The csc Graded Lie Group as a Quantum Group

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Abstract : The quantum group as an object belonging to $\mathbf{Gr}^{o}_{k}$, the dual category of category of Z_2- graded Lie groups E(G,**g**), E(G′,**g**′), ... with co-morphisms Σ^{-1} : E(G,**g**) $\longleftarrow$ E(G′,**g**′), given by faithful group homomorphisms is constructed. The Hopf structure associated with E(G,**g**), and the irreducibility property of E(G,**g**)–modules are also discussed.

I. Introduction

The development of deformation of algebraic structures[1,2] and subsequent analysis of cohomological properties associated with such structures[3] have become a forerunner in our understanding of quantum theory. In Faddeev's hand, quantum group appears as an abstraction of the developments of the theory of quantum integrable dynamical systems.[4] The kinship between the braid structure of R–matrices and the deformation of algebraic groups were later established.[5] In 2–dimensional conformal field theory, operator product algebra possessing a symmetry group structure displays the product coefficients depending upon the secondary fields which in turn can be explained only in terms of primary fields. However, the tensor product of the symmetry algebra associated with the primary fields is undefined. This difficulty was cirumvented by a new structure of the symmetry algebra which possesses tensor product corresponding to the operator product algebra.[6] The new symmetry algebra (K–Hopf algebra)

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supplemented with the quasi-triangular conditions (quasi Hopf algebra, weak quasi–Hopf algebra...) were later identified with quantum groups. Quantum group can be visualised also as deformation of the function algebra Func ($\mathbf{g}^*$) into a non-commutative, co-commutative Hopf algebra[7] in the spirit of Kostant-Kirrilov Scheme.

In the sequel, we will consider K, a fixed field (R or C) of characteristic zero, graded means Z_2–graded unless stated otherwise, $\mathbf{Gr}_k$ (resp. $\mathbf{Alg}_k$, $\mathbf{Comm\ Alg}_k$, $\mathbf{Hopf}_K$, $\mathbf{Comm\ Hopf}_k$, $\mathbf{SLie}_k$) for the category of csc graded Lie groups (resp. K–algebras, commutative K–algebras, K–Hopf algebras, commutative K-Hopf algebras, Z_2–graded Lie algebras), the dual category of the category of K–Hopf algebras by $(\mathbf{Hopf}_K)^o$ etc. For notational convenience, we will denote csc graded Lie group E(G,**g**) by **G** also.

Our material is organised as follows.

In Section II, we quickly recall the definition of a Hopf algebra (A, M, $\eta, \Delta, \varepsilon, \gamma$) defined over the base field K, where $M : A \otimes A \longrightarrow A$ (multiplication), $\eta : K \longrightarrow A$ (unit), $\Delta : A \longrightarrow A \otimes A$ (commultiplication), $\varepsilon : A \longrightarrow K$ (co-unit) and $\gamma : A \longrightarrow A$ (antipode) satisfying the associativity, existence of unit element, co-associativity and co-unitary properties. We explicitly construct E (G,**g**) as the smash product (semi direct product) of the K–group ring of G, K(G) and the universal enveloping algebra of the Z_2–graded Lie algebra $\mathbf{g} = \mathbf{g}_o + \mathbf{g}_1$. We note that K(G) and U(**g**), E(G,**g**) possess K–Hopf and super K–Hopf algebra structures respectively.[8]

In Section III, we construct the covariant functor

Kos : $\mathbf{SLie}_k \longrightarrow \mathbf{Gr}_k$ defined by

1. Kos ($\mathbf{g}$) $=$ E (G,$\mathbf{g}$)

2. Kos ($\sigma : \mathbf{g} \longrightarrow \mathrm{g}'$) $= \Sigma :$ E (G,$\mathbf{g}$) $\longrightarrow$ E (G$'$,$\mathbf{g}'$),

where E (G,$\mathbf{g}$) $\in$ ob ($\mathbf{Gr}_k$) and morphisms Σ given by faithful group homomorphisms etc. The exactness property of the functors Kos and Kos^{-1} is also discussed. In Section IV, we establish the Theorem :

Theorem : 1 : Let V be a reducible (resp. completely reducible, irreducible) $\mathbf{g}$-module. Then, V has the same reducibility property as $\mathbf{G}$-module.

