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CONFERENCES

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Utopian Mathematics or Wishful Teaching*

Kwadrat Rynku wpisany w koło Plant –
To geometria wesoła.
W ten sposób Kraków rozwiązał
Kwadraturę koła.
Jan Sztaudynger (1904–1970)

The Market Square inscribed
In Planty – where town and gown.
You see that squaring the circle
Is possible – in Kraków alone.
(translation by the author of the present article)

When you teach your son, you teach your son's son (the Talmud)

1. Why?
2. Problems with problems;
3. My favourite start-up;
4. The equation thinks for us;
5. Leg godt means play well;
6. Can you wonder?
7. Wild or tame? Miles to go.

The article is based on the author's short talk on the session. It is then rather concise.

*the discussion during the didactic session of the Jubilee Congress of the Polish Mathematical Society, Kraków, September 2019.

Key words: wishful teaching, creative attitude, journey across mathematics, generalization, solution, equation.

1 Why did I participate in the panel?

To answer the question, I refer to the wisdom of the Talmud, mentioned above. This, what will happen in 50 years time, is being determined right now, in our grammar schools. The societies where the teachers have low social status and are badly paid are on the best way to collapse.

What do I mean by a "wishful teaching", or by an "utopian mathematics"? Roughly speaking, a new style of teaching – different from what we have practised. I call it "wishful-teaching", because I do not think it is possible to implement it. And yet...

Regardless if good or bad, we have been teaching mathematics practically in the same way "for ages". The core curriculum changes a little, something is removed, something is added. What remains, is a well-known routine. The teacher gives a problem to the students. Students are to solve it. When they are done, the next problem comes and so forth. Of course solving problems must be preceded by some preliminaries. In the university education it is simply a lecture. Professional educations call a "lectio ex cathedra" – "acroamatic" method. The teacher speak, the pupils listen. Hardly anybody sees another way of teaching. Most likely it does not exist at all.

The reader of the paper may think that the author wants to present his revolutiary approach to teaching. This is not the case. The hell is full of teachers who tried to introduce too revolutionary ideas of. Rapid changes always strike back.

> La découverte est le privilege de l'enfant: l'enfant qui n'a pas peur encore de se tromper, d'avoir l'air idiot, de ne pas faire sérieux, de ne pas faire comme tout le monde.
>
> (Alexander Grothendieck, 1928–2010)

> Discovery is the privilege of the child: the child who has no fear to be once again wrong, of looking as an idiot, of not being serious, of not doing things like everyone else.
>
> (Alexander Grothendieck, op. cit.)

Philosophers sometimes say that all of their work consists of remarks on Plato only. It can be said (with the same degree of exaggeration) that theory of education includes comments on Socrates only. He compared the role of a teacher to the one of a midwife: she helps the child to come into the world whereas the teacher does the same with student's thoughts. Since the beginning of common education (the middle of the 18^{th} century in Prussia, then Switzerland and Austria), the generations of educationists have been trying to turn mathematics into a pleasant subject and to make children learn it eagerly. The heuristic methods have been usually suggested. The best example is vulgar fractions. The pupils break chocolate bar into pieces, cut apples, add pieces of pizza and in this way discover the rules of arithmetic of fractions. To determine the volume of a cone we first fill three identical conical containers with water. Then we pour the water out of the cones to a cylindrical barrel and conclude that the volume of the cone is one third of the cylinder with the same base and altitude. Every teacher is well aware of such tricks. Here is another brilliant example.

Let us imagine that along the circumference of a triangle ABC an old-fashion steam locomotive moves. At each station A, B, C it must be turned (in a special device) by an appropriate angle. On the closed loop it makes a full turn, equal to the sum of the angles at the vertices.

The reader is well aware of the radical changes occurred in teaching mathematics in Europe half a century ago, starting from the 1970s. The story started in about 1935 when a group of French mathematicians, under a joint name Nicolas Bourbaki, start to built mathematics in an entirely new way. Certainly, they did not claim that two plus two was not four any longer. It was about the proportions between general and specific. A simple example clears it up. Everybody knows the definitions of injective and surjective functions. We teach the students that the most important property of an injective function is that it does not glue different points together, whereas for surjective functions the characteristic property is that the image of the domain equals codomain. These properties are important in the applications to engineering, economics and the like. For the Bourbakists the crucial fact would be that injective and surjective are dual notions in view of the category theory. Injective objects have left inverse, surjective ones have the right inverse.; therefore surjective + injective = one-to-one map, i.e., an isomorphism. The two are dual, symmetric, they match as the two sides of the same coin. As you see, "general" wins over "specific". Numbers, functions or even spaces of any dimension do not count. Important are general forms and highly abstract ideas. The category theory (mentioned above and having clear philosophical reference to Aristotle's ideas of categorization), has a nickname "abstract nonsense" – it is too general to have any straightforward application. And yet it has a big impact on mathematics.

> C'est dire que s'il y a une chose en mathématique qui (depuis toujours sans doute) me fascine plus que toute autre, ce n'est ni « le nombre », ni « la grandeur », mais toujours la forme. Et parmi les mille-et-un visages que choisit la forme pour se révéler a nous, celui qui m'a fasciné plus que tout autre et continue a me fasciner, c'est la structure cachée dans les choses mathématiques.
>
> (Alexander Grothendieck)

> If there is one thing in mathematics that fascinates me more than anything else (and doubtless always has), it is neither "number" nor "size", but always form. And among the thousand-and-one faces whereby form chooses to reveal itself to us, the one that fascinates me more than any other and continues to fascinate me, is the structure hidden in mathematical things.
>
> (Alexander Grothendieck), op. cit

The Bourbaki movement developed rapidly in the years after WWII. Alexander Grothendieck (1928–2014) is considered its most prominent representative. The Bourbakists wanted to see everything in the possibly most general context. For example, addition of specific objects is not important, but it is important to study such algebraic structures, where addition is feasible. The interesting properties of a triangle are not essential – transformations and groups of transformations are essential. Educationists of 1960s really believed that teaching children (even in kindergarten) foundations of set theory will be an enormous progress. Indeed, children grasped this knowledge quite well – contrary to teachers, but it effectively dampened spontaneous joy of discovery of geometrical truths. We slowly retreated to the old methods. In spite of all TI, our teaching of mathematics at school is well anchored in the 19^{th} century. It does not mean it is bad (look at the classical music, it cannot be forgotten). Before throwing away old junk to the garbage, let us check if is indeed obsolete. Let us recall the story with old winyl gramophone disks?

Let me finally recall two obvious truths. One of them served as the motto of the paper. The other one is a revelation of our century. Our own children may work in such occcupations which do not exist yet. They have no name, the future is foggy as never before. Any yet our elementary duty is to prepare our beloved small kids to survive in the unknown. How to do it? The role of a teacher will increase, not decrease, at least in the societies, which care.

In the USA, the universities give more jobs than agriculture. In terms of the employees, the biggest workshop in Warsaw is the Technology University.

2 Problem with problems

We teach mathematics by means of problems. School knowledge of mathematics is a skill to solve problems (deutsch: Aufgabe, po polsku zadanie, po rosyjsku задача, en français un problème): to prove, to calculate, to show. Show what you can do. *Hic Rhodos, hic salta.* Only on a very high level the situation changes: to be a state professor you are expected either to solve a difficult problem (rather a series of them) or appear with some general, deep ideas.

