Proc. Inst. Wrk.shps. Int. Conf. – Essays in Geometric Group Theory No. 9, 2009, pp. 105–124.

Kac-Moody Groups as Discrete Groups

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Abstract. This survey paper presents the discrete group viewpoint on Kac-Moody groups. Over finite fields, the latter groups are finitely generated; they act on new buildings enjoying remarkable negative curvature properties. The study of these groups is shared between proving results supporting the analogy with some S-arithmetic groups, and exhibiting properties showing that they are new groups.

Keywords. Kac-Moody group, Tits system, pro-p group, lattice, arithmetic groups, algebraic group, Bruhat-Tits building, hyperbolic building, commensurator superrigidity, linearity.

Subject Classification: 22F50, 22E20, 51E24, 53C24, 22E40, 17B67

Introduction

Kac-Moody groups were initially designed to generalize algebraic groups [Tit87]. They share many properties with the latter groups, mainly of combinatorial nature. For instance, they admit BN-pair structures [Bou81, IV.2], and actually a much finer combinatorial structure – called *twin root datum* – formalizing the existence of root subgroups permuted by a (possibly infinite) Coxeter group [Tit92] (survey papers on this are for instance [Tit89] and [Rem04a]). Moreover Kac-Moody groups and some twisted versions of them are expected to be the group side in the classification of a reasonable class of buildings, the so-called Moufang 2-spherical twin buildings. Our intention is not to go into detail about this; nevertheless, we briefly recall these facts in order to emphasize that the goal of this paper is to present a significant change of viewpoint on Kac-Moody groups.

The geometric counterpart to the group combinatorics of a BN-pair is the existence of an action on a remarkable geometry: a building. To any Coxeter system (W, S) is attached a simplicial complex Σ on the maximal simplexes of which the group W acts simply transitively [Ron89, §2]. A *building* is a simplicial complex, covered by subcomplexes all isomorphic to the same Σ – called *apartments*, and satisfying remarkable incidence properties: any two *facets* (i.e. simplices) are contained in an apartment, and given any two apartments $A, A' \simeq \Sigma$, there is a simplicial isomorphism $A \simeq A'$ fixing $A \cap A'$ – see [Bro89, p. 77], and also [Ron89, §3] for the chamber system approach.

In order to define a general Kac-Moody group Λ, we need an integral matrix A satisfying properties much weaker than those defining Cartan matrices in the classical sense of complex semisimple Lie algebras [Tit87, Introduction]. We also need to choose a ground field (which will always be a finite field $\mathbf{F}_q$ in what follows). Writing down a presentation of Λ would require a lot of combinatorial and Lie algebra material we wouldn't use later – see [Tit87, Subsect. 3.6] and [Rem02b, Sect. 9] for details. It is a basic result of the theory that to a Kac-Moody group is naturally attached a pair of twin buildings (via BN-pairs) on the product of which it acts diagonally (§5.1.2 of the present paper). The standard example of such a group is $\Lambda = \mathbf{G}(\mathbf{K}[t, t^{-1}])$ for $\mathbf{G}$ a semisimple group over a field $\mathbf{K}$. Now, if we choose the ground field to be $\mathbf{F}_q$, the latter example is an arithmetic group in positive characteristic, and in this case the buildings alluded to above are the Bruhat-Tits buildings of the non Archimedean semisimple Lie groups $\mathbf{G}(\mathbf{F}_q((t)))$ and $\mathbf{G}(\mathbf{F}_q((t^{-1})))$.

The latter examples are a very special case of Kac-Moody groups, called of *affine type*, but many other cases are available. On the one hand, this suggests one to see Kac-Moody groups as generalized arithmetic groups over function fields, which leads to natural questions, e.g. asking whether some classical properties of discrete subgroups of Lie groups are relevant or true. On the other hand, it can be shown that some new buildings can be produced thanks to Kac-Moody groups, and this leads to asking whether the groups attached to exotic buildings are themselves new. Note that «new» in this context means that the buildings are neither of spherical nor of affine type, i.e. don't come from the classical Borel-Tits (resp. Bruhat-Tits) theory on algebraic groups over arbitrary (resp. local) fields.

From the point of view of metric spaces, Kac-Moody theory is an algebraic way to construct spaces with non-positively curved, often hyperbolic, distances and admitting highly transitive isometry groups. The algebraic origin of these groups enables to obtain interesting structure results for various isometry groups (discrete or much bigger). Therefore, in the case of hyperbolic Kac-Moody buildings, techniques from group combinatorics such as BN-pairs and from hyperbolic spaces «à la Gromov» can be combined. This leads us to say that the general trend to study finitely generated Kac-Moody groups is from algebraic and combinatorial methods to geometric and dynamical ones. In this paper, we explain for instance how the theory of finitely generated Kac-Moody groups, i.e. Kac-Moody groups over finite fields, naturally leads to studying uncountable totally disconnected groups generalizing semsimple groups over local fields of positive characteristic, groups which we call *topological Kac-Moody groups*. We are especially interested in proving that the groups we obtain are new in general, by proving that they cannot be linear over any field. In short, the study of finitely generated Kac-Moody groups is shared between proving classical properties by comparing them to (linear) lattices of non Archimedean Lie groups, and finally standing by a difference with the classical situation to disprove linearities.

This approach is not new, since for lattices of products of trees M. Burger and Sh. Mozes managed to prove many classical properties of lattices (among which the normal subgroup theorem) but proved that an important difference is the possibility to obtain non residually finite groups, which is impossible for finitely generated linear groups [BM97], [BM00a], [BM00b]. The main application is the construction of the first finitely generated torsion free simple groups. We note also that Y. Shalom's work [Sha00] shows that thanks to representation theory, many properties can be proved for irreducible lattices of products of general locally compact groups. Since trees are one-dimensional special cases of buildings, the previous references encourage us to think that finitely generated Kac-Moody groups will produce interesting examples of groups which are not linear but handleable via their diagonal actions on products of buildings.

