GLOBAL HEAT KERNEL BOUNDS VIA DESINGULARIZING WEIGHTS

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Abstract. We study the integral kernels of semigroups which need not be ultracontractive by transferring them to appropriately chosen weigted spaces where they become ultracontractive. Our construction depends mainly on two assumptions: the classical Sobolev imbedding and a "desingularizing" (L^1, L^1) bound on the weighted semigroup.

1. Introduction and Main Results. In this paper we are concerned with a generalization of singular heat kernel bounds in abstract setting. Our paper essentially contains a singular case, i.e. when the standard bounds are not valid (rather than simply the standard methods do not apply). In a special case of Schrödinger semigroups our abstract results imply a stronger version of [MS] for critical potentials of $c|x|^{-2}$ type.

Let $(M, d\mu)$ be a measurable space with σ-finite measure and $A \geq 0$ be a selfadjoint operator on the (complex) Hilbert space $L^2 = L^2(M, d\mu)$ with the inner product $\langle f, g\rangle := \int_M f\bar{g}d\mu$. Let $Q_\nu(A)$, $\nu \geq 0$ denote the Hilbert space $(\mathcal{D}(A^{1/2})$, $(f,g)_{Q_\nu} := \langle A^{1/2}f$, $A^{1/2}g\rangle + \nu\langle f, g\rangle)$. Then $Q_1(A) \subseteq L^2 \subseteq Q_1'(A)$.

We first consider the most common case of A possessing the Sobolev imbedding property:

$$Q_\nu(A) \subseteq L^{2j} \text{ for some } \nu \geq 0 \text{ and } j > 1 \tag{1}$$

but such that $e^{-tA}|L^1 \cap L^2$, $t > 0$, cannot be extended by continuity to a bounded map on L^1 and the ultracontractivity estimate

$$\|e^{-tA}f\|_\infty \le c_t\|f\|_1 \ , \ f \in L^1 \cap L^\infty \ , \ t > 0$$

is not valid.

In this case we will assume that there exists a family φ of weights, i.e. functions $\{\varphi_s\}_{s>0}$ on M such that for all $s > 0$

$$\varphi_s \ , \ 1/\varphi_s \in L^2_{\text{loc}}(M, d\mu) \tag{2}$$

and there is a constant c_1 independent of s such that, for all $0 < t \le s$,

$$\|\varphi_s e^{-tA} \varphi_s^{-1} f\|_1 \le c_1\|f\|_1 \ , \quad f \in D_s \ , \tag{3}$$

where $D_s := \varphi_s L^\infty_{\text{com}}(M)$.

Let $c_S > 0$ denote the constant in the inequality

$$\|f\|^2_{Q_\nu} \ge c_S\|f\|^2_{2j} \ , \quad f \in \mathcal{D}(A^{1/2}) \ , \tag{1'}$$

which exists due to (1).

Our first main result is the following

Theorem A. *In addition to (1)-(3) assume that*

$$\inf_{s>0, x\in M} |\varphi_s(x)| \ge c_0 > 0 \ .$$

Then, for all $t > 0$ *and a.e.* $x, y \in M$,

$$|e^{-tA}(x,y)| \le Ct^{-j'}|\varphi_t(x)\varphi_t(y)| \ , \tag{5}$$

where $C = C(c_1, c_0, c_S, j), \quad j' = j/(j-1)$.

In applications of Theorem A to concrete operators the main difficulties are in verification of the assumption (3). It is not easy to establish (3) even in the regular case (i.e. $\varphi \equiv 1$): general second order elliptic and parabolic operators produce non contractive L^1 -semigroups (propagators). In fact, the failure in establishing (3) (with $\varphi \equiv 1$)

from the first principals had been for a long time the main obstacle in adopting the most fundamental in the area Nash method (see [Se 2,3] and also the proof of Corollary 2).

We apply Theorem A to the Schrödinger operators. The modeling operator $-\Delta-\beta V_0$, $V_0(x) = \frac{(d-2)^2}{4}|x|^{-2}$, $0 < \beta \le 1$, is of a special interest because the potential exhibits critical local and global behaviour. This circumstance attracted great attention (see e.g. [KPS], [BS], [BV], [LS], [SV], [BG], [CM], [Se 1,3], [MS], [DD], [BFT]). In a considerably simpler case of bounded potentials behaving at infinity like βV_0 for $\beta < 1$ various heat kernel estimates were obtained in [DS], [Zh].

The following is our main result for operator $-\Delta - \beta V_0$, $0 < \beta \le 1$.

Theorem 1. *Let* $H^- = -\Delta \dot{-} \beta V_0$, $0 < \beta < 1$ *be the form sum of* $-\Delta$ *and* $-\beta V_0$ *in* $L^2(\mathrm{R}^d, dx)$, $d \ge 3$. *If* $\beta = 1$ *define* H^- *to be the strong resolvent* L^2*-limit of* $-\Delta \dot{-} \beta V_0$ *as* $\beta \nearrow 1$. *Define weights* $\varphi_\sigma^-(t,x) \in C^2(\mathrm{R}^d \backslash \{0\})$ *by*

$$\varphi_\sigma^-(t,x) = \begin{cases} \left(\frac{\sqrt{t}}{|x|}\right)^\sigma & \text{if } |x| \le \sqrt{t} \\ \frac{1}{2} & \text{if } |x| \ge 2\sqrt{t} \end{cases}$$

and $1/2 \le \varphi_\sigma^-(t,x) \le 1$ *for* $\sqrt{t} \le |x| \le 2\sqrt{t}$, *where* $\sigma := \frac{d-2}{2}(1 - \sqrt{1-\beta})$. *Then, for all* $t > 0$ *and all* $x, y \in \mathrm{R}^d \backslash \{0\}$,

$$e^{-tH^-}(x,y) \le c t^{-\frac{d}{2}} \varphi_\sigma^-(t,x) \varphi_\sigma^-(t,y) .$$

Remarks. 1. Except for the Gaussian factor the global upper bound is sharp in the sense that σ is the best possible exponent.

2. The choice of weights in Theorem 1 implies that operators $\varphi e^{-tA} \varphi^{-1} : L^\infty_{\mathrm{com}} \to L^1_{\mathrm{loc}}$ and $A = H^-$ are bounded from L^p into L^p *only* for $p = 1$.

3. Our proof of Theorem 1 does not essentially differ in the critical $(\beta = 1)$ and non-critical cases.

Next, we discuss the desingularizing method in a different situation. To motivate the discussion let us consider the operator $-\Delta + V$ on R^d, $d \ge 3$ with a non-negative potential. The corresponding heat kernel, $Z_V(t,x;s,y)$, satisfies the Gaussian upper bound

$$Z_V(t,x;s,y) \le \Gamma_{t-s}(x-y)$$

for all $t > s$ and a.e. x , $y \in \mathrm{R}^d$, where

$$\Gamma_t(z) = (4\pi t)^{-d/2} \exp(-|z|^2/4t) \equiv e^{t\Delta}(z,0) \ .$$

This bound holds as soon as the heat kernel can be rigorously defined, e.g. for any $V \in L^1_{\mathrm{loc}}(\mathrm{R}^{d+1})$. On the other hand the Gaussian lower bound

$$e^{-tw} c_1 \Gamma_{c_2(t-s)}(x-y) \leq Z_V(t,x;s,y)$$

$$(c_1 > 0 \ , \ c_2 \geq 1 \ , \ w \geq 0)$$

holds under some additional assumptions on V . The most general sufficient condition seems to be the following: $V \in \mathcal{K}^p_d =$ the parabolic Kato class [MS]. In the case of time independent potentials this condition reads as follows

$$\inf_{\lambda>0} \|(\lambda - \Delta)^{-1} V\|_\infty < \infty \ ,$$

and is also necessary for the Gaussian lower bound to be valid [MS], [Se1]. Thus any potential $V \geq 0$ which violates it makes the Gaussian upper bound fundamentally rough (not feasible). Inevitably the following question arises. What is a proper form of the upper heat kernel bound if, for instance $V(x) = |x|^{-2}(\log(e + |x|^{-1})^{-\gamma} + W$, $\frac{2}{d} < \gamma \leq 1$, $W \in \mathcal{K}^p_d$ with $\inf_\lambda \|(\lambda - \Delta)^{-1}|W|\|_\infty = 0$?

Theorem B below provides conditions which can be readily verified for appropriate weights depending on the choice of the potential.

In [MS] we considered operator $H^+ = -\Delta + \beta V_0$, $0 < \beta < 1$ and proved that $e^{-tH^+}(x,y) \leq c_T t^{-\frac{d}{2}-l} \varphi(x)\varphi(y)$, $0 < t \leq T$, where $\varphi \in C^2(\mathrm{R}^d)$, $\varphi(x) = |x|^l$ if $|x| \leq 1/2$, $\varphi(x) = 1$ if $|x| \geq 1$ and $l := \frac{d-2}{2}(-1 + \sqrt{1+\beta})$.

Here we obtain a sharp bound for all $\beta > 0$ and $t > 0$ by making use of the following abstract result.

Let $(M, d\mu)$ be a measurable space with σ -finite measure and let A be a non-negative selfadjoint operator on $L^2(M, d\mu)$ such that

i) $e^{-tA_1} := (e^{-tA}|L^1 \cap L^2)^{\mathrm{clos}}_{L^1 \to L^1}$, $t \geq 0$ is a C_0 semigroup of bounded operators, i.e.

$$\|e^{-tA_1}\|_{1\to 1} \leq c_1 \ , \quad t \geq 0 \ .$$

ii) e^{-tA} is ultracontractive, i.e.

$$\|e^{-tA_1}\|_{1\to\infty} \leq c_2 t^{-j'} \ , \quad t > 0$$

for some $(j' > 1)$.