In our times, the future is unknown as never before. Therefore a creative attitude in life would be not to deal with problems which somebody has given, but rather creating new ideas. Look at the scene from a typical classroom.

The teacher: Here is problem number one. Please solve.

A student (students): done.

The teacher: Please solve problem 2.

The students: Done.

The teacher: Be quiet, please. Listen to problem 3.

Bartek: Well, if you order...

The students: Done.

The teacher: Children, quiet. Problem 4 is the following...

Bartek (whispering to his friend): Does mathematics have to be so boring?

Among many mathematical competitions in Poland there were some based on a different approach. The competitors had to elaborate a certain topic, using relevant literature. In the pre-Internet times (young people think it was when dinosaurs existed on the Earth) they were fairly interesting, while now the participants just copy what they have found. I remember one boy wrote a paper about separable topological spaces without understanding a word. Let me recall: a topological space is said to be separable, iff it contains a countable dense subset.

> Question: Who is the most prominent sage of our times as well as the most fertile author?
> Answer: A fellow named Copy Paste.

Wacław Sierpiński (1882–1969) said that inventing problems in science is more important than solving them. As far as his main field of research is considered, i.e., number theory, it is really the case, but much of this approach remains true even outside of mathematics and even outside science.

I have practised this many times with students and teachers with ambivalent results. A short presentation, what was it about, is one of aims of the paper.

Have a look look at figure 1. Only the teacher with the mathematical preparation (and imagination) will have an idea to ask children the question: how many bales of hay are there here and how to calculate it the fastest (the answer: add $1 + 4$ and $2 + 3$; two fives make ten). How many bales will there be in the next layer? Do you think that they will also be placed in such an equilateral triangle? Let's look at figure 2. Only the teacher with the mathematical imagination will come to the idea to ask children how would they calculate the volume of wood. No, not "at home". Here, on the spot. Only the teacher with the mathematical (or may artistic) background will have the idea to discuss linear perspective. "Ann, why do people seem to be so small from the distance?"

Figure 1. Bales of hay in the north of Poland.

Figure 2. Felling in the Kampinos Forest.

Figure 3. An isosceles triangle made of coins. Invent at least 5 problems dealing with the configuration. Example. In such a triangle of size n (i.e., each side has n coins), how many points of tangency it there? Where are they (on any of the coins)?

Figure 4. I dream of ice-cream for 2.50 PLN. The coins are 0.01 PLN. Determine the heights of the pyramids which will allow me to fulfill my dream.

In the already forgotten book from 1950s, "Parkinson's Law", its author C. Northcote Parkinson, jokingly suggested testing candidates to managerial posts in public service for skills of persuasion. They were to make the board of the Baptist Congress to dance rock-and-roll in public. Such an idea came to my friend's mind once. We were then young mathematicians, on holidays in a mountain chalet in a scenic Five Lakes Valley in the Tatra mountains.

We made a kind of night seminar: who will talk about mathematics in the most interesting way late evening, in the dark. The current used to be turned off at ten. After many years I appreciate this somewhat crazy didactic idea. Would Socrates himself have to come up with it? Probably... yes.

> Do jasnych dążąc głębi, nie mógł trafić w sedno,
> Śledź pewien, obdarzony naturą wybredną.
> Gdziekolwiek się ruszył, wszystko nadaremno:
> Tu jasno, lecz płytko. Tam głębia, lecz ciemno.
>
> Tadeusz Kotarbiński, the President of Polish Academy of Sciences in 1957-1962.
>
> There was a herring with a peculiar taste,
> Who searched the ocean for a deep, well-lit space.
> "Give up, you fool", said an old wise shark.
> "The light shines only in the shallows, each deep is dark."
>
> (Translation by the author of the present article.)

3 My favorite journey across mathematics

My favourite *start-up* is "a parallelogram of midpoints." Let us connect the midpoints of the sides of a quadrangle, not necessarily convex. What we obtain, is a parallelogram. The proof that it is indeed so, is very simple – just notice that the sides of the inscribed polygon are parallel to the respective diagonals of the quadrangle.

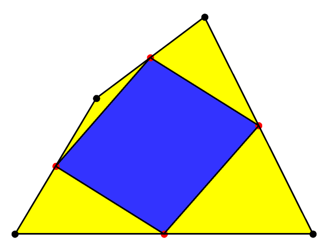

Figure 5. Parallelogram of midpoints. It is obtained by joining the midpoints of the sides of a quadrangle.

Having drawn this on the board, I pose a problem to the kids (or relevant listeners). Invent your own problem, using the picture as a background or association, or an illustration or a comment or... What comes to your mind?

The first suggestions are usually to calculate the area and the circumference. The students are eager to solve it... and they cannot. Why? Let me formulate the problem precisely.

Problem. Given a rectangle with the sides, say, a, b, c, d, find the area and circumference of the parallelogram of midpoints.

You see, of course, why the students fail. Of course you do! However, it is a stariting point for a short discussion of rigid polygons with obvious answer that the only rigid polygons are triangles.

Let us go on. Strangely enough, the students did not ask very natural questions: when the parallelogram becomes a rhombus or a rectangle. They just experienced something new. Do not solve the problem, but invent it. This is something unexpected for the pupils. When done, the students are usually pleased and proud of their "own" discovery.

Hardly ever anybody from the class suggests looking for the inverse theorem. The idea of the inverse property seems to be difficult. To my regret, my students in an engineering school do not understand it, either.

Let me recall that the inverse is usually not necessarily unique. In our case, the most natural possibility goes as follows. Assume a parallegoram P is given. Does there exist a quadrangle for which P is "the parallelogram of midpoints". Maybe there are even more than one? Find them all.

Another way to formulate the inverse was once found by a clever student. Let P be a quadrangle and let R be a parallelogram inscribed in P. Are the vertices of R the midpoints of the sides of P?

Putting geometry aside for a while, I turn to arithmetic.

In figure 6, everybody guesses easily what numbers are in the midpoints of the sides (i.e. arithmetic means of the vertex numbers). Do you (dear students) see an analogy between the geometrical problem and the arithmetical one?

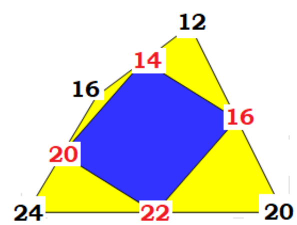

Figure 6. Geometry $\longrightarrow$ arithmetic.

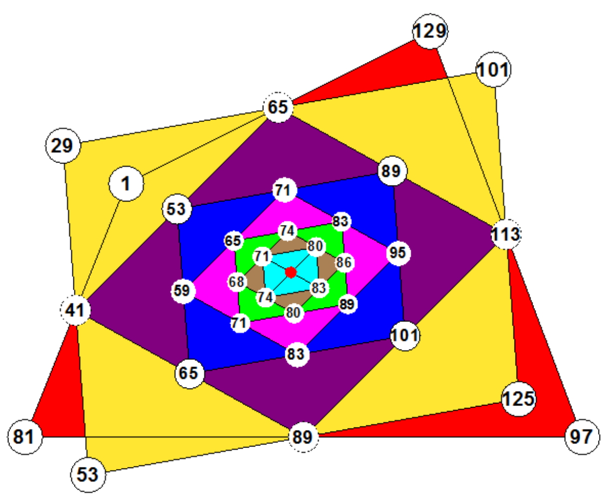

Figure 7. Geometry, arithmetic, calculus, programming. What happens at infinity?