In Section V, we discuss the conditions under which **G** is quantizable[9]. Our results in this direction can be summarised by

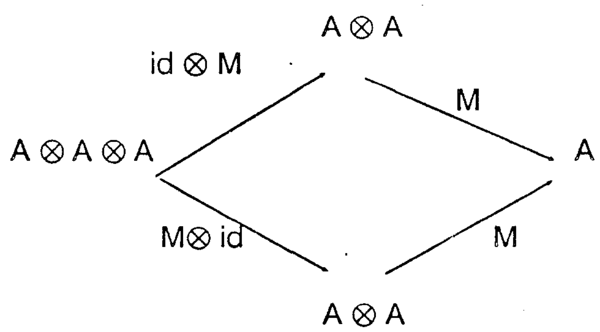
Theorem : 2 : Let **G** be a csc Z_2-graded Lie group and let $\mathbf{g}$ be the corresponding Z_2-graded Lie algebra. Let $H^2(\mathbf{G},\mathbf{G}) = \{0\}$. Then **G** is quantizable.

Theorem : 3 : $\mathbf{QGr}_k = (\mathbf{Hopf}_k)^{\circ}$, the dual category of category of Hopf algebras where any $\mathbf{G} \in$ ob ($\mathbf{QGr}_k$) is called a quantum group with co-morphism $\Sigma^{-1} : \mathbf{G}' \longrightarrow \mathbf{G}$ is a faithful (?) group homomorphism.

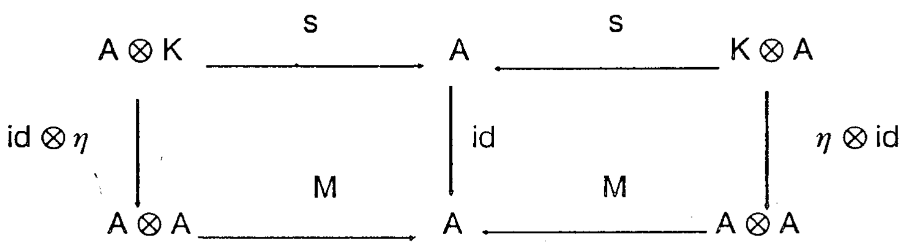
II. Hopf algebra and construction of E(G, g)

Definition : 1: The Hopf algebra (A, M, η, Δ, ε, γ) is a vector space over the field K of characteristic zero satisfying the following properties represented by the commutative diagrams:

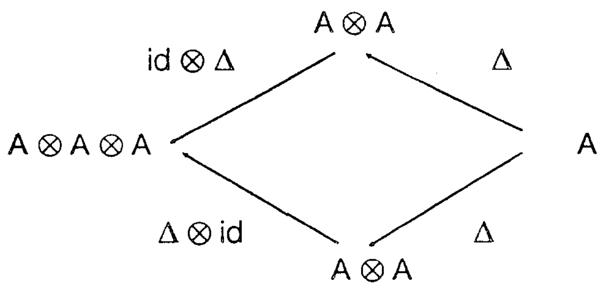
1. associativity :



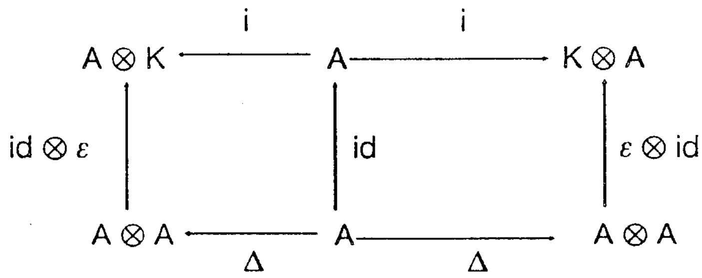
2. existence of unit:



