$

where g is the gravitational acceleration. Mass conservation to first order

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}_1 + \rho_1 \nabla \cdot \mathbf{u}_0 + \mathbf{u}_0 \cdot \nabla \rho_1 + \mathbf{u}_1 \cdot \nabla \rho_0 = 0 \tag{120}$$

We drop terms with $\mathbf{u}_0$ as the equilibrium system is static and $\mathbf{u}_0 = 0$. The gradient $\nabla \rho_0$ only contains a z component.

$$\frac{\partial \rho_1}{\partial t} + u_{z1} \frac{\partial \rho_0}{\partial z} + \rho_0 \nabla \cdot \mathbf{u}_1 = 0 \tag{121}$$

Euler's equation for momentum conservation, again to first order

$$\frac{\partial \mathbf{u}_1}{\partial t} = \frac{\rho_1}{\rho_0^2} \frac{dp_0}{dz} \hat{\mathbf{z}} - \frac{1}{\rho_0} \nabla p_1 \tag{122}$$

and we have ignored the variation of the gravitational acceleration. We have used the fact that the unperturbed pressure gradient is in the z direction.

We have a system that in which two directions x, y differ from the other, z, where gradients in the ambient pressure and density exist. Let our velocity perturbation

$$\mathbf{u}_1 = (u, v, w). \tag{123}$$

Let our perturbations be functions of z that are wavelike in the other two dimensions,

$$u, v, w, \rho_1, p_1 \propto \exp i(\omega t - k_x x - k_y y) \tag{124}$$

with

$$k_\perp \equiv \sqrt{k_x^2 + k_y^2}. \tag{125}$$

The equation for conservation of mass (equation 121) becomes

$$i\omega \rho_1 + w \frac{d\rho_0}{dz} + \rho_0 \left[-i(k_x u + k_y v) + \frac{dw}{dz} \right] = 0 \tag{126}$$

Euler's equation (equation 122) becomes

$$\begin{aligned} \omega u &= \frac{1}{\rho_0} k_x p_1 \\ \omega v &= \frac{1}{\rho_0} k_y p_1 \\ i\omega w &= \frac{\rho_1}{\rho_0^2}\frac{dp_0}{dz} - \frac{1}{\rho_0}\frac{dp_1}{dz} \end{aligned} \tag{127}$$

We can use the first two expressions above to remove u, v from equation (126).

$$i\omega\rho_1 + w\frac{d\rho_0}{dz} - i(k_x^2 + k_y^2)\frac{p_1}{\omega} + \rho_0\frac{dw}{dz} = 0 \tag{128}$$

We now relate pressure perturbations to density perturbations. We assume that perturbations are locally adiabatic. This condition can be written

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + (\mathbf{u}\cdot\nabla)\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{129}$$

Expanding this

$$\frac{\partial p}{\partial t} - c_s^2\frac{\partial \rho}{\partial t} + (\mathbf{u}\cdot\nabla)p - c_s^2(\mathbf{u}\cdot\nabla)\rho = 0 \tag{130}$$

where $c_s^2 = \gamma p/\rho$. To first order

$$\frac{\partial p_1}{\partial t} - c_s^2\frac{\partial \rho_1}{\partial t} + w\frac{dp_0}{dz} - c_s^2 w\frac{d\rho_0}{dz} = 0 \tag{131}$$

or

$$i\omega(p_1 - c_s^2\rho_1) + w\left(\frac{dp_0}{dz} - c_s^2\frac{d\rho_0}{dz}\right) = 0. \tag{132}$$

Remember that c_s^2 depends on z! The right terms can be written in terms of the Brunt-Väisälä frequency (equation 118)

$$i\omega(p_1 - c_s^2\rho_1) + w\frac{\rho_0 c_s^2 N^2}{g} = 0. \tag{133}$$

Use hydrostatic equilibrium

$$g = -\frac{1}{\rho_0}\frac{dp_0}{dz} \tag{134}$$

and the Brunt-Väisälä frequency

$$-\frac{d\rho_0}{dz} = \frac{N^2\rho_0}{g} + \frac{g\rho_0}{c_s^2} \tag{135}$$

we can remove vertical derivatives of ρ_0 and p_0. Pulling together our equations (equation 128,127 and 133)

$$i\omega\rho_1 - w\rho_0\left(\frac{N^2}{g} + \frac{g}{c_s^2}\right) - ik_\perp^2\frac{p_1}{\omega} + \rho_0\frac{dw}{dz} = 0 \tag{136}$$

$$i\omega w + \frac{g\rho_1}{\rho_0} + \frac{1}{\rho_0}\frac{dp_1}{dz} = 0 \tag{137}$$

$$i\omega(p_1 - c_s^2\rho_1) + w\frac{\rho_0 c_s^2 N^2}{g} = 0 \tag{138}$$

The above involve vertical derivatives of w and p_1 but not ρ_1 so we can reduce the problem to two coupled first order equations involving p_1 and w. These can be studied by themselves with all variables depending on z.