Exercise. Invent interesting problems connected with the situation depicted in figure 7.

Examples. By slight abuse of language let arithmetization mean the operation which replaces the numbers a, b, c, d by their arithmetic means, as above. Denote the operation by A. Let *parallelogram of four numbers* a, b, c, d mean such a quadruple that $a + c = b + d$.

1. For any quadruple, its aritmetization is a parallegoram. For a given parallellogram Q, there is 1-parameter family of quadruples with $\mathrm{A}(P) = Q$, one of them being parallelogram.
2. Prove that the aritmetization decreases the standard deviation. You may use a very simple arguments or use heavy artillery, like Muirhead inequalities.
3. What would you call a deltoid of four numbers? A rectangle? A square? What is the aritmetization of a deltoid?
4. What happens at the infinity (at both sides)? In particular, is it possible to avoid both negative numbers and fractions?

I always expect that the students will suggest, sooner or later, to pass to other polygons than quadrangles. I happens rarely. Usually it is me who has to propose. If this is the case, then I start with triangles. For the reader of this article the triangle case is an easy exercise. Nevertheless, it casts light on a more general situation. Let me skip the pentagon case and pass to hexagons (figure 8). Please work out the case of octagons.

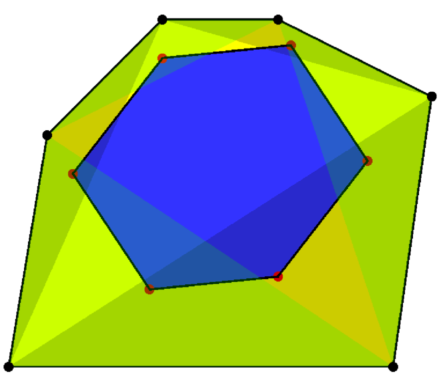

Figure 8. The vertices of the hexagons are centroids of the relevant triangles. The opposite sides of the hexagon are parallel to each other. Can you find some interesting properties of such hexagons? Do not ask, what the problem is. Set the problem yourself.

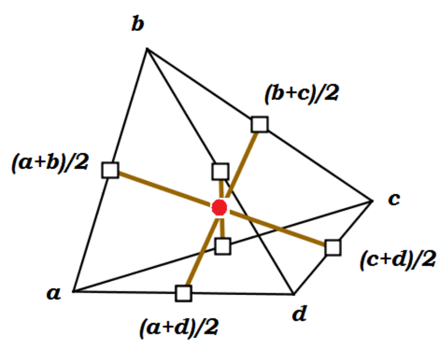

Figure 9.

In figure 9 we see (do we?) a tetrahedron. What problems come to your minds? Set them. The first property is easy to see: three bimedians of tetrahedrons meet at one point which the mass centre of the tetrahedron. *A bimedian* is a line segment joining the midpoints of opposite sides.

However, you may say that figure 9 shows a quadrilateral with lines joining the midpoints. What for a planar theorem you get?

Let us then continue. Here is a tetrahedron (represented by its net), figure 10.

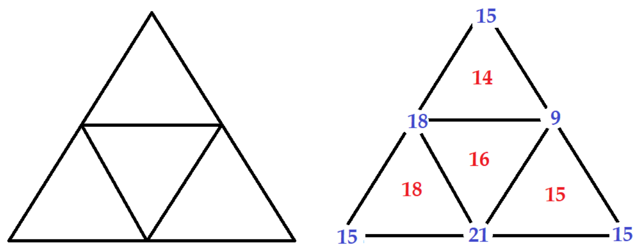

Figure 10.

Problem. On each face of a tetrahedron a number was written. Is it possible to place some numbers at vertices in such a way that the number on each side in the arithmetic mean of the adjacent vertices? Is the solution unique? Generalize for another mean (e.g, geometric) and another solid (a cube, for example).

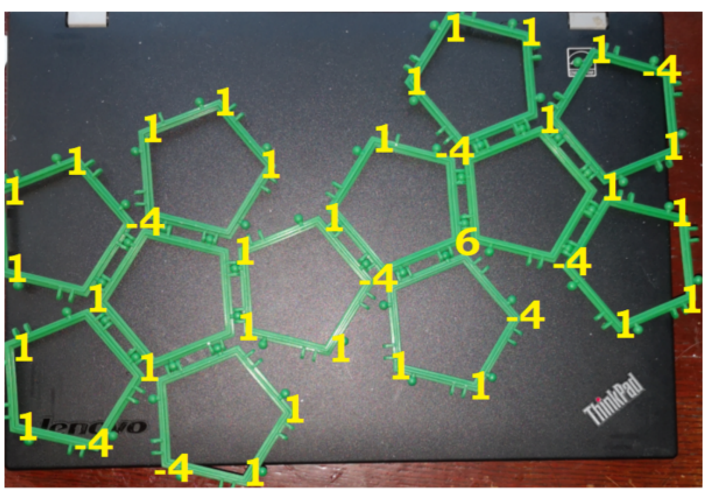

Figure 11. The sums on each face equals zero.

In figure 11 a solution of the simplified problem for the dodecahedron. Namely, the data (i.e., the numbers at the faces) are all zero.

Let me make a diggression. People in Poland are fond of collecting mushrooms in the forests. The most valuable one is called "prawdziwek" ("a true mushroom"), auf Deutsch Steinpilz, the Polish-English dictionary (the one of Kosciuszko foundation) calls it *boletus.* Its smell cannot be compared to… anything else. That is why it is an important ingredient in the traditional Christmas Eve cousine. (*as a tradition, we fast on* 24^{th} *of December. MacDonalds are empty. We also leave one place at the Christmas table for the somebody we long for*). We know that *prawdziwek* does not want to grow alone. Whenever you find it, look for another one in the neighborhood. This is, so speak, an axiom for mushrooms-hunters.

The same with mathematical theorems. Whenever you find one, look for its cousins and relatives. This is what the student in my "wishful-thinking" style of education should acquire.

Exercise. According to a legend, 9-years old Gauss managed to calculated the sum of integers from 1 to 100 in a clever way. If you do not know it, please find it. Where? Please do not ask questions which have immediate simple answers. Apply this idea to calculate the sum of consecutive odd numbers and look for further generalization.

Hence, having discussed numbering faces and vertices, let us turn to the edges. We flow into a vast, open area. Quite literarily, we flow in. The numbers on the edges can be best interpreted as flows: of electric current, of water in pipes, of railway cars on rails, of skiers on ski-lift, and finally of bits of information along a network. Let us start with electricity.

In graph theory, the resistance distance between two vertices of a simple connected graph, G, is equal to the total resistance of the system between two points on an electrical network, constructed so as to correspond to G, with each edge being replaced by a 1 ohm resistance. It is a metric on graphs. How to determine such a metric?

How to determine such a metric? See examples in figure 12 and 14.

By conductance we mean the inverse (the reciprocal) of the resistance. Then the total resistance of resistance units in series is equal to the sum of their individual resistances, whereas the conductance of resistance unit in parallel is equal to the sum of their individual conductance.