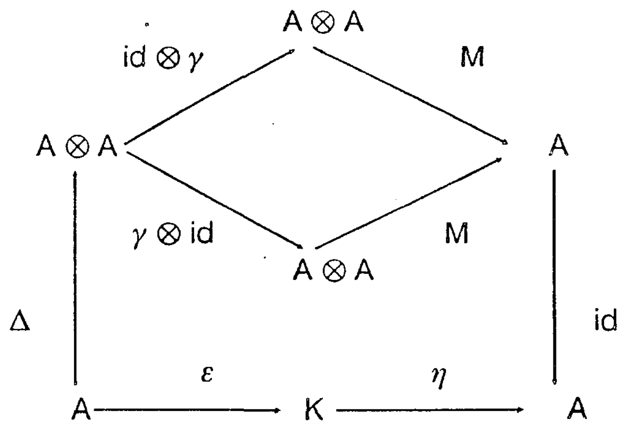
3. coassociativity:



4. existence of counit:



5. existence of antipode :



Here $s : A \otimes K \to A$ and $i : A \to A \otimes K$ denote the scalar multiplication and the inclusion $x \to x \otimes 1$ (with 1 the multiplicative unit of K) respectively.

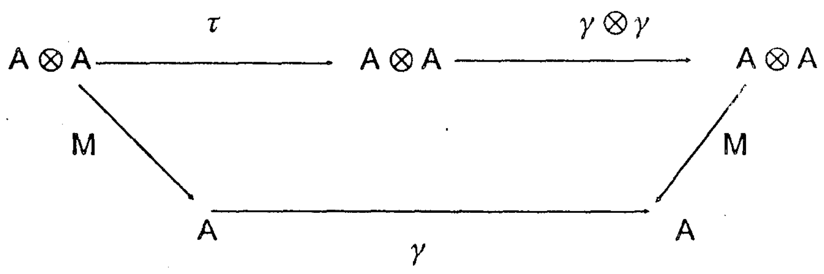
The antipode is an antihomomorphism, i.e., $\gamma(x \otimes y) = \gamma(y) \otimes \gamma(x)$ for all $x, y \in A$.

If τ is a permutation map

$\tau : A \otimes A \to A \otimes A$

i.e., $x \otimes y \mapsto y \otimes x$ (2.1)

then, we have the following commutative diagram :



Definition : 2 : Let $\mathbf{g} = \mathbf{g}_o + \mathbf{g}_1$ be a finite dimensional Z_2-graded Lie algebra over K of characteristic zero. Let U(**g**) be the universal enveloping algebra of **g**, i.e., U(**g**) = T(**g**)/**J**, where T(**g**) is the tensor algebra over **g** and **J** is the two-sided ideal of T(**g**) defined by elements of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y], x, y \in \mathbf{g}. \qquad (2.2)$$

The quotient map

$$T(\mathbf{g}) \longrightarrow U(\mathbf{g})$$

is injective for **g** and as usual $\mathbf{g} \subseteq U(\mathbf{g})$. U(**g**) is a Z_2-graded co-commutative Hopf algebra satisfying

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \Delta(1) = 1 \otimes 1,$$

$$\varepsilon(x) = 0, \quad \eta(\xi) = \xi.1, \quad \varepsilon(1) = 1,$$

$$\gamma(x \otimes y) = (-1)^{|x||y|}.\gamma(y) \otimes \gamma(x),$$

$$\gamma(1) = 1, \quad \gamma(x) = -x, \qquad (2.3)$$

where the graded tensor product is given by

$$(x \otimes y)\ (z \otimes w) = (-1)^{|y||z|} (xz \otimes yw),\ x, y, z, w \in \mathbf{g},\ \xi \in K. \qquad (2.4)$$