Theorem B. *In addition to i), ii) assume that there exists a one-parameter family* ψ *of weights* $\psi_s(x)$, $s > 0$, *such that*

B$_1$) $\psi_s(x)$, $\psi_s(x)^{-1} \in L^2(M \backslash N, d\mu)$ *for all* $s > 0$, *where* N *is a closed set.*

B$_2$) *There is a constant* $\tilde{c}_1$ *independent on* s *such that, for all* $t \leq s$,

$$\|\psi_s e^{-tA} \psi_s^{-1} f\|_1 \leq \tilde{c}_1 \|f\|_1 \quad f \in D_s ,$$

where $D_s := \psi_s L^\infty_{\text{com}}(M \backslash N, d\mu)$.

B$_3$) *For some* $\varepsilon \in]0,1[$ *and any* $s > 0$ *there are constants* $\hat{c}_i = \hat{c}_i(\varepsilon)$, $i = 1, 2$ *and a measurable* $\Omega^s \subset M$ *such that*

(a) $|\psi_s(x)|^{-\varepsilon} \leq \hat{c}_1$ *for all* $x \in M \backslash \Omega^s$.

(b) $|\psi_s(\cdot)|^{-\varepsilon} \in L^{q'}(\Omega^s)$ *and* $\||\psi_s(\cdot)|^{-\varepsilon}\|_{L^{q'}(\Omega^s)} \leq \hat{c}_2 s^{j'/q'}$, *where* $q' = \frac{2}{1-\varepsilon}$.

Then, for all $t > 0$ *and a.e.* $x, y \in M$,

$$|e^{-tA}(x,y)| \leq c t^{-j'} |\psi_t(x) \psi_t(y)| .$$

We apply Theorem B to the Schrödinger operator $H^+ = -\Delta \dot{+} \beta V_0$, $\beta > 0$ on $L^2(\mathrm{R}^d, dx)$, $d \geq 3$.

Theorem 2. *Define weights* $\psi = \psi^+(s,x) \equiv \psi_\ell^+(s,x)$ *as* $C^2(\mathrm{R}^d \backslash \{0\})$ *functions* $\psi \leq 2$ *such that* $\psi^+(s,x) = \left(\frac{|x|}{\sqrt{s}}\right)^\ell$ *if* $|x| \leq \sqrt{s}$, *where* $\ell = \frac{d-2}{2}(-1 + \sqrt{1+\beta})$, *and* $\psi^+(s,x) = 2$ *if* $|x| \geq 2\sqrt{s}$, *and such that* $1 \leq \psi \leq 2$, $|\nabla \psi| \leq c/\sqrt{s}$, $|\Delta \psi| \leq c/s$ *for* $\sqrt{s} \leq |x| \leq 2\sqrt{s}$. *Then, for all* $t > 0$ *and* $x, y \in \mathrm{R}^d$,

$$e^{-tH^+}(x,y) \leq c t^{-d/2} \psi_\ell^+(t,x) \psi_\ell^+(t,y) .$$

We remark that lower bounds on $e^{-tH^\mp}(x,y)$ can be obtained by combining Theorems 1 and 2 with the inequalities

$$e^{t\Delta}(x,y) \leq (e^{-tH^-}(x,y))^\nu (e^{-t(-\Delta \dot{+} \frac{\nu}{1-\nu}\beta V_0)}(x,y))^{1-\nu} ,$$
$$e^{t\Delta}(x,y) \leq (e^{-tH^+}(x,y))^{\nu_1} (e^{-t(-\Delta \dot{-} \frac{\nu}{1-\nu}\beta V_0)}(x,y))^{1-\nu_1}$$

which are valid for all $\nu \in]0,1[$ and $\nu_1 \in]0,(1+\beta)^{-1}[$ (see e.g. [MS]).

Corollary 1. *In the assumptions of Theorems 1 and 2 for any* $\varepsilon\in]0,\beta/2[$ *there are constants* $c^{\mp}(\varepsilon)>0$ *and* $c_{\mp}(\varepsilon)>0$ *such that, for all* $t>0$ *and* $x,y\in\mathrm{R}^d\backslash\{0\}$,

$$c^-(\varepsilon)t^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{c_-(\varepsilon)t}}\psi^+_{\hat\ell}(t,x)^{-1}\psi^+_{\hat\ell}(t,y)^{-1}\le e^{-tH^-}(x,y)$$
$$c^+(\varepsilon)t^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{c_+(\varepsilon)t}}\varphi^-_{\hat\sigma}(t,x)^{-1}\varphi^-_{\hat\sigma}(t,y)^{-1}\le e^{-tH^+}(x,y)$$

where $\hat\ell=\hat\sigma=\frac{d-2}{2}\left(\frac{\beta}{2}-\varepsilon\right)$.

The lower *on-diagonal* bounds can be improved considerably.

Corollary 2. *In the assumptions of Theorems 1 and 2 there are constants* $c^{\mp}>0$ *such that, for all* $t>0$ *and* $x\in\mathrm{R}^d\backslash\{0\}$,

$$c^-t^{-\frac{d}{2}}\varphi^-_{2\sigma}(t,x)\le e^{-tH^-}(x,x)$$
$$c^+t^{-\frac{d}{2}}\psi^+_{2\ell}(t,x)\le e^{-tH^+}(x,x).$$

Theorem 1 and Corollary 2 imply that the on-diagonal upper and lower heat kernel bounds are sharp.

The upper bounds from Theorems 1 and 2 can be supplied with the Gaussian factors.

Corollary 3. *In the assumptions of Theorems 1 and 2, for any* $c_{\mp}>4$ *there are constants* $c^{\mp}$ *such that, for all* $t>0$ *and* $x,y\in\mathrm{R}^d$,

$$e^{-tH^-}(x,y)\le c^-\varphi^-_\sigma(t,x)\varphi^-_\sigma(t,y)t^{-d/2}e^{-\frac{|x-y|^2}{c_-t}}$$
$$e^{-tH^+}(x,y)\le c^+\psi^+_\ell(t,x)\psi^+_\ell(t,y)t^{-d/2}e^{-\frac{|x-y|^2}{c_+t}}.$$

Our next result is in the framework of symmetric Markov semigroups.

Theorem C. *Let* $(M,d\mu)$ *be a measurable space with* σ *-finite measure. Let* A *be a selfadjoint bounded from below operator on* $L^2(M,d\mu)$ *such that the semigroup* e^{-tA} , $t>0$ *is positivity preserving. Also assume that*

C $_1$) *The bottom of the spectrum* $E:=\inf\sigma(A)$ *is an eigenvalue and the corresponding eigenfunction (ground state)* $\varphi\ge 0$ a.e. .

C $_2$) $Q_1(A-E)\subseteq L^{2j}$ *for some* $j>1$.

C $_3$) $1/\phi\in L^2_{\rm loc}$ *and* $c_1\phi^{-1}\le(c_2+A)^{\alpha/2}$ *(in the sense of the quadratic forms) for some*

constants $c_1 > 0$, $c_2 \geq -E$ *and* $\alpha > 0$.
Then, for all $t \in]0, T]$ *and a.e.* $x, y \in M$,

$$e^{-tA}(x, y) \leq c_T t^{-j'-\alpha}\phi(x)\phi(y) \ . \tag{6}$$

Also, for any $\varepsilon > 0$ *there exists a sufficiently large* T *such that the following two-sided inequality*

$$(1-\varepsilon)e^{-tE}\phi(x)\phi(y) \leq e^{-tA}(x, y) \leq (1+\varepsilon)e^{-tE}\phi(x)\phi(y) \tag{7}$$

holds for all $t \geq T$ *and a.e.* $x, y \in M$.

Theorem C can be viewed as a far reaching generalization of the well known bound

$$e^{t\Delta_\Omega}(x, y) \leq C_T t^{-1-\frac{d}{2}}\phi_0(x)\phi_0(y) \quad (0 < t \leq T)$$

for the Dirichlet operator $-\Delta_\Omega$ on a C^2 smooth bounded region $\Omega \subset \mathrm{R}^d, d \geq 3$ (see [Da]). In this case the assumption C $_2$) is valid for $j = \frac{d}{d-2}$ and is equivalent to Sobolev imbedding $W_0^{1,2}(\Omega) \subset L^{2j}(\Omega)$. Therefore, $E_0 := \inf \sigma(-\Delta_\Omega) > 0$ is the first simple eigenvalue, $-\Delta_\Omega\phi_0 = E_0\phi_0$, $\phi_0 \geq 0$. Thus C $_1$) is verified. The Hopf boundary lemma, i.e. $\phi_0 \geq c_0\delta(x)$ for some $c_0 > 0$ and $\delta(x) := \mathrm{dist}(x, \partial\Omega)$, together with the Hardy inequality $-\Delta_\Omega \geq c\delta^{-2}$ imply that C $_3$) holds with $c_2 = 0$ and $\alpha = 1$.