One can do a *local* analysis assuming that wavelengths are smaller than the atmosphere scale height. In this case we can assume that ρ_1, p_1, w all depend on $e^{ik_z z}$ and require that $k_z \gg g/c_s^2$. In this limit we find a local dispersion relation

$$\omega^4 - (N^2 + k^2c_s^2)\omega^2 + N^2k_\perp^2c_s^2 = 0 \tag{139}$$

with

$$k^2 = k_\perp^2 + k_z^2 \tag{140}$$

There are two separated regimes on a plot of ω^2 vs $k_\perp^2$ for allowed wave propagation (see Figure 4). There is a higher frequency set with $\omega^2 > N^2$ which are known as p-waves and a lower frequency set which are known as g-waves. The acoustic or p-waves have phase velocities (ω/k) greater than the sound speed and the buoyancy or g-waves have phase velocities below the sound speed.

In the limit of high ω, we find $\omega^2 \sim N^2 + k^2c_s^2$.

In the limit of high ω, k, we find $\omega^2 \sim k^2c_s^2$ and the waves are acoustic.

In the limit of low ω, we find $\omega^2 \sim \frac{N^2k_\perp^2c_s^2}{N^2+k^2c_s^2}$.

In the limit of low ω, k, we find $\omega^2 \sim k^2c_s^2$ and the waves are acoustic.

In the limit of low ω and high k we find $\omega^2 \sim \frac{N^2k_\perp^2}{k^2}$.

Some notes: Apparently low g waves are incompressible, (but why?) and the phase velocity is perpendicular to the group velocity (make this clear?). Why is this important?

If there are boundary conditions then there is a discrete set of modes. I have tried to be careful here not to use the word ‘mode’ instead of the word ‘wave’. Above we have a local dispersion relation. Prior to that we had coupled equations that depended on depth in the atmosphere. For a whole body one would not necessarily use a plane parallel approximation but instead consider the equations as a function of radius, and if the system is rotating, as a function of latitude. Certain frequencies

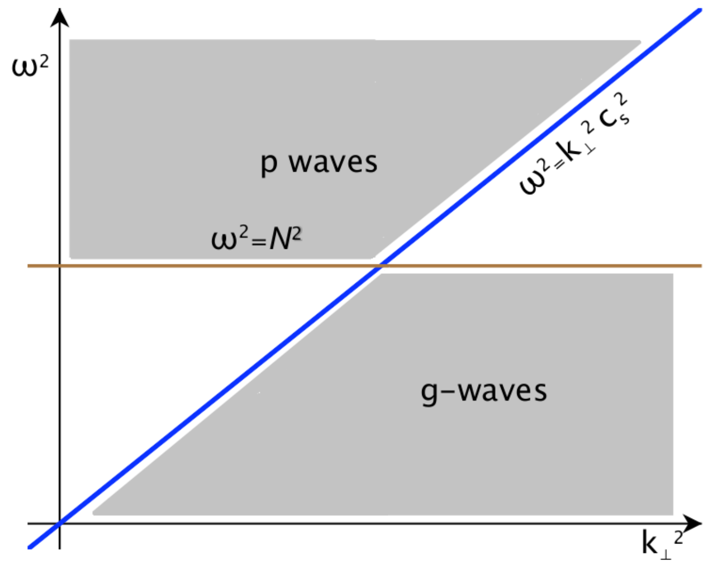

FIGURE 4. The grey regions show the allowed regions for wave propagation based on the local dispersion relation for waves in a plane parallel atmosphere. Here N is the Brunt-Väisälä frequency and c_s the sound speed. Waves propagating with $\omega > N$ are p-waves and those propagating with $\omega < N$ are g-waves.

will resonate and these can be called modes. Observations of the spectrum of modes can be used to probe the structure of the object. For example thousands of modes have been measured on the Sun. The study of these waves and modes is called *helioseismology*. The internal structures of the Sun, Earth and stars are tightly constrained by the properties of the modes of oscillation.

7. Acknowledgements

Instability at an interface following Clarke and Carswell. Thermal instability following Pringle and King. Convective instability following Clarke and Carswell. Waves in atmospheres following Pringle and King.