This paper is organized as follows. In Sect. 5.1, it is shown why Kac-Moody groups over finite fields should be seen as generalizations of some S-arithmetic groups in positive characteristic. It is also explained how they provide new buildings and why Kac-Moody groups should be expected to be new groups. In Sect. 5.2, we are interested in a specific class of hyperbolic buildings. In this context, we can produce non-isomorphic Kac-Moody groups with the same buildings, and discrete groups which are close to Kac-Moody groups, but with several ground fields: they have strong non-linearity properties. In Sect. 5.3, we are interested in totally disconnected groups generalizing semisimple groups over local fields, arising as closures of non-discrete actions of Kac-Moody groups on buildings. We quote the existence of a nice combinatorial structure, as well as a topological simplicity result for them. In Sect. 5.4, we sketch the proof of complete non-linearity of some Kac-Moody groups. This is where we use the topological groups of the previous section. We actually mention that there are some Kac-Moody groups all of whose linear images are finite, whatever the target field. In Sect. 5.5, we ask some questions about the various groups previously defined in the paper. We conjecture the non-linearity of a wide class of finitely generated Kac-Moody groups and the abstract simplicity of topological Kac-Moody groups.

The author expresses his deep gratitude to the organizers of the conference 《Geometric Group Theory》 (Guwahati, Assam – India), organized by the Indian Institute of Technology Guwahati and the Indian Statistical Institute, and supported by the National Board of Higher Mathematics. Meenaxi Bhattacharjee managed to make a tremendous human and scientific event out of a mathematical conference.

5.1. Generalized arithmetic groups acting on new buildings

This section is mainly dedicated to quoting results supporting the analogy between Kac-Moody groups over finite fields and $\{0; \infty\}$-arithmetic groups over function fields [Rem02a, Sect. 2–3]. The arguments are: the existence of a discrete diagonal action on a product of two buildings (5.1.1), cohomological finiteness properties (5.1.2) and continuous cohomology vanishings (5.1.3). In the last two subsections (5.1.4 and 5.1.5), we provide arguments showing that Kac-Moody theory does provide interesting new group-theoretic/geometric situations.

5.1.1. *Discrete actions on buildings and finite covolume*

A Kac-Moody group is generated by those of its root groups which are indexed by simple roots and their opposites (in finite number), and by a suitable maximal torus normalizing them [Tit87, §3.6]. Since over $\mathbf{F}_q$ all these groups are finite, we obtain:

Fact 5.1.1. Any Kac-Moody group Λ over any finite field is finitely generated.

From now on, Λ is a Kac-Moody group defined over $\mathbf{F}_q$. Recall that a group action on a building is *strongly transitive* if it is transitive on the inclusions of a chamber in an apartment. Combining [Tit87, Subsect. 5.8, Proposition 4] and [Ron89, Theorem 5.2], we obtain:

Fact 5.1.2. To Λ are attached two isomorphic, locally finite buildings $X_\pm$, each of them admitting a strongly transitive Λ-action.

This fact is fundamental to understanding Kac-Moody groups: the geometry of the buildings is the basic substitute for a natural structure on Λ arising from infinite-dimensional algebraic geometry. In the specific case of an S-arithmetic group $\Lambda = \mathbf{G}(\mathbf{F}_q[t, t^{-1}])$ (for $S = \{0; \infty\}$), the buildings $X_\pm$ are the Bruhat-Tits buildings of $\mathbf{G}(\mathbf{F}_q((t)))$ and $\mathbf{G}(\mathbf{F}_q((t^{-1})))$. In general, it is still true that for an arbitrary Kac-Moody group, the diagonal action on the product of buildings $X_- \times X_+$ is discrete [Rem99]. Moreover the action has a nice convex fundamental domain contained in a single apartment and defined as an intersection of roots (seen as half-apartments) [Abr97, §3, Corollary 1]. The next step in the analogy with arithmetic groups consists in asking whether the finitely generated group Λ is a *lattice* of $X \times X_-$, meaning that the locally compact group $\mathrm{Aut}(X_-) \times \mathrm{Aut}(X_+)$ moded out by the image of Λ carries a finite invariant measure [Mar91, 0.40]. This is the main result of [CG03] and [Rem99]:

Theorem 5.1.3. *Assume the Weyl group W of Λ is infinite and denote by $W(t) = \sum_{w \in W} t^{\ell(w)}$ its growth series. Assume that $W\left(\frac{1}{q}\right) < \infty$. Then Λ is a lattice of $X \times X_-$ for its diagonal action, and for any point $x_- \in X_-$ the stabilizer $\Lambda(x_-)$ is a lattice of X. These lattices are never cocompact.*

In the arithmetic case of $\mathbf{G}(\mathbf{F}_q[t, t^{-1}])$, the Weyl group has polynomial growth (it is virtually abelian because it is an affine reflection group), so the condition $W\left(\frac{1}{q}\right) < \infty$ is empty and $\mathbf{G}(\mathbf{F}_q[t, t^{-1}])$ is always a lattice of $\mathbf{G}(\mathbf{F}_q((t))) \times \mathbf{G}(\mathbf{F}_q((t^{-1})))$, whatever the value of the prime power q. This is a particular case of a well-known result in reduction theory in positive characteristic [Beh69], [Har69].

5.1.2. *Finiteness properties*

Cohomological finiteness is a very hard problem for arithmetic groups in positive characteristic [Beh03], which makes a sharp difference with the number field case. Nevertheless, it is natural to expect that some results, similar to those which are known for arithmetic groups in the function field case, should hold for Kac-Moody groups. This is indeed the case, up to taking into account more carefully the sub-matrices in the generalized Cartan matrix defining the group – this is the theme of P. Abramenko's book [Abr97]. The following theorem sums up Theorems 1 and 2 from [Abr03] in a slightly different language. For instance, we introduce the parabolic subgroups (defined by means of BN-pairs in [loc. cit.]) via the group action on the buildings: such a subgroup is a facet fixator. Note also that the below quoted results are still valid in the more general context of abstract *twin BN-pairs* [Abr97, §1].

Theorem 5.1.4. *Let Λ be a Kac-Moody group over $\mathbf{F}_q$, and let Γ be a facet fixator. Let Σ be the Coxeter complex of the Weyl group W, i.e. the model for any apartment in the building $X_\pm$ of Λ. Let R be a chamber of Σ and let Π be the set of reflections in the codimension one faces of R.*

(i) *If any two reflections of Π generate a finite group and if $q > 3$, then Γ is finitely generated; but it is not of cohomological type FP_2, in particular not finitely presented, whenever some set of three reflections in Π generates an infinite subgroup of W.*
(ii) *If any three reflections of Π generate a finite group and if $q > 6$, then Γ is finitely presentable; but it is not of cohomological type FP_3 whenever some set of four reflections in Π generates an infinite subgroup of W.*

When any two reflections of Π generate a finite group, the Kac-Moody group is called 2-*spherical*. Along with the Moufang property [Ron89, 6.4] and twinnings [Abr97, Definition 3], this notion plays a major role in the classification of buildings with infinite Weyl groups. In [Abr97] some further results are available; they deal with the higher cohomological finiteness properties FP_n and F_n [Bro87, VIII.5]. According to §5.1.1, finite presentability is the first finiteness property to be considered for the group Λ itself. A result due to P. Abramenko and B. Mühlherr [AM97] is available:

Theorem 5.1.5. *With the same notation as above, if any two reflections of Π generate a finite group and if $q > 3$, then Λ is finitely presentable.*

For instance, finite presentability holds for Kac-Moody groups whose buildings have chambers isomorphic to hyperbolic regular triangles of angle $\frac{\pi}{4}$ and $q > 3$, but never holds for those whose buildings are covered by chambers isomorphic to a regular right-angled r-gon, $r \geq 5$ (we will see in 5.1.4 that such Kac-Moody groups exist in both cases).

5.1.3. *Continuous cohomology and Kazhdan's property (T)*

By results of J. Dymara and T. Januszkiewicz, being 2-spherical also implies useful continuous cohomology vanishing for automorphism groups of buildings. The result below is a special case of [DJ02, Theorem E].

Theorem 5.1.6. *Let Λ be a Kac-Moody group over $\mathbf{F}_q$, defined by a generalized Cartan matrix A of size $n \times n$. Let $m < n$ be an integer such that all the principal submatrices of size $m \times m$ of A are Cartan matrices (i.e. of finite type). Then for $1 \leq k \leq m-1$ and $q \gg 1$, the continuous cohomology groups $\mathrm{H}^k_{\mathrm{ct}}(\mathrm{Aut}(X_\pm), \rho)$ vanish for any unitary representation ρ.*

The first cohomology case is extremely useful since vanishing of $\mathrm{H}^1_{\mathrm{ct}}(G, \rho)$ for any unitary representation ρ is equivalent to property (T) [dlHV89, Chap. 4]. Therefore, when Λ is 2-spherical, Theorem 5.1.6 implies property (T) for the full automorphism groups $\mathrm{Aut}(X_\pm)$ with $q \gg 1$, hence for their product, and finally for any lattice in this product, by S.P. Wang's Theorem [Mar91, III, Theorem 2.12]. The above result says in particular that many Kac-Moody groups have property (T), a fundamental property for lattices of higher-rank simple algebraic groups [Mar91, III].

5.1.4. *Hyperbolic examples*

The existence of buildings with prescribed shapes of apartments and links around vertices is well-known in many cases [Ron86], [Bou97]. Some examples lead to interesting full automorphism groups and lattices [BP00], and some other examples have surprisingly few automorphisms, though they are quite familiar since they are Euclidean and tiled by regular triangles [Bar00]. The result below [Rem02c, Proposition 2.3] shows that some of these buildings are both relevant to Kac-Moody theory and to (generalized) hyperbolic geometry [BH99, III.H]. This enables to mix arguments of algebraic and geometric nature in the study of the corresponding Kac-Moody groups.

Proposition 5.1.7. *Let P be a polyhedron in the hyperbolic space $\mathbb{H}^n$, with dihedral angles equal to 0 or to $\frac{\pi}{m}$ with $m = 2, 3, 4$ or 6. For any prime power q, there is a Kac-Moody group whose building has constant thickness equal to $q+1$ and where the apartments are all tilings of $\mathbb{H}^n$ by P; this building is a complete geodesic $CAT(-1)$-metric space.*

Recall that the *thickness* at a codimension 1 cell C of a building is the number of maximal cells containing C. Note that according to G. Moussong [Mou88], any Coxeter group acts discretely and cocompactly on a $CAT(0)$-space, which provides a good metric realization of the associated Coxeter complex [Ron89, Chap. 2] (in which only spherical facets are represented). Moreover, the metric space under consideration is $CAT(-1)$ if and only if the Coxeter group is Gromov-hyperbolic (which is a weaker property in general [GdlH90, Chap. 3]). Using G. Moussong's non-positively curved complex as a model for apartments, M. Davis [Dav97] proved the existence of a $CAT(0)$-metric realization for any building, which is $CAT(-1)$ whenever the Weyl group of the building is Gromov-hyperbolic. In the latter case, the metric complex is usually not as nice as a hyperbolic tiling as in Proposition 5.1.7.

5.1.5. *Easy non-linearities*

On the one hand, in 5.1.1-5.1.3 we have quoted arguments supporting the analogy between arithmetic groups over function fields and Kac-Moody groups over finite fields. On the other hand, 5.1.4 shows that the geometries are certainly new since hyperbolic buildings can be obtained. Still, it is not proved at this stage that the groups are not well-known groups, only with new actions: what about arguments showing that the groups are new? This question leads to rigidity problems, i.e. proving that groups acting naturally on some geometries cannot act via a big quotient on another geometry. A way to attack the problem is to prove that the groups are not linear over any field. Easy non-linearities are summed up in the following [Rem02a, Theorem 4.6]:

Theorem 5.1.8. *Let Λ be a Kac-Moody group over $\mathbf{F}_q$ with infinite Weyl group, and let Γ be a facet fixator in Λ. Then Γ always contains an infinite group of exponent p. Therefore Γ cannot be linear over any field of characteristic $\neq p$.*

Exhibiting an infinite group of exponent p is made possible thanks to arguments on Kac-Moody root systems and pairs of parallel walls in infinite Coxeter complexes. Then, elementary Zariski closure and algebraic group arguments imply that an infinite group of exponent p cannot be linear over any field of characteristic different from p [Mar91, Lemma VIII.3.7].