Definition : 3 : The group ring K(G) is a Hopf algebra. K(G) is a free abelian group generated by elements of the form (r,g) or r.g, $r \in K$, $g \in G$ such that

$$r' (r.g) = (r.g) r' = (rr'.g),$$

$$(r.g) + (r'.g) = ((r + r').g),$$

$$(r.g) . (r'. g') = (rr'. gg'),\ r, r' \in K,\ g, g' \in G . \qquad (2.5)$$

Further, we have

$$\Delta (x) = x \otimes x,$$

$$\varepsilon (x) = 1,$$

$$\gamma (x) = x^{-1},\ x \in K (G). \qquad (2.6)$$

We note that the antipode γ is just the inverse. Let ad : $\mathbf{g}_o \times \mathbf{g} \to \mathbf{g}$ be the adjoint mapping restricted to $\mathbf{g}_o$. Then, ad exponentiates to π such that π : $G \times \mathbf{g} \to \mathbf{g}$, i.e., π (g) is a graded Lie algebra automorphism for any $g \in G$. Then, π uniquely extends to a representation.

$$\pi : G \times U(\mathbf{g}) \to \mathbf{aut}\ U(\mathbf{g}) \qquad (2.7)$$

i.e., G operates as a group of automorphism of U (**g**).

Definition : 4 : Let E (G,**g**) = K (G) # U(**g**) be the smash product of

K(G) with U(**g**) with respect to π. E(G,**g**) is a co-commutative Z_2-graded Hopf algebra with antipode.

1. As a graded vector space, E (G,**g**) = K (G) ⊗ U (**g**). We denote g ⊗ x by g # x, g ∈ K (G), x ∈ U (**g**).

2. E (G,**g**) is an algebra having K(G) and U (**g**) as sub algbras such that

   $x \rightarrow g\, x\, g^{-1} = \pi\,(g)\, x,\ g \in K\,(G),\ x \in U(\mathbf{g}).$

   i.e., $(g \# x)\,(g' \# x') = (gg' \# x\, \pi\,(g)\, x'),$

   $$\eta_{K(G)\,\#\,U(g)} = \eta_{K\,(G)} \otimes \eta_{U\,(g)}\,. \tag{2.8}$$

3. w.r.t the diagonal map Δ, the elements of K(G) are group-like and the elements of U(**g**) are primitive,

   i.e., $\Delta\,(g \# x) = \Sigma_{(g)(x)}\ (g_{(1)} \# x_{(1)}) \otimes (g_{(2)} \# x_{(2)})$

   $$\varepsilon\,(g \# x) = \varepsilon\,(g)\,\varepsilon\,(x). \tag{2.9}$$

   (We note Δ, ε are K-algebra morphisms).

   4. The anitpode γ is defined by

   $$\gamma_{\,E(G,\mathbf{g})} = (\gamma_{\,K(G)} \otimes \gamma_{\,U(\mathbf{g})}\,(\pi \otimes 1)\,(\gamma_{\,K(G)} \otimes 1 \otimes 1)\,(\tau \otimes 1)\,(1 \otimes \Delta) \tag{2.10}$$

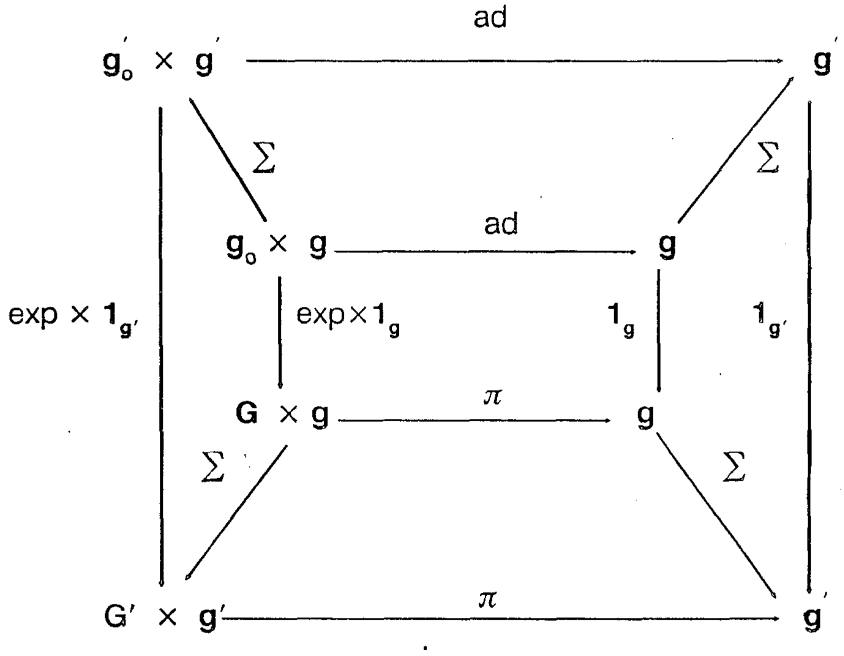
   i.e., for g ∈ K (G) and x ∈ U(**g**), we have