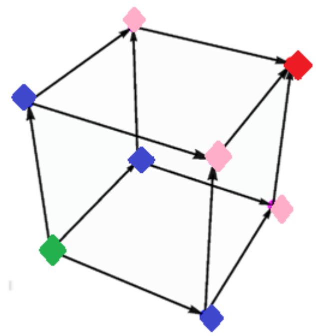

Figure 12. Hexahedron made of wire. The current flows in the left-bottom corner and flows out the opposite one.

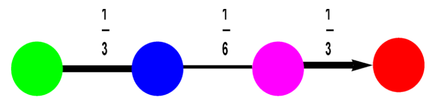

Figure 13. Figure 13 illustrates the net of figure 12. We see resistors connected in series and in parallel. The total resistance equals $\frac{5}{6}$.

Instead of electricity, let railway cars go along connections, skiers on ski lifts or bites of information on along a network. Figure 15 shows the situation where the inlet (source) is at the point named Początek and the outlet (in the theory of group actions also being called *sink*) at three point Koniec. The source and the sink are then two ends of the same edge of the polyhedron. Let us solve the problem (i.e. to find the total conductance) by means of raiway cars travelling along the net. Let me recall that conduct of each segment is 1, which means that one car per unit of time. Two cars will need two time units and so forth. I am a station manager, but also a dispatcher in the whole district. I am to send a transport from the station Początek to the station Koniec in the shortest time. Out of 24 train cars, I send fourteen straight from Początek to Koniec.

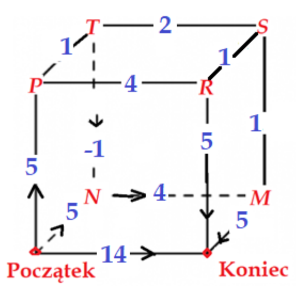

Figure 14.

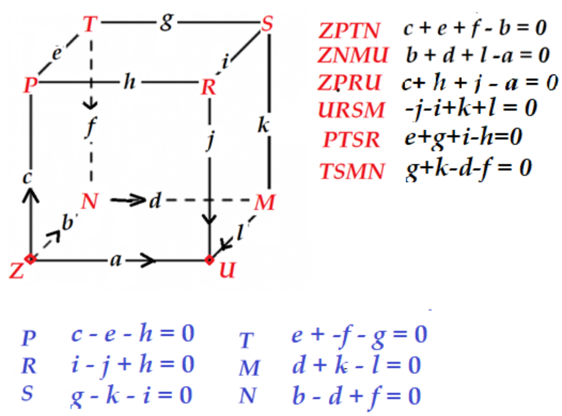

Figure 15. Kirchhoff's laws.

They need 14 time units. I send five train cars to P, five to $N\ldots$ and then along the arrows. Regardless of the route, they all meet at the final station after 14 time units. What did I do? 24 cars needed 14 time units to arrive at the destination. The conductress is then $\frac{24}{14}$, i.e., resistance is $\frac{7}{12}$.

Let me recall that the we are still within the exursion which started from the parallelogram of midpoints. Where do we meet “midpoints” yet? Clear – dealing with dual solids. The polyhedron with vertices in centres of faces of a cube is an octahedron and vice versa. The same with the dodecahedron and the isosahedron. The tetrahedron is self-dual. The discussion of the duality is interesting, but long. In the excursion the teachers plays the role of a guide (like in the mountains). He leads the tourists up to some point and shows the way up hill. “Sorry, today it is not possible go any further. Have a nice hike tomorrow with a colleague of mine”. What I mean, is for example the method to determine resistance metric by a high-level linear algebra, using a so called pseudo-inverse matrix of the Laplacian of the graph.

> Czy musisz wszystko zrozumieć? Uszanuj obecność tajemnicy.
> To, co niewiadome, nadaje rzeczom głębi.
>
> Do you have to understand everything? Respect the mystery.
> The unknown gives depth to things.
>
> Ryszard Kapuściński (1932–2007),
> a Polish essayist in his *Lapidaria, ed. Czytelnik*, 1997.

4 The equation thinks for us

> Man kann dem Gefühl nicht entrinnen, dass mathematische Formeln eine unabhängige Existenz und eine eigene Intelligenz besitzen, dass sie klüger sind als wir, ja sogar klüger als ihre Entdecker, und dass wir mehr aus ihnen bekommen, als wir ursprünglich in sie hineingesteckt haben.
>
> One cannot escape the feeling that mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers and we can obtain from there more that we have put in.
>
> Heinrich Hertz (1847–1894)

This is one of my favorite feature in teaching mathematics. Every experieneced teacher knows many examples and here I just mention the issue. Let me only point out that the ”railway” solution of the problem depicted in figure 15 is brilliant and effective. The regular way leads via Kirchhoff’s laws, or rather via their special case. In the situation as in figure 15, the sum of the incoming currents must be equal the sum of the outcomes, at each vertex of the network. The algebraic sum of the current in a loop is zero. As it results from simple linear algebra, all the solutions of this system of equations are proportional to the one illustrated in figure 15. Let's have a closer look at it. We see that $f = -1$. What does it mens? No doubt – it shows that the current flows in the other direction. The equations sends a message: *You were wrong. Correct. Please reverse the arrow in the picture.*

Example. Find the point between Earth and Moon where the gravitation force of our planet is equal to that of our romantic satellite.

The problem is very easy, but extremely nice. The mass of the Earth is 81 times the mass of the Moon. Let assume the distance between the two planets to be 1 (let's upgrade our natural satellite to be a planet in this problem) and let x be the point we look for (putting zero at the Earth). The attracting force is proportional to the inverse of the distance, hence $\frac{81}{x^2} = \frac{1}{(1-x^2)}$ and two roots occur: $x = \frac{9}{10}$, or $x = \frac{11}{10}$. There are two such points: between the two *planets*, close to the Moon... and behind it. The equation was wiser than we and recalled: hey, don't forget about the other point.

Exercise. Find more examples when the equation was wiser than a human. Did your students understand?

5 *Leg godt* means play well

A well-known company, producing toy plastic bricks, offers a basic set and exercises for first-graders. The set consists of several pieces and suggestions what to build. One of the recommended exercises is: On a given board of size 8×8 pins, build a snake of the length 14, using six pieces. The snake has to have a white head (in the set there is a block with "the eye") and it must not touch itself. The children willingly rush to do the task, most of them wants to build several ones at once. For the first-graders this is the end of the play. With older students we may go further and introduce them in problems of coding and classifying things. I used to play with first-graders of grammar school as well as with sophomores of the Technology University (as an exercise in programming).

To classify all snakes, a good method of presentation, or coding, or a shorthand is needed. We have also to precise the rules. Let us assume that a new brick is placed on the left or the right at the last one, i.e. not like in figure 16.

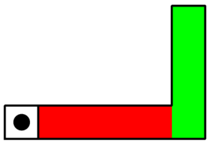

Figure 16. Wrong. Do not place blocks in this way.

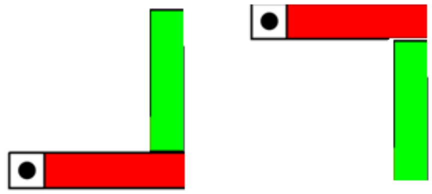

Figure 17. Correct. The rule "place the next one on the left or on the right" simplifies the classification.

Figure 18. Correct.

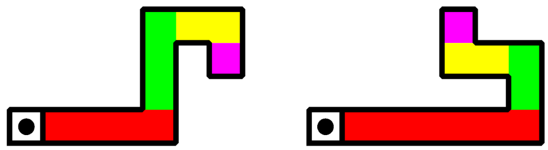

Figure 19. The snakes the length of 11 (has 11 unit squares).

In picture 19 the snake on the left is 4 L 3 R 2 R 1, on the right 6 L 2 L 2 R 1. What does it mean? We may assume that the first segment of the snake is always to the right behind the head (otherwise we may rotate it). Then 4 L 3 R 2 R 1 means: fours pins, left turn, 3 pins, right turn, 2 pins, right turn, 1 pin. Please decode the other snake.

Another way to code the snake is to use N E W S, the geographical directions, as here: E5N3E2S.

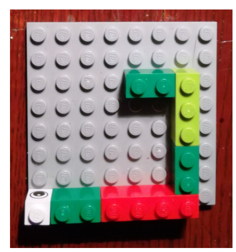

Figure 20. A friendly repail, 6 L 5 L 2.

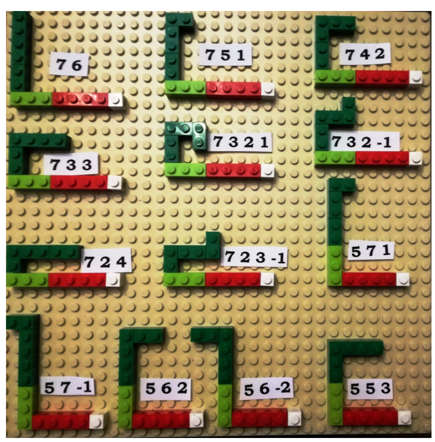

Figure 21. Excerpt from the work of the university students – first play then learn. Set up several snakes to see the rules and how to avoid, e.g. forbidden self-intersections of the reptile, like $4-3-3-3$.

In higher mathematics, we meet the notion of partition of a number. *The partition* of a natural number n is its presentation in the form of the sum of elements (usually ordered non-increasingly).
$5=5$, $5=4+1$, $5=3+2$, $5=3+1+1$, $5=2+2+1$, $5=2+1+1+1$, $5=1+1+1+1+1$.

They can be presented with the so-called Young's diagrams.
The process of constructing "snakes" helps the children in grammar school to grasp some of the key words in the mathematical education: add, count, how long, longer than. On the higher level of the play we may meet simple algorithms and rudiments of an algorithmic thinking. Young diagrams are helpful tools in the advanced linear algebra. The reader is encouraged to type in 'plethysm' e.g. at the Wolphram page.

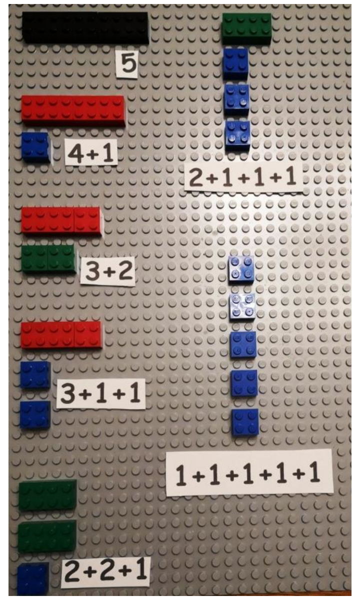

Figure 22. Young diagram illustrating partition of the number 5. Can you construct Young diagrams for the numbers 6 and 7? They require much more bricks.

6 Can you wonder?

Du mußt verstehn!
Aus Eins mach' Zehn,
Und Zwei laßgehn,
Und Drei mach' gleich,
So bist Du reich.
Verlier' die Vier!
Aus Fünf und Sechs,
So sagt die Hex',
Mach' Sieben und Acht,
So ist's vollbracht:
Und Neun ist Eins,
Und Zehn ist keins.
Das ist das Hexen-Einmal-Eins!

Johann Wolfgang Goethe,
Faust. Die Hexenküche

You shall see, then!
From one make ten!
Let two go again,
Make three even,
You're rich again.
Take away four!
From five and six,
So says the Witch,
Make seven and eight,
So it's full weight:
And nine is one,
And ten is none.
This is the Witch's one-times-one!

Translation by Morris Kline

Another motif in my approach to teaching is based on my naive belief that onyone is capable of reasoning and is sensitive to the intellectual charm of mathematical proofs.

The first idea resembles “reverse engineering”: having a machine, a tool, a construction, find out what it was used for.

Can we do in another way? For example: here is the proof. Find the theorem.

I will answer to those of the readers who think that I got crazy because of the heat or my old age in the following way. There are different mathematical minds. One can solve each problem (Wacław Sikorski), the other one is bad at problems but he understands things deeply (Roman Sikorski). Why should the other one be worse?

Here is an example. Having the proof, find the most general theorem, where the proof applies.

And here is a similar task for (theoretically) primary school. Practically for us (mathematicians). You have to be mature enough to deal with problems. Firstly: what does the word “why” mean in mathematics? Because it can be proved? Yes, but a deeper answer is: is there any more general property from which every special case is a special result. A typical problem: Why is the divisibility rule by 3 the same as by 9 and quite similar to the divisibility rule by 11?

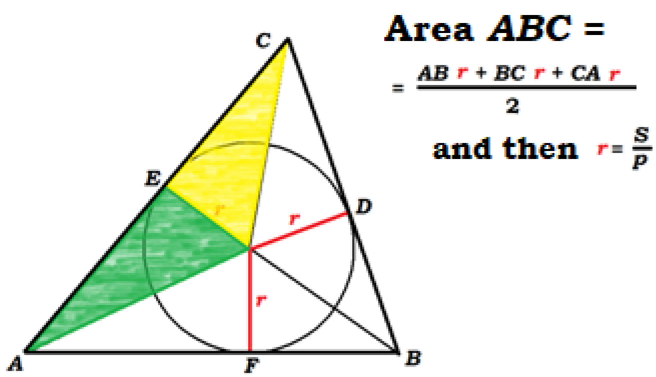

Figure 23. Look at it. Generalizing the proof, find a more general theorem.

Hence: look at the picture and deduce what theorem it illustrated. Formulate precisely the assumptions and the thesis. For example: what is interesting in figure 24 ? What theorem can you derive out of it?

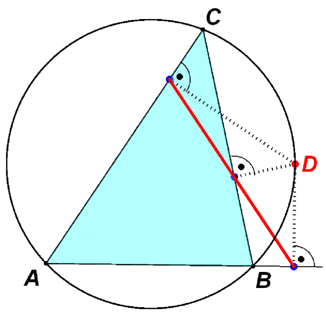

Figure 24. Simpson's theorem.

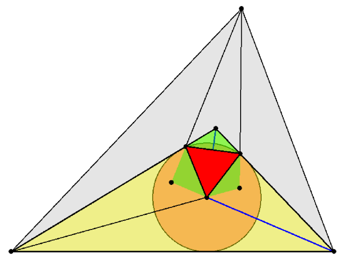

Figure 25. Morley's theorem.