5.2. Generalized Kac-Moody groups. Non-isomorphic groups with the same building

In this section, we concentrate on buildings whose apartments are tilings of the hyperbolic plane. Since we are interested in understanding new groups and geometries, it is natural to consider the case of Fuchsian Weyl groups, corresponding to the simplest exotic Kac-Moody groups. The results are mainly of algebraic and combinatorial flavour; they emphasize the role of conditions previously introduced to classify buildings. Abstract isomorphisms between Kac-Moody groups with the same hyperbolic building can be naturally factorized (5.2.1). This is an analogy with the spherical building case, but a difference comes when exhibiting some non-isomorphic groups with the same building (5.2.2). Moreover the local structure of right-angled Fuchsian buildings is particularly simple: this enables one to construct groups which are very close to Kac-Moody groups, but so to speak defined over several ground fields (5.2.3). Back to geometric group theory, this provides lattices of buildings of arbitrarily large rank satisfying a strong non-linearity property (5.2.4).

5.2.1. *Factorization of abstract automorphisms*

We say that a building is *Fuchsian* if its Weyl group is the reflection group of a tiling of the hyperbolic plane (in which case the latter tiling is a pleasant model for apartments). According to Poincaré's Theorem [Mas88, 4.H], Fuchsian tilings are nice metric realizations of Coxeter complexes (the most familiar ones after Euclidean tilings). Elementary facts from hyperbolic geometry enable one to prove the following [Rem02c, Theorem 3.1]:

Theorem 5.2.1. *Let G and G' be two Kac-Moody groups defined over the same finite field $\mathbf{F}_q$ of cardinality $q \geq 4$. Assume that the associated buildings are all isomorphic ($\star$) either to the same locally finite tree, ($\star\star$) or to the same Fuchsian building with regular chambers. Then, up to conjugacy in G, any abstract isomorphism from G to G' is the composition of a permutation of the simple roots and possibly a global opposition of the sign of all roots.*

This factorization result is close to R. Steinberg's classical result on finite groups of Lie type, saying that an abstract automorphism of such a group is the product of a Dynkin diagram automorphism, a ground field automorphism and an inner automorphism [Car72, Theorem 12.5.1]. Stronger factorizations of isomorphisms have recently been obtained by P.-E. Caprace and B. Mühlherr [CM05a], [CM05b].

5.2.2. *Several isomorphism classes*

Let R be a right-angled r-gon of the hyperbolic plane $\mathbb{H}^2$. For any integer $q \geq 2$ there is a unique building $I_{r,q+1}$ whose apartments are Poincaré tilings of $\mathbb{H}^2$ by R, and such that the link at each vertex is the complete bipartite graph of parameters $(q+1, q+1)$ [Bou97, 2.2.1]. Here the *link* of a vertex is a small ball around it, seen as a graph. We call such a building the *right-angled Fuchsian building* of parameters r and $q+1$. These buildings are interesting because they locally look like products of trees, making them simple combinatorially, but globally their Weyl group is irreducible and Fuchsian, leading to remarkable rigidity properties [Bou97], [BP00]. By uniqueness and Proposition 5.1.7, for each $r \geq 5$ the building $I_{r,q+1}$ comes from a Kac-Moody group whenever q is a prime power. Using Theorem 5.2.1, it can be proved that there are abstractly non-isomorphic Kac-Moody groups with the same building [Rem02c, §4, Proposition]:

Corollary 5.2.2. *Let $q \geq 4$ be a prime power and let $r \geq 5$ be an integer. Then there are several abstract isomorphism classes of Kac-Moody groups whose associated buildings are the same $I_{r,q+1}$.*

This result is in contrast with the spherical case, where according to J. Tits' classification, a spherical building of rank ≥ 3 uniquely determines a field and an algebraic group over this field [Tit74]. Here the rank r of the building is an arbitrary integer ≥ 5. This is an argument explaining why in the classification of Moufang twinnings [MR95], [Mue99], the buildings must be assumed to be 2-spherical: Kac-Moody groups with $I_{r,q+1}$ buildings obviously do not satisfy this property.

5.2.3. *Mixing ground fields*

There is an even stronger argument showing that being 2-spherical is a necessary condition for a Moufang twin building to be part of a reasonable classification (where the group side would be given by Kac-Moody groups and their twisted versions). Indeed, some generalizations of Kac-Moody groups with several ground fields can be constructed, provided the buildings have apartments tiled by regular hyperbolic right-angled r-gons. Here《generalization》means that the groups satisfy the same combinatorial axioms as Kac-Moody groups (namely, those of twin root data [Tit92]). We have [RR06, Theorem 3.E]:

Theorem 5.2.3. *A right-angled (Fuchsian) building belongs to a Moufang twinning whenever its thicknesses at panels are cardinalities of projective lines.*

The above quoted reference actually covers more general buildings, namely those buildings whose Weyl group is a right-angled Coxeter group (not necessarily Fuchsian). In the case of trees, a more theoretical construction is sketched by J. Tits in his Notes de Cours au Collège de France [Tit90, §9]. Still, the rank of the infinite dihedral Weyl group there is always equal to 2, whereas in the above result it can be any integer ≥ 5.

5.2.4. *Stronger non-linearities*

Besides its combinatorial interest, the above construction can be seen as a way to produce lattices of hyperbolic buildings with remarkable non-linearity properties. Recall that according to Theorem 5.1.8 the characteristic p of the ground field of a Kac-Moody group prevents this group from being linear over any field of characteristic $\neq p$. Therefore, mixing fields of different characteristics in Theorem 5.2.3 should enable one to produce groups which are not linear over any field. This is indeed the case [RR06, Theorem 4.A]:

Theorem 5.2.4. *Let $r \geq 5$ be an integer and $\{\mathbf{K}_i\}_{i \in \mathbf{Z}/r}$ be a family of fields, among which are two fields with different positive characteristics. Let Λ be a group defined as in Theorem 5.2.3 from these fields and let Γ be a chamber fixator. Then, any group homomorphism $\rho : \Gamma \to \prod_{\alpha \in A} \mathbf{G}_\alpha(\mathbf{F}_\alpha)$ has infinite kernel, whenever the index set A is finite and $\mathbf{G}_\alpha$ is a linear algebraic group defined over the field $\mathbf{F}_\alpha$ for each $\alpha \in A$.*

Note that the result is stronger than plain non-linearity since mixing ground fields is also allowed at the (linear) right hand-side of the representation. Moreover the kernel is not only non-trivial, but always infinite. From the purely Kac-Moody viewpoint, this result says that the truly difficult case of non-linearity is when the characteristic of the target algebraic group is the characteristic of the ground field of the Kac-Moody group. This case is reviewed in Sect. 5.4.