   $$\gamma_{\,E(G,\mathbf{g})}\,(g \# x) = \gamma_{\,K(G)}\,(g_{(1)})\,\gamma_{\,U(g)}\,(x) \# \gamma_{\,K(G)}\,(g_{(2)}). \tag{2.11}$$

Thus one has $\gamma(g) = g^{-1}, \gamma(x) = -x$ and $\varepsilon(g) = 1, \varepsilon(x) = 0.$ (2.12)

Theorem : 4 : Let $\sigma : \mathbf{g} \to \mathbf{g}'$ be a faithful finite dimensional matrix representation of $\mathbf{g}$, i.e., dim $\mathbf{g}' < \infty$. Let $\sigma_o = \sigma|_{\mathbf{g}_o} : \mathbf{g}_o \to \mathbf{g}'_0$. Further, exp $\sigma_o : G \to G'$. Let exp σ_o : K (G) → K (G′). Also, by universality of U($\mathbf{g}$), let σ define an isomorphism U(σ) : U($\mathbf{g}$) → U($\mathbf{g}'$). Let Σ = exp $\sigma_o \otimes$ U(σ) : K (G) $\otimes$ U($\mathbf{g}$) ⟶ K (G′) $\otimes$ U($\mathbf{g}'$). Then, Σ defines a unique finite dimensional and faithful matrix representation of E (G,$\mathbf{g}$).

Proof: From the commutative diagram,



it can be shown that Σ commutes π and U(σ) = $\Sigma|_{U(\mathbf{g})}$ preserves the degree of $\mathbf{g}$. Hence the theorem.For a detailed derivation, see ref. 8.

III. Exactness of Kos and Kos^{-1}

Let $\mathbf{SLie}_K$ and $\mathbf{Gr}_K$ be the categroies of finite dimensional Z_2-graded Lie algebras and csc graded Lie groups of finite type respectively.

Let Kos : $\mathbf{SLie}_K \longrightarrow \mathbf{Gr}_K$ be defined by

$$\text{Kos}\,(\mathbf{g}) = E\,(G,\mathbf{g}),$$

$$\text{Kos}\,(\sigma : \mathbf{g} \to \mathbf{g}') = \Sigma : E\,(G,\mathbf{g}) \longrightarrow E(G',\mathbf{g}'). \qquad (3.1)$$

Let O, O' be the zero objects in $\mathbf{SLie}_K$ and $\mathbf{Gr}_K$ respectively such that

$$O' = K(e) \# K \qquad (3.2)$$

with O being the trivially graded Lie algebra. Further, $O \in \text{hom}\,(\mathbf{g}^{(1)}, \mathbf{g}^{(2)})$, $O' \in \text{Hom}\,(\mathbf{G}^{(1)}, \mathbf{G}^{(2)})$, where $\mathbf{g}^{(1)}, \mathbf{g}^{(2)} \in \text{ob}\,(\mathbf{SLie}_K)$. $\mathbf{G}^{(1)}, \mathbf{G}^{(2)} \in \text{ob}\,(\mathbf{Gr}_K)$ are called zero morphisms defined by

$$O' : \mathbf{G}^{(1)} \longrightarrow \mathbf{G}^{(2)}, \text{ i.e., } (r.g \# x) \mapsto (r.e \# c) \qquad (3.3)$$

where $g \in G$, $x \in U(\mathbf{g})$, $c \in K$ being the constant term in the expansion of elements of $U(\mathbf{g})$ in Poincare'-Birkoff-Witt basis while O is the trivial natural map in $\mathbf{SLie}_K$.