Look at figure 27. What do you see? The Olympic Circles? Excellent! You must have a very good mathematical imagination. Why? Look – there are no circles at all. And yet you saw them? This is what mathematics is all about! Invent some other mathematical problems connected with the picture.

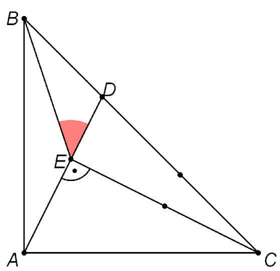

Figure 26. Given a picture, invent a problem to solve. Generalize.

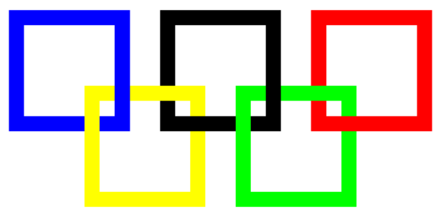

Figure 27. Do you see a mathematical problem here? What do you see in the picture?

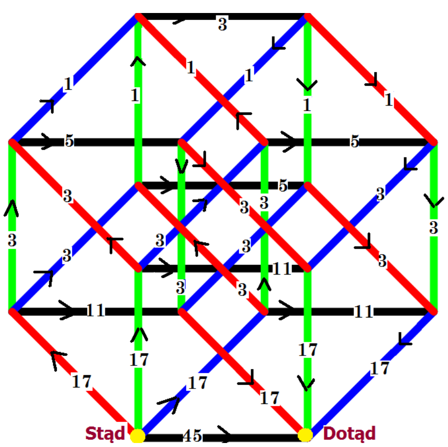

Figure 28.

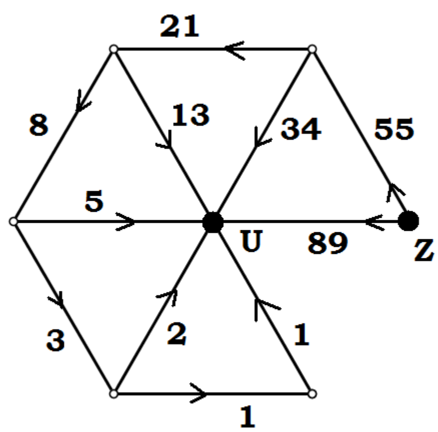

Figure 29.

I guess you recognize the tesseract, i.e, the four dimensional analogon of a cube. The carriages are transported from the initial station Stąd to the destination Dotąd. We see that 96 carriages are transported in 45 time units. Therefore the conductance of the whole network is $\frac{32}{15}$, i.e. the electric resistance is $\frac{15}{32}$. Recall that in the 3-dimensional case it was $\frac{12}{7}$, resp. $\frac{7}{12}$.

Exercise (not easy). Prove that if the inlet and outlet are in adjacent vertices of an n-dimensional hypercube, then the resistance of the network equals

$$\frac{2^n - 1}{n \cdot 2^{n-1}}.$$

In the picture you see a graph called a fan with $n = 5$ feathers. Find (guess) a general theorem for funs with n feathers. Prove it by induction.

Dziwna rzecz a prawdziwa. Oto i algebra.
Z wyjętego poezji stworzona jest żebra.
Dlatego ku jednym i tym samym stronom
Kierują swe oczy i wieszcz i astronom.

(Józef Bohdan Zaleski, 1802–1886, a romantic poet)

On the eighth day God took a rib from Poetry,
And created Algebra (what about Geometry?)
That's why astronomers share the prophets' sight.
"Aha," said God and smiled. "I knew I was right."

Translation by the author of the present article.

What interesting can you see in figure 30? A young fellow received the silver medal in the Polish edition of the EUCYS competition for an interesting answer to this question and the second prize of the popular mathematical magazine *Delta* for the best student's discovery in 2017.

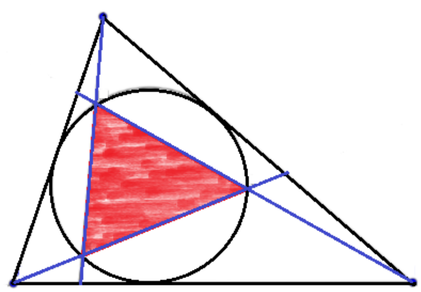

Figure 30. Kappa triangle.

Exercise. Match the theorem to picture 31. The clue: two triangles EGC and AEC have a ratio of angles $3 : 4 : 5$. The triangles ECG and AEC are *tonic triangles* respectively the scale of C major and parallel to it A minor (the details to be found elsewhere).

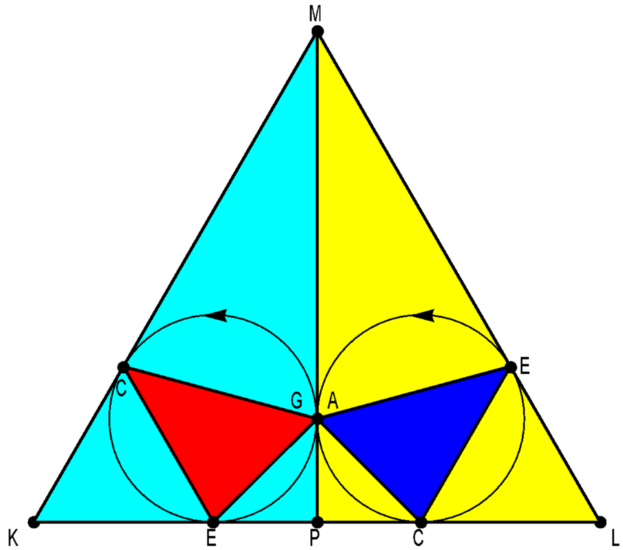

Figure 31. Tonic triangles of parallel scales C-major and A-minor are symmetric.

Problem. Read the proof in figure 32 that the sum of consecutive odd numbers is a square of a natural number. Precisely: $1 + 3 + \cdots + (2n - 1) = n^2$.

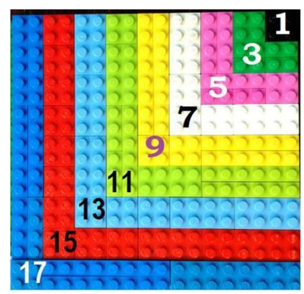

Figure 32. The sum of consecutive odd numbers is a square.

Certainly, the purists will say that this is no proof, as the mathematical proof is a sequence of sentences resulting logically from one another. Should we be purists? If we are convinced, why to bother about a formal proof? Do you remember Winnie-the-Pooh visiting Rabbit? The latter asks the Pooh: do you prefer *honey or condensed milk to your bread*? Happy Pooh answers gently: *Both (...) but do not bother about the bread, please.*

One has to be very careful with mathematical proofs. *If you carry melted lead, you'd better watch your step.* The necessity of formal proofs can be understood in the following example. Let us consider the theorem:

The number $2^n - 2$ is divisible by n iff $n > 1$ is a prime number.

Let us then imagine than the judge in the criminal court is intelligent and quite capable of mathematics. Moreover, let us imagine that the defendant can establish an alibi only if the theorem is false. Otherwise he may be sentenced and hanged. "Your Honour" says the prosecutor, no doubt that the theorem is true and the guy is guilty. We may check, Your Honour, one by one. It is true for 2, for 3, for 5, for 6, for 7, please, look at the out print from our newest computers. We calculated all numbers up to 337. If something is true in more than 300 consecutive cases, it must be true for the rest of them. “I agree” nods the judge.