5.3. Generalized algebraic groups over local fields

A Kac-Moody group Λ acts discretely on the product of its buildings, but its action on a single factor is no longer discrete. Therefore it makes sense to take the closure $\bar{\Lambda} < \mathrm{Aut}(X_\pm)$ of the image of a Kac-Moody group in such a non-discrete action. The result is called a *topological Kac-Moody group* (the kernel of the Λ-action on $X_\pm$ is the finite center of Λ). In the classical case $\Lambda = \mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])$, $X_\pm$ is the Bruhat-Tits building of $\mathrm{SL}_n(\mathbf{F}_q((t)))$ or $\mathrm{SL}_n(\mathbf{F}_q((t^{-1})))$, respectively. If $\mu_n(\mathbf{F}_q)$ denotes the n-th roots of unity in $\mathbf{F}_q$, the image $\Lambda/Z(\Lambda)$ of $\mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])$ under the action on $X_\pm$ is $\mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])/\mu_n(\mathbf{F}_q)$ and the completions $\bar{\Lambda}$ are $\mathrm{PSL}_n(\mathbf{F}_q((t)))$ and $\mathrm{PSL}_n(\mathbf{F}_q((t^{-1})))$, respectively. In fact, there are many arguments to compare a topological Kac-Moody group with a semisimple group over a local field: existence of a combinatorial structure refining Tits systems (5.3.1), group-theoretic characterization of chamber-fixators in terms of pro-p subgroups (5.3.2), topological simplicity

(5.3.3). Still, as for discrete groups we have phenomena suggesting that some so-obtained totally disconnected groups are new (5.3.4). We finally quote a result exhibiting the coexistence of non-linear non-uniform lattices and uniform lattices embedding convex-cocompactly in hyperbolic spaces (5.3.5).

5.3.1. *Refined Tits systems*

The structure of a *refined Tits system* is due to V. Kac and D. Peterson [KP85]; it is a generalization of a split BN-pair, a notion from the theory of finite groups of Lie type. The difference with a plain Tits system is the formalization in the axioms of the existence of an abstract unipotent subgroup in the Borel subgroup. We have [RR06, Theorem 1.C]:

Theorem 5.3.1. *Let Λ be a Kac-Moody group over the finite field $\mathbf{F}_q$ of characteristic p. The associated topological Kac-Moody group $\bar{\Lambda}$ admits a refined Tits system, and the Tits system gives rise to a building in which any facet-fixator is, up to finite index, a pro-p group.*

It is well-known that a group acting strongly transitively on a building admits a natural Tits system [Ron89, Theorem 5.2], so the point in the first assertion lies in the difference between a Tits system and a refined Tits system. Moreover standard properties of double cosets in Bruhat decompositions imply that topological Kac-Moody group are compactly generated [Rem04b, Corollary 1.B.1]. It is explained in [Rem04b, 1.B.1] why refined Tits systems for $\bar{\Lambda}$ imply the existence of a lot of torsion in facet fixators – called *parahoric subgroups*. This explains why the analogy with semisimple groups over local fields is relevant only when the local field has the same characteristic p as the finite ground field of Λ (parahoric subgroups of semisimple groups over finite extensions of $\mathbf{Q}_l$ are virtually torsion free). In analogy with the classical case of Bruhat-Tits theory [BT72], [BT84], we call *Iwahori subgroup* a chamber fixator in $\bar{\Lambda}$.

5.3.2. *Iwahori subgroups*

Back to the case $\Lambda = \mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])$, let v be a vertex in the Bruhat-Tits building of $\mathrm{SL}_n(\mathbf{F}_q((t)))$. Then its fixator is isomorphic to $\mathrm{SL}_n(\mathbf{F}_q[[t]])$, and for some chamber containing v the corresponding Iwahori subgroup is the group of matrices in $\mathrm{SL}_n(\mathbf{F}_q[[t]])$ reducing to upper triangular matrices modulo t. Moreover the first congruence subgroup of $\mathrm{SL}_n(\mathbf{F}_q[[t]])$, i.e. the matrices reducing to the identity modulo t, is a maximal pro-p subgroup whose normalizer is the above Iwahori subgroup. In the general Kac-Moody setting, the result below [Rem04b, Proposition 1.B.2] is a generalization of [PR94, Theorem 3.10].

Proposition 5.3.2. *The Iwahori subgroups are group-theoretically characterized as the normalizers of the pro-p Sylow subgroups.*

In general, the analogue of the first congruence subgroup of a vertex fixator must be defined as the (pointwise) fixator of the star around the vertex under consideration.

5.3.3. *Topological simplicity*

In view of the simplicity of adjoint algebraic simple groups over large enough fields, the theorem below [Rem04b, Theorem 2.A.1] is not surprising:

Theorem 5.3.3. *For $q \geq 4$, a topological Kac-Moody group over a finite field is the direct product of finitely many topologically simple groups, with one factor for each connected component of its Dynkin diagram.*

This result is extremely useful when extending abstract representations of finitely generated Kac-Moody groups to continuous representations of topological Kac-Moody groups (Theorem 5.4.3). The arguments of the proof are basically: a normal subgroup in an irreducible Tits system either is chamber-transitive or acts trivially on the associated building [Bou81, IV.2], Iwahori subgroups are virtually pro-p, and a generating set for a Kac-Moody group can be chosen in a finite collection of finite subgroups of Lie type (which are perfect whenever $q \geq 4$). It is reasonable to think that topological Kac-Moody groups are in fact abstractly simple (Question 5.5.7).

5.3.4. *Non-homogeneous Furstenberg boundaries*

When the building $X_\pm$ has hyperbolic apartments, the existence of many hyperbolic translations leads to an interesting connection with topological dynamics. Let Y be a compact metrizable space and let us denote by $\mathscr{M}^1(Y)$ the space of probability measures on Y. Recall that $\mathscr{M}^1(Y)$ is compact and metrizable for the weak-$*$ topology. If Y admits a continuous action by a topological group G, we say that Y is a *Furstenberg boundary* for G if it is G*-minimal* and G*-strongly proximal* [Mar91, VI.1.5]. The first condition says that any G-orbit in Y is dense and the second one says that any G-orbit closure in $\mathscr{M}^1(Y)$ contains a Dirac mass. The arguments of [Rem04b, Lemma 4.B.1] give:

Lemma 5.3.4. *Let Λ be a Kac-Moody group whose buildings have apartments isomorphic to a hyperbolic tiling. Then the asymptotic boundary $\partial_\infty X_\pm$ is a Furstenberg boundary for any closed automorphism group of $X_\pm$ containing $\bar{\Lambda}$.*

This existence of non-homogeneous boundaries is new with respect to semisimple Lie groups (archimedean or not), since in the latter case any Furstenberg boundary is equivariantly isomorphic to a flag variety of the group [GJT98, 9.37].

5.3.5. *Coexistence of two kinds of lattices*

We close this section by stating a result which says that some topological Kac-Moody groups contain lattices of surprisingly different nature [RR06, Proposition 4.B].

Proposition 5.3.5. *There exist topological Kac-Moody groups $\bar{\Lambda}$ over $\mathbf{F}_q$ which contain both non-uniform lattices which cannot be linear over any field of characteristic prime to q, and uniform lattices which have convex-cocompact embeddings into real hyperbolic spaces. The limit sets of the latter embeddings often have Hausdorff dimension > 2.*

It follows from [BM96, Corollary 0.5] that if a Kac-Moody group Λ is S-arithmetic and such that $\bar{\Lambda}$ is a higher-rank simple Lie group, then any lattice of $\bar{\Lambda}$ fixes a point in each of its actions on proper $CAT(-1)$-spaces. Therefore the above phenomenon is excluded in the classical algebraic situation, unless the building of $\bar{\Lambda}$ is a tree, meaning that the Weyl group Λ is infinite dihedral (of rank 2 as a Coxeter group). The rank r of a Fuchsian building $I_{r,q+1}$ may be chosen arbitrarily large ≥ 5.

5.4. Non-linearity in equal characteristics

In view of the easy non-linearity results (Theorem 5.1.8), the remaining linearity to disprove is for Kac-Moody groups over finite fields of characteristic p into linear groups of characteristic p. But some Kac-Moody S-arithmetic groups such as $\mathrm{SL}_n(\mathbf{F}_q[t, t^{-1}])$ are linear in equal characteristic. Therefore the question is rather, for any finite field of characteristic p, to find examples of Kac-Moody groups that cannot be linear over any field of characteristic p either. The main idea is to use some superrigidity property (5.4.1) to show that the existence of a faithful abstract homomorphism from a finitely generated Kac-Moody group into an algebraic group implies the existence of an embedding of a topological Kac-Moody group into a non-Archimedean simple group (5.4.2). The existence of such an embedding is expected to be simpler to disprove because it comes with an embedding of the vertices of the (possibly exotic) Kac-Moody building to a Euclidean one, which enables one to take advantage of the incompatibility between hyperbolic and Euclidean geometries. For some Kac-Moody groups with Fuchsian buildings this is indeed the case (5.4.3). The same circle of ideas allows us to show a complementary result: a non-faithful representation from a finitely generated Kac-Moody group most of the time has a virtually solvable image (5.4.4). Combining the latter results leads to groups all of whose linear images are finite (5.4.5).

5.4.1. *Commensurator superrigidity*

Let us first recall that the *commensurator* of a group inclusion $\Delta < G$ is the group:

$$\mathrm{Comm}_G(\Delta) = \{g \in G \mid \Delta \cap g\Delta g^{-1} \text{ has finite index in both } \Delta \text{ and } g\Delta g^{-1}\}.$$

The next theorem is basically due to G.A. Margulis [Mar91, VII.5.4], and the formulation below can be found in [Bon, Theorem 1]. G.A. Margulis proved it when G is a semisimple group over a local field. A (certainly non-exhaustive) list of later contributions is the following: in [AB94], the semisimple group G is replaced by a group containing an amenable subgroup $P < G$ similar to a minimal parabolic subgroup; in [Bur95] the existence of P is replaced by the existence of a substitute for a Furstenberg boundary; in [BM02] such boundaries are constructed for compactly generated groups G; and the double ergodicity Theorem in [Kai02] shows that suitable boundaries are available for any locally compact second countable group, via Poisson boundary theory. Different approaches lead to similar results: [Mar94] by means of equivariant generalized harmonic mappings and [Sha00] by means of representation theory.

Theorem 5.4.1. *Let G be a locally compact second countable topological group, $\Gamma < G$ be a lattice and Λ be a subgroup of G with $\Gamma < \Lambda < \mathrm{Comm}_G(\Gamma)$. Let k be a local field and H be a connected almost k-simple algebraic group. Assume that $\pi : \Lambda \to H_k$ is a homomorphism such that $\pi(\Lambda)$ is dense in the Zariski topology on H and $\pi(\Gamma)$ is unbounded in the Hausdorff topology on H_k. Then π extends to a continuous homomorphism $\bar{\Lambda} \to H_k/Z(H_k)$, where $Z(H_k)$ is the center of H_k.*

A more difficult superrigidity result consists in extending representations of lattices (instead of their commensurators) to continuous representations of the ambient topological group (to algebraic representations when so is the ambient group). This is a more difficult result which requires stronger assumptions (e.g. higher rank for algebraic ambient groups), and again the main ideas are due to G.A. Margulis. The first results in positive characteristic were proved by T.N. Venkataramana [Ven88].

5.4.2. *Embedding theorem*

For the following theorem, we were inspired by a paper due to A. Lubotzky, Sh. Mozes and R.J. Zimmer [LMZ94], where superrigidity is used to disprove linearities for commensurators of tree lattices. Trees are one-dimensional buildings, and Kac-Moody theory naturally leads to lattices for buildings of arbitrary dimension. We have [RR06, 1.B Lemma 2]:

Lemma 5.4.2. *Let Λ be any Kac-Moody group over $\mathbf{F}_q$ and let Γ be any negative facet fixator in Λ. Then: $\Lambda < \mathrm{Comm}_{\bar{\Lambda}}(\Gamma)$.*

Therefore it makes sense to use commensurator superrigidity because when $q \gg 1$, Γ is a lattice of $\bar{\Lambda}$ by Theorem 5.1.3. This enables to obtain [Rem04b, §3, Theorem]:

Theorem 5.4.3. *Let Λ be a Kac-Moody group over the finite field $\mathbf{F}_q$ of characteristic p with $q > 4$ elements, with infinite Weyl group W and buildings X_+ and X_-. Let $\bar{\Lambda}$ be the corresponding Kac-Moody topological group. We make the following assumptions:*

(TS) *the group $\bar{\Lambda}$ is topologically simple;*
(NA) *the group $\bar{\Lambda}$ is not amenable;*
(LT) *the group Λ is a lattice of $X_+ \times X_-$ for its diagonal action.*

Then, if Λ is linear over a field of characteristic p, there exists:

- *a local field k of characteristic p and a connected adjoint k-simple group $\mathbf{G}$,*
- *a topological embedding $\mu : \bar{\Lambda} \to \mathbf{G}(k)$ with Hausdorff unbounded and Zariski dense image,*
- *and a μ-equivariant embedding $\iota : V_{X_+} \to V_\Delta$ from the set of vertices of the Kac-Moody building X_+ of Λ to the set of vertices of the Bruhat-Tits building Δ of $\mathbf{G}(k)$.*

Conditions (TS) and (LT) are satisfied whenever the Weyl group W of Λ is irreducible and $q \gg 1$. Note that the conclusion of Theorem 5.4.3 provides a continuous extension but it is not clear that this topological homomorphism is a closed map.

Indeed, there is still work to do [Rem04b, Lemma 3.C] and, besides the topological simplicity of $\bar{\Lambda}$, the arguments used for that are of combinatorial nature: basically the Bruhat decomposition of $\bar{\Lambda}$ with respect to an Iwahori subgroup.

5.4.3. *Non-linear Fuchsian Kac-Moody groups*

In the case of some Kac-Moody groups with Fuchsian buildings, it can be proved that the associated topological groups cannot be closed subgroups of any simple non-Archimedean Lie group, thus implying the non-linearity of the involved finitely generated groups [Rem04b, Theorem 4.C.1].

Theorem 5.4.4. *Let Λ be a countable Kac-Moody group over $\mathbf{F}_q$ with right-angled Fuchsian associated buildings. Assume that any prenilpotent pair of roots not contained in a spherical root system leads to a trivial commutation of the corresponding root groups. Then for $q \gg 1$, the group Λ is not linear over any field.*

The condition on commutation of root groups is technical: prenilpotency of pairs of roots is relevant to abstract root systems of Coxeter groups [Tit87], [Rem02b, 1.4.1]. It is not very restrictive and actually a weaker assumption may be required – see the remark after [Rem04b, Lemma 4.A.2]. Seeing roots as half-apartments, a pair of roots is prenilpotent if and only if the walls of the roots intersect or if a root contains the other.

The main idea at this stage is to use a dynamical characterization of parabolic subgroups proved by G. Prasad in a paper on strong approximation [Pra77, Lemma 2.4] in the algebraic case, e.g. the case of the target $\mathbf{G}(k)$ of the continuous extension $\mu : \bar{\Lambda} \to \mathbf{G}(k)$. In the case of hyperbolic buildings, one must first say that a parabolic subgroup is by definition a boundary point fixator (in analogy with the symmetric space or Bruhat-Tits building case), and then show that the dynamical characterization holds too [Rem04b, Lemma 4.B.2]. We now have a dynamical round about to the non-existence of an algebraic structure on $\bar{\Lambda}$, and this enables one to show that under the continuous homomorphism μ, parabolic subgroups go to parabolic subgroups. The contradiction comes when one notes that, thanks to the strong dynamics of a Fuchsian group on the asymptotic boundary of the hyperbolic plane, the dynamical analogues of unipotent radicals on the left hand-side are not normalized by the parabolic subgroups containing them, whereas it is the case by definition on the right hand-side of μ.

5.4.4. *Virtual solvability of non-faithful linear images*

In 5.4.2, the starting point is a faithful abstract representation from a finitely generated Kac-Moody group. Starting from non-faithful representations is interesting too, since it can be shown that in this case the images are solvable up to finite index. In other words, when the kernel is non-trivial, it is big [Rem05, Theorem 11]:

Theorem 5.4.5. *Let Λ be a Kac-Moody group over the finite field $\mathbf{F}_q$ of characteristic p, with connected Dynkin diagram and $q \gg 1$. Let $\rho : \Lambda \to \mathrm{GL}_n(\mathbf{F})$ be a linear representation. If ρ is not faithful, the group $\rho(\Lambda)$ is virtually solvable. In particular if Λ is Kazhdan, $\rho(\Lambda)$ is finite.*

Recall that by Theorem 5.1.6, many Kac-Moody groups over $\mathbf{F}_q$ are Kazhdan whenever $q \gg 1$. The above theorem is proved thanks to the same kind of arguments as for Theorem 5.4.3. In order to be in a position to apply Theorem 5.4.1, we have to take the Zariski closure of the image $\rho(\Lambda)$ and to mod out by the radical $R(\overline{\rho(\Lambda)}^Z)$ of the latter algebraic group. It then suffices to show that the image of Λ in $\overline{\rho(\Lambda)}^Z / R(\overline{\rho(\Lambda)}^Z)$ is finite, which can be done thanks to Burnside's theorem [Jac89, 4.5, Exercise 8]. A way to fulfill the unboundedness assumption on the image of Λ (in the Hausdorff topology) is to use tricks from Tits's paper on the existence of free groups in linear groups [Tit72].

5.4.5. *Groups all of whose linear images are finite*

From the previous result, it can be shown that some Kac-Moody groups with hyperbolic buildings don't have any infinite linear image [Rem05, Theorem 16].

Theorem 5.4.6. *There is an integer N such that for any Kac-Moody group Λ over $\mathbf{F}_q$ whose buildings have apartments isomorphic to the tessellation of the hyperbolic plane by regular triangles of angle $\frac{\pi}{4}$, if $q \geq N$ then any linear image of Λ is finite, whatever the target field.*

The proof roughly goes as follows. By Theorem 5.4.5, it is enough to show that some Kac-Moody groups enjoying Kazhdan's property (T) are not linear over any field. According to Theorem 5.1.6, the groups as in the theorem are Kazhdan, and by twin root datum arguments it can be proved that such groups contain Kac-Moody-like subgroups to which the proof of Theorem 5.4.4 applies.

5.5. Conjectures and questions

In this final section, we propose a few questions about finitely generated and totally disconnected Kac-Moody groups. Here, implicit in the word *totally disconnected* is that the group under consideration is uncountable and not endowed with the discrete topology.

5.5.1. *Finitely generated groups*

For discrete groups, we think that the class of non-linear Kac-Moody groups is much wider than the one for which the property has been proved so far.