A more sophisticated example covered by Theorem C is the following. Again, let Ω be a C^2 smooth bounded region in R^d and let $0 \leq V \in L^1_{\mathrm{loc}}(\Omega)$ be form bounded with relative bound $\beta < 1, i.e. V \leq \beta(-\Delta_\Omega) + \hat{c}$. Due to the KLMN-theorem [Ka, Ch. VI] one can define the selfadjoint operator $H^- = -\Delta_\Omega \dot{-} V$ associated with quadratic form

$$h_-[f, g] := \langle \nabla f, \nabla g \rangle - \langle V^{1/2}f, V^{1/2}g \rangle \ , \quad \mathcal{D}(h_-) = W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \ .$$

The imbedding C $_2$) with $j = \frac{d}{d-2}$ $(d \geq 3)$ holds due to the definition of H^- and hence $E^- := \inf \sigma(H^-)$ $(> -\hat{c})$ is the first simple eigenvalue, e^{-tH^-} , $t \geq 0$ is postivity preserving and the ground state $\phi_- \geq 0$ on Ω , which proves C $_1$). Since $H^- + \hat{c} \geq (1-\beta)(-\Delta_\Omega) \geq (1-\beta)c\delta^{-2}$ and $e^{-E^-}\phi_- = e^{-tH^-}\phi_- \geq e^{\Delta_\Omega}\phi_- \geq \tilde{c}\delta$ we may conclude by making use of the Hopf lemma that C $_3$) holds with $\alpha = 1$, $c_2 = \hat{c}$ and $c_1 =$

$\tilde{c}^{-1}e^{-\hat{c}}\sqrt{(1-\beta)c}$. Thus, according to Theorem C, (6) holds for $A = H^-$ with $\alpha = 1$ and $j' = \frac{d}{2}$:

$$e^{-tH^-}(x,y) \le const_T t^{-1-\frac{d}{2}}\phi_-(x)\phi_-(y) \ . \tag{8}$$

Let us note that if $0 \le V$ belongs to the elliptic Kato class with the corresponding norm $\inf_{\lambda>0}\|(\lambda-\Delta_\Omega)^{-1}V\|_\infty < 1$, then ϕ_- is also bounded and, moreover, we can show that there is a constant $c > 0$ such that $c\phi_0 \le \phi_- \le c^{-1}\phi_0$ and hence from (8) we obtain a more valuable bound

$$e^{-tH^-}(x,y) \le c_T t^{-1-\frac{d}{2}}\phi_0(x)\phi_0(y) \ . \tag{9}$$

Also, the Gaussian factor $\exp\left(-\frac{|x-y|^2}{ct}\right), c > 0$ can be added to the R.H.S. of (9).

But this is not the case for form bounded potentials because this class contains fairly singular potentials such as $c_1\delta^{-2}(x) + c_2|x-x_0|^{-2}$, $x_0 \in \Omega$ with suitably small constants $c_i = c_i(\beta) > 0$. The best information about possible singularities of ϕ_- is this: $\phi_- \in L^p(\Omega)$ for any $p < p'(\beta) := \frac{d}{d-2}\cdot\frac{2}{1-\sqrt{1-\beta}}$ (see also [LS], [Se2]).

Now let us discuss the case of $H^+ = -\Delta_\Omega \dot{+} V$, $0 \le V \in L^1_{\text{loc}}(\Omega)$. Except for C$_3$) the assumption of Theorem C are satisfied for $A = H^+$. Indeed, since $e^{-tH^+}|f| \le e^{t\Delta_\Omega}|f|$, C$_2$) is trivially valid and hence $E^+ := \inf\sigma(H^+) > 0$ is the first simple eigenvalue and the ground state $\phi_+ \ge 0$ on Ω. Thus the only non-trivial hypothesis is C$_3$), because the inequality $\phi_+ \ge c\delta$ $(c > 0)$ is no longer available (though it does hold for the elliptic Kato potentials without any restriction on its Kato norm). But if C$_3$) holds, then one would have according to Theorem C the following bound:

$$e^{-tH^+}(x,y) \le C_T t^{-\alpha-\frac{d}{2}}\phi_+(x)\phi_+(y) \ . \tag{10}$$

In conclusion we remark on possible magnitude of the constant α from (10) and behaviour of ϕ_+ near the boundary.

Fix $x_0 \in \Omega$ and set $V_0 = \frac{(d-2)^2}{4}|x-x_0|^{-2}$. By the standard regularity theory the ground state ϕ_+ for $H^+ = -\Delta_\Omega \dot{+} \beta V_0$, $\beta > 0$ is a smooth function on $\Omega\backslash\{x_0\}$ behaving near x_0 like $|x-x_0|^\ell$, $\ell = \frac{d-2}{2}(-1+\sqrt{1+\beta})$. Its behaviour near the boundary is similar to ϕ_0. Thus $\alpha = \max(1,\ell)$. In general, however, the picture is not so simple. For instance, for $V_0(x)=\sum\limits_{i=1}^{\infty}\frac{c_i}{|x-x_i|^2}$ with suitably small c_i and $\mathrm{dist}(x_i,\partial\Omega)\to 0 \quad (i\to\infty)$ the boundary behaviour of ϕ_+ is quite different from that of ϕ_0 .

2. Proofs of Theorems A,B and C.

Our proofs of the theorems are built on an idea of J. Nash [Na].

Remark-Notation. Set $L^2_\varphi := L^2(M,\varphi^2 d\mu)$ and define the unitary mapping $\Phi : L^2 \to L^2$ by $\Phi f = \varphi f$. Then the operator $A_\varphi = \Phi^{-1}A\Phi$ of domain $\mathcal{D}(A_\varphi)=\Phi^{-1}\mathcal{D}(A)$ is selfadjoint on L^2_φ and $\|e^{-tA_\varphi}\|_{2\to 2,\varphi} = \|e^{-tA}\|_{2\to 2}\le 1$ for all $t\ge 0$. Here and below the subscript φ indicates that the corresponding quantities are related to the measure $\varphi^2 d\mu$.

Proof of Theorem A. Let $f=\varphi^{-1}h$, $h\in L^\infty_{\mathrm{com}}$, so that $f\in L^2_\varphi$. Let $u_t = e^{-t(A_\varphi+\nu)}f$. Then $\varphi u_t = e^{-t(A+\nu)}\varphi f$ and

$$\begin{aligned}\langle (A_\varphi+\nu)u_t,u_t\rangle_\varphi &= \|A^{1/2}e^{-t(A+\nu)}\varphi f\|_2^2+\nu\|e^{-t(A+\nu)}\varphi f\|_2^2\\ &\ge c_S\|e^{-t(A+\nu)}\varphi f\|_{2j}^2\\ &\ge c_S\|e^{-t(A+\nu)}\varphi f\|_2^{2(1+\frac{1}{j'})}\|e^{-t(A+\nu)}\varphi f\|_1^{-2/j'}\\ &= c_S\langle u_t,u_t\rangle^{1+1/j'}\|\varphi^{-1}\varphi e^{-t(A+\nu)}\varphi^{-1}\varphi^2 f\|_1^{-2/j'} \ ,\end{aligned}$$

where we have used $(1')$ and Hölder inequality.

By the definition of u_t , $-\frac{d}{dt}u_t = (A_\varphi+\nu)u_t$. Hence $-\frac{1}{2}\frac{d}{dt}\langle u_t,u_t\rangle_\varphi = \langle (A_\varphi+\nu)u_t,u_t\rangle_\varphi$. Setting $w:=\langle u_t,u_t\rangle_\varphi$ and using (4) we have

$$\frac{d}{dt}(w^{-1/j'})\ge \frac{2}{j'}c_S(c_0^{-1}\|\varphi e^{-t(A+\nu)}\varphi^{-1}\varphi^2 f\|_1)^{-2/j'} \ .$$

By our choice of f , $\varphi^2 f=\varphi h\in D$. Therefore we may apply (3) . It follows

$$\frac{d}{dt}(w^{-1/j'})\ge \frac{2}{j'}c_S\left(\frac{c_1}{c_0}\|f\|_{1,\varphi}\right)^{-2/j'} e^{t\nu 2/j'} \ .$$

Integrating this inequality over $[0,t]$, where $\varphi=\varphi_s$, $s\ge t$, gives

$$\|e^{-tA_{\varphi_s}}f\|_{2,\varphi_s}\le ct^{-j'/2}\|f\|_{1,\varphi_s} \ , \quad 0<t\le s \ .$$

Since $f\in\varphi^{-1}L^\infty_{\mathrm{com}}$ and $\varphi^{-1}L^\infty_{\mathrm{com}}$ is a dense subspace of L^1_φ , the last inequality yields

$$\|e^{-tA_{\varphi_s}}\|_{1\to 2,\varphi_s}\le ct^{-j'/2} \ , \quad 0<t\le s \ ,$$

and (5) follows.

Let us note that there is no connection between the above proof of Theorem A and the Beurling-Deny theory. Moreover, the assumption $A = A^*$ is not crucial for the result, though one would also have to assume (3) for e^{-tA^*} .

Proof of Theorem B. Setting $u_t = e^{-tA_{\psi_s}} f$, $f \in D_s$, we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\langle u_t, u_t\rangle_\psi &= \langle A_{\psi_s} u_t, u_t\rangle_\psi \\ &= \langle A^{1/2}\psi u_t, A^{1/2}\psi u_t\rangle \\ &\geq c_S \|\psi u_t\|_{2j}^2 \\ &\geq c_S \frac{\langle u_t, u_t\rangle_\psi^{2r}}{\|\psi u_t\|_q^{2(2r-1)}} \end{aligned}$$

where $q = \frac{2}{1+\varepsilon}$ and $2r = \frac{(1+\varepsilon)j-1}{j\varepsilon}$.

We have used above the imbedding $Q_0(A) \subseteq L^{2j}$, equivalent to ii), and then Hölder inequality. B $_3$) allows us to estimate $\|\psi u_t\|_q$ as follows

$$\begin{aligned} \|\psi u_t\|_q &= \|e^{-tA}\psi_s f\|_q = \|e^{-tA}|\psi_s|^{-\varepsilon}|\psi_s|^{2/q} f\|_q \\ &\leq \hat{c}_1 \|e^{-tA}\|_{q\to q}\|f\|_{q,\psi} + \||\psi_s|^{-\varepsilon}\|_{L^{q'}(\Omega^s)} \cdot \|e^{-tA}\|_{1\to q} \cdot \|f\|_{q,\psi} \\ &\leq (\hat{c}_1 c_1 + \hat{c}_2 c_2 (s/t)^{j'/q'})\|f\|_{q,\psi} \ . \end{aligned}$$

Setting $w := \langle u_t, u_t\rangle_\psi$ and using the last estimate, we have

$$\frac{d}{dt} w^{1-2r} \geq \frac{2c_S}{2r-1}(\hat{c}_1 c_1 + \hat{c}_2 c_2 (s/t)^{j'/q'})^{-2(2r-1)} \|f\|_{q,\psi}^{-2(2r-1)} \ .$$

Integrating this differential inequality yields

$$\|u_t\|_{2,\psi_s} \leq ct^{-j'\left(\frac{1}{q}-\frac{1}{2}\right)}\|f\|_{q,\psi_s} \ , \quad 0 < t \leq s \ . \tag{11}$$

Rewriting B $_2$) in the form $\|u_t\|_{1,\psi_s} \leq \tilde{c}_1 \|f\|_{1,\psi_s}$ and using (11) we obtain (see remark below)

$$\|u_t\|_{2,\psi_s} \leq ct^{-j'/2}\|f\|_{1,\psi_s} \ , \quad 0 < t \leq s \ ,$$

thus completing the proof of Theorem B.

Remark 1. *Let* $(P^t, t \geq 0)$ *be a semigroup on* $L^1 = L^1(M, d\mu)$ *. If, for some* $1 < q < 2$ *,* $\nu > 0$ *,* c_1 *and* c_2 *,*

$$\|P^t h\|_1 \leq c_1 \|h\|_1 \text{ and } \|P^t h\|_2 \leq c_2 t^{-\nu} \|h\|_q$$

for all $t > 0$ *and* $h \in L^1 \cap L^2$ *, then*

$$\|P^t h\|_2 \leq c t^{-\nu/(1-\varepsilon)} \|h\|_1 \ , \quad t > 0 \ , \quad h \in L^1 \cap L^2 \ ,$$

where $\varepsilon = 2/q'$ *,* $c = c_1 (2^\nu c_2)^{1/(1-\varepsilon)}$ *.*

Indeed, the semigroup property, the hypotheses and Hölder inequality imply

$$\begin{aligned}
\|P^{2t} h\|_2 &\leq c_2 t^{-\nu} \|P^t h\|_q \\
&\leq c_2 t^{-\nu} \|P^t h\|_2^\varepsilon \|P^t h\|_1^{1-\varepsilon} \\
&\leq c_2 c_1^{1-\varepsilon} t^{-\nu} \|P^t h\|_2^\varepsilon \|h\|_1^{1-\varepsilon}
\end{aligned}$$

and hence

$$(2t)^{\nu/(1-\varepsilon)} \|P^{2t} h\|_2 / \|h\|_1 \leq \hat{c} (t^{\nu/(1-\varepsilon)} \|P^t h\|_2 / \|h\|_1)^\varepsilon .$$

Setting $R_T := \sup_{t \in]0,T]} \ (t^{\nu/(1-\varepsilon)} \|P^t h\|_2 / \|h\|_1)$, one has $R_{2T} \leq \hat{c} R_T^\varepsilon$. But $R_T \leq R_{2T} \leq (2T)^{\varepsilon\nu/(1-\varepsilon)} (\|h\|_q / \|h\|_1)^\varepsilon$ so that $R_T \leq \hat{c}^{1/(1-\varepsilon)}$ and the required bound follows.

Assertions similar to that in Remark 1 are standard in the theory of elliptic operators of the second order (cf. [VSC, p.9]).

Proof of Theorem C. Denote by $\Phi f = \phi f$ the unitary map $\Phi : L^2_\phi \to L^2$. Set $\tilde{A} = \Phi^{-1}(A - E)\Phi$, $D(\tilde{A}) = \Phi^{-1} D(A)$. Since $\phi \in L^2$, one sees that $1 \in L^2_\phi$ and $e^{-t\tilde{A}} 1 = 1$, $t > 0$. Since $\phi \geq 0$ and e^{-tA} is positivity preserving, $e^{-t\tilde{A}}$ is positivity preserving. Therefore $e^{-t\tilde{A}}$ is a symmetric Markov semigroup. It is well known that the semigroups $(e^{-t\tilde{A}} | L^2_\phi \cap L^r_\phi)^{\text{clos}}_{L^r_\phi \to L^r_\phi}$ are strongly continuous on L^r_ϕ for all $1 \leq r < \infty$. The corresponding generators will be denoted by $-\tilde{A}_r$.

We will need the following general fact.

Proposition 1 [LS]. *Let* $(e^{-tB}, t \geq 0)$ *be a symmetric Markov semigroup acting on* $L^2(M, d\mu)$ *. If* $0 \leq u \in D(B_r)$ *for some* $r \in]1, \infty[$ *, then* $u^{r/2}, u^{r-1} \in D(B^{1/2})$ *and*

$$\langle B_r u, u^{r-1} \rangle \geq 4 \frac{r-1}{r^2} \|B^{1/2} u^{r/2}\|_2^2 .$$

Lemma 1. $\|e^{-t\tilde{A}}\|_{2\to 4,\phi} \le const_T t^{-(\alpha+j')(\frac{1}{2}-\frac{1}{4})}$ *for all* $0 < t \le T$.

Proof. Set $u_t := \exp[-t(\tilde{A} + E + c_2)]u_0$, $u_0 \in L^4_\phi$ where $c_2 \ge -E + 1$. Then $-\frac{d}{dt}u_t = (\tilde{A} + E + c_2)u_t$ and $-\langle \frac{d}{dt}u_t, u_t^3\rangle_\phi = \langle(\tilde{A}_4 + E + c_2)u_t, u_t^3\rangle_\phi$. By Proposition 1,

$$-\frac{d}{dt}\|u_t\|^4_{4,\phi} \ge 3\|(\tilde{A} + E + c_2)^{1/2}u_t^2\|^2_{2,\phi} \ .$$

Using that Φ is unitary and setting $w := \|u_t\|^4_{4,\phi}$ it follows

$$-\frac{d}{dt}w \ge 3\langle(A + c_2)^{1/2}\phi u_t^2 \ , \ (A + c_2)^{1/2}\phi u_t^2\rangle$$

(here we are using assumption C 2) and a choice of $c_2 \ge -E + 1$)

$$\ge 3c_S\|\phi u_t^2\|^2_{2j}$$

(here we are using Hölder inequality)

$$\ge 3c_S\frac{w^{1+1/j'}}{\|\phi u_t^2\|_1^{2/j'}} \ .$$

Thus

$$\frac{d}{dt}(w^{-1/j'}) \ge 3c_S(j')^{-1}\|\phi u_t^2\|_1^{-2/j'} \ .$$

By C 3) and the analyticity of e^{-tA} ,

$$\begin{aligned}\|\phi u_t^2\|_1 &= \langle e^{-t(A+c_2)}\phi u_0 \ , \ \phi^{-1}e^{-t(A+c_2)}\phi u_0\rangle \\ &\le c_1^{-1}\langle e^{-t(A+c_2)}\phi u_0 \ , \ (A + c_2)^{\alpha/2}e^{-t(A+c_2)}\phi u_0\rangle \\ &\le \text{const.}\, t^{-\alpha/2}\|\phi u_0\|_2^2 \ .\end{aligned}$$

Integrating the inequality

$$\frac{d}{dt}(w^{-1/j'}) \ge \text{const.}\, t^{\alpha/j'}\|u_0\|_{2,\phi}^{-4/j'}$$

over $[0, t]$ yields

$$w^{-1/j'} \ge \text{const.}\, t^{1+\alpha/j'}\|u_0\|_{2,\phi}^{-4/j'} \ ,$$

or, equivalently,

$$\|u_t\|_{4,\phi} \le ct^{-(\alpha+j')/4}\|u_0\|_{2,\phi} \ ,$$

which proves the lemma.

Next, Lemma 1 implies via duality that

$$\|e^{-t\tilde{A}}\|_{\frac{4}{3}\to 2,\phi} \le const_T t^{-(\alpha+j')(\frac{3}{4}-\frac{1}{2})} , \quad 0 < t \le T. \tag{12}$$

The ultracontractivity estimate

$$\|e^{-t\tilde{A}}\|_{1\to\infty,\phi} \le const_T t^{-\alpha-j'} , \quad 0 < t \le T$$

follows now from (12) and Remark 1 after the proof of Theorem B. Since $e^{-t\tilde{A}}(x,y) = e^{-t(A-E)}(x,y)\phi(x)^{-1}\phi(y)^{-1}$, the required in Theorem C bound (6) follows.

Finally, examining the above proof of (6) one easily obtains the following global in time estimate

$$\|e^{-t\tilde{A}}\|_{1\to\infty,\phi} \le c(\varepsilon)t^{-\alpha-j'}e^{\varepsilon(E+c_2)t}$$

valid for any $\varepsilon \in]0,1]$. Now the second assertion of Theorem C follows from this global bound and Theorem 4.2.5 in [Da].

3. m**-sectorial forms and contractivity criterions.**

Our proofs of Theorems 1 and 2 are based on some general facts concerning m-sectorial forms on the (complex) Hilbert space $L^2 = L^2(\Omega, dx)$, where $\Omega \subseteq \mathrm{R}^d$ is an open set, related to formal differential operators of the form $\varphi(-\Delta)\varphi^{-1}$.

Let $b : \Omega \to \mathrm{R}^d$ be a vector-valued function from $[L^2_{\mathrm{loc}}(\Omega)]^d$ such that, for some real constants $0 < \beta < 1$ and c_β,

$$\langle bh, bh\rangle \le \beta\langle \nabla h, \nabla h\rangle + c_\beta\langle h, h\rangle , \quad h \in C_0^\infty(\Omega) ,$$

or shortly

$$b^2 \le \beta(-\Delta_\Omega) + c_\beta . \tag{13}$$

Define a sesquilinear form t_b on L^2 by

$$t_b[u,v] = \langle \nabla u, \nabla v\rangle - \langle bu, bv\rangle + \langle \nabla u, bv\rangle - \langle bu, \nabla v\rangle ,$$
$$D(t_b) = W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) .$$

Set $t_b^*[u,v] := \overline{t_b[v,u]}$, $\mathrm{Re}t_b := \frac{1}{2}(t_b + t_b^*)$, $\mathrm{Im}t_b := \frac{1}{2\sqrt{-1}}(t_b - t_b^*)$. Then

$$\mathrm{Re}t_b[u,v] = \langle \nabla u, \nabla v\rangle - \langle bu, bv\rangle \ ,$$
$$\mathrm{Im}t_b[u,v] = \frac{1}{\sqrt{-1}}(\langle \nabla u, bv\rangle - \langle bu, \nabla v\rangle) \ ,$$

and hence

$$t_b = \mathrm{Re}t_b + \sqrt{-1}\mathrm{Im}t_b \ ,$$

where both forms $\mathrm{Re}t_b$ and $\mathrm{Im}t_b$ are symmetric.

Using (13) one easily concludes that the form t_b is m-sectorial and that the operator H_b associated with t_b has the following property:

$$(\lambda + H_b)^{-1} = B^{-1/2}(1+\sqrt{-1}G)^{-1}B^{-1/2} \ , \ \lambda > c_\beta \ , \tag{14}$$

where $B = \lambda - \Delta_\Omega \dot{-} b^2$ is the operator associated with $\mathrm{Re}t_b + \lambda$ and (with a minor abuse of notation) $G = -\sqrt{-1}B^{-1/2}(b\cdot\nabla + \nabla\cdot b)B^{-1/2}$ is a bounded symmetric operator (see [Ka, Ch. VI, Theorem 3.2]).

Let $b_n : \Omega \to \mathrm{R}^d$, $n = 1,2,\ldots,$ be another vector-valued functions such that $b_n^2 \leq b_{n+1}^2 \leq b^2$ a.e. and $b_n \to b$ a.e. as $n \to \infty$. Let H_{b_n} be the operator associated with t_{b_n} . Then

$$(\lambda + H_{b_n})^{-1} \ \underset{L^2}{\overset{s}{\longrightarrow}} \ (\lambda + H_b)^{-1} \text{ as } n \to \infty \tag{15}$$

(meaning a strong convergence in L^2).

The latter is a direct consequence of formula (14), assumption $b_n \to b$ a.e. and of the following fact:

$$B_n^{1/2}u \to B^{1/2}u \text{ strongly in } L^2 \text{ as } n \to \infty \tag{16}$$

for all $u \in \mathcal{D}(B^{1/2}) = \mathcal{D}(B_n^{1/2}) = W_0^{1,2}$, where $B_n := \lambda - \Delta_\Omega \dot{-} b_n^2$ (see [Ka, Ch. VIII, Theorem 3.11]).

In turn, (15) is equivalent to the convergence

$$e^{-tH_{b_n}} \ \underset{L^2}{\overset{s}{\longrightarrow}} \ e^{-tH_b} \text{ as } n \to \infty \tag{15'}$$

uniformly in $t \in [0,1]$ (see [Yo, Ch. IX, Sect. 12]).

Next, let $a : \Omega \to \mathrm{R}^d$ be a vector-valued function from $[L^2_{\mathrm{loc}}(\Omega)]^d$ such that pointwise a.e.

$$a^2 \leq (1-\varepsilon)W + \bar{c} \tag{17_1}$$

for some $W \in L^1_{\mathrm{loc}}(\Omega)$ and real constants $\varepsilon \in]0,1[$ and $\bar{c}$.

Define form $\tau[u,v]$ on $D \times D$, where $D := W^{1,2}_0 \cap \mathcal{D}(|W|^{1/2})$, by

$$\tau[u,v] = \langle \nabla u, \nabla v \rangle - 2\langle au, \nabla v \rangle + \langle W^{1/2}_{||} u \ , \ |W|^{1/2} v \rangle \ ,$$

where $W^{1/2}_{||} := |W|^{1/2}\mathrm{sgn}W$.

Using (17_1) we conclude that τ is m-sectorial with the vertex $\geq -\frac{\bar{c}}{1-\varepsilon}$ and $C^\infty_0(\Omega) \times C^\infty_0(\Omega)$ is a core of τ.

The following result is crucial for all subsequent considerations.

Proposition 2. *Let $\mathcal{J}$ denote the m-sectorial operator associated with τ. In addition to (17_1) assume that*

$$a^2 \leq \gamma(-\Delta_\Omega) + c_\gamma \tag{17_2}$$

for some real constants $\gamma < 1$ and c_γ. Let $\mathcal{V} \geq 0$ be a potential such that

$$W - \mathcal{V} \geq -\omega$$

pointwise a.e. for some real constant ω. Set $\mathcal{V}_m := \mathcal{V} \wedge m$, $m = 1, 2, \ldots$. Then

i) $(e^{-t(\mathcal{J}-\mathcal{V}_m)}, \ t \geq 0)$ *are postivity preserving semigroups.*

ii) For all $t > 0$ and $f \in L^1 \cap L^2$,

$$\|e^{-t(\mathcal{J}-\mathcal{V}_m)} f\|_1 \leq e^{t\omega}\|f\|_1 \ .$$

iii) $e^{-t(\mathcal{J}-\mathcal{V}_m)}$ *extends by continuity to a C_0 semigroup on $L^1(\Omega)$ for each m and strong $L^1 - \lim_m e^{-t(\mathcal{J}-\mathcal{V}_m)} =: e^{-t(\mathcal{J}-\mathcal{V})_1}$ exists and determines a C_0 semigroup of quasi contractions, i.e.*

$$\|e^{-t(\mathcal{J}-\mathcal{V})_1}\|_{1\to 1} \leq e^{t\omega} \ , \quad t > 0 \ . \tag{18}$$

Proof. We first *claim* that $(e^{-t\mathcal{J}}, t \geq 0)$ is positivity preserving and that $e^{-t\mathcal{J}^*}[L^2 \cap L^\infty] \subseteq [L^2 \cap L^\infty]$. One possible way to verify the claim is to make use of the following abstract criterions.

Criterion 1. *Let* $(e^{-tA}, t \geq 0)$ *be a* C_0 *semigroup of contractions on* $L^2(M, d\mu)$ *. Then it is positivity preserving if and only if it is real, i.e.* $e^{-tA}\mathrm{Re}L^2 \subseteq \mathrm{Re}L^2$ *, and*

$$\langle Af, f \vee 0\rangle \geq 0 \qquad for \quad all \quad f \in D(A) \cap \mathrm{Re}L^2 \ .$$

Criterion 2. *[BP]. Let* $(e^{-tA}, t \geq 0)$ *be a* C_0 *semigroup on* $L^2(M, d\mu)$ *. Then*

$$\|e^{-tA}h\|_\infty \leq \|h\|_\infty \qquad for \quad all \quad h \in L^2 \cap L^\infty \ and\ t > 0$$

if and only if

$$\mathrm{Re}\langle Af, f - f_\wedge\rangle \geq 0 \qquad for \quad all \quad f \in D(A),$$

where $f_\wedge := (|f| \wedge 1)\mathrm{sgn}f$, $\mathrm{sgn}f := \frac{f}{|f|}$ *if* $f \neq 0$ *and* $= 0$ *if* $f = 0$.

Using assumption (17_1) the proof of the claim based on Criterions 1 and 2 is straightforward.

Let us verify, for example, that $e^{-t\mathcal{J}^*}[L^2 \cap L^\infty] \subseteq L^2 \cap L^\infty$. Set $A = \mathcal{J}^* + \lambda$, $\lambda \geq \frac{\bar{c}}{1-\varepsilon}$, where $\bar{c}$ and ε are from (17_1). Let $f \in \mathcal{D}(A)$. Then $f \in W_0^{1,2}(\Omega)$ and, since $f - f_\wedge = [(|f| - 1) \vee 0]\frac{f}{|f|}$, also $f - f_\wedge \in W_0^{1,2}(\Omega)$. Therefore

$$\langle Af, f - f_\wedge\rangle = \langle \nabla f, \nabla(f - f_\wedge)\rangle - 2\langle \nabla f, a(f - f_\wedge)\rangle + \langle (W + \lambda)f, f - f_\wedge\rangle \ .$$

Setting $\chi := (|f| - 1) \vee 0 \equiv (|f| - 1)_+$ and using that $\mathrm{Re}(\bar{f}\nabla f) = |f|\nabla|f|$ it follows

$$\mathrm{Re}\langle Af, f - f_\wedge\rangle = \langle \nabla f, \frac{\chi}{|f|}\nabla f\rangle + \langle \nabla|f|, \nabla\chi\rangle - \langle \nabla|f|, \frac{\chi}{|f|}\nabla|f|\rangle$$
$$- 2\langle \nabla|f|, a\chi\rangle + \langle (W + \lambda)|f|, \chi\rangle \ .$$

Since $\langle \nabla f, \frac{\chi}{|f|}\nabla f\rangle - \langle \nabla|f|, \frac{\chi}{|f|}\nabla|f|\rangle = \langle \frac{\chi}{|f|}, \frac{(\eta\nabla\zeta - \zeta\nabla\eta)^2}{|f|^2}\rangle$, where $\zeta = \mathrm{Re}f$, $\eta = \mathrm{Im}f$, it follows using (17_1) that

$$\begin{aligned}\mathrm{Re}\langle Af, f - f_\wedge\rangle &\geq \langle \nabla\chi, \nabla\chi\rangle - 2\langle \nabla\chi, a\chi\rangle + \langle (W + \lambda)|f|, \chi\rangle \\ &= \langle \nabla\chi - a\chi, \nabla\chi - a\chi\rangle + \langle (-a^2 + W + \lambda)\chi, \chi\rangle + \langle (W + \lambda), \chi\rangle \\ &\geq 0 \ .\end{aligned}$$

In order to prove the assertion ii) of Proposition 2 set $f_t = e^{-t(\mathcal{J}^* - \mathcal{V}_m)}f$, where $0 \leq f \in L^2 \cap L^\infty$. Then applying the claim above yields $f_t \geq 0$ and $f_t \in L^\infty$. Therefore, since $f_t \in D(\mathcal{J}^*) \subseteq W_0^{1,2}$, it easily follows that f_t^{r-1} and $f_t^{r/2}$ are also in $W_0^{1,2}$ for all $r > 2$ and hence

$$
\begin{aligned}
-\frac{1}{r}\frac{d}{dt}\langle f_t^r \rangle &= \langle (\mathcal{J}^* - \mathcal{V}_m) f_t, f_t^{r-1} \rangle \\
&= 4\frac{1}{rr'}\langle \nabla f_t^{r/2}, \nabla f_t^{r/2} \rangle - \frac{4}{r}\langle a f_t^{r/2}, \nabla f_t^{r/2} \rangle - \langle (W - \mathcal{V}_m) f_t^r \rangle ,
\end{aligned}
$$

where $r' := \frac{r}{r-1}$. Setting $v := f_t^{r/2}$ and $J := \|\nabla v\|_2^2$, and using assumptions (17_2) and $W - \mathcal{V} \geq -\omega$, it follows

$$
-\frac{d}{dt}\|v\|_2^2 \geq -r\omega\|v\|_2^2 + 4\left(\frac{1}{r'}J - \frac{\gamma}{2\varepsilon_1}J - \frac{\varepsilon_1}{2}J - \frac{c_\gamma}{2\varepsilon_1}\|v\|_2^2\right) .
$$

Choosing $\varepsilon_1 = \sqrt{\gamma}$ it follows

$$
-\frac{d}{dt}\|v\|_2^2 \geq -\left(r\omega + c_\gamma\sqrt{\frac{1}{4\gamma}}\right)\|v\|_2^2 + 4\left(\frac{1}{r'} - \sqrt{\gamma}\right) J
$$

and, since $\gamma < 1$ for r large enough $\frac{1}{r'} - \sqrt{\gamma} > 0$, it follows

$$
-\frac{d}{dt}\|v\|_2^2 \geq -\left(r\omega + c_\gamma\sqrt{\frac{1}{4\gamma}}\right)\|v\|_2^2 .
$$

The latter yields

$$
\|f_t\|_r \leq e^{\left(\omega + \frac{c_\gamma}{r}\sqrt{\frac{1}{4\gamma}}\right)t}\|f\|_r .
$$

Letting $r \to \infty$ and using the continuity of $r \mapsto \|\cdot\|_r$, one has

$$
\|f_t\|_\infty \leq e^{\omega t}\|f\|_\infty
$$

which proves ii). Finally, assertion iii) follows from ii) by means of Fatou lemma.

4. Schrödinger semigroups on $\mathrm{R}^d, \quad d \geq 3.$

Remark - Definition of H^- *.* For $0 < \beta < 1$, define H^- to be the form sum $-\Delta \dot{-} V$. The latter definition is justified due to the famous Hardy inequality

$$
\|\nabla h\|_2^2 \geq \frac{(d-2)^2}{4}\||x|^{-1}h\|_2^2 , \quad h \in C_0^\infty(\mathrm{R}^d) .
$$

In this cases the hypothesis (1) holds because

$$
Q_0(H^-) = Q_0\big((\beta - 1)\Delta\big) \subset L^{2j} , \quad j = \frac{d}{d-2} .
$$

For $\beta = 1$ set $H^- := s - L^2 - R - \lim_{\beta \nearrow 1} H^-(\beta V_0)$ (the strong resolvent limit). The operator $H^- = H^-(V_0)$ is selfadjoint, non-negative and $C_0^\infty(\mathrm{R}^d)$ is dense in $Q_1\big(H^-(V_0)\big)$. Hypothesis (1) now holds using a Hardy type inequality due to Mazja [Ma, Section 2.1.6]

$$\|\nabla h\|_2^2 \geq \frac{(d-2)^2}{4} \||x|^{-1} h\|_2^2 + c\|h\|_{2j}^2 \ , \quad h \in C_0^\infty(\mathrm{R}^d)$$

with $c > 0$, $j = \frac{d}{d-2}$.

It is also clear that $(e^{-tH^-}, t \geq 0)$ is positivity preserving and symmetric.

Definition of desingularizing weights. For any $s > 0$ define weight $\varphi = \varphi^-(s,x) \equiv \varphi_\sigma^-(s,x)$ as a $C^2(\mathrm{R}^d \backslash \{0\})$ function $\varphi \geq 1/2$ such that $\varphi^-(s,x) = \left(\frac{\sqrt{s}}{|x|}\right)^\sigma$ for all $x \in B_{\sqrt{s}} := \{x \in \mathrm{R}^d \ : \ |x| \leq \sqrt{s}\}$, where $\sigma = \frac{d-2}{2}(1 - \sqrt{1-\beta})$, and $\varphi^-(s,x) = 1/2$ for all $x \in \mathrm{R}^d \backslash B_{2\sqrt{s}}$, and such that $1/2 \leq \varphi \leq 1$, $|\nabla \varphi| \leq \frac{c}{\sqrt{s}}$, $|\Delta \varphi| \leq \frac{c}{s}$ for $x \in B_{2\sqrt{s}} \backslash B_{\sqrt{s}}$.

Proof of Theorem 1. Due to the preceeding remark and the definition of weights in order to prove Theorem 1 it suffices to verify assumption (3) of Theorem A for $A = H^-$ and $\varphi = \varphi_\sigma^-(s,x)$.

We will first treat the case of $\beta < 1$. The case of $\beta = 1$ requires minor changes and we attend it at the end.

Define $b = \frac{\nabla \varphi}{\varphi}$, $\varphi = \varphi_\sigma^-(s,x)$. It follows from the definition of desingularizing weights that $b^2 \leq \beta V_0 + \frac{c_0}{s}$ for some real constant c_0 and all $s > 0$. Therefore

$$b^2 \leq \beta(-\Delta) + \frac{c_0}{s} \ .$$

For any $n \geq 1$ define

$$\varphi_n = \begin{cases} n & \text{if } \varphi \geq n \\ \varphi & \text{if } 1/n \leq \varphi \leq n \\ 1/n & \text{if } \varphi \leq 1/n \end{cases} \quad \text{and} \quad b_n := \frac{\nabla \varphi_n}{\varphi_n} \ .$$

Then $b_n \to b$ a.e., $b_n^2 \leq b_{n+1}^2 \leq b^2$ and hence, setting $H_0(\varphi_n) := H_{b_n}$, $H_0(\varphi) := H_b$, $(15')$ holds, i.e.

$$e^{-tH_0(\varphi_n)} \xrightarrow[L^2]{s} e^{-tH_0(\varphi)} \text{ as } n \to \infty \ . \tag{19}$$

Next, we claim that

$$\varphi_n e^{t\Delta} \varphi_n^{-1} = e^{-tH_0(\varphi_n)} \tag{20}$$

for all $n \geq 1$ and $t \geq 0$.

Indeed, $\varphi_n e^{t\Delta}\varphi_n^{-1}$ is a C_0 semigroup on $L^2 = L^2(\mathrm{R}^d, dx)$. Let F denote the negative of its generator. Then $\varphi_n(\lambda-\Delta)^{-1}\varphi_n^{-1} = (\lambda+F)^{-1}$ for any $\lambda > 0$. Set $u = (\lambda+F)^{-1}f$, $f \in L^2$. Since $\varphi_n^{-1}u = (\lambda-\Delta)^{-1}\varphi_n^{-1}f$, it follows $\varphi_n^{-1}u \in W^{2,2}$ and $(\lambda-\Delta)\varphi_n^{-1}u = \varphi_n^{-1}f$. Therefore

$$\langle(\lambda-\Delta)\varphi_n^{-1}u, \varphi_n v\rangle = \langle f, v\rangle \ , \quad v \in W^{1,2} \ .$$

Since $\varphi_n v \in W^{1,2}$, it easily follows from the last equality

$$\langle -\Delta\varphi_n^{-1}u, \varphi_n v\rangle = \langle \nabla\varphi_n^{-1}u, \nabla\varphi_n v\rangle$$

or, equivalently,

$$t_{b_n}[u,v] = \langle f - \lambda u, v\rangle \ .$$

Since $v \in W^{1,2}$ is arbitrary, it follows using the last equality and the definition of $H_0(\varphi_n)$ that $u \in \mathcal{D}(H_0(\varphi_n))$ and $H_0(\varphi_n)u = f - \lambda u$. Therefore $\mathcal{D}(F) \subset \mathcal{D}(H_0(\varphi_n))$ and $H_0(\varphi_n) \supset F$. But $-H_0(\varphi_n)$ and $-F$ are both the generators and hence $H_0(\varphi_n) = F$. Consequently (20) is proved.

Now let $f \in L^2$ and $g \in L^\infty_{\mathrm{com}}$. Then

$$\lim_n \langle \varphi_n e^{t\Delta}\varphi_n^{-1}f, g\rangle = \langle e^{t\Delta}\varphi^{-1}f, \varphi g\rangle$$

and by (19), $\langle e^{-tH_0(\varphi)}f, g\rangle = \langle e^{t\Delta}\varphi^{-1}f, \varphi g\rangle$. The latter shows that $e^{t\Delta}\varphi^{-1}f \in \mathcal{D}(\varphi) = \{h \in L^2; \varphi h \in L^2\}$ and that $\varphi e^{t\Delta}\varphi^{-1}f = e^{-tH_0(\varphi)}f$.

Hence the following representation formula holds:

$$e^{-tH_0(\varphi)} = \varphi e^{t\Delta}\varphi^{-1} \ , \quad t \geq 0 \ .$$

Since $V_m := (\beta V_0) \wedge m$, $m = 1, 2, \ldots,$ are bounded operators, we also have

$$e^{-t(H_0(\varphi)-V_m)} = \varphi e^{-t(-\Delta-V_m)}\varphi^{-1} \ , \quad t > 0 \ . \tag{21}$$

Next, consider the form $\tau[u,v]$ with $a = b \left(= \frac{\nabla\varphi}{\varphi}\right)$ and $W = \frac{-\Delta\varphi}{\varphi}$. Then $t_b = \tau$ on $C_0^\infty(\mathrm{R}^d) \times C_0^\infty(\mathrm{R}^d)$ and the latter is a core of t_b and τ. Therefore $\mathcal{J} = H_0(\varphi)$. Also, $W-$

$V_m \geq -\frac{c}{s}$, $\left(\frac{1-\sqrt{1-\beta}}{1+\sqrt{1+\beta}}\right) W \geq b^2 - \frac{c}{s}$. Applying Proposition 2 yields $\|e^{-t(H_0(\varphi)-V_m)}f\|_1 \leq e^{\frac{c}{s}t}\|f\|_1$, and due to (21)

$$\|\varphi_s e^{-t(-\Delta - V_m)}\varphi_s^{-1} f\|_1 \leq e^{\frac{c}{s}t}\|f\|_1 \ , \quad f \in L^1 \cap L^2 \ , \quad t > 0$$

where c is an absolute constant.

Finally, since $e^{-t(-\Delta - V_m)} \underset{L^2}{\overset{s}{\longrightarrow}} e^{-tH^-}$ and in fact $e^{-t(-\Delta - V_m)}|f| \nearrow e^{-tH^-}|f|$ a.e. as $m \to \infty$, it follows

$$\|\varphi_s e^{-tH^-}\varphi_s^{-1} f\|_1 \leq e^{c}\|f\|_1 \ , \quad f \in L^1 \cap L^2 \ , \quad 0 < t \leq s \ . \tag{22}$$

We may now apply Theorem A . This completes the proof of Theorem 1 in the case that $\beta < 1$.

The case of $\beta = 1$. Set $H^-_{(\varepsilon)} := -\Delta \dot{-} (1-\varepsilon)V_0$, $\varepsilon > 0$. Since now $|V_0 - b^2| \leq \frac{c}{s}$, the assumption (17_2) holds but with $\gamma = 1$, *namely* : $b^2 \leq -\Delta + \frac{c_0}{s}$. On the other hand the crucial estimate ii) of Proposition 2 holds for $f_t = e^{-t(\mathcal{J}^* - \mathcal{V}_m)}f$, because now $\mathcal{V}_m = (1-\varepsilon_0)V_0 \wedge m$, $W - \mathcal{V} \geq \varepsilon V_0 - \frac{c}{s}$, and hence $-\frac{d}{dt}\|f_t\|_r^r \geq -\frac{c}{s} r\|f_t\|_r^r$ for all r such that $\frac{1}{2r}\left(r' - \frac{1}{r'}\right) \leq \varepsilon$. Therefore $\|f_t\|_\infty \leq e^{\frac{c}{s}t}\|f\|_\infty$. The latter means that c in (22) does not depend on $\varepsilon > 0$. Finally, by the definition of H^- we have $e^{-tH^-_{(\varepsilon)}} \underset{L^2}{\overset{s}{\longrightarrow}} e^{-tH^-}$ (as $\varepsilon \searrow 0$) . Hence (22) also holds in the case that $\beta = 1$.

Proof of Theorem 2. Set $b = \nabla \log \psi_s$, $V = \beta V_0$, $\beta > 0$. Then $\mathrm{div} b = \frac{\Delta \psi_s}{\psi_s} - b^2$ and $V - b^2 \geq -\frac{c_1}{s}$, $V + \frac{-\Delta\psi_s}{\psi_s} \geq -\frac{c_2}{s}$, $s > 0$. Define the sesquilinear form t by

$$t[u,v] := \langle \nabla u, \nabla v\rangle + \langle (V - b^2)u, v\rangle + \langle \nabla u, bv\rangle - \langle bu, \nabla v\rangle \ , \ D(t) = W^{1,2} \times W^{1,2} \ .$$

It is easy to see that t is m-sectorial. Let $H^+(\psi)$ denote the operator associated with form t . Setting $B = \lambda - \Delta + (V - b^2)$, $\lambda > \frac{c_1}{s}$, $\mathcal{D}(B^{1/2}) = W^{1,2}$ and (with a minor abuse of notation) $G = -\sqrt{-1}B^{-1/2}(b \cdot \nabla + \nabla \cdot b)B^{-1/2}$ it follows

$$(\lambda + H^+(\varphi))^{-1} = B^{-1/2}(1 + \sqrt{-1}G)^{-1}B^{-1/2} \ .$$

Using this formula and an approximation argument similar to that in the proof of Theorem 1 it follows that

$$\psi e^{-tH^+}\psi^{-1} f = e^{-tH^+(\psi)} f \ , \quad f \in D_s = \psi_s L^\infty_{\mathrm{com}} \ .$$

Next we prove that

$$\|e^{-tH^+(\psi_s)}f\|_1 \leq e^{\frac{c}{s}t}\|f\|_1 \ , \quad 0 < t \leq s \ .$$

The latter follows by a straightforward verification of Criterion 2.

Indeed, let $A = H^+(\psi_s) + \lambda$, $\lambda \geq \frac{c_1 \vee c_2}{s}$. We have to show that $\mathrm{Re}\langle A^* f, f - f_\wedge\rangle \geq 0$ for all $f \in \mathcal{D}(A^*)$. Since $f \in \mathcal{D}(A^*) \subseteq W^{1,2} \Rightarrow f - f_\wedge \in W^{1,2}$, it follows

$$\mathrm{Re}\langle A^* f, f - f_\wedge\rangle \geq \langle \nabla\chi, \nabla\chi\rangle - 2\langle \nabla\chi, b\chi\rangle + \langle (V - b^2 - \mathrm{div} b + \lambda)|f|, \chi\rangle \ .$$

Using equality $-2\langle\nabla\chi, b\chi\rangle = \langle \chi, (\mathrm{div} b)\chi\rangle$ yields

$$\begin{aligned} \mathrm{Re}\langle A^* f, f - f_\wedge\rangle \geq & \langle \nabla\chi, \nabla\chi\rangle + \langle (V - b^2 + \lambda)\chi, \chi\rangle \\ & + \langle (V - b^2 - \mathrm{div} b + \lambda), \chi\rangle \\ \geq & \langle \left(\lambda - \frac{c_1}{s}\right)\chi, \chi\rangle + \langle \left(\lambda - \frac{c_2}{s}\right)\chi\rangle \\ \geq & 0 \ . \end{aligned}$$

The latter shows that in the case that $A = H^+$ the hypotheses B $_1$) and B $_2$) of Theorem B are valid.

We next fix $\varepsilon \in]0, \frac{d}{d+2\ell}[$ and set $\Omega^s := \{x \in \mathrm{R}^d ; |x| \leq \sqrt{s}\}$. Then by definition

(a) $\psi_s(x)^{-1} \leq 1$ for all $x \in \mathrm{R}^d \setminus \Omega^s$.

(b) $\|\psi_s(\cdot)^{-\varepsilon}\|_{L^{\frac{2}{1-\varepsilon}}(\Omega^s)} = c_d \left(\int_0^{\sqrt{s}} \left(\frac{|x|}{\sqrt{s}}\right)^{-\ell_\varepsilon \frac{2}{1-\varepsilon}} |x|^{d-1} d|x| \right)^{\frac{1-\varepsilon}{2}}$
$= c(d,\ell,\varepsilon) s^{\frac{d}{2}\frac{1-\varepsilon}{2}} = c(d,\ell,\varepsilon) s^{j'/q'}$, $j' = \frac{d}{2}$, $q' = \frac{2}{1-\varepsilon}$.

This verifies the hypothesis B $_3$) of Theorem B and completes the proof of Theorem 2.

We remark on the main difference between operators $H^-(\varphi)_r$ and $H^+(\psi)_r$: the generators $-H^+(\psi)_r$, $1 \leq r \leq 2$, are well defined, while $H^-(\varphi)_r$ make sense only for $r = 1$.

Proof of Corollary 1. The Trotter product formula and Hölder inequality imply that

$$e^{t\Delta}(x,y) = e^{-t(-\Delta - \nu V + \nu V)}(x,y) \leq (e^{-tH^-}(x,y))^\nu (e^{-t(-\Delta \dot{+} \frac{\nu}{1-\nu}V)}(x,y))^{1-\nu}$$

where $V = \beta V_0$, $0 < \beta \leq 1$ and for all $0 < \nu < 1$.

Applying Theorem 2 we have

$$e^{-t(-\Delta+\frac{\nu}{1-\nu}V)}(x,y) \le ct^{-d/2}\psi^+_{\ell_\nu}(t,x)\psi^+_{\ell_\nu}(t,y)$$

with $\ell_\nu = \frac{d-2}{2}(-1+\sqrt{1+\gamma\beta})$, $\gamma = \frac{\nu}{1-\nu}$. Therefore

$$e^{-tH^-}(x,y) \ge c_\nu \Gamma_{\nu t}(x,y)\big(\psi^+_{\ell_\nu}(t,x)\psi^+_{\ell_\nu}(t,y)\big)^{-\frac{1-\nu}{\nu}} .$$

Since $\psi^+_{\ell_\nu}(t,x)|B_{\sqrt{s}} = \left(\frac{|x|}{\sqrt{s}}\right)^{\ell_\nu}$, it follows $\big(\psi^+_{\ell_\nu}(t,x)\big)^{1/\gamma} = \psi^+_{\hat\ell}(t,x)$ on $B_{\sqrt{s}}$, where $\hat\ell = \ell_\nu/\gamma = \frac{d-2}{2}\frac{\beta}{1+\sqrt{1+\gamma\beta}}$ is a decreasing function of γ . This proves the first estimate in Corollary 1. A similar argument applies to $e^{-tH^+}(x,y)$.

Proof of Corollary 2. Let $\nabla^+ := -\nabla - 2b$, $b = \frac{\nabla\varphi}{\varphi}$ and $\tilde A_0 = (\nabla^+)\nabla$ be the selfadjoint operator associated with the closure of $a_0[u,v] = \langle \nabla u, \nabla v\rangle_\varphi$ initially defined on $C_0^\infty(\mathrm{R}^d)$. We will use the following representation of $\tilde H^- = \Phi^{-1}H^-\Phi$, where $\Phi f = \varphi f$ and $\varphi = \varphi^-_\sigma(s,x)$

$$\tilde H^- = (\nabla^+)\nabla + W, \quad W := \frac{-\Delta\varphi}{\varphi} - \beta V_0, \quad |W| \le \frac{c}{s}.$$

It follows from the Trotter product formula that pointwise a.e.

$$e^{-\frac{c}{s}t}e^{-t\tilde A_0}|f| \le e^{-t\tilde H^-}|f| \le e^{\frac{c}{s}t}e^{-t\tilde A_0}|f| \ for\ \ all\ t \le s.$$

Therefore, letting $p(t,x,y) = e^{-t\tilde A_0}(x,y)$, we obtain the following important bound

$$p(t,x,y) \le ct^{-\frac{d}{2}}, \quad 0 < t \le s. \tag{23}$$

In order to simplify the procedure below we reformulate the problem by working with regular weights and potentials by simply setting $\varphi(\sqrt{x^2+\mu})$ instead of $\varphi(x)$ and $\frac{-\Delta\varphi}{\varphi}$ instead of βV_0 . We then will obtain the required estimates with constants independent on $\mu > 0$, and will let μ tend to zero afterwards. Note that $p(t,x,y)$ and its time and spatial derivatives have regular behaviour. In particular, $p(t,x,y)$ not only satisfies (23) but also enjoys the *qualitative* Gaussian lower and upper bounds, $\langle p(t,x,\cdot)\rangle_\varphi = 1$, and weighted analogs $\mathcal{Q}$, $\mathcal{M}$ and $\mathcal{N}$ of Nash functions are well defined, namely:

$$\mathcal{Q}(t) := -\langle p\log p\rangle_\varphi \equiv -\int_{\mathrm{R}^d} p(t,x,y)\log p(t,x,y)\varphi^2(y)dy, \quad 0 < t \le s.$$

$$\mathcal{M}(t) := \langle |x-\cdot| p(t,x,\cdot)\rangle_{\varphi} \equiv \int_{\mathrm{R}^d} |x-y| p(t,x,y)\varphi^2(y)dy.$$

$$\mathcal{N}(t) := \langle (\nabla p)^2/p\rangle_{\varphi} \equiv \int_{\mathrm{R}^d} (\nabla_y p(t,x,y))^2/p(t,x,y)\varphi^2(y)dy.$$

Our main goal is to prove the Nash entropy estimate (NEE):

$$-C_- \leq \mathcal{Q}(t) - \tilde{\mathcal{Q}}(t) \leq C_+,$$

where $\tilde{\mathcal{Q}}(t) := \frac{d}{2}\log t$ and $C_{\mp}$ are constants independent on μ.

¿From (23) it follows that $\mathcal{Q}(t) \geq \tilde{\mathcal{Q}}(t) - C_-$ and hence we are left to prove only the upper bound. Following Nash [Na] we have

$$\tfrac{d}{dt}\mathcal{M}(t) = \langle |x-\cdot| \tfrac{d}{dt} p(t,x,\cdot)\rangle_{\varphi} = -\langle |x-\cdot| (\nabla^+)\nabla p\rangle_{\varphi} = \langle \nabla |x-\cdot|, \nabla p\rangle_{\varphi}$$

and hence $\frac{d}{dt}\mathcal{M}(t) \leq \sqrt{\mathcal{N}(t)}$. Since $\frac{d}{dt}\mathcal{Q}(t) = \mathcal{N}(t)$ and $\mathcal{M}(0) = 0$, we have

$$\mathcal{M}(t) \leq \int_0^t \sqrt{\tfrac{d}{d\tau}\mathcal{Q}(\tau)}d\tau.$$

We estimate the last integral by using Hölder inequality, integration by parts and the L.H.S. of (NEE) as follows

$$\int_0^t \sqrt{\tfrac{d}{d\tau}\mathcal{Q}(\tau)}d\tau \leq \sqrt{\int_0^t \tau^{-1/2}d\tau}\sqrt{\int_0^t \sqrt{\tau}d\mathcal{Q}(\tau)} \leq \sqrt{2t(\mathcal{Q}(t) - \tilde{\mathcal{Q}} + d + C_-)}.$$

Therefore,

$$\mathcal{M}^2(t) \leq 2t(\mathcal{Q}(t) - \tilde{\mathcal{Q}}(t) + C).$$

On the other hand $p\log p \geq -np - e^{-1-n}$ for all real n. Setting $n = m + k|x-\cdot|$ with $k > 0$ and integrating over spatial variables yields $\mathcal{Q}(t) \leq m + k\mathcal{M}(t) + e^{-1-m}\langle e^{-k|x-\cdot|}\rangle_{\varphi}$. Using the latter inequality, that $\langle e^{-k|x-\cdot|}\rangle_{\varphi} \leq C(k^{-d} + s^{d/2})$ and letting $m = C - d\log k$ and $k\mathcal{M} = d$, we obtain that $\mathcal{Q}(t) \leq C + d\log(\mathcal{M}(t) + \sqrt{s})$. For $s/2 \leq t \leq s$ it follows

$$e^{(\mathcal{Q}(t) - \tilde{\mathcal{Q}}(t))/d} \leq Ct^{-1/2}(\mathcal{M}(t) + \sqrt{s}) \leq C\sqrt{\mathcal{Q}(t) - \tilde{\mathcal{Q}}(t) + C}.$$

The latter yields the R.H.S. of (NEE).

In turn, the R.H.S. of (NEE), the reproductive property of $p(t,x,y)$ and Jensen inequality combined yield

$$p(2t,x,x) \geq e^{\langle p(t,x,\cdot)\log p(t,x,\cdot)\rangle_\varphi} = e^{-\mathcal{Q}(t)} \geq Ct^{-\frac{d}{2}},$$

or $e^{-2tH^-}(x,x) \geq C\varphi^-_{2\sigma}(t,x)t^{-\frac{d}{2}}$. Thus Corollary 2 is proven for e^{-tH^-}. A similar argument works for e^{-tH^+}.

Remarks. 1. As soon as (NEE) is obtained Corollary 3 can be proven by repeating the corresponding proof of the Gaussian upper bound in [Se2] for the "simplest" case of the uniformly elliptic operator $\nabla \cdot a \cdot \nabla$.

2. Due to Corollary 3 it becomes possible to exploit the L^1-perturbation techniques [Se3] and to establish weighted Gaussian upper heat kernel bound in the case of $-\Delta - \beta V_0 + a\cdot\nabla + V$, $\beta \leq 1$, with a and V from (the weighted) Nash and Kato classes respectively.

3. The problem of improving Corollary 1 remains open.

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