7th International Conference on the Anthropological Theory of the Didactic (CITAD7)

19-23 Jun 2022
Bellaterra, Barcelona
Spain

Table of contents

Entering into questioning the world: a case of teaching Mathematics with primary school teachers-students

Camila de Oliveira da Silva[1] and Marilena Bittar[2]

[1] Faculdade de Educação, Universidade Federal de Mato Grosso do Sul, Brazil; camimatt@gmail.com

[2] Programa de Pós-Graduação em Educação Matemática, Universidade Federal de Mato Grosso do Sul, Brazil; marilenabittar@gmail.com

This text concern the ongoing doctoral research, of which we present a first study and research paths (SRP) experienced by a group of teachers and future primary school teachers. Towards the paradigm of questioning the world, the entry into the research took place with a teaching case, supported by the studies of Judith Shulman. We will describe some contributions of this 'entry to the game', as well as didactic gestures that allow us to analyze some necessary changes in the transition of paradigms during the execution of this first SRP.

Este texto se refiere a una investigación doctoral, en curso, de la cual presentamos un primer recorrido de estudio e investigación (REI) experimentado por un grupo de profesores y futuros profesores de escuela primaria. Hacia el paradigma del cuestionamiento del mundo, el ingreso a la investigación se dio con un caso didáctico, sustentado en los estudios de Judith Shulman. Describiremos algunos aportes con esta 'entrada al juego', así como gestos didácticos que nos permitan analizar algunos cambios necesarios en la transición de paradigmas durante la ejecución de este primer REI.

*Ce texte fait référence à une recherche doctorale en cours, à partir de laquelle nous présentons un premier parcours d'études et de recherche (PER) vécu par un groupe de enseignants et futurs enseignants de l'école primaire. Visant le paradigme questionnement du monde, l'entrée dans la recherche s'est faite par un cas d'enseignement, appuyé sur les travaux de Judith Shulman. Dans ce texte nous décrivons quelques apports de cette « entrée en jeu », et des gestes didactiques qui permettent d'analyser quelques changements nécessaires dans la transition des paradigmes lors de l'exécution de ce premier PER.**Keywords: Type the keywords here, the first letter of the first keyword (only) is capital, there is a comma between keywords and a dot at the end. Use 3 to 5 Keywords. If possible, use reference keywords found on http://eric.ed.gov/?ti=all.*

Context and focus the research

This article refers to some questions of a doctoral research, in progress, within the problem of the teacher education who teach or will teach mathematics in the primary school.

In several countries, research is carried out with a focus on problems related to the teaching profession and, in a special way, on the mathematical training of the teacher, as in Cirade (2006), Bosch and Gascón (2009) and Ruiz-Olarría (2015). In Brazil, fragilities in mathematics training in several

Pedagogy courses have been highlighted by authors such as Curi (2004; 2020), Passos (2020) and Fiorentini (2002). Such research discusses how the graduates of these courses experience difficulties with basic mathematical and didactic knowledge for teaching, thereby reinforcing the need to seek for strategies beyond a curricular process which is apparently plastered by several institutional restrictions, since in most courses there is only one discipline dedicated to the teaching of mathematics. Such observations lead us to question about possible ruptures in this process: What training devices can be used to provide teachers-students with favorable conditions for teaching, in mathematics, since their university education?

Supported by the studies of the epistemological program in didactics of mathematics and, in a special way, the ATD, we seek elements that support our research hypothesis: a proposal based on the perspective of the paradigm questioning the world (PQW) can cause changes on praxeological mathematical and didactic equipment of those teachers-students.

The ATD has the potential to support a proposal that seeks to change paradigms, as described by Chevallard (2009). It is necessary to break with the paradigm of visiting works, which is a pedagogical one that is in line with a more traditional perspective of teaching, in which is the teacher who presents the entire material so that the student can get to know it. Thus, Chevallard proposes to move towards the PQW. In his view students collectively carry out an investigation based on a given question (proposed under the guide of the teacher or by the study community itself - students and teachers). Starting from a generating question (Q_0) that will bring about the study of different works, when accessing different media, it will be possible to raise different questions and partial answers to be validated from a relevant milieu, which will lead to the final answer (R♥). This process is materialized through study and research paths (SRPs), a didactic tool constituted by a model of questions and answers, characterized by the power of Q_0 to generative a durable study, in the emergence of new questions and answers to be investigated, and based on didactic gestures described as dialectic of investigation (CHEVALLARD, 2011). Allied to the didactic functions (mesogenesis, chronogenesis and topogenesis), the dialectics allow us to identify and describe possible changes that prevail in the functioning of the didactic system present in the execution of an SRP, as well as other theoretical elements that we deem necessary in the analysis of this process.

For the development of the research, we formed a group of approximately 22 volunteer participants, undergraduates and graduates in Pedagogy from all over the country. The meetings were online through the google meet platform and started in October 2021.

An entrance to question the world of Mathematics in the early years?

Q_0: a problematic situation - Alice's teaching case.

In order to build a path that could favor the entrance of the group participants in the experience of an SRP and that, at the same time, put on the scene, in a more 'open' way, some problems[1] that can be evidenced in the teaching profession, we elaborated a brief "fictional" story, having as character the teacher Alice.

> *Alice is a teacher in the early years whose main source of work is the textbook. She has always taught, without difficulty, the subject of even and odd numbers. However, in a class episode, trying to do something different, she takes to the room some packages containing several colored socks, in order to introduce the aforementioned concept. When the students are divided into groups, a group of students asks her if zero is odd or even, since all socks are different and do not form any pair. Another group working with an even number of socks, separated some socks, claiming that it would not be possible to pair with those socks. And in a third group, a visually impaired student claims, unlike his colleagues, that there was an even number of socks in that package, since when pairing up there were no socks left. Not knowing how to answer the students' questions and feeling desperate with such confusion generated in the class, the teacher is "saved" by the break bell. Due to the school routine, she cannot communicate with the coordination and with the other colleagues, and that is why she comes to us asking for help...*

Figure 1. Alice's teaching case synthesis

This was created in the light of the proposal of teaching cases (Shulman, 2002), which arise from the classroom experiences of teachers who participate in a training process. This author reinforces that in the art of this creation "ask teachers to focus on the surprises that occur in their classrooms rather than their celebrations, it can be quite threatening. But our goal is to make such surprises opportunities for learning, rather than occasions to avoid" (Ibid., p.4).

Therefore, we have Alice's teaching case as a problematic situation and identify it as Q_0 together with inquiries that allowed triggering a study proposal with the group of teachers-students: what elements are evidenced in Alice's case? How can we help her? This choice was made for two reasons. Firstly, because of its potential to contribute to the devolution of the problem to the group (as expressed by Brousseau), and to the necessary cognitive imbalance with mathematical, didactic, and pedagogical issues that were often implicit in the teaching case. Secondly, Q_0 had the potential to favor the path towards to the PQW, due to the questions it could raise and lead them to inquiry dynamic to help Alice. This first movement of chronogenesis took place through the drawing of a questions-answers map (Q-A map), a priori carried out by the researchers, which was confronted with the map a posteriori (Figure 2), based on the group's productions. We intended to work with the Q-A map as tool in teacher education, inspired by Florensa et al. (2021), which would allow us to discuss with teachers-students, its potential to interpret, plan and manage a given situation in the classroom, as well as to trigger possible learning with the objects of knowledge at stake.

[1] For example, the difficulties encountered with mathematical concepts in the exercise of the profession, the lack of dialogue between pedagogical coordination and teachers due to the dense school routine, among other aspects.

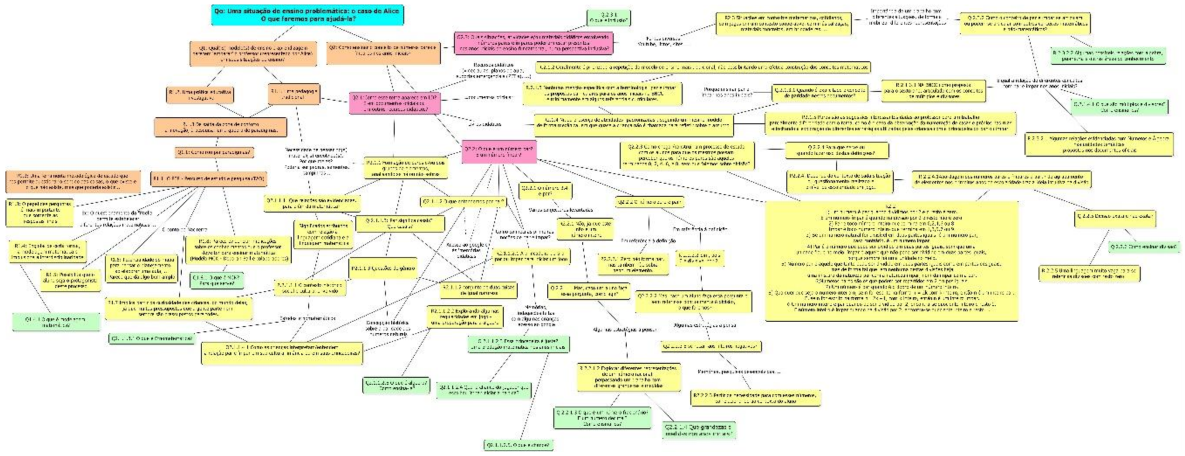


Figure 2. **Sketch (unfinished)[2] of the first Q-A map of the group**

The choice of odd and even numbers was made because it was a rich topic of work with teachers in previous experiences and because it was possible to start and complete this first SRP in just five sessions, time available before the school year ended.

The dynamics took place with the presentation of Alice's case, in the first meeting with the group, in the form of a story to be told. This allowed the researchers to trigger Q_0 together with inquiries: what elements are evidenced in Alice's case? How can we help her? With this, a dialogue was established about the teaching case and the participants were punctuating their ideas that moved from issues arising from the noosphere, which were of a curricular and school order, to the mathematical object present there. At the same time, the researchers raised some questions that invited everyone to contribute, such as: what do we mean by pair? What is an even number? Some participants spontaneously searched the internet for some elements of answers, as well as resorted to their didactic memories. Faced with the questions raised and the dialogue produced, one of the researchers, occupying the position of scribe, organized the issues at stake and in agreement with the participants, all of them chose to focus on the study of three 'big' questions so that we could, together, seek and validate responses obtained in different media that have already started to be present there. Thus, the participants were divided into three working groups, with the help of the researchers, to study the following questions arising around the teaching of this mathematical object in the primary school. They are: $Q_{2.1}$. How does this theme appear in textbooks? And in official documents or other teaching resources? $Q_{2.2}$. What is an even number? And an odd number? $Q_{2.3}$. What situations, activities and/or teaching materials involving even and odd numbers could be present in the early years of elementary school, from an inclusive perspective?

[2] Map produced with the questions and answers given by the whole group before the last work session, when some answers still needed to be resumed.

A little of this movement can be observed in Figure 3 which shows a zoom with a part of the map of questions and answers produced by the group. We express, in pink, the entry of derived questions studied in the three working groups.

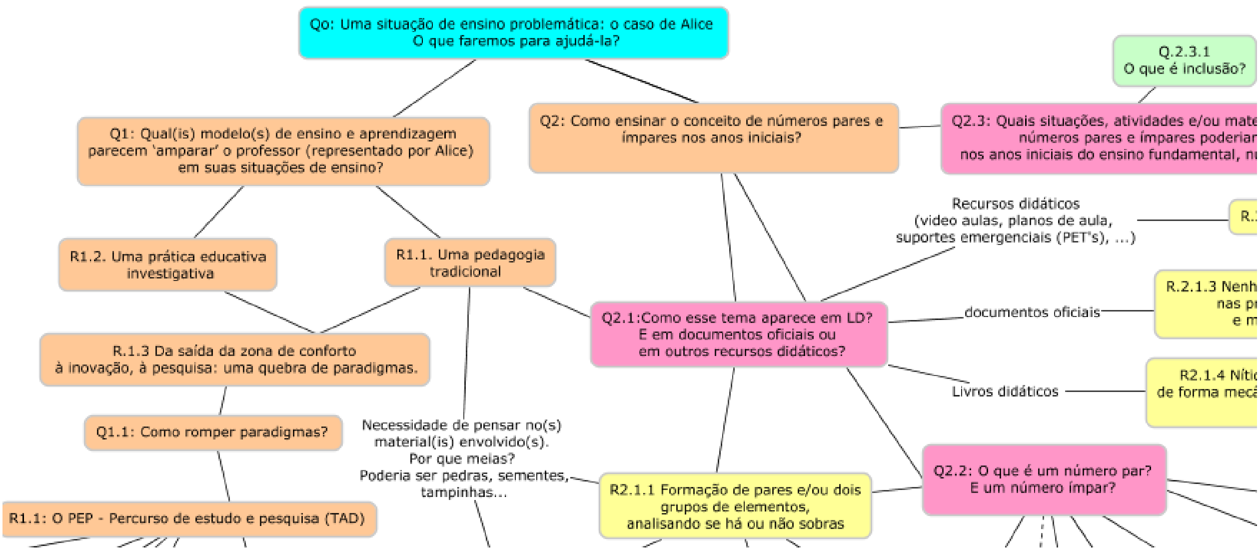


***Figure 3*. Part of the first Q-A map expressed in Figure 2**

On the other hand, the green color symbolizes the questions that appeared, but have not been studied, due to the time available to study this SRP. The questions marked in orange emerged by the researchers in the dialogue with the participants, the ones in yellow were produced collectively. In addition, the entry with the theme of even and odd numbers also shows us a range of possibilities with other mathematical themes that can be articulated and dedicated in a practice of study, in a non-fragmented way, as is commonly done nowadays.

In the group responsible for $Q_{2.1}$, we observed a significant change in terms of topogenesis in relation to the individual-collective dialectic. This is because they assumed a greater responsibility of study and investigation to answer the question posed, also enabling them with a certain vision of a skydiver when studying how the theme was officially proposed throughout the primary school (6 to 10 years) till the beginning of middle school (11 years).

As for aspects of mesogenesis, the main media accessed by participants came from internet sites, but also from video classes, papers and textbooks. A case to consider is when participants seek to understand whether or not the number 3,4 would be even. As a result of research by the group that investigated $Q_{2.2}$, a small text appears from a renowned university in Brazil, in which there is a definition and explanation of an even number, but which does not explicitly include the term: "interger number". The dialogue established in the group showed that the praxeological equipment of the participants was insufficient to understand that the text read was linked to the theory of numbers which, in turn, deals only with whole numbers.

It is at this point when we look at the evolution of the milieu in the work of participants, in the media-milieu dialectic, and we ask ourselves: how can they assess the reliability of a given media, if they still do not have enough elements in their praxeological equipment to allow such an evaluation? Here we note that the attention given to the role of the researcher is essential, since the constituted milieu will not always be enough to question it. It is not a matter of evaluating the media for the group, but of mediating, with questions, what is "found".

In line with the reception-diffusion dialectic of the study process carried out by the working group, we have been asking ourselves: How to carry out a collective study with the Q-A map tool, in an inclusive way? This has been a challenge in the research, since one of the participants is visually impaired. We understand that restrictions arising especially from the levels of Society and Pedagogy.

Discussion

The teaching case proved to be relevant to provoke cognitive imbalance in teachers-students, allowing them to enter the game of knowledge production. This first path, which was collectively constructed by the group's participants, showed potential for us to gradually advance towards PQW. Furthermore, this case potentiated the entry into the study dynamics of an unfinished SRP, since Alice's case emanated a discussion with several issues related to the teaching profession.

However, we are still asking ourselves about a reality that is very present in this research and in Brazilian society as a whole: how to favor the reception-diffusion of knowledge, without causing micro-exclusions, to teachers-students with visual impairment, having the production of Q-A maps as a training tool? Thus, between potentialities and challenges, the story continues with other SRPs being covered by the group.

References

Bosch, M. & Gascón, J. (2009). Aportaciones de la teoría antropológica de lo didáctico a la formación del profesorado de matemáticas de secundaria. In M.J. González, M.T. González & J. Murillo (Eds.), *Investigación en educación matemática* XIII (pp. 89-113). Santander: SEIEM.

Cirade, G. (2006). *Devenir professeur de mathématiques: entre problèmes de la profession et formation en IUFM*. Les mathématiques comme problème professionnel. 453 f. (Doctoral Tesis). Université Aix-Marseille I.

Chevallard, Y. (2009). La notion de PER: problèmes et avancées. *Conférence à l'IUFM de Toulouse 28/04/09*. http://yves.chevallard.free.fr.

Chevallard, Y. (2011). *Éléments de didactique du développement durable*. Enseignement donné en 2010-2011 en licence de sciences de l'éducation. Université de Provence. http://yves.chevallard.free.fr.

Curi, E. (2020). O conhecimento do professor para ensinar Matemática nos anos iniciais: indicações de pesquisas, reflexões e desafios. In Jr Traldi, A., Tinti, D.S., & Ribeiro, R.M. Formação de professores que ensinam matemática: processos, desafios e articulações com a educação básica. *Sociedade Brasileira de Educação Matemática – Regional São Paulo 2020*, São Paulo.

Fiorentini, D., Nacarato, A. M., Ferreira, A. C., Lopes, C. S., Freitas, M. T. M., & Miskulin, R. G. S. (2002). Formação de Professores que ensinam matemática: um balanço de 25 anos da pesquisa brasileira. *Educação em Revista* (pp. 137-160). Belo Horizonte: UFMG.

Florensa, I., Bosch, M. & Gascón, J. (2021). Question–answer maps as an epistemological tool in teacher education. *Journal of Mathematics Teacher Education*, 24(2), 203–225.

Passos, C.L.B. (2020). Formação matemática em um curso de Pedagogia:

dilemas e perspectivas. In Jr Traldi, A., Tinti, D.S., & Ribeiro, R.M. Formação de professores que ensinam matemática**:** processos, desafios e articulações com a educação básica. *Sociedade Brasileira de Educação Matemática – Regional São Paulo 2020*, São Paulo.

Ruíz-Olarría, A. (2015). *La Formación Matemático-Didáctica del Profesorado de Secundaria*. De las Matemáticas por Enseñar a las Matemáticas para la Enseñanza. 372 f. (Doctoral Tesis). Universidad Autónoma de Madrid.

Shulman, J. H (2002). Happy Accidents: cases as oportunities for teacher learning. Annual Meeting of the American Educational Research Association, New Orleans.

How can ATD concepts contribute to teacher training in the face of the changes caused by the Covid-19 pandemic?

José Vieira do Nascimento Júnior[1], Saddo Ag Almouloud[2], Edmo Fernandes Carvalho [3], and Viviana Oliveira Mateus[4]

[1]Universidade Estadual de Feira de Santana, Brasil; jvnjunior@hotmail.com

[2]Universidade Federal da Bahia, Brasil; saddoag@gmail.com

[3]Universidade Federal do Oeste da Bahia, Brasil; edmofc@gmail.com

[4]Colégio Estadual Polivalente de Camaçari, Brasil; vivianaoliveira2301@gmail.com

Keywords: Teacher training, chemical kinetics, pandemic.

Introduction

This article is a contribution that arises from the need to train chemistry teachers in a scenario of changes in the educational system on a global level caused by the effects of the Covid-19 pandemic, with emphasis on the mandatory closure of schools, which anticipated full remote teaching.

The investigation occurred during a school semester in a chemistry teacher training class at the State University of Feira de Santana, in northeastern of Brazil, during the pandemic, where an epistemological reference model was applied to the topic of chemical reaction kinetics, for training in the year 2021. The observations regarding the proposed activities were analyzed in the light of concepts from the Anthropological Theory of Didactics (ATD). This was done through the analysis of theoretical-experimental tasks in a predominantly remote environment and according to the notion of webbing (Trouche & Drijvers, 2014).

The didactic modeling applied was in terms of the theoretical proposals of Chevallard (2002, 2009, 2020) and Barquero et al. (2013). Mediation involved resources from the Study and Research Paths (SRP) methodology. The notion of praxeology (t,τ,θ,Θ) and praxeological equipment were also introduced for the analyses between the model interaction and the subjects (future teachers), regarding the fulfillment of the tasks.

Objective

Train chemistry teachers by articulating teaching in kinetics and web resources.

Route Description

The choice of tasks was guided by the scientific knowledge that tells us the answer to the initial question, which is in the field of kinetics, *i.e.* Q_0: "how does the purity of a material vary in time?" Questions such as "how to study kinetic laws? “How to analyze the experimental data and promote the process of webbing in class?” also guided our investigation.

Methodology

The research took place in two moments, between the years 2019 and 2020, over the course of one semester, in theoretical-practical classes, with seven (07) participants. In the pandemic period the remote classes took place via Google Meet and www.schoology. The software www.titrAB.fr mediated the study of the topic kinetics of the chemical equilibrium. Other resources employed were internet access, Office 2019, data-show, and pH glassware potentiometer – along the face-to-face stage. The research method adopted was based on the construction and application of tasks, and observation of the data generated.

The fundamental concepts, principles, and formalism of kinetics were presented throughout the encounter with the tasks. They were revisited when studying the techniques and exploring the technological-theoretical environment. The work with the different techniques in different tasks was performed by the students as they exercised the ability to elaborate questions and answers around Q_0.

The formalism involved ordinary differential equations (ODEs), whose variables are species concentration and time, see table 1.

Table 1: Ordinary differential equations and Kinetic Parameters

Ordinary differential equations	Kinetic Parameters
$\left(\frac{C_A}{C_{A,0}}\right)^{-n+1} = 1 + {C_{A,0}}^{n-1}(n-1)kt \quad for\ n \neq 1$ $lnc_A = lnc_{A,0} - kt \quad for\ n = 1$	$n: reaction\ order$ $C_A: concentration\ of\ species$ $c_{A,0}: cinitial\ concentration\ of\ species$ $t: time$

Results

Four participants accessed the online classes via smart-phones and three via PCs. At the first moment the provisional answers, $R^{\lozenge}$, indicated different paths to the answer $R^{\heartsuit}$.

The next moment a group of three students proposed question q_1 derived from the initial question from internet searches: "According to health agencies, the shelf life of a medicine corresponds to a purity of 90%. According to the kinetic data for the decomposition of *Paracetamol* ®, in table 2, determine the shelf life of this drug in months. The objective of this activity was to develop in the students, praxeologies linked to the theme through the ability to elaborate questions that help to find the answers $R^{\heartsuit}$ to question Q_0, thus improving its praxeological equipment - bringing into play relevant aspects of the object such as possible techniques, seen below, from the dialectic of questions and answers.

Table 2: kinetics of decomposition of pure paracetamol

Time/ month	C_A/mmolL^{-1}	Ln C_A	$(1/C_A)/(mmol^{-1}L)$

0	2,719	1,000	0,368
1	2,612	0,960	0,383
2	2,586	0,950	0,387
3	2,509	0,920	0,399
4	2,459	0,900	0,407
10	2,138	0,760	0,468
15	1,855	0,618	0,539
20	1,664	0,509	0,601
25	1,448	0,370	0,691
30	1,276	0,244	0,784

Source: (Ball 2005, p. 724).

Among the answers of two students for q_1, we highlight those represented in figure 1, constructed on graph paper.

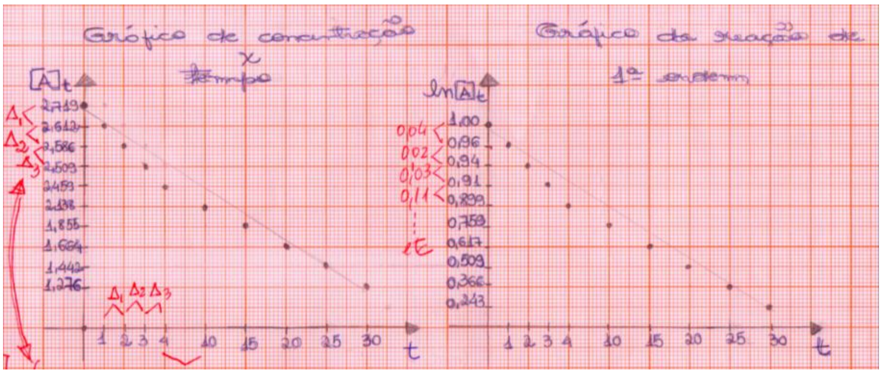


Figure 1: Student's answer to q_1

The representation in figure 1 was compared with the representation prepared by the teacher, obtained with the help of the Excel application, in figure 2, which follows.

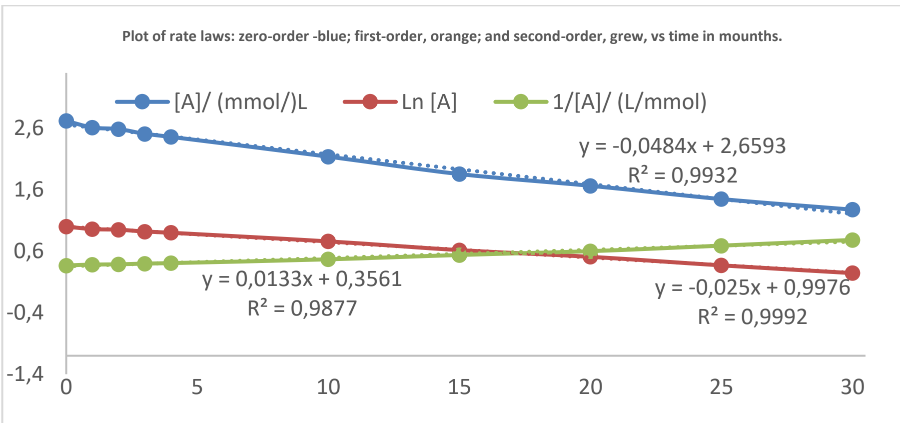


Figure 2: Teacher's answer to q_1

Comparing figures 1 and 2 it is possible to notice gaps in the level of students' representations in solving q_1. The notion of scale was not applied in the students' treatment. The students' representation shows that their relationship with objects involving these graphical elements (dimension, scale, quantities, magnitudes, and units) lacks more mastery - they have not yet internalized the elements of a graphical representation, which has implications for the algebraic treatment, although the relationship with the object is not empty $R(u, o) \neq \emptyset$, in this case.

In the search for $R^{\heartsuit}$ the students with the help of the teacher formulated more derived questions besides q_1; we elaborated $q_{2,}$ $q_{1,1}$, $q_{1,2}$,etc.

Table 3: Derivatives questions of the path

$q_{1,1}$: how do you define and determine the order (n) of the decomposition reaction of Paracetamol?
$q_{1,2}$: what is the physical meaning of the speed constant (k)?
q_2: how can you determine the rate constant of the reaction?
$q_{2,1}$: how can i determine the shelf life of this drug t90, in months, from the values of k and n?

The answers to $q_{1,1}, q_{1,2}, q_2$, and $q_{2,1}$ were later incorporated in the construction of a question-answer map, indicating how to get to the answer $R^{\heartsuit}$ to Q_0.

This collective construction helped in the construction of tasks whose answers served as elements of analysis and theoretical reflection about the evolution of the students' object universe throughout the semester. Next, we present praxeological elements of the activities in the form of praxeological elements formulated in the process, see table 4.

Table 4: Tasks and their respective techniques

<table>
<tr><td>t_a</td><td>Demonstrate how velocity and temperature relate in reaction below. Follow the kinetic data from task t_a.
$\text{HO}\bullet + \text{CH}_2\text{BrCl} \rightarrow \text{Produto}$
<table>
<tr><td>Ensaio</td><td>$T/^oC$</td><td>$K10^{13}$ $/cm^3 moléculas^{-1}s^{-1}$</td></tr>
<tr><td>1</td><td>393,6</td><td>6,91</td></tr>
<tr><td>2</td><td>127,0</td><td>2,54</td></tr>
<tr><td>3</td><td>24,6</td><td>1,11</td></tr>
</table></td></tr>
<tr><td>$t_{a,1}$
τ_a e τ_E</td><td>Calculate the value of the Activation Energy of the reaction. Exploring the obtained line. Use the algebraic and Excel techniques: τ_a e $\tau_{Excel(E)}$.</td></tr>
<tr><td>$t_{a,1,1}$
τ_E</td><td>Organize a table containing the variables lnk e $1/T$.</td></tr>
</table>

$t_{a,1,2}$	From the graph in Excel obtain the equation of the line, its angular coefficient, and linear coefficient.
t_b	From the graph in Excel estimate the values of the activation energy and the pre-exponential Arrhenius factor (A).
$t_{b,1}$ τ_a	determine the rate constant of the radical hydroxylation reaction of *chlorobromomethane* at 370 K from the kinetic data. Here the algebraic technique is the only possible one - we have a punctual praxeology.
t_c	Build an acid-base titration curve on www.titrAB.fr and compare it with the curve made in Excel from empirical data. Compare the two techniques: the titrAB - τ_{TIT} -, and Excel - $\tau_{Excel(E)}$.

Praxeology of the task t_c

Among the tasks we highlight the genuinely technological one, task t_c. Its technique, by simulation in titrAB - τ_{TitrAB} was built in this application. The acid-base curve and other data from the titrAB.fr application page can be found in Figure 3. From the analysis of the curves obtained in situ, in Excel, and in the titrAB software - different techniques - it was possible to establish distinct technologies for each. The two techniques [τ_{TitrAB} and τ_{Excel}], employed in the fulfillment of t_c reveals at least punctual praxeologies: [$t_1, \tau_{TitrAB}, \theta_1, \Theta$] and [$t_1$, τ_{Excel}, θ_1, Θ], or a local one: [$t_1, \tau_{1i}, \theta_1, \Theta$] (ALMOULOUD, 2015).

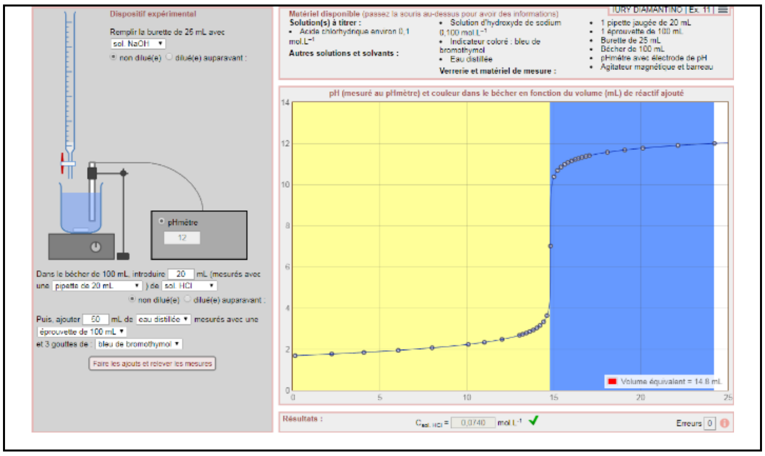


Figure 3: Simulation of titration in the titrAB between acetic acid, CH_3COOH and NaOH

From task t_c it was also observed that students mobilized two techniques: one online in titrAB and one volumetric. To perform the latter, the simulator provided information that enabled the preparation of the experiment protocol. In this sense, prior knowledge of chemistry such as the law of dilution of titration and equivalence were mobilized. This was evident in the reports - in the pre-calculation - when they used the relation $C_1V_1 = C_2V_2 = k\ (constante)$;[C = molar concentration; V = volume;

1: acid solution; 2: basic solution], besides expressing themselves using concepts from the acid-base theories (Arrhenius, Bronsted-Lowry, Lewis). These findings reveal that there were articulations between the theoretical [θ/Θ] and technical [t/τ] blocks in performing tc. The presence of concepts and definitions such as acidic and basic strength, *p*H and concentration, and the notion of chemical equivalence point was in 6/7 of the responses. This evidences that the τ_{TitrAB}technique favored access to these theoretical elements.

Final considerations

Among the results of the course we highlight that webbing was crucial to continue the training of chemistry teachers during the Covid-19 pandemic. Although there were restrictions on non-face-to-face teaching, precarious access of students to the content and limitations due to the lack of face-to-face communication, an evolution in the praxeological equipment of the participants was verified at the end of the course. This was evidenced by the articulations between the practical and theoretical blocks, revealed in the answers and argumentations in the fulfillment of the tasks, in addition to their mobilization and their involvement in the dialectic of questions and answers.

The didactic organization proposed allowed the participants to reflect on the objects starting from a question Q_0, not always easy to be solved, resorting to revisions, deepening, raising questions related to the question, searching for the answer in several sources, mobilizing several web resources, building techniques that lead to the answer $R^{\heartsuit}$, without losing sight of the totality of the curriculum.

It was possible to highlight some gaps that will serve as a basis for future training paths and professional practice: students were sometimes unfamiliar with this proposed model - ecologically unusual, from a "question of the world". Here the most important thing was not to use the techniques per se, but to prepare them to develop praxeological organizations and didactic praxeologies around chemical kinetics.

In this sense, certain concepts from ATD such as praxeology, praxeological equipment and resources of SRP were essential to understand how training in chemistry can occur, even under adverse conditions as in the pandemic.

References

Almouloud, S. A. (2015). Teoria Antropológica do didático: metodologia de análise de materiais didáticos. *Unión: Revista Iberoamericana de Educación Matemática.* Ebro, Espanha: , n. 42, p. 09-34.

Ball, D. W. (2005). *Físico-Química*. São Paulo: Pioneira Thomson Learning, ISBN 85-221-0417-5, 2005.

Barquero, B. F., Bosch, M. Gascón, J. (2013). Las tres dimensiones del problema didáctico de la modelización matemática. *Educ. Matem. Pesq.*, São Paulo, v.15, n.1, pp.1-28.

Chevallar, Y. (2009). La notion de PER: problèmes et avancées, In: *<http://yves.chevallard.free.fr/spip/spip/IMG/pdf/La_notion_de_PER_problems_et_avancees.pdf>*.

Chevallard, Y. (2002) Organiser l'étude. Cours 3 - Structures & Fonctions. *Actes de la XIième Ecole d'été de didactique des mathématiques*. Grenoble, La Pensée Sauvage

Chevallard, Y. (2020). Some sensitive issues in the use and development of the anthropological theory of the didactic. *Educ. Matem. Pesq***.**, São Paulo, v.22, n. 4, pp. 013-053. Access by http://dx.doi.org/10.23925/1983-3156.2020v22i4p013-053.

Trouche, L; Drijvers, P. (2014). Webbing and orchestration. Two interrelated views on digital tools in mathematics education. Teaching Mathematics and Its Applications. Oxford, v. 33, 193-209, 2014.

Is this object mathematical, para-, or proto-? A didactic application of the epistemological profile

Koji Otaki[1] and Hiroaki Hamanaka [2]

[1]Hokkaido University of Education, Japan; otaki.koji@k.hokkyodai.ac.jp

[2] Hyogo University of Teacher Training, Japan

Abstract. *We propose here a methodological tool for creating and elaborating our own reference epistemological models of different works involved in mathematical activities. It is based on a fundamental distinction of objects of mathematics introduced by the didactic transposition theory: the mathematical, the paramathematical, and the protomathematical. According to our initial hypothesis, any work in mathematics can possess multiple statuses for instances observing it who can be persons, positions, and institutions, e.g., "mainly mathematical but paramathematical just a little". Such possible multiplicity of works can be clarified through an application of Bachelard's idea of epistemological profile.*

Resumen. *Proponemos aquí una herramienta metodológica para crear y elaborar nuestros propios modelos epistemológicos de referencia de diferentes obras involucradas en actividades matemáticas. Se basa en una distinción fundamental de objetos de las matemáticas introducida por la teoría de la transposición didáctica: le matemático, le paramatemático y le protomatemático. De acuerdo con nuestra hipótesis inicial,, cualquier obra matemática puede poseer múltiples estatus para las instancias que la observan, que pueden ser personas, posiciónes e instituciones,, por ejemplo, "principalmente matemático pero solo un poco paramatemático". Tal posible multiplicidad de obras puede esclarecerse mediante una aplicación de la idea bachelariana de perfil epistemológico.*

Keywords: paramathematical object, protomathematical object, instantial relation, judgement

Introduction: on three open concepts

This poster presentation aims to provide a theoretical resource for accomplishing epistemological analysis of mathematical knowledge, that is, creating, elaborating, discussing, and criticizing different reference epistemological models. Such a resource is related to a typology of "mathematical" objects—which seems to be often forgotten—introduced in the didactic transposition theory: *mathematical*, *paramathematical*, and *protomathematical* (Chevallard, 1985/1991). We used this typology in a paper presented in the last CITAD (6th, 2018) for clarifying the process of the growing of logical knowledge in mathematical activities (Hamanaka & Otaki, 2020). On the one hand, we are aware of the usefulness of this distinction of objects through our own researching experience. In our view, the notions of protomathematical and paramathematical objects become more and more important in didactics involved in the paradigm of questioning the world. It is because they seem to allow us to analyze applied- or mixed-mathematical activities. On the other hand, it is difficult for us to identify the status of different objects of mathematics, e.g., negative number, division, proportionality, and so on. One reason for this is that the ideas of the mathematical, the paramathematical, and the protomathematical are given as the kind of "open concepts" in Chevallard (1985/1991), which is Bourdieu's notion for emphasizing that every well-described definition of a given concept appears at the (never-coming) last part of theorization, through the dialectic of tentative theoretical organizations where the concept lives and its various applications. In fact, the authors' identifications of the status of objects of mathematics often differed. Then, we recently started to use a diagram for our negotiation. It is a relatively free variation of the *epistemological profile* introduced by Gaston Bachelard. We want to share it in this presentation.

"Mathematical" objects

Before introducing the idea of epistemological profile, let us give a minimum and *provisional* definitions of mathematical, paramathematical, and protomathematical objects for discussion. These definitions are created by our interpretation through reading Chevallard (1985/1991, Chap. 4) and Brousseau (1997), and applying the notions to our actual research based on more recent ATD framework of the *praxeology model* of human activities (cf. Chevallard, 2019).

- Mathematical object: the stake and target of study in a given mathematical activity: e.g., triangle. Praxeologically speaking, it occupies a niche in the *theoretical* division Θ.
- Paramathematical object: the tool and instrument of study in a given mathematical activity: e.g., geometrical construction. It lives in the *technological* part θ of a given mathematical praxeology.
- Protomathematical object: the contextual and infrastructural entities in a given mathematical activity: e.g., ruler. The *technical* district τ is its habitat.

An important thing is that these distinctions are *not* the substances of different objects but the functions of them in a certain activity. Of course, any mathematical activity also involves many objects of *nonmathematical* kinds, pedagogical (e.g., class), school (e.g., classroom), and so on. Moreover, there are objects which go up grades in the history of mathematical experience, whether short or long does not matter.

Epistemological profile of the object of mathematics

The French philosopher and epistemologist Gaston Bachelard, at the second chapter of his book *La philosophie du non*, introduces the notion and diagram of *epistemological profile* for describing his cognitive state—or *instantial relation R* in the ATD sense—about several physical concepts like mass: $R(\mathrm{GB}, o)$, where GB and o denote Gaston Bachelard and the concept of mass respectively. He insists—*judges* in the ATD sense—that his relations to them are complexes of some conceptions—i.e. *sub-relations*—based on different "backgrounds", naïve realism, Newtonian mechanics, and so on: $\mathrm{GB} \vdash (R(\mathrm{GB}, o) \cap \mathcal{R}_1 \neq \emptyset) \wedge (R(\mathrm{GB}, o) \cap \mathcal{R}_2 \neq \emptyset) \wedge \ldots$; where $\mathcal{R}_1$ denotes the union of the *realist* relations of GB to all the objects in GB's *cognitive universe* $\Omega(\mathrm{GB})$, and $\mathcal{R}_2$ follows to the same manner. And then, he confesses such sub-relations has different degrees of familiarity for him. The distribution of his familiarity is represented by a kind of bar graphs (Bachelard, 1940/1966, p. 42; p. 44).

Let us apply the epistemological profile to analysis of our own *epistemological judgement* about *objects* involved in school mathematics denoted by σ, in which $R_\Sigma(\sigma) \neq \emptyset$, where Σ denotes any *school*. And, we assume that the union $\mathcal{R}_\Sigma$ of all the relations of Σ in mathematics could be divided to the four parts of the mathematical $\dot{\mathcal{R}}_\Sigma$, the paramathematical $\ddot{\mathcal{R}}_\Sigma$, the protomathematical $\dddot{\mathcal{R}}_\Sigma$, and the nonmathematical $\tilde{\mathcal{R}}_\Sigma$. Then, any identification of the status of a given relation $R_\Sigma(\sigma)$ by a given (epistemological) instance $\hat{\imath}$ can be described by the variations of the following forms: $\hat{\imath} \vdash (R_\Sigma(\sigma) \cap \dot{\mathcal{R}}_\Sigma \neq \emptyset) \wedge (R_\Sigma(\sigma) \cap \dddot{\mathcal{R}}_\Sigma \neq \emptyset)$; $\hat{\imath} \vdash R_\Sigma(\sigma) \cap \tilde{\mathcal{R}}_\Sigma \neq \emptyset$; and so on. Besides, we replace the bar graph with the dot plot style with totally ten dots. Maybe an example is more useful than formalization. Table 1 is a possible profile of the concept of fraction at secondary school.

Fraction at secondary school	
Judgement	**Points (Total 10)**
Mathematical	★★★★★
Paramathematical	★★
Protomathematical	★★★
Nonmathematical	

Table 1. An epistemological profile of the concept of fraction at secondary school.

Apparently, a fraction may be clearly identified as a mathematical object, which is a target for calculation. However, in some situations, fractions are recognized as the paramathematical object of the operation of division. In addition, the fraction is often regarded as not a mathematical concept of number—that is, (positive) rational number—but a type of “numerals” as a kind of ordered pair of numerator and denominator called fractional expression, which can be identified as protomathematical reality. Moreover, the name of fraction can mean a set $\mathbb{Q}$ or an algebraic system $(\mathbb{Q}, +, \times)$ which is also protomathematical at school, even though it is (para)mathematized at university.

Examples

Now, let us give some examples of the use of the epistemological profile of the object of mathematics. We analyze some objects respectively and independently. The results of such analyses show some differences between our models, whereby you could understand that the epistemological triviality of school mathematics is illusion. Let us emphasize here that our following analyses are very intuitive, that is to say, there is no methodological framework for analysis. What we want to argue in this poster presentation is not methodological matters for making reference epistemological models, but the possibility of a new epistemological framework which postulates that any “mathematical” work could have multiple statuses at the same time.

The concept of derivative at secondary school

Mathematical: as the object of study in differential analysis

Paramathematical: a tool for studying functions

Protomathematical: graphic image based on the intuitive definition of limit

Derivative at secondary school

Judgement	The first author KO	The second author HH
Mathematical	★★★★	★★★★
Paramathematical	★★★★	★★★★★
Protomathematical	★★	★
Nonmathematical		

Table 2. Epistemological profiles of the concept of derivative at secondary school.

The concept of proportionality at elementary school

Mathematical: as some magnitudes e.g., density and speed

Paramathematical: a tool for studying variations

Protomathematical: an implicit pattern of variations

Proportionality at elementary school		
Judgement	**KO**	**HH**
Mathematical	★★	★★★★
Paramathematical	★★★★★	★★★★
Protomathematical	★★★	★★
Nonmathematical		

Table 3. Epistemological profiles of the concept of proportionality at elementary school.

The concept of implication at secondary school

Mathematical: a stake in the study of logic (material implication)

Paramathematical: an inferential tool for studying mathematics

Protomathematical: a property of subsumptions in the Benn diagram

Nonmathematical: conditional conjunctions in natural language

Implication at secondary school		
Judgement	**KO**	**HH**
Mathematical		★★★
Paramathematical	★★★	★★★
Protomathematical	★★★★★	★★★★
Nonmathematical	★★	

Table 4. Epistemological profiles of the concept of implication at secondary school.

The concept of implication at secondary school

Paramathematical: a tool for measuring angles

Protomathematical and nonmathematical: a tool for the measuring as a proto- or non-mathematical activity.

Protractor at university		
Judgement	**KO**	**HH**
Mathematical		
Paramathematical		★
Protomathematical	★★★★★	
Nonmathematical	★★★★★	★★★★★★★★★

Table 5. Epistemological profiles of the instrument of protractor at university.

Final remarks: as a tool for epistemological vigilance

In this short paper for poster presentation, we gave an epistemological tool for studying the nature of mathematical knowledge, the epistemological profile. Before closing our discussion, we want to reflect on the epistemological obstacle with which we are encountering in our collective work. On the one hand, the first author has been educated in teacher education institution before entering didactic research, and thereby is likely obsessed with the school view of mathematics. On the other hand, the second author is originally a pure-mathematician in algebraic topology, who tends to incarnate dominant epistemology in the institution of scholarly mathematics. From the ATD point of view, either of such possession, whether school or scholarly, could spoil didactic research. Thus, we must not stop epistemological vigilance. In our opinion, the epistemological profile is a good tool for describing our own view of mathematics. And then, it raises the possibility of productive communication between persons or institutions who have different praxeological equipments about mathematics; e.g., mathematicians and mathematics educators.

Acknowledgments

This work is supported by JSPS KAKENHI (19K14196 & 21H00924).

References

Bachelard, G. (1940/1966). *La philosophie du non : Essai d'une philosophie du nouvel esprit scientifique*. Les Presses universitaires de France.

Brousseau, G. (1997). *Theory of didactical situations in mathematics. Didactique des mathématiques 1970-1990* (edited and translated by N. Balacheff, M. Cooper, R. Sutherland, V. Warfield). Kluwer Academic Publishers, Dordrecht.

Chevallard, Y. (1985/1991). *La transposition didactique: Du savoir savant au savoir enseigné*. Grenoble, France: La Pensée Sauvage.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114.

Hamanaka, H. & Otaki, K. (2020). Generating the raison d'être of logical concepts in mathematical activity at secondary school: Focusing on necessary/sufficient conditions. In *Educação Matemática Pesquisa*, *22*(4), 428–453.

Besoins praxéologiques pour l'organisation de l'étude des probabilités

Clément Durringer, Marie-Hélène Lécureux and Jean-Pierre Bourgade

Université Toulouse 2 Jean Jaurès, France ; clement.durringer@univ-tlse2.fr

EFTS, Université Toulouse 2 Jean Jaurès, France ; marie-helene.lecureux@univ-tlse2.fr

EFTS, Université Toulouse 2 Jean Jaurès, France ; jean-pierre.bourgade@univ-tlse2.fr

Keywords: Mathematics education, probability, secondary school mathematics, course organization, experimentation and theorization.

Besoins praxéologiques pour la mathématisation

L'étude des probabilités en l'absence d'une infrastructure statistique se résume rapidement à une « naturalisation » (Chevallard & Wozniak, 2011) du calcul des probabilités qui devient alors un objet d'étude en soi, détaché des systèmes qu'il est susceptible de modéliser – ou plutôt, sans que le *coût* important de cette modélisation soit pleinement assumé. L'analyse réalisée par Chevallard et Wozniak (2011) de ce manque infrastructurel a permis de pointer un besoin praxéologique de la profession (Artaud, 2019) relatif aux *mathématiques* de la variabilité statistique. Nous souhaitons, dans ce travail, indiquer un autre besoin qui porte sur les praxéologies *didactiques* de la profession, en mettant en évidence que le caractère expérimental du *travail de mathématisation*, et en particulier la maîtrise de techniques dialectiques de l'expérimentation et de la déduction (Kim, 2015, p. 184), s'ils sont nécessaires pour construire en classe une mathématisation de la variabilité statistique, ne sont vraisemblablement pas disponibles dans l'équipement praxéologique de la profession.

L'expérimentation et la théorisation dans les manuels : un exemple

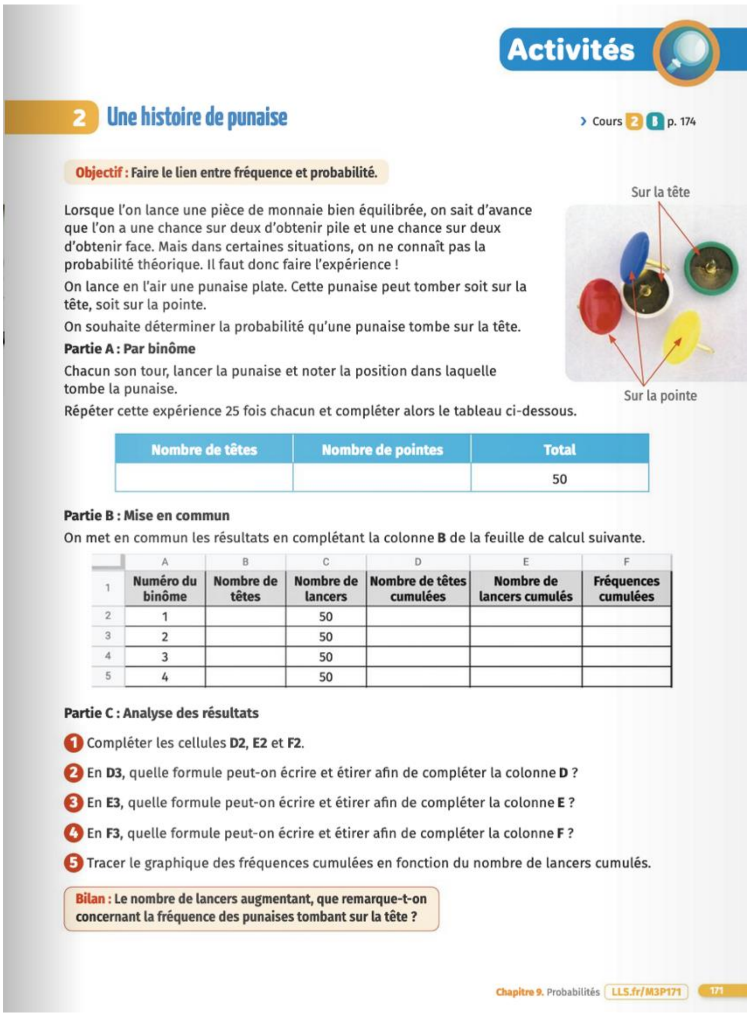
Les manuels ne sont qu'un matériau, parmi d'autres, pour servir à l'enseignement des mathématiques ; la plupart des enseignants les utilisent comme ressources pour choisir des exercices et des problèmes, mais certains les utilisent également pour concevoir des "activités d'introduction" ou les notes de cours qui s'inscriront sur les cahiers des élèves. Nous faisons le choix d'étudier ici un manuel, en essayant de décrire, à côté des composantes praxéologiques présentées (Wijayanti & Winsløw, 2017), les indices d'organisations possibles de leur étude : il s'agit de dégager non seulement l'environnement technologico-théorique relatif aux probabilités, mais aussi les pistes fournies pour la réalisation en classe du *moment de la construction de l'environnement technologico-théorique* (Chevallard, 1999), en l'occurrence la présentation dans un manuel d'un *protocole expérimental*.

Description d'un protocole expérimental

Le *Manuel collaboratif* (Godebin, 2021) propose une « activité » intitulée « Une histoire de punaise » dont l'objectif affiché est de « faire le lien entre fréquence et probabilité » (Figure 1). Les programmes de cycle 4 de 2020, les attendus de fin d'année ainsi que les repères de progression associés à ces programmes dans le cadre de la loi « pour l'école de la confiance » de 2019, ne prévoient pas une étude de la stabilisation des fréquences avant la classe de 3^{e} (Ministère de l'éducation nationale et de la jeunesse, 2019a, 2019b, 2019c, 2019d, 2020). Ce manuel de 3[e], publié en 2021, se doit donc de prendre en charge l'étude de ce phénomène et de son lien à la notion de probabilité.

Les auteurs situent bien l'enjeu : « dans certaines situations [différentes en cela de celle où on lance une pièce bien équilibrée], on ne connaît pas la probabilité théorique. Il faut donc [*sic*] faire l'expérience ! ». L'expérience proposée est celle d'un lancer de punaise, expérience pour laquelle on propose deux issues : "tomber sur la pointe" et "tomber sur la tête" de la punaise. L'objectif est précisé à nouveau : « on souhaite déterminer la probabilité qu'une punaise tombe sur la tête ».

Le *protocole expérimental* proposé est le suivant : dans un premier temps, « par binôme », les élèves lancent la punaise 50 fois et notent le nombre de têtes et le nombre de pointes obtenus. Une mise en commun est ensuite réalisée, qui



Activités

2 Une histoire de punaise

› Cours 2 3 p. 174

Objectif : Faire le lien entre fréquence et probabilité.

Lorsque l'on lance une pièce de monnaie bien équilibrée, on sait d'avance que l'on a une chance sur deux d'obtenir pile et une chance sur deux d'obtenir face. Mais dans certaines situations, on ne connaît pas la probabilité théorique. Il faut donc faire l'expérience !

On lance en l'air une punaise plate. Cette punaise peut tomber soit sur la tête, soit sur la pointe.

On souhaite déterminer la probabilité qu'une punaise tombe sur la tête.

Partie A : Par binôme

Chacun son tour, lancer la punaise et noter la position dans laquelle tombe la punaise.

Répéter cette expérience 25 fois chacun et compléter alors le tableau ci-dessous.

Nombre de têtes	Nombre de pointes	Total
		50

Partie B : Mise en commun

On met en commun les résultats en complétant la colonne **B** de la feuille de calcul suivante.

	A	B	C	D	E	F
1	Numéro du binôme	Nombre de têtes	Nombre de lancers	Nombre de têtes cumulées	Nombre de lancers cumulés	Fréquences cumulées
2	1		50			
3	2		50			
4	3		50			
5	4		50			

Partie C : Analyse des résultats

1. Compléter les cellules **D2**, **E2** et **F2**.
2. En **D3**, quelle formule peut-on écrire et étirer afin de compléter la colonne **D** ?
3. En **E3**, quelle formule peut-on écrire et étirer afin de compléter la colonne **E** ?
4. En **F3**, quelle formule peut-on écrire et étirer afin de compléter la colonne **F** ?
5. Tracer le graphique des fréquences cumulées en fonction du nombre de lancers cumulés.

Bilan : Le nombre de lancers augmentant, que remarque-t-on concernant la fréquence des punaises tombant sur la tête ?

Chapitre 9. Probabilités LLS.fr/M3P171 171

Figure 1: Une activité pour mathématiser la variabilité statistique (Godebin, 2021)

donne lieu à la production d'une feuille de tableur et au calcul d'effectifs et de fréquences cumulés pour l'ensemble des lancers de la classe. Enfin, l'« analyse des résultats » conduit à « tracer le graphique des fréquences cumulées en fonction du nombre de lancers cumulés ». Le « bilan » de l'activité prend la forme d'une question : « Le nombre de lancers augmentant, que remarque-t-on concernant la fréquence des punaises tombant sur la tête ? »

Enjeux liés à la théorisation

L'enjeu posé (pour déterminer la probabilité d'un événement, dans certains cas, il faut expérimenter) est doublement problématique :

- pour quelle raison *n'*est-il *pas* nécessaire d'expérimenter dans des cas comme le lancer d'une pièce équilibrée ? Et n'est-ce *vraiment* pas nécessaire ?
- en quoi « expérimenter » permet-il de *déterminer* la probabilité d'un événement ?

Le premier aspect est déjà largement documenté chez Chevallard et Wozniak (2011). Le deuxième est essentiel au bon fonctionnement de l'activité comme dispositif didactique : si l'on veut déterminer la probabilité à partir d'expérimentations, il faut établir, auparavant ou chemin faisant, un lien entre probabilité (notion théorique qui existe dans le *modèle*) et les observations qui portent sur le *système modélisé*. Il y a à ce niveau plusieurs points à préciser.

La loi *expérimentale* des grands nombres (Chevallard & Wozniak, 2011), soit l'observation de la stabilisation progressive des fréquences cumulées dans la réalisation d'une expérience à deux issues répétée un grand nombre de fois, conduit à l'idée d'une convergence (au sens naïf, non mathématisé, du terme) des fréquences vers « une » valeur qui « est » la probabilité de l'événement.

La détermination de *cette* valeur est impossible expérimentalement : il y a toujours un choix à opérer (pourquoi préférer 0,5 à 0,499 pour la probabilité de faire pile ? 1/6 à 0,16 pour la probabilité de faire 3 avec un dé équilibré ?). Ce choix peut s'étayer sur des arguments de simplicité ou de symétrie, arguments « rationnels » (Lévy, 1970) que les données expérimentales (la proximité des valeurs des fréquences des différentes issues) viennent en retour renforcer.

Un *protocole expérimental* dont l'objectif serait de « faire le lien entre fréquence et probabilité » doit favoriser la prise en charge d'une discussion sur ce que la *théorisation* (la *probabilité*) doit conserver du *système* (les *fréquences*), et sur les conditions et contraintes à poser sur cette théorisation.

Topos et (im)possibilités de théorisation

Prenons par la fin : « Le nombre de lancers augmentant, que remarque-t-on concernant la fréquence des punaises tombant sur la tête ? » Le *topos* laissé aux élèves est ici très large puisqu'une question aussi ouverte appelle en droit une infinité de réponses possibles (« c'est un nombre entre 0 et 1 », « il y a souvent plus de deux chiffres après la virgule », « la fréquence est parfois, mais rarement, proche de 0,9 », « à partir d'un certain moment, le premier chiffre après la virgule est toujours le même », « ça bouge tout le temps », « ça bouge de moins en moins », etc.). Le graphique crée des conditions pour la sélection des réponses pertinentes (celles qui indiquent une stabilisation des fréquences), réduisant par là même le *topos* des élèves. La question séminale (« quelle est la probabilité qu'une punaise tombe sur la tête ? ») n'est pas reprise pour orienter l'étude.

D'ailleurs, cette question n'offre pas de prise à l'expérimentation tant que le lien entre probabilité et fréquence n'est pas opéré *a minima*, par exemple en indiquant la raison d'être de la probabilité comme *moyen de produire une technique* pour déterminer le *nombre de réalisations* d'un événement lors de la reproduction multiple d'une expérience aléatoire (Gnedenko & Khintchine, 1964, p. 9-10 ; Chevallard & Wozniak, 2011). Le protocole ne pose pas la question de la détermination du *nombre* *de réalisations* de l'événement « la punaise tombe sur la tête », mais bien de sa *probabilité*. Ne sachant les fonctions que l'on souhaite faire endosser à la notion de probabilité, comment en produire une définition satisfaisante ?

Dans l'énoncé de l'activité, la seule précision sur la nature de l'objet « probabilité » semble se réduire à une pseudo-définition de la probabilité comme proportion de chances : « lorsque l'on lance une pièce de monnaie bien équilibrée, on sait d'avance que l'on a une chance sur deux d'obtenir pile […] mais dans certains cas on ne connaît pas la probabilité théorique » (nous soulignons). L'intérêt de la probabilité serait ainsi de quantifier la proportion de « chances » de succès lors qu'une expérience est réussie. L'expérience mentionnée (« lorsque l'on lance une pièce […] on sait d'avance que… ») ne différencie pas la réalisation unique d'une expérience de sa réalisation un grand nombre de fois ; la question du type de connaissance que représente précisément, dans le cas d'un lancer *unique*, le fait de savoir que « l'on a une chance sur deux d'obtenir pile », n'est pas abordée et les raisons de multiplier des expérimentations en très grand nombre, comme y invite le protocole, ne sont dès lors pas claires. Ou plutôt, l'étude de ces raisons est rejetée hors du *topos* des élèves, voire même hors de la classe : c'est le professeur, ou le concepteur de l'activité, qui affirment l'adéquation du protocole expérimental à l'objectif annoncé, et qui réduisent par là-même l'étendue du travail de production de *logos* qui pourrait résulter de l'analyse de la question posée et de la recherche d'un protocole expérimental adapté pour y répondre.

Nous pouvons désormais reprendre l'analyse du « bilan » et observer que le protocole expérimental proposé ne permet pas de répondre de façon pertinente à la question posée (qui est trop ouverte) ni à la question séminale (quelle est la probabilité que la punaise tombe sur la tête) parce que

- la notion de probabilité est insuffisamment caractérisée par les fonctions qu'elle permettrait de remplir (estimer un *nombre* de succès lors de répétitions *multiples*) ;
- la convergence (au sens naïf) des fréquences est prise comme une raison de postuler une probabilité unique, plutôt que comme la manifestation du besoin d'*approcher* ces fréquences par une valeur unique (inversion entre système et modèle : c'est le système qui fournit des approximations du modèle – ce qui devient vrai dans d'autres états du modèle, mais est incorrect lors de la première construction de la théorisation de la variabilité statistique) ;
- par conséquent, la question de la détermination de la valeur de la probabilité est pressentie comme la détermination de la valeur « limite » d'une suite de fréquences et non comme portant sur la recherche de critères rationnels (voire mathématiques) pour poser, conventionnellement, la valeur de cette probabilité de telle sorte qu'elle constitue cependant une bonne approximation des fréquences observées.

Discussion

Le protocole expérimental étudié ici relève d'une philosophie empiriste inassumée qui réduit la théorisation à l'observation : il suffit de remarquer (« que remarque-t-on […] ? ») pour théoriser ; cela revient, dans les faits, à renvoyer le travail de théorisation dans le *topos* du professeur, voire même dans un ailleurs étranger à la classe, tandis que le *topos* des élèves est restreint à la mise en œuvre d'expériences dont ils ne peuvent concevoir eux-mêmes la nature ni les éléments qu'elles pourront amener au milieu pour l'étude de la variabilité statistique.

La définition de la probabilité qui est proposée dans le même manuel (Godebin, 2021, p. 174) : « lors d'une expérience aléatoire, on peut associer, à chacune des issues de l'univers, un nombre appelé probabilité », ainsi qu'une propriété qui indique que « lorsque l'on répète un très grand nombre de fois une expérience aléatoire, la fréquence d'apparition d'une issue se rapproche d'une fréquence "limite" qui est la probabilité de cette issue », reproduisent la co-présence d'une théorisation formelle et d'une approche empiriste qu'il faudrait travailler conjointement et non comme illustration l'une de l'autre. L'activité censée faire le « lien » entre fréquence et probabilité se fonde sur ce lien lui-même, et la définition purement formelle de la probabilité comme « nombre » ne permet pas de produire une telle « propriété » que la loi des grands nombres par voie déductive. Le protocole expérimental n'est pas utilisé pour réaliser une dialectique de l'expérimentation et de la déduction qui conduirait à passer de la manifestation phénoménale de la stabilisation des fréquences observées à la postulation d'une *loi expérimentale des grands nombres*. Le processus d'élaboration de la définition des probabilités ne donne par conséquent lieu à aucune dialectique du structurel et du fonctionnel (Kim, 2015, p. 187) : une définition formelle est donnée sans lien avec les fonctions qu'on peut lui faire jouer. Tout ceci contribue à rendre manifestes des besoins praxéologiques de la profession relatifs à la réalisation des fonctions de l'étude au moyen de techniques dialectiques relevant des dialectiques de la mathématisation (Kim, 2015).

Références

Artaud, M. (2019). Besoins praxéologiques de la profession : le cas des grandeurs et de leur mesure. *20e école d'été de didactique des mathématiques*. Autrans, Grenoble.

Chevallard, Y. (1999). Analyse des pratiques enseignantes et didactique des mathématiques : l'approche anthropologique. Dans R. Noirfalise (Éd.), *Analyse des pratiques enseignantes et didactique des mathématiques* (pp. 91-118). Clermont-Ferrand : IREM.

Chevallard, Y. & Wozniak, F. (2011). Un cas d'infrastructure manquante, statistique et probabilités en classe de 3e. In Bosch, et al. (Éds), Un panorama de la TAD, pp. 23-32.

Gnedenko, B. V., Khintchine, A. Ia., (1964). *Introduction à la théorie des probabilités*. Paris : Dunod.

Godebin, P. (2021). Manuel collaboratif. Maths 3e. Lyon : Lelivrescolaire.fr Éditions

Kim, S. (2015). *Les besoins mathématiques des non-mathématiciens : quel destin institutionnel et social ? : études d'écologie et d'économie didactiques des connaissances mathématiques*. Thèse de doctorat. Aix-Marseille Université, France.

Lévy, P. (1970). *Quelques aspects de la pensée d'un mathématicien. Souvenirs mathématiques. Considérations philosophiques.* Paris : Albert Blanchard.

Ministère de l'éducation nationale et de la jeunesse, (2019a). *Attendus de fin d'année en 5e en mathématiques*. https://eduscol.education.fr/document/14044/download

Ministère de l'éducation nationale et de la jeunesse, (2019b). *Attendus de fin d'année en 4e en mathématiques*. https://eduscol.education.fr/document/14056/download

Ministère de l'éducation nationale et de la jeunesse, (2019c). *Attendus de fin d'année en 3e en mathématiques*. https://eduscol.education.fr/document/14068/download

Ministère de l'éducation nationale et de la jeunesse, (2019d). *Repères annuels de progression au cycle 4 en mathématiques*. https://eduscol.education.fr/document/14080/download

Ministère de l’éducation nationale et de la jeunesse, (2020). *Programme du cycle 4 en vigueur à la rentrée 2020*.
https://eduscol.education.fr/document/621/download

Wijayanti, D., & Winsløw, C. (2017). Mathematical practice in textbooks analysis: Praxeological reference models, the case of proportion. *REDIMAT*, 6(3), 307-330. doi: 10.1783/redimat.2017.2078

La théorie des rapports pour penser le « travail sur l'erreur »

Elsa Puchaczewski

Université Toulouse II (INSPÉ), France ; elsa.puchaczewski@etu.univ-tlse2.fr

Mots-clés : Erreurs, Moment de travail, Milieu herbartien

Keywords: Mistakes, Praxeological work moment, Herbartian milieu

L'erreur perçue comme un levier d'apprentissage

"Dédramatiser l'erreur" est, en mathématiques dans le secondaire, une injonction gouvernementale présente dans les documents à disposition du professeur. Ainsi peut-on lire dans le programme des classes de lycée (élèves de 15-18 ans, Bulletin Officiel, 2019) :

> [L'élève] ne doit pas craindre l'erreur, mais en tirer profit grâce au professeur, qui l'aide à l'identifier, à l'analyser et la comprendre. Ce travail sur l'erreur participe à la construction de ses apprentissages.

En dehors de la dimension psychologique, le travail sur l'erreur semble constituer un "levier" pour l'apprentissage des mathématiques. Dans la production scientifique internationale, on relève un intérêt certain pour les erreurs (*mistakes*, ou *errors*) des élèves ; le regard se porte vers les mauvaises représentations (*misconceptions*) que les élèves se font des mathématiques qu'ils ont apprises, ou bien vers les erreurs typiques (*error patterns*) et l'identification de leurs causes (par exemple : Aksoy & Yazlik, 2017). Souvent, on reconnaît à l'erreur une importance centrale dans les processus d'apprentissage, et on détermine les conditions de vie de classe qui doivent prévaloir pour que les élèves puissent s'autoriser à procéder par essais et erreurs (« students' trial and error behavior », Dalehefte, Seidel, & Prenzel, 2012) et puissent apprendre de leurs erreurs (« actual opportunities to learn from errors », Dalehefte, Seidel, & Prenzel, 2012). Les théories cognitives ou neuroscientifiques de l'apprentissage ménagent une place importante aux erreurs des élèves, en soulignant l'importance de l'erreur pour l'apprentissage (Boaler, 2013, p. 149) :

> Research has shown that mistakes are important opportunities for learning and growth, but students routinely regard mistakes as indicators of their own low ability. […] Dweck proposes that every time a student makes a mistake in mathematics, new synapses are formed in their brain [*personal communication by Dweck to Boaler*]. When students think about why something is wrong, new synaptic connections are sparked that cause the brain to grow. [… Therefore] students and teachers should value mistakes and move from viewing them as learning failures to viewing them as learning achievements. […students'] mistakes should be valued for the opportunities they provide for brain development and learning.

L'objectif principal de cette communication est d'identifier la place que l'on peut accorder à l'erreur au cœur des processus d'étude, outre son utilité reconnue pour l'apprentissage. D'après les théories des moments de l'étude et de la cognition développées dans le cadre de la Théorie anthropologique du didactique (TAD ; Chevallard, 2022) certaines fonctions de l'étude sont en relation directe avec l'apprentissage, alors que d'autres visent avant tout l'enrichissement du milieu d'étude de la classe. Nous supposons que l'invisibilisation des usages possibles de l'erreur - définies *comme des assertions non conformes à ce qui est attendu* par l'institution au sein de laquelle on se trouve - pour la réalisation de cette dernière classe de fonctions contribue au refoulement du didactique (Chevallard, 2012) autant qu'il en est l'un des effets.

Moment du travail praxéologique et refoulement du didactique

Si l'erreur est exploitée par une instance, elle acquiert une visibilité et une existence au sein de cette instance : nous pouvons expliquer cela comme un ajout des erreurs exploitées au schéma herbartien.

Le schéma herbartien

Considérons une question *Q* ("La somme de deux nombres impairs est-elle toujours paire ?", ou "Comment convertir des données satellites en images numériques ?"), étudiée au sein d'une institution (l'École, ou l'entreprise qui traite la question). Nous notons $X=\{x_1, ... , x_n\}$ l'ensemble des individus qui ont pour mission d'étudier *Q* (pour reprendre nos exemples, il s'agirait de l'ensemble des élèves, ou de l'ensemble des développeurs d'une équipe de l'entreprise chargée du traitement des images). Lorsque cet ensemble est aidé dans son étude par une ou plusieurs personnes, on désignera par y_i chacune de ces personnes et *Y* leur ensemble (toujours selon les exemples, *y* désignera le professeur, ou le manager de l'équipe s'il existe ; *Y* pourrait représenter, dans le deuxième exemple, un groupe de développeurs venus d'un autre département de l'entreprise pour aider l'équipe *X* à entreprendre l'étude de *Q*). On appelle alors système didactique, et on note $S(X, Y, Q)$ le système constitué par le groupe formé par les étudiants (terme pris au sens littéral : les personnes qui étudient) *X*, et les aides à l'étude *Y*, dans le but d'étudier la question *Q*.

Étudier une question *Q*, c'est donc se donner les moyens d'y apporter une réponse *R*. Pour y parvenir, un système didactique $S(X, Y, Q)$ va s'appuyer sur un ensemble d'œuvres qui lui permettront de produire une réponse cohérente au regard de ces œuvres. Par exemple, il pourra produire ou analyser des données *D*, se poser des questions intermédiaires *Q*', y apporter des réponses *R*', ou bien étudier des œuvres produites par d'autres individus, dans d'autres institutions. On appelle l'ensemble des œuvres qui sont mises à contribution lors de l'étude d'une question *Q* le *milieu didactique*, ou *milieu de l'étude*. On peut schématiser le fait que le système didactique permet l'émergence du milieu didactique *M* par la flèche ascendante dans le schéma herbartien incomplet ci-dessous :

$$S(X, Y, Q) \rightarrow M$$

C'est la richesse du milieu qui permet au collectif d'étude [*X*, *Y*] d'acquérir une meilleure compréhension de la question étudiée, ainsi que de produire des hypothèses et surtout de les confronter aux œuvres ainsi saillantes.

En effet, les différentes œuvres qui vont apparaître dans le milieu au fur et à mesure de l'avancée de l'étude vont être tantôt produites par différents *médias* et pourront être confrontées par [*X*, *Y*] à d'autres œuvres prises comme des "morceaux de nature", des milieux, du fait de leur indifférence (supposée ou contrôlée) quant à la véridicité des œuvres en question. Le fait, pour une œuvre, de "résister" à différents milieux, renforce d'autant plus sa vraisemblance que les milieux auront été choisis selon des critères de diversité, de solidité éprouvée dans d'autres enquêtes, etc. Il est de règle générale que les milieux viennent à jouer un rôle de média : telle conjecture soumise à un milieu le conduit à exprimer une assertion dont la vérité n'est pas acquise et qui sera à son tour soumise à l'assaut de différents milieux (parmi lesquels des éléments qui pouvaient auparavant jouer le rôle de média).

Ce processus de *dialectique des médias et des milieux* (DMM), au cœur de la production praxéologique, permet d'asseoir la validité des techniques, de produire des éléments technologico-théoriques, etc. La DMM est donc un élément fondamental de la production praxéologique, et on comprend qu'un collectif d'étude soit d'autant plus sûr des réponses qu'il produit qu'il sera parvenu à se constituer un milieu plus riche.

Une hypothèse qui survit à une dialectique vigoureuse des médias et des milieux peut devenir une partie de la réponse *R* produite par le système didactique en réponse à la question posée. La flèche descendante dans le schéma herbartien semi-développé ci-dessous représente le fait que la réponse *R* découle d'un travail du système didactique qui consiste en la construction d'un milieu (la *mésogenèse*) adéquat et en son exploitation (notamment par la DMM, mais pas seulement) :

$$[S(X, Y, Q) \rightarrow M] \rightarrow R.$$

Parmi les œuvres qui constituent le milieu, des *erreurs*, qui sont des œuvres en tant que produits d'une action humaine intentionnelle et dont l'étude participe à une meilleure compréhension de la question posée, et à une circonscription plus précise de la réponse à apporter.

Moments de l'étude

Si les erreurs peuvent donc appartenir au milieu de l'étude, nous pouvons nous demander lors de quels moment celles-ci sont potentiellement exploitées.

Une partie du travail de production d'une réponse à *Q* est consacrée à la genèse d'éléments praxéologiques : il s'agit de réaliser les moments de *première rencontre*, *exploratoire*, et *technologico-théorique* (Chevallard, 1999, 2002).

L'étude peut aussi conduire à la réalisation de fonctions qui concernent plutôt la relation entre ces éléments praxéologiques et l'institution (et ses sujets) au sein de laquelle évolue le système didactique : le système *S*(*X*, *Y*, *Q*) cherche à les "officialiser" lors du *moment d'institutionnalisation* ; il en accroît sa maîtrise, et travaille les praxéologies lors du *moment de travail praxéologique* ; il les évalue, ainsi que leur maîtrise par les étudiants lors du *moment d'évaluation*.

Lors du moment de travail praxéologique, les productions des étudiants peuvent s'éloigner de ce que l'institution attend : soit parce que ce qu'ils produisent n'est pas conforme aux éléments praxéologiques élaborés auparavant et présents dans le milieu ; soit parce que ces éléments praxéologiques eux-mêmes sont en certains points *défectueux* : les praxéologies produites "fonctionnent" mal. La réalisation du moment de travail, pour être pleinement réalisée, doit pouvoir prendre en charge ces deux dimensions, en mettant en évidence d'une part les écarts entre les productions et la praxéologie de référence, d'autre part les éventuels défauts de cette praxéologie de référence. Ces deux dimensions permettent pour l'une d'approfondir l'étude, et pour l'autre de favoriser spécifiquement l'apprentissage.

Moment de travail : étude et apprentissage

On définit le rapport (Chevallard, 2020, 2022) d'une personne à un objet *o*, noté *R*(*x*,*o*), comme l'ensemble des relations qu'elle noue à cet objet : ses croyances, ses connaissances et ses manipulations sur *o*, etc. Le rapport *R*(*x*,*o*) peut être vide, il peut évoluer, etc. Au sein d'une institution, les rapports personnels à certains objets d'intérêt institutionnel doivent souvent se conformer à un rapport "officiel", dont l'institution attend que chaque personne en fasse son propre rapport à cet objet : il s'agit du *rapport institutionnellement attendu* à *o*.

La théorie des rapports permet d'analyser le moment de travail praxéologique en deux de ses fonctions : une *fonction cognitive* qui consiste à œuvrer à la modification des rapports personnels des élèves à une praxéologie $\wp$ en cours d'étude :

> [Le moment] du travail de la technique [en fait : de la praxéologie], doit […] *accroître la maîtrise que l'on en a* (Chevallard, 1999, p. 112, nous soulignons).

À l'École, on peut comprendre cet accroissement comme un signe de l'évolution (temporelle) des rapports personnels dans le sens d'une plus grande conformité à un rapport institutionnellement attendu. La TAD désigne comme *apprentissage* une telle évolution (Chevallard, 2020).

Une deuxième fonction du moment de travail est de "faire travailler" $\wp$ elle-même, ce qui requiert des praxéologies d'étude spécifiques :

> [Le moment de travail praxéologique] doit […] *améliorer la technique en la rendant plus efficace et plus fiable* (ce qui exige généralement de retoucher la technologie élaborée jusque-là), […] ce moment de mise à l'épreuve de la technique suppose en particulier un ou des *corpus de tâches* adéquats qualitativement aussi bien que quantitativement (Chevallard, 1999, p. 112, nous soulignons).

C'est en réalité à un enrichissement du milieu herbartien que vont contribuer ces « corpus de tâches », milieu auquel $\wp$ est soumis dans le cadre de la DMM.

Les "erreurs" vont alors apparaître comme des symptômes de l'inadéquation des rapports *R*(*x*, $\wp$) personnels à $\wp$, et donc d'un apprentissage encore incomplet, mais aussi comme des manifestations de l'inachèvement de $\wp$: à ce titre les erreurs pourraient être introduites dans le milieu herbartien comme des œuvres qui devraient être étudiées afin de "faire travailler la praxéologie$\wp$.

Or, si la fonction cognitive du moment de travail est reconnue et travaillée dans les institutions scolaires (c'est le "travail sur l'erreur" mis en avant par l'institution scolaire), la deuxième fonction l'est drastiquement moins ; en effet, le travail autour de l'erreur n'est réalisé que pour autant qu'il est supposé favoriser l'évolution des rapports *personnels*, c'est-à-dire l'apprentissage. Le travail sur l'erreur comme *œuvre* dont l'étude contribue à la poursuite de la production des praxéologies est essentiellement un travail *didactique* : la contrainte du *refoulement du didactique* est particulièrement bien manifestée par la réduction des fonctions du moment de travail à sa seule fonction cognitive :

> […] dès que l’on entend étudier le didactique, on se trouve devant un phénomène massif, qui est au cœur de nos sociétés : l’immense majorité des *discours* et des *textes* qui parlent du monde social *ignorent le didactique*. […] Nous exprimerons ce fait lourd en disant que, dans la mise en scène ordinaire du monde social, il y a *refoulement du didactique* […]. Tout se passe en vérité comme s’il n’était convenable de parler ni du « quelque chose » que *y* prétend aider *x* à apprendre (le contenu de savoir qui est l’enjeu de l’interaction didactique), ni du quelque chose que *y* fait pour cela (les « gestes » didactiques qu’il accomplit à propos de ce contenu) (Chevallard, 2012, p. 11).

En négligeant la part du travail praxéologique qui ne s’attache pas directement à faire évoluer les rapports des élèves, on *refoule* un certain nombre de « gestes didactiques » ; il est notable que ce sont les gestes qui, précisément, n’ont pas pour vocation directe d’œuvrer à l’apprentissage. Tout se passe comme si la préoccupation pour *l’apprentissage* obérait toute perception de la part du didactique qui ne lui serait pas vouée.

Fonctions de l’erreur dans des documents institutionnels

La résolution de problèmes est une catégorie de situations didactiques intéressante pour notre propos puisqu’elles peuvent être utilisées tant pour produire de nouvelles connaissances que pour travailler des praxéologies complexes déjà partiellement élaborées. Un document institutionnel (Ministère de l’éducation nationale, 2022) fournit un environnement technologico-théorique riche pour la profession à propos de la résolution de problème. L’exploration de ce document apporte des informations sur le rapport de ses auteurs à la question des fonctions des erreurs :

> [La résolution de problèmes] participe pleinement à la construction même des notions [mathématiques] et de leur ancrage [… Les apprentissages mathématiques] bénéficient aussi des démarches réflexives autour des erreurs, ainsi que des retours d’informations reçus lors de tentatives de trouver la solution.

Un appel de note renvoie à l’ouvrage de Dehaene (2018) qui indique qu’« apprendre, c’est minimiser ses erreurs », et explicite ce rapport à l’erreur en s’appuyant sur une comparaison avec les mécanisme d’apprentissage d’algorithmes de type réseaux neuronaux, pour lesquels l’apprentissage se fait par ajustement de paramètres en vue de minimiser la probabilité d’erreurs, l’apprentissage se faisant par une approche essais-erreurs. La métaphore sert ensuite de base pour développer le point de vue suivant sur les fonctions de l’erreur dans l’apprentissage :

> La procédure est simple : je tente une réponse, on me dit ce que j’aurais dû répondre, je mesure mon erreur, et je corrige tous mes paramètres afin de la réduire. À chaque étape, je ne fais qu’un tout petit pas, une petite correction dans la bonne direction. C’est pourquoi apprendre une activité complexe, comme jouer à Tetris, exige d’appliquer cette recette plusieurs milliers, millions, voire milliards de fois. (Dehaene, 2018).

Cette fonctionnalisation de l’erreur pour l’apprentissage est reprise dans (Ministère de l’éducation nationale, 2022, p. 17) :

> Le retour sur l’erreur est également essentiel en résolution de problèmes. En rendant possible la prise de conscience du décalage entre les propres attentes de l’élève et les conséquences de ses

actions, cela favorise une remise en cause et une évolution de ses conceptions ou de ses stratégies pour se retrouver plus en phase avec le retour de l'environnement.

La fonction du travail de l'erreur n'est envisagée que sous l'angle cognitif (« évolution de ses conceptions [*logos*] ou de ses stratégies [*praxis*] » et non comme un moyen de travailler les praxéologies elles-mêmes.

L'exemple suivant donnera une idée d'un tel travail des praxéologies. Dans une classe de 5e (élèves de 12-13 ans), une professeure fournit la définition suivante des angles alternes-internes comme angles « situés "à l'intérieur" de deux droites […], de part et d'autre d'une sécante [à ces deux droites] ». Lors d'une séance ultérieure, une élève identifie la configuration (*d*) sur la figure comme représentant deux angles alternes-internes.

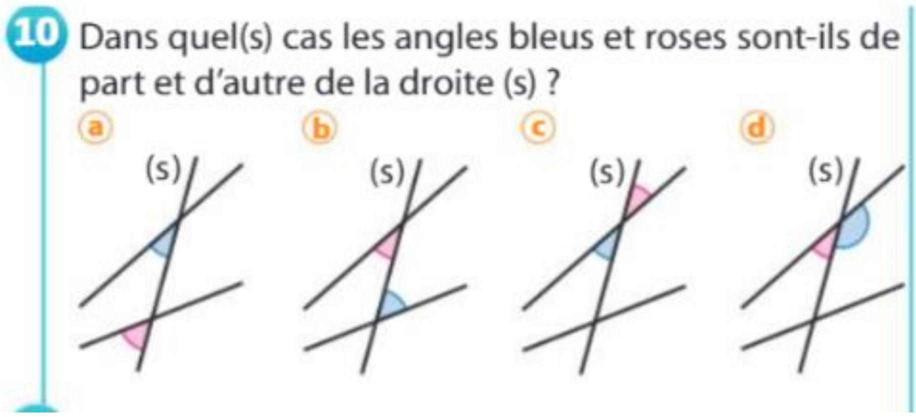


Figure 1 : Un exercice sur les angles alternes-internes (Barnet, 2016, p. 206)

Il y a "erreur" puisque les angles codés *ne sont pas* alternes-internes. Deux attitudes sont alors possibles : renvoyer sans autre forme de procès à l'élève que sa compréhension des angles alternes-internes est inadéquate et qu'il faut "bien sûr" que les angles ne soient pas en outre adjacents. On pourrait par ailleurs reconnaître dans cette erreur un symptôme, non pas d'un rapport non conforme de l'élève, mais bien de la praxéologie institutionnalisée elle-même et donc procéder à une modification de la définition qui tienne compte de l'obstacle rencontré par l'élève. Il faudrait alors justifier cette modification en faisant référence au milieu ébauché lors de la réalisation du moment exploratoire : il fallait tracer une parallèle à une droite donnée, connaissant l'angle qu'elle faisait avec une autre droite, en n'utilisant que le rapporteur et la règle non graduée, et la technique ébauchée nécessitait la production de deux angles égaux n'ayant pas "même sommet".

Remerciements

Cette publication a été réalisée dans le cadre d'une recherche en Master à l'INSPE Toulouse Occitanie Pyrénées, avec le soutien des Directions Recherche et Formation et de la SFR-AEF

Références

Aksoy, N. C., & Yazlik, D. O. (2017). Student Errors in Fractions and Possible Causes of These Errors. *Journal of education and training studies*, *5*(11), 219-233.

Artaud, M. (2007). La TAD comme théorie pour la formation des professeurs. Structures et fonctions. In L. Ruiz-Higueras, A. Estepa & F. J. García (Eds.), *Sociedad, escuela y matemáticas. Aportaciones de la teoría antropológica de lo didáctico (TAD)* (pp. 241–259). Jaen, Spain: Publicaciones de la Universidad de Jaén.

Barnet, C. (2016). *Mission Indigo, Maths 5^e^*. Paris : Hachette.

Boaler, J. (2013). The mindset revolution that is reshaping education. *Forum*, 55, 143– 152.

Bulletin Officiel. (2019). *Bulletin officiel spécial n°1 du 22 janvier 2019*.

Chevallard, Y. (1999). Analyse des pratiques enseignantes et didactique des mathématiques : l'approche anthropologique. Dans R. Noirfalise (Éd.), *Analyse des pratiques enseignantes et didactique des mathématiques* (pp. 91-118). Clermont-Ferrand : IREM.

Chevallard, Y. (2004). Vers une didactique de la codisciplinarité. Notes sur une nouvelle épistémologie scolaire. *Journées de didactique comparée 2004*, Lyon.

http://yves.chevallard.free.fr /spip/spip/article.php3?id_article=45

Chevallard, Y. (2012). *Didactique fondamentale M1 2011-2012*.

http://yves.chevallard.free.fr/spip/spip/IMG/pdf/DFM_2011-2012_Module_1_LD_.pdf

Chevallard, Y. (2020). *Humble séminaire 2019-2020, Séance 2*.

http://yves.chevallard.free.fr/spip/spip/IMG/pdf/HS_2019-2020_-_1_a_5.pdf

Chevallard Y. (2022) On the Genesis and Progress of the ATD. In: Chevallard Y. et al. (Eds) *Advances in the Anthropological Theory of the Didactic*. Birkhäuser, Cham. https://doi.org/10.1007/978-3-030-76791-4_1

Dalehefte, I. M., Seidel, T., & Prenzel, M. (2012). Reflecting on learning from errors in school instruction: Findings and suggestions from a Swiss-German video study. In *Human fallibility* (pp. 197-213). Springer, Dordrecht.

Organizaciones matemáticas relativas al Teorema de Pitágoras: descripción en términos de sus componentes y grado de completitud

Yesica Eugenia Torres[1] and Verónica Parra[2]

[1]EES N°5. Argentina; eugeniat@live.com.ar

[2]Universidad Nacional del Centro de la Provincia de Buenos Aires (UNCPBA), Núcleo de Investigación en Educación en Ciencia y Tecnología (NIECyT), Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina; vparra@niecyt.exa.unicen.edu.ar

Palabras clave: Organización matemática, grados de completitud, teorema de Pitágoras.

Introducción.

Este trabajo presenta los resultados de una tesis de grado (Torres, 2020) inducida por una problemática propia de la profesión docente: el equipamiento praxeológico útil (y necesario) del que deben disponer los profesores de matemática del nivel secundario (Bosch y Gascón, 2009). Este equipamiento se compone de praxeologías que, junto a otra serie de recursos, le permiten al profesor diseñar y gestionar sus clases. Entre los recursos utilizados por los profesores, el libro de texto o manual escolar es uno de los muy frecuentemente utilizados. Siguiendo esta línea, nos centramos en el Teorema de Pitágoras en el nivel secundario, enfocándonos en las praxeologías propuestas para enseñar reconstruidas a partir de un conjunto de 34 libros de texto destinados al nivel escolar medio argentino. Resulta importante desarrollar este análisis no sólo para describir las praxeologías, sino también porque muchos docentes, como ya lo mencionamos, utilizan los libros como uno de sus recursos principales, y en algunas ocasiones, el único, para el desarrollo de las clases en el aula y para la preparación fuera de ellas.

Metodología.

Este trabajo se propone:

1. identificar las tareas, las técnicas, tecnologías y teorías que componen las praxeologías propuestas, entorno al Teorema de Pitágoras, en un conjunto de 34 libros de texto pertenecientes al Ciclo Básico (1ro, 2do y 3er año) destinados al nivel secundario argentino (estudiantes entre 13 y 16 años);
2. agrupar esas tareas en tipos de tareas y éstos, a su vez, en géneros de tareas;
3. clasificar estas praxeologías tratando de delimitar si se tratan de puntuales, locales, regionales o globales y
4. dar indicios sobre su grado de completitud.

La selección de estos 34 libros de texto se debe al acceso a los mismos por parte de la investigadora. Se trata de libros editados entre los años 1975 y 2016. Los libros fueron rotulados con una M_i, con i desde 1 hasta 34.

La Tabla 1 permitió identificar, reconstruir y detallar la organización matemática (OM) propuesta en cada libro (M_i) a partir de la identificación de sus componentes: tareas, técnicas, tecnologías y teorías.

M_i	Tarea (t_i)	Técnicas (τ_{ij})	Tecnología (θ_{ijk})	Teoría (Θ_{ijk})

Tabla 1: Descripción e identificación de las OM en términos de sus componentes

A continuación, se detalla qué fue considerado como "tarea", "técnica", "tecnología" y "teoría".

Tareas (t_i): se consideró una tarea a los ejercicios/actividades/problemas que se abordan o estudian y proponen en cada libro, ya sean resueltos o no, y se rotuló con la letra t. El subíndice "i" refiere al número de tarea del M_i correspondiente.

Técnicas (τ_{ij}): se consideró una técnica a las herramientas que se construyen y utilizan para abordar ese problema y se rotuló con la letra τ. El subíndice "ij" refiere a la técnica número j que permite resolver la tarea número i.

Tecnología (θ_{ijk}): se consideró una tecnología a las descripciones y explicaciones que se proponen en el libro como una manera de explicar o de justificar las técnicas empleadas y/o propuestas. Se rotuló con la letra θ. El subíndice "ijk" refiere a la tecnología/teoría número k de la técnica j que permite resolver la tarea número i.

Teoría (Θ_{ijk}): se consideró una teoría a aquellas justificaciones y/o explicaciones que se explicitaban o inferían como maneras de justificar las tecnologías. Se rotuló con la letra Θ. El subíndice "ijk" refiere a la tecnología/teoría número k de la técnica j que permite resolver la tarea número i.

La tabla 1 permitió detallar cada tarea precisada en cada uno de los libros, así como las técnicas, tecnologías y teorías. Luego, esas tareas se agruparon en tipos de tareas. El criterio para este agrupamiento fue identificar aquellas tareas que se resuelven con una misma técnica. Esto permitió determinar los diferentes tipos de tareas propuestos en cada libro, las técnicas asociadas a esos tipos y en caso de explicitarse, las tecnologías y teorías. Luego, esos tipos se agrupan en géneros de tareas. El criterio fue, tal como lo indica Chevallard (1999), considerando un verbo en infinitivo.

Para determinar el nivel de OM, se consideran los 4 niveles de praxeologías formulados por Chevallard (1999): puntuales, locales, regionales y globales. Luego, nos cuestionamos sobre el grado de completitud de esas praxeologías matemáticas. Para esto, utilizamos los indicadores formulados por Fonseca (2004) generando descriptores para cada uno de ellos. Finalmente, la Tabla 2 permitió identificar cuál o cuáles de estos indicadores podían detectarse (o no) en cada OM reconstruida. Para ello, se consideran en las primeras columnas, cada tipo de tarea, las técnicas y tecnologías propuesta en cada caso. Luego, en las columnas siguientes, los descriptores de cada uno de los siete indicadores, donde se colocan cruces para determinar cuál o cuáles de ellos estarían o no presentes.

M_i	T	τ	θ	OML1		OML2		OML3		OML4		OML5		OML6		OML7	
				$OML1_1$	$OML1_2$	$OML2_1$	$OML2_2$	$OML3_1$	$OML3_2$	$OML4_1$	$OML4_2$	$OML5_1$	$OML5_2$	$OML6_1$	$OML6_2$	$OML7_1$	$OML7_2$

Tabla 2: Grados de completitud de una OM ya reconstruida

Resultados.

De los 34 libros, 9 están destinados al primer año (estudiantes de 13 años); 19 textos, a segundo año (estudiantes de 14 años) y 6 libros, a tercer año (estudiantes de 15 años). En los del primer año, en general, el Teorema de Pitágoras se propone después de la unidad de construcciones geométricas y antes de la unidad de números enteros; en segundo año, mayoritariamente, después de la unidad de cuadriláteros y antes de la unidad de probabilidad y estadística y, en tercer año, antes de la unidad de sistemas de ecuaciones y después de la unidad de movimientos en el plano.

A partir de la Tabla 1 se obtuvo un total de 201 tareas agrupadas en 11 tipos de tareas diferentes (T_h, con h desde 1 a 11):

T_1: Calcular el valor de la hipotenusa/diagonal, lado/s, catetos, apotema, altura del triángulo (122 tareas);

T_2: Calcular el valor del área, perímetro, superficie lateral, distancia (54 tareas);

T_3: Identificar triángulos (2 tareas);

T_4: Identificar el valor de los lados (1 tarea);

T_5: Determinar la veracidad o falsedad de las proposiciones (2 tareas);

T_6: Determinar si se cumplen las condiciones para construir un triángulo (5 tareas);

T_7: Determinar las ternas pitagóricas y a que lados corresponden los datos (2 tareas);

T_8: Comprobar la relación pitagórica y decidir si los triángulos son rectángulos (4 tareas);

T_9: Comprobar el valor de un cateto en la terna pitagórica (4 tareas);

T_{10}: Construir triángulos rectángulos (4 tareas) y

T_{11}: Construir cuadrados (1 tarea).

Claramente, hay una preponderancia de dos géneros: T_1 con 122 tareas y T_2 con 54 tareas.

Los 11 tipos de tareas, se agruparon en 5 tipos de genero de las tareas (G_n, con n desde 1 a 5):

G_1: Calcular (conformado por T1 y T2, 176 tareas)

G_2: Identificar (conformado por T3 y T4, 3 tareas)

G_3: Determinar (conformado por T5, T6 y T7, 9 tareas)

G_4: Comprobar (conformado por T8 y T9, 8 tareas)

G_5: Construir (conformado por T10 y T11, 5 tareas)

El gráfico 1 representa esta distribución:

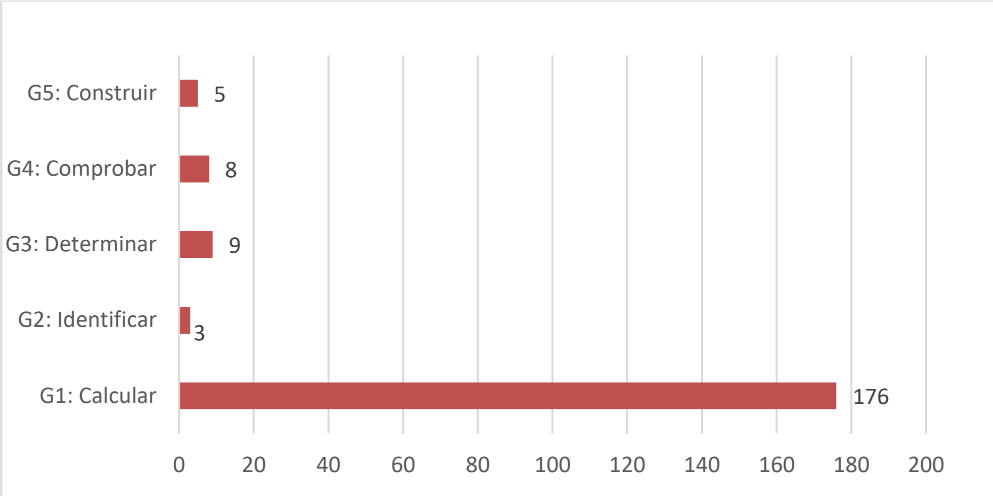


Gráfico 1: Cantidad de tareas por géneros de tareas

Las técnicas rondan, por ejemplo, en: reemplazar datos en figuras de análisis para saber qué lado está faltando (T_1); descomponer figuras de análisis para comprobar el teorema (T_3); evaluar la propiedad triangular para hacer posible la construcción del triángulo (T_6); descomponer y relacionar las ternas pitagóricas. (T_8); reemplazar valores en la expresión pitagórica (T_9); construir a través de datos el triángulo pedido (T_{10}), entre otras. Las tecnologías han sido inferidas ya que no se explicitan en los textos considerados. Poco se dice de las justificaciones, explicaciones y/o producciones de tal o cual técnica, salvo por el mismo Teorema de Pitágoras, donde, por momentos actúa de tecnología al resolver las tareas asociadas precisamente a su uso en la determinación de la longitud de un cateto o hipotenusa. Este nivel de tecnología nos conduce a concluir en OM de nivel local.

La Tabla 2 permitió concluir sobre el grado de completitud. Los descriptores que se identifican son cinco. Conviene aclarar que esta identificación no se ha considerado en sentido fuerte. Es decir, se ha considerado presente tal o cual descriptor si existe algún indicio del mismo:

$OML1_1$: Los tipos de tareas y técnicas aparecen integrados. Se ha considerado una “presencia” de este indicador pues para cada tipo de tarea es posible determinar al menos una técnica que permite resolver las tareas de ese tipo.

$OML2_1$: Para cada tipo de tareas existen diferentes técnicas. En este caso, se han identificado y/o inferido para algunas tareas, dos posibles maneras de resolverlas. Por ejemplo: el M_6 propone una tarea donde hay ternas de longitudes de los lados de triángulos y entonces, los estudiantes deben decidir cuáles de esas ternas corresponden a un triángulo rectángulo. En este caso, se propone el Teorema de Pitágoras como una posible técnica a usar, es decir, comprobando (o no) la relación pitagórica entre esas longitudes y también, se propone como técnica alternativa, el cálculo de los ángulos y de esta forma, resolver la tarea en función de la identificación (o no) de un ángulo recto.

$OML3_2$: Los objetos matemáticos son independientes de los objetos (ostensivos) que se utilizan para representarlos. Cabe aclarar que, el mismo, se identifica en muy pocos casos.

$OML5_1$: La OM contiene tareas matemáticas que permiten interpretar el funcionamiento de las técnicas. Se ha considerado aquí esta presencia pues hay algunas tareas que ponen a prueba los alcances y limitaciones del Teorema de Pitágoras. Por ejemplo: el M_1 propone una tarea donde hay diversos triángulos que contienen solamente la longitud de dos lados (y no la del tercero). Se debe señalar, en esos triángulos, el ángulo recto (si lo contiene) y calcular la longitud del tercer lado. Algunos de esos triángulos no son rectángulos y, entonces, los estudiantes deben advertir que no pueden usar el teorema de Pitágoras para calcular la longitud del tercer lado. He aquí lo que interpretamos como un "alcance y limitación" de la técnica.

$OML6_2$: La OM contiene tipos de tareas que requieren un proceso de modelación matemática. Se considera en este caso una presencia de este indicador cuando hay alguna tarea que considera un "contexto" para la misma. De esta forma, se ha considerado la modelización en un sentido muy débil.

Conclusiones

El análisis del conjunto de los 34 libros de texto arrojó un total de 201 tareas, agrupadas en 11 tipos de tareas diferentes y éstos, a su vez, asociados en cinco géneros de tareas. El mayor porcentaje de esas 201 tareas corresponden al género "G_1: Calcular" y dentro de él, a sus dos tipos tareas T_1 (122 tareas) y T_2 (54 tareas). Esto da indicios de praxeologías sesgadas al cálculo con una cantidad insignificante de tareas que pongan en juego otros aspectos de la matemática, tales como las verificaciones, la identificación de datos, construcciones, etc. Con respecto a las tecnologías y teorías, es decir, al nivel de las justificaciones, las primeras han sido algunas inferidas, puesto que no se explicitan en los textos y las teorías, completamente ausentes. Se concluye entones en la identificación de praxeologías con una amplia preponderancia del bloque práctico-técnico sobre el bloque tecnológico-teórico, que casi podría decirse está completamente ausente, salvo por esas inferencias tecnológicas. Considerando el análisis a partir de los indicadores de Fonseca (2004), se concluye que estas OM tienen un bajo grado de completitud. Sería interesante, además, poder comparar nuestra clasificación con las categorías formuladas por Salone (2015) quien propone una categoría para las fuentes de referencia potencialmente utilizables en una clase a partir de análisis de textos públicos producidos por instituciones didácticas. Esta categoría contempla 6 valores de codificación, de 0 a 5, desde un elemento discursivo no interpretado (código 0) hasta fuentes externas (código 5).

A partir de los resultados obtenidos y, más allá de los objetivos específicos de este trabajo, es importante propiciar una reflexión sobre las praxeologías propuestas en los libros de texto pues, como ya se mencionó, éstos constituyen una parte importante de los recursos de los profesores, tanto para la planificación como para el desarrollo de las clases.

Referencias

Bosch, M., & Gascón, J. (2009). Aportaciones de la Teoría Antropológica de lo Didáctico a la formación del profesorado de matemáticas de secundaria. En González, M.J; González, M.T; Murillo, J (Eds.). *Investigación en Educación Matemática XIII* (pp. 89- 113). Santander: SEIEM.

Chevallard, Y. (1999). El análisis de las prácticas docentes en la teoría antropológica de lo didáctico. *Recherches en Didactique des Mathématiques*, 19 (2), 221-266.

Fonseca, C. (2004). *Discontinuidades matemáticas y didácticas entre la enseñanza secundaria y la enseñanza universitaria* (Tesis doctoral). Universidad de Vigo.

Salone, J-J. (2015). Les références praxéologiques dans les systèmes didactiques (Tesis doctoral). Universidad Aix-Marseille. https://www.theses.fr/2015AIXM3021

Torres, Y. (2020). *Organizaciones Matemáticas propuestas para el nivel secundario relativas al Teorema de Pitágoras: una descripción desde la Teoría Antropológica de lo Didáctico* (Tesis de grado). Universidad Nacional del Centro de la Provincia de Buenos Aires.

About the Mapping of Personal Practice Paths in the ATD

Jany Goulart[1], Hamid Chaachoua[2], Luiz Márcio Farias[3] and Avenilde Romo-Vázquez[4]

[1]DEXA, State University of Feira de Santana, Brazil; jssgoulart@uefs.br

[2]LIG, University of Grenoble Alpes, France; Hamid.Chaachoua@imag.fr

[3]IHAC, Federal University of Bahia, Brazil; lmsfarias@ufba.br

[4]Department of Educational Mathematics, Cinvestav, Mexico; avenilderv@yahoo.com.mx

Abstract. *By studying the personal work of the students, we introduced the notion of Personal Practice Paths, idealized from the conception of personal praxeologies and praxeological recombination outlined by the praxeological dynamics inherent to the task resolution process. The study was done with students of a mathematics teaching degree course from a Brazilian university. As a methodological approach, we adopted a link-up between didactic engineering and ATD concepts, which allowed us to think about the emergence of a new dialectic that underlies the firstfruits of the Personal Study and Research Paths.*

Keywords: Anthropological theory of didactics, personal praxeologies, personal practice paths.

Résumé. *En étudiant le travail personnel des étudiants, nous avons introduit la notion de Chemins de Pratiques Personnels idéalisés à partir de la conception de praxéologies personnelles et de recombinaisons praxéologiques esquissées par la dynamique praxéologique inhérente au processus de résolution des tâches. L'étude avait pour protagonistes des étudiants d'un cours de licence en mathématiques d'une université brésilienne. Comme voie méthodologique, nous avons adopté la jonction entre l'ingénierie didactique et des concepts de la TAD, ce qui nous a permis de penser à l'émergence d'une nouvelle dialectique qui fonde les prémices des parcours d'étude et de recherche personnels.*

Mots clés: Théorie anthropologique du didactique, praxéologies personnelles, chemins de pratiques personnels.

Resumen. *Estudiando el trabajo personal de los estudiantes introdujimos la noción de Caminos de Prácticas Personales idealizados a partir de la concepción de las praxeologías personales y de las recombinaciones praxeológicas esbozadas por la dinámica praxeológica inherente al proceso de resolución de tareas. El estudio tuvo como protagonistas a estudiantes de un curso de licenciatura en matemáticas de una universidad brasileña. Como camino metodológico, se adoptó el cruce entre la ingeniería didáctica de los conceptos de la TAD, lo que permitió pensar en el surgimiento de una nueva dialéctica que funda los inicios de los Caminos de Estudio e Investigación Personal.*

Palabras clave: Teoría antropológica, de lo didáctico, praxeologías personales, caminos de prácticas personales.

Introduction.

This text aims to create a scenario that stimulates theoretical discussions on elements that emerged in the development of the doctoral thesis[1].The core of the investigation consisted of highlighting the functioning and use of the Hybrid Learning Environment (HLE), as a didactic device (CHEVALLARD, 2002), immersed in a context that involves the teaching and learning of vector algebra within the scope of a mathematics teaching degree course of a Brazilian university.

One of the pillars on which the research was based corresponds to the conception of Chevallard (1996), who highlights changes in personal relationships with object *O*, which, consequently, results in learning. In these terms, we assumed that personal relationships, in interaction with the object of knowledge, produce changes in the subjects' praxeological baggage, resulting in changes in the student's praxeological equipment (CHEVALLARD, 2009).

In the same sense, Chevallard (2007, p. 4) clarifies that "personal and institutional relations are linked to the study of deformations, reforms, and transformative formations, notably by the generalization of the notion of didactic transposition and the notion of institutional transposition", which reveals an institutional configuration that must be analyzed under different study levels.

From this perspective, we stimulated changes through actions produced within the HLE, which we defined as a space consisting of a variety of *media*[2] that enabled a transit between different *media* that were linked to the transformations in the *milieu.* In addition, Goulart and Farias (2018) consider that by providing students with various media - different from conventional ones -, we can integrate a flexible approach (DREYFUS, 1991) to the detriment of mathematical rigidity (SILVA *et al.*, 1999). This, we intended to identify, implement, and formalize a kind of complementation of teaching tools in the vector context, amalgamating new ones to existing ones.

For this, we used methodologically some phases of didactic engineering (ARTIGUE, 1996) mixed with ATD concepts formalized in T4TEL (CHAACHOUA, 2018) through the Task Type Generators – (Geradores de Tipo de Tarefa – GT), which had its Variable System fed by different combinations of the Didactic Devices {Manipulable Materials - MM, Pencil Paper - PP, GeoGebra - GG} from the HLE.

It is relevant to highlight that the GTs are determined by a type of task and a system of variables. For Chaachoua and Bessot (2016), this new object allows structuring a set of types of tasks for research purposes. In this perspective, a GT has the following configuration: GT = [Action Verb; Fixed Complement; System of Variables]. Note that (Action Verb; Fixed Complement) define a type of task T, so we can note GT: [T; SV]. Thus, we undertake the metaphorical interpretation that Task Type Generators GT act as a "machine" that produces types of tasks, in which the Hybrid Learning Environment provided "fuel" for operational functioning in interaction with the media, similar to the

[1] A didactic proposal for teaching vector algebra based on the T4TEL model in the context of a Hybrid Learning Environment – HLE

[2] See Chevallard (2007).

Media Milieu Dialectic (CHEVALLARD, 2007), regulated by the values of the variables inherent to the nature of each Didactic Device {MM, PP, GG}.

From this context, Kaspary (2020, p. 268) proposes the Task Type Subgenerators denoted by: "Let GT: [T, SV] and GT': [T', SV']. GT' is a GT task type subgenerator of GT tasks if T' ⊆ T e SV' ⊆ SV." Thus, we had to be aware of the behavior of a potential learning path conducted by the values of the variables at the core of the GT. Based on those elements, it was feasible to discuss the specificities of an a priori analysis, aimed at predicting which techniques $\tau_1, \tau_2, \dots, \tau_n$ the students would employ during the resolutions of a given type of task T.

During experimentation, we felt the need to fragment the analyses into three levels, general, partial, and individual. The first level was revealed through a broader configuration in terms of the praxeologies (CHEVALLARD, 1998) of the student groups. However, at the partial level, we reduced our scope, and focused on some students' personal praxeologies (CROSET & CHAACHOUA, 2016). On the individual scale, we sought greater detail in the resolution of a task presented by a student that evoked developments to theorize about the Personal Practice Paths - PPP and therefore idealize the *Individual-Individual Dialectic*, which can be added to the range of Dialectics or "gestures of study and investigation" (CHEVALLARD, 2007) in the organization of the specific actions that lead the Study and Research Paths – SRP and or Personal Study and Research Paths - PSRP.

The PSRP allow an individualized analysis based on the choices and productions of a student facing a generating question Q_0. In this way, more accurate identification possibilities open up, as if we were putting a lens on implicit elements, which will compose a more reliable description of the PPP, since, in general, the minutiae are not fully revealed in group work.

Personal Practice Paths – PPP.

Before defining what we conceive as PPP, we initially proposed what Chevallard (1996, p. 117) signaled: "thought develops based on metaphors (…). It is not, therefore, a priori, in any way illegitimate to think about theories and models in terms of image and representation." In these terms, we will use some analogies to communicate the essence of what we design as PPP preliminarily.

First, we established the following analogy: Personal Practice Paths generate opportunities for researchers and teachers to assume attributions that resemble football game analysts because, from an attentive observation, they identify and unravel the specificities of strategies in the executions of each bid or play, i.e., they aim to determine why and how actions happen during the game (task resolution).

That said, we aimed to understand the strengths and weaknesses (conditions and restrictions) of the teams (group of students), a portion of the team (part of this group) or a player (student) in particular, which refers to the different levels of analysis: general, partial, and individual so that we can establish more robust and consistent game strategies (task resolution).

Moreover, personal praxeologies (CROSET & CHAACHOUA, 2016) are built within one or more institutions according to different dynamics, as underlined by Kaspary, Chaachoua, and Bessot (2018). From this conception, confronting a task t belonging to a specific type of task T allows the student to evoke techniques (τ_i), $i = 1, 2, \dots, n$, and implement them without difficulties if T is a type of family/routine task for him/her. However, if the type of task T is not immersed in the usual praxeological repertoire of this subject, the accomplishment of this task can go through several attempts of adaptations, mobilizing techniques (τ_i), $i = 1, 2, \ldots, n$, that may not lead to the effective management of calculations until an eventual association with a technique with potential adequacy[3] arises.

As an example, imagine that the task *t: Show that if M and N are midpoints of the segments AC and BC, respectively, then* |*AB*|*=2*|*MN*|, was proposed to two students. As depicted in Figure 1,

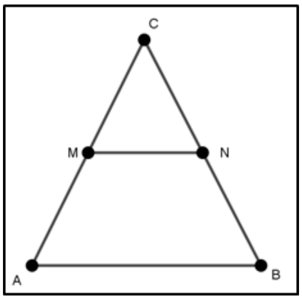


Figure 1: Triangle ABC

The first, making use of vector algebra presented the following resolution:

$$\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AB} + \overrightarrow{BN} = \frac{1}{2}\overrightarrow{CA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\overrightarrow{CA} + \overrightarrow{BC}) + \overrightarrow{AB} =$$

$$\frac{1}{2}(\overrightarrow{BC} + \overrightarrow{CA}) + \overrightarrow{AB} = \frac{1}{2}\overrightarrow{BA} + \overrightarrow{AB} = -\frac{1}{2}\overrightarrow{AB} + \overrightarrow{AB} = (1 - \frac{1}{2})\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AB}.$$

Then, $|\overrightarrow{MN}| = \left|\frac{1}{2}\overrightarrow{AB}\right| = \frac{1}{2}|\overrightarrow{AB}|$, i.e., $|AB| = 2|MN|$.

Thus, the technique can be described as: (vector sum), (midpoint), (vector multiplication by scalar), (distributive property), (opposite vector).

While, the second student resorted to the concepts of plane geometry, presented in resolving terms:

$$\frac{|AB|}{|MN|} = \frac{|CA|}{|MA|} = 2\frac{|MA|}{|MA|} = 2, \text{ i.e., } |AB| = 2|MN|$$

In these terms, the description of the technique is: (similarity of triangles) and (notion of midpoint). This aspect dialogues directly with Chaachoua and Bessot (2016, p. 3) conception, in the theoretical framework "T4TEL, a technique is defined as a sequence of task types". These authors also signal that: "The scope of a technique τ is the set of tasks performed by τ, denoted by P(τ)." (CHAACHOUA & BESSOT, 2018, p. 121).

From this perspective, the process can be described as a set of techniques according to the time of the praxeological construction (CHEVALLARD, 2009) and modeled through the Personal Practical Paths related to a task t_i: $PPP\ (t_i) = \{\tau_{i_1}, \tau_{i_2}\ \tau_{i_3}, \ldots, \tau_{i_k}\}$, as was addressed in the Goulart (2021), by linking all stages of the resolution process managed by the student or better, of his/her praxis(t_i, τ_{i_k}).

[3] The term "adequacy" means that the student judges that the technique allows solving the task in a way that is consistent with what the institution demands.

This sequence of techniques, in association with their ingredients and gestures, connects to the conception of dynamics of the personal praxeological constructions and one of the forms of praxeological recombination in which a logos is maintained, but the praxis is modified $(\Pi^*) \oplus \Lambda$ (CHEVALLARD, 2018). This allowed us to characterize this trajectory as the union of all the techniques used to achieve a solution for t_i, i.e., $\bigcup_{k=1}^{n_i}(t_i, \tau_{ik})$ in which a task is fixed (t_i) and the techniques undergo variations.

We infer that the student can find other tasks (t_i) of a given type T in which the same or different techniques are mobilized to generate approximations or achieve the result expected by the institution. Hence, we reveal the modeling of the Personal Practice Paths for a given type of task T: PPP(T) = $\bigcup_{i=1}^{m} \bigcup_{k=1}^{n_i}(t_i, \tau_k)$, as illustrated in Figure 2:

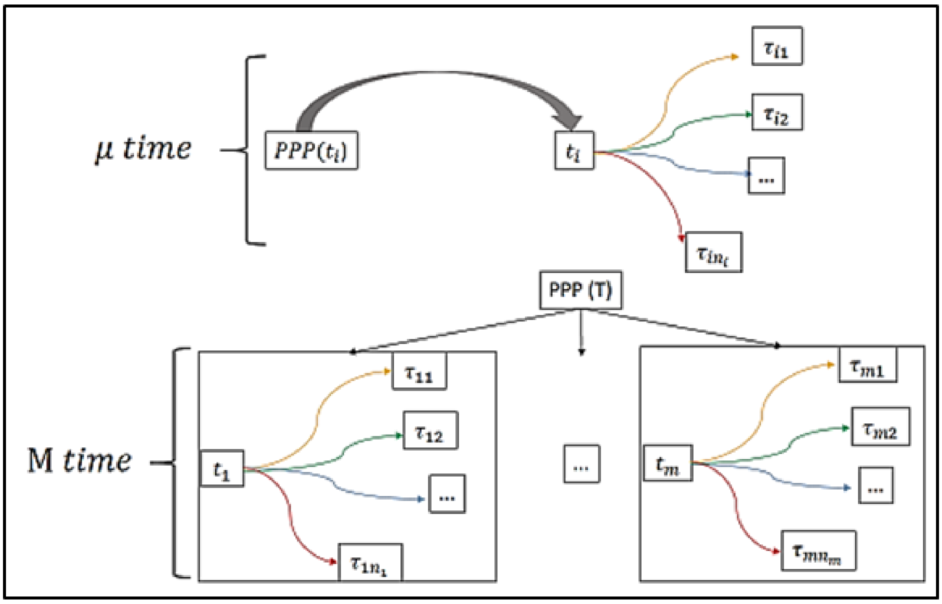


Figure 2: Personal Practice Paths PPP ($_{ti}$) or PPP (T)

It is equivalent to saying that this idea joins the conception of praxeological dynamics described by Kaspary, Chaachoua, and Bessot (2020), which made us conceive that the processes that come from the PPPs behave analogously to "a sequencing of photos" recorded at different times. These alternations of images allow the researcher to observe the process of praxeological movement or transit at different times (μ *tempo*) or longer time intervals (M *tempo*) associated with task types T.

This also makes us infer that the PPP can be modeled taking as influence and inspiration the definition of mathematical sequence (finite) defined as a function $f: \mathbb{N}' \subset \mathbb{N} \longrightarrow \mathbb{R}^q$, with a finite $\mathbb{N}'$, which, to each natural number $n \in \mathbb{N}'$, associates q-duple of real numbers $f(n)$. In our case, having fixed a task t_j, at each instant μ_i there is a relation with a given number of techniques described in terms of praxeological variations that occur in the management of the calculations performed by the students.

Thus, in the search for the details inherent to the students' praxelogical repertoire, ostensibly communicated by different techniques used in the resolution of tasks, a resolutive work was fostered through the GT, which made it possible to identify the existence of the PPPs, from the perspective of a more refined observation lens. Schematically, as in Figure 3:

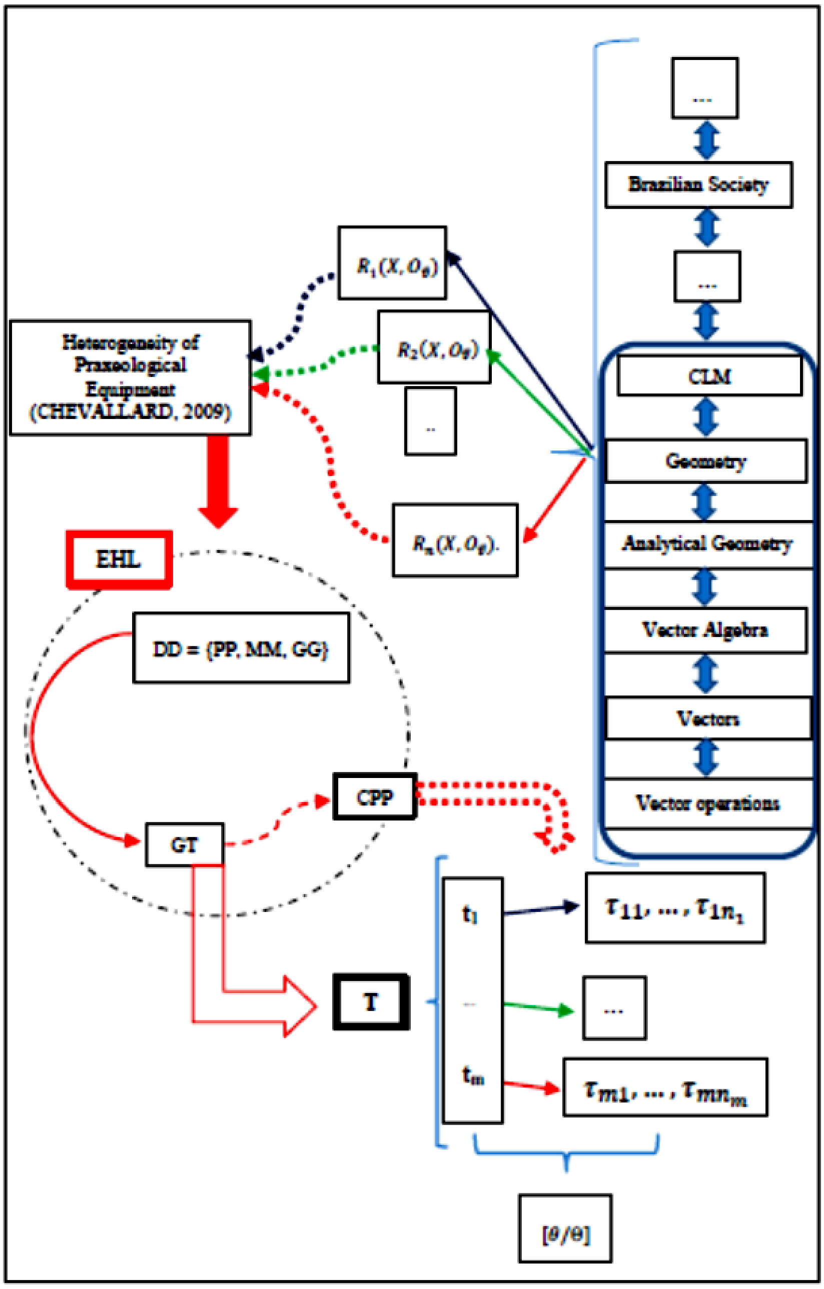


Figure 3: Design of the interconnections between research elements

An excerpt of the experimentation.

The experimental aspects of the research will not be fully described. However, we will provide elements that will lead the reader to overview the directions of the investigation. The example dealt with in this text consists of a vectorial task, managed, and solved by some students, according to techniques characterized by didactic devices[4], in this case, Manipulable Materials – MM with records

[4] Didactic devices materialize techniques (CHEVALLARD, 2002, p.2)

on Pencil Paper – PP. However, due to space limitation, we will not report the students' productions related to the GeoGebra – GG software and the exclusive use of PP, in favor or denser descriptions of the techniques used in each resolution.

At this stage, mini-courses were recommended and developed for the study of vectors contextualized through tasks that integrated the Numerical, Algebraic and Geometric domains – NAG (FARIAS, 2010). So, students were invited to participate in this didactic endeavor. We offered twenty vacancies, of which eighteen were filled, and held the meetings off-shift. It is noteworthy that vector algebra is part of the curricular component Analytical Geometry syllabus that makes up the 2nd-semester grid of the teaching degree course.

We then designed a Didactic Organization as described by Goulart (2021), in terms of the arrangement of the variables $V_1 = \{GG, MM, PP\}$, through the devices established in the course with its derivations and branches, as summarized in Figure 4.

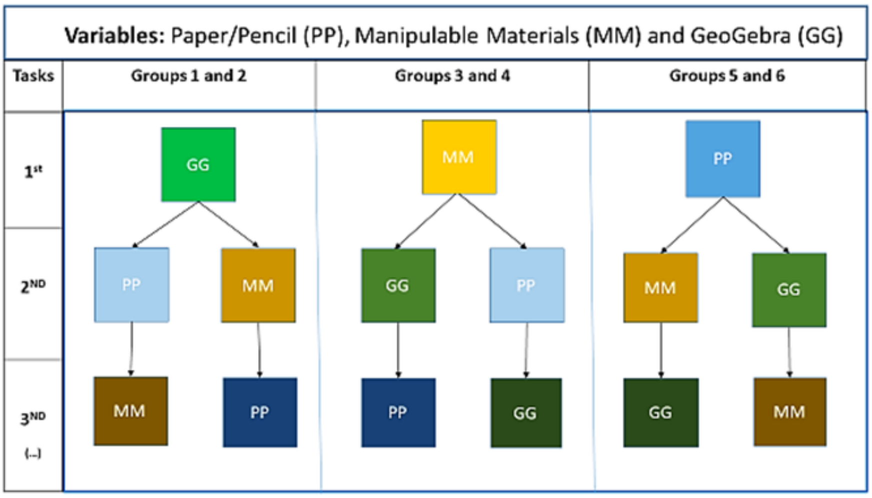


Figure 4: Drawing of the arrangements of the variables in terms of the devices

From this outline, there was a structuring, revealed due to transformations that reflected in the expansion of the number of tasks, which required different techniques because, as Chevallard (1989, p.93) indicates, we are generally limited to a "small number of techniques recognized institutionally, excluding possible alternative techniques (...)".

In favor of these alternative techniques, we presented tasks to students that dealt in essence with vectors, Mathematical Organization - OM, which, according to Costa, Arlego, and Otero (2015, p. 146), "provides students with basic and indispensable tools to model several physical events mathematically". Beyond the physical phenomena, the vector calculus or the vector algebra, in their different nuances, reverberates in other mathematical domains.

As a result, we can assume that a rupture emerges with something that is disseminated in the institutional scope, because "instead of confronting the student with a type of task for which there are several techniques, the institution organizes the study through several types of tasks for which there is only one technique." (CHAACHOUA, 2010, p. 6, our translation[5]), i.e., students should recognize to which type of tasks they belong and what techniques are pertinent to use to obtain an answer accepted or not by the institution.

That said, in this text, our attention will turn to task 5, as denoted in the Goulart (2021). Beyond managing calculations developed by the students, in agreement with the institution, we will consider the ostensives of departure, development, and arrival, according to a given time ($\mu\ tempo$) of observation and analysis, which made us reflect and assume as inspiration to theorize the PPPs. Thus, it was a task that presents in its approach vectors in space, usually denoted by R^3, with the following statement de t_5: *In the ABCD tetrahedron, E is the midpoint of the DC segment, and F satisfies* $\overrightarrow{BF} = \frac{1}{4}\overrightarrow{BE}$. *Suppose* $\vec{a} = \overrightarrow{AB}; , \vec{b} = \overrightarrow{AC}$ *e* $\vec{c} = \overrightarrow{AD}$. *Get* $\overrightarrow{AF}$ *as a linear combination of* $\vec{a}$, $\vec{b}$ *and* $\vec{c}$.

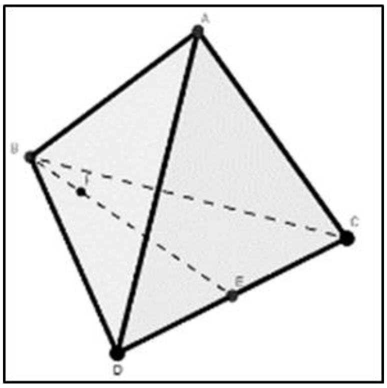


Figure 5: Representation that integrates the statement of task 5

We will highlight the Task Type Subgenerator (KASPARY, 2020) that derived from GT_{t5} =*[Get; a vector as a linear combination of a base;*V_1,V_2, V_3, V_4,V_5, V_6, V_7*]*[6], which guided the production of tasks according to the value assumed by each variable, in line with the devices, proposing conditions to the environment (*milieu*) to promote articulations between vector concepts and properties with manipulative techniques that triggered changes in the personal relationships of the subjects in interaction with vectors.
Thus, $G'T_{t5(MM)}$ as described in Table 1:

Table 1. Task Type Subgenerator $G'T_{t5(MM)}$

$G'T_{t5(MM)}$ = [*Obter; um vetor como combinação linear de uma base em MM*; V_2, V_3, V_6, V_7]			
V_2 - *Nature of MM (straws, elastics, strings, EVA, among others).*	V_3- *Auxiliary instruments aggregated to MM (scissors, glue, studs, Styrofoam plate, etc.).*	V_6- *Available PP instruments (ruler, squares, millimeter squared paper, etc.).*	V_7- *Configuration (plane, space, rectangular, triangular, ...).*

[5] Ainsi, plutôt que de confronter l 'elève à un type de tâches pour lequel il existe plusieurs techniques, l 'institution organise l'étude à travers plusieurs types de tâche pour lesquels il y a une seule technique.

[6] See Goulart (2021)

At this stage, the productions should reproduce a tetrahedron in a tangible environment, to then obtain $\overrightarrow{AF}$, which alters the material instantiation of this object. This allowed us to observe that the variable V_7 assumes another value since even the previous questions were developed in the plan, i.e., a change in the configuration. Thus, the attention turns to alternating images by sequencing some records evidenced in excerpts from a student's production, as shown in Figure 6,

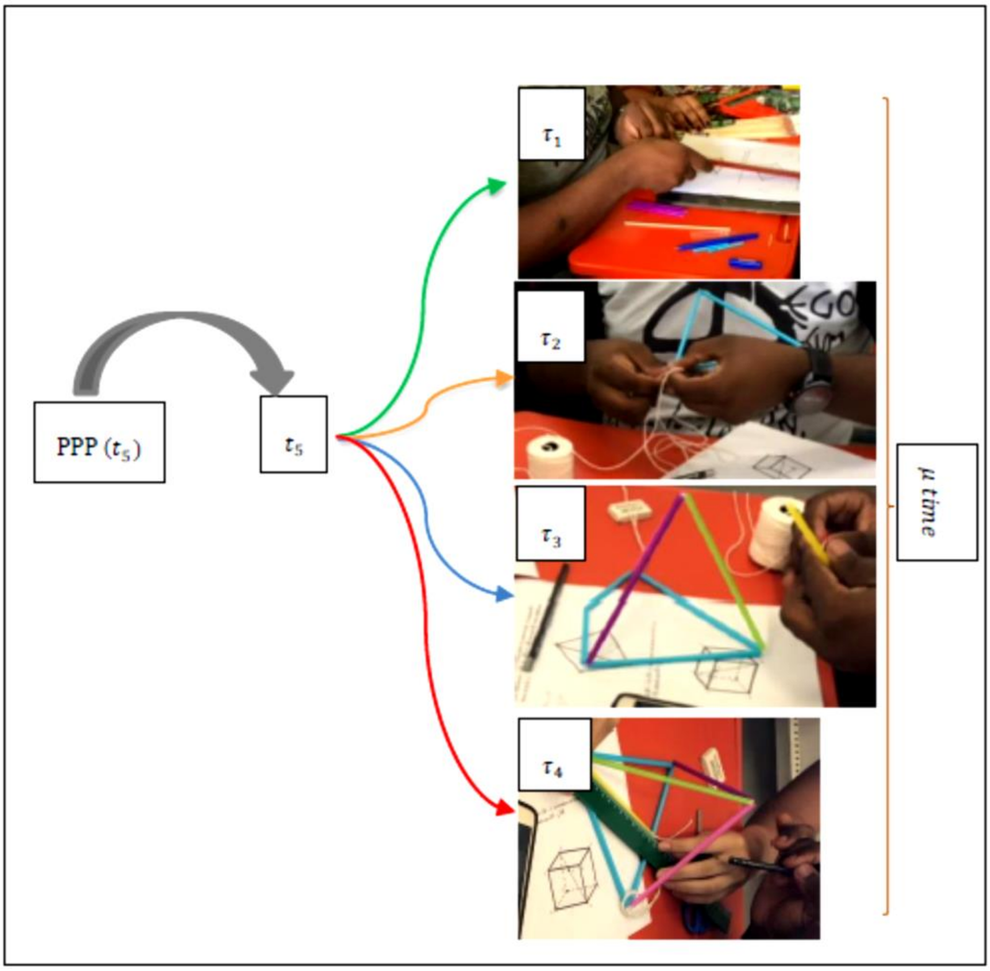


Figure 5: Characteristic elements of the PPP$_{(t_i)}$ in terms of μ $_{tempo}$

Thus, we infer that PPPs allow regulating the levels of praxeological analyses according to a given time fraction *(micro - μ or macro -M)*, in accordance with the research motivations. However, it is relevant to point out that this exemplification is presented as a fragment of the entire process and that the adjustments of the model dialogue directly with the objectives that the researcher aims to achieve. In summary, when observing the images, it is possible to undertake some interpretations: at the moment $\mu\tau_1$, we can note elements specific to one of the moments of study specified by Chevallard et al. (2001). In $\mu\tau_2$, we can observe the representation of one of the triangles that make up the tetrahedron. Yet, in $\mu\tau_3$, a three-dimensional prototype appears and in $\mu\tau_4$ there is a need to use the ruler and pencil paper belonging to V_6, as shown in Table 1.

In these terms, it is feasible to verify that the time interval of observations records appears as another variable that will allow regulating, calibrating, or tuning the objectives inherent to the investigation with the records of the ostensive changes observed in the images, which enable us to achieve details that are not always reached when we examine the extremes (departure and arrival) of a resolution.

Emergence of some developments for research in Didactics.

From the PPP, we glimpse one more theoretical element that will certainly contribute to future studies in the context of Didactics. It is a gesture of study conceived as an *Individual-Individual Dialectic,* which has as its conceptual inspiration the reflexive property of the relation of equipolence between oriented segments that can constitute one of the levels of analysis of the PPP, which can be characterized as a solitary moment of knowledge production of the (student ~ student) in line with the Dialectic of the Individual and Collective as in the dialogue for the construction of personal (CROSET & CHAACHOUA, 2016) and group praxeologies.

In this sense, we see the possibility of Study and Research Paths that contemplate the personal and individual aspect of a subject, in particular, returning to the metaphor that contributed to the characterization of our investigative design as an analysis of the performance of a single player in a soccer match. We think that by having access to the instants of personal praxeologies that occur according to specific dynamics when solving a task, it is possible to examine thoroughly how students manage their calculations, i.e., it will be feasible to describe the current state of their praxeological equipment when faced with certain types of tasks – *T*, which will allow the identification of problematic points that restrict an "adequate" resolution process. This consists of presenting to the student strategies that will enable the player (student) to develop actions that result in "goal" (good resolution management). Once this is done, it will facilitate the insertion of themes, subjects or contents that are not yet part of the subject's praxeological baggage. We also cannot forget that the change in techniques is reflected in the evolution of the technological aspect characterized as a strong point of the study.

Therefore, this study reinforces that the ATD is a theory that allows developments, motivated by new questions and by connexions with other theoretical frameworks such as T4TEL (CHAACHOUA, 2018) contributing to a broader understanding of didactic phenomena.

References

Artigue, M. (1996). Engenharia Didática. *In*: Brun, J. *Didática das Matemáticas.* Tradução Maria José Figueiredo. Lisboa: Instituto Piaget. p. 193-217.

Chaachoua, H. (1998). Géométrie dans l'espace. Le point sur la lecture des dessins pour les élèves en fin de collège. *Petit x*, (48), 37–68, 1998.

Chaachoua, H.; Bessot, A.(2016). Introduction de la notion de variable dans le modele praxéologique. *Congrès pour la Théorie Anthropologique du Didactique*, 5. 2016, Espagne. Atas [...]. Castro-Urdiales, Espagne, 2016.

Chaachoua, H. (2010). La praxéologie comme modèle didactique pour la problématique EIAH. Etude de cas: la modélisation des connaissances des élèves. *Séminaire National de Didactique Des Mathématique*. Paris, France. p. 82. https://hal.archives-ouvertes.fr/hal-01282325/documen.

Chaachoua, H. (2018). T4TEL un cadre de référence didactique pour la conception des EIAH. *In: Séminaire de Didactique des Mathématiques de L'ardm. 2018. Atas* [...]. Paris. (pp. 5-22). https://ardm.eu/wp-content/uploads/2018/10/Préactes-ARDM-fevrier2018.

Chaachoua, H., & Bessot, A. (2018). A noção de variável no modelo praxeológico. *In*: *A Teoria Antropológica do Didático: princípios e fundamentos.* Almouloud, S. A.; Farias, L. M. S.; Henriques, A. (org.). Curitiba: CVR. p. 119-133.

Chevallard, Y. (1996). Conceitos Fundamentais da Didática: as perspectivas trazidas por uma abordagem antropológica. In: Brun, J. *Didática das Matemáticas*. Tradução Maria José Figueiredo. Lisboa: Instituto Piaget, 1996.

Chevallard, Y. (1998). Analyse des pratiques enseignantes et didactique des mathématiques: l'approche anthropologique. In: L'UNIVERSITE D'ETE, 1998, p. 91-118. *Actes* [...]. Clermont-Ferrand, France: IREM. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Analyse_des_pratiques_enseignantes.pdf.

Chevallard, Y. *et al*. (2001). *Estudar Matemática: o elo perdido entre o ensino e a aprendizagem.* Tradução Daisy Vaz de Moraes. Porto Alegre: Artes Médicas.

Chevallard, Y. (2002). Nouveaux dispositifs didactiques au collège et au lycée: raisons d'être, fonctions, devenir. Communication aux *Journées de la commission inter-IREM Didactique.* Paru dans les actes correspondants (pp. p.1-26). IREM, Dijon.

Chevallard, Y. (2007). Passé et Présent de la Théorie Anthropologique du Didactique. En L. Ruiz-Higueras, A. Estepa y F. J. García (eds.), *Sociedad, Escuela y Matemáticas*. *Aportaciones de la Teoría Antropológica de la Didáctico* (pp. 705-746). Universidad de Jaén, http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=134.

Chevallard, Y. (2009). Théorie Anthopologique du Didactique & Ingénierie Didactique du Développement. In: *Journal du Seminaire TAD/IDD – 3*; pp. 1-39. <http://yves.chevallard.free.fr/spip/spip/IMG/pdf/journal-tad-idd-2009-2010-3.pdf>.

Croset, M. C. & Chaachoua, H.(2016). Une réponse à la prise en compte de l'apprenant dans la TAD: la praxéologie personnelle. *Recherches en Didactique des Mathématiques*, Grenoble, 36-2.

Costa, V. A., Arlego, M., & Otero, M.R. (2015). Las dialécticas en un Recorrido de Estudio e Investigación para la enseñanza del Cálculo Vectorial en la Universidad. *Revista de Formación e Innovación Educativa Universitaria. 8*(3), 146-161,.

Dreyfus, T. (1991). Advanced mathematical thinking processes. In: TALL, D. (ed.). *Advanced mathematical thinking* (pp. 25-41.. Dordrecht: Kluwer.

Farias, L. M. S. (2010). *Étude des interrelations entre les domaines numérique, algébrique et géométrique dans l'enseignement des mathématiques au secondaire: une analyse des pratiques* *enseignantes en classes de troisième et de seconde*. Tese (Doutorado) – Universidade de Montpellier 2, France.

Goulart, J. S. S.; Farias, L. M. S. (2018). Um diálogo sobre a Teoria Antropológica do Didático – TAD intermediado por um curso introdutório sobre os vetores. *Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática*, *20*(3) jan. http://revistas.pucsp.br/emp/article/view/40111.

Goulart, J. S. S. (2021). *Uma proposta didática para o ensino da álgebra vetorial fundamentada pelo modelo T4TEL no contexto de um ambiente híbrido de aprendizagem.* Tese (Doutorado) Universidade Federal da Bahia em covênio com a Universidade Estadual de Feira de Santana.

Kaspary, D. (2020). *La noosphère, un lieu de tension pour le curriculum: étude didactique de la mise en place d'un système d'évaluation de manuels scolaires sur l'étude du champ additif à l'école primaire.* Université Grenoble Alpes; Thèse de doctorat. Universidade Federal de Mato Grosso do Sul, 2020. Français. https://hal.archives-ouvertes.fr/tel-03100691/.

Kaspary, D. *et al*.(2020). Ou apporte la Notion de Portee d'une Technique a l'etude de la Dynamique Praxeologioue*. Annales de Didactique et de Sciences cognitves* , vol. 25 p.243-269.

Silva, A. *et al*. (1999). O Currículo de Matemática e as Actividades de Investigação. In: Abrantes, P.; Ponte, H. Fonseca; L. Brunheira (org.). *Investigações matemáticas na aula e no currículo* (pp. 69-85.). Lisboa: APM e Projeto MPT.

D'un modèle didactique de référence pour la position de professeur

Michèle Artaud

ADEF, Aix-Marseille Université, France; artaudm.inspe@gmail.com

Abstract. The civilisational constraint of the denial of the didactics leads, on the one hand, to a pedagogical reduction of didactical gestures and, on the other hand, to their epistemological reduction. This makes it difficult to explain techniques for carrying out moments of study which are both sufficiently generic and contain ingredients of specificity linked to the works studied. It is this problem that we will address in this paper –without pretending to solve it– through the analysis of examples from the teaching of mathematics in secondary education in France (students aged 11 to 18) and from the training of mathematics teachers. In particular, we will examine the praxeological needs for the direction of the study of the teacher's position that emerge through the analysis of these examples.

Résumé : La contrainte civilisationnelle du déni du didactique conduit, d'un côté, à une réduction pédagogique des gestes didactiques et, d'un autre côté, à leur réduction épistémologique. Cela rend difficile l'explicitation de techniques de réalisation des moments de l'étude qui, à la fois, soient suffisamment génériques et comportent des ingrédients de spécificité liées aux œuvres étudiées. C'est ce problème que nous aborderons dans cette communication – sans prétendre le résoudre – à travers l'analyse d'exemples issus de l'enseignement des mathématiques dans l'enseignement secondaire en France (élèves de 11 à 18 ans) et de la formation des professeurs de mathématiques. Nous examinerons en particulier les besoins praxéologiques de direction d'étude de la position de professeur qui se font jour à travers l'analyse de ces exemples..

Observer et analyser le didactique

Gestes didactiques et moments de l'étude

Lors d'une conférence aux journées sur la TAD qui se sont tenues à Barcelone en 2016, nous avions exploré les questions auxquelles les chercheurs en didactique, dans des disciplines autres que les mathématiques, apportent une réponse produite par la TAD, et nous avions mis en évidence que les questions relatives aux organisations de l'étude en étaient largement, pour ne pas dire totalement, absentes (Artaud, 2016a). Cela rappelle, s'il en était besoin, la nécessité de développer les infrastructures didactiques : nous nous proposons d'y contribuer ici en examinant la question des gestes didactiques à travers l'analyse de moments de l'étude.

Le didactique se niche à l'intersection de–- et à l'interaction entre – l'œuvre à étudier dont la nécessité de l'étude peut surgir d'une question, d'une part, et des gestes accomplis par une instance pour aider une autre instance (éventuellement elle-même) à étudier cette œuvre ou cette question, d'autre part. Ces gestes didactiques, qui peuvent relever de l'ensemble des niveaux de codétermination didactique – du plus spécifique au plus générique –, s'inscrivent dans des praxéologies didactiques, d'étude et de direction d'étude, et nous plaidons depuis plusieurs années pour que l'élucidation de ces praxéologies soit organisée, pensée, à partir du modèle des moments de l'étude (Artaud, 2011 ; Artaud

& Cirade, 2021). En resituant les gestes didactiques dans tel ou tel épisode du moment de l'étude où ils apparaissent, on voit que, d'un côté, les mêmes types de tâches pourraient être mobilisés dans différents moments de l'étude tout en nécessitant d'être accomplis avec des techniques différentes suivant le moment de l'étude à la réalisation duquel il concourt – variations de techniques qui dépendent bien entendu de beaucoup d'autres facteurs – et que, d'un autre côté, certains gestes didactiques pourraient être mobilisés pour réaliser certains moments de l'étude et pas d'autres..

Une praxéologie d'analyse de la réalisation d'un épisode d'un moment de l'étude

C'est dans cette perspective que nous présenterons ci-dessous quelques analyses de réalisation d'épisodes de moments de l'étude. Une première analyse a déjà été faite, permettant de dégager, d'une part, la praxéologie mathématique ou le fragment de praxéologie mathématique enjeu de l'étude durant le processus d'étude considéré, d'autre part, les moments de l'étude réalisés et leur découpage (éventuel) en différents épisodes.

Chacun des épisodes considérés a été observé dans une séance en classe à laquelle nous avons eu accès par le biais d'un compte rendu d'observation. Ces séances ont été réalisés par des professeurs stagiaires de mathématiques durant leur formation, et les analyses que nous présentons n'ont pas été produites en position de chercheur mais en position de formateur – il s'agit généralement d'analyses construites à partir de propositions d'étudiants. Le but poursuivi est de donner aux élèves professeurs des matériaux pour travailler à la construction d'un équipement praxéologique, en position de professeur de l'enseignement secondaire, compatible avec le paradigme de questionnement du monde – ce qui supposera aussi d'évaluer ces analyses et de les développer. Dans cette perspective, la technique élaborée comprend principalement quatre axes d'analyse : identifier l'entité praxéologique qui est produite, travaillée, évaluée ou institutionnalisée ; identifier le dispositif dans lequel on se place pour réaliser cet épisode du moment de l'étude considéré ; expliciter le rôle et le topos[1] respectifs de la position de professeur ou de la position d'élève ; expliciter le milieu d'étude qui permet la production, le travail, l'évaluation ou l'institutionnalisation de la praxéologie mathématique ou du fragment de praxéologie mathématique enjeu de l'étude.

L'environnement technologico-théorique de cette technique comprend pour l'essentiel les notions de praxéologie, de milieu, de rôle et de topos, ainsi que le modèle des moments de l'étude et le schéma herbartien (Chevallard, 2002 & 2019). C'est notamment ce dernier qui permet de justifier la présence des quatre axes d'analyse. Puisque le schéma $(S(X\ ; Y\ ; Q) \rightarrow M) \hookrightarrow R^{\heartsuit}$ modélise le processus d'étude d'une question Q amenant à une réponse $R^{\heartsuit}$ et qu'un (épisode d'un) moment de l'étude est une partie de ce processus, la réalisation d'un (épisode d'un) moment de l'étude doit contenir ce que font X et Y dans l'étude de Q – ce qui est pris en charge ici par l'intermédiaire des notions de rôle et de topos

[1] On rappelle que, dans l'accomplissement d'une tâche *collaborative t*, « chacun des acteurs x_i [doit] effectuer certains gestes, dont l'ensemble constitue alors son *rôle* dans l'accomplissement de *t*, ces gestes étant à la fois différenciés (selon les acteurs) et coordonnés entre eux par la technique τ mise en œuvre collectivement. Certains de ces gestes seront regardés alors comme des tâches à part entière, *t'*, dans l'accomplissement desquelles x_i agira (momentanément) *en autonomie relative* par rapport aux autres acteurs de la tâche. L'ensemble de ces tâches, sous-ensemble du rôle de x_i lorsque *t* est accomplie selon τ, est nommé alors le *topos* de x_i dans *t*. » (Chevallard, 1999, p. 247)

–, le milieu pour l'étude auquel le système se confronte ainsi que les ingrédients de R produits – réponse qui est dans ce contexte une praxéologie mathématique.

Gestes didactiques susceptibles d'analyser les moments de l'étude

Nous allons présenter trois analyses d'épisodes, deux relatifs au moment exploratoire et un relatif au moment technologico-théorique, que nous compléterons de quelques remarques évaluatives ; nous rappelons qu'elles sont construites en formation à partir de propositions d'étudiants. Cela nous permettra ensuite de dégager certains gestes didactiques afin de développer quelques aspects que nous avons mentionnés dans la première partie.

Deux analyses

Voici une première analyse. L'observation est relative à une séance en classe de 3e (élèves de 14-15 ans) portant sur le plus grand commun diviseur.

> L'épisode observé du moment exploratoire est effectué par le biais du dispositif suivant : une activité dont le but est de réaliser un nombre de bouquets identiques avec un certain nombre de fleurs de deux types différents. Il s'agit alors de calculer le plus grand diviseur commun correspondant au nombre maximal de bouquets de ce type composés de fleurs issues des deux types. Ce dispositif permet d'explorer deux spécimens, un premier cas où l'on dispose de 252 marguerites et 144 roses et un deuxième cas où l'on dispose de 20 marguerites et 21 roses.
>
> Afin de réaliser cet épisode du moment exploratoire, la classe doit savoir qu'un nombre est divisible par un autre si le reste de la division euclidienne du premier par le second vaut 0, les principaux critères de divisibilité (par 2, 3, 5, 11), ou encore que pour qu'un nombre soit divisible par un produit de facteurs, il est nécessaire qu'il soit divisible par l'un des facteurs, suffisant qu'il soit divisible par chacun des facteurs ; il s'agit du milieu. La détermination de l'ensemble des diviseurs d'un nombre fait partie du topos de l'élève.
>
> Finalement, durant cet épisode, le rôle de l'élève est de chercher la solution à la question posée alors que le professeur pose des questions qui scandent l'avancée du temps de l'étude et dégage les étapes de la technique. Le professeur est celui qui dirige la classe puisqu'il interroge la classe et envoie des élèves au tableau. Il valide et invalide les réponses que ceux-ci fournissent, ces réponses venant enrichir le milieu.

On voit apparaître dans cette analyse les quatre axes proposés dans la technique d'analyse. Le type de tâches étudié est mentionné, mais aucune indication ne figure sur la technique ; le dispositif est rappelé de façon détaillée, les œuvres du milieu pour l'étude également, mais les questions intermédiaires ou les données ne le sont pas. Notons qu'il apparaît également que, dans la dialectique des médias et des milieux réalisée dans la classe, si les réponses des élèves jouent le rôle de média, il n'est pas certain qu'elles jouent le rôle de milieu. Pour amener ces réponses dans le milieu pour l'étude, le professeur pose des questions (non précisées, on l'a dit) et envoie des élèves au tableau. Les conditions de production des réponses par les élèves sont assez opaques, excepté la mise à disposition des éléments du milieu pour l'étude explicités.

Voici une deuxième analyse. L'observation est cette fois celle d'une séance de quatrième (élèves de 13-14 ans) dans laquelle il s'agit de produire la propriété de la position du centre de gravité d'un triangle sur une médiane à partir de la question de la construction du troisième sommet, C, d'un triangle, connaissant un côté [AB] et le centre de gravité G.

> La séance observée contient deux épisodes du moment exploratoire de longueur inégale réalisé dans le cadre d'une activité. Le premier épisode explore le type de tâches à partir du spécimen fourni par la feuille d'activité, et d'un spécimen tracé au tableau par le professeur. Le professeur instaure un dialogue avec les élèves qui aboutit à la question « peut-on construire les médianes ? ». Cette question reçoit une réponse positive produite par les élèves. Une nouvelle question émerge : « où se situe C sur la médiane ? », qui n'est pas véritablement explorée, de même qu'une proposition d'un élève de faire glisser la règle pour avoir une approximation de la solution. Le topos de la position d'élève comprend ainsi la proposition de voies de solution, le tracé des médianes tandis que le topos du professeur comprend principalement la validation des propositions des élèves et leur éventuelle correction (pas systématique). Le milieu comprend, outre les deux questions signalées précédemment, l'OM autour du tracé de la médiane, la règle et le compas, ainsi que la feuille d'activité comprenant le spécimen du type de tâches et la figure tracée au tableau, le fait que G est intérieur au triangle. Un deuxième épisode du moment exploratoire a lieu une fois que la propriété donnant la position du centre de gravité sur les médianes a émergé. C'est P qui en prend l'initiative en affirmant « maintenant, je pense qu'on va pouvoir arriver à trouver le point C ». La production de la technique fait partie du topos de l'élève. Le milieu est identique à celui de l'épisode précédent.

Par contraste avec l'analyse précédente, le dispositif n'est pas détaillé mais les questions intermédiaires ou les données le sont davantage. Il apparaît également que, dans la dialectique des médias et des milieux à l'œuvre dans la classe, les réponses des élèves jouent le rôle de média mais pas celui de milieu. Pour amener ces réponses dans le milieu pour l'étude, le professeur pose des questions et on ne sait pas si des élèves viennent les exposer au tableau. Le fait que la production de réponses aux questions posées par le professeur se situe dans le topos de l'élève peut sous-entendre que du temps de travail en autonomie est donné mais ce n'est pas explicité. On peut noter également que, dans la réalisation des deux épisodes, les œuvres du milieu d'étude sont dans l'ensemble endogènes, soit qu'elles fassent partie de ce qui est « déjà connu », soit qu'elles soient produites par la classe dans la séance. C'était également le cas dans la première analyse proposée.

Des gestes didactiques

Les deux analyses précédentes permettent ainsi de mettre en évidence les gestes didactiques suivants, que l'on pourrait souhaiter mieux caractériser :

- Créer des conditions qui influent sur la production par la position d'élève de réponses $R^\lozenge$ à la question étudiée ou aux questions secondaires suscitées par son étude ;
- Créer des conditions qui influent sur l'existence et l'importance de la mise en œuvre d'une dialectique des médias et des milieux dans le topos de la position d'élève ;
- Intégrer une œuvre dans le milieu d'étude, qui peut faire partie du topos du professeur ou de celui de l'élève ou encore être un type de tâches collaboratif.

Une partie de la technique de création de conditions favorisant l'intégration dans le topos de la position d'élève de la production de réponses $R^{\lozenge}$ ou encore de la mise en œuvre, partiellement au moins, d'une dialectique des médias et des milieux par la position d'élève repose sur le développement du milieu pour l'étude. Ainsi par exemple, dans les deux épisodes précédents, on peut noter que le milieu pour l'étude ne comporte qu'un petit nombre d'exemplaires de tâches du type de tâches à étudier : deux spécimens dans chaque épisode. Cette disponibilité d'un milieu d'étude « suffisamment riche » n'est cependant pas suffisante : il faut encore que des conditions de son exploitation soient réunies – on pourrait citer par exemple la constitution préalable par le professeur d'un réseau de questions cruciales.

Le troisième geste didactique identifié est crucial. On peut noter à cet égard que, dans les deux cas, donner à chaque élève ou à chaque petit groupe d'élèves de la classe un spécimen différent du type de tâches permet d'augmenter la base expérimentale et d'accroitre à la fois la possibilité d'émergence de techniques et la possibilité de les mettre à l'épreuve sans trop augmenter le temps d'horloge consacré au moment exploratoire ; il en va de même de la mise à disposition d'un tableur et d'un logiciel de géométrie dynamique. Soulignons que ces deux gestes – augmenter la base expérimentale en donnant à chaque élève ou à chaque petit groupe d'élèves de la classe un spécimen différent du type de tâches enjeu de l'étude ; mettre à disposition un système informatique permettant de mettre à l'épreuve les techniques – peuvent être vus comme classiques, voire comme « allant de soi », mais que c'est leur intégration fonctionnelle comme sous-geste du geste de direction d'étude « Intégrer une œuvre dans le milieu pour l'étude » d'un moment exploratoire par la position de professeur qui nous intéresse ici. Le fait que, dans chaque cas, on soit dans un dispositif d'AER partant d'une situation matérialisant une raison d'être de l'organisation mathématique enjeu de l'étude est également une condition plus qu'importante.

Une troisième analyse

L'observation est cette fois relative à une séance en classe de 6e (élèves de 11-12 ans) portant sur la médiatrice d'un segment (Artaud, 2016b). Les élèves disposent déjà de la définition de la médiatrice d'un segment comme droite perpendiculaire au segment passant par le milieu du segment, et savent la construire avec la règle graduée et l'équerre. Il s'agit dans la séance de faire émerger une organisation mathématique autour du type de tâches « Construire l'ensemble des points équidistants de deux points donnés, A et B ». Ce type de tâches sera accompli en construisant la médiatrice de A et de B à la règle et au compas, soit en construisant deux points équidistants de A et de B, C et D, l'un au-dessus du segment et l'autre au-dessous et en traçant la droite [CD]. Cette construction est justifiée par la caractérisation de la médiatrice de [AB] comme ensemble des points équidistants de A et de B. L'activité qui permet de faire émerger cette organisation mathématique est reproduite ci-après (figure 1).

Dans un stand d'une kermesse on a installé un jeu d'adresse. Il s'agit de lancer sur une planche une bille métallique entre deux gros aimants. Si la bille reste toujours à égale distance des aimants elle peut traverser la planche et l'on gagne la partie. Si elle se rapproche de l'un des aimants, celui-ci attire la bille qui vient se coller dessus et la partie est perdue.

On a représenté la planche et les deux aimants (A et B) sur le dessin ci dessous.
Tracer le trajet de la bille pour gagner la partie.

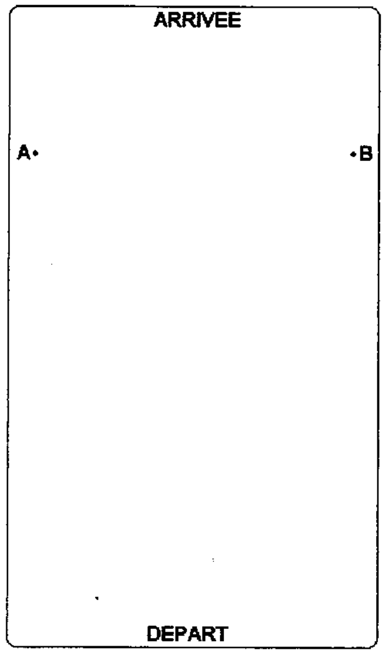


***Figure 1.* Une activité pour l'étude de la construction de l'ensemble de points équidistants de deux points donnés.**

Voici l'analyse de la réalisation du moment technologico-théorique.

> Dans la séance observée, deux épisodes du moment technologico-théorique sont réalisés. Le premier prend place dans l'étude de l'activité et se situe dans le topos de la position de professeur. D'abord quand celui-ci annonce à un élève qui propose de tracer la perpendiculaire à [AB] passant par le milieu pendant le travail de tracé des sommets des triangles « oui, mais on va d'abord tracer point par point et quand on rejoindra les points on verra que, effectivement, c'est la perpendiculaire passant par le milieu », dégageant ainsi au moins que l'ensemble cherché est une droite qui a certaines propriétés. Puis, en fin de travail, quand il annonce que l'ensemble cherché est la médiatrice de [AB] et qu'il formule la propriété dégagée. Le second a lieu lors de la synthèse, une fois la propriété écrite sous la forme « La médiatrice du segment [AB] est l'ensemble des points équidistants de A et de B ». Le professeur tient alors un discours venant justifier la propriété : « Alors, ça marche dans les deux sens, ça veut dire, si vous avez un point de la médiatrice, vous êtes sûrs qu'il est à la même distance de A et de B ; et on peut l'utiliser dans l'autre sens : si vous voulez un point à égale distance de A et B, il faudra aller le chercher sur la médiatrice, il n'y a pas d'autres points. Dès qu'on s'écarte de la médiatrice, on est plus près soit d'un point soit de l'autre. » Il fait ensuite remarquer que la médiatrice du segment [AB] est un axe de symétrie de [AB] ; c'est celui qui passe par le milieu. Donc si on prend un point C sur la médiatrice de [AB], quand on trace le symétrique de [AC], il passe par C – comme il est sur l'axe, il n'a pas bougé – et il

passe par B parce que A et B sont symétriques ; si on plie, on aura bien distance AC égale à la distance AB.

Cette analyse comporte comme les précédentes les quatre axes proposés mais de façon moins explicite et plus « narrative ». Le milieu d'étude comme les topos des deux positions d'élève et de professeur notamment sont moins dégagés.

Il faut noter à l'égard du milieu pour l'étude que, d'un côté, si la construction d'un point équidistant de deux points donnés comme sommet d'un triangle isocèle de base le segment d'extrémités les deux points donnés, le compas et la feuille d'activité sont des œuvres en faisant partie, les autres œuvres susceptibles d'en faire partie comme les réponses des élèves ne sont pas nettement intégrées ; et que, d'un autre côté, certaines ont un rôle théorique dans l'organisation mathématique construite comme la définition de la médiatrice ou encore la symétrie axiale et ses propriétés, ce qui peut gêner leur identification comme partie du milieu de l'étude.

Le topos de la position d'élève paraît réduit, c'est celui de la position de professeur qui prend le dessus. Par conséquent la dialectique des médias et des milieux est réduite à peau de chagrin dans le topos de la position d'élève, d'autant que le milieu d'étude ne comprend qu'un spécimen du type de tâches. On notera cependant la convocation d'un milieu déductif, même s'il n'est pas véritablement assumé, par le biais du discours « si on prend un point C sur la médiatrice de [AB], quand on trace le symétrique de [AC], il passe par C – comme il est sur l'axe, il n'a pas bougé – et il passe par B parce que A et B sont symétriques ; si on plie, on aura bien distance AC égale à la distance AB » : il déduit des propriétés de la symétrie le fait que si un point est sur la médiatrice d'un segment, il est à égale distance des extrémités de ce segment – l'activité n'a justifié « expérimentalement » que la réciproque – l'expérience a porté sur un seul spécimen.

L'analyse proposée permet encore de mettre en évidence qu'une partie du moment technologico-théorique prend place dans le dispositif de synthèse, qui a plutôt vocation à mettre en forme l'organisation mathématique construite, et donc à réaliser le moment d'institutionnalisation (Chevallard, 2002).

Retour sur les gestes didactiques

Reprenons le geste didactique « Intégrer une œuvre dans le milieu pour l'étude ». Dans ce moment technologico-théorique, selon l'analyse proposée, il apparaît réalisé avec une technique différente de celles que nous avons vues à l'œuvre dans l'analyse des moments exploratoires, puisqu'il fait partie du topos de la position de professeur. Cela ne semble pas résulter d'une impossibilité des élèves de produire des réponses à la question de la justification de la propriété qui a émergé puisque cette question n'est pas posée. On peut donc davantage y voir une difficulté de la position de professeur quand il s'agit de créer des conditions favorables à l'existence d'une praxéologie autour du type de tâches « justifier une assertion » dans le topos de la position d'élève lors du moment technologico-théorique. Notons que, dans le cadre d'un moment de travail d'une organisation mathématique, cette difficulté – si elle existe – ne sera sans doute pas de même nature puisque l'assertion à justifier relèvera le plus souvent d'un type de tâches pour lequel les élèves disposent d'une technique. Pour le dire autrement, les réponses $R^\Diamond$ produites par la position d'élève à la question Q ne comprennent pas d'ingrédients technologiques.

Vers un modèle didactique de référence pour la position de professeur

Les analyses précédentes ainsi que les commentaires faits à leur propos nous paraissent dessiner un embryon d'infrastructure d'un modèle didactique de référence pour la position de professeur. On obtient en effet des éléments sur des gestes didactiques permettant la réalisation des moments de l'étude, et notamment sur l'intégration d'une œuvre dans le milieu pour l'étude, geste qui parait crucial pour aller vers le paradigme du questionnement du monde.

Les éléments notés sont principalement situés au niveau de codétermination de la pédagogie, même si certains, comme l'utilisation d'un logiciel de géométrie dynamique pour augmenter la mise à l'épreuve et la base expérimentale relèvent du niveau de la discipline. La contrainte civilisationnelle du déni du didactique conduit sans doute à une réduction pédagogique de ces gestes et rend difficile l'explicitation de techniques de réalisation des moments de l'étude qui, à la fois, soient suffisamment génériques et comportent des ingrédients de spécificité liées aux œuvres étudiées. Certains éléments de spécificité sont liés aux AER, voire aux PER, dans lesquels certains des moments sont réalisés, qu'il faudrait donc davantage mettre en lumière en les accompagnant des raisons d'être des organisations mathématiques (OM) étudiées. Certains autres sont relatifs aux OM qui sont produites et qui font partie du modèle praxéologique de référence. L'analyse des séances ou des séquences mettant au jour au préalable cette OM, comme nous l'avons dit plus haut, cela handicape sans doute le développement de cet axe dans l'analyse des moments.

Il conviendrait bien évidemment de considérablement développer et systématiser les analyses de réalisation de moments de l'étude, mais aussi de les confronter, pour à la fois mettre au jour les gestes didactiques mais aussi pour faire surgir les besoins praxéologiques de la position de professeur à cet égard (Artaud, 2021). Ce que nous avons présenté met en lumière principalement des besoins praxéologiques liés à la mise en œuvre d'une dialectique des médias et des milieux vigoureuse et intégrée au topos de la position d'élève dans le moment technologico-théorique et le moment exploratoire. Dans le cadre de mémoires de master de recherche en didactique des mathématiques que nous avons dirigés (Alfieri, 2019 ; Megherbi, 2021), des travaux ont mis en évidence une différence à cet égard entre le moment technologico-théorique et le moment exploratoire : dans des conditions assez favorables, avec un professeur formé à la TAD dans le cadre d'un master recherche notamment, l'intégration de la dialectique des médias et des milieux est plus aisée dans le moment exploratoire que dans le moment technologico-théorique, même si des choses sont possibles dans ce moment. La reprise des résultats en les examinant à travers le prisme du geste didactique « amener une œuvre dans le milieu de l'étude » nous paraît pouvoir permettre d'élucider plus finement les différences entre les deux moments à cet égard.

Références

Alfieri, C. (2019). *Diriger l'étude dans le cadre du paradigme du questionnement du monde : une étude de cas sur l'optimisation des grandeurs* (Mémoire de master). Université d'Aix-Marseille.

Artaud, M. (2011). Les moments de l'étude, un point d'arrêt du didactique ? Dans M. BOSCH et al. (Éds), Un panorama de la TAD, pp. 141-162, Barcelone, Espagne : CRM.

Artaud, M. (2016a). Éléments d'écologie de la TAD dans le continent didactique. Conférence aux *journées sur la théorie anthropologique du didactique*, Barcelone, Espagne, 4-5 juillet 2016. Non publié.

Artaud, M. (2016b). Former des enseignants par l'analyse des praxéologies : La question de la réalisation des moments de l'étude. In Calmettes, B., Carnus, M., Garcia-Debanc, C., & Terrisse, A. (Éds.), Didactiques et formation des enseignants : Nouveaux questionnements des didactiques des disciplines sur les pratiques et la formation des enseignants. Presses universitaires de Louvain (197-206). Tiré de http://books.openedition.org/pucl/9079

Artaud, M. (2021). Des grandeurs et de leur mesure : Besoins praxéologiques de la position de professeur et leur satisfaction. In H. Chaachoua, A. Bessot et al. (Éds). Nouvelles perspectives en didactique : point de vue de l'élève, questions curriculaires, grandeurs et mesure Vol. 1 (pp. 197-226). Grenoble, France : La pensée sauvage.

Artaud, M. & Cirade, G. (2021). La TAD comme milieu pour l'étude de l'activité des institutions didactiques. Caminhos da Educação Matemática em Revista (Online), 11(1), 388-411. https://aplicacoes.ifs.edu.br/periodicos/caminhos_da_educacao_matematica/article/view/762

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. Recherches en didactique des mathématiques, 19(2), 221-266. https://revue-rdm.com/1999/l-analyse-des-pratiques/

Chevallard, Y. (2002). Organiser l'étude. 1. Structures & Fonctions. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot & R. Floris (Éds), Actes de la XI^{e} École d'été de didactique des mathématiques. La pensée sauvage. http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=52

Chevallard Y. (2019). On using the ATD: Some clarifications and comments. Educação Matemática Pesquisa, 21(4), 1-17. https://doi.org/10.23925/1983-3156.2019v21i4p001-017

Mergherbi, D. (2021). *Écologie de la dialectique des médias et des milieux dans la réalisation d'un moment technologico-théorique : le cas de la définition des fonctions sinus et cosinus en première* (Mémoire de master). Université d'Aix-Marseille.

Didactic Transposition Circle:
A proposal for complementing an essential tool of ATD

Dániel Katona

MTA-ELKH-ELTE Research Group in Mathematics Education, Alfréd Rényi Institute of Mathematics; & Doctoral School of Mathematics, ELTE Eötvös Loránd University, Budapest, Hungary; danikatona@gmail.com

The present article offers a complemented version of the model of the theory of didactic transposition, named 'the didactic transposition circle' (DTC). The main modification is that a transposition sub-process from learned knowledge to scholarly knowledge was inserted into the original unidirectional model, complementing it into a circular one. Preliminary elements of the rationale for the introduction and for the application of the new model is presented, after a brief but detailed literature review on the original model.

Keywords: Anthropological theory of the didactic, didactic transposition, didactic transposition circle.

Introduction

The present article intends to initiate a possible research direction within the Anthropological Theory of the Didactic (ATD) on the development of one of its core theoretical construct: didactic transposition (DT). The aim is solely to present the theoretical problem, the rationale for this presentation, and also to propose (only) the core of the direction of this possible theoretical development—complementing the model of didactic transposition, by a step from learnt to scholarly knowledge, into the so called 'didactic transposition circle'—basically in order for invite researches to present more theoretical arguments (for or against), to raise problems in connection to this proposal and also to present (the so far missing) illustrative examples. Before presenting the proposed model, a short, but somewhat comprehensive survey is presented on the development and also on the role of the Theory of DT (TDT), mostly for the reason to see what has been published in the English language literature (for the English speaking reader) on this theoretical construct, to "know" the model that is to be complemented. This comprehensive survey is therefore regarded as a necessary step of this theoretical developmental process, which has been initiated by the present paper.

Theory of Didactic Transposition– a short historical survey

The role of DT in the development of ATD and in the conceptualization of Didactics of Mathematics

ATD underpins a prevailing, progressive research tradition in (and outside) inquiry-based mathematics education (IBME) (Artigue & Blomhøj, 2013), being one of the intermediate-level theoretical frameworks for task-design in mathematics education (Watson & Ohtani, 2015, pp. 31–36), with a growing community, recently publishing its own comprehensive casebook (Bosch et al., 2019). Although, as the theory is growing and endowing its researchers with new tools and theoretical constructions to be focused on—such as study and research paths (SRP), see e.g. (Chevallard, 2006; Bosch, 2018; Bosch & Winsløw, 2015)—early fundamental constructs in the cradle of the theory always remain influential in the process of giving birth to these new constructs and in interpreting them. In ATD, such early fundamental construct is the TDT. See e.g. (Chevallard, 1981, 1989).

In 1975, the French sociologist, Michel Verret introduced the concept ‘didactic transposition’ (Verret, 1975) for describing the transformation of objects (knowledge) entering the teaching and learning processes. Based on Verret’s ideas, Chevallard, as part of his contribution to the conceptualization and forming of didactics of mathematics as a new field of research, presents the concept “didactic transposition of knowledge” as central in his understanding of this newborn field of science (Chevallard, 1981, p. 151). The consideration of this phenomenon DT was proposed to be one of the “criteria to be imposed upon any scientific approach to educational matters” (Chevallard, 1981, p. 147). Chevallard gave his first course on the subject “at the ‘First Summer School in Didactics of Mathematics’ in Chamrousse (France, 7-19 July, 1980)” (Bosch & Gascón, 2006, p. 52). His elaborated ‘theory of didactic transposition’ was published in the 1980s. See e.g. (Chevallard, 1989).

This, the born of TDT was the moment, when a more comprehensive theory − that encompasses TDT − ATD was born. The theoretical framework and research programme that later became known as ATD was initiated “with the study of didactic transposition processes” (Bosch & Gascón, 2014, p. 67), TDT was “the germ of” ATD (Bosch & Gascón, 2006, p. 54).

The concept didactic transposition

The concept DT, appears under slightly different names in the literature on its way to be settled, with a more and more crystallised, but in its essence constant meaning (‘definition’).

It is called ‘didactic transposition of knowledge’ or ‘didactic processing’ in (Chevallard, 1981), referring to the process when “a didactic system maintains itself in an almost continuous inflow and outflow of that particular material, knowledge” and “the knowledge to be imparted within the system gradually becomes obsolete and is eventually discarded; while new knowledge, or old knowledge in new form, enters curricula” (Chevallard, 1981, p. 150), or in other words from the same paper, referring to only a part of this process (that knowledge is changing): a process in which “a given mathematical subject induces changes in the subject itself in order to make that mathematical subject teachable, that is compatible with the structure of the didactic system and the didactic processes”. (Chevallard, 1981, p. 152). That is, knowledge enters and exists the didactic system and when it enters, it is changed to fulfil its purpose there; knowledge is continuously moving and changing across and within systems. When the focus is on a didactic system, the process is called didactic transposition of knowledge. The main idea behind this concept, that knowledge as being part of a didactic system is different that of being outside, because it was transformed and transposed. This transposed knowledge is called “knowledge to be taught” (Chevallard, 1981, p. 152).

In Chevallard’s later paper on ‘didactic transposition theory’ from 1989, the author again confirms the essential importance of the process ‘didactic transposition’ as an object of the field of mathematics education research by claiming that “a major problem of the didactics of mathematics” is about “the processing of knowledge within the teaching system”(Chevallard, 1989, p. 154). The distinction of the concepts ‘knowledge used’ and ‘knowledge taught’ − expressing the main idea that knowledge shall be considered as a something being transformed − are also presented in this article (Chevallard, 1989, p. 155), along with the slightly different versions: “scholarly body of knowledge” “taught body of knowledge" (p. 157). With the use of the first pair, we are presented with another form of the 'definition' of 'didactic transposition of knowledge': "The transition from knowledge regarded as a tool to be put to use, to knowledge as something to be taught and learnt" (Chevallard, 1989, p. 156). The postulate that these processes are conducted and legitimized by proper (educational) institutions are also important part of the whole anthropological perspective. Although the term "knowledge to be taught" is also presented (Chevallard, 1989, p. 158.), the distinction between 'knowledge to be taught' and 'taught knowledge' is not discussed there.

In the reviewal work on the initial 25 years of research on 'didactic transposition' (Bosc & Gascón, 2006) the authors present a description of DT in more details, though otherwise describing the same procedure depicted beforehand:

> ... what is being taught at school ('contents' or 'knowledge') is ... something generated outside school that is moved — 'transposed' — to school ... transpositive work needs to be carried out so that something that was not made for school changes into something that may be reconstructed inside school. (Bosc & Gascón, 2006, p. 53)

This transpositive work means that bodies of knowledge produced in e.g. scholarly institutions are "selected, delimited, reorganised and, thus, redefined" (Bosc & Gascón, 2006, p. 55).

Another reformulation in by Chevallard is: "...didactic transposition theory ... considers knowledge as a changing reality" (Chevallard, 2007, p. 132).

The model of DT with the different 'forms of knowledge' at the corresponding institutions

In the same work on the initial 25 years of research on DT mentioned before (Bosch & Gascón, 2006), the authors present the following model of DT, illustrated in Figure 1.

> A distinction is established among: the 'original' or 'scholarly' mathematical knowledge as it is produced by mathematicians or other producers; the mathematical knowledge 'to be taught' as it is officially designed by curricula; the mathematical knowledge as it is actually taught by teachers in their classrooms and the mathematical knowledge as it is actually learnt by students and that can be considered at the same time the end of the didactic process and also the starting point of new ones. (Bosch & Gascón, 2006, pp. 55–56)

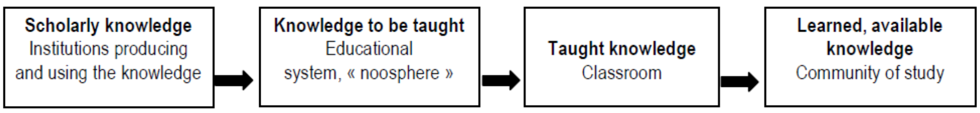


Figure 1: The model of the didactic transposition process in (Bosch & Gascón, 2006, p. 56)

A basically the same model can be found in (Chevallard & Bosch, 2014, p.170), illustrated by Figure 2, and in (Bosch & Gascón, 2014, p. 70).

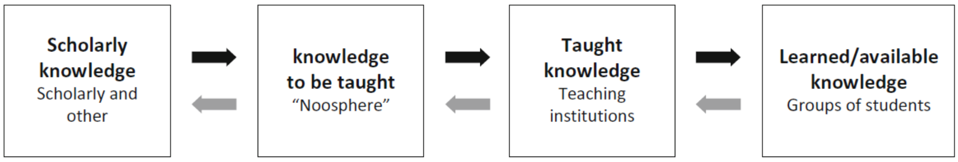


Figure 2: The model of the didactic transposition process in (Chevallard & Bosch, 2014, p. 171)

A part of this model appears in (Chevallard, 2019), highlighting only one element of the transposition process, the one which is the most relevant in what the French perspective and ATD brought to the scene of mathematics education research. The only difference to the previously discussed models is the equation/ highlight of the strong relationship between ‘knowledge to be used’ and ‘scholarly knowledge’, which is, being otherwise an interesting and important problem, not in the focus of the present paper:

> In order to become teachable (and learnable) in some school system Σ, a piece of knowledge $\mathscr{k}$ has to be “transposed” from some hypothetical scholarly world to adapt to conditions specific to the schools σ ∈ Σ. ‘knowledge to be used’ / scholarly knowledge → ‘knowledge to be taught’ (Chevallard, 2019, p. 76.)

In the Glossary section of (Bosch et al., 2019), instead of highlighting the different “forms” of knowledge in the transposition process, such as e.g. ‘knowledge to be taught’, by having separate entries for them, ‘only’ an entry for ‘transposition’ exists, being ‘solely’ a reference to the entry ‘praxeologies and praxeological analysis’, in which transposition is indirectly defined as something through which a praxeology is changed, and as something that reflects the institution dependent nature of praxeologies. As the present paper does not intend to discuss the concept praxeology and the interpretation of the formation of new knowledge in terms of (re)formulation of praxeologies, this issue is not discussed here. For the present discussion, the important note is that no new model for the transpositional processes between the different forms of knowledge, no new figure is presented in this recent work on ATD.

The Didactic Transposition Circle (DTC) – Complementing the DT model with the transitional step from ‘learned’ to ‘scholarly’ knowledge

As can be seen from section 2, the process of DT has phases, some of which are highlighted and discussed in an elaborated way in ATD. However there are phases, which are completely neglected, although may be important for understanding some type of teaching phenomena. The transposition process of taught knowledge into scholarly knowledge seem to be completely neglected so far. Involving this phase of the transposition of knowledge, regarding teaching and learning, is resulted in a circle of the transposition process, in the proposal of model DTC, giving a more complete ‘picture’ of the phenomenon.

In the DTC model, distinct terms are applied for the different phases of the whole (circular) process, namely, for the sub-processes, that reflects their main role in the corresponding social procedures or realities, as can be seen in Figure 3: didactic transposition, teaching transposition, learning transposition, and scholarly transposition. The incorporation of the latest one into the model is to be the main contribution of the present paper.

The whole circulating procedure may be called ‘the didactic transposition of knowledge through generations’, reflecting Brousseau’s idea of understanding “Didactique as what is specific to the transmission of a piece of knowledge from one generation to another” (Brousseau, 2008, p. 253), or—avoiding the misunderstanding that may arise from the fact that the whole process is named the same as one of its sub-processes—the ‘evolution of knowledge’.

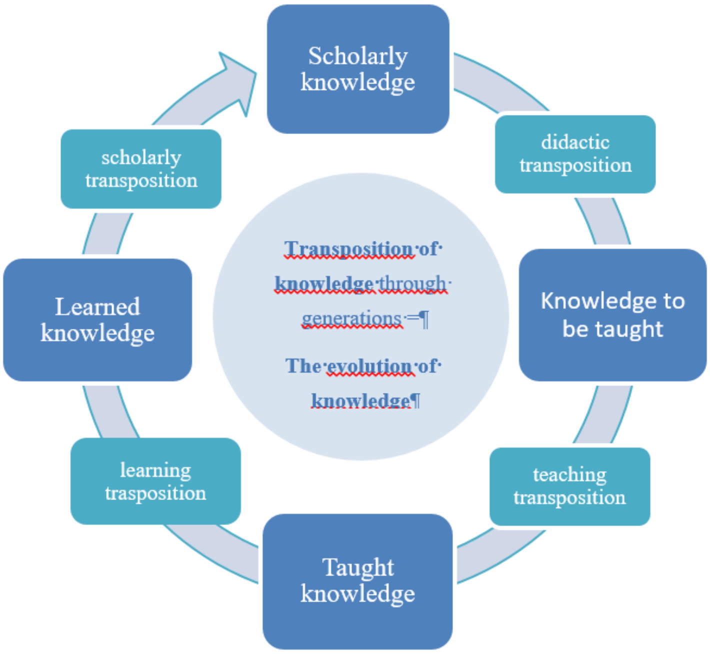


Figure 3: The model of the didactic transposition circle (or cycle)

Discussion and Conclusion − Rationale for the introduction of the DTC model

Didactics to be concerned with all forms of the diffusion of knowledge

Chevallard in his papers proposed a ‘definition’ for didactics as a field of science: “the science of the diffusion of knowledge in any social group” (Chevallard, 2006, p. 22) or “in any institution” (Chevallard, 2007, p. 133). Therefore, if there exist a sub-process of mathematical knowledge diffused, or transposed from teaching institutions to scholarly institutions (of mathematics, engineering, etc.) didactics of mathematics ought to be concerned with this sub-process too.

Bosch and Gascón translated how Chevallard formulated the commitment of didactics in 2005: “didactics is devoted to study the conditions and restrictions under which praxeologies start living, migrating, changing, operating, dying, disappearing, reviving, etc., within human groups” (Bosch & Gascón 2006, p. 60). This formulation of the scope of didactics also supports that modelling the diffusion of knowledge (or praxeologies) within the ATD framework shall rather be open to incorporate the transpositional step with which the DT model has been implemented into the DTC one.

Further enlargement of the empirical unit of analysis within the field of Didactics

According to Bosch and Gascón, one outstanding merit of the introduction, development and application of TDT was that it enlarged the empirical unit of analysis within the field of didactics as a science (Bosch & Gascón, 2006, pp. 55–58). This has been done by first of all directing (more) attention to the knowledge element of the didactic system, and also by inviting an anthropological approach, an institutional perspective into the didactics of (not only) mathematics, an approach focusing (partly) on transformational and transpositional processes of knowledge within different institution of societies. A particularly important part of this enlargement was the claim for the necessity of analysing the so called 'scholarly knowledge', besides the knowledges to be taught, taught and learned.

However, this new perspective, according to its communicated models described in section 3, focused solely on transpositions from scholarly to educational institutions, accentuating one, even if the perhaps most decisive direction. The DTC model further enlarges the enlarged scope of didactic analysis, by laying a focus of the opposite direction of how newborn, or revitalized 'scholarly knowledge' is influenced by 'taught knowledge'. Therefore, the direction of the development by the present complementation of the original DT model into the DTC one seems to be solely the "natural" continuation of the scope-broadening journey began in the 1980s in the cradle of ATD.

Research into the birth of scholarly knowledge

The aforementioned scope-broadening procedure navigates the research focus not solely on transpositional processes of knowledge between institutions, but thereby, on those institutions, and the knowledge "of" those institutions which gained less research attention before, particularly on institutions where scholarly knowledge is born and organised.

"… It is also necessary to analyse — examine minutely and break down — the spontaneous models of the 'scholarly knowledge' that are taken for granted in educational institutions."(Bosch & Gascón, 2006, p. 56.)

A breakdown analysis of scholarly knowledge shall also take into consideration those procedures which play a role in the birth, organization and reorganization of the elements of this knowledge. Therefore, if knowledge learned (and also the ones designed to be taught and taught − indirectly) has any effect, by any procedure on the nascency (birth), organization and reorganization of scholarly knowledge, these procedures shall be part of the object of study by Didactics as a field of research. If these effects and procedures are transpositional in nature, in the way they claimed to be (by e.g. ATD) in the case of − for instance – the transposition from scholarly knowledge to knowledge to be taught, then the necessity for the complementation of the DT model into DTC one is justified.

Accordingly, research is needed on the supposed transpositional nature of the aforementioned effects and procedures in order for justifying the existence of such transposition from learned to scholarly knowledge.

The supportive role of taking into consideration the transposition from learned to scholarly knowledge in understanding the opposite direction and therefore supporting didactic engineering

In every scholarly institutions, the persons who are responsible there in the production of scholarly knowledge were all students before, in educational institutions, where they may have learnt knowledge that is relevant in this production of scholarly knowledge. In the case of mathematics, mathematician scholars were all students of mathematics before, and the knowledge they learnt in their former educational institution may quite probably be used, in a transposed form, or play a role more indirectly in the process of producing new knowledge at their latter scholarly institution. This impact of learned knowledge on the production of scholarly knowledge is therefore created through researchers' (mathematicians') personal relationships, rooted (partly) in educational institutions and reshaped in different institutions. Scholarly knowledge being born is therefore partly rooted in knowledge learned and taught before, however through personal relationships. The relationship of these personal relationship, and so whether a more institutional relationship and process may also be detected between educational and scholarly institutions, regarding a transformation of knowledge from the previous to the latter one needs to be further studied.

Furthermore, in the didactic engineering processes of task- and curriculum design—as understood in (Artigue, 2014; Barquero & Bosch, 2015)—during the development of any teaching material, or in any contribution to the DT processes from the established scholarly knowledge to knowledge to be taught, the possible effect of the designed teaching material to future production of scholarly knowledge shall also be taken into account.

The possibly endless formation of learned knowledge

In his paper on TDT, Chevallard defines teaching as a "process by which people who do not know some knowledge will be made to learn it, and thereby come to know it" (Chevallard, 1989, p. 156), that is, the didactics (of mathematics) concerns itself with a process by which students get to know something (about mathematics) they did not know before. If this process of getting to know is only partly completed in the particular educational institution, and it also continues after, for instance when taking part in the producing of new knowledge in a scholarly institution, then didactics (of mathematics) shall also concerns itself with those transformational and transpositional processes of knowledge that occurs from educational to scholarly and within scholarly institution. In other words, if the process of knowledge being transposed from 'knowledge to be taught' to 'learned knowledge' does not end within a particular educational institution, but it continues after and also takes place in a scholarly institution, than didactics may also concern the (supposedly existing) transpositional processes from the educational to the scholarly institution.

Farness and closeness of scholarly and educational institutions – a possible need for partial change in perspective

"The process of didactic transposition starts far away from school, in the choice of the bodies of knowledge that have to be transmitted" (Bosch & Gascón, 2006, p. 53). For understanding the transposition of 'scholarly knowledge' into 'knowledge to be taught', it is indeed important to emphasise the farness of the body of knowledge from the one that is part of a didactic system, to understand that e.g., if M stands for mathematics, then teaching M doesn't simply mean teaching M, its rather teaching M', where M' is transposed from M. However, in the current stage of the development of a field of research − didactics of mathematics – roughly four decades after the introduction of the concept DT (for shaping the basic paradigmatic nature of the then evolving discipline), it is also reasonable to understand the closeness of these bodies of knowledge and especially to put more focus on the 'other side of the coin': the possible role of the body of knowledge that was designed to be taught in the formation of the “new” and the reformation of “older” scholarly bodies of knowledge.

Acknowledgements

This study was funded by the Scientific Foundations of Education Research Program of the Hungarian Academy of Sciences.

References

Artigue, M. (2014). Didactic engineering in mathematics education. In S. Lerman (Ed.), *Encyclopaedia of mathematics education* (pp. 159–162). Springer.

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM, 45*(6), 797–810. https://doi.org/10.1007/s11858-013-0506-6

Barquero, B., & Bosch, M. (2015). Didactic engineering as a research methodology: From fundamental situations to study and research paths. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education: An ICMI study 22* (pp. 249–271). Springer.

Bosch, M. (2018). Study and research paths: A model for inquiry. In B. Sirakov, P. Ney de Souza, & M. Viana (Eds.), *Proceedings of the International Congress of Mathematicians Rio de Janeiro* (Vol. 3, pp. 4001–4022). World Scientific Publishing.

Bosch, M., Chevallard, Y., García, F. J., & Monaghan, J. (Eds.). (2019). *Working with the Anthropological Theory of the Didactic in mathematics education: A comprehensive casebook*. Taylor & Francis.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin, 58*, 51–65.

Bosch, M., & Gascón, J. (2014). Introduction to the Anthropological Theory of the Didactic (ATD). In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 67–83). Springer.

Bosch, M., & Winsløw, C. (2015). Linking problem solving and learning contents: The challenge of self-sustained study and research processes. *Recherches en Didactique des Mathematiques, 35*(3), 357–399.

Brousseau, G. (2008). Research in mathematics education. In M. Niss (Ed.), *Proceedings of the 10th International Congress on Mathematical Education* (pp. 244–254). Roskilde University.

Chevallard, Y. (1981). The Didactics of mathematics: Its problematic and related research. *Recherches en Didactique des Mathématiques, 2*(1), 146–157.

Chevallard, Y. (1989). On didactic transposition theory: Some introductory notes. In H. G. Steiner & M. Hejny (Eds.), *Proceedings of the International Symposium on Selected Domains of Research and Development in Mathematics Education* (pp. 51−62). University of Bielefeld.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *European Research in Mathematics Education IV: Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 21–30). Fundemi IQS.

Chevallard, Y. (2007). Readjusting didactics to a changing epistemology. *European Educational Research Journal, 6*(2), 131–134. https://doi.org/10.2304/eerj.2007.6.2.131

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114.

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 170–174). Springer.

Verret, M. (1975). *Le temps des études*. Libraire Honoré Champion.

Watson, A., & Ohtani, M. (Eds.). (2015). *Task design in mathematics education: An ICMI study 22*. Springer.

Is it the same twelfth?
Questioning an unquestioned principle

Christine Chambris

CY Cergy Paris Université, Université de Paris, Univ Paris Est Creteil, Univ. Lille, UNIROUEN, LDAR, F-95000 Cergy-Pontoise, France, email@google.com

Abstract. The aim of this paper is to investigate theoretical component of praxeologies, a component that has often been overlooked. It goes back to a series of research, both in the French and the Anglo-Saxon contexts. It focuses on research in the domain of arithmetic, including Douady and Perrin-Glorian pioneer didactic engineering and works on multiplicative reasoning in cognitive psychology.

Résumé. L'objectif de cet article est d'étudier la composante théorique des praxéologies, une composante qui a souvent été négligée. Il revient sur une série de recherches, tant dans le contexte français qu'anglo-saxon. Il se concentre sur les recherches dans le domaine de l'arithmétique, notamment l'ingénierie didactique pionnière de Douady et Perrin-Glorian et les travaux sur le raisonnement multiplicatif en psychologie cognitive.

Keywords: praxeology, logos, arithmetics.

Introduction

This paper investigates a question posed in the first axis: "Is the full import of the theory [ATD] recognised and drawn upon?" Chevallard, Bosch and Kim start their 2015 paper as follows: "In this study we examine the meaning and scope of a key concept of ATD which, paradoxically, since the inception of this theory [ATD], seems to have been consistently overlooked: that of theory [□]." (p.2614) For instance, in early texts that include praxeologies (e.g., Chevallard, 1994, Artaud, 1998), theory component is sometimes said to be "évanouissante" (vanishing). Much more attention seems to have been paid to theory [□] recently, e.g., Bosch, Gascon and Trigueros (2017) specify its critical role in praxeologies in a text related to research praxeologies. They write "[theory] gives meaning to the problems, allows for the interpretation of the techniques and serves as a basis to the technological descriptions and justifications. In this sense, the theory provides the main notions, assumptions and unquestioned principles of the research." (p. 40) Might the overlooking of □ be due to the difficulty to question unquestioned principles, or even to identify them as actual constraints? This would be paradoxical because an essential means provided by ATD, and one of its foundational features, is the ability to interpret what looks natural as actually often institutional (or cultural).

One of the aspects of New Math reform of the years 1955-1975 is the introduction of New Math in schools. This results in transposition of set theory as scholarly knowledge for arithmetic knowledge to be taught, especially in the domain of numbers. Set-theory might be "evanescent" today (Radford, 2021), and this mathematics background seems to be a shared principle in most literature in mathematics education research, and in many arithmetic curricula. What kind of traces does this transposition leave in current praxeologies? Is it possible to question this principle? If so, could this shed new light on some old research results?

In this paper, I'm rereading several works related to arithmetic in primary education (i.e., in the sense of numbers, operation, ratio, etc.) that draw upon a diversity of theoretical perspectives and I question the unquestioned principle mentioned above.

Several stories in the French context and elsewhere

On the occasion of teacher education lectures, mathematics education researcher Butlen (e.g., 2018) evoked a classroom episode. He described the scene as follows. During a teaching course on fractions, grade-4-or-5 students had to fold a sheet of paper in 12 equal parts. The teacher was guiding them: folding in three in one direction, then in four in the other. What the students did. However, at a moment, a student showed that he had folded in the other directions (fig. 1).

Figure 1: Two divisions of a rectangle in 12 equal parts

Then, a student asked: "Is it the same twelfth?" One kid –let's call her Joe- discretely said to Butlen (who was observing the class) "Of course, it is the same, there are twelve of each".

What happened then? The teacher proposed to take one twelfth of each kind and to recompose the first shape into the other. This done, the pieces did not fit well and teacher concluded the two twelfths were not the same! Joe was very disappointed. She said "Yet, I was sure they were the same". Joe, who seemed to master the notion at stake before the episode, might have lost something after.

Butlen promoted then the idea that teachers should learn various ways for validating: pragmatic vs theoretic; and that, in this case, validation should have been theoretical. In ATD, this means Butlen's view on the problem is in the domain of didactical praxeologies. Yet, the theoretical validation proposed is: There are 12 twelfths, 12 twelfths is one. Hence, the two twelfths are equal.

If such task exists in schools, its technology seems to be: 12a=12b, thus a=b. It requires the use of regularity of multiplication of rational numbers by whole numbers, or a general algebraic rule of division. This episode questions the moment such knowledge is taught in elementary schools.

This story strongly resonates with two episodes in French literature of didactics.

Douady's view

Douady (1980) reports a similar episode. The rectangle is shared in 3x5 and Joe's name is Nicolas. Interestingly, even though the aim was initially similar (parts of different shapes have same area), Douady concludes that conceptual means to interpret difference between conjecture and experimental facts were not available in students.[1]

[1] That said, Douady's didactical engineering was typically successful. She was working with Perrin-Glorian, and a few years later, Perrin-Glorian attempted to reproduce success in weak classes, with 'ordinary teachers'. Replication was not successful (Perrin-Glorian, 1993).

Perrin-Glorian's observations

Perrin-Glorian (1993) reports the following episode. Grade 6 students had to make "fourths of various shapes". They proposed figures 1 to 3 (fig. 2). Prompted to justify whether figure 1 presented fourths, they did not know. Researchers then asked them to make eighths of various shapes; students then made figures 4-5. This enabled them to justify fourths of figure 1. When they then were asked to say what happens when welding eighths of type 4 and 5 (i.e., 2x1/8=1/4), they did not know and said: "we have to pave" (fig.6).

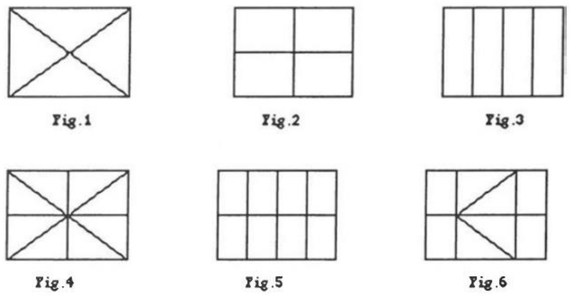


Figure 2: Fourths and eighths of various shapes (Perrin-Glorian, 1993, p. 73)

Perrin-Glorian considers that while students "were supposed to be working at the level of numbers", they were still thinking in term of paving. She links this to teachers' practices: They decontextualize brutally. Perrin-Glorian suggests teachers should include several levels of decontextualisation: e.g., from material to evocation of material, from a kind of quantity (i.e., length) to another (i.e., area).

An episode in India

Rahaman and Subramaniam (2016) reported a difficulty in the domain of area teaching and learning (grades 6-7). First, students had to draw 15cm^2 rectangles. A student proposed to draw a 6x5 rectangle and to cut it vertically. Then teacher asked students to draw a 30 cm^2 (6cmx5cm) rectangle and to divide it in two equal parts. The following problem then emerges in the classroom: When the rectangle is cut in two triangles, does each of them have 15cm^2? Indeed, 15 units are no longer visible in the triangles. Whereas the teacher argues since the two triangular halves of the 6×5 rectangle are congruent, the area must be half of 30, not all students were convinced and some students reconfigured the units (fig. 3).

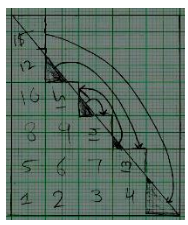


Figure 3: Paving for seeing units (ibid., p.4)

Indian authors interpret this difficulty "as a gap between the spatial understanding and the numerical understanding" (p.3), an acknowledged difficulty (e.g., Battista, 2007) in learning of area. Battista (2007) suggests students should be able to reason with different forms of units. The former reasoning is multiplicative thinking, and the latter is additive thinking, they add.

The difficulty presented by Rahaman et al. is similar to that of Perrin-Glorian's story. In the case of Butlen-Douady, there is no possibility to pave. This suggests Butlen's technique, which is implicitly shared by other French researchers, might be difficult to teach. Moreover, it is important to note that, Perrin-Glorian suggests the difficulty is not specific to the kind of quantity area.

The support of the cognitive psychology

Anglo-Saxon research in measurement and arithmetic teaching and learning largely develops in the field of cognitive psychology. An amount of research on multiplicative reasoning developed with set-theory for numbers as mathematics background (e.g., Confrey & Harel, 1994). Chambris, Coulange, Rinaldi and Train (2021) specifically investigated Thompson and colleagues' work that includes a perspective on magnitudes.

Cognitive psychology in the U.S.A.

Based on Piagetian's schemes analyses, Thompson, Carlson, and coll. (2014) identified five levels of meaning for a "quantity's magnitude" (p.2). These levels are thus situated in person's conceptions.

- Level 1: awareness of size (quantities can be smaller or greater)
- Level 2: measure magnitude (a quantity is formed with a number of identical pieces: the greater number, the greater the quantity without considering size of pieces)
- Level 3: Steffe's magnitude (multiplicative relationship: a quantity can be three times greater than another, and reciprocally three times smaller)
- Levels 4 and 5 require level 3 with additional characteristics.

In Rahaman et al. and Perrin-Glorian's episodes, the "additive" or "paving" interpretations can be situated at level 2, and "multiplicative" or "number" interpretations at level 3. Similarly the impossibility to see similar twelfths (or areas of equal sizes) in Butlen-Douady's episodes reflects thinking at level 2.

Towards institutional re-interpretation of difficulties

Interestingly, Thompson does not reject institutional perspective. Indeed, based on international surveys such as TIMMS, he assumed Japanese and Korean teaching systems were much more efficient in multiplicative reasoning teaching than the USA. Then, e.g., in a comparative study (textbooks and teachers' knowledge), Thompson, Hatfield, et al. (2017) established that co-variational reasoning that requires multiplicative quantities was deeply included in South Korean curriculum.

Questioning the unquestioned

As indicated in the introduction, one of the aspects of New Math reform is the introduction of New Math in reformed curricula. Though set-theory might be "evanescent" today (Radford, 2021), this mathematics background is likely to have become an unquestioned principle in most current literature in mathematics education research, and in many arithmetic curricula. Is there an alternative to set-theoretical constructions of number sets?

Resources in mathematics over time

It is largely known that huge changes of epistemological nature occur in mathematics in 19th century. Yet, the twofold nature of the changes (Otte, 2007) is not often recalled. Firstly, the paradigm of axiomatization changed: from idealization of reality to no contradiction principle. Secondly, up to mid-19th century, mathematics was a science of quantity (e.g., Gandon, 2009), arithmetization of analysis corresponded to a change of base object from quantities into whole numbers, and later into sets[2]. Regarding each of these issues, there is to say. On the one hand, even though it is not fully consensual today –e.g., Bourbaki (1984) more or less rejects the idea-, philosophers and historians of mathematics (e.g., Otte) and major mathematicians (e.g., Thom, 1970) consider idealization of reality remains a fundamental way of working for mathematicians. On the other hand, several mathematicians (e.g., by chronological order, Burali-Forti, Lebesgue, Kolmogorov, Freudenthal, Whitney, Rouche) developed axiomatizations of quantities that resonate with classical view on quantities as in Euclid's Elements, i.e., common notions (Book I) and Book V, especially in an educational perspective.[3] It is important to note that Burali-Forti and Whitney (1968) not only axiomatized quantities but supplemented their approach to quantities with quantity-based constructions of number sets (from whole numbers to real numbers). Furthermore, at the time idealization of reality as an axiomatization paradigm, there were arithmetic treatises that presented knowledge to be taught, including numbers, based on quantities (Bronner, 2008, Neyret, 1995). Bezout's treatise of arithmetic published in 1764 goes up to fractions. Euclid's Elements takes place in this tradition for geometry. So, what is a quantity mathematically speaking?

What is a quantity?

The mathematical definition of a quantity is summary. A kind of quantity is a set of elements called 'quantities', including a total order relationship, a commutative and associative addition, and a specific twofold property P that links addition and order (not all definitions are presented similarly and this makes difference in the definition of order. That said, they all share these properties as first tenets).

- P-g (for growing): for any a, b, if it exists c, a=b+c then a>b,
- P-s (for supplement): for any a, b, if a>b then it exists c, a=b+c.

[2] Gandon (2009) explains quantities took a paradoxical place in history of mathematics at the end of 19h century: They were both disappearing and being actively formalized.

[3] Cases of Lebesgue and Burali-Forti could be discussed but it is not really important for this paper. The former because his book is sometimes considered as a justification of rejection of quantities in the 1970s and it's true he does not axiomatize quantities, the latter because his theoretical construction is not only a pedagogical proposal, but also constitutes an alternative proposition for fundaments of numbers, published in first edition of Peano's formulaire. It seems Peano did not appreciate the construction (Gandon, 2009).

There are additional properties (i.e., axioms) for divisibility, continuity, but I consider the core is above. Indeed, set N verifies these properties and not the other properties implied by the additional axioms.

What do the basic tenets mean? Interpreting them in terms of idealization of reality even though they are formulated in a formal manner provides a 'sensible' or 'phenomenological' meaning (fig. 4). The difference in size provides subtraction (line 4, in green): The whole is greater than the part (Euclid's common notion).

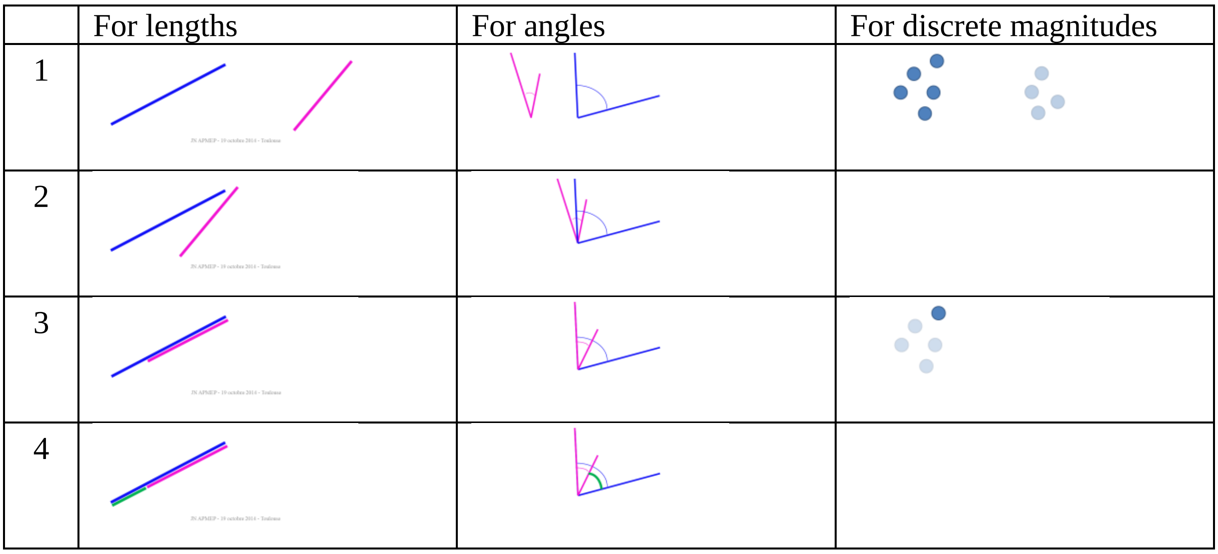


Figure 4: Basic link between order and addition in quantities

What is the definition useful for? What kind of implication might this background imply?

A proof

1.1.

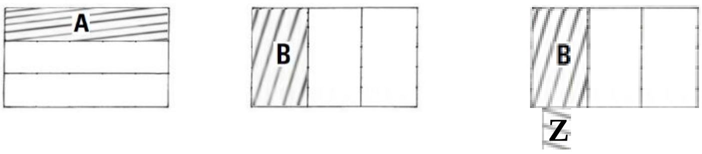


Figure 5. Two different divisions in three equal parts of the same object (U) and an additional part Z to form A from B.

Let's consider quantity u shared, for instance, in 3 equal parts in two different ways (fig. 5). Are A and B equal?

If not, one is greater than the other -say A-. This means it is possible to find a piece Z such that is possible to recompose A with B and Z (P-s). Then three pieces B form U, and there is a need of three pieces Z, to form U with A. This is impossible! (Similarly B cannot be greater than A). My idea is that this reasoning is based on idealization of reality.

In a more formal manner, we can say: a and b are two quantities that are such that: a+a+a=b+b+b or 3.a=3.b. Are a and b equal? Responding yes, means regularity of multiplication by whole number:

this is not an axiom of quantities. Can the equality be proved? Let’s suppose they are unequal. According to the total order property, inequality of two quantities implies one is greater than the other, e.g., $a>b$. So, there is a quantity z such that: $a=b+z$. Thus $a+a+a =(b+z)+(b+z)+(b+z)$, associativity and commutativity of addition imply that $a+a+a =(b+b+b)+(z+z+z)$. Consequently, $u=u+v$, with $v=z+z+z$. Property P implies then that $u>u$. Contradiction. Thus, $a>b$ is absurd. Similarly, $b>a$ is absurd. Consequently, if $3.a=3.b$ then $a=b$.

What can we learn from this?

Discussion and implications

Of course, theory component of mathematics praxeologies is not restricted to a series of axioms and theorems. For instance, didactical knowledge constraints tasks: e.g., measurement of areas in the teaching of fractions. In French curriculum, such geometrical-arithmetical tasks aim at making visible insufficiency of whole numbers and need of ‘new numbers’. They are included in arithmetic praxeologies. Their theory component thus includes didactic features. To some extent, in a quantity theory perspective, such didactical knowledge is included in mathematical theory (Chambris & Visnovska, 2021). This balance is to be explored further.

Towards a mathematical view on multiplicative reasoning?

The proof (section 3.3) suggests a mathematical means to convince students of the ‘equality of twelfths’, of fifteenths, of halves. The reductio ad absurdum (raisonnement par l’absurde in French) provides a new view on quantities. Indeed, being at this point, there is no longer the need to recompose A into B to prove they have equal size. It becomes possible, and meaningful, to say that A (similarly to B) is three times smaller than U, because A and B have become sizes independent of the shapes. My understanding of Steffe magnitude is exactly this kind of relationship between A and U, a ‘multiplicative relationships’.

In other words, such reasoning suggests a mathematical interpretation of psychological ‘multiplicative reasoning’. From another perspective, it provides multiplicative reasoning a mathematical existence. It is important to note that, though it can be written formally, this proof is meaningful in the paradigm of idealization of reality. This might newly nurture relationships between mathematics and cognitive psychology.

A praxeology to develop?

Of course there is a distance from a proof written on a sheet of paper by an expert of mathematics up to a learnt praxeology. Yet, this proof suggests some steps, in equal sharing situations, that could lead to ‘understanding multiplicative relationships’:

1. Recomposing shapes into other shapes. Then observing the shapes, and understanding the changes: a part missing here could be filled by a part in excess there, a kind of compensation. Acknowledging that sometimes, it is possible to conclude.
2. Acknowledging that, sometimes, it is not possible to conclude, especially because, doing several times the same thing do not give the same result.

3. This actually provides a theoretical problem the students may be required to solve. The proof presented in section 3.3 appears to provide a solution.

This raises questions. For instance, is there a step before which students should be convinced the size should be equal (like Nicolas)? Which step?

Existence of quantity (in the sense of 3.2) remains mathematically invisible without questioning the unquestioned principles, i.e., the theoretical approach of numbers. What emerges from the equality of sizes is that a third, a twelfth are units (i.e., quantities) that can then be iterated (i.e., multiplied) and used to measure other quantities. Even though numbers remain inherited from set-theory, their actual generation is better formalized as number of times.

Meaning of task, interpretation of technique

An important role about theory (Θ) is that it “gives meaning to the problem”, “allows interpretation of the techniques”, and “serves as a basis to the technological descriptions and justifications”. In section 3, quantity resembles an intermediate layer between ‘real world’ and numbers. This layer enables specific reasoning, provides specific technologies and techniques.

In numbers constructions with set theory, this layer does not exist. French works presented in section 1.1 go directly from geometrical objects to numbers. Even a possible meaning of the task might be “what number does the shape represents?” Meaning of the geometrical shapes could be to represent numbers. In the quantity theory, geometrical shapes are not representations of numbers: they are quantities to be compared to U. Given two different sharing, in the former case, the task can be ‘what is their measures?’; in the latter case, it can be ‘do they provide parts of equal sizes?’ Given two theoretical environment, in the former case, the technique is to use regularity of multiplication of fraction by whole numbers, whereas in the latter case, it is to reason with size, possibly using P and reductio ad absurdum, establishing regularity of multiplication, and finally use regularity. Contrary to the quantity perspective, the “twofold sharing task” might be not really meaningful in the set-theory perspective.

It is also interesting to note that even in the Indian case, that takes place in a lesson on “quantities” (area), reasoning presented in section 3 does not appear. This suggests further study of transposition of area in the geometrical domain.

Investigating theory components in successful teaching contexts

In Douady and Perrin-Glorian resources (1986), there are many tasks such that comparisons of numbers like 1/5 and 1/6, or 5/6 and 4/5. Based on set-theoretical construction, this requires same denominators, because order is defined based on addition. Based on order of quantities , it can be proved: Let’s suppose $1/6 \geq 1/5$, then $5/6 \geq 5/5$, i.e., $5/6 \geq 1$. Thus $5/6+1/6>1$. Absurd. Despite the difficulty presented in section 1.1 from (Douady, 1980), students learnt extremely well in this teaching experiment. Sections 2 and 3 suggest it is actually impossible to teach fraction without understanding of quantity as a multiplicative relationship. This suggests to further investigate what the actual theory component of the taught praxeology was in the first implementation of the didactic engineering (this might be possible to get some old data). Such investigation could nurture complex issues in mathematics education such as replication of studies.

Theoretical horizon (Ma & Kessel, 2018) of Chinese arithmetic (in the context studied by Liping Ma, 1999) seems to have common points with the one of classical arithmetic (Chambris, 2021). Thompson evidences prominent teaching of co-variational reasoning in countries like Japan and Korea. These works suggest further exploration of mathematical praxeologies in these countries, including the interpretation of tasks, techniques, and technologies through the lens of a theoretical component that may have been neglected until now.

References

Artaud, M. (1998). Introduction à l'approche écologique du didactique. L'écologie des organisations mathématiques et didactiques. In M. Bailleul, et al. (Eds.), *Actes de la IXe école d'été de didactique des mathématiques* (pp. 101–139). Caen: ARDM et IUFM.

Battista, M. T. (2007). The Development of Geometric and Spatial Thinking. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Charlotte, NC: Information Age.

Bosch, M., Gascón, J., & Trigueros, M. (2017). Dialogue between theories interpreted as research praxeologies: The case of APOS and the ATD. *Educational Studies in Mathematics, 95*(1), 39–52.

Bourbaki. (1984). *Éléments d'histoire des mathématiques*. Paris: Masson.

Bronner, A. (2008). La question du numérique dans l'enseignement du secondaire au travers des évolutions curriculaires. In A. Rouchier & I. Bloch (Eds.), *Perspectives en didactique des mathématiques* (pp. 17–45). Grenoble: La pensée sauvage.

Butlen, D. (2018). *L'enseignement des mathématiques à l'école primaire—12/12/2018*. https://www.academie-sciences.fr/fr/Seances-publiques/enseignement-mathematiques-ecole-primaire.html

Chambris, C. (2021). Raisons d'être des grandeurs. Le cas de l'arithmétique à l'école élémentaire. In H. Chaachoua, A. Bessot, et al. (Eds.), *Perspectives en didactique des mathématiques*. (Vol. 1, pp. 169–196). Grenoble: Pensée sauvage.

Chambris, C., Coulange, L., Rinaldi, A.-M., & Train, G. (2021). Unités (relatives) pour les nombres et le calcul à l'école. In H. Chaachoua, et al. (Eds.), *Perspectives en didactique des mathématiques* (Vol. 2, pp. 373–396). Grenoble: Pensée sauvage.

Chambris C., Visnovska, J. (2021) On the history of units in French elementary school arithmetic: The case of proportionality. *Historia Mathematica*.

Chevallard, Y. (1994). Ostensifs et non-ostensifs dans l'activité mathématique. *Actes Du Séminaire 1993-1994*, 190–200. http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=125

Chevallard, Y., Bosch, M., & Kim, S. (2015). *What is a theory according to the anthropological theory of the didactic?* 2614–2620. Prague: Charles University.

Confrey, J., & Harel, G. (1994). Introduction. In G. Harel & J. Confrey (Eds.), *Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. vii–xviii). Suny Press.

Douady, R. (1980). Approche des nombres réels en situation d'apprentissage scolaire (enfants de 6 à 11 ans). *Recherches En Didactique Des Mathématiques, 1*(1), 77–111.

Douady, R., & Perrin-Glorian, M.-J. (1986). *Liaison École-Collège: Nombres décimaux*. Paris: IREM.

Gandon, S. (2009). Relations et quantités chez Russell (1897-1913). Université Blaise Pascal, Clermont.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Ma, L., & Kessel, C. (2018). The Theory of School Arithmetic: Whole Numbers. In M. Bartolini-Bussi & X. Sun (Eds.), *Building the Foundation* (pp. 439–463). Springer.

Neyret, R. (1995). *Contraintes et déterminations des processus de formation des enseignants: Nombres décimaux, rationnels et réels dans les IUFM*. UJF-Grenoble 1, France.

Otte, M. (2007). Mathematical history, philosophy and education. *Educational Studies in Mathematics*, *66*(2), 243–255.

Perrin-Glorian, M.-J. (1993). Questions didactiques soulevées à partir de l'enseignement des mathématiques dans des classes «faibles». *Recherches En Didactique Des Mathématiques, 13*(1–2), 5–118.

Radford, L. (2021). Davydov's concept of the concept and its dialectical materialist background. *Educational Studies in Mathematics, 106*(3), 327–342.

Rahaman, J., & Subramaniam, K. (2016). *Constructing the concept of area measurement in a classroom*. 1–8. ICME 13, Hambourg, Allemagne.

Thom, R. (1970). Les Mathématiques "Modernes": Une erreur pédagogique et philosophique ? *L'Age de La Science, 3*, 225–242.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes. *Epistemic Algebraic Students: Emerging Models of Students' Algebraic Knowing, 4*, 1–24.

Thompson, Patrick W., Hatfield, N. J., Yoon, H., Joshua, S., & Byerley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *The Journal of Mathematical Behavior, 48*, 95–111.

Whitney, H. (1968). The mathematics of physical quantities. Part I: mathematical models for measurement. *The American Mathematical Monthly, 75*(2), 115–138.

La TAD et l'étude des questions didactiques et pédagogiques face aux situations sociales de transformation qui ne cessent de travailler et de renouveler les sociétés humaines

Caroline Ladage[1], Marine Dintrich[1], Hugo Paris[1] and Sophie Péninque[1]

[1]UR 4671 ADEF, Aix Marseille University, France, caroline.ladage@univ-amu.fr, marine.dintrich@etu.univ-amu.fr, hugo.paris@insa-lyon.fr, sophie.peninque@etu.univ-amu.fr

Abstract. We present a questioning scheme based on the Anthropological Theory of the Didactic, which is the object of a research programme whose issue is the identification of knowledge, and more generally of praxeologies, useful for answering professional questions emerging from societal, technological and ecological transformations (linked to school inclusion, the environment, digital technology, etc.). We present three doctoral research projects on the questions that teachers face in the organisation of their training courses in the face of the societal challenges of inclusive schools and the ecological transition of engineering professions.

Keywords: Didactics in the workplace, societal transformations, inclusive school.

Résumé. Nous présentons un schéma de questionnement basé sur la Théorie Anthropologique du Didactique, faisant l'objet d'un programme de recherche dont la problématique est l'identification des savoirs, et plus généralement des praxéologies, utiles pour répondre à des questions de métier émergeant des transformations sociétales, technologiques, écologiques (liées à l'inclusion scolaire, l'environnement, le numérique…). Nous présentons trois recherches doctorales sur les questions qui se posent à des enseignants et formateurs dans l'organisation de leurs formations face à ces enjeux sociétaux de l'école inclusive et de la transition écologique des métiers de l'ingénieur.

Mots clés : Didactique en milieu professionnel, transformations sociétales, école inclusive

Resumen. Presentamos un esquema de interrogación basado en la Teoría Antropológica de lo Didáctico, que es objeto de un programa de investigación cuyo tema es la identificación de conocimientos, y más generalmente de praxeologías, útiles para responder a las preguntas profesionales que surgen de las transformaciones sociales, tecnológicas y ecológicas (vinculadas a la inclusión escolar, al medio ambiente, a la tecnología digital, etc.). Presentamos tres proyectos de investigación doctoral sobre las cuestiones que se plantean a los profesores en la organización de sus formaciones frente a los retos sociales de la escuela inclusiva y la transición ecológica de las profesiones de ingeniería.Type your abstract here.

Palabras clave: Didáctica en el trabajo, transformaciones sociales, escuela inclusiva.

La TAD pour étudier les lieux situés au-delà des frontières habituelles des didactiques

Dans le cadre d'un programme de recherche sur la didactique en milieu professionnel, qu'il soit scolaire ou relevant de la formation professionnelle, nous nous intéressons aux problèmes didactiques et pédagogiques, souvent ignorés, qui apparaissent en des lieux situés au-delà des frontières habituelles des didactiques et des sciences de l'éducation et de la formation. Ces problèmes émergent dans des situations sociales de transformations et de transitions sociétales – du fait notamment du développement du numérique, de l'école inclusive, des défis socio-écologiques, … – qui ne cessent de travailler et de renouveler les sociétés humaines, auxquelles l'éducatif et le didactique demeurent coextensifs.

Face à ces problèmes, la professionnalisation des enseignants est contrainte de se construire en grande partie en référence à l'expérience personnelle du terrain, ce qui pose de nombreuses questions. En nous appuyant sur le cadre théorique de la TAD, nous centrons nos recherches sur les questions suivantes : Quelle est la qualité des praxéologies mises en œuvre à titre personnel ? Quels sont les effets de la faible confrontation des pratiques au sein de collectifs d'enseignants ? Comment s'opèrent les choix d'institutionnaliser certaines des praxéologies mises en œuvre au niveau des programmes de formation ? La question de savoir comment dépasser le niveau « local » ou, plus souvent, « ponctuel » de ces praxéologies afin de constituer des modèles didactiques de référence partagés apparaît dans de nombreuses situations professionnelles comme problématique.

En référence à l'une des définitions de la didactique donnée par Yves Chevallard (2010) comme étant « la science des conditions et des contraintes de la diffusion (et de la non-diffusion) des praxéologies au sein des institutions de la société », nous étudions les conditions de la diffusion sociale des connaissances pour comprendre les transitions dans la structuration des métiers de l'enseignement dans ces contextes professionnels particuliers et la manière dont s'organisent les évolutions du curriculum dans les formations à ces métiers. Dans chacun des contextes étudiés dans le cadre de notre programme de recherche, la problématique étudiée est le questionnement à mettre en œuvre afin d'interroger les praxéologies existantes ou à créer au sein de certaines institutions sociales, dans l'optique de les enseigner. Le repérage et le questionnement des praxéologies porte à la fois sur les contenus à enseigner et les gestes professionnels pour ce faire. Il implique un travail de didactique en milieu professionnel ainsi que la formation d'enseignants dans des domaines nouveaux, non habituels dans une institution scolaire.

Les contextes que nous présentons dans cette publication et qui illustrent ce programme de recherche, sont d'une part le dispositif national de l'école inclusive, et d'autre part l'introduction de questions socio-écologiques dans les formations à vocation professionnalisantes comme les écoles d'ingénieur. Nous emploierons le terme de *contexte-témoin* pour souligner le fait que les contextes à l'étude constituent des témoins, ou encore fournissent des indices, de l'existence des phénomènes étudiés : les besoins en formation aux transpositions didactiques de praxéologies relevant non pas du cadre habituel des disciplines scolaires, mais de praxéologies relevant plus généralement de l'activité humaine quelle que soient les institutions qui les font vivre, voire qui en auraient besoin.

Contextes-témoins pour questionner les formations face aux transformations sociétales

Nous présentons trois recherches doctorales, dont l'objet partagé est la problématique de l'identification de modèles didactiques de référence dans la formation des enseignants, dans les cas où les contenus praxéologiques des champs éducatifs et professionnels considérés se situent à la périphérie des champs (domaines) praxéologiques faisant l'objet de programmes d'enseignement ou de référentiels de formation généralement bien établis. Les contextes à l'étude, le service national de l'école inclusive et la formation d'ingénieurs face à l'introduction de questions socio-écologiques, sont ainsi pris comme témoins pour étudier les problématiques rencontrées.

Notons que cette situation périphérique – qui concerne aussi bien les gestes professionnels que les contenus à enseigner –, est marquée par une tension entre la volonté de prolonger les praxéologies existantes et la conscience de la nécessité de les adapter, voire de créer des praxéologies nouvelles, ce qui pose la question du déplacement des « frontières praxéologiques » (Ladage, 2008).

> Une frontière organise l'espace […] et y exprime de façon spécifique des contraintes qui pèsent sur les personnes et les institutions agissant dans cet espace. En règle générale, du point de vue des collectifs anciennement stabilisés, enracinés […], on est d'un côté ou de l'autre d'une frontière, dont le franchissement ne va pas de soi. (Ladage, 2008, p. 268)

En nous intéressant plus particulièrement à la position de l'enseignant aux prises avec ces problématiques dans ces contextes-témoins, notre objectif est de mettre en lumière la difficulté personnelle et institutionnelle, non seulement à faire évoluer son équipement praxéologique, mais aussi à déplacer les *frontières praxéologiques* vers des champs de connaissances (autrement dit des champs ou domaines praxéologiques) pour lesquels initialement l'on ne se pensait pas concerné, voire légitime (du fait de sa formation initiale par exemple).

> Quand celui-ci [le déplacement de frontière] se réalise, il est souvent furtif, momentané – on ne s'installe pas à demeure, en règle générale, de l'autre côté d'une frontière. Si l'on est un frontalier, qui ne cesse de franchir telle ou telle frontière, il arrive qu'on le cache, comme un fait mal compris, mal reçu, étrange comme peut l'être un étranger. (p. 268)

La question du déplacement de(s) frontière(s) praxéologique(s) est complexe, « une frontière praxéologique ne saurait se traiter par l'ignorance, notamment au plan didactique : car une telle frontière ne s'abolit pas aisément. Son franchissement, personnel ou institutionnel, est toujours un problème didactique délicat. » (p. 268). Il peut nécessiter des déconstructions de praxéologies existantes, ou des efforts importants pour engager des domaines praxéologiques inconnus, relevant parfois d'autres métiers, ou restant à créer. Ces problématiques concernent aussi bien les transformations nécessaires des gestes didactiques et pédagogiques, tel qu'en témoignent les situations de l'école inclusive, que les questions qui se posent aux enseignants face aux transformations du curriculum des formations. La question est d'autant plus vive lorsque les enseignants sont eux-mêmes amenés à contribuer à l'identification des contenus à enseigner du fait de la nouveauté/sensibilité du champ praxéologique concerné, ou des gestes didactiques et pédagogiques appropriés à mettre en œuvre.

Il est utile de rappeler ici que la notion de *frontière praxéologique* a initialement été étudiée et mise à l'épreuve dans une recherche doctorale sur la didactique de la recherche d'information sur Internet (Ladage, 2008). Le domaine praxéologique de la recherche d'information sur Internet (RII) constitue un exemple de domaine praxéologique complexe, à l'intersection de nombreuses disciplines[1] et à l'émergence récente. Il est marqué par des praxéologies labiles au gré des évolutions technologiques. Il concerne un public très large (dans les sphères éducatives et professionnelles, mais aussi dans la sphère privée), qui, le plus souvent, a appris la RII « sur le tas » ou à partir de modèles praxéologiques élémentaires intégrés récemment dans le curriculum scolaire. Au regard du développement fulgurant d'Internet et des transitions sociétales qu'il génère, de tels apprentissages de la RII apparaissent insuffisamment problématisés et travaillés du point de vue didactique. L'étude de cette problématique à la lumière de la théorie anthropologique du didactique a permis de mettre en lumière un schéma de questionnement pouvant concerner de nombreuses situations professionnelles face aux évolutions sociétales, aux transitions numériques et écologiques.

Pour l'illustrer l'intérêt de questionner la construction des équipements praxéologiques des personnes et des institutions à la lumière de la notion de « frontière-praxéologique », nous prenons appui sur différents contextes-témoins. Celui des situations engendrées par le développement du service public de l'école inclusive, qui suscite des besoins importants de déplacement de « frontières praxéologiques » au regard des adaptations souvent complexes des gestes professionnels à déployer, dans des situations de classe ordinaire, pour accueillir des élèves en situation de handicap et à besoin éducatif particulier (EBEP). Dans cette perspective nous questionnons l'équipement praxéologique des enseignants, les programmes de formation et les ressources didactiques et pédagogiques qui leur sont proposés ou qui seraient à construire pour l'apprentissage de la mise en œuvre de l'inclusion scolaire des EBEP.

Un autre contexte-témoin est celui d'une école d'ingénieur face à la pression pour faire évoluer le curriculum de formation des ingénieurs, appelant la conception de modèles didactiques inédits, tant du point de vue des praxéologies de référence émanant de la société, que de celui des modèles pédagogiques idoines pouvant assurer leur enseignement.

Dans les contextes-témoins à l'étude, le milieu professionnel évolue et subit des transformations auxquelles les enseignants sont confrontés, nécessitant des adaptations de leurs modèles praxéologiques de référence, souvent avant même que ces gestes et modèles puissent faire l'objet de formations dédiées.

L'école primaire et la question de la formation des enseignants.

La recherche doctorale de Marine Dintrich met en lumière que du fait que l'inclusion scolaire est un service récent, aussi bien la formation que la recherche n'en sont qu'au début d'une construction de réponses aux difficultés rencontrées lors de l'accueil des élèves en situation de handicap en classe

[1] L'analyse praxéologique menée dans le cadre de cette recherche repère notamment les disciplines et domaines suivants : l'histoire, l'économie, la politique, la documentation, le droit, les mathématiques, la physique et les technologies numériques, l'informatique, l'Internet, le World Wide Web, l'anglais, … (Ladage, 2008, p. 279-456).

ordinaire dans l'école primaire en France. Or dans l'approche didactique que nous adoptons, la construction de réponses à ces questions nécessite le renforcement de la dialectique existant entre formation et recherche.

Il y a une réelle difficulté à former à un métier lorsqu'il n'y a pas eu de travaux empiriques sur la professionnalisation et les questions qui se posent aux personnes exerçant le métier. De la même façon, la formation étant actuellement lacunaire, il est difficile pour la recherche de réaliser des études sur cet objet récent qui relève encore le plus souvent d'initiatives personnelles du fait des enseignants eux-mêmes, voire des équipes pédagogiques (Dintrich, Ladage & Hache, en cours).

Le cadre de référence de la TAD révèle et questionne dans ce contexte la prédominance de rapports personnels aux questions que suscite l'école inclusive, sur des rapports institutionnels qui peinent à se construire et à être diffusés auprès des professeurs des écoles.

L'école secondaire et la question de l'inclusion des élèves à haut potentiel intellectuel.

Dans le cadre de sa recherche doctorale, Sophie Péninque s'interroge sur les équipements praxéologiques des enseignants de l'école secondaire pour assurer l'inclusion des collégiens à haut potentiel en classe ordinaire. L'étude de la population d'élèves à haut potentiel intellectuel (EHPI) et de leurs besoins éducatifs particuliers se situe dans un contexte de recherche contrastant avec celui du contexte-témoin l'école primaire, du fait qu'il existe sur ces élèves une littérature scientifique abondante en particulier à l'international et relevant de différents champs disciplinaires. Des recherches existent aussi bien sur les caractéristiques cognitives, physiologiques et comportementales particulières des EHPI, que sur des expérimentations de modalités didactiques et pédagogiques de prise en charge adaptée. Nous notons la tendance pathologisante du HPI (et donc la prédominance des champs de la psychologie dans la recherche et sur le terrain), ainsi que les approches plutôt centrées sur le sujet et beaucoup moins sur l'élève (du point de vue de sa position au sein d'un système didactique). Or la recherche doctorale menée découvre là encore, et malgré la profusion de littérature scientifique, un faible développement de la dialectique entre formation et recherche pour les professeurs de collège accueillant des EHPI. On retrouve auprès de cette population de professeurs la même prédominance de rapports personnels aux questions que pose le service de l'école inclusive.

En s'appuyant sur la TAD, la thèse défend que pour intégrer l'EHPI dans la classe, l'enseignant devrait pouvoir l'identifier en s'appuyant sur une connaissance des particularités des différents profils d'EHPI, et pouvoir connaître les modalités pédagogiques et didactiques adaptées aux besoins repérés. Dans les conditions actuelles, comment s'y prennent-ils pour intégrer ces élèves dans la classe ? Qu'est-il possible pour les enseignants de savoir sur et de faire avec ces élèves aujourd'hui ? Grilles, fiches, vadémécum, séminaires, etc.[2], qu'apportent les ressources pour la personnalisation des parcours des élèves à haut potentiel développées à destination des professeurs ?

[2] À titre d'exemple l'on peut consulter la section dédiée aux ressources pour la personnalisation des parcours des élèves à haut potentiel sur le site officiel de l'Éducation nationale : https://eduscol.education.fr/1188/ressources-pour-la-personnalisation-des-parcours-des-eleves-haut-potentiel.

Le contexte témoin d'une école d'ingénieur

La recherche doctorale d'Hugo Paris cherche d'une part à comprendre les processus de problématisation des questions socio-écologiques dans l'enseignement supérieur français et comment s'emparer de ces enjeux dans les formations proposées par une école d'ingénieur. D'autre part elle s'interroge sur la manière d'accompagner les enseignants dans cette tâche.

> Face à l'ambition d'adresser ces questions vives portant sur le rôle social de l'ingénieur et ses pratiques, on peut s'interroger sur la manière dont les enseignants de ces écoles identifient les contenus et les compétences pertinentes pour leurs étudiants. Dans un contexte où ces savoirs et ces pratiques font l'objet de développements scientifiques et professionnels soutenus, comment les enseignants actualisent-ils leurs propres connaissances et pratiques ? (Paris, Saïd-Touhami, & Ladage, 2022).

La recherche s'appuie sur le cadre de référence de la TAD pour enquêter sur la complexité et l'imbrication des domaines praxéologiques pouvant être en jeu, d'autant que les questions socio-écologiques et de durabilité sont des questions sensibles. Cela notamment parce qu'elles sont souvent politiques et susceptibles d'impliquer chacun de nous. Elles font par ailleurs fréquemment l'objet de déclarations polémiques, sans que les connaissances à leur sujet soient stabilisées et suffisamment diffusées ce qui a des conséquences sur les enseignants mais également sur les étudiants et leurs attentes. Elle met en avant la manière dont les différentes praxéologies sont prises dans des rapports de pouvoir qui compliquent le partage de praxéologies par exemple en essentialisant les frontières entre les disciplines, notamment entre sciences humaines et sociales et sciences naturelles. La TAD invite à questionner les différentes perspectives en jeu : celle des praxéologies en matière d'écologie elles-mêmes (individuelles ou collectives) ; celle de l'équipement praxéologique de l'enseignant ; celle de l'institution de formation…

Quelles théorisations en TAD pour comprendre la didactique en milieu professionnel ?

Les trois recherches doctorales discutées dans cette publication s'appuient principalement, même si ce n'est pas de manière exclusive, sur différentes théories constitutives de la TAD comme cadre de référence théorique structurant les recherches menées dans leurs différentes phases : la construction des questionnements, les différents types d'enquêtes et de méthodologies de recherche engagés[3], la formulation des résultats. Nous en présentons succinctement les plus importantes.

L'analyse didactique et l'échelle de niveaux de codétermination didactique

Chevallard prolonge la définition rappelée plus haut de la science du didactique comme l'étude des conditions et contraintes de la diffusion sociale des connaissances, avec la théorie dite de l'échelle des niveaux de codétermination didactique (Chevallard, 2010). L'étude des conditions et contraintes qui pèsent sur le didactique à chaque niveau de l'échelle offre un cadre de référence solide pour

[3] Des schémas de recherche similaires ont été proposés dans les thèses de Cécile Redondo (2018), Christine Pintus (2018), Corinne Manceau (2019) et Christine Rivier-Perret (2020).

structurer et approfondir l'analyse de ce qui est favorable ou de ce qui peut gêner la diffusion des praxéologies ou des champs praxéologiques à l'étude, à tous les niveaux de l'échelle : le système didactique, l'école, la société, la civilisation, voire l'humanité. Toujours dans les termes de la TAD, une telle étude vise la découverte de l'écologie d'un système didactique et l'étude de l'économie du didactique œuvrant dans de ce même système, comme gestes didactiques pour assurer son bon fonctionnement.

Du point de vue des méthodologies mises en œuvre pour ces recherches, une telle compréhension anthropologique du didactique nécessite une démarche d'enquête, consubstantielle à une attitude de questionnement et de problématisation du monde, non seulement au niveau du fonctionnement des systèmes didactiques eux-mêmes, mais aussi, et au préalable, au niveau des faits d'apprentissage ou de non-apprentissage des champs praxéologiques considérés à tous les niveaux de la société. Notons qu'une part importante de ces enquêtes s'appuie sur Internet pour le recueil de données (Ladage, à paraître) en complément à d'autres outils idoines pour mener à bien l'enquête sur les activités et les métiers de l'humain. Notons que la TAD considère l'enquête comme l'une des praxéologies les plus essentielles du chercheur en didactique, qui à aucun moment ne peut être considérée comme achevée (Ladage & Chevallard, 2011 ; Ladage, 2017 ; Ladage & Redondo, 2021), et se prolonge tout au long des recherches menées.

L'enjeu de ces enquêtes, est de contribuer à la connaissance et la compréhension de l'écologie et de l'économie institutionnelles de champs praxéologiques de façon générale, et, plus particulièrement pour ce qui concernent les trois recherches en question, les phénomènes de transformation qu'ils endurent ou portent en eux du fait des transitions sociétales.

La théorie du rapport au savoir et l'analyse praxéologique

L'exploration des conditions et contraintes qui pèsent sur le didactique invite à la mise en œuvre d'analyses des phénomènes découverts au cours des enquêtes menées. L'analyse des rapports (absents, existant ou à construire) que les personnes et les institutions entretiennent aux objets relevant des champs praxéologiques étudiés constitue une dimension importante du travail du didacticien. Avec la théorie du rapport au savoir Chevallard (2003) soulignait « le fait massif de la fragmentation institutionnelle et personnelle de la connaissance, appréciée à la diversité des rapports personnels et institutionnels observables. » (Chevallard, 2007, p. 4). Cette fragmentation de la connaissance, et des praxéologies plus généralement, est bien celle à l'origine des questionnements des trois recherches doctorales, dont les enquêtes menées confirment l'idée d'un « bric-à-brac indistinct, changeant d'individu à individu et d'institution à institution » (p. 5), menant à constater une relativité importante des contenus d'enseignement.

La théorie des rapports offre au chercheur un outil puissant pour l'analyse de situations sociales et un langage permettant de les décrire. Il s'agissait pour Chevallard (2007) de « modéliser la structure de l'activité humaine » ce qui le conduira à proposer l'idée de réaliser des analyses praxéologiques à partir de la notion de *praxéologie* et d'*équipement praxéologique*. L'analyse praxéologique s'inscrit dans le prolongement de l'analyse des rapports, elle entre dans le détail des rapports aux objets et de l'activité humaine en termes de types de tâches, des techniques et des discours justificatifs les accompagnant. Elle constitue un outil, autant dans le repérage et la compréhension du développement des praxéologies au sein de la société, que dans l'analyse fine de l'organisation didactique qu'elles nécessitent pour être diffusées.

Dans nos travaux, les deux théorisations se complètent du fait de leurs manières distinctes de nommer les phénomènes, facilitant de ce fait des mises en perspectives différentes entre l'observation des gestes professionnels (et la formation à ces gestes), l'identification des praxéologies à enseigner et les techniques pédagogiques idoines.

Il est à ce stade essentiel d'étudier dans ces contextes particuliers comment ces travaux peuvent nourrir la construction de modèles praxéologiques de référence dont les acteurs des terrains concernés pourraient se saisir pour faire évoluer leurs pratiques professionnelles. Au-delà des visées exploratoire et compréhensive de nos recherches sur les champs praxéologiques existants, il est indispensable d'y associer une démarche d'étude proactive et expérimentale pour répondre aux besoins de constructions didactiques inédites, dépassant les habitus praxéologiques. Afin de répondre aux phénomènes de transformation des sociétés humaines, l'éducatif et le didactique ne peuvent en effet pas rester dans un mode d'étude rétroactif (Ladage & Chevallard, 2011), en se contentant de reproduire l'existant. C'est la raison pour laquelle les recherches doctorales menées s'intéressent à l'expérimentation, là où les terrains de recherche le permettent, de modèles didactiques et pédagogiques nouveaux. Il est alors essentiel de comprendre les rouages de la construction d'un projet didactique et de son modèle praxéologique de référence.

Apprendre à construire un modèle praxéologique de référence

En étudiant l'une des « Leçons de didactique » de Chevallard pour tenter de comprendre comment faire pour construire un projet didactique d'un champ nouveau, nous retenons que :

> lorsqu'on veut étudier un champ nouveau de l'éducation, il convient de disposer d'un modèle didactique de référence. Celui-ci suppose en particulier un modèle praxéologique de référence, qui modélise les contenus praxéologiques du champ éducatif considéré, et un modèle pédagogique de référence. (Chevallard, 2010)

Ainsi pour construire un modèle didactique de référence Chevallard identifie trois grandes questions au cœur de la didactique de ce champ praxéologique nouveau $\mathcal{A}$[4] :

> Q_1. Que sont ou que pourraient être les praxéologies personnelles et institutionnelles de $\mathcal{A}$? En d'autres termes, quels *types de tâches T* accomplir ? Et *comment*, c'est-à-dire selon quelles *techniques* τ ? Et encore *pourquoi*, c'est-à-dire en vertu de quelles *technologies* θ et de quelles *théories* Θ ?
>
> Q_2. Par quels systèmes de conditions et de contraintes, c'est-à-dire à travers quelles successions de situations, telle praxéologie de $\mathcal{A}$ parvient-elle ou pourrait-elle parvenir à s'intégrer à l'équipement praxéologique de telle institution ou de telle personne ?

[4] Il est intéressant de noter que Chevallard a donné cette leçon dans le cadre d'un enseignement sur l'éducation au développement durable à destination d'étudiants se destinant au professorat des écoles.

Q_3. Quelles positions *p* de quelles institutions *I* sont-elles concernées ou pourraient-elles l'être, dans des conditions adéquates, par quelles praxéologies *P* du domaine d'activité @ ?

Nous constatons que pour apporter des réponses à ces questions il faut mener les enquêtes et analyses didactiques et praxéologiques présentées plus haut. Mais au-delà de l'observation de l'existant, il faut également mener ce travail exploratoire des praxéologies et des champs praxéologiques pour identifier celles et ceux pouvant intéresser, de près ou de loin, le développement de connaissances du champ nouveau. Ce travail est au cœur du travail du didacticien, comme l'a montré l'étude sur les praxéologies utiles à la recherche d'information sur Internet citée plus haut.

Sans pouvoir entrer ici dans le détail des techniques d'exploration pouvant être mises en œuvre, il faut souligner l'importance d'un changement de paradigme (de rapport au monde) nécessaire non seulement au chercheur en didactique, mais également à l'enseignant. En effet, dans de nombreux cas, l'attente d'une transposition didactique des praxéologies à enseigner ou d'un référentiel officiel des gestes professionnels à mettre en œuvre, est trop longue et n'arrive que longtemps après la confrontation des enseignants aux problématiques du terrain. Les réformes en éducation et en formation autour des champs praxéologiques nouveaux ne sont que rarement accompagnées d'une diffusion de modèles praxéologiques et didactiques opérationnels dès la mise en vigueur des lois qui les instituent.

Il est de ce fait primordial de développer auprès de chaque professionnel en éducation et en formation « une attitude à reconnaître la “problématicité” des situations vécues ou observées, c'est-à-dire à soulever des questions à leur propos. » (Chevallard, 2013). Le fait que l'on rencontre sur le terrain des rapports d'abord personnels, ponctuels et locaux peut alors être analysé sous un regard nouveau, à condition que soit développé un rapport à la connaissance solidement construit. Ce sont les outils théoriques de la TAD qui offrent là encore un cadre d'analyse pour le vérifier, en particulier par l'observation de la mise en œuvre d'une suite de dialectiques de la recherche et de l'étude (Chevallard, 2010 ; 2007 ; Bosch, Chevallard, García, & Monaghan, 2019). Il reste à savoir comment transposer cette attitude de questionnement du monde, en d'autres termes, et en prenant la position de l'enseignant, comment enseigner et encourager l'attitude de problématisation auprès des enseignants dans le système de l'éducation nationale ou encore auprès des futurs ingénieurs ?

Conclusion et perspectives

Les travaux d'enquête et d'analyse réalisés dans le cadre de la TAD contribuent à une meilleure compréhension des phénomènes observés et apportent au moins en partie des éléments de réponse aux questions de recherche posées sur les conditions et contraintes déterminant les équipements praxéologiques institutionnels et personnels des professionnels dans les contextes-témoins que nous étudions. Sur les chemins des enquêtes accomplies d'autres cadres de référence théorique ont été explorés, ce qui a permis des interactions et dialogues d'abord avec d'autres théories didactiques. C'est le cas des théorisations proposées notamment par Pastré, Mayen et Vergnaud (2006) sur l'utilité de la notion de *schème* de Piaget comme outil d'analyse de l'activité humaine. Des interactions ont également été réalisées avec les travaux qui s'inscrivent dans d'autres champs disciplinaires, notamment avec la psychologie différentielle. Les recherches dans ce champ ont permis d'enrichir la compréhension du phénomène des EHPI et sa portée sociale. Du point de vue didactique elles justifient l'intérêt de prendre en compte la variabilité étudiée et observée chez les EHPI. Cet exemple encourage l'adoption de cadres de référence théorique et de méthodologies de recherche à même de prendre en compte cette diversité disciplinaire, ce que la TAD et le paradigme de l'enquête semblent bien en mesure de proposer.

References

Bosch, M., Chevallard, Y., García, F. J., & Monaghan, J. (Eds.) (2019). Working with the Anthropological Theory of the Didactic in Mathematics Education: A Comprehensive Casebook. Routledge.

Chevallard, Y. (2003). Approche anthropologique du rapport au savoir et didactique des mathématiques. Dans Maury S. & M. Caillot (Éds), *Rapport au savoir et didactiques*, p. 81-104. Éditions Fabert.

Chevallard, Y. (2007). Passé et présent de la théorie anthropologique du didactique. Conférence plénière au premier congrès international sur la théorie anthropologique du didactique. Dans L. Ruiz-Higueras, A. Estepa, & F. Javier Garcia (Éds.), *Sociedad, Escuela y Mathemáticas. Aportaciones de la Teoría Antropológica de la Didáctico*, p. 705-746, Universidad de Jaén.

Chevallard, Y. (2010). La didactique, dites-vous ? *Éducation et didactique*, *4*(1), 139-148.

Chevallard, Y. (2013). *Eléments de didactique du DD 2012-2013*. Enseignement donné en 2012-2013, Éducation au développement durable, licence science de l'éducation de l'université d'Aix-Marseille. http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=213

Dintrich, M., Ladage, C., & Hache, C. (en cours, 2022). La formation initiale des enseignants du premier degré à l'inclusion scolaire des élèves en situation de handicap : le point de vue de formateurs d'INSPE. *Éducation et socialisation*.

Paris, H., Saïd-Touhami, F., & Ladage, C. (2022). Actualiser ses connaissances : l'évolution professionnelle des enseignants d'école d'ingénieur autour des enjeux de durabilité. Regards croisés en didactique en milieu professionnel. *6e Colloque international de didactique professionnelle, Entre travail et formation : regards croisés sur les questions actuelles de la formation professionnelle.* Lausanne 2022.

Ladage, C. (2008). Étude sur l'écologie et l'économie des praxéologies de la recherche d'information sur Internet. Une contribution à la didactique de l'enquête codisciplinaire. Thèse de doctorat de l'Université Aix-Marseille I.

Ladage, C. (2017). *Enquêter pour savoir. La recherche d'information sur Internet comme solution et comme problème*. Coll. Paideia. Presses universitaires de Rennes.

Ladage, C. (à paraître, 2022). Enquêter sur Internet : Quels recueils de données ? Dans B. Albéro & J. Thiévenaz (Éds.). *Traité de méthodologie de la recherche en sciences de l'éducation et de la formation : Enquêter dans les métiers de l'humain*.

Ladage, C. & Chevallard, Y. (2011). Enquêter avec l'Internet. Études pour une didactique de l'enquête. *Éducation & Didactique*, 5(2), 85-115.

Ladage, C. & Redondo, C. (2021). Which methodological paths of inquiry on the Internet for research in didactics? *Zeitschrift Bildungsforschung*, DOI: 10.25539/bildungsforschun.v0i2.418.

Pastré, P., Mayen, P. et Vergnaud, G. (2006). La didactique professionnelle. *Revue française de pédagogie*, 154, 145-198. https://doi.org/10.4000/rfp.157.

Pintus, C. (2018). La contribution de la communication par le web au développement de praxéologies professionnelles infirmières. Thèse de doctorat de l'université Aix-Marseille.

Manceau, C. (2019). La formation professionnelle en prison : la transposition institutionnelle et l'environnement didactique en question. Thèse de doctorat de l'université Aix-Marseille.

Redondo, C. (2018). Étude didactique des praxéologies de l'éducation au développement durable : Des fondements épistémologiques des pédagogies de l'EDD. Thèse de doctorat de l'université Aix-Marseille.

Rivier-Perret, C. (2020). Approche didactique de l'influence du numérique sur la formation des métiers artisanaux : le cas du métier d'ébéniste. Thèse de doctorat de l'université Aix Marseille.

Modélisation de parcours d'apprentissage, adaptables à l'apprenant, dans un EIAH

Sébastien Jolivet[1, 2] and Brigitte Grugeon-Allys[3]

[1]IUFE, Université de Genève, Suisse ; sebastien.jolivet@unige.ch

[2]LDAR, Université de Paris, France

[3]LDAR, Université Paris Est Créteil, France ; brigitte.grugeon-allys@u-pec.fr

Abstract : The research presented in this paper is part of a project to design a learning environment for mathematics in secondary schools (11 - 15 years old). We consider three issues in this work: the consideration of the learner in the environment, the description and production of exercises in the platform, and finally the linking of these two elements through learning paths adaptable to the learner. For the consideration of the learner we have introduced the notion of personal logos. The resources are described and produced using a knowledge model. Finally, the learning paths are structured in relation to the praxeological needs of the students, without however aiming to satisfy them completely in the environment.

Keywords: learning path; personal logos; learning environment; resources; knowledge model

Résumé : Le travail présenté dans cette contribution s'inscrit dans un projet de conception d'un environnement numérique d'apprentissage des mathématiques au collège (11 – 15 ans). Nous abordons trois aspects de ce travail : la prise en compte de l'apprenant dans l'environnement, la description et la production des exercices dans la plateforme, et enfin la mise en relation de ces deux éléments au travers de parcours d'apprentissage adaptables à l'apprenant. Pour la prise en compte de l'apprenant nous proposons la notion de logos personnel. Les ressources sont décrites et produites à l'aide d'une modélisation du savoir. Enfin, les parcours sont construits en relation avec les besoins praxéologiques des élèves, sans toutefois viser à les satisfaire complétement dans l'environnement.

Mots clé : parcours d'apprentissage ; logos personnel ; EIAH ; ressources ; modélisation du savoir

Introduction

Le développement d'environnements numériques d'apprentissage est un sujet important dans le monde éducatif depuis de nombreuses années. Si certains sont fondés sur des travaux de didactique (Cabri Géomètre, Aplusix, SmartEnseigno…) de nombreux autres ne les prennent pas en compte.

Dans cette contribution nous présentons l'exploitation de la TAD lors du travail de conception d'un environnement numérique d'apprentissage (ENA) permettant de travailler l'algèbre et la géométrie du collège. Ce travail a été réalisé dans le cadre du projet MindMath, d'une durée de trois ans, réunissant différents partenaires, scientifiques (le LDAR et le LIP6[1]) et industriels (Tralalère,

[1] Respectivement Laboratoire de Didactique André Revuz et Laboratoire d'Informatique de Sorbonne Université

Cabrilog, Domoscio, Breakfirst). Notre travail a notamment visé à définir les fondements didactiques de l'environnement.

Dans la plateforme il s'agit de proposer à l'apprenant des ressources ; organisées en parcours, adaptées à son profil et à son activité mathématique ; ainsi que des rétroactions épistémiques. Le premier objectif nécessite de disposer de ressources, conformes aux attendus institutionnels, décrites de manière à pouvoir les mettre en relation avec les besoins d'apprentissage de l'apprenant. Les objectifs poursuivis dans le projet sont donc divers et ambitieux.

Notre propos dans cette contribution est principalement d'expliciter comment nous avons conçu les ressources présentes dans l'environnement afin, notamment, de pouvoir les organiser au sein de parcours et les proposer à l'apprenant. Pour cela, dans la deuxième section, nous précisons notre problématique, notamment en situant notre travail par rapport à l'existant selon trois axes : la description didactique de ressources, les environnements numériques d'apprentissage (ENA) et la prise en compte de l'apprenant. Dans la troisième section nous revenons plus particulièrement sur la prise en compte de l'apprenant dans un ENA ce qui nous permet de préciser certaines délimitations, de cette contribution en particulier et du projet en général. Dans la quatrième section nous présentons les éléments théoriques mis en jeu pour modéliser les ressources. La cinquième section vise à définir la modélisation des parcours d'apprentissage adaptables à l'apprenant mettant ainsi en relation les troisième et quatrième section. La conclusion est l'occasion de proposer quelques perspectives et de revenir sur certaines limites identifiées au fil du texte.

Problématique

Dans les différents environnements numériques d'apprentissage il s'agit de mettre à disposition de l'apprenant des ressources. Elles peuvent être de différentes natures : cours, exemple, exercices, environnement de géométrie dynamique ou de calcul, etc. Afin de pouvoir organiser ces ressources pour permettre à l'apprenant d'y accéder, et éventuellement permettre au système de les recommander, il est nécessaire d'y associer des métadonnées. La question de la description didactique de ressources de type exercices est au cœur du travail de Jolivet (2018). Dans (Jolivet et al., 2022) il est proposé un panorama des standards de description des ressources éducatives. Nous ne revenons pas sur ce travail mais nous retenons la nécessité de disposer d'une modélisation du savoir pour décrire les ressources.

Pour produire une telle modélisation nous avons choisi d'exploiter l'approche praxéologique de la TAD et la notion de modèle praxéologique de référence (MPR, voir Bosch & Gascón, 2005). Nous exploitons en particulier le cadre T4-TEL[2], développé par Chaachoua (2018). Une des motivations de ce choix est l'ancrage initial de T4-TEL dans le domaine des Environnements Informatiques pour l'Apprentissage Humain (EIAH). Pour définir notre MPR nous nous sommes appuyés sur des travaux épistémologiques et didactiques et sur la modélisation des types de tâches à l'aide des générateurs de types de tâches (Chaachoua et al., 2019). Nous donnons d'autres motivations à l'utilisation d'un MPR

---

[2] T4 fait référence aux quatre T du quadruplet praxéologique, TEL est l'acronyme de Technology Enhaced Learning, équivalent anglophone d'EIAH.

au fil du texte. Par exemple dans la quatrième section nous présentons comment il nous permet de caractériser les ressources. Un exemple est donné dans le domaine de l'algèbre.

Nous revenons maintenant rapidement sur les environnements numériques d'apprentissage. Il en existe actuellement de très nombreux annoncés comme permettant l'apprentissage des mathématiques[3]. Cependant, ceux qui sont fondés sur un travail didactique sont nettement moins nombreux. Dans cette catégorie on peut penser aux environnements Aplusix (Nicaud et al., 2006) et Pépite (Grugeon-Allys et al., 2012) en algèbre. En géométrie on signalera QED-Tutrix (Font et al., 2018) et Cabri Géomètre. Ces environnements sont de types divers (De Vries, 2001) : exerciseurs, logiciel de diagnostic, Intelligent Tutoring System, micro-monde.

Lors du travail de conception d'un ENA, de nombreux aspects sont à prendre en compte mêlant des considérations liées à la recherche et d'autres liées à des aspects éditoriaux, technologiques ou économiques. Pour n'en citer que certains liés à des domaines de recherche il y a par exemple des aspects relevant de l'ergonomie, d'autres de l'IHM, d'autres encore de la génération de traces numériques, etc. Pour nous, l'objectif est d'assurer la prise en compte de différents enjeux didactiques, permettant à l'ENA de contribuer au travail sur les besoins d'apprentissage de l'apprenant ; évidemment en assurant leur compatibilité avec les autres enjeux.

Au regard des éléments présentés dans cette section, nous sommes donc confrontés à une triple problématique :

- proposer une description des ressources permettant de les caractériser didactiquement et de les produire de manière effective,
- caractériser l'apprenant de manière implémentable,
- mettre en relation l'apprenant et les ressources et exploiter l'activité de l'apprenant pour actualiser son profil.

L'apprenant, ses besoins d'apprentissage, nos ambitions et nos limites

Comme évoqué ci-dessus notre objectif est de définir et implémenter des ressources didactiquement pertinentes, mais aussi de pouvoir proposer ces ressources au sein de parcours, parcours qui doivent être adaptés à l'apprenant. Ceci nécessite tout d'abord que nous définissions les besoins de l'apprenant. Dans les environnements d'apprentissage, l'apprenant peut être pris en compte selon différentes dimensions : l'engagement, la motivation, les émotions, etc. Nous n'abordons pas ces aspects et nous nous concentrons sur la question de ses besoins d'apprentissage.

Indépendamment du contexte d'un ENA la question des besoins d'apprentissage d'un apprenant peut être abordé à l'aide de la TAD en évaluant l'écart entre son rapport personnel et le rapport institutionnel visé aux praxéologies à étudier. Cet écart permet de définir les besoins praxéologiques de l'apprenant, qui vont donner lieu à la mise en place d'un dispositif d'étude pour les satisfaire. Ces besoins peuvent être de natures diverses allant de la construction de nouvelles praxéologies à

[3] Une recherche sur n'importe quel "store" d'un smartphone avec les mots clés "mathématiques + collège" permet d'obtenir plusieurs centaines de réponses.

l'adaptation ou l'amélioration de praxéologies existantes. D'une part il sera nécessaire de mettre en œuvre les six moments de l'étude, au sens défini par Chevallard (2002) et d'autre part, par nature, ces besoins sont propres à chaque individu puisque son rapport personnel est le fruit de ses rencontres, au sein de diverses institutions, avec diverses praxéologies et du travail d'articulation qu'il aura opéré, ou non, entre elles, des décalages importants existants entre praxéologies au cours des transitions institutionnelles (Grugeon, 1997 ; Grugeon-Allys & al, 2012)). Les praxéologies à travailler par l'élève visent donc :

- d'une part à négocier des ruptures d'ordre épistémologique (Vergnaud, 1990) et construire des raisons d'être des notions en jeu,
- et, d'autre part, à poursuivre la construction d'éléments technologico-théoriques pour résoudre des tâches du domaine nécessitant la convocation de différents types de tâches pour agréger des praxéologies ponctuelles et locales (Castela, 2008).

En ce qui concerne le domaine algébrique, Vergnaud et al. (1988) définissent la double rupture épistémologique entre l'arithmétique et l'algèbre en lien avec l'équivalence de l'égalité et les nouveaux objets de l'algèbre, expressions algébriques, formules et équations et la symbolisation algébrique nécessaire pour résoudre les problèmes. Nous développons nos arguments appuyant cette définition.

D'un point de vue institutionnel, ces ruptures peuvent être mis en relation avec les décalages des rapports institutionnels, souvent au passage d'une institution à l'autre, induits par la transposition didactique en jeu dans les curriculums et les praxéologies à enseigner. L'appui sur un MPR du domaine mathématique permet de caractériser ces décalages (Bosch et al., 2004 ; Grugeon-Allys et al., 2012). D'autres phénomènes peuvent participer aux difficultés des élèves : au-delà des effets liés aux curriculums, les praxéologies apprises dépendent fortement des praxéologies enseignées et des discours technologiques développés, ou non, par les enseignants pour justifier les techniques. Les praxéologies enseignées peuvent être complètes ou non, muettes, faibles ou fortes selon la dialectique entre les composantes *praxis* et *logos* (Wozniak, 2012).

Du point de vue du sujet cognitif, les élèves conceptualisent les notions en réalisant des tâches issues des domaines mathématiques dans les curriculums visant à développer des raisons d'être des nouveaux objets et des éléments technologico-théoriques. L'apprentissage dépend de l'activité développée par l'élève selon les énoncés des tâches, les mises en fonctionnement des connaissances anciennes et nouvelles mises en jeu, sous forme d'application directe ou non, et parfois des adaptations à la charge des élèves (Castela, 2008; Robert, 1998, 2010). Au-delà des praxéologies mobilisées, ces trois critères permettent de caractériser les tâches et de définir leur complexité.

Dans des ENA, le besoin d'apprentissage, défini au regard d'une référence épistémologique du domaine mathématique travaillé et pour une institution donnée, est une notion présentée par Grugeon-Allys et al. (2012) puis Pilet (2015).

Grugeon (1997) définit un modèle de l'apprenant, repris dans Grugeon-Allys et al. (2012), qui modélise de façon intelligible le rapport personnel de l'élève à un savoir donné dans une institution donnée, nommé profil de l'élève. Celui-ci décrit les principaux traits des activités effectives de l'élève en termes de cohérences sur le domaine de savoir étudié, selon les praxéologies locales du domaine. Ces activités mettent parfois en jeu des éléments technologico-théoriques anciens ou erronés. Une description de l'activité de l'élève au niveau microscopique des techniques (composante *praxis*), pour chaque tâche d'une praxéologie locale, ne permet pas une synthèse significative et opératoire des activités effectives de l'élève sur le domaine de savoir. Le modèle de l'apprenant repose sur une description de l'activité de l'élève au niveau macroscopique des technologies (composante *logos)* pour chaque praxéologie locale du domaine et permet une vision synthétique du rapport personnel de l'élève à un savoir donné dans une institution. Cette démarche permet davantage une opérationnalité du modèle de l'apprenant pour concevoir des parcours d'apprentissage adaptés aux besoins praxéologiques de l'élève.

Grugeon-Allys (2016) a formalisé ce modèle dans le cadre de la TAD. L'étude du développement des apprentissages s'appuie sur l'analyse des praxéologies apprises de l'élève, praxéologies convoquées au cours de la réalisation de tâches. En effet, contrairement à Croset & Chaachoua (2016), nous ne prenons en compte, parmi les praxéologies personnelles, que celles correspondant à des praxéologies institutionnelles à enseigner, pour lesquelles l'élève a développé des techniques et technologies correctes ou erronées sur le long terme. Nous faisons donc référence aux praxéologies apprises. Dans l'enseignement primaire et secondaire, l'élève a rencontré des praxéologies ponctuelles travaillées avec des technologies anciennes puis nouvelles et les a plus ou moins amalgamées, de façon plus ou moins idoine, au regard des praxéologies institutionnelles attendues. L'enjeu est de situer les praxéologies apprises relatives à un savoir mathématique au regard des praxéologies à enseigner et enseignées, compte tenu de la transposition didactique. À partir d'une approche globale de l'enseignement et de la caractérisation d'un MPR du domaine mathématique, nous cherchons à situer l'environnement technologico-théorique (le *logos)* des praxéologies apprises d'un élève spécifique au regard de celles attendues pour un élève générique.

Dans le cadre du projet MindMath, nous nous appuyons sur ces différents travaux et, pour une institution donnée, relativement à une praxéologie locale donnée, nous définissons une typologie des rapports personnels de l'élève aux praxéologies qu'il est censé avoir apprises, en décrivant *a priori* des types de *logos*[4] *personnels* à partir d'une étude épistémologique du domaine de savoir en lien avec les institutions en jeu, les rapports personnel et institutionnel à un savoir donné, pour les situer selon les rapports institutionnels relatifs aux domaines numérique / algébrique développés au collège. Ainsi nous distinguons trois types de *logos personnel* : le type relevant d'un *logos* « ancien », celui relevant d'un *logos* en cours de construction et celui relevant d'un *logos* idoine. Nous spécifions les types de *logos* pour chacune des praxéologies constitutives du domaine mathématique.

Dans le cas de l'algèbre élémentaire, il s'agit des praxéologies : calculer, modéliser, représenter, prouver (Grugeon-Allys, 2016). Le premier type (par exemple arithmétique en ce qui concerne l'algèbre) privilégie de façon dominante des savoirs et technologies anciens (notion d'équivalence absente dans des praxéologies de calcul, processus de généralisation et modélisation absents et preuve numérique prégnante, avec des erreurs relevant d'une rupture d'ordre épistémologique entre

[4] Le terme mode technologico-théorique était utilisé dans Grugeon-Allys (2016)

l'arithmétique et l'algèbre non négociée). Le deuxième type relevant d'un *logos* incomplet en cours de construction met en jeu de façon dominante des savoirs visés par l'institution mais avec des technologies ne permettant pas toujours de contrôler des techniques personnelles, les praxéologies ponctuelles étant insuffisamment amalgamées (aspect syntaxique dominant dans des praxéologies de calcul, faible amalgamation des praxéologies de modélisation, de preuve ou de calcul, ce qui permet la réussite dans la réalisation de tâches d'application directe, mais qui conduit souvent à des difficultés dans la réalisation de tâches plus complexes convoquant différents sous-types de tâches dont l'identification est laissée à la charge de l'élève). Le troisième type de logos relevant du *logos* idoine dans l'institution met en jeu les éléments technologico-théoriques attendus dans la réalisation des tâches.

Ces types de *logos personnel* caractérisent des éléments technologico-théoriques mis en jeu de façon dominante dans la variété des techniques / technologies apprises, correctes ou erronées, utilisées par l'élève pour réaliser différentes tâches selon les praxéologies locales et qui peuvent être anciens ou conformes à l'institution dans laquelle l'élève apprend. Ces types de *logos personnels* donnent une description pertinente des besoins praxéologiques à travers les différences repérées entre le *logos* dominant des élèves et le *logos* institutionnel visé.

Nous structurons le *modèle didactique de l'apprenant* par un *n*-uplet de types de *logos* pour les praxéologies constitutives d'un domaine mathématique. Un n-uplet peut faire apparaître une dynamique entre les besoins praxéologiques de l'élève en algèbre, selon les types de *logos* en jeu pour les praxéologies, calculer, modéliser, représenter, prouver (par exemple, (Ancien, Ancien, Incomplet, Ancien) ou (Incomplet, Incomplet, Incomplet, Ancien)) et donner des pistes pour déterminer un parcours d'apprentissage selon les besoins praxéologiques mais aussi les points d'appui praxéologiques repérés. Dans le cas d'un type de logos relevant d'un *logos* incomplet en cours de construction, nous lui associons une typologie de classes d'erreurs. Nous décrivons le profil d'un élève spécifique comme le pourcentage de réussite d'exercices du parcours complété par le *n*-uplet[5].

Ceci nous permet de spécifier deux délimitations importantes de notre travail. Tout d'abord, concernant le travail de l'apprenant avec l'ENA : nous ne visons pas à prendre en charge l'intégralité des moments de l'étude. Il est essentiellement pensé pour instrumenter[6] différentes réalisations du moment « du travail de l'organisation mathématique et plus précisément de la technique ». Pour participer à la satisfaction des besoins praxéologiques d'un apprenant, il doit donc nécessairement être articulé avec des apports externes, prévus par l'enseignant dans l'organisation de l'étude des

[5] Le n-uplet de types de *logos personnel* caractérisant le profil de l'élève au départ de l'utilisation de l'ENA est défini à partir de l'analyse des réponses au test dévaluation diagnostique Pépite selon le niveau scolaire (Grugeon-Allys et al., 2022)

[6] Il est aussi important de noter qu'en dehors d'expérimentations contrôlées, les usages d'un ENA échappent totalement aux concepteurs. Nous explicitons donc dans quelles perspectives d'usages nous avons conçu cet environnement sans préjuger des orchestrations qui seront mises en œuvre par les enseignants ou des possibilités d'un éventuel usage autonome de l'apprenant.

praxéologies visées. Nous précisons aussi que l'ENA proposé s'inscrit plutôt dans le paradigme de la visite des œuvres que dans celui de l'exploration du monde.

Par ailleurs, toujours dans le contexte d'un ENA, les éléments disponibles pour prendre en compte l'apprenant peuvent avoir deux origines : des éléments initiaux obtenus par déclaration et/ou diagnostic et des éléments liés à une interprétation de l'activité de l'apprenant à partir des traces numériques. La grande diversité des apprenants amenés à utiliser l'ENA, nécessite de considérer des utilisateurs apprenants génériques auxquels nous associons les différents profils, basé sur leur *logos personnel*. Par contre, la mise en relation initiale entre un apprenant et un profil, puis l'actualisation des profils au cours de l'usage, ne sont pas détaillées dans ce texte mais sont appuyées sur les travaux antérieurs présentés dans les références précédentes, une présentation des calculs permettant cette actualisation est détaillée dans (Grugeon-Allys et al., 2022).

Les ressources et leur caractérisation dans MindMath

Dans le cadre du projet, nous avons d'abord proposé une description des tâches à implémenter dans la plateforme. Ce travail est avant tout fondé sur une analyse épistémologique et didactique des savoirs en jeu.

Précisons que dans le cadre du projet, pour des raisons techniques liées à la plateforme, un exercice est constitué d'une unique tâche à réaliser. La description d'une tâche permet donc, sans considérations supplémentaires, de définir un exercice.

Concrètement, la description des tâches doit permettre de :

- produire les exercices de manière effective,
- identifier les exercices équivalents,
- définir des parcours,
- guider la production et la décision des rétroactions.

Les deux premiers objectifs nécessitent de définir les ensembles de tâches considérées comme “identiques à l'aléatoire près” afin que l'apprenant puisse travailler plusieurs fois un type de tâches sans avoir deux tâches identiques à résoudre.

Pouvoir définir des parcours nécessite de caractériser les praxéologies mobilisées dans la tâche et notamment les éléments technologico-théoriques. Ceci doit permettre *in fine* de :

- De déterminer leur conformité aux attendus de l'institution.
- D'anticiper la manifestation, pour la tâche donnée, des différentes classes d'erreurs identifiées *a priori*. En particulier celles liées à la non-perception des sauts technologiques associés aux ruptures d'ordre épistémologique.
- D'identifier la mobilisation d'éléments technologico-théoriques issus d'autres praxéologies, locales ou régionales (voir Figure 1), et ainsi de caractériser la complexité de la tâche.

Pour atteindre ces objectifs nous nous sommes appuyés sur le modèle T4TEL et plus précisément sur les générateurs de type de tâches (Chaachoua et al., 2019). A partir des nombreux travaux antérieurs de didactique de l'algèbre (voir les références dans Jolivet, Lesnes-Cuisiniez, et Grugeon-Allys,

2021) nous avons défini une structuration praxéologique du domaine algébrique et des relations entre les différents niveaux de cette structure (Figure 1). Dans les différentes OML nous avons défini des générateurs de types de tâches. Quatre pour l'OMR des expressions algébriques : calculer (dont réduire, développer, factoriser), représenter, modéliser, prouver. Deux pour l'OMR des équations : mettre en équation, résoudre une équation du 1er degré.

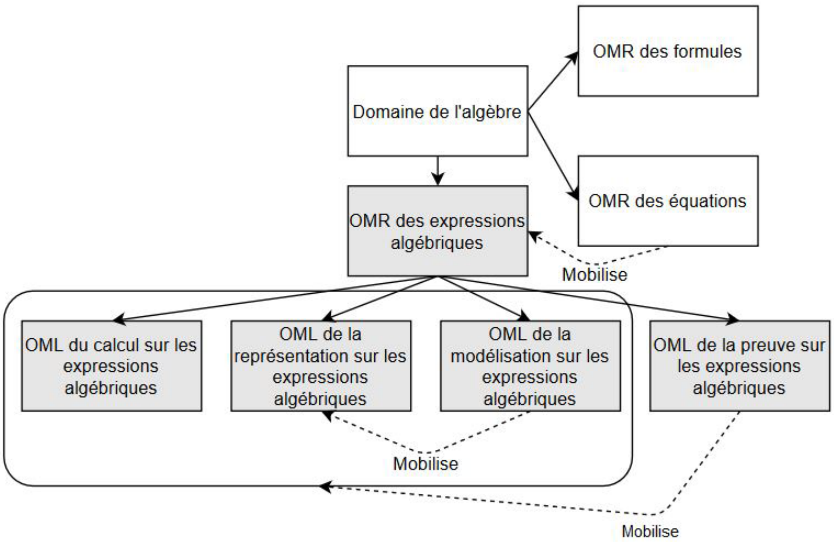


Figure 1 : structuration du domaine de l'algèbre (d'après Jolivet et al., 2021)

Pour définir ces générateurs nous avons distingué différentes fonctions pour les variables et leurs valeurs conformément à l'approche proposée par Chaachoua & Bessot (2019). Afin de pouvoir construire des parcours nous avons tout d'abord défini les variables, et organisé leurs valeurs, pour rendre compte des différentes techniques et technologies. De plus, nous avons choisi des variables pour rendre compte de la portée des techniques. Ceci nous permet d'identifier les types de tâches relatifs aux ruptures épistémologiques évoquées plus haut. Enfin, d'autres variables permettent de jouer sur la complexité des tâches d'un type de tâches (Jolivet, Lesnes-Cuisiniez, et Grugeon-Allys, 2021). Par exemple, pour le générateur de types de tâches « résoudre une équation du premier degré », nous avons défini une première variable qui est la forme de l'équation. Les différentes valeurs (Figure 2) nous permettent de différencier les équations qui vont pouvoir être résolues avec une technique arithmétique (*Ft*1) et celles qui nécessitent la mise en œuvre d'une technique algébrique (*Ft2*.2) (Tableau 1).

Une autre variable permet de caractériser l'ensemble des solutions. Cette variable permet de délimiter la portée de certaines techniques. Par exemple une solution dans {0 ; 1 ; 2 ; 3 ; 4 ; 5} rend possible une technique de résolution du type « identification d'une racine évidente » (*Ft*2.1), alors qu'une solution non décimale, associée à une équation de la forme $ax+b=cx+d$ permet de garantir qu'il est nécessaire de mobiliser une technique algébrique pour résoudre l'équation. Deux autres variables, les valeurs possibles des coefficients (Figure 3) et la nécessité éventuelle d'une réécriture (réduction, développement) de l'équation avant la résolution (*Ft*3, *Ft*4) (Tableau 1), permettent de caractériser la complexité de la tâche. Cette complexité est liée d'une part à l'ensemble de nombres dans lequel les calculs vont être réalisés et, d'autre part, à la mobilisation ou non de l'OML de calcul sur les expressions algébriques. La spécification des coefficients possibles de manière plus précise qu'un simple ensemble de nombres (par exemple ℕ) est liée au besoin de pouvoir produire effectivement les équations en conservant des expressions « raisonnables » à manipuler par les apprenants. Ces choix de valeurs sont liés aux institutions dans lesquelles la plateforme est appelée à être utilisée, ils pourraient évidemment être différents dans une autre institution.

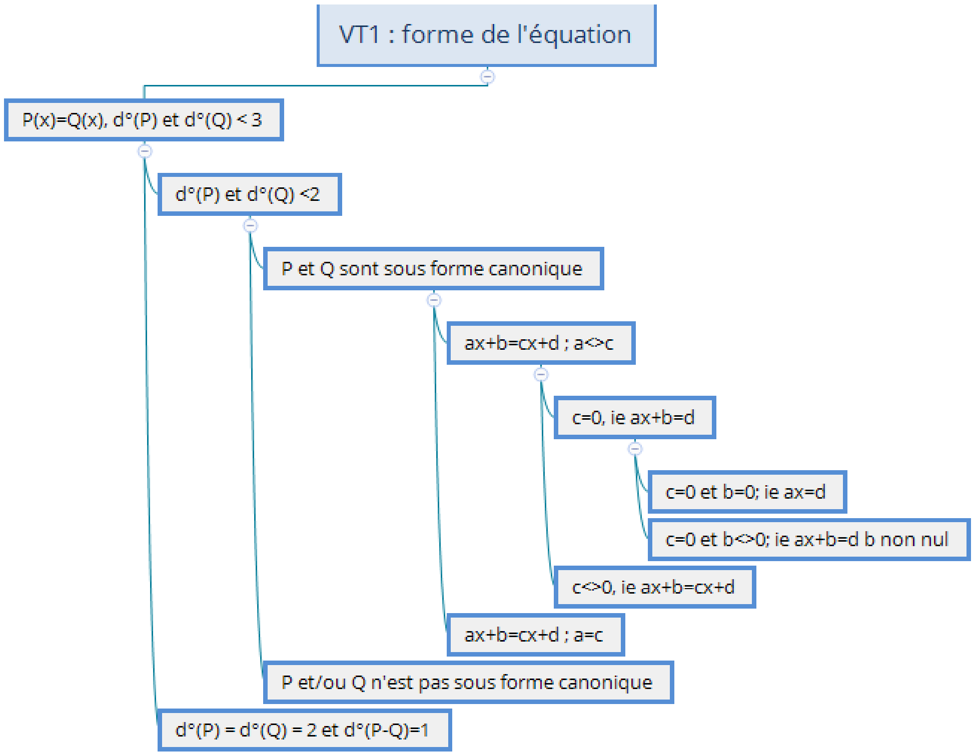


Figure 2 : variable "forme de l'équation" pour le générateur "résoudre une équation du 1er degré"

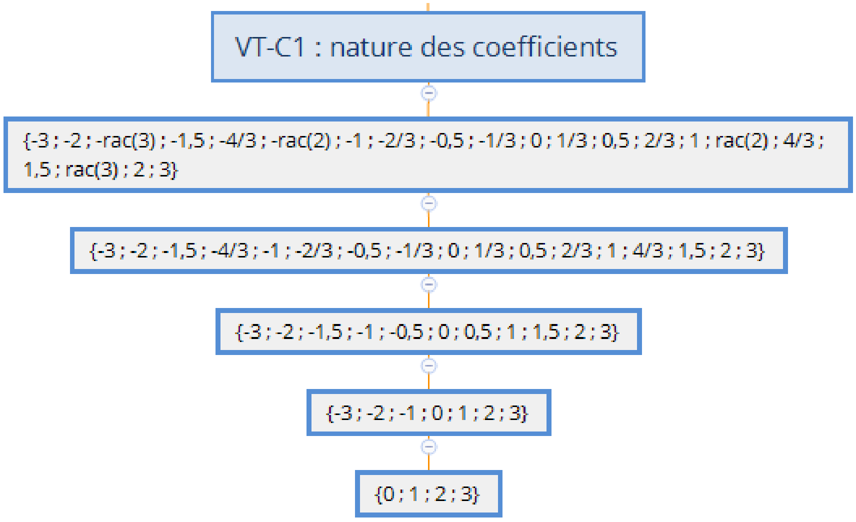


Figure 3 : variable "coefficients de l'équation" pour le générateur "résoudre une équation du 1er degré"

Il est important de noter, dans notre perspective, qu'un générateur de types de tâches permet de produire des types de tâches et non pas, comme nécessaire pour l'ENA, des tâches. Dans le cadre du projet, nous avons choisi d'identifier des sous-types de tâches ayant des fonctions spécifiques, permettant de mettre en relation types de tâches et besoins praxéologiques. Il s'agit de types de tâches qui sont choisis pour permettre de définir, soit des tâches liées aux ruptures d'ordre épistémologiques, soit pour rendre compte de divers niveaux de complexité. Notamment pour pouvoir échanger avec nos partenaires, nous avons choisi de nommer ces types de tâches particuliers *familles de tâches*[7]. Ce sont ces familles de tâches qui permettent la production effective des tâches.

En particulier deux tâches d'une même famille de tâches mettent en jeu les mêmes éléments technologico-théoriques et sont de même complexité.

Par exemple dans le tableau 1 nous proposons cinq familles de tâches issues du générateur "résoudre une équation du 1[er] degré".

Tableau 1 : Exemples de familles de tâches du générateur "résoudre une équation du 1[er] degré"

	VT1 (forme équation)	VT2 (solution)	VT3 (coefficients)	VT4 (manipulation)
$Ft1$	$ax+b=c$	$\mathbb{Z}$	{-3 ; -2 ; -1 ; 0 ; 1 ; 2 ; 3}	Non
$Ft2.1$	$ax+b=cx+d$	$\mathbb{Z}$	{-3 ; -2 ; -1 ; 0 ; 1 ; 2 ; 3}	Non
$Ft2.2$	$ax+b=cx+d$	$\mathbb{Q}$	{-3 ; -2 ; -1 ; 0 ; 1 ; 2 ; 3}	Non
$Ft3$	$ax+b=cx+d$	$\mathbb{Q}$	{-3 ; -2 ; -1,5 ; -1 ; -0,5 ; 0 ; 0,5 ; 1 ; 1,5 ; 2 ; 3}	Non
$Ft4$	$ax+b=cx+d$	$\mathbb{Q}$	{-3 ; -2 ; -1,5 ; -1 ; -0,5 ; 0 ; 0,5 ; 1 ; 1,5 ; 2 ; 3}	Oui

Ainsi les tâches $3x+3=-3$ (t_1) et $-2x+3=5$ (t_2) appartiennent toutes les deux à $Ft1$ et pourront être proposées indistinctement pour travailler le type de tâches défini par $Ft1$.

Si la tâche (t_2) peut être résolue par une technique arithmétique la tâche $3x+3=-2x+1(t_3 \in Ft2.1)$ $3x+3=-2x+1$ (t_3 $\in Ft2.1$) nécessite pour sa part une technique algébrique. Elles peuvent donc être exploitées pour travailler sur la rupture épistémologique. Enfin la tâche $1,5(x-2)=x+3(t_4 \in Ft4)$ permet aussi de travailler la technique algébrique mais avec un niveau de complexité supérieur car elle mobilise le développement qui est dans l'OMR de la manipulation des expressions algébriques.

Le MPR intègre aussi les techniques et les éléments du *logos* liés aux sous-types de tâches produites à partir des générateurs. Nous associons aussi les classes d'erreurs identifiées *a priori* à partir des travaux de didactique.

Conclusion : la construction d'un MPR, permettant de structurer le savoir du domaine jusqu'aux familles de tâches, nous permet de décrire les tâches prescrites à l'apprenant dans la plateforme et les

[7] Il ne s'agit pas d'un nouvel objet que nous proposons d'introduire dans la TAD, il s'agit bien, comme écrit, d'un sous-type de tâches au sens usuel du terme. Cependant ces sous-types de tâches jouent un rôle particulier dans notre travail et afin de les identifier aisément nous les nommerons familles de tâches dans la suite du document.

classes d'erreurs en lien avec ces tâches. Les tâches sont décrites non seulement par rapport aux savoirs mobilisés mais aussi en termes de complexité de l'activité visée. Les tâches étant décrites nous abordons maintenant la question de la modélisation de l'apprenant pour un domaine de savoir donné.

Modéliser des parcours d'apprentissage pour mettre en relation ressources et besoins praxéologiques de l'apprenant

Dans les troisième et quatrième sections nous avons présenté les profils d'apprenant basé sur les types de *logos personnel* et la description des tâches dans l'environnement MindMath. Dans cette section nous présentons brièvement la manière dont ces deux éléments sont mis en relation dans la plateforme par une modélisation de parcours d'apprentissage définis *a priori*.

Les parcours sont organisés autour des objectifs institutionnels, des analyses épistémologiques du domaine de savoir et des types de *logos personnel*. Notre parti pris est que les apprenants doivent impérativement rencontrer, de manière significative, les types de tâches qui sont au cœur des attendus institutionnels, selon le niveau scolaire.

L'appui sur le rapport personnel de l'apprenant au savoir *via* les types de *logos personnels* relatif aux praxéologies locales, permet d'ajuster le point d'entrée dans le parcours d'apprentissage. En cas de rupture épistémologique non négociée par l'élève dans le passage de l'arithmétique à l'algèbre, des tâches de type donné mettant en jeu une portée de technique arithmétique, puis par substitution puis algébrique sont proposées afin d'amener l'élève à négocier cette rupture, en s'appuyant à la fois sur les raisons d'être des nouveaux objets travaillés, l'élaboration des éléments technologico-théoriques lors de l'enseignement et des feedbacks proposés à la suite des réponses des élèves. D'autre part la modélisation du savoir, et la description des tâches, permet, en cas de difficultés d'un élève d'envisager de lui proposer des tâches relevant d'une autre OML, voir d'une autre OMR (voir les relations identifiées dans la Figure 1).

Dans le tableau 2 nous proposons un canevas des parcours proposés. On peut noter qu'un apprenant qui a un type de *logos personnel* ancien va se voir proposer des tâches relevant d'une technologie ancienne puis des tâches l'amenant à (re)rencontrer la rupture et enfin des tâches correspondant au cœur du savoir visé. *A contrario*, un élève qui dispose déjà d'un type de *logos personnel* idoine va être confronté directement à des tâches liées au cœur du savoir visé puis va rencontrer des tâches plus complexes, allant jusqu'à nécessiter la mobilisation d'autres OMR. Pour ces dernières, selon le niveau d'amalgamation des praxéologies de l'apprenant, nous sommes à nouveau dans une situation où l'ENA devra nécessairement s'articuler avec une intervention externe.

Tableau 2 : canevas global d'organisation des parcours (d'après Jolivet & al., 2021)

Type de *logos* relatif à une praxéologie locale travaillée dans l'OMR	Sous-types de tâches pouvant être résolues avec une	Sous-types de tâches pivot pour négocier la rupture	Sous-types de tâches cibles : mobilisation de la technologie visée	Sous-types de tâches cibles plus complexes en restant dans	Sous-types de tâches cibles avec convocation de

	technologie ancienne			l'OMR travaillée	praxéologies d'autres OMR
Ancien non idoine	10%	20%	50%	20%	
Incomplet		15%	50%	20%	15%
Idoine			40%	40%	20%

Au fil de la réalisation des tâches, l'apprenant va se voir proposer deux types de retours de l'environnement. En réponse à son activité ponctuelle, des rétroactions épistémiques (Luengo, 2009), c'est-à-dire des commentaires sur les erreurs qui relient ces erreurs à des besoins praxéologiques relatifs au logos, lui sont affichées. Par ailleurs, suite à la réalisation de plusieurs tâches d'un ensemble de sous-types de tâches, le type de *logos personnel* est réévalué au regard des technologies mobilisées dans les techniques utilisées pour leur réalisation (Grugeon-Allys & al., 2022). Par ailleurs, selon la nature des erreurs réalisées, il peut[8] se voir proposer des tâches relevant d'une autre OMR, par exemple des tâches relatives à la manipulation des expressions algébriques s'il s'avérait en échec sur les développements proposés lors de la résolution d'équations alors qu'il a un niveau de maitrise satisfaisant de la résolution des équations.

D'autre part, les réalisations successives des tâches et l'utilisation faite des rétroactions épistémiques vont permettre de réévaluer régulièrement le type de *logos personnel* de l'apprenant pour chaque praxéologie locale et ainsi de réviser ses besoins praxéologiques.

Il est à noter que le grain du parcours résulte de celui fixé pour décrire les sous-types de tâches. Ainsi, en fonction des considérants institutionnels et/ou épistémologiques il est toujours possible de raffiner les parcours en introduisant de nouvelles variables ou de nouvelles valeurs.

Conclusion et perspectives

Dans cette contribution nous avons présenté les fondements didactiques permettant d'articuler ressources et apprenant dans un environnement numérique d'entrainement aux mathématiques. Les aspects liés aux rétroactions épistémiques n'ont pas été développés dans ce texte mais sont présentés en détail dans (Jolivet, Yessad, et al., 2021) et dans (Jolivet et al., soumis). Si nous avons illustré notre propos avec des exemples dans le champs de l'algèbre, les aspects relatifs à la géométrie sont l'objet d'une partie de la thèse de (Lesnes-Cuisiniez, 2021) et sont aussi repris dans (Jolivet, Lesnes-Cuisiniez, et al., 2021). Les aspects concernant le diagnostic en algèbre sont développés dans Grugeon-Allys et al., 2022).

Si nous avons précisé à différentes reprises que l'ENA trouve son utilisation naturelle dans le cadre du moment du travail de la praxéologie, il nous semble que sa conception permet d'envisager son

[8] Cette fonctionnalité n'est pas implémentée pour le moment dans la plateforme, mais la modélisation didactique permettant ce fonctionnement est réalisée.

potentiel d'instrumentation de l'enseignant pour aborder ou outiller le travail lié à d'autres moments. En particulier, la construction des parcours qui permet à l'apprenant de rencontrer des types de tâches mobilisant des praxéologies situées en amont et en aval des ruptures épistémologiques peut motiver le moment de la construction du bloc technologico-théorique. De même, la présence de rétroactions épistémiques, peut être un soutien au travail d'institutionnalisation.

Au-delà du travail didactique réalisé pour définir les fondements de la plateforme MindMath, ce projet a permis d'introduire de prometteuses collaborations avec le LIP6 (laboratoire d'informatique de Sorbonne Université). Ces échanges ont débouché d'une part sur la représentation du savoir modélisé dans une ontologie, à l'image du travail évoqué par Chaachoua dans (Chaachoua, 2018). D'autre part la décision des rétroactions, et toutes les incertitudes associées, amènent à explorer une articulation entre apports de la didactique et algorithme d'intelligence artificielle.

Cette articulation IA – didactique est aussi prévue, dans le cadre du projet, au niveau des parcours, avec des algorithmes d'*adaptive learning* qui doivent permettre l'évolution des parcours et leur adaptation aux besoins praxéologiques de l'apprenant. Nous ne développons pas cet aspect. Les parcours définis *a priori* que nous avons présentés sont en revanche un moyen important d'initialisation des algorithmes d'*adaptive learning*.

Références

Bosch, M., Fonseca, C., & Gascon, J. (2004). Incompletitud de las organizaciones matematicas locales en las instituciones escolares. *Recherche en didactique des mathématiques*, *24*(2), 205-250.

Bosch, M., & Gascón, J. (2005). La praxéologie comme unité d'analyse des processus didactiques. In A. Mercier & C. Margolinas (Éds.), *Balises pour la didactique des mathématiques : Cours de la 12e école d'été de didactique des mathématiques* (p. 107-122). La Pensée Sauvage, Grenoble.

Castela, C. (2008). Travailler avec, travailler sur la notion de praxéologie mathématique pour décrire les besoins d'apprentissage ignorés par les institutions d'enseignement. *Recherche en didactique des mathématiques*, *28*(2), 135-182.

Chaachoua, H. (2018). T4TEL, un cadre de référence didactique pour la conception des EIAH. In J. Pilet & C. Vendeira (Éds.), *Actes du séminaire de didactique des mathématiques 2018* (p. 8-25). IREM de Paris - Université Paris Diderot; hal-02421410. https://hal.archives-ouvertes.fr/hal-02421410/document

Chaachoua, H., & Bessot, A. (2019). La notion de variable dans le modèle praxéologique. *Educação Matemática Pesquisa : Revista do Programa de Estudos Pós-Graduados em Educação Matemática*, *21*, 234-247. https://doi.org/10.23925/1983-3156.2019v21i4p234-247

Chaachoua, H., Bessot, A., Romo, A., & Castela, C. (2019). Developments and functionalities in the praxeological model. In M. Bosch, Y. Chevallard, F. Javier Garcia, & J. Monaghan (Éds.), *Working with the anthropological theory of the didactic : A comprehensive casebook* (Routledge, p. 41-60). https://www.routledge.com/Working-with-the-Anthropological-Theory-of-the-Didactic-in-Mathematics/Bosch-Chevallard-Garcia-Monaghan/p/book/9780367187705

Chevallard, Y. (2002). Organiser l'étude. 3. Ecologie & régulation. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Éds.), *Actes de la XIe école d'été de didactique des mathématiques* (p. 41-56). La Pensée Sauvage, Grenoble. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Organiser_l_etude_3.pdf

De Vries, E. (2001). Les logiciels d'apprentissage : Panoplie ou éventail ? *Revue française de pédagogie*, 105-116.

Font, L., Richard, P. R., & Gagnon, M. (2018). Improving QED-Tutrix by Automating the Generation of Proofs. *Electronic Proceedings in Theoretical Computer Science*, *267*, 38-58. https://doi.org/10.4204/EPTCS.267.3

Grugeon, B. (1997). Conception et exploitation d'une structure d'analyse multidimensionnelle en algèbre élémentaire. *Recherches en didactique des Mathématiques*, *17*(2), 167-209.

Grugeon-Allys, B. (2016). Modéliser le profil diagnostique des élèves dans un domaine mathématique et l'exploiter pour gérer l'hétérogénéité des apprentissages en classe : Une approche didactique multidimensionnelle. *e-JIREF*, *2*(2), 63-88.

Grugeon-Allys, B., Chenevotot-Quentin, F., & Pilet, J. (2021). Using didactic models to design adaptive pathways to meet students' learning needs in an on-line learning environment. In *Mathematics Education in the Age of Artificial Intelligence—How Artificial Intelligence can serve mathematical human learning?* (Springer, p. 157-188).

Grugeon-Allys, B., Pilet, J., Chenevotot-Quentin, F., & Delozanne, E. (2012). Diagnostic et parcours différenciés d'enseignement en algèbre élémentaire. *Recherches en Didactique des Mathématiques, Numéro spécial hors-série, Enseignement de l'algèbre élémentaire: bilan et perspectives*, 137-162.

Jolivet, S. (2018). *Modèle de description didactique de ressources d'apprentissage en mathématiques, pour l'indexation et des services EIAH* [Thèse de doctorat, Université Grenoble Alpes]. https://tel.archives-ouvertes.fr/tel-02079412/document

Jolivet, S., Chaachoua, H., & Desmoulins, C. (2022). Modèle de description didactique d'exercices de mathématiques. *Recherche en Didactique des Mathématiques*, *42*(1).

Jolivet, S., Lesnes-Cuisiniez, E., & Grugeon-Allys, B. (2021). Conception d'une plateforme d'apprentissage en ligne en algèbre et en géométrie : Prise en compte et apports de modèles didactiques. *Annales de didactique et de sciences cognitives*, *26*, 117-156.

Luengo, V. (2009). *Les rétroactions épistémiques dans les Environnements Informatiques pour l Apprentissage Humain* [HDR, Université Joseph Fourrier - Grenoble 1]. https://hal.archives-ouvertes.fr/docs/00/69/98/02/PDF/Luengo_2009.pdf

Nicaud, J.-F., Chaachoua, H., & Bittar, M. (2006). Automatic calculation of students' conceptions in elementary algebra from Aplusix log files. In M. Ikeda, K. D. Ashley, & T.-W. Chan (Éds.), *8th International Conference on Intelligent Tutoring Systems* (p. 433-442). Springer.

Pilet, J. (2015). Réguler l'enseignement en algèbre élémentaire par des parcours d'enseignement différencié. *Recherches en Didactique des Mathématiques*, *35*(3), 271-312.

Robert, A. (1998). Outils d'analyse des contenus mathématiques à enseigner. *Recherches en didactique des mathématiques*, *18*(2), 139-190.

Robert, A. (2010). Formation professionnelle des enseignants de mathématiques du second degré. *Repères-IREM*, *80*, 87-103.

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en didactique des mathématiques*, *10*(23), 133-170.

Vergnaud, G., Cortes, A., & Favre-Artigue, P. (1988). Introduction de l'algèbre auprès de débutants faibles. Problèmes épistémologiques et didactiques. In G. Brousseau, M. Hulin, & G. Vergnaud (Éds.), *Actes du colloque de Sèvres. Didactique et acquisition des connaissances scientifiques* (p. 259-279). La Pensée Sauvage, Grenoble.

Wozniak, F. (2012). Des professeurs des écoles face à un problème de modélisation : Une question d'équipement praxéologique. *Recherches en didactique des mathématiques*, *32*(1), 7-55.

On the theory of praxeologies

Josep Gascón[1] and Pedro Nicolás[2]

[1]Autonomous University of Barcelona, Faculty of Mathematics, Spain; gascon@mat.uab.cat

[2]University of Murcia, Faculty of Education, Spain; pedronz@um.es

We propose a general description of (a part of) the theory of a praxeology, and illustrate it with an example. This general description should contribute to provide more complete praxeological descriptions, and to show to what extent any inquiry is a modelling activity.

Keywords: Praxeology, Theory, System, Model.

1. Research question

Since the introduction the idea of "praxeology" in the ATD (Chevallard, 1999), this notion has been extensively used to carried out a fine-grained analysis of human activity and the corresponding byproducts. The idea of praxeology is used to describe the activity of a certain individual, typically in parallel to many other individuals inside a shared institution, and to show the symbiotic relationship between praxis, made of *types of tasks* (T) and *techniques* (τ), and what could be naively called 'theoretical knowledge', made of *technology* (θ) and *theory* (Θ).

In many works of the ATD one finds a description of the theory of a mathematical praxeology representing a specific part of mathematics. For instance, proportionality (Bosch, 1994; García, 2005), elementary algebra (Bolea, 2002; Ruiz-Munzón, 2010), numeral systems (Sierra, 2006), elementary differential calculus (Lucas, 2015), or real numbers (Licera, 2017). However, a general description of the theory of a praxeology is not yet at hand. Thus, for instance, in the theory of the didactic moments (Chevallard, 2002), which provides an account of the genesis and development of some praxeological ingredients along a study process, little is said about the theory part, generally embedded in the expression "construction of the technological-theoretical block".

As postulated by the ATD, every human activity (and their creations) can be described in terms of praxeologies. In particular, both mathematical activity and mathematics can be described in terms of praxeologies. If, in our praxeological description, we only make explicit T, τ and θ, then we will not be able to understand in which sense a large part of mathematics, the theoretical part (formed by objects, relations, interpretations, axioms, theorems, arguments, etc.), appears and develops. Actually, if all the ingredients of Π are organically entangled, then we cannot properly speak of T, τ and θ without speaking of Θ. Indeed, Θ is a condition of possibility for T, τ and θ, because in those elements Θ is already given and assumed.

Our aim in this work is to sketch a possible general description of Θ, or at least, of a part of Θ, the one devoted to the conceptualization of the portion of the world the praxeology deals with. We think this will contribute to make more precise the idea that adidactic mathematical activity (and even any adidactic inquiry activity), can always be ragarded as a modelling activity. An idea that underlies the paradigm of mathematical modelling (Gascón & Nicolás, 2021).

Once we have a language to refer to the components of Θ, we will be able to provide a full *synchronic description* (which present the praxeology as static object), but also a full *diachronic description* of Π (which present the praxeology as dynamic object, evolving along a study process). It will also be possible to describe how a praxeology gives rise to new praxeologies, how the *inter-praxeological links* are stablished along a study process. Unfortunately, there is no space here to incorporate our account of Θ in a presentation of a general description neither of the inner diachronic description of a praxeology nor of the inter-praxeological links. This will be developed somewhere else. In this work we will propose the diachronic description.

2. On the description of the theory

2.1. Praxeolgies as models of systems

A praxeology Π=(T, τ, θ, Θ) is a model of someone's relation with a certain kind of portion of the world, a certain *type systems*, $\mathscr{S}$. The idea of 'system' is a basic one in this work. We just would like to say that, typically, a system is made of objects satisfying certain properties and in certain relations with each other, but the idea will not be further analysed. Nevertheless, we will give some examples of the types of systems corresponding to certain praxeologies, briefly named with a label.

Table 1: praxeologies and systems

Praxeology	**System**
cardinality (involving natural numbers, operations, etc.)	comparison of finite collections
length (involving decimal numbers, operations, etc.)	comparison of segments
area (involving polygons, formulas, etc.)	comparison of surfaces
volume (involving polyhedra, formulas, etc.)	comparison of bodies
probability (involving frequencies, Laplace's account, Kolmogorov's axioms, conditional probability, etc.)	comparison of events or outputs of random experiments
ring theory	different intra-mathematical systems comprising a large variety of objects of different nature, like numbers, functions, matrices, etc.

In general, the foundations of the mathematical activity go back to the study, through a mathematical modelling processes, of certain types of extra-mathematical systems, those formed comparison of objects with respect to a certain magnitude. As mathematics evolve, there is an increasing degree of reflexivity, and the study of intra-mathematical systems becomes more frequent.

2.2. The ingredients of the theory

We propose to regard the theory, Θ, of a praxeology as a 4-tuple, Θ=($\mathscr{O}$, $\mathscr{N}$, $\mathscr{E}$, $\mathscr{A}$), were $\mathscr{O}$ is the *ontological* component, $\mathscr{N}$ is the *nomological* component, $\mathscr{E}$ is the *epistemological* component, and $\mathscr{A}$ is the *axiological* component. In few words, we could say that: $\mathscr{O}$ provides an ontological description of $\mathcal{S}$, it tells us which are the *ontos* of $\mathcal{S}$; $\mathcal{N}$ tells us which are the laws, the *nomos*, that govern $\mathcal{S}$; $\mathcal{E}$ states criteria for an argument to be *valid* according to Π, and $\mathcal{A}$ states the reason why someone (the agent whose activity is described by Π) care about the type of tasks of Π, why they are regarded as worthy or valuable.

Let us present a more detailed description of the ingredients of $\mathcal{O}$:

1. *L* is a language, including: a set Ω of *variables* (refering to the objects of $\mathcal{S}$), a set of *constants* (refering to certain distinguished objects of $\mathcal{S}$), *properties* that the objects might have, and possible *relations* between the objects. As it is customary in logic, the science that study the logos, *properties* can be characterized as subsets of the set of variables formed by all the objects satisfying that property, and *relations* can be characterized as subsets of certain cartesian products of the set of variables.

2. *Int* is an interpretation of *L* in terms of $\mathcal{S}$. In particular, *Int* must say which are the objects of $\mathcal{S}$ the variables refer to, which are the individual objects the constants refer to, and what does it mean for each property or relation to hold.

3. *Axioms* is a list of statements expressing certain formal constraints satisfied by the elements of *L*. The axioms do not express anything that has been discovered along the praxeological development Π, but rather something which works as a starting point.

The nomological component, $\mathcal{N}$, is a made of *laws*. Each law is a statement expressed in terms of *L*, and, contrary to what happens with axioms, those laws are not a starting point in Π, but the conclusions of a *valid arguments* with premises regarded as true in Π. There is a strong link between the epistemological component $\mathcal{E}$ and the nomological component $\mathcal{N}$, as $\mathcal{N}$ only uses those arguments allowed by $\mathcal{E}$.

In symbols, this is how we propose to describe the theory Θ of a praxeology Π:

$\Theta=(\mathcal{O}, \mathcal{N}, \mathcal{E}, \mathcal{A})=((L, Int, Axioms), \mathcal{N}, \mathcal{E}, \mathcal{A})$.

2.3. Some remarks

Remark 1: We claim that our account of Θ is suitable for any praxeology, not just a mathematical one. In particular, we think that our account of Θ could be useful for the description of didactic praxeologies. In future works we would like to explore it.

Remark 2: $\mathcal{A}$ states why the type of tasks T is important. For instance: why is it important to master the comparison of collections (resp. segments, surfaces, bodies, events, etc.) with respect to the magnitude cardinality (resp. length, area, volume, probability)? In turn, to explain why T is important, $\mathcal{A}$ has to states why the tokens of T, the specific tasks $t \in T$ are themselves important. Indeed, the value of the $t \in T$ is, after all, the reason why Π exists. This value is typically placed in the upper parts of the scale levels of didactic codetermination. In few words, $\mathcal{A}$ speaks of the anthropological or sociological motivations behind the existence Π. Notice that, in the case of a didactic praxeology, $\mathcal{A}$ refers to the purpose of education.

Remark 3: $\mathcal{O}$ is a condition of possibility for Π itself to exist. Indeed, we need a minimal conceptualization of $\mathcal{S}$ in order to start considering any type of task at all. Frequently, after getting deeper into the study of that type of task, we need to reconsider the components of $\mathcal{O}$ taken for granted in a first step. These reconsiderations are a part of the construction of the theoretical block, included in one of the didactic moments (Chevallard, 2002).

Remark 4: What distinguishes an axiom from a law is that the first is part of the conceptual landscape taken from granted in Π, whereas the second is a daring statement which, at a certain point of the praxeological development, demands an argument. Thus, whether a statement is an axiom or law is not intrinsic to the statement itself, but relative to Π and the institution in which Π lives.

Remark 5: Examples of laws are theorems, in mathematical praxeologies, or scientific laws, in scientific praxeology. But also, in other kind of praxeologies, a law is any belief supported by an argument, in a very general sense. For instance, in my dietary praxeology, a law could be “To eat vegetables would provide me with necessary vitamins”, supported by an argumentum ad verecundiam (experts say so).

Remark 6: The epistemological principles $\mathcal{E}$ state criteria for an argument to be valid, and also criteria about what laws could be assumed without having an argument in the terms L, but a different kind of argument (for instance, an argumentum ad verecundiam). For example, sometimes in Secondary Education, when studying planar geometry, one assumes the Pythagorean Theorem as a law (a ‘true’ but non-evident statement) without having an argument for it other than the fact that teachers and handbooks say it is true. It is certainly an argument: if everybody say it is true, either it is really true or there is a global conspiracy for me to believe that it is true; and Ockham’s razor and common sense advises us against the second option. But this is not an argument stated in the language L of planar geometry.

Remark 7: Our description of Θ does not intend to be the ‘interior’ description that someone would do of her own theory of world, but just an ‘exterior’ description. Actually, the same happens with the very idea of ‘praxeology’, which does not intend to grasp the so-called emic but rather an etic point of view (Harris, 1976). In particular, even if we express the elements of L in terms of sets and relations, we do not claim that the protagonist of the praxeology regards herself as doing formal set theory.

3. Development of the praxeological model about the magnitude cardinality

Let us describe, with an special emphasis on Θ, some aspects of the possible praxeologies which gradually model the comparison of collections with respect to the magnitude *cardinality*.

We choose the topic of comparing and measuring collections because it has been thoroughly studied in didactics from the point of view of tasks, techniques and technology. See for instance, (Briand, 1993), (Sierra, 2006), (Nicolás, 2014a, 2014b) (García & Sierra, 2015). Now we would like to show how our general account of Θ can be used to give a description of a *praxeology of direct comparison* of collections, Π_0, as well as the description of a *praxeology of measurement* of collections, Π_1, and to show the link between them.

3.1. The praxeology Π_0 of direct comparison

To describe the $\mathcal{S}$ modelled by Π_0 we necessarily use already the ontological component $\mathcal{O}(\Pi_0)$ of Π_0. The objects and the relations involved in $\mathcal{S}$ are given by the language $L(\Pi_0)$ and the corresponding interpretation $Int(\Pi_0)$ of Π_0. In other words, a first description of $\mathcal{S}$ is already the ontological component of an initial praxeology.

Let us describe the **language $L(\Pi_0)$**:

- The set Ω of *variables* is a non empty set.
- The set of *constants* is empty.
- The list of *properties* is empty.
- *Relations*: two binary relations, < and ~.

Let us now give the **interpretation, $Int(\Pi_0)$,** of that language:

- Variables refer to finite collections of objects.
- For two finite collections, x and y, the relation "x < y" is true if, after *pairing* (making pairs formed taking first one element of x and second one element of y, without using again already paired elements), all the elements of x have been used and there are still some elements of y unpaired. If x < y is true we say that x is *smaller than* y.
- For two finite collections, x and y, the relation "x ~ y" is true if, after *pairing*, all the elements of x and y are parts of one pair. If x ~ y is true we say that x is *equivalent to* y.

The axioms $Axioms(\Pi_0)$: the relations < and ~ are not constrained by any axiom.

Types of tasks $T(\Pi_0)$: Notice that each relation is linked to a type of task. Indeed, the relation < (resp. ~) is linked to the following type of task $T_<$ (resp. $T_\sim$): "given x and y, do we have x < y (resp. x ~ y)?".

The axiological component $\mathcal{A}(\Pi_0)$: the comparison of specific collections in order to watch possessions and to ensure fair or suitable distribution (there are plenty of examples of those situations) explains the value of $T_<$ and $T_\sim$.

Techniques $\theta(\Pi_0)$: Notice that the interpretation of < is already a technique, which could be said to be the *standard technique* $\tau_<$ for the type of tasks $T_<$. Similarly, the interpretation of ~ is already a technique, which could be said to be the *standard technique* $\tau_\sim$ for the type of tasks $T_\sim$.

Technology $\theta(\Pi_0)$: The standard techniques have a limited scope as they do not provide always a solution. Sometimes, for physical reasons, one cannot make pairing, for example because the two collections are not simultaneously reachable. This is a praxis problem which forces us to improve Π_0.

3.2. The praxeology Π_1 of indirect comparison through measurement

This praxeology proposes an enlargement $L(\Pi_1)$ of the $L(\Pi_0)$ with the idea of *unit of measure* (the very idea of *unit*, in our case) and a *measurement map* from collections to numbers (numbers organized in the *the unary numeral system*, in our case). This enlargement is motivated by certain laws which can be already expressed with the $L(\Pi_0)$:

The nomological component $\mathcal{N}(\Pi_1)$: we can consider the following laws

- L_1 (total comparison): For all x and y, we have either x < y, or y < x or x ~ y, and those options are mutually exclusive.
- L_2 (reflexivity): For all x, we have x ~ x.
- L_3 (symmetry): For all x and y, if x ~ y then y ~ x.
- L_4 (transitivity of equivalence): For all x, y and z, if x ~ y and y ~ z, then x ~ z.
- L_5 (transitivity of comparison): For all x, y and z, if x < y and y < z, then x < z.
- L_6 (replacement in comparison): For all x, y and z, if x ~ y and y < z, then x < z.
- L_7 (replacement in comparison): For all x, y and z, if x < y and y ~ z, then x < z.

These laws are conclusions of arguments using as premises just the interpretation of < and ~. For instance, an argument supporting L_1 could be as follows: either we leave elements of x unpaired, elements of y unpaired, or no element is left unpaired, and it is not possible, for instance, to leave at the same time elements unpaired both of x and y, because in this case we still could make more pairs.

The language $L(\Pi_1)$: We enlarge $L(\Pi_0)$ with two kinds of new objects, the idea of *tally mark* and the idea of *number*. A tally mark is just a written symbol, I. A *number* is a string of copies of the tally mark: I, II, III, IIII, etc. All the numbers are regarded as constants in *L*. We write $\mathbb{N}_u$ for the set of numbers (the "u" stands for "unary numeral system"). We also enlarge the $L(\Pi_0)$ with the so-called *measurement map* μ from Ω to $\mathbb{N}_u$. Finally, we also consider in $\mathbb{N}_u$ a binary relation, <.

Let us now give the **interpretation, $Int(\Pi_1)$,** of that language:

- Interpretation of I: the tally mark, actually, does not denote any real object of the world, but the ideal object of 'unit', the basic ingredient of which collections are made.
- Interpretation of any number: a number, in the unary numeral system $\mathbb{N}_u$, is a collection of tally marks, a collection of ideal objects. Thus, numbers are ideal collections whose elements are represented one by one[1]. The fact of being ideal implies that numbers are not spacetime located, and this is useful in order to overcome the limitations of techniques considered in Π_0.
- We have that μ(x)=III holds if x ~ III, that is, x is equivalent to the collection of copies of I that form the string III.

[1] As numeral system change along History, numbers stop being symbolic *literal* representations of ideal collections and become something more complex.

- ~~We have that I…I < I…I holds when I…I < I…I regarded as collections of copies of I.~~

The axioms *Axioms*(Π_1): none.

Types of tasks T(Π_1): Same as in Π_0.

The axiological component $\mathscr{A}$(Π_1): Same as in Π_0. Perhaps the importance of solving $T_<$ and $T_\sim$ is underlined in Π_1 by the fact that, regarding Π_0, the ontological and the nomological components complicate considerably in exchange for a good technique.

The nomological component $\mathscr{N}$(Π_1): After the introduction of numbers and the measurement map, we enlarge the nomological component of Π_1 with the following laws:

- L_8 (measurement reflect and respect equivalence): For all x and y, we have x ~ y if and only if μ(x) = μ(y),
- L_9 (measurement reflect and respect comparison): For all x and y, we have x < y if and only if μ(x) < μ(y).

Let us show an argument for the law L_8: assume x ~ y and that μ(x) = III. Then x ~ III, and, after L_4, we will have that y ~ III, and so μ(y) = III. Conversely, assume μ(x) = μ(y). Say, μ(x) = III = μ(y). Then x ~ III ~ y, and so, after L_4, we have x ~ y. Notice that in this argument we use the law L_4 as a premise.

Let us show an argument for the law L_9: assume x < y and that μ(x) = I…$^{\text{n times}}$…I and that μ(y) = I… $^{\text{m times}}$…I. Then I…$^{\text{n times}}$…I ~ x and after L_6 we have I…$^{\text{n times}}$…I < y. Similarly, after L_7 we have I…$^{\text{n times}}$…I < I… $^{\text{m times}}$…I. Notice that in this argument we use the laws L_6 and L_7 as a premises.

Techniques θ(Π_1): We still have the standard techniques, $\tau_<$ and $\tau_\sim$, of **θ(Π_0)** based on pairing but, moreover, the laws L_8 and L_9 help us to solve $T_<$ and $T_\sim$ with new techniques consisting in using auxiliar ideal collections (that is to say, numbers) in order to compare real collections indirectly.

3.3. Warning

Even if our example is expressed in terms of variables, constants, relations, axioms, etc. it does not deal with formal set theory, but with very basic mathematics, typical in formal infant education. As we say in § 2.3. Remark 7, we do claim that our description is the 'interior' description that someone would do of himself when comparing finite collections. Just as nobody would describe his own activity in terms of types of tasks, techniques, technology and theory, nobody would describe his theoretical considerations in terms of language, interpretations, axioms, nomological, epistemological and axiological components. But still, we think that all those ingredients are useful in order to understand human activity from the 'exterior' point of view of research in didactics.

4. Conclusion

Our analysis of the theory of a praxeology allows to distinguish several praxeological ingredients which, as our example shows, seem important in order to grasp certain features of the inner dynamics of the whole praxeology. In particular, we have shown how without the ontological component one cannot even formulate the types of tasks, and without the axiological component the type of tasks seems to appear out of the blue. Also, we have pointed out that, typically, relations have a linked type of tasks (namely, “is that relation satisfied by those objects?”), and that the interpretation of the relations already gives standard techniques for those types of tasks. Moreover, we have shown how limitations in the praxis encourage the enrichment of the nomological component which, in turn, help to enlarge the ontological component and to create new, more satisfactory, techniques.

In future works we would like to show how our analysis of the theory of a praxeology helps to discern different links between praxeologies, something useful in order to get a better understanding of praxeological development along a study and research path.

Finally, our description of the theory of a praxeology contributes to make precise the idea that every activity of study comprises a certain model (both at the level of praxis and logos) of a certain portion of the world,. This idea is at the basis of the paradigm of mathematical modelling (Gascón & Nicolás, 2021).

References

A Briand, J. (1993). *L’énumération dans le mesurage des collections* [Doctoral dissertation, Université de Bordeaux I]. HAL theses. https://tel.archives-ouvertes.fr/tel-00494623

Bolea, P. (2003). *El proceso de algebrización de organizaciones matemá-ticas escolares.* Prensa de la Universidad de Zaragoza.

Bosch M. (1994). *La dimensión ostensiva en la actividad matemática. El caso de la proporcionalidad* [Unpublished doctoral dissertation]. Universitat Autònoma de Barcelona.

Chevallard, Y. (1999). L’analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–226.

Chevallard, Y. (2002). Organiser l’ètude. Structures & fonctions. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, R. Floris (Eds.), *Actes de la 11e ècole d’été de didactique des mathématiques* (pp. 3–22). La Pensée Sauvage.

García, F. J. (2005). *La modelización como instrumento de articulación de la matemática escolar: De la proporcionalidad a las relaciones funcionales* [Unpublished doctoral dissertation]. Universidad de Jaén.

García, F. J., & Sierra, T. Á. (2015). Modelos epistemológicos de referencia en el análisis de la actividad matemática en libros de texto: el caso del número en la Escuela Infantil. In C. Fernández, M. Molina and N. Planas (Eds.), *Investigación en Educación Matemática XIX* (pp. 299–307). SEIEM.

Gascón, J., & Nicolás, P. (2021). Paradigm crisis in the step from tertiary to secondary mathematics education. *International Journal of Mathematical Education in Science and Technology*. Advance online publication. https://doi.org/10.1080/0020739X.2021.2005169

Harris, M. (1976). History and Significance of the Emic/Etic Distinction. *Annual Review of Anthropology*, *5,* 329–350.

Licera, R.M. (2017). *Economía y ecología de los números reales en la Enseñanza Secundaria y la Formación del Profesorado* [Unpublished Doctoral dissertation]. Pontificia Universidad Católica de Valparaíso.

Lucas, C. (2015). *Una posible «razón de ser» del cálculo diferencial elemental en el ámbito de la modelización funcional* [Unpublished Doctoral dissertation]. Universidad de Vigo.

Ruiz-Munzón, N. (2010). *La introducción del álgebra elemental y su desarrollo hacia la modelización funcional* [Unpublished Doctoral dissertation]. Universitat Autònoma de Barcelona.

Sierra, T. A. (2006). *Lo matemático en el diseño y análisis de organizaciones didácticas. Los sistemas de numeración y la medida de magnitudes* [Doctoral dissertation, Universidad Complutense de Madrid]. E-Prints Complutense Repositorio Institucional de la UCM. https://eprints.ucm.es/id/eprint/7373/

Practical memory and positional and formative pathways

Yves Matheron

IREM and Institut de Mathématiques de Marseille, Université d'Aix-Marseille, France; yves.matheron@free.fr

Abstract: Using recent notions of positional and formative pathways developed in ATD, this text is part of a complementary work for the modelling of didactic memory previously proposed by the author. This work begins with the notion of practical memory.

Keywords: Practical memory, positional and formative pathways, praxemic extension.

The "memory" object in didactics.

Position

We have previously proposed (Matheron, 2001), and then developed (Araya & Matheron, 2015), a model for didactic memory. It is divided into three categories: practical, ostensive and S-knowledge memories. This text returns to the concept of practical memory alone, based on the notion of pathway, which has recently appeared in ATD (Chevallard, 2021). Below is a reminder of the anthropological positioning adopted for the notion of memory in didactics.

The fact that *homo sapiens* has a more or less efficient system for remembering and forgetting is considered in this text as a given that is functionally specified according to the activities in which he engages; activities of study for didactic institutions[1]. Similarly, we do not question the complex physiological system that allows *homo sapiens* to speak and interact verbally in the study of mathematics. Nor do we question the biomechanics of his prehensile hand, allowing interaction with various objects (pen, keyboard, compass, etc.) and enabling mathematical work.

The interest in memory, or rather in the combination of remembering and forgetting in relation to objects o, which presupposes an indexing on time t and an immersion in institutions, aims at understanding functional processes required by the study: first the constitution of a milieu, whether personal or institutional, as well as the institutionalization.

First questions

The expression of a memory in didactics, i.e. the recollection or forgetting at a time t_2, of certain dimensions of a relationship $R(\hat{\imath}, \wp, t_1)$ established by an instance $\hat{\imath}$ –a person or an institutional position– at an earlier time t_1 to elements of a praxeology $\wp$, presupposes the existence at t_2 of a situation, within an institution, where $\wp$ will be solicited in one way or another.

Identifying such an expression of a memory, or the lack of it, requires the existence of an institutional position assigned to *an observing and evaluating instance* $\hat{v}_2$ of $R(\hat{\imath}, \wp, t_2)$. Note that we can have $\hat{\imath} = \hat{v}_2$ in the case of a self-assessment by a person $\hat{\imath}$ who states that they remember, that they encounter difficulties in remembering or that they have forgotten. It is thus possible to define,

[1] Other non-didactic and unsolicited fields study human memory: psychology, neuroscience, etc.

in a first approach, recollection as the possibility in t_2 of the expression of a relation $R(\hat{\imath}, \wp, t_2)$ judged by $\hat{v}_2$ to be consistent with that, $R(\hat{\imath}, \wp, t_1)$, which $\hat{v}_2$ assumes to have been previously established by $\hat{\imath}$ in t_1; and forgetting by its opposite[2].

Some initial questions can be asked. Given an instance $\hat{\imath}$ assumed by $\hat{v}_2$ to have established in t_1 a relation $R(\hat{\imath}, \wp, t_1)$ to certain elements of a praxeology $\wp$, what elements of $\wp$ can the instance $\hat{v}_2$ expect to solicit in $\hat{\imath}$ in t_2? On the basis of which observations will the evaluating instance $\hat{v}_2$ pronounce in t_2 the evaluation of $R(\hat{\imath}, \wp, t_2)$? What gestures will it make to solicit $R(\hat{\imath}, \wp, t_1)$ in t_2? This last question will be temporarily addressed in this text only from the perspective of practical memory.

The permanence of $\wp$ between t_1 and t_2 has so far been assumed. Is it because of the fact that the advance of didactic time between t_1 and t_2 may have modified elements of the $\{\tau, \theta, \Theta\}$ block for the same type of tasks T around which $\wp$ was organised in t_1[3]? The analysis then requires a finer sequencing of the time interval $[t_1, t_2]$.

Some consequences

Seen from the didactic institution, the attribution of recollection and forgetting and the observation of this phenomenon, now appear to be a function of three variables: depending on time t, positions ($\hat{\imath}$, $\hat{v}$, etc.) and praxeologies $\wp_j$. To this relativity, a fourth dimension must be added. It brings in an external $\hat{w}$ instance, which can be the didactician who "is interested in what they think about what an other instance $\hat{\imath}$ thinks, or what $\hat{\imath}$ thinks about what such an instance $\hat{\jmath}$ thinks, etc." (Chevallard, 2019).

Evaluating the relationship $R(\hat{\imath}, \wp, t_2)$ by $\hat{v}_2$ within the didactic system, and by $\hat{w}$ outside it, requires the collection of observables. The intention concerning such a collection is then either didactic –for example in the case where $\hat{v}_2$ wishes to perform a gesture allowing to direct the continuation of the didactic process– or finalised by analysis in the case where $\hat{w}$ observes the didactic system or observes $R(\hat{\imath}, \wp, t_2)$ and $\hat{v}_2$ observing and/or acting from $R(\hat{\imath}, \wp, t_2)$.

In the above case, the relationship $R(\hat{\imath}, \wp, t_2)$ is observed and evaluated by other instances than the instance $\hat{\imath}$ which uses this report to carry out its work within the didactic system, *without necessarily being animated by the intention to show it*. In this case, it is a memory for practices stemming from mathematical praxeologies $\wp_j$ or their components, i.e. *a practical memory*. But the expression of a relation to $\wp$ can be *intentionally shown*, for example from a position $\hat{v}_2$ occupied by a teacher or a student who then evokes $R(\hat{v}_2, \wp, t_2)$; the latter being able to be merged with or be close to the expected institutional relation $R_I(\wp, t_2)$. It is then *an ostensive memory* because it shows itself with the insistence conferred on an imposition or sharing.

[2] When there is no need to specify further, a praxeology or some elements of a praxeology will be noted indistinctly by $\wp$.

[3] You may consider the changes in the primary school curriculum in the techniques for the tasks of obtaining the quotient and remainder in the division of two integers: repeated addition or subtraction from the divisor to the dividend in the lower grades, framing by two consecutive multiples of the divisor thereafter, and the 'classical' algorithm of division in the last grade.

The distinction based on a 'compliance/non-compliance' type of dichotomy of the relationship to the components defining $\wp$ from the analysis and evaluation in t_2, conducted by a third party evaluator $\hat{w}$, needs to be nuanced. Having established a relationship $R(\hat{\imath}, \wp, t_1)$, $\wp$ may be part of the praxeological equipment or, more modestly, the cognitive equipment of $\hat{\imath}$ in t_1[4].

If $\wp$ is part of $\hat{\imath}$'s praxeological equipment in t_2, what about the nature of this relationship? What kinds of tasks related to $\wp$ can $\hat{\imath}$ still perform in t_2, and through what techniques? An example will later show a change in the praxeological equipment. It is seen, from a $\hat{w}$ position, as a case of forgetting a technique for a type of tasks T in favour of another one to continue performing T.

Modelling the practical memory of an instance

Preliminary remarks

In order to constitute a milieu, a didactic constraint specific to the standard school form results in minimising, or even prohibiting, the provision of media in the classroom: objects, persons, institutions that are repositories of an external memory (Leroi-Gourhan, 1964). In return, it feeds a school paradigm: to create a milieu and act on and with it, the essential must come from recourse to the capacity of memory attributed to the positions $\hat{T}$ (teacher) and $\hat{S}$ (student). There is then the temptation to reduce memory in didactics to psychology, confusing the positions $\hat{T}$ and $\hat{S}$ with the individuals who occupy them.

When the role of media is attributed to the persons occupying a position $\hat{T}$, they use towards others, in position $\hat{S}$ for example, an ostensive memory relating to the elements of the praxeologies contained in the Institutionally Offered Curriculum (IOC) which they presuppose also belong to the Personally Lived Curriculum (PLC) of those to whom they address themselves5. They therefore presuppose an equivalence between the objects of the ICOs and the PLCs. The fiction of the coincidence of didactic and learning times (Chevallard, 1985 & 1991) is doubled with the fiction of an equivalence of the relationships to objects within the two curriculums.

The resulting consequence is that the teacher and students are supposed to be able to select in common, thanks to their memory, the same elements useful for the production of the new situation for one (T), for its apprehension and resolution for the others (Ss).

The practical memory

The "memory for the practice of a praxeology $\wp$", or simply "practical memory" is the memory observed by an evaluating instance $\hat{w}$ (we can have $\hat{w} = \hat{v}$) about the memory solicited by an

---

[4] Let's recall that the *cognitive equipment* of $\hat{\imath}$ as it can be seen by an instance $\hat{w}$, consists of the set of pairs formed by the objects o with which, from the point of view of $\hat{w}$, $\hat{\imath}$ has established a non-empty relation:
$\Gamma_{\hat{w}}(\hat{\imath}) = \{(o \,;\, R(\hat{\imath}, o)) \,/\, R(\hat{\imath}, o) \neq \emptyset\}$.

The praxeological equipment of $\hat{\imath}$ seen by an instance $\hat{w}$, is the set of couples formed by the praxeologies $\wp$ and the relations of $\hat{\imath}$ to these praxeologies seen non-empty by $\hat{w}$:

$\Gamma^{\blacklozenge}_{\hat{w}}(\hat{\imath}) = \{(\wp \,/\, R(\hat{\imath}, \wp)) \,/\, R(\hat{\imath}, \wp) \neq \emptyset\}$.

[5] For the distinction between ICO and PLC, see Chevallard (2021).

instance $\hat{\imath}$ (student, institution, teacher, parent helping their child, etc.), engaged in the accomplishment of a task identified as belonging to this praxeology.

Given a praxeology $\wp$, ***practical memory*** is that part of $R(\hat{\imath}, \wp, t_1)$ which, for an observer-evaluator $\hat{w}$, manifests itself in some way –in which case, $\hat{w}$ will speak of recollection–, or cannot manifest itself –in which case, $\hat{w}$ will speak of forgetting–, at $\hat{\imath}$ having to resort to $\wp$ in t_2. This definition calls for some remarks.

To an evaluating instance $\hat{w}$, observing an instance $\hat{\imath}$ engaged in a mathematical activity, more often than not hardly appears the pratico-technical block $\{T, \tau\}$ of a praxeology $\wp$, whether $\wp$ is mathematical or didactic. This is due to a property of institutions and people: *the economy of cognitive energy* (Douglas, 1986 & 1999). When one engages in the work of a task of a type one knows, one resorts to the technique one knows without holding a discourse. The technology justifying the technique is found in the fact that the task is adequately performed; there is no need to say anything more from the institutional position from which this work takes place.

When the task becomes problematic for an instance $\hat{\imath}$, i.e. when $\hat{\imath}$ lacks an adequate technique, or when $\hat{\imath}$ has to teach another instance $\hat{\imath}'$, it comes back to the technological elements associated with a technique; either for its elaboration in the problematic case, or for didactic reasons.

The memory observed in t_2 by $\hat{w}$ is that of the constituent gestures of $R(\hat{\imath}, \wp, t_1)$, for a task engaging in $\hat{\imath}$ elements of $\wp$ in t_2 which are actualised (or not) in instance $\hat{\imath}$ (person or institutional position). It can be relative to the technique or techniques for a given type of task, to the ostensives objects[6] allowing to engage and to carry out the gestures required for the technique, to the associated technological elements which allow to produce, to justify and to make this technique understandable.

Institutional speed of a formative pathway; consequences on practical memory and prototypical example

In didactical systems, in mathematics in particular, the succession of mathematical's organisations is very rapid. The expression of people's practical memory, memories and forgetfulness, is then largely constrained by the institutions –Halbwachs (1925) speaks of "social frameworks"– that organise the scrolling of the praxeological organisations they have attended. An observation made by a $\hat{w}$-didactician will provide an example.

An ancient observation...

The episode (Matheron, 2001) is summarised below. The correction of an exercise in class is filmed by $\hat{w}$ on 5 February: solve the equation $\ln x^2 + \ln x = 2$. The curriculum then only allowed for the teaching of the properties of the logarithm: namely $\ln x = 1 \Leftrightarrow x = e$. Exponentials and fractional exponents were not yet taught. The solution technique is then as follows:

$\ln x^2 + \ln x = 2 \Leftrightarrow 2\ln x + \ln x = 2 \Leftrightarrow 3\ln x = 2\ln e \Leftrightarrow \ln x^3 = \ln e^2 \Leftrightarrow x^3 = e^2 \Leftrightarrow x = \sqrt[3]{e^2}$.

[6] "We call *ostensives* objects those objects which have a material, sensitive form for us, which is in any case arbitrary. A material object (a pen, a compass, etc.) is an ostensive object". (Chevallard, 1994).

A student on the board solves the equation properly up to the step $x^3 = e^2$ and stumbles when he has to write $\sqrt[3]{e^2}$; the writing $e^{\frac{2}{3}}$ was not then known, as has been said.

On 25 March, exponential and fractional exponents having been taught, this film is shown outside the classroom by $\hat{w}$ to two students, *G* and *B*, present on 5 February. Then comes the episode $x^3 = e^2 \Leftrightarrow x = \sqrt[3]{e^2}$. $\hat{w}$ asks *G* and *B* how they had overcome the difficulty. They state that they had arrived at the step $\ln x = \frac{2}{3}$ and had logically concluded $x = e^{\frac{2}{3}}$. Then they come to their senses, remembering that using this technique was not possible since neither the definition of the exponential function, and thus $\ln x = a \Leftrightarrow x = e^a$, nor fractional exponents, were taught and known to them at the time.

… and its analysis from the recent notion of the formative pathway

In didactical systems, the bet is taken that a person *x* having occupied a position p_1 where they will have studied a praxeology $\wp_1$ –that is, where they will have enriched their praxeological equipment with an element π_1 relating to $\wp_1$– will be able to occupy the consecutive position p_2.

They will be able to increase in p_2 their praxeological equipment by an element π_2 relating to a praxeology $\wp_2$. An Institutionally Offered Curriculum (IOC) is seen from a position $\hat{w}$ as organising a succession of positions that person *x* can come to occupy in order to study praxeologies $\wp_1$, $\wp_2$, …, $\wp_n$; i.e., a positional pathway $\dot{p}_{\hat{w}} = (p_0, p_1, \ldots, p_n)$. To this positional pathway corresponds for *x*, and is seen by $\hat{w}$, a set of praxeological equipments; that is, a formative pathway $\tilde{p}_{\hat{w}} = (\pi(p_0), \pi(p_1), ..., \pi(p_n))$, again noted $\tilde{p}_{\hat{w}} = (\pi_0, \pi_1, ..., \pi_n)$ (Chevallard, 2021).

The analysis of the episode concerning the interaction with *G* and *B* on the logarithmic equation allows us to identify several positions and relations to praxeologies $\wp$ distinguished into $\wp_{\text{without exp}}$ ($\wp$ without the exponential) and $\wp_{\text{with exp}}$ ($\wp$ with the exponential). Let therefore praxeological equipments $\pi_i = Rp_i(\wp_i)$, seen by $\hat{w}_i$.

- p_1 : position of *G* and *B* having worked in t_1 (5 February) the solution of the equation $2\ln x + \ln x = 2$; relation to $\wp_{\text{without exp}}$ noted $Rp_1(\wp_{\text{without exp}})$
- p_2 : student position occupied in t_2 (25 March); relation to $\wp_{\text{with exp}}$ noted $Rp_2(\wp_{\text{with exp}})$
- p_3 : position of *G* and *B* questioned in t_3 (25 March +) by $\hat{w}$ about the correction they attended in t_1; in position $\hat{v} = \hat{w}$ waits for the students to mention p_1 and the relation in p_1 to $\wp_{\text{without exp}}$ noted $R_{\hat{v}}\{Rp_3\ [Rp_1(\wp_{\text{without exp}})]\}$
- p_4 : position of *G* and *B* questioned in t_4 (25 March ++) about the techniques for solving $2\ln x + \ln x = 2$ that they used in p_1; the students mention p_3 and the relation in p_3 to $\wp_{\text{with exp}}$, denoted $Rp_4\ [Rp_3(\wp_{\text{with exp}})]$
- p_5 : in t_5 (25 March +++) the $\hat{w}_{G\&B}$ students occupy a position $\hat{v}'$ which allows them to evaluate impossible p_2 in p_1: a 'memory gap' which relates to their relation to $\wp_{\text{without exp}}$, noted as $R_{\hat{v}'}\{Rp_4\ \neg[Rp_1(\wp_{\text{without exp}})]\}$

Everything happens as if, seen from observers $\hat{w}_i$ in evaluative positions $\hat{v}$ and $\hat{v}'$, only $R_{G,B}(\wp_{\text{with exp}})$ remained from the formative pathways $\tilde{p} = (\wp_{\text{without exp}}, \wp_{\text{with exp}})$ of *G* and *B*.

The praxeology $\wp_{\text{with exp}}$, which succeeds $\wp_{\text{without exp}}$, consists in using $\ln x = a \Leftrightarrow x = e^a$. One step is enough when three more were needed engaging techniques built from the technological element "ln isomorphism of ($\mathbb{R}_+^*$, ×) onto ($\mathbb{R}$, +)". The decrease in technical cost is ensured by ostensives objects whose activation allows an economy of gestures.

The rapidity of the scrolling of praxeologies, organised by the IOC, influences in turn the formative pathway to which people *x* are subjected. More economical praxeologies for the same type of tasks have replaced old praxeologies in their Personal Lived Curriculum (PLC).

Within a formative pathway, the integration of new, more economical praxeologies $\wp_i$ into the praxeological equipment of an instance $\hat{\imath}$ (person or institutional position) can be seen, from a $\hat{w}$ position (here $\hat{w} = G, B$), as the forgetting of technical elements of older praxeologies for the same type of tasks.

Producing something new from a practical memory

The praxemic extension

In an adidactic situation, deprived of recourse to media, how, within exploratory didactic moments and the drafting of a technique, can pupils construct a new praxeology answering the question that has been devolved to them?

An ordinary example...

In order to teach the rule of signs for the product of relative numbers to pupils of the 4th grade (13-14 years old), we assume that it is possible for them to extend to $\mathbb{D}$ the distributivity of multiplication with respect to addition in $\mathbb{N}$ that they know. To do this, they are first asked to calculate the product 3.8 × (4.7 – 14.7). They calculate the parenthesis: 3.8 × (-10), a product they come up against. A debate ensued, leading to four answers: 38, -38, 0.38 and 3.8.

A student replies: "3.8 × (4.7 – 4.7) = 3.8 × (-10) = -38 because we keep the sign of the largest number and that is -10". He thus calls upon part of his praxeological equipment on the addition of the relation R($\wp$add); in particular, part of the block (τ, θ), established at times ti prior to the present time t_2, as part of a personal milieu enabling him to produce his answer in t_2.

... and what can be said about it

When it is a question of extending, for another use, a part of the practice coming from a given institution, i.e. a praxeme, this way of doing things can be called praxemic extension. Praxemic extension consists in using a praxeme in the framework of the construction of another praxeological organisation than the one from which it originates[7].

The pattern is then as follows. Given $\wp_1 = (T_1, \tau_1, \theta_1, \Theta_1)$, an answer to a question Q_1 to which a relation $R(\hat{\imath}, \wp_1, t_1)$ has been established by $\hat{\imath}$, $\hat{\imath}$ wishes to construct an answer $A^\heartsuit$ to a question Q_2.

[7] A praxeme is, from an institutional point of view, considered as a generic unit or part of the practice within a pratico-technical block (T, τ), without the technologic-theoretical (θ, Θ) of (T, τ, θ, Θ) from which it is derived being mobilised or even evoked (Matheron, 2015). The case of this student shows the erroneous justification of his praxemic extension: '[...] the sign of the greatest number and it is -10'.

To do this $\hat{\imath}$ engages in the construction of a milieu $M_2 = \{A^{\Diamond}_i, W_j, D_k, Q_l,\}$ where one of the works W_j contains the technique τ_1. In other words, in t_2, $\hat{\imath}$ mobilises part of $R(\hat{\imath}, \tau_1, t_2)$ in order to start constructing the practical-technical block (T_2, τ_2) that will allow it to obtain $A^{\heartsuit}$, while the technological-theoretical block (θ_2, Θ_2) remains unquestioned for the moment.

It is possible that $\hat{\imath}$ tests in t_3, later than t_2, the limit of the scope of the part of τ_1 used for T_2 or the erroneous way of its use in relation to the realization of T_2. He must then elaborate a technological element to justify the non-effectiveness of τ_1 for T_2, leading to the rejection or improvement of τ_1 for T_2. In other words $\hat{\imath}$ engages in an adidactic process or situation of validation/invalidation of τ_1 for T_2, which produces a possibly provisional technological element θ_2, to be improved or rejected later.

Consequence on the possible adidactic: an example[8]

In order to constitute a milieu for the start of a new question, it is necessary that memories relating to old praxeologies, i.e. praxeological equipment, can be temporarily engaged. From the point of view of teacher *Y*'s position, the a priori bet of a praxeological extension by some of the students *X* justifies the construction and devolution of a situation that the teacher considers as possibly adidactic.

An example taken from a SRC proposal shows the support constituted by a possible praxemic extension for adidactic moments. To mentally calculate 35748 + 27489 – 27492, which is problematic for pupils in $\hat{\imath}$ position, the bet is taken by *Y* that $\hat{\imath}$ will resort in t_2 to the memory of a generic technique τ of "borrowing" used in t_1[9], when learning subtraction in primary school.

For example, to calculate 41 – 25, one "borrows" a ten from 4 tens and calculates the difference in the units row: 11 – 5. A generic part of τ is used, i.e. the "borrowing praxeme", in t_2:

35748 + 27489 – 27492
= 35745 + (**3** + 27489) – 27492
= 35745 + 27492 – 27492
= 35745 which is equivalent to subtracting 3, noted -3, from 35748.

The mobilization in t_3 of the "borrowing praxeme" allows the elaboration of a first technique for an addition task, initially seen as problematic by $\hat{\imath}$ during a first meeting with 7 + (-2): 7 + (-2) = 5 + 2 + (-2) = 5 + 0 = 5.

In t_2 and t_3, a generic part of τ is used to construct $\tau_{2\&3}$, without questioning their legitimacy; a work carried out later in a technological-theoretical moment.

[8] This example could refer to the situations of action and formulation in didactic situation theory.
[9] The expression "a generic technique" means that students will use one of several additive decomposition techniques present in their praxeological equipment, amalgamated during their formative pathways. They will adapt it to the task they are facing. For more information, see: http://educmath.ens-lyon.fr/Educmath/ressources/ressources-pour-la-classe/per-une-entree-dans-l-algebre-par-les-nombres-relatifs/

References

Araya, A. & Matheron, Y. (2015). Un modèle pour l'évocation des connaissances en classe de mathématiques. Micro-cadre institutionnel de la mémoire didactique, *Recherches en Didactique des Mathématiques,* 35/1, 37-67.

Chevallard, Y. (1985, ed. 1991). *La transposition didactique*. Grenoble: La Pensée sauvage.

Chevallard, Y. (1994). *Résumé des séances de l'UV de didactique des mathématiques, licence de mathématiques*, département de mathématiques, faculté des sciences de Luminy, Université d'Aix-Marseille II, unpublished.

Chevallard, Y. (2019). *Journal de l'Humble séminaire 2019-2020, séances de 1 à 5*. Retrieved from http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=223

Chevallard, Y. (2021). La question curriculaire à la lumière de la TAD: défigement praxéologique et questionnement du monde. In H. Chaachoua & A. Bessot (Eds.), *Nouvelles perspectives en didactique: le point de vue de l'élève, questions curriculaires, grandeur et mesure* (pp. 93-111). Grenoble: La Pensée Sauvage éditions.

Douglas, M. (1986, ed. 1999). *Comment pensent les institutions*. Paris: Editions La découverte et Syros.

Halbwachs, M. (1925, ed. 1994). *Les cadres sociaux de la mémoire*. Paris: Albin Michel .

Leroi-Gourhan, A. (1964). *Le geste et la parole II, La mémoire et les rythmes*. Paris: Albin Michel.

Matheron, Y. (2001). Une modélisation pour l'étude didactique de la mémoire, *Recherches en Didactique des Mathématiques*, 21/3, 207-245.

Matheron, Y. (2015). L'extension praxémique, un des moteurs possibles de la chronogenèse. In D. Butlen & al. (Eds.), *Rôles et places de la didactique et des didacticiens des mathématiques dans la société et dans le système éducatif* (pp. 597-600). Grenoble: La Pensée Sauvage éditions.

Studying, learning and mesogenesis at the light of the theory of relations

Jean-Pierre Bourgade[1]

[1]EFTS, Université Toulouse 2 Jean Jaurès, France; jean-pierre.bourgade@univ-tlse2.fr

A major constraint in mathematics education research is the repression of didactics, as expressed in the confusion between study and learning. We propose to study the extent to which theories of cognition and moments of study allow for a better differentiation between study and learning. This yields a new insight into some difficulties of the profession of teacher. It enables to understand why the specificity of study and research activities (SRA) is not well taken into account in practical implementations by trainee or experienced teachers; the confusion between learning-oriented and mesogenetic functions is at the core of a misconception of SRAs.

Keywords: Mathematics education, mathematics activities, repression of the didactic, learning theories, didactics.

Introduction

In our societies, a close connection is usually established between two a priori distinct aspects of human activities—studying and learning. This phenomenon is but an effect of a wider, pervasive constraint that operates at the level of western societies, the constraint of *the repression of the didactic* (Chevallard, 2012, p. 11):

> We have indicated that there is didactic in any social situation in which some instance (person or institution) plans to do (or does) *something* in order to make some instance learn *something*. […] a massive phenomenon, [is] at the heart of our societies: the vast majority of *discourses* and *texts* that speak of the social world ignore the didactic. […] We will express this heavy fact by saying that […] there is a *repression of the didactic*. […] When the "subject" of the didactic project has been roughly indicated, there is generally no mention of how this project will take concrete form. In fact, everything happens as if we could only talk about the first "something"—the didactic interaction, the didactic "gestures" to be accomplished—on the condition that we expel the second "something".

As a matter of fact, restricting the analysis of study to learning issues is a way of "expelling" the studied object from the analysis of its learning conditions. In particular, the genesis of praxeologies, their institutionalization in mathematics classes is not taken into account.

We propose an in-depth analysis of the possible joint use of the cognitive theory of ATD and of the theory of study functions, to demonstrate a definite delineation between study and learning. This theorization of learning is used to analyze effects of the repression of the didactic in trainee teachers' understanding of the so-called quaternary structuration of the study.

Confusion between studying and learning

The anthropological theory of the didactic has developed several bodies of theory (Chevallard, 2022): the praxeological theory attempts to provide tools for the analysis of human works; the theory of didactic functions (or moments) aims at describing didactic praxeologies and, thereby, processes of study (Chevallard, 1999, 2002; Artaud, 2007, 2011); the theory of cognition embedded in the ATD notably includes a theorization of learning. However, the learning theory in ATD is not just another theorization of psychological or neural processes: the definition of learning in ATD is rather minimalist and relativistic in essence. In particular, learning is not taken *a priori* as an absolute fact: it is an observable, which under certain circumstances *and from a definite position*[1], can be interpreted as learning.

Relations and learning

The theory of cognition and the theory of didactic moments are well-known parts of the theorization of the didactic in the anthropological approach (Chevallard, 1999, 2002, 2020, 2022; Artaud, 2011). We will restrict ourselves to elementary features of the theory of cognition: consider a person x, or an institutional position (p, I), and an object; the relation of x, or (p, I), to o is denoted by $R(x, o)$, or $R_I(p, o)$, and it represents the many ways in which x or p deal with o: what x or p know about o, what they may or may not do with or about o, their likes or dislikes about features of o, etc. In the very specific situation where o represents a mathematical praxeology, the relation of a person x to o will include his or her knowledge about the types of tasks that may be realized in relation to o, about the techniques it may involve to realize them; it will also include the way x understands these techniques and how x imagines they are justified, etc. If x is a student in a mathematics class, the relation of x to o may be in a more or less acute accordance to what is expected, that is, to what some other person or position expects—say the person y who sits in the position π of professor of mathematics in a given class I.

Evidently, someone might object that the relation of y to $R(x, o)$, that is $R(y, R(x, o))$, could be incorrect, or that the relation $R(y, R_I(s, o))$ of y to what any student (in fact, the position s of student in the class I) should "know" about o is not correct. In any case, y refers himself to a standard relation to both o and $R_I(s, o)$. This standard relation we will denote by $\bar{R}_S(s, o)$: it is the *standard relation* to o, expected from anyone willing to occupy the position s of student in the institution S (School).

As an autonomous scientific field, didactics defines as the study of situations where a person or an institutional position makes specific gestures in order to 'derange' (Chevallard, 2022) conditions so that the relation of a person or an institutional position to a given object may evolve. As a consequence, personal learning is defined in the theoretical framework of relations as the *evolution* of a person's relation to an object, in such a direction that the relation's fit to a standard relation is acknowledged[2] as better in the end. Therefore, the theorization of learning in ATD is grounded in the

[1] This relativization of learning consists in noting that 'learning' is *always* assessed by some person or institutional position, be it the learner him or herself. There is no absolute way to assess learning. This relativization is but one example of the *instantial excentration* recently introduced in ATD (Chevallard, 2020).

[2]For the sake of simplicity, we voluntarily do not emphasize on the fact that this acknowledgement is expressed by some person or institutional position, and is not "absolute."

study of the conditions and constraints that allow a certain person to improve, in another person's view, his or her relation to a given object.

Didactic functions and quaternary structuring of the study

It is an important elaboration in ATD that any human action can be modelled as the realization of a type of tasks, according to a technique that can be justified, explained, produced by a technological-theoretical environment; such a model is a *praxeology*, formed by a *praxis*—a type of tasks and a technique to realize it—and a *logos* that is functionally related to the *praxis*: the functions of *logos* are to justify, explain, produce the *praxis*. Accordingly, the study of *any* object can be considered as the study of a set of praxeologies related to that object. The inherently praxeological nature of objects led to the observation that study, conceived of as a means to create conditions and constraints for learning, can be understood as the realization of six functions or moments: the moments of the study, or didactic moments (Chevallard, 1999, 2002; Artaud, 2011). In a given study process, these functions can be realized in several episodes, at different degrees of achievement; it may even happen that one or other function be not realized—and actually, it often happens. These functions are closely related to praxeological components. In the frame of teachers' training in France, it is sometimes (in Toulouse and Marseille, for instance) advised to follow a specific structuring of the study. Schematically, the study should begin with a didactic device created in order to facilitate the realization of three of the didactic moments (moment of first encounter with the type of tasks, moment of the exploration of the type of tasks and of the emergence of the technique, technological-theoretical moment): that is, the moments that allow for the *genesis* of a new praxeology, as (part of) an answer to a generating question. The study of this question must be performed in a relative (didactical) autonomy from the part of the students. Such a didactic device is an *Activité d'étude et de recherche*, a study and research activity (SRA). It is followed by the realization of the institutionalization moment, then by the moment of the praxeological work and by the moment of evaluation of the praxeology and of the students' mastery of it. This yields a quaternary structuring of the study:

I. SRA (moment of first encounter, exploratory moment, technological-theoretical moment)

II. Synthesis (moment of institutionalization)

III. Exercises (moment of praxeological work)

IV. Evaluation (moment of evaluation)

The main function of the three moments realized in an SRA is to facilitate the *genesis of praxeologies*. In institutions as different as schools, companies, factories, administrations, families, groups of friends, etc., praxeologies can be produced and soon given up, or on the contrary, become part of the core praxeological equipment of the subjects of the institution. In this case, the praxeology becomes somewhat 'official,' institutional, and people have to train using it, and they even may be evaluated as regards their capability to use it. The didactic function by the realization of the which the praxeology is set up as an 'official' praxeology in the institution, and takes part in the expected relation of subjects to the objects it deals with, is the *moment of the institutionalization of the praxeology*.

Study and learning in ATD

A study process is supposed to yield a substantial move in the relations of students to the studied object. Somehow paradoxically, ATD also emphasizes on an important feature of teacher-student relationships: the teacher has no means to impose learning; supposing that the teacher is capable of enforcing learning, is at best a denial of the intellectual, psychological, or even human autonomy of the subjects of an institution, at worst the reflection of the teacher's attitude of omnipotence and his or her refusal to take into account the student's desire. This is yet another facet of ATD's rejection of any psychological approach to learning and its emphasis on ecological issues; two important questions, in ATD, related to learning are: do didactic functions intend to achieve student's learning? are all didactic functions supposed to realize conditions fit to learning to the same extent? We will discuss these questions after analyzing some trainee teachers' views on institutionalization.

Trainee teachers' relation to learning and study

In 2021-2022, trainee teachers attending the so-called master MEEF[3] at the Toulouse INSPÉ[4] were asked to explain their institutionalization techniques, and to describe the way these techniques took account of what comes before (SRA), and of what comes after, institutionalization (praxeological work, evaluation). They were also asked the following question:

> At the end of [an SRA], do the students' relations conform to the standard relation to the studied mathematics, such as the teacher views it?

This question was designed to assess the students' understanding of the functions of didactic moments relative to personal learning.

Institutionalized relation as a 'mean' of personal relations

Many students had an opinion close to the following[5]:

> After the SRA has taken place, personal relations will normally have evolved, but will still be different from the standard relation: students will have had different research approaches, will have had their own rhythm and will not have interpreted everything in the same way.

Or,

> This [institutionalized] relation was chosen by the teacher, but it takes into account the diversity and difference in level of the students in the class.

The idea is that institutionalization takes students' relations into account to produce the institutionalized relation. Another student claimed that the institutionalized relation would be "denser" for a "strong" class, coarser for a "weak" class. Yet another considered institutionalization as a process "meant to homogenize all personal relations". In their view of institutionalization, SRAs

---

[3] Teaching, education and training professions, *métiers de l'enseignement, de l'éducation et de la formation*.

[4] National higher institute of teaching and education, *institut national supérieur du professorat et de l'éducation*

[5] Unless otherwise stated, quotations are taken from students' written answers.

are there to enrich the students' personal relations, institutionalization processes will produce a sort of 'mean' of these personal relations.

Emergence of praxeologies *vs* learning praxeologies

This point of view on institutionalization depends on a conception of SRA as a device designed for facilitating *learning*. Far from limiting the functions of SRA to the genesis of praxeological elements, many trainee teachers consider that the goal of an SRA is to ensure students' *understanding* of the emerging praxeology or of the activity itself. SRAs are therefore understood as the onset of a *learning* process:

> it is obvious that students' relations will have changed after the SRA. […] the SRA will have made it possible for their relation to *evolve towards the expected relation* (our emphasis).

Even for beginner teachers who are conscious of the functions of an SRA, the connection to learning is straightforward:

> an SRA allows for the realization of the moment of first encounter, the exploratory moment, the technological-theoretical moment. So, at the end of this stage, the students have a better *understanding* of the studied praxeology (our emphasis.)
>
> When institutionalizing, it is possible for the students to use the contextualized example that was worked on previously. Indeed, we observed that this decontextualization could be quite difficult for a number of students, even though the activity seemed to be *understood* (our emphasis.)

SRA as problem-solving

A striking side-effect of a learning-oriented focus is a form of didactical individualism: an SRA is conceived of as an individual activity, a 'problem,' which can be more or less efficiently overcome. Some teachers even mention the institutionalization process as the 'correction' of the SRA. Others mention that institutionalization is a moment for the teacher to 'validate' the mathematical praxeology that has been emerging during the SRA. The SRA is therefore driven towards a more traditional device, the 'problem solving' device, the didactic functions of the which can strongly differ from the SRA's.

The individualistic approach to SRAs has effects on the institutionalization; either most of the students have reached a satisfactory degree of elaboration in the process of building a new mathematical praxeology, or differences among students on this part remain important. In the first case,

> if the student has *passed the SRA* perfectly, then his personal relation at the end of the SRA is already close to the 'expected' relation.

Then, institutionalization appears as the 'generalization' of the content of most of personal relations to the praxeology under construction. In the problem-solving perspective on SRAs, the expected relation will be institutionalized by reference to these relations, only going further in reference to the teacher's standard relation conceived as the "correct answer to the SRA." The teacher bases the institutionalization on the analysis of the students' personal relations, such as they are at the end of the SRA, with two consequences: a tendency to 'force' the praxeological production during the SRA, a tendency to fill gaps during institutionalization.

Institutionalization as gap filling and validation

One of the main gaps to be filled is, in general, the *logos* part of the praxeology under construction: trainee teachers often consider that the production of *praxis* is within the reach of students, while the production of *logos* is too difficult and has to be taken over by the teacher. Asked about the potential interest of exercising after the SRA and before the synthesis, some teachers considered that exercises just after the SRA would give students the opportunity to "identify the several types of tasks involved and use their *praxis*. They will later meet the expected *logos* during the institutionalization." The institutionalization moment is therefore mixed with the technological-theoretical moment and there is a confusion between 'elaborate technological-theoretical elements (including definitions and theorems),' or 'justify a *praxis*' (technological-theoretical moment), and 'give definitions and theorems,' or even 'give adequate wordings of definitions and theorems' (institutionalization moment). This is of course related to the problem-solving approach to SRAs: the idea that institutionalization is a process used to 'validate' a praxeology, directly conflicts with the fact that the validation function is clearly attributed to the technological-theoretical moment, not to the institutionalization function.

Institutionalization as a study process

The materials we have just analyzed were used as a means to delineate the personal relation of trainee teachers to institutionalization, and to the organization of the study in general; on the whole, trainee teachers consider that:

- the institutionalized relation is independent from the SRA, which is but a preliminary work that, at best, offers a glimpse at the reasons why a particular mathematical object has to be learned;
- since students are not able to achieve a full-study of a given praxeology in autonomy, and to modify accordingly their relation to this praxeology, the institutionalization cannot base itself on the personal relations of students after SRA; *therefore*, it must be based on the teacher's standard relation, and prepared before the SRA.

Institutionalization is therefore functionally disconnected from the genesis of praxeologies; SRAs are conceived of either as *learning* devices or as mere "introductory" activities. In both cases, the institutionalization process is reduced to replacing a void or unachieved personal relation to the studied object by the standard relation elaborated by the teacher.

Main functions of didactic moments

This is grounded on the idea that the main function of the praxeological genesis moments is not only to *build* new praxeologies, but to help students' relations to these praxeologies move in a suitable direction. In fact, personal relations of students have no reason to evolve during an SRA—and no reason *not* to evolve: only, this aspect is *functionally independent* of the main functions of these moments[6]. Similarly, the *main* function of the institutionalization moment is *not* to help student have their relations conform to the institutionalized relation—nor is it to prevent them to: the main function is to point out a specific relation to some praxeologies (eventually, to help producing this relation), and to present it as the from-now-on expected relation to these praxeologies, in a specific institution—the mathematics class. In other words: the moments of praxeological genesis and of institutionalization create conditions for the construction and the institutionalization of new praxeologies, which is secondarily a condition for learning—but it cannot be expected that these moments be enough for students to *learn* anything notable about these praxeologies.

Didactic systems and mesogenesis

When an SRA is realized, and yields genuine results (identification of a type of tasks, emergence of a technique, production of technological-theoretical elements), where do these praxeological elements hide if not in students' personal relations? Here is the key point; the individualistic approach to study leads to neglecting an important feature of didactic processes. Whenever a teacher wants to introduce the students to some mathematical object, say the resolution of linear systems, the question he addresses the class may be: "How to find the values of several quantities connected with each other through linear relations?" In fact, this question is not the first that students will hear from the teacher: he or she will rather ask a question Q_g that will generate the study of a situation which can be solved only, or with an obvious improvement, by modelling it with a system of linear equations. This question generates a study and research activity, in the process of which several works W_i will be discovered or produced by the didactic system: $S([X, y], Q_g) \Rightarrow M_g=\{W_1, W_2, \ldots, W_k\}$. Based on this *milieu* M_g, the collective of study will eventually build an answer R to question Q_g, which, in our example, will include a set of values for some prior unknown quantities:

$\{S([X, y], Q_g) \Rightarrow \{W_1, W_2, \ldots, W_k\}\} \Rightarrow R.$

Of course, the goal of the teacher is not reached yet: having solved the problem posed by Q_g is not the same as having produced a praxeology for solving any similar problem (that is for realizing tasks of the same type). However, if the functions of praxeology genesis have been realized, elements of the expected praxeology must have been produced and actually form part of the works W_i. The institutionalization process will consist in a study process, generated by another question, the question Q: "How to find the values of several quantities connected with each other through linear relations?" The question Q_g, the answer R and the works W_i constitute a *milieu* M for the study of Q:

$$S([X, y], Q) \Rightarrow M = \{Q_g, R, W_1, W_2, \ldots, W_k\}.$$

The institutionalization process must lead to the construction of a relation to a praxeology, starting from the praxeological elements that have been gathered during the SRA and that constitute the *milieu*

[6] On the contrary, the moment of praxeological work—though not limited to this dimension (Puchaczewski, 2022)—is partially dedicated to the *training* of students, with a view towards a better *mastery* of the studied praxeology (Chevallard, 1999, 2002).

M. The answer provided to Q is therefore a *praxeology*, or, better said, a *relation* to the praxeology about which y has him- or herself developed a *standard relation*:

$$\{S([X, y], Q) \Rightarrow \{Q_g, R, W_1, W_2, \ldots, W_k\}\} \Rightarrow R_{[X, y]}(s, \wp).$$

Therefore *two* didactic systems are at work: $S([X, y], Q_g)$ and $S([X, y], Q)$. From a curricular point of view, $S([X, y], Q)$ comes first: it is because y knows the students have to learn about $\wp$, that is about question Q, that he or she addresses the question Q_g to them. From a didactic viewpoint, Q_g comes first, both temporally and rationally: Q_g is a *raison d'être* (see e.g., Bourgade, 2019) for Q. In any case, *two* questions are studied, and the teacher should keep in mind that two didactic systems have to exist. Most of time, however, the didactic autonomy (*topos*) of students is very limited in the didactic system $S([X, y], Q)$, that is during institutionalization. The reasons for this *topos* restriction are manyfold, but we can sort out at least two main factors:

- institutionalization is conceived as 'the teacher's part;'
- little or no effort is made, during the SRA, to keep in mind the structural dependence of $S([X, y], Q)$ to $S([X, y], Q_g)$.

An SRA can be used to enrich the *milieu* of the study, for instance with praxeological elements. The institutionalization process can be based on the exploitation of the *milieu* in order to characterize an emerging praxeology as constituting part of the henceforth expected relation to the generating question of the SRA. This would require practical tools for keeping track of the *milieu*: notebooks, paperboards, laptops, etc., tools which are not part of the praxeological equipment of the mathematics class in our times.

Conclusion

On the whole, it is probably a dominant constraint of today's society that molds more specific professional praxeologies: the prominence of individualistic spontaneous theorizations can only contribute to the erasing of the collective dimension of study, and in particular of some possible ways to realize mesogenesis. Such individualistic theorizations derive from a more generic opposition, which engages both philosophical and political dimensions, between the individual and the collective, and the idea that freedom of thought, liberalism, creativity, etc., need individualistic approaches to policy and education (Pinto, 2009). Indeed, SRA managing is often conducted as with as many parallel, independent didactic systems as there are students in the class; the outcome of such devices is, if not learning, at least an individual *understanding* of praxeological elements rather than the collective, *mesogenetic*, gathering of indices in view of the later production of a collective relation to the collective synthesis of these praxeological elements.

As regards theorizations of the didactic, the use of the theory of relations is a good means to question the connection of didactic devices and functions, to learning and understanding. We hope to have illustrated the fact that the confusion between studying and learning is an important manifestation of the repression of the didactic.

We have mainly addressed the issue of SRA and institutionalization in the frame of a quaternary structuring of the study, for reasons of relative compatibility with the current praxeological equipment of the profession of teacher. However, the institutionalization function is probably one of the more important sources of difficulty in the management of study and research paths (SRP, for a recent review, see Bosch, 2018; institutionalizations issues of modelling praxeologies have been briefly addressed in Florensa, Barquero, Bosch & Gascón, 2019), be it praxeologically open or finalized. In the latter case, the fact that the designers of the SRP *know* what praxeologies they want student to get acquainted with, produces the same sort of difficulties as with SRAs: the possibility exists that managers of such SRPs unconsciously neglect the importance of institutionalization as a study process of its own. In the former case, open SRPs may lead to unpredicted praxeologies, and institutionalization is important for at least two reasons: first reason, the long-term development of the SRP makes it necessary to regularly produce syntheses of what has been achieved; second reason, what to institutionalize may not be obvious for participants in the SRP. In this case, the questions "what have we learned, and is it important, and what for?" take on considerable importance, since the evolution of the SRP depends on the answer given them.

The herbartian scheme has already been explored as a useful means to describe study and research paths, together with questions-answers (Q-A) maps (Bosch, 2018; Barquero, Bosch & Gascón, 2008; Florensa, Barquero, Bosch & Gascón, 2019). However, such questions-answers maps generally limit themselves to *mathematically driven* issues (treatment of subcases, or questions derived from other questions in trophic chains): the possibility to include in Q-A maps such questions as those which drive institutionalization should be given attention.

Acknowledgment

The author would like to thank Michèle Artaud for making the importance of the phenomenon of the repression of the didactic clear to him—and, of course, for the many discussions he had with her on this and many others topics.

References

Artaud, M. (2007). Some conditions for modelling to exist in mathematics classrooms. In *Modelling and applications in mathematics education* (pp. 371-378). Springer.

Artaud, M. (2011). Les moments de l'étude : Un point d'arrêt de la diffusion ? In M. Bosch et al. (Eds.), *Un panorama de la TAD* (pp. 141–162). Centre de Recerca Matemàtica.

Barquero, B., Bosch, M., & Gascón, J. (2008). Using research and study courses for teaching mathematical modelling at university level. In *Proceedings of the fifth congress of the European society for research in mathematics education* (Vol. 5, pp. 2050-2059). Larnaca, Cyprus: University of Cyprus.

Bosch, M. (2018). Study and research paths: A model for inquiry. *Proceedings of the Inter- national Congress of Mathematicians*. Rio de Janeiro, 1–9 August (vol. 3, pp. 4001–4022).

Bourgade, J.-P. (2019). À la recherche de raisons d'être pour les structures de groupe : quelques praxéologies pour la profession de professeur. *Recherches En Didactique Des Mathématiques*, *39*(2), 131–166. https://revue-rdm.com/2019/a-la-recherche-de-raisons-detre-pour-les-structures-de-groupe-quelques-praxeologies-pour-la-profession-de-professeur/

Chevallard, Y. (1999). Analyse des pratiques enseignantes et didactique des mathématiques : l'approche anthropologique. Dans R. Noirfalise (Éd.), *Analyse des pratiques enseignantes et didactique des mathématiques* (pp. 91-118). Clermont-Ferrand : IREM.

Chevallard, Y. (2002). Organiser l'étude. Structures & fonctions. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la 11[e] école d'été de didactique des mathématiques* (pp. 3–22). La pensée sauvage.

Chevallard, Y. (2012). *Didactique fondamentale M1 2011-2012*.
http://yves.chevallard.free.fr/spip/spip/IMG/pdf/DFM_2011-2012_Module_1_LD_.pdf

Chevallard, Y. (2020). *Humble séminaire 2019-2020, Séance 2*.
http://yves.chevallard.free.fr/spip/spip/IMG/pdf/HS_2019-2020_-_1_a_5.pdf

Chevallard, Y. (2022). On the Genesis and Progress of the ATD. In: Chevallard, Y. *et al*. (Eds) *Advances in the Anthropological Theory of the Didactic*. Birkhäuser, Cham. https://doi.org/10.1007/978-3-030-76791-4_1

Florensa, I., Barquero, B., Bosch, M., & Gascón, J. (2019). Study and research paths at university level: managing, analysing and institutionalizing knowledge. *Eleventh Congress of the European Society for Research in Mathematics Education*, Utrecht University, Feb 2019, Utrecht, Netherlands. Hal-02422617

Pinto, L. (2009). *Le collectif et l'individuel. Considérations durkheimiennes*, Paris : Raisons d'agir.

Puchaczewski, E. (2022). La théorie des rapports pour penser le « travail sur l'erreur ». CITAD7.

Teachers' epistemic ethos and mathematical modelling teaching

Floriane Wozniak

EFTS, University Toulouse Jean Jaurès

Abstract. This text focuses on the epistemic part of teachers' practices and the collective dimension of their praxeologies. Considering what brings together and founds the common identity of teachers of a given school discipline, four founding concepts of the anthropological theory of the didactic are used: *object, relation, institution, person*. The notion of ethos in sociology is used to define the *epistemic ethos of teachers* (Wozniak, 2019) to articulate personal and institutional relationship. In a second part, the relation to modelling of French teachers is studied through the observation of the resolution of the same problem at different teaching levels. In this way, the study of the teachers' epistemological work reveals part of their epistemic ethos.

Résumé. Ce texte s'intéresse à la part épistémique des pratiques des professeurs et à la dimension collective de leurs praxéologies. Considérant ce qui rassemble et fonde l'identité commune de professeurs d'une discipline scolaire donnée, quatre concepts fondateurs de la théorique anthropologique du didactique sont utilisés : *objet, rapport, institution, personne*. La notion d'ethos en sociologie est convoquée pour définir l'*ethos épistémique des professeurs* (Wozniak, 2019) afin d'articuler rapport personnel et rapport institutionnel. Dans une seconde partie, le rapport à la modélisation des professeurs français est étudié à travers l'observation de la résolution d'un même problème à différents niveaux d'enseignement. C'est ainsi que l'étude du travail épistémologique des professeurs permet de révéler une part de leur ethos épistémique.

Liste des editeurs (Eds)

Advances and challenges in the anthropological theory of the diactic (pp. xx-yy)

VII congrès international de la TAD (Bellaterra, 19-23 juin 2022)

Axe 1. *Analyse et évaluation des usages de la TAD dans la recherche et la Formation en didactique*

Editorial, año

1. Introduction

Today, in France, a primary or secondary school teacher no longer uses the theory of proportions[1] to solve a proportionality problem. They prefer to use the properties of linearity[2] or the "unit rate" which reveals the proportionality coefficient. The taught praxeologies evolve because "the objects of teaching are victims of didactic time, they are subjected to an erosion, to a "moral" wear, which implies their renewal during a cycle of study". (Chevallard, 1991, p. 68, my translation).

This phenomenon of obsolescence leads to curricular changes and teachers teach the current mathematical praxeologies, even if they know other more efficient ones. The practices of a particular teacher are also those of the social group to which he or she belongs, the mathematics teachers at a particular period and particular level of teaching.

To study this dialectic between the individual and the collective, four fundamental concepts of the anthropological theory of the didactic can be used: *object*, *relation, institution, person* (Chevallard, 1992). An *object* refers to everything that can exist for someone, whether the point of view is material or non-material. The set of interactions of an individual with this object constitutes his or her personal *relationship* to this object. Personal relationships are built through the frequentation of *institutions* which, by the mere fact of this frequentation, impose ways of doing and thinking to their subjects according to their occupied place:

> Given an object o, an institution I, and a position p in I, we call *institutional relationship* to o in position p, denoted by $R_I(p, o)$, the relation to the object o which should ideally be that of the subjects of I in position p. Saying that x is a good subject of I in position p is equivalent to say that $R(x, o) \cong R_I(p, o)$, where the symbol $\cong$ refers to the conformity of the personal relationship

[1] Bezout (1868, p. 103) names geometric proportion: « The fundamental property of the geometric proportion is that the product of the extremes is equal to the product of the means; for example, in the ratio 3:15::7:35, the product of 35 by 3, and that of 15 by 7, are also 105." (my, translation)

[2] If f is a linear function, the additive and multiplicative linearity properties are respectively: $f(x_1+x_2) = f(x_1) + f(x_2)$ and $f(kx) = k\, f(x)$. The property of multiplicative linearity is used to deduce from $f(a) = b$ that $f(1) = b/a$ (unit rate).

> of *x* to the institutional relationship in position *p*. (Chevallard , 2003, p.2, my translation).

A *person* is modelled by the set of an individual and the set of their personal relationships to objects at a given moment; their singularity is thus determined by the diversity of their subjection to the institutions they attend. This is what the responses of Tunisian students from a rural and Muslim[3] culture reveal at the end of a physics lesson on lightning[4]:

> Student A: "Then the scientific explanation is real and we have to admit it even if it opposes our beliefs."
> Student B: "I think it is a punishment from God ... a punishment for those who don't give to the poor. It shows that God is very powerful and can reach us at any time."
> Student C: "Maybe it's a wrath of God. Well I don't know [...] I think they are both true [...]. To what degree is one true and the other false? I don't know."
> Student D: "[The physics teacher] is trying to convince us [...]. I think physics is wrong to eliminate the religious explanation."
> Student E: "God even encouraged us to understand things in a rational way. The scientific explanation is then real and we have to admit it, even if it opposes our beliefs." (Caillot, 2014, pp. 11-12, my translation).

These responses illustrate the continuum between assumed subjection to an institution (A chooses science, B and D choose religion), the difficulty of choosing between two subjections (student C) and the effort to reconcile two institutional relationships that are nevertheless opposed (student D).

There are therefore at least as many ways of knowing an object as there are institutions in which this object finds its place. And the personal relationship to an object arises as much from the plurality of institutions frequented as a subject, as from the plurality of positions that the same individual may have occupied within an institution. One does not know fractions in the same way, depending on whether one is a primary school

[3] “It is He who makes you see the lightning (which inspires you) fear and hope; The thunder glorifies Him with its praise." (Sura 13, verses 12 and 13, Qur’an)” Maury & Caillot (2003) quoted in Caillot (2014).

[4] Only the students' responses are included here, the full quote would have been too long.

pupil, a secondary school pupil or a university student preparing to become a mathematics teacher.

The point of view adopted in this text is to go beyond the singularity of individuals to consider the common part of teachers' practices in relation to the knowledge to be taught and the taught knowledge, which we call the *teachers' epistemological work*. We do not consider what differentiates and distinguishes their practices but what they share and unites them in their condition as teachers. How can the diversity of institutional relationships be encompassed to constitute a common "whole"? How can we talk about and study this common part of teachers' practices in relation to knowledge? What determines this common part that makes the identity of teachers? These are the questions that this text attempts to answer.

The first part of this text introduces the notion of the teacher's epistemological work and defines the epistemic ethos in reference to the notion of ethos developed by sociologists. The second part deals with teachers' relationship to modelling. The conclusion returns to the relevance of going beyond singularity to understand what underlies teachers' practices.

2. Epistemological work and epistemic ethos

The study of the difficulties of disseminating the results of research in didactics, and in particular of didactic engineering, is old (Artigue, 1986). It aims to reveal the conditions and constraints under which the didactic system functions. Lalina Coulange and Brigitte Grugeon-Allys (2008) studied the implementation of the "bordered square" situation in the third grade (14–15-year-old students). The aim is to count the grey squares enclosing a square grid (Figure 1) according to the number of white squares.

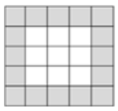


Figure 1. bordered square 3

The authors show how the teacher empties the tasks of their didactic potential by adding questions that reduce the students' topos or by developing technological discourses that they describe as "legal": "one must do...", "one cannot...". Pupils thus build a formal rather than functional relationship with algebra. The observed practices are explained by a personal component (the teacher would have a formal relationship to algebra) and a pedagogical constraint (the approach of exams reduces the time available).

Sylvie Coppé, Brigitte Grugeon-Allys and Julia Pilet (2016, p. 63, my translation) return to this situation which

> can motivate the introduction of algebraic expressions, the role of the counter-example to invalidate a false assertion by giving a place to the algebraic/numeric dialectic; and also the use of the distributive property of multiplication over addition in its syntactic and semantic aspects.

They note, however, that

> work on calculation programmes that precede the situation of the bordered square and an articulation between moments of first encounter, institutionalisation and constitution of knowledge around the property of distributivity are decisive in the success of this activity to favour the pupils' learning. If these conditions are not put in place, this situation may not be exploited with interest for both the pupils and the teacher. (op. cit., p. 63, my translation).

The authors move away from a contingent analysis to an ecological analysis. They note that this situation is present in France in several scientific publications, ministry resource documents, various school manuals and brochures for teachers:

> The descriptions give little indication of the scenario of the different phases, the difficulties in managing them, or the role of the teacher, which is most often implicit both in terms of time management and the management of the processes of devolution, validation and institutionalisation. (op. cit., p. 74, my translation).

They conclude on the need for collaborative research between teachers and researchers to produce documents that explain the mathematical and didactic issues of the situation. The difficulties of dissemination thus reveal the praxeological needs of teachers (Wozniak, 2019).

When teachers design and carry out their teaching, they elaborate at the same time an organization of knowledge and a didactic organization. If the finality of his praxeologies are the gestures of the study, their matter is made of the knowledge to be taught and the knowledge for teaching. Thus, the teacher's work is nourished, implicitly or explicitly, by epistemological questions:

> Defined narrowly, epistemology is the study of knowledge and justified belief. As the study of knowledge, epistemology is concerned with the following questions: What are the necessary and sufficient conditions of knowledge? What are its sources? What is its structure, and what are its limits? As the study of justified belief, epistemology aims to answer questions such as: How we are to understand the concept of justification? What makes justified beliefs justified? Is justification internal or external to one's own mind? Understood more broadly, epistemology is about issues having to do with the creation and dissemination of knowledge in particular areas of inquiry. (Steup, 2018).

The teacher's work, which I therefore qualify as epistemological, is made up of all the praxeologies which contain, even in a naive or spontaneous way, a part of the study of the processes of production, formation, development, transformation, organisation and transmission of mathematical objects or a part of the characterisation of the nature of the mathematical activity itself and of its objects. It is made up of mathematical and didactic dimensions articulated by an epistemological dimension (Figure 2).

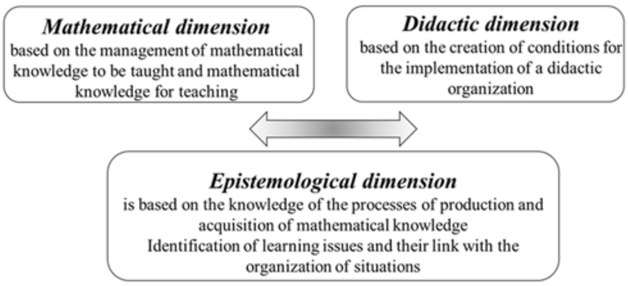


Figure 2. The three dimensions of the teacher's epistemological work.

The epistemological dimension makes the link between mathematical objects and the gesture of the study of these objects. Studying the conditions for the dissemination of didactic engineering is therefore studying the conditions for the epistemological work required to implement them. If we return to the situation of the bordered square, it appears that contrary to the initial hypotheses of Lalina Coulange and Brigitte Grugeon-Allys (2008), the personal component probably plays little part in the difficulties of its implementation in class. The practices of the observed teacher do indeed testify to a formal relationship to algebra, but this relation does not characterize this particular teacher, since the work of Sylvie Coppé, Brigitte Grugeon-Allys and Julia Pilet (2016), based on observations of practices and the analysis of various media, attests that it is widely shared.

Seeking to understand why teachers do what they do, questioning what determines their practices should therefore lead to going beyond the singularity of individuals, as sociologists of the professions do: "even if their mastery develops over the course of different personal histories, the knowledge and know-how used by the members of the same profession comprise a common element that is much greater than what differentiates them". (Champy, 2012, p.76, my translation). And this is in addition to the project of transforming the teaching profession into a fully-fledged profession, which is discussed in axis 3 of this conference.

This is how I propose to think about the articulation between personal and institutional relationships based on the notion of ethos developed in sociology:

> Ethos is presented as one of the interpretative concepts that make it possible to grasp a recurrence of behaviour on the part of actors sharing the same social insertion. Above all, it has a heuristic vocation for thinking about the relationship between collective history and the logics of action, insertion into a social milieu and practices, and this from a structurationist rather than deterministic perspective. (Fusulier, 2011, p. 107, my translation).

The ethos of teachers is thus made up of various praxeologies, some of which may be pedagogical, for example, when they participate in the life of their school. But, considering the praxeologies related to the teachers' epistemological work, I will limit myself to the study of the epistemic part of the ethos of teachers. I call it the *epistemic ethos* of teachers (Wozniak, 2019). It is the union of the institutional relationships to the objects $R_I(p, o_k)$ mobilised to carry out the teachers' epistemological work. What is this epistemic ethos made of? How can it be studied? How can we go beyond the individual to reach the collective?

For Bernard Zarca (2012, p. 13, my translation),

> A professional ethos can be found in the practices, representations and individual judgements of the members of the profession and, even more so, in collective practices, representations and judgements, crystallising in hierarchies, of prestige or power, and objectifying in modes of organising exchanges, controlling work, distinguishing excellence, etc.

Thus, studying the ethos of mathematicians, he discovered that pure and applied mathematicians criticise each other on their relation to the world but agree on the role of demonstration in mathematics:

> The formers criticise the latter in the name of the universalist ideal of science. They sometimes denounce their compromises with capitalism or with a particular state and taunt their lack of disinterestedness. They are taunted in return for their lack of a sense of reality or their naivety, their exaggerated and narcissistic elitism. (op. cit., pp.132- 133, my translation) […] There is a very broad consensus among mathematicians, pure or applied, on the idea that there can be no authentic mathematics that is not demonstrative, that the hypothetico-deductive proof that characterises them ensures the certainty of its results. (op. cit., p. 223, my translation).

Let us return, one last time, to the situation of the bordered square. Lalina Coulange and Brigitte Grugeon-Allys (2008) mention in their analysis algebra as a modelling tool and ask: "In the end, in the absence of an explicit link with the modelling problematic of the "bordered square" situation, one can wonder if there is a difference between the work done by the pupils during this session and the accomplishment of simple and isolated tasks around factoring." My hypothesis is that the observed modelling praxeologies are not the consequence of a formal relationship to algebra. It is a tenuous relationship to modelling that induces a formal relationship to algebra. It is therefore the relation to modelling that we will now consider.

3. Epistemological work and teaching modelling

Problem solving and inquiry-based education have become structuring elements of school curricula under the impulse of international assessments such as TIMSS or PISA (Barquero et al., 2018). Since 2015 in France, mathematics curricula from primary schools to the end of secondary school (children aged 6 to 18) declare to contribute to the "development of six major skills: research, modelling, representation, calculation, reasoning and communication". A paradigm shift is at work. It is about teaching mathematics to solve certain types of problems but also about teaching modelling as a process where the steps to be taken are as important as the final result to be achieved.

I take as a praxeological reference model of the modelling process the one developed by Yves Chevallard (1989). He identifies three interacting steps: definition of the system to be studied (which may be a mathematical or extra-mathematical situation); construction of the mathematical model to study the system, i.e. linking of the variables of the system; mathematical work in the model. The process is cyclical because the mathematical work in the model may lead to re-evaluate the relevance of the elements chosen to define the system. Moreover, a reversibility of the modelling relationship is always possible: "The relationship of the system to the model can indeed be reversed; the system can appear, in reverse, as a model of its model" (op. cit., p. 56).

In a teaching situation, the mathematical knowledge at stake is commonly revealed through the teacher's discourse, which updates, describes, explains, justifies, questions and validates the knowledge aimed for. A previous work (Wozniak, 2012) has revealed the problem of the teachers' lack of words and concepts to teach modelling processes. This led me to define three kinds of mathematical praxeologies in teaching situations: *mute*, *weak* and *strong praxeologies*.

A mute praxeology is one with its *praxis* component visible only: no explicit discourse to describe techniques used to carry out the types of tasks. In a weak praxeology, the *logos* component is visible with a limited discourse for the description of the technique based an incomplete formulation. Finally, a strong praxeology links its two explicitly components, the *praxis* and the *logos*, dialectically.

My observations (Wozniak, 2012), like those on the bordered square, show that, in French primary and secondary schools, modelling praxeologies are mute praxeologies revealing missing components in the teachers 'praxeological equipment.

The study I am now considering is part of a research programme on the identification of didactic determinants that weigh on the teacher's epistemological work. Considering a didactic institution, an object of knowledge O_k and the associated mathematical praxeologies, four cases can occur (table 1), depending on whether these praxeologies are active or not for a subject and whether O_k is in the institution's curriculum.

	Inactive mathematical praxeologies associated with O_k	Active mathematical praxeologies associated with O_k
O_k outside the curriculum	(1)	(2)
O_k inside the curriculum	(4)	(3)

Table 1. Mathematical praxeologies associated with *O* and curriculum.

The subject considered in the table 1 may be a student or a teacher. Taking as a subject a schoolteacher, if O_k is the normal distribution[5], we are in case 1 because he or she does not necessarily know what it is and does not have to teach it; if O_k is the probability of an event[6], we are in case 2 because he or she knows what it is and O_k is not a part of the curriculum at primary school; if *Ok* is the addition of integers, we are in case 3. Case 4 occurs whenever a new object of knowledge is introduced into the curriculum that the teachers have not studied in their personal curriculum. As with the recent introduction of basic algorithms and programming in the French curriculum.

In order to evaluate the weight of the curriculum on the teachers' epistemological work, a female schoolteacher and two female secondary

[5] In France, schoolteachers have a non-scientific background and the normal distribution is in the science curriculum of secondary education.

[6] The calculation of the probability of an event is part of the curriculum for pre-service schoolteachers.

school teachers were asked to have their students study the same modelling problem.

The observation of practices - without intervention or indication of how to conduct the study - was completed by interviews before and after the teaching sessions. The chosen problem is the cake box problem (Chappaz, Michon, 2003) often used in pre-service teachers (Barquero, Bosch, Wozniak, 2019). Boxes are built from the folding of a sheet of paper (Figure 3) and the relationship between the dimensions of the sheet and the dimensions of the box must be determined.

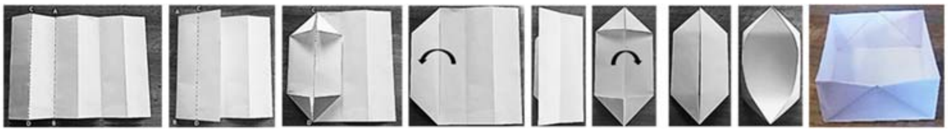


Figure 3. Folding to build a box.

If we consider algebra as an object of knowledge, the first three cases in Table 1 are represented: P1 teaches at the end of primary school and her personal relationship to algebra is old and inactive (case 1), P2 is a at the beginning of secondary school, her relationship to algebra is active (case 2) and P3 teaches mathematics after the introduction of algebra (case 3). There is not enough space to present the details of the study (see Wozniak, Cattoën (submitted) for a detailed analysis). Nevertheless, the main results can be summarised as follows.

All three teachers expressed in the interviews their difficulties in clarifying what the skill "modelling" covers and its relation with the proposed activity: "it's super vague" (P2), "I always had trouble" (P3) while P1 confuses modelling and model making: "reproducing something on a smaller scale".

None of the teachers perceived that, depending on the orientation of the sheet, two boxes of different sizes could be made with the same rectangular sheet. However, in the classrooms, both types of folding were present, generating confusion and misunderstanding without the teachers being able to identify their origin. The lack of analysis of the type of folding short-circuits the necessary work of defining the system in the modelling process. This work could have been done by providing sheets of different formats. The work that makes the system explicit is missing. There is no comparison of techniques to validate the built models. The modelling process is never made explicit, and mute modelling praxeologies have been observed.

Generalization is not an aim even at the level where algebra is taught. The secondary school teachers (P2 and P3) consider that a question is missing that explicitly asks: "Make a conjecture, what is the relationship between the dimensions of the sheet and the dimensions of the box?" Thus, the teachers'relation to the teaching of mathematics is revealed: the interest of making the pupils perceive the reason for the existence of knowledge through their emergence during problem solving is not envisaged. Algebra as a tool for modelling by generalising the studied cases could not be perceived by the pupils, it must necessarily be explicitly requested by the teacher.

The teacher's epistemological work has three dimensions. The unidentification of two solutions according to the orientation of the sheet during the folding is part of the mathematical dimension of this work and has an impact on its didactic dimension by not allowing to anticipate possible difficulties or misunderstandings and by not favouring the work of defining the system in the modelling process. The epistemological dimension articulates modelling as mathematical activity and modelling as a teaching object. To solve the problem, a mathematical model is created but if modelling is not identified as a mathematical object, it cannot be taught as a process. If it is a tool that works like a black box, it cannot be an object of study. Thus, the epistemological dimension is revealed by a relation to modelling and to school mathematical activity through the discourses and practices observed. As in Wozniak (2012), it is about teaching solutions rather than teaching ways to construct them.

These observations at different levels of teaching aimed to identify how the curriculum influences the teacher's choices in leading the study of this type of problem. Even if the teachers develop discourses in accordance with the curriculum of the level where they intervene, common practices were observed. Because they have the same difficulties, I hypothesise that the lack of mathematical and didactic knowledge about the modelling process is a more important didactic determinant than the curricula.

Finally, I will note that after two years of the global COVID pandemic, we all have in mind the way epidemiological models have been disseminated and understood by governments, journalists, and the population. The teachers' relationship to modelling is part of a relation to mathematical models at the level of society, if not civilisation, in the scales of didactic codeterminacy (Chevallard, 2002).

4. Conclusion

Teachers' epistemic ethos is the set of institutional relationships to the objects of knowledge in the position of teacher. It is made up of the praxeologies associated with the teachers' epistemological work based on a theory - implicit or explicit - of the knowledge to be taught and a theory of the learning of this knowledge within a given institution at a given time. Uncovering this epistemic ethos means identifying what is shared without prejudging what is shared, it means recognising what makes the identity of mathematics teachers. This makes it possible to understand why teachers do what they do and to reveal their praxeological needs for teaching. To conclude, I will use an analogy that I have presented elsewhere (Wozniak, 2019)

> When a doctor auscultates a patient, he recognises and treats the disease regardless of the subject and regardless of the place where the auscultation is carried out. It is not a particular individual he is examining, but symptoms he is looking for. If a "good" doctor takes care of the person with her singularity - which can lead to personalising a treatment - he is first of all an efficient doctor because he recognises and treats the disease. To carry the metaphor further, I would say that if a "good" teacher addresses a singular student by adapting his relationship to him - considering as much as possible his affect or his social, educational, family environment, etc. - he is first of all an effective teacher because he knows the conditions for establishing a new relation to the objects of knowledge. It is in this sense that it is reasonable to think that a professionalisation of (mathematics) teachers requires a depersonalisation and a decontextualization of their practices.

Because progress can only be made collectively, identifying teachers' epistemic ethos seems to be a fruitful project.

References

Artigue, M. (1986) L'étude de la dynamique d'une situation de classe. Une approche de la reproductibilité. *Recherches en Didactique des mathématiques*, 7 (1), 5-62.

Barquero, B., Bosch, M., Wozniak, F. (2019). Modelling praxeologies in teacher education: the cake box. In U.T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Eleventh Congress of the European Society for Research in Mathematics Education*, Utrecht University, Utrecht, Netherlands. https://hal.archives-ouvertes.fr/hal-02408705

Barquero, B., Florensa, I., Jessen, B., Lucas, C., Wozniak, F. (2018). The external transposition of inquiry in mathematics education: impact on curriculum in different countries. In *ICMI Studies 24. School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities*. University of Tsukuba, Japan. (pp. 189-197).

Bezout, É. (1821). Traité d'arithmétique à l'usage de la marine et de l'artillerie. Paris : Librairie pour les sciences. 9e édition.

Caillot, M. (2014). Les rapports aux savoirs des élèves et des professeurs. In M.-C., Bernard, A., Savard, C., Beaucher (dir.). *Le rapport aux savoirs : une clé pour analyser les épistémologies enseignantes et les pratiques de classe* (pp. 7-18). Québec : Livres en ligne du CRIRES.

Champy, F. (2009). *La sociologie des professions*. Paris : PUF.

Chappaz, J., Michon, F. (2003). La boîte du pâtissier. *Grand N*, 72, 19-32.

Chevallard, Y. (1985). *La transposition didactique* (1991). Grenoble : La Pensée Sauvage

Chevallard, Y. (1989) Le passage de l'arithmétique à l'algèbre dans l'enseignement des mathématiques au collège. Deuxième partie. Perspectives curriculaires : la notion de modélisation. *Petit x*, 19, 43-72.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique : perspectives apportées par une approche anthropologique. *Recherches en didactique des mathémati*ques, 12(1), 73-112.

Chevallard, Y. (2002). Organiser l'étude. 3. Écologie & régulation. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Éds), *Actes de la XI^{e} école d'été de didactique des mathématiques* (pp. 41-56). Grenoble : La Pensée Sauvage.

Chevallard, Y. (2003) Approche anthropologique du rapport au savoir et didactique des mathématiques, In S. Maury S. & M. Caillot (Éds), *Rapport au savoir et didactiques* (pp. 81-104). Paris : Éditions Fabert.

Coppé, S., Grugeon-Allys, B, Pilet, J. (2016). Conditions pour diffuser des situations issues de la recherche en didactique des mathématiques : l'exemple du carré bordé, *Petit x*, 102, 57-80.

Coulange, L., Grugeon-Allys, B. (2008) Pratiques enseignantes et transmission de situations d'enseignement en algèbre, Petit x, 78, 5-23.

Fusulier, B. (2011). Le concept d'ethos. *Recherches sociologiques et anthropologiques*, 42(1), 97-109.

Steup, M. (2018). Epistemology, *The Stanford Encyclopedia of Philosophy* (Summer 2018 Edition), Edward N. Zalta (ed.). https://plato.stanford.edu/archives/sum2018/entries/epistemology/

Wozniak, F. (2012). Des professeurs des écoles face à un problème de modélisation : une question d'équipement praxéologique. Recherches en Didactique des Mathématiques, 32(1), 7-55.

Wozniak, F. (2019). Fondements du travail épistémologique du professeur. *Recherches en didactique des mathématiques*, 39(1), 15–50.

Wozniak, F., Cattoën, M.-O. (soumis). L'algèbre comme outil de modélisation : étude comparée aux cycles 3 et 4 en France. *Revue québécoise de didactique des mathématiques*.

Zarca, B. (2012). *L'univers des mathématiciens. L'éthos professionnel des plus rigoureux des scientifiques*. Rennes: Presses universitaires de Rennes.

Using ATD to enable strategic decisions in institutions: the case of the infrastructures at the Haute-Ecole Pédagogique du Valais

Julie Jovignot

Haute-Ecole Pédagogique du Valais, Suisse; julie.jovignot@hepvs.ch

Abstract. *This paper aims at discussing an institutional strategic tool built from the framework of anthropological theory of the didactic. We explain how, based on the concepts of relationships (personal, institutional) but also thanks to the co-determination scale, we build a tool dedicated to the management of the Haute-Ecole Pédagogique du Valais. This tool will not only allow decisions to be taken for the development of new infrastructures but also to orient the whole institutional strategy. This paper describes the methodology for the construction and analysis of this tool.*

Keywords: anthropological theory of the didactic, institutional strategy, management, institutional culture, infrastructures.

Résumé *Ce papier a pour objectif la mise en discussion d'un outil institutionnel stratégique construit à partir du cadre de la théorie anthropologique du didactique. Nous expliquons comment, à partir des concepts de rapports (personnels, institutionnels) mais aussi grâce à l'échelle de co-détermination, nous construisons un outil dédié à la direction de la Haute-Ecole Pédagogique du Valais. Cet outil permettra non seulement de prendre des décisions pour le développement de nouvelles infrastructures mais également d'orienter l'ensemble de la stratégie institutionnelle. Cette communication décrit la méthode de construction et d'analyse de cet outil.*

Mots-clefs : théorie anthropologique du didactique, stratégie institutionnelle, management, culture institutionnelle, infrastructures.

Resumen. *Este trabajo tiene como objetivo discutir una herramienta estratégica institucional construida desde el marco de la teoría antropológica didáctica. Explicamos cómo, basándonos en los conceptos de relaciones (personales, institucionales) pero también gracias a la escala de codeterminación, construimos una herramienta dedicada a la gestión de la Haute-Ecole Pédagogique du Valais. Esta herramienta no sólo permitirá tomar decisiones para el desarrollo de nuevas infraestructuras, sino también orientar toda la estrategia institucional. Este documento describe la metodología para la construcción y el análisis de esta herramienta*

Palabras clave: teoría antropológica de la didáctica, estrategia institucional, gestión, cultura institucional, infraestructuras.

Introduction

The Haute-Ecole Pédagogique du Valais (HEP-VS), created in 2000 on the basis of the normal school, is in a pivotal phase of its history. Indeed, it receives more and more students who wish to train as teachers, whether in primary, secondary or specialised education. At the moment, one element that strongly hinders its development is the size and functionality of the premises. For example, there is no room capable of accommodating entire classes of primary school students. There is also no working space for students. Furthermore, there are not enough rooms for all students to have classes at the same time, which has implications for timetables and also for pedagogical and didactic aspects.

For example, four years ago, teachers were asked to adopt a hybrid teaching strategy, in particular to make up for the lack of space. Thus, we realise that institutional constraints such as infrastructure can have a significant impact on pedagogical and didactic aspects, forcing the redesign of curricula and teaching methods.

The cantonal authorities, aware of the need for new infrastructure for the HEP-VS, have started to implement an infrastructure project. However, the management of the HEP-VS, aware that physical choices can have a strategic impact on teaching and the quality of training in the long term, decided that the HEP-VS should be able to propose its own vision of infrastructures. To this end, it mandated me, as a project manager, to help evaluate the infrastructure needs. This mandate leads us to the following research question **Q_0**: *What strategy modelling tool should be built to model the infrastructure needs of a tertiary education institution?*

In order to provide some answers to this question, in the first part we will describe the theoretical frameworks that we believe could be relevant for the construction of this tool. These two parts will allow us, on the one hand, to give the methodology of data collection and, on the other hand, the methodology of data analysis. Finally, in the conclusion, we will show how this tool allows us to answer our research question.

Theoretical tools

A frame for life

Designer Crawford (2014) explains how she has enabled a paradigm shift in design by putting people at the centre: “the studio has challenged the strongly style-driven and entirely male-dominated architecture industry with a humanistic approach to desigm; an approach that has meant challenging the system of how buildings are built, as well as changing the design approach” (Crawford, 2014, p. 27). She further explains that

> Design is essentially a constructed reality. It needs to be built around the life that will be lived within (or with) it. Around real needs and activities. The designer must put human experience at the core of the design process, so that the result is a physical manifestation of human behaviour. (Crawford, 2014, p. 61)

Thus, in order to build a tool that would allow the modelling of infrastructure needs, following Crawford, we start from the principle that it is the users' experiences that must be taken into account. However, in a teaching and research institution such as the HEP-VS, this experience appears to be guided by the way in which the various actors of the institution live their functions, be it teaching, research or service provision. In short, our modelling tool should make it possible to analyse how the actors of the institution live in the institution and what prevents them or, on the contrary, what allows them to develop. In a way, we would like to model the overall ecology of the institution today and where it could be headed in order to deduce the infrastructure needs.

In Candy (2020), we interviewed university teachers about their personal and institutional relationships (Chevallard, 2003) to tertiary education and more specifically to the teaching of abstract algebra. Through these interviews, we were able to highlight elements of relationships that strongly guided teaching styles. We also highlighted that when professors were offered freedom from constraints, they proposed changes in the strategic decisions of their universities. For example, some professors proposed the grouping of modules, others better coordination between lecturers and TDs, or highlighted the constraints of professorial competitions on their teaching. In the same work, the use of the didactic co-determination scale (Chevallard, 2002) made it possible to model and analyse the constraints and conditions that had an influence on institutional relationships. Furthermore, Gueudet, Bosch, diSessa, Kwon and Verschaffel (2016) highlight the use of the didactic co-determination scale in work on transitions. We note that, although we are not aware of any such use, the results of this research could enable institutions to make strategic decisions to facilitate transitions (e.g. changing official curricula). Thus, these two theoretical tools seem to us to be suitable for modelling strategy within the HEP-VS and we describe them in more detail below.

Personal and institutional relationships

For Chevallard (2003), the personal relationship of an individual to an object is made up of all the interactions between this individual and this object. For example, in the personal relationship of a user to the infrastructure, we will find teaching or maintenance situations that have created needs, such as the need for a digital blackboard or a technical room for example. The different people evolving within the HEP-VS and even the different positions that can be occupied by the staff invite us to define the notion of institution and position. Chevallard defines the notion of institution as:

> un dispositif social « total », qui peut certes n'avoir qu'une extension très réduite dans l'espace social (il existe des « micro-institutions »), mais qui permet - et impose - à ses sujets, c'est à dire aux personnes x qui viennent y occuper les différentes positions p offertes dans [l'institution], la mise en jeu de manières de faire et de penser propres. (Chevallard, 2003, p. 2)

It should be noted that the HEP-VS can, in this sense, be considered as an institution. The staff can occupy several positions which derive from the law concerning the Haute-Ecole Pédagogique du Valais. These are as follows: teaching position, research position, administrative position, technical position, intermediate body position, pedagogical animator position, internship supervisor position, student position and management position. Furthermore, due to the specificity of the HEP-VS specifications, it is possible that the same staff may belong to several positions and we will therefore have to take this into account in the following.

We will therefore seek to assess the personal relationships of HEP-VS users to the infrastructure. This should enable us to model the institutional relationships in the various positions identified. It is these institutional relationships that will allow us, in particular, to determine the institution's needs in terms of infrastructure.

However, if it is the global relationship to infrastructures that is to be studied, it is obvious that the study of relationships in different positions in a more local way must be studied. Indeed, for example, if the majority of professors have a monumentalist relationship (Chevallard, 2015) to teaching, they may prefer amphitheatres to teach their courses, unlike professors who would like to teach in the paradigm of questioning the world and would have other needs, particularly for research work or sharing. Thus, questioning the relationship to infrastructures will necessarily require questioning relationships to other objects in connection with the institutional role of each. Furthermore, this highlights the interdependence of constraints at different levels (e.g. institutional or pedagogical) and therefore the need to use the scale of co-determination to enable the analysis of these different relationships and the impact on the infrastructures to be developed.

Co-determination scale

In order to use the concept of rapport as a tool for highlighting professional and training needs, we need to describe the constraints and conditions that weigh on institutional actors. Thus, we will use the co-determination scale of didactics (Chevallard, 2002) which will allow us to characterise the level of constraints and conditions and thus to propose solutions adapted to the ecology of the institution.

Figure 1 - Levels of the co-determination scale (Chevallard, 2002) in play in our study

It should also allow us to identify the constraints that are linked to the infrastructure and to analyse the impact of their changes at different levels of granularity of scale. Such a study can allow management to analyse the impact of infrastructure decisions on the overall teaching, research and service delivery strategy of the HEP-VS and vice versa.

Methodology for designing the questionnaire

In order to collect the personal relationships of the users of the infrastructures and to be able to distinguish elements of the institutional relationship to the infrastructures, we opted for a questionnaire intended for all these users.

The overall methodology for developing this questionnaire is as follows: firstly, we developed a first version based on the different positions that staff may occupy. In Candy (2020), we noted that asking questions in which respondents are asked to free themselves from the constraints that weigh on them makes it possible, on the one hand, to highlight these constraints and, on the other hand, to obtain elements of their personal relationships to the objects questioned.

We are not going to directly question the relationship of users to infrastructures. Indeed, we assume that this material constraint, which is placed at the level of the institution in the scale of co-determination, has an impact on the level of pedagogy and perhaps even of the discipline. Indeed, one could imagine that certain disciplinary organisations are not teachable due to lack of material. Yet, in the context of creating new infrastructures, we are given permission to let the lower levels of discipline and pedagogy guide the creation of an Institution-level constraint: infrastructures. Thus, following Crawford (2014) we no longer place the building at the centre of our thinking but the human activity that takes place within it. These different activities can be modelled through existing institutional positions. Therefore, in the questionnaire, we will seek to interrogate the relationships of users in the different positions they may occupy to objects constituting their institutional position.

The field of research has a special status at the HEP-VS since, although research was already taking place at the HEP-VS, the research and development unit has recently undergone a complete overhaul to meet the requirements of a tertiary institution. The processes associated with this reorganisation do not yet appear to have stabilised. Thus, research-related issues are more related to a future institutional relationship than to an already established institutional relationship.

In the following section, we present the first version of the questionnaire. Then the next part will consist of the analysis of the results of the first consultations. Finally, in the last part, we will present the final version of the questionnaires.

First version of the questionnaire

A first version of the questionnaire, intended for faculty, was developed taking into account the principles outlined above. An English translation of this first version can be found in Figure 2.

Personal relationship to working conditions

1. If you were completely free to choose your working arrangements what would they be? Detail here according to the nature of your activities (teaching, research, administrative and strategic).
2. What currently prevents you from working in these ways?
3. If you were completely free of all constraints (institutional, family, etc.) how would you organise your working time?
4. What prevents you from organising your work in this way now?
5. If you were completely free of all constraints (institutional, family, etc.) where would you work?
6. What prevents you from working in this way at present?

Personal relationship to teaching

7. What does teaching at the Haute Ecole Pédagogique mean to you?
8. If you were completely free of all constraints, how would you organise your teaching and, more broadly, the students' curriculum?
9. What prevents you from organising yourself in this way at present?
10. If you are appointed to the teaching strategy at the HEP, what are the first two measures you take?
11. What would prevent you from taking these steps now?

Personal relationship to student support

12. In your opinion, what are the ideal conditions to encourage students' involvement in their training?
13. What is currently hindering this engagement?
14. What would you need to encourage exchanges with students?

Personal relationship to research

15. If you are a researcher. What is your vision of your mission as a researcher at HEP in 10 years (teamwork, service to the city, service provision, research innovation, ...)?
16. If you are a researcher. What would help you to achieve this vision?
17. If you are a researcher. On the contrary, what would hinder it?
18. What will a scientific conference look like in 10 years?

Assessment of the infrastructure

19. What aspect(s) of the current infrastructure do you think should be maintained?
20. What aspect(s) of the current infrastructure do you think should be improved as a priority?

Figure 2 – first version of questionnaire

Passation des questionnaires tests

The faculty version of the questionnaire was sent to three members of the faculty in order to test whether the answers to the questions could provide elements for modelling the functioning of the institution and the relationship to the infrastructure. Of these three professors, one is a German speaker. The analysis of the answers allowed the creation of a new questionnaire which was submitted to the board of directors and then to the delegates responsible for sustainable development within the institution.

The analysis of all these data leads us to make several modifications to the questionnaire, which are detailed below:

- the questions on constraints are grouped as much as possible in the initial questions in order to reduce the "size" of the questionnaire so as not to discourage respondents.

- Question 7, which was intended to elicit the personal relationship to teaching, does not provide any more information than the previous answers and is therefore deleted.

- Question 17. on the future of the research congresses does not seem to be relevant for information on the structure. One respondent answered "?", another spoke of a hybrid conference and the third of a plenary with sub-groups. These succinct answers do not seem to provide any additional information and question 17 was therefore deleted to shorten the questionnaire.

- If the aim is to model the institutional relationship to work and structures, it seems interesting to ask the respondents about their site of origin (the HEP-VS is a bilingual institution with a German-speaking and a French-speaking site), their age group and their seniority in the institution. This will make it possible, in particular, to group together the different institutional relationships and to link them, if necessary, with the history of the HEP-VS. Indeed, the first staff were recruited from the field and trained in the former teacher training college, which did not have a tertiary or academic culture. Since the foundation of the HEP-VS, the staff recruited is more and more academically qualified (at least a Master's degree) and the HEP-VS meets the criteria of a Haute-Ecole, a tertiary training institute. Thus, we hypothesise that different institutional relationships can be obtained which coexist within the institution.

- After consultation with the people in charge of sustainable development at the HEP-VS, it was decided that questions on sustainable development would not be directly indicated, but that they would participate in the analysis of the results from a sustainable development perspective.

- Finally, in view of the complexity of the staff's specifications, it was decided that there would be only one questionnaire and that staff would be able to choose the questions they answer according to their functions.

Final version of the questionnaires

The final version of the questionnaire for employees is in the form of a branching questionnaire, with employees only seeing the questions that relate to working conditions, student support and student profile.

Personal Report to Teacher Education

1. If you were completely free of all constraints, how would you organise your activity as a trainer and, more generally, teacher training? What prevents you from doing this at the moment?
2. If you are appointed as the person in charge of teacher training at the HEP-VS, what are the first two measures you would take ? What would prevent you from taking these measures at the moment?

Personal relationship to student support

3. In your opinion, what are the ideal conditions to encourage the involvement of student teachers? What is currently hindering this engagement?

Personal relationship to research

4. What is your vision of research at the HEP-VS? What would favour the realisation of this vision ? On the contrary, what would hinder it ?

Personal report to the administration

5. If you were completely free of all constraints, how would you organise your administrative work? What prevents you from doing so at present?
6. If you are appointed as strategic manager for the administration at the HEP-VS, what are the first two measures you take ? What would prevent you from taking these steps at the moment?

Personal relationship to technology

7. If you were completely free of all constraints, how would you organise your work related to technology? What prevents you from doing this at the moment?
8. If you are appointed as the strategic manager for technology at HEP-VS, what are the first two steps you take? What would prevent you from taking these measures at present?

Personal report to the pedagogical animation

9. If you were completely free of all constraints, how would you organise your work related to pedagogical animation? What prevents you from doing this at the moment?
10. If you are appointed as strategic leader for pedagogical animation at the HEP-VS, what are the first two measures you would take ? What would prevent you from taking these measures at present?

Personal relationship to services

11. If you were completely free of all constraints, how would you organise your work relating to services? What prevents you from doing so at present?
12. If you are appointed as the strategic manager for services at the HEP-VS, what are the first two steps you would take? What would prevent you from taking these measures at the moment?

Personal relationship to working conditions

13. If you were completely free to choose your working conditions, what would they be? Detail here according to your activities. What prevents you at present from working according to these arrangements?
14. If you were completely free of all constraints how would you organise your working time? What prevents you from organising your work in this way at the moment?

15. If you were completely free of all constraints in which location(s) would you work? What prevents you from working in this way at present?

Assessment of the infrastructure

16. What aspect(s) of the current infrastructure do you think should be maintained?
17. What aspect(s) of the current infrastructure do you think should be improved as a priority?

Profile

18. Age group (25-/30-40/40-50/50-60/60+)
19. Length of time in the institution (-5 years / 5-10 years / 10-20 years/+20 years)
20. You are (educational facilitator, lecturer/teacher, administrative staff, technical staff, tutor, proxy, administrative staff)

Figure 3 – questionnaire for HEP-VS staff

Personal relationship to teaching

1. When you are appointed as the strategic leader of teacher education at HEP-VS, what are the first two steps you take? What would prevent you from taking these steps at the moment?
2. If you had to organise the training yourself, what working methods would you choose (synchronous or asynchronous distance learning, face-to-face, lectures, tutorials, in-school training, etc.)?

Personal relationship to student support

3. What do you consider to be the ideal conditions to encourage your commitment to your training? What is currently hindering this commitment?

Report on the infrastructure

4. What aspect(s) of the current infrastructure do you think should be maintained?
5. What aspect(s) of the current infrastructure do you think should be improved as a priority?

Personal relationship to working conditions

6. If you were completely free of all constraints, how would you organise your training time? What prevents you from doing so at present?
7. If you were completely free of all constraints in which location(s) would you work for training? What prevents you from working in this way at the moment?

Profile

8. Field of study (BP/FS/MAES/PCEO/PIRACEF/CAS)
9. Employed (employed in teaching (% to be specified)/not employed/employed in something else (% to be specified))
10. Age range (25-/30-40/40-50/50-60/60+)

Figure 4 – questionnaire for the student body

Methodology for analysing the questionnaire

The sample concerned

Infrastructure choices will impact the life and development of the institution for at least one or two decades. Thus, it is advisable to rely on the expertise of the institution's subjects in order to assess not only current needs, but also future needs.

In agreement with the management, it was decided that all staff would be consulted. This represents, with the educational facilitators, about one hundred people. As for the students and field workers, considering the chosen methodology, it did not seem possible to process all the feedback. It was therefore decided that we would interview students and fieldworkers deemed representative by the course leaders. Most of the time, these are the members of the student committees of the different courses or the elected representatives of the field actors.

Data analysis

As Mayring (2014, p. 39) explains “content analysis is not a standardized instrument that always remains the same; it must be fitted to suit the particular object or material in question and constructed especially for the issue at hand”.

We have chosen to analyse the questionnaires using an analysis grid. The choice of themes and categories in this grid is, in the first place, guided by the results of an a priori analysis of this questionnaire. Mayring (2014) explains that while the use of categories fixes what will be extracted from the data, it is very useful for the exploitation and comparison of the data. Furthermore, the grid will evolve during the first analyses as categories or coding may not have been anticipated by the a priori analysis.

For our analysis of the questionnaires, we will therefore use what Mayring (2014) calls a 'parallel' method as it employs both a deductive and an inductive method: some themes, some categories are fixed in advance and are based on the theoretical frameworks while others may emerge when reading the responses to questionnaires.

A priori analysis and methodology of the construction of the grid

In this section, we present the first analysis grid, which will be completed inductively when processing the data. In Table 1, the headings are in bold, with IC standing for "inductive category".

Subcategory	Theme	Categories
	Prescribed curriculum	
Links with the professional field	Professional status of students	Training in employment
		Training without a job
		Hybrid training
	Duration of internships	IC

	Position of the practice training	
	Modalities of evaluation of the practice	IC
Elements of the external didactic transposition process	Modalities of the transposition process	Harmonised process with a check on the consistency of the training.
		Process in charge of the module leader.
		The process is the responsibility of the disciplinary teams.
	Presence of mandatory institutional elements in the product of external didactic transposition	IC
Elements related to the partition in modules	Nature of the modules	Disciplinary modules
		Interdisciplinary modules
	Changes in the distribution of ECTS credits	IC
	Changes in the division into modules	IC
	Subjects to be taught	IC
Internal didactic transposition and the process of internal didactic transposition		
Evaluation of modules		Continuous monitoring
		MCQ
		Oral
		Tabletop examination
Teaching methods	Distribution of hours	Separation of lectures and practical courses
		Lectures only
		Course and practice not distinguished
	Teaching aids	Role play
		Training based on video or script analysis
		Training with practice analysis
	Physical modalities	Presential
		Remote
		Hybrid
		MOOC

	The place of research in the IDT and its process	IC
	The place of epistemology in IDT and its process	IC
	Didactic devices	Study and research course
		Team Academy
Educational and institutional arrangements		
Aspects techniques	Equipment of the premises	IC
	Digital infrastructure	IC
	Student equipment	IC
Organisation of semesters	Timetable	IC
	Institutional exchange	IC
Elements that concern infrastructure directly		
Physical aspects		IC
Digital aspects		IC

Table 1 - A priori analysis grid for personal relationship to teaching

We proceed in the same way for all parts of the questionnaire. We will analyse our data using this grid, which will be completed when processing the data according to the parallel method presented by Mayring (2014). This grid also allows us to highlight the links between the questionnaire and aspects related to infrastructure or strategy in general.

Conclusion

We would like to point out that one part of the grid directly concerns infrastructure. Indeed, the "assesment the infrastructure" part of this questionnaire will lead people to point directly to infrastructure elements. However, we note that we do not refer directly to infrastructure in the other parts, so we will obtain a broader picture of the relationships in different positions to their work objects of the staff. There are two reasons for this methodological choice. The first, as we have seen, is to place the users of the premises and their needs at the centre of the design and not the other way around (Crawford, 2014). The second reason is strategic: who better than the various actors in the institution, thanks to their expertise, to determine what the institution of tomorrow might become? Thus, the analysis of these responses should allow us to model the different institutional relationships to objects and possible future institutional relationships. In this way, the tool constructed in this paper can provide a model for the management, which can use it to orientate its strategy.

Moreover, the choice of this strategy modelling tool seems perfectly in line with an institutional culture of participatory construction. Indeed, in recent years the HEP-VS has put its strengths in building participation within its institution. Thus, this type of modelling tool, in educational institutions, could allow a participatory development of the strategy. During our meeting with the management, one of the risk factors identified was that some respondents to the questionnaire may feel aggrieved that their views have not been taken into account. However, the analysis methodology, which is based on the difference between personal and institutional relationships, makes it possible to qualify this individual character in participation in favour of an institutional character. Thus, it mitigates one of the major risk factors of participatory institutional policies.

As such, this tool seems to us to be an interesting strategic tool based on theoretical frameworks appropriate for educational institutions. However, we see limitations to its use. First of all, this tool is extremely costly in terms of processing and analysis. Thus, it does not seem possible to use it regularly. However, in the context of projects of the importance (strategic, financial, sustainable) of this one, the cost of the analysis seems appropriate. It would be advisable to find an adaptation of the tool allowing for continuous evaluation. On the other hand, we are a member of the institution and we have had many functions that have led us to model it (head of domains in the institutional accreditation, head of practical training, project manager). Thus, the categories, whether inductive or deductive, are refined by direct knowledge of the institution, which biases the effectiveness of the tool. In order to evaluate this tool in another framework, it will be necessary to test and adapt it in other institutions, which will be one of our projects if the results of this analysis are satisfactory from a strategic point of view.

References

Candy, J. (2020). Etude de la transposition didactique du concept d'idéal : écologie des savoirs et problématique de l'entrée dans la pensée structuralise en France et en Suisse romande [Thèse de doctorat]. Université de Montpellier, Montpellier.

Chevallard, Y. (2002). Organiser l'étude. 3. Ecologie & régulation. In J.-L. e. a. Dorier (Ed.), Actes de la XIème Ecole d'Eté de Didactique des Mathématiques (pp. pp. 41–56). Grenoble : La Pensée Sauvage.

Chevallard, Y. (2003). Approche anthropologique du rapport au savoir et didactique des mathématiques. In Rapport au savoir et didactiques (pp. 81–104). Université René Descartes Paris 5 : S. Maury & M. Caillot.

Chevallard, Y. (2015). Teaching Mathematics in Tomorrow's Society: A Case for an Oncoming Counter Paradigm. In S. Cho (Ed.), The Proceedings of the 12th International Congress on Mathematical Education (pp. 173–187). Cham : Springer International Publishing.

Crawford (2014) Crawford, I. (2014) A frame for life. The designs of StudioIlse. Rizzoli, New-York.

Gueudet, G., Bosch, M., diSessa, A. A., Kwon, O. N., & Verschaffel, L. (2016). Transitions in Mathematics Education. Hamburg : Springer International Publishing. Consulté sur http://link.springer.com/10.1007/978-3-319-31622-2 doi:10.1007/978-3-319-31622-2.

Mayring (2014) Qualitative Content Analysis : theoretical foundation, basic procedures and software solution. Klagenfurt. Consulté sur https://nbnresolving.org/urn:nbn:de:0168-ssoar-395173.

A practical approach to an SRP in university statistics

Laura Fernández-Ruano, Vanessa Serrano, Judith Báguena, Jordi Cuadros and Meritxell Sáez

IQS, Universitat Ramon Llull, Spain; laura.fernandez@iqs.url.edu, vanessa.serrano@iqs.url.edu, judith.baguena@iqs.url.edu, jordi.cuadros@iqs.url.edu, meritxell.saez@iqs.url.edu

An SRP approach was implemented in two university statistics courses. Its implementation was analysed through surveys and interviews held with the students, and the personal perception of the instructor. The analysis of the results indicates in what aspects the strategy followed, with a marginal SRP run in parallel to the lectures, promoted a move towards the paradigm of questioning the world. The experience was well-received by the students and positively affected the teacher's practice. It also helped identify institutional restrictions such as the lack of time for the development of the activity.

Keywords: Statistics education, higher education, study and research paths, anthropological theory of the didactic.

Introduction

In statistics, data contextualisation is essential. The mechanistic teaching model proposes class examples and problems that include real cases related to the context of the student's degree programme (chemistry, chemical engineering, and industrial engineering in the case presented). When following that model, data are provided after a short problem formulation, and the students' resolution is usually produced without considering the context provided. The goal is to apply the knowledge acquired by the students to realistic situations so as to demonstrate the use of the concepts taught (Fonseca et al, 2020), but a breach with the real world remains.

Project-based learning (PBL) goes a step further, and tries to involve students in the process of statistical investigation (Santana & Gómez-Blancarte, 2019). The idea is to engage them by offering them the opportunity to answer questions of their choice through an independent investigation based on existing data (Dierker et al., 2018). Nevertheless, a recent critical review (Markulin, Bosch & Florensa, 2021) of PBL in statistics shows that in numerous implementations there are no considerations about the selection of knowledge, skills, and dispositions that determine the teaching and learning process. PBL is used as a means towards achieving a goal: learning a specific set of contents using a project.

The anthropological theory of the didactic (ATD) (Chevallard, 1992) provides a theoretical framework and proposes study and research paths (SRPs) to overcome the monumentalisation (Chevallard, 2006) characteristic of traditional education –including PBL– both at the cognitive and procedural levels (Boigues et al., 2013).

In an SRP, one starts from a generating question that is open both for the students and the instructors. The students, guided by the instructor, study that generating question proposing new derived questions, searching for answers in all resources available (the Internet, papers, manuscripts, books, etc.), and validating and exploiting the information and knowledge obtained in order to elaborate partial answers, until they get to a final resolution of the generating question (Winsløw et al., 2013). The instructor adopts a guiding role to a greater or lesser extent, depending on the students' progress in each particular situation. In this new paradigm, the students search, (with the instructor’s help), for new concepts as a resource to provide answers to the questions formulated, and not only because of their intrinsic interest.

The objective of using SRPs is to study the possibilities of starting to change the didactic paradigm in higher education from traditional lectures based on the “visiting monuments” (or contents) paradigm towards a new paradigm based on “questioning the world”. The main challenge is to determine the conditions that make it possible to develop SRPs in higher education subjects, and to analyse the restrictions encountered related to subject contents, university type, or educational organisation.

The experimentation here presented was performed in the subjects of statistics in the BSc degrees in Chemistry (DC), Chemical Engineering (DCE), and Industrial Technologies Engineering (DITE) in the IQS School of Engineering at Universitat Ramon Llull during academic year 20-21. It constitutes a first approximation to an SRP. The strategy followed by the instructor, given practical constraints, was to avoid changing the lectures and to develop the SRP in parallel. Feedback was obtained through surveys and interviews.

The statistics subject in DITE is taught in the second semester of the first year, while in DC and DEC it is taught in the second semester of the second year. Although both subjects correspond to 6 ECTS credits, the effective teaching hours are different. In the academic year considered, the DITE students had 32 hours of class with the entire group of students (a total of 56 students), and 4 hours in a split class (50-50% approx.). The DC and DEC students had 39 hours of class all together (41 students in total). The general syllabus is the same for the three degrees, as shown in the right-hand column of Figure 4. In the three degrees, the evaluation consists of a partial exam, the work performed throughout the SRP (pre-reports, a final report and a presentation), and a final exam. Figure 1 shows the competencies assessed in each case.

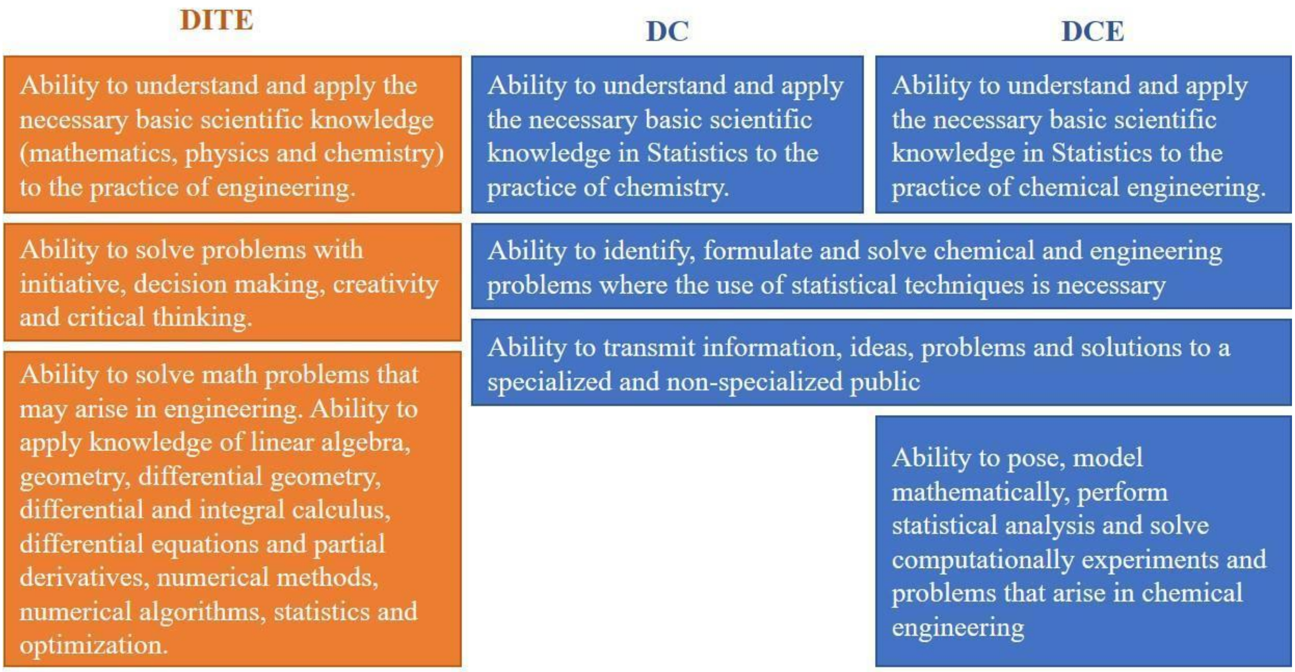


Figure 1: Assessed competencies in the statistics subjects (Source: IQS internal documents)

In this context, the aim of this work is to assess the feasibility of this approach and investigate its ecology.

Design of the SRP

A priori analysis

Since statistics is the science of data, searching or simulating, pre-processing, and filtering data are as important as the later computations performed to answer the question to be solved. When working on each topic separately, in an orderly and closed manner, the real meaning of statistics as a tool to find answers may be forgotten. Providing the students with a realistic open question and an appropriate context is therefore crucial.

Competency assessment is another challenging task. The only way to assess if the students are able to "identify, formulate and solve […] problems where the use of statistical techniques is necessary" (see Figure 1), is when they are the ones that identify the problem, collect data and formulate the appropriate statistical hypotheses.

Furthermore, students generally consider each subject as a closed box, and instructors usually assess their learning in the same manner. This is not a realistic situation because the different subjects are interconnected. In the particular case of statistics, its instrumental nature makes this even more evident and shows the necessity of working using a global perspective.

An SRP offers answers to all the above-mentioned issues because "it is a didactic mechanism [...] that is designed based on the seeking of answers to questions that, to be solved, require the construction of a more or less complex series of complete and articulated praxeologies" (Boigues et al, 2013).

Hence, the educational question that the lecturer proposed in order to address the above challenges is how to pose a problem that requires data gathering and processing, in order to answer an initial real open question. It should involve asking the appropriate questions and making use of the information acquired during theory classes, but also additional knowledge obtained independently.

Instead of proposing a radical change in the didactic contract (from traditional teaching to teaching exclusively by means of SRPs), the SRP is performed in parallel with traditional lectures as a first approach to the new contract.

Path description

The starting point was the framing question "What is the air quality like?". From here, each team proposed its own generating question to ensure they were genuinely interested in the investigation. The proposed generating question was supervised to ensure that it was appropriate for the original teaching purpose. The instructors made sure that the derived questions allowed an important part of the syllabus to be covered, and that the amount of work required corresponded to the subject's programme.

Figure 2 shows a brief description of the first stages of the SRP. It shows how the generating question (Q0, with examples of some questions proposed) produced other derived questions. Q1 made the students look for open-source data that were obtained in different formats that needed investigation. Once the data were downloaded, the students had to assess their usefulness with regard to Q0 by determining what variables the data contained (Q2), what they meant, and what data pre-processing was required (Q2,i). Since not all the information gathered was necessary, the students had to perform a selection (Q3,i).

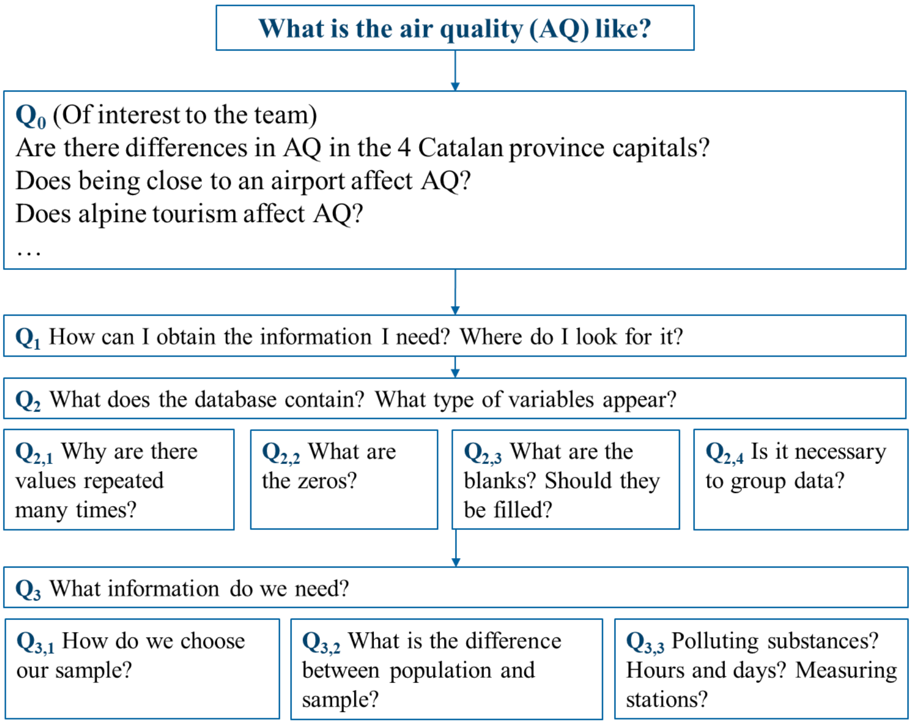


Figure 2: Beginning of the path

Class organisation

The IQS available platform, Microsoft Teams, was used for communication and follow-up of the SRP. All the students had access to a common workspace. In the General channel, there was a document ("Equipos proyecto.docx") with basic information on each team (members, used data, considered variables, and temporal framework). There was another channel for consultation meetings, and yet other channels corresponding to the different topics investigated for all the teams to collaborate. Moreover, each team had a private channel that included the team members and the instructor (see Figure 3).

The different teams were chosen by the students and had between 4 and 7 members. The 56 first-year students from DITE formed 10 teams and the 41 second-year students from DC/DEC formed 9 teams.

SRP organisation

Activities and tasks were recommended to the teams in order to guide their work. They handed in pre-reports that were used by the instructor to provide specific feedback to each team. The most frequent errors or issues were discussed during the class sessions.

The Teams channel provided for discussion was another source of information about the work they were performing. There was a fixed consultation day, although it was flexible around pre-report deadlines. The consultation meetings were performed in a common channel in order to include students from different teams. Additionally, students were asked to keep track of their work by using weekly logbooks.

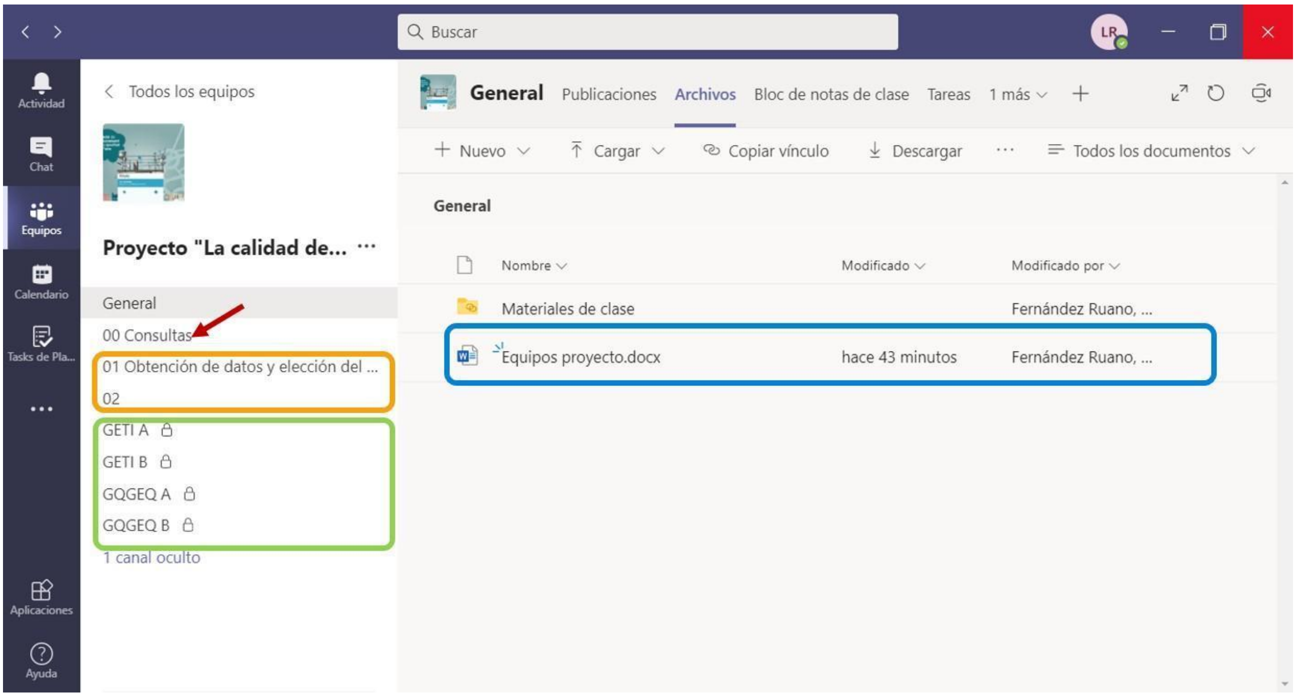


Figure 3: "Air quality" project workspace

If a team found the solution to a question (or proposed a good idea to study it) by themselves, the team members would present it to the whole group in the classroom. Otherwise, the instructor gave answers, or provided resources to facilitate the resolution of the problem.

In the theory classes, not all the information necessary for the project was given. Some examples were as follows: (1) sampling theory, necessary for data collection, was only covered superficially in class. The students decided how to perform it to ensure that the data were representative. It involved selecting the following according to the question investigated: polluting substances analysed, monitoring stations considered, and temporal framework of the measurements; (2) data pre-processing (repeated values, zeros, blanks, normalisation, baseline) was not explicitly discussed in class; (3) not previously introduced visualisations and analysis of asymmