# Local actions on graphs and semiprimitive groups

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# Automorphisms of graphs

$\Gamma$ a locally finite, simple, connected graph.

Vertex set $V\Gamma$, edge set $E\Gamma$, arc set $A\Gamma$

$\mathrm{Aut}(\Gamma)$ is the group of all automorphisms of $\Gamma$.

# Symmetry conditions

Given $G \leqslant \mathrm{Aut}(\Gamma)$ then $G$ is

| | |
|---|---|
| vertex-transitive: | transitive on $V\Gamma$ |
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Edge-transitive but not vertex-transitive implies that $\Gamma$ is bipartite and $G$ has two orbits on vertices.

# How many automorphisms?

$G \leqslant \mathrm{Aut}(\Gamma)$ arc-transitive or edge-transitive.

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- If $\Gamma$ is finite and $G$ is vertex-transitive, then by the Orbit-Stabiliser Theorem $|G| = |V\Gamma||G_v|$, so also bound $|G|$.
- If $\Gamma$ is infinite then $|G_v|$ is bounded if and only if $\mathrm{Aut}(\Gamma)$ has finitely many conjugacy classes of discrete arc/edge-transitive subgroups.

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Note that if $\Gamma$ has valency $d$ and $G$ is arc-transitive then also have $|G_v| = d|G_{vw}|$.

# A Theorem of Tutte

(1947,1959)

Theorem Let $\Gamma$ be a connected cubic graph with an arc-transitive group $G$ of automorphisms such that $G_v$ is finite. Then $|G_v| = 3.2^s$ for some $s \leq 4$.

# Structure of stabilisers

Djoković and Miller (1980): Determined the possible structures of finite vertex and edge-stabilisers for cubic arc-transitive graphs:

- Only 7 possibilities for the pair $(G_v, G_e)$ with $e = \{u, v\}$.
- In particular, $G$ is a quotient of one of 7 finitely presented groups.

# Possibilities for $(G_v, G_e)$

| $s$ | $G_v$ | $G_e$ |
|---|---|---|
| 1 | $C_3$ | $C_2$ |
| 2 | $S_3$ | $C_2 \times C_2$ or $C_4$ |
| 3 | $S_3 \times C_2$ | $D_8$ |
| 4 | $S_4$ | $D_{16}$ or $QD_{16}$ |
| 5 | $S_4 \times C_2$ | $(D_8 \times C_2) \rtimes C_2$ |

# Applications

Conder and Dobcsányi (2002): Determined all cubic arc-transitive graphs on at most 768 vertices:

- $|\,\mathrm{Aut}(\Gamma)| \leqslant 768.48 = 36864$
- So need to find all normal subgroups of index at most 36864.

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- $|\mathrm{Aut}(\Gamma)| \leqslant 768.48 = 36864$
- So need to find all normal subgroups of index at most 36864.
- Conder has subsequently enumerated all such graphs on at most 10,000 vertices.

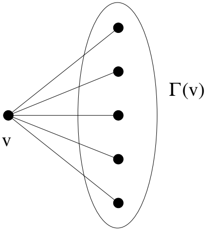
# Edge-transitive

Goldschmidt (1980): Determined the possible structures of finite pairs $(G_u, G_v)$ for adjacent vertices $u, v$ in cubic edge-transitive graphs:

- only fifteen possibilities
- $|G_v| \leqslant 384$

# Local actions

$\Gamma(v)$ is the set of neighbours of $v$.




$G_v^{\Gamma(v)}$ is the permutation group induced on $\Gamma(v)$ by $G_v$, called the local action of $G_v$.

If $G$ is vertex-transitive then all the $G_v^{\Gamma(v)}$ are isomorphic.

# Local actions

$\Gamma$ connected, $G \leqslant \mathrm{Aut}(\Gamma)$ vertex-transitive

- Given a permutation group $L$, we say that the pair $(\Gamma, G)$ is locally $L$ if $G_v^{\Gamma(v)} \cong L$ for all vertices $v$.
- Given some permutation group property $\mathcal{P}$, we say that $(\Gamma, G)$ is locally $\mathcal{P}$ if $G_v^{\Gamma(v)}$ has property $\mathcal{P}$ for all vertices $v$.

# Weiss Conjecture

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Weiss Conjecture (1978): There is some function $f(d)$ such that for every locally primitive pair $(\Gamma, G)$ of valency $d$ and $G_v$ finite we have $|G_v| \leqslant f(d)$.

- Tutte's result is that $f(3) = 48$.

# Graph-restrictive

Verret: We say that $L$ is graph-restrictive if there is a constant $C$ such that for all locally $L$ pairs $(\Gamma, G)$ with $G_v$ finite, we have that $|G_v| \leqslant C$.

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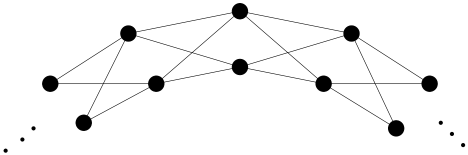
- Tutte's result is that $C_3$ and $S_3$ are graph-restrictive.
- The Weiss Conjecture asserts that every primitive group is graph-restrictive.

# A nonexample

## Wreath graphs



$\mathrm{Aut}(\Gamma) = S_2 \,\mathrm{wr}\, D_{2n}$

$\mathrm{Aut}(\Gamma)_v^{\Gamma(v)} = D_8$

$|\,\mathrm{Aut}(\Gamma)_v| = 2^{n-1}.2$

# An equivalent definition

$G_v^{[i]}$ is the kernel of the action of $G_v$ on the set of all vertices at distance at most $i$ from $v$.

$G_{vw}^{[1]}$ is the kernel of the action of $G_{vw}$ on $\Gamma(v) \cup \Gamma(w)$, where $\{v, w\}$ is an edge.

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Lemma If $\Gamma$ is connected and $G_v^{[i]} = G_v^{[i+1]}$ for some $i$, then $G_v^{[i]} = 1$.

Lemma $L$ is graph-restrictive if and only if there is some constant $k$ such that for all locally $L$ pairs $(\Gamma, G)$ with $G_v$ finite, we have $G_v^{[k]} = 1$.

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Tutte: For cubic arc-transitive graphs with $G_v$ finite we have $G_v^{[3]} = 1$.

# Some graph-restrictive groups

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- Sami (2006): $D_{2n}$ for $n$ odd.
- Potočnik, Spiga, Verret (2012): $\mathrm{GL}(2, p)$ acting on the set of nonzero vectors of $\mathrm{GF}(p)^2$.

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Trofimov, Weiss (1995): $\mathrm{PSL}_n(q)$ acting on $m$-spaces is graph-restrictive.

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- Call $G$ quasiprimitive if every nontrivial normal subgroup is transitive.
- Call $G$ semiprimitive if every nontrivial normal subgroup is transitive or semiregular.
  (A permutation group $H$ is semiregular on $\Omega$ if $H_\alpha = 1$ for all $\alpha \in \Omega$, that is, free.)

# Semiprimitive groups

Initially studied by Bereczky and Maróti (2008) (motivated by an application from universal algebra and collapsing monoids).

Examples include:

- primitive and quasiprimitive groups;
- regular groups;
- Frobenius groups (that is, all nontrivial elements fix at most one point);
- $\mathrm{GL}(n, p)$ acting on the set of nonzero vectors of $\mathbb{Z}_p^n$.
- Any locally quasiprimitive, vertex-transitive group of automorphisms of a non-bipartite graph. (Praeger 1985)

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PSV conjecture: A transitive group is graph-restrictive if and only if it is semiprimitive.

# The edge-transitive case

$\Gamma$ edge-transitive but not vertex transitive. Edge $\{v, w\}$

Say $(\Gamma, G)$ is locally $[L_1, L_2]$ if $G_v^{\Gamma(v)} \cong L_1$ or $L_2$ for all vertices $v$.