Conjecture 5.5.1. If the Weyl group W is Gromov-hyperbolic and if $q \gg 1$, then the Kac-Moody group Λ is not linear.

Recall that according to G. Moussong, in the class of Coxeter groups hyperbolicity is equivalent to acting discretely and cocompactly on a $CAT(-1)$-space [Mou88]. The non-linearity proof of Sect. 5.4 deals with groups Λ whose Weyl groups have pleasant Coxeter complexes since the latter complexes are Fuchsian tilings. In general, Moussong's complex is defined by isometric gluings of cells and is less easy to understand, so one may have to use arguments of combinatorial nature on Kac-Moody root systems at some points. Note also that one should get rid of the technical condition on prenilpotent pairs of roots in the statement of Theorem 5.4.4.

Another natural question is:

Question 5.5.2. Can the assumption «$q \gg 1$» be removed in non-linearity results?

In other words: is there a generalized Cartan matrix A and a prime number p such that the corresponding group is linear over $\mathbf{Z}/p$ but no longer linear over $\mathbf{F}_{p^r}$ for large enough r? Note that for Λ to be a lattice of the product of its buildings – which is a crucial argument in our proof - the value of q is important (Theorem 5.1.3). So if the answer to the above problem is yes, the non-linearity proof should require new ideas. Moreover a counter-example due to P. Abramenko shows that for buildings whose chambers are regular hyperbolic triangles of angles $\frac{\pi}{4}$, facet fixators in some groups over $\mathbf{F}_2$ or $\mathbf{F}_3$ are not finitely generated [Abr03, Counter-example 1, Remark 2], hence cannot have property (T) [Mar91, Theorem III.2.7]. But according to Theorem 5.1.6, for $q \gg 1$ the groups do enjoy Kazhdan's property (T).

The most general question on non-linearity for Kac-Moody groups is:

Problem 5.5.3. Find necessary and sufficient conditions for the non-linearity of a finitely generated Kac-Moody group, only in terms of the generalized Cartan matrix defining the group.

Of course, if the generalized Cartan matrix A is of affine type, meaning that the Weyl group is a Euclidean reflection group, then the corresponding Kac-Moody groups are S-arithmetic groups, hence are linear. A restatement is: are there other linear examples than the affine ones?

Finally, there is another problem which comes from the analogy with the situation of lattices, namely the problem of arithmeticity. Of course when algebraic structures are available, the definition is well-known [Zim84, Definitions 6.1.1 and 10.1.11]. In the general context of an inclusion $\Delta < G$ of a discrete subgroup Δ in a locally compact group G, it has now become classical to say that Δ is *arithmetic* in G if by definition $\mathrm{Comm}_G(\Delta)$ is dense in G (5.4.1). This is reasonable since by Margulis's commensurator criterion, a lattice in a non-compact simple Lie group is arithmetic in the classical sense if and only if it is so in the above sense [Zim84, Theorem 6.2.5]. This definition makes sense in the case where the topological group G is the isometry group of some metric space, e.g. of some building with enough automorphisms. Since Kac-Moody buildings are close in many respects to Bruhat-Tits buildings, it is natural to ask:

Question 5.5.4. Let X_+ be the positive building of some Kac-Moody group Λ over $\mathbf{F}_q$. Is a negative facet fixator in Λ arithmetic in $\mathrm{Aut}(X_+)$? More generally, which lattices, not necessarily relevant to Kac-Moody groups, are arithmetic in $\mathrm{Aut}(X_+)$?

Note that by definition of a topological Kac-Moody group (Sect. 5.3, introduction), any negative facet fixator Γ is arithmetic in $\bar{\Lambda}$ since by Lemma 5.4.2, $\mathrm{Comm}_{\bar{\Lambda}}(\Gamma)$ contains Λ. Here is a list of known arithmeticity results:

(i) Cocompact lattices in a locally finite tree T are all arithmetic in $\mathrm{Aut}(T)$ [Liu94].
(ii) The Nagao lattice $\mathrm{SL}_2(\mathbf{F}_q[t^{-1}])$ is a non-uniform arithmetic lattice in the full automorphism group of the Bruhat-Tits tree of $\mathrm{SL}_2(\mathbf{F}_q((t)))$ [Moz99]. This was later extended to some Moufang twin trees, which enables to deal with non-linear non-uniform tree lattices [AR06].
(iii) Uniform lattices in some hyperbolic buildings, not necessarily from Kac-Moody theory [Hag03].

See the end of [AR06] for more precise questions.

We finish by another question about linearity, which deals with facet fixators in finitely generated Kac-Moody groups instead of the full latter groups.

Question 5.5.5. Let Λ be a Kac-Moody group over $\mathbf{F}_q$. Are there non-linear facet fixators in Λ?

This question seems to be more difficult because the latter groups are always residually finite (linearity of finitely generated groups is often disproved by disproving residual finiteness, i.e. using the Mal'cev theorem). To see the residual finiteness in this case, it suffices to take subgroups fixing combinatorial balls of increasing radius around the given facet; the intersection of these groups is trivial since it fixes pointwise the building. Constructing and studying non-linear finitely generated groups acting on $CAT(-1)$ metric spaces is a very interesting problem. Note that the above facet fixators are often, but not always, finitely generated.

5.5.2. *Totally disconnected groups*

Theorem 5.4.3 shows that non-linearity of some finitely generated groups roughly amounts to non-linearity of much bigger topological groups. In the study of topological Kac-Moody groups in Sect. 5.3 appear an interesting class of pro-p subgroups. A way to disprove some linearities would be to answer the following:

Question 5.5.6. Which topological Kac-Moody groups contain non-linear pro-p subgroups?

Forgetting discrete groups, the question is interesting in its own right: for instance, the linearity of free pro-p subgroups is a question so far admitting only partial answers (E. Zelmanov).

In the classical case, topological Kac-Moody groups correspond to groups of rational points of adjoint semisimple groups, so it makes sense to ask the following:

Question 5.5.7. Are topological Kac-Moody groups direct products of abstractly simple groups?

Theorem 5.3.3 only says that the latter groups are direct products of topologically simple groups. F. Haglund and F. Paulin, elaborating on J. Tits' proof for trees [Tit70], showed the abstract simplicity of full automorphism groups of many hyperbolic buildings [HP98, Theorems 1.1 and 1.2].

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