Obviously, we have

$$\text{Kos}\,(O) = O', \qquad (3.4)$$

where O, O' stand for the zero objects on the zero morphisms in the relevant category.

Following the notions of epics, monics, kernels and cokernels as in Maclane[10], we have

Lemma : 1. Kos and Kos^{-1} map epics → epics and monics → monics.

Proof : Let us assume σ is epic in $\mathbf{SLie}_K$ and let $\Sigma = \text{Kos}(\sigma)$. Then by definition,

$$\alpha\sigma = \alpha'\sigma \Rightarrow \alpha = \alpha' \;\forall\; \alpha, \alpha' \in \mathbf{SLie}_k.$$

Let Ξ and Ξ' be maps in $\mathbf{SLie}_k : \Xi.\Sigma = \Xi.\Sigma'$.

Applying Kos^{-1}, we readily see that

$$Kos^{-1}(\Xi) = Kos^{-1}(\Xi') \Rightarrow \Sigma \text{ is epic.}$$

We can analogously show that Kos and Kos^{-1} map monics to monics.

Theorem : 5 : Kos and Kos^{-1} are exact functors.

Proof : It is sufficient to prove that

$$\text{Kos}(\text{Ker}\,\alpha) = \text{Ker}(\text{Kos}\alpha) \tag{3.5a}$$

and $$\text{Kos}(\text{Coker}\,\alpha) = \text{Coker}(\text{Kos}\alpha) \tag{3.5b}$$

for an arbitrary morphisms $\alpha \in \mathbf{SLie}_k$ and similar properties for Kos^{-1}.

For $\chi \in \text{Ker}\alpha$, we have $\alpha\chi = 0$ and

$$\alpha\beta = 0 \Rightarrow \beta = \chi\beta' \;\forall\; \beta \in \mathbf{SLie}_k$$

and some $\beta' \in \mathbf{SLie}_k$. It follows immediately that Kos α, Kos$\aleph = 0'$. Now, let us assume $\text{Kos}(\alpha)\beta = 0'$ for some $\beta \in \mathbf{SLie}_k$. Then, if $\beta = Kos^{-1}(\beta)$, we have $\alpha\beta = 0$ and hence $\beta = \aleph\beta'$ for some $\beta' \in \mathbf{SLie}_k$. Hence $\beta = \text{Kos}(\aleph)$. Kos

(β').Thus, $\aleph \in \text{Ker}\alpha \Rightarrow \text{Kos}(\aleph) \in \text{Ker}(\text{Kos}\alpha)$. The reverse implication follows similarly proving

$$\text{Kos}(\text{Ker}\,\alpha) = \text{Ker}(\text{Kos}\alpha)$$

and $\text{Kos}^{-1}(\text{Ker}\alpha) = \text{Ker}(\text{Kos}^{-1}\alpha)$.

Corollary : Objects and quotient objects are preserved under Kos and Kos^{-1} as direct sums.

Remark : It is evident from the above corolloary that one has equivalence class of subobjects and quotient objects. In the sequel, one can fix the choice out of each equivalence class such that Kos and Kos^{-1} preserve this choice. In other words, given σ in $\mathbf{SLie}_k$, $\Sigma = \text{Kos}(\sigma)$. Further, we have selected objects, Ker σ, KerΣ, such that Kos (Ker σ) = KerΣ.

IV. Reducibility of E(G, g)-modules

Let V be a graded vector space over K and let **end V** denote the graded Lie algebra of graded endomorphisms of V. Let $\mathbf{g} \in \mathbf{SLie}_k$ and $\mathbf{G}$ = E (G, **g**) = Kos (**g**)..

Definition : 5: V is said to be a **g**-module if there exists a morphism $\sigma \in$ hom **(g, end V)**.