According to law, the verdict belongs to the Jury, which is then closeted togeter and deliberates. The case seems to be obvious, but one member of the Jury still objects, as in the famous movie “Twelve angry men” of Sidney Lumet from 1957; Henry Fonda starring. “Let us check some five more numbers, just to be sure.... Gosh, $341 = 13 \cdot 31$ (hence is composite) whereas $\frac{2^{341}-2}{341} =$ 1313633279869679888899954724741608669335164206654835981818117894215 788100763407304286671514789484550 is clearly an integer. The Jury gives an unanonymous verdict: Not guilty. The guy is freed, as in the movie I mentioned.

Exercise. Look at figure 33. Find the general formula for the numbers going north.

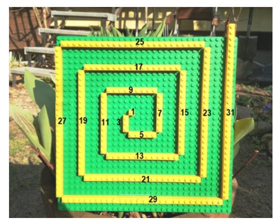

Figure 33. Archimedean spiral.

7 From fun to learning

One cannot to be a psychologist to know that the positive attitude to the performed work has an incredible influence on effects. We can work non-stop on something we like – even several hours a day. It is truism that one of the teacher’s most important virtues is a skill to get a student interested in a given issue.

A lot of, a lot of books have been written on “through fun to learning.” The case is obvious. In pursuit of new forms, I reached for blocks to build polyhedrons. The simple task: build a tetradecahedron! attracted a lot of interest at the conference of the Association for Mathematical Education. The participants showed extraordinary creativity. The elaboration of the results will take me a lot of months.

In his widely known (only in Poland, I am afraid) novel “Syzyfowe prace” by Stefan Żeromski (the action is located in the 1880s), a student named Karol Radek forgot how to derive formulas for the volume of a truncated pyramid. It is nicely presented in the film of 2000 by Paweł Komorowski, the formulas in the blackboard seem to be true; Gustaw Lutkiewicz (1924–2017) played the role of the professor, Bartek Kasprzykowski played the role of Radek.

Figure 34. Wyprowadź wzór na ostrosłup ścięty = derive the formula for the truncated pyramid.

No comments. It is happened one and a half century ago. Is anyone of todays high schools students capable to do this?

8 Stopping by Woods on a Snowy Evening

The woods are lovely, dark and deep,
But I have promises to keep,
And miles to go before I sleep,
And miles to go before I sleep.
(Robert Frost, 1922)

Children are already born who will live up to the 22^{nd} century. In few years they appear at schools. On the other hand, there are some people living who remember the 19^{th} century… from grandparents' tales.

The majority of students is forced to put smartphones aside in the lesson **during the class**. It seems to us that it is right. But maybe we should follow in accordance with the saying: "if you cannot fight them, join them." How to make a smartphone a friend not an enemy? I just give one example. I do not know if it is worth to be followed.

Quite, please. Take out sheets of paper and pencils. Switch on your smartphones. Not out. Switch on. You are to solve the problem.

On the 28^{th} of February 2020 at 11 pm the plane takes off from LAX to SYD. It flights at the speed of 500 miles an hour. When is it going to be over the Equator? Which day and at what time will it land at its destination? Find all data needed in the Internet.

The pupils (of age 12–14) needed a quarter of an hour to calculate the time of travel. They quickly identified that LAX means Los Angeles, and SYD is Sydney. They found the distance between these cities: 7511 miles. Thus, the journey lasted $\frac{7511}{500} = 15.022$ hours. They were clever enough to drop 0.0222 of an hour. Then real difficulties occurred: what does '15 hours from 23 pm'. Some did not pay attention to a trap, connecting with the leap year. Not everybody knew about "the international date line" so that I had to explain it in detail. As a byproduct the students got a lesson in geography.

Let me point out that the global network was desribed with amazing details by the founder of the science fiction literature, Polish writer Stanisław Lem (1921–2005), in his novel *Powrót gwiazd* (*Return from the stars*), 1959. The novel anticipates global Internet, electronic papers and tablet computers. A Dutch author Hugo Brandt Corsius (known as Hugo Battus, 1935–2014) made a brilliant remark on the verge of the computer era: De computer is niet de steen, maar de slijpsteen der wijzen. (The computer is not the philosopher's stone but the philosopher's whetstone.) (Battus, 1989).

I am sure that the upcoming generation of teachers will cope with smartphones, and the like, with global network, and break in the new inventions as cowboys used to break in wild horses. Good luck.

Appendix. Wishful teaching/Can you wonder? Hard mathematics out of simple toys.

Problem. Construct a tetrahedron of table tennis balls. Is it possible to colour the surface balls to obtain four isometric solids? Generalize the problem to any dimension.
Literature: any textbook dealing with spatial geometry.

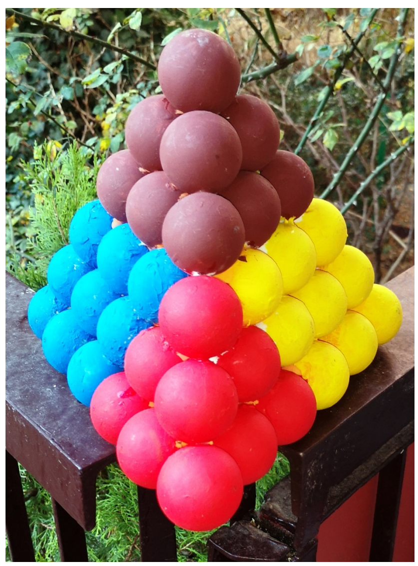

Figure 35.

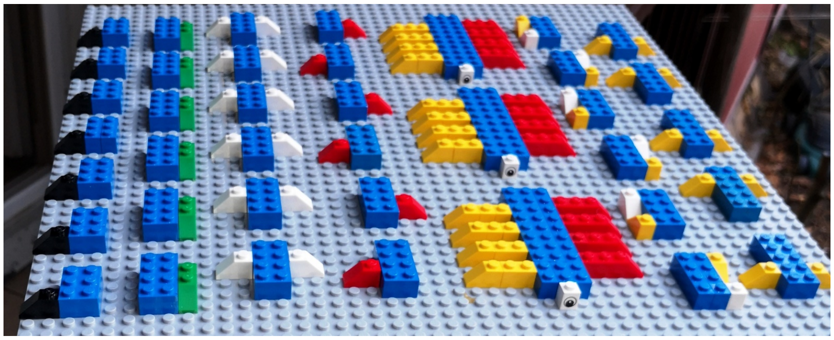

Figure 36. A glance in the group theory.

Figure 37. Gaussian primes.

Seven generators of the frieze group, illustrated by a Lego construction and water sports. From left to right: the group **p1** (canoe singles, the paddle on one side), the group **p1m1** (representing by a ferry attached to the rope), the group **p11m** (swimming butterfly stroke), the group **p11g** (kayak singles, two-hand paddle, alternating sides; also: swimming crawl), the group **p2** (represented by the most beautiful competition of all sports: eights race in rowing), the group **p2mg** (for which the author does not see a clear reference to water sports) and (utmost right) the group **p2mm** (reflecting the moves of the butterfly swimmers making a reverse turn at the wall of the pool).
Exercise. Classify the moves of the swimmer in the classic stroke. Is the backstroke equivalent to crawl? Express the move of a single-oar and of double-oar scull in terms of the above generators.
Literature: find "Friese groups" in your viewer. For the reader born in the former millennium: take e.g. Coxeter books *Geometry*.

In some books (Guy, 1981) Gausian prime numbers are depicted. Here is an appropriate Lego construction. Lets recall that, in general, an element of a PID-ring (i.e., the ring where all the ideals are generated by one element) is called a prime if it generates a prime ideal (i.e, such that if the product ab is in I, then either a or b is in I, too). The Gaussian number $a+bi$ is prime iff it cannot be expressed as a product to two Gaussian integers, none of them invertible. For example, let us show, that $4+i$ is prime. The norm $|4+I|^2 = 17$ is prime integer, hence it cannot be decomposed. On the other hand, $3+i$ is not prime, since

$$3+i = -i(1+i)(1+2i).$$

Exercise. Find a mistake in the above picture. Hint. Look at the two bottom rows. What about the numbers $7-15i$, $-7-15i$? Are they prime or composite? Is it possible that one of them is prime and the other is not?

Figure 38. A collection of polyhedra with 14 faces, constructed by the participants of the seminar of the Society for Mathematical Education in Sielpia, Poland, October 2019. (fot. Alicja Gawrońska).

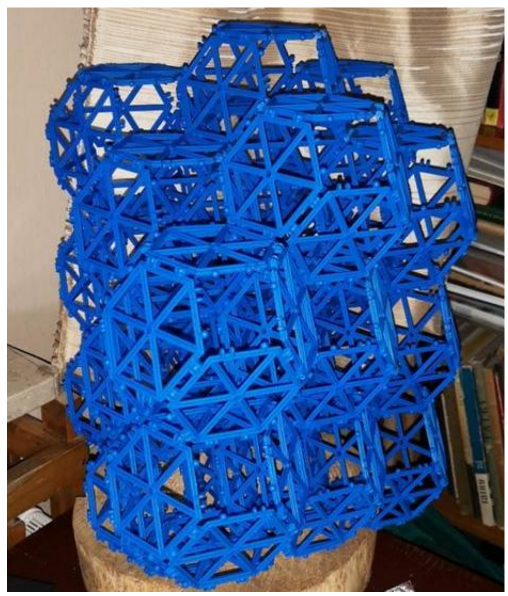

Figure 39. Filling the space with truncated octahedrons.

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Matematyka utopijna*

S t r e s z c z e n i e

Dlaczego „utopijna" i jeżeli tak, to jaki jest sens o niej pisać?

Mniej więcej chodzi o to, że „od zawsze" nauczamy matematyki w podobny sposób. Dyskusje dotyczące zmian na ogół dotyczą spraw drugorzędnych. Zmienia się trochę podstawa programowa, coś jest usuwane, coś dodawane, specjaliści toczą boje o kotangens (mówić o nim w szkole, czy nie), o zapis, o proporcje między ścisłością a kolokwializmem. Od początku oświaty ludowej (połowa XVIII wieku) pokolenia dydaktyków próbowały zrobić coś, by matematyka nie była dla uczniów przedmiotem nielubianym. Dodatkowo – truizmem jest, że mamy przygotować dzieci i młodzież do zawodów, których

*dyskusja na sesji dydaktycznej z okazji Jubileuszowego Zjazdu Polskiego Towarzystwa Matematycznego, wrzesień 2019 r.

jeszcze nie ma i nie wiadomo, jakie będą. Jak to robić? Uczyć inaczej? No, dobrze, ale jak? Konkretnie!

Szkoła jest, była i musi być lekko konserwatywna. Z kilku powodów. Musimy odróżnić chwilowe mody od stałych trendów. Wynalazki muszą się przyjąć i upowszechnić. Najważniejsze jest jednak to, że każdą zmianę trzeba wprowadzać ostrożnie i przygotowywać do niej nauczycieli; a więc osoby dorosłe, które nie uczą się już łatwo.

Moja propozycja dotyczy zmiany podejścia do rozwiązywania zadań. Matematyki uczymy na zadaniach. Szkolna (a i w dużej części uniwersytecka) znajomość matematyki to umiejętność rozwiązywania zadań.

Po pierwsze: czy można inaczej? Zauważmy, że twórcza postawa w życiu nie polega właśnie na rozwiązywaniu problemów, które ktoś nam da, tylko na realizacji własnych. Takiej postawy najtrudniej nauczyć. Autor tych słów nie twierdzi, że umie to robić, chociaż stara się w swoim nauczaniu taką postawę wytworzyć. Próbowałem wielokrotnie z uczniami i nauczycielami.

Spójrzmy na rys. 1. Tylko nauczyciel z przygotowaniem matematycznym wpadnie na to, by zadać dzieciom pytania: ile jest tu zwojów siana i jak to najłatwiej obliczyć. Ile zwojów będzie w następnej warstwie?

Lubię takie zadania, czy raczej ciągi zadań. Zaczynamy od kilku prostych kresek i łatwej obserwacji; kończymy na zadaniach ze współczesnej matematyki, którą nazywam „uniwersytecką". Trudności narastają; w pewnym momencie rozważania stają się za trudne … niemalże dla każdego. Moim ulubionym start-upem jest „równoległobok środków". Łączę środki boków czworokąta. Powstaje równoległobok. Po narysowaniu mówię: proszę ułożyć zadanie na ten temat, mogą być luźne skojarzenia. Do czego jeszcze ilustracją mógłby być ten rysunek? Co tu widzisz ciekawego? Godnego zainteresowania ze strony matematyka? W artykule pokazuję, że zaczynając od 4 klasy szkoły podstawowej („równoległobok środków") można niepostrzeżenie dojść na wyżyny współczesnej matematyki.

Drugi kierunek jest podobny. Przypomina „inżynierię odwrotną", która polega na tym, że próbujemy dociec, do czego służyła maszyna, którą mamy. Nazywam to: „odkryj twierdzenie". Podaję kilka przykładów typu: jakie twierdzenie ilustruje ten rysunek; ułóż zadanie na kanwie tego rysunku; ułóż zadanie ogólniejsze; naśladując podany dowód sformułuj i udowodnij odpowiednie twierdzenie, tylko bardziej ogólne.

Wiele inspiracji można znaleźć w propozycjach dydaktycznych sprzed 90 lat. Równie inspirujące może być zagadnienie wymuszone wszechobecną cyfryzacją: jak zrobić smartfon przyjacielem, a nie wrogiem nauczania? Jest to być może najważniejsze zadanie dydaktyczne, temat pracy dla dydaktyków co najmniej na lata, stulecia, które nam się już dobrze rozgościło.

Dlaczego mówię tu o matematyce „utopijnej”? No, bo jednak tak uczyć nie będziemy. Ale nie trzeba być psychologiem, żeby wiedzieć, że pozytywny stosunek do wykonywanej pracy ma kolosalny wpływ na efekty. Nad czymś, co nam się podoba, możemy pracować bez wytchnienia po kilkanaście godzin dziennie. Truizmem jest też, że jedną z ważniejszych zalet nauczyciela jest umiejętność zainteresowania ucznia danym zagadnieniem.

To jest drobny przyczynek do innego modelu nauczania. Daję ci skrzynkę z narzędziami. Czy umiesz je wykorzystać?

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