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Goldschmidt-Sims Conjecture: If $L_1$ and $L_2$ are primitive then there is a constant $C$ such that if $(\Gamma, G)$ is locally $[L_1, L_2]$ with finite vertex stabilisers then $|G_{vw}| \leq C$.

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Morgan, Spiga, Verret (2015): If either $L_1$ or $L_2$ is not semiprimitive then there is no bound on $|G_{vw}|$ for a locally $[L_1, L_2]$ pair $(\Gamma, G)$ with finite stabilisers.

# Variation on Thompson-Wielandt

Given an edge $\{v, w\}$, $G_{vw}^{[1]}$ is the kernel of the action of $G_{vw}$ on $\Gamma(v) \cup \Gamma(w)$.

Thompson-Wielandt Theorem: If $(\Gamma, G)$ is a locally primitive pair with $G_v$ finite and $\{v, w\}$ is an edge, then $G_{vw}^{[1]}$ is a $p$-group for some prime $p$.

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- van Bon (2003): Still holds if $(\Gamma, G)$ is locally quasiprimitive.
- Spiga (2012): Still holds if $(\Gamma, G)$ is locally semiprimitive.

# Plinths

$G \leqslant \mathrm{Sym}(\Omega)$, transitive.

Define a plinth of $G$ to be a minimal transitive normal subgroup of $G$.

- Every finite transitive group has a plinth.
- If a group has a transitive minimal normal subgroup it is a plinth.
- Any regular normal subgroup is a plinth.

# Properties of plinths of primitive groups

$G$ primitive with minimal normal subgroup (plinth) $N$:

- $N$ is characteristically simple and so in finite case $N \cong T^k$, for some finite simple group $T$.
- $C_G(N)$ is semiregular.
- $G$ has at most two plinths
- If $M$ is a second plinth then $N \cong M$ and both $N$ and $M$ are regular.

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O'Nan–Scott Theorem for primitive groups, quasiprimitive groups.

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Theorem (G-Morgan) Let $K$ be a plinth of a semiprimitive group.

- If $K$ is a nonregular plinth then $K$ is perfect and unique.
- If $L$ is a another plinth then every plinth is contained in $KL$ and every nonregular normal subgroup contains $KL$.

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Bereczky, Maróti (2008): A finite soluble semiprimitive group has a unique plinth, it is regular, and contains every intransitive normal subgroup.

# “Topological” plinths

$\Gamma$ an infinite, locally finite, nonbipartite graph.

Let $G$ be a non-discrete, vertex-transitive, locally quasiprimitive closed subgroup of $\mathrm{Aut}(\Gamma)$. Note that $G$ is semiprimitive.

Define $G^{(\infty)} = \cap_{L<G} L$, for $L$ open and of finite index.

Burger-Mozes (2000): Let $N$ be a closed normal subgroup of $G$. Then either:

- $N$ is nondiscrete and contains the transitive group $G^{(\infty)}$, or
- $N$ is discrete and acts freely with infinitely many orbits.

Moreover, $G^{(\infty)}$ is topologically perfect.

# Multiple plinths

Theorem (G-Morgan) Let $G$ be semiprimitive with distinct plinths $K$ and $L$. Then

- $\overline{G} = G/(K \cap L)$ acts primitively on the set of $(K \cap L)$-orbits
- there exists a characteristically simple group $X$ such that $L/(K \cap L) \cong K/(K \cap L) \cong X$.

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Corollary Finite plinths have the same set of composition factors.

Theorem (G-Morgan) If $G$ is a finite semiprimitive group with multiple plinths then $G$ is graph-restrictive. ($G_{uv}^{[1]} = 1$)

# Nilpotent plinths

Let $L$ be a finite semiprimitive group with a nilpotent plinth $K$.

Theorem (G-Morgan (2015)) Let $(\Gamma, G)$ be a locally $L$ pair with $G_v$ finite and valency coprime to 6. Then $L$ is graph-restrictive. $(G_{vw}^{[1]} = 1)$

- Also give detailed information about what a counterexample with valency not coprime to 6 must look like.
- Analogous to Weiss's results for primitive affine groups