Now, the graded Lie group corresponding to **end V** is given by **G aut V** = Kos **(end V)** = K (**aut** $V_0 \otimes$ **aut** V_1) # U **(end V)**. We know that U**(end V)** and hence **G aut V** act in a natural manner onV. We have thus

Definition : 6: V is said to be a **G**-module if there exists a morphism

$\Sigma \in$ hom **(G, G aut V).**

Lemma : 2: V is a **g**-module $\Leftrightarrow$ V is a **G-** module the proof follows trivially from the above definitions.

A module V over **g** (resp. **G**) is said to be faithful if the map σ (res.Σ) in definition 5 (resp.definition 6) is monic.

Lemma : 3: Kos preserves faithful modules.

Proof : Follows trivially from Lemma 1.

Lemma : 4: Let W be a **g** – sub-module of V. Then, W is a sub-module of V.

Proof : Follows from the irreducibility of Kos.The proof of following Lemma is equally trivial .

Lemma : 5: W is a direct summand of V as a **g**–module $\Leftrightarrow$ W is a direct summand of V as **G**–module. Thus we have the theorem:

Theorem : 1: Let V be a reducible (resp. completely reducible, irreducible) **g**–module. Then, V has the same reducibility property as **G**–module.

V. E(G, g) a quantum group

To demonstrate that **G** $\in$ ob(**QGr**$_k$), we recourse to some of the well-known results existing in the literature. Consequently, proofs of Lemmas that either trivially follow or have been proved elsewhere.

Lemma : 6: Let **g** be a finite dimensional Z_2-graded Lie algebra of the compact, connected and simply connected (csc) Z_2-graded Lie group **G**. Then, H^q (**g**, **g**) $\approx$ H^q (**G**, **G**) and the ring H (**g**, **g**) $\approx$ H (**G**, **G**).

This Lemma can be established as a generalisation of the well-known Theorem due to Chevally and Eilenberg.[11]

Lemma : 7: If **g** is strongly semisimple, then H^q (**g**, **g**) $= \{0\}$, $q > 1$. Consequently, H^2 (**g**, **g**) $= \{0\}$.

Theorem : 2: Let **G** be a csc and compact Z_2 –graded Lie group. Then, any $x \in$ **g** is integral iff H^2 (**O, R**) defined by the symplectic structure on the orbit **O** = **G**. x is integral (integrability condition). In other words, **G** is said to be quantizable if H^2 (**G, G**) $= \{0\}$.

The proof of the Theorem 2 has been discussed by Kostant and Auslander for solvable Lie group[12] and later it was generalised to semi-simple Lie groups to obtain the IRS (Harish-Chandra representations) associated with the quantizable orbits.[13]

Theorem 3 is a direct consequence of our construction; namely $\mathbf{QGr_k}$ $=$ **(super** $\mathbf{Hopf_k)^0}$**,** the dual category of the category of super Hopf algebras[14]. Any **G**$\in$ ob ($\mathbf{QGr_k}$) is a quantum group with the morphism Σ^{-1} **: G** $\leftarrow$ **G′** defined by the faithful (?) group homomorphism.

Remark : 2: The quasitriangular super Hopf algebra can be identified with quantum group, i.e. $R \in A \otimes A$:

1. $R\,\Delta\,(g)R^{-1} = \Delta'\,(g)$. where

   $\Delta'\,(g) = \tau\,\Delta\,(g) = \sum(-1)^{|g_{(1)}|\,|g_{(2)}|} g_{(2)} \otimes g_{(1)}$,

   $\Delta\,(g) = \sum g_{(1)} \otimes g_{(2)}$, $g \in A$:

2. $(\Delta \otimes \mathrm{id})\, R = R_{13}\; R_{23}$;

3. $(\mathrm{id} \otimes \Delta)\, R = R_{13}\, R_{12}$.

Thus, we have

$$R_{12}\;(\Delta\otimes \mathrm{id})\,(R) = (\Delta'\otimes \mathrm{id})\,(R)\,R^{12}.$$

In a future communication, we will report on a detailed construction of R for E (G, **g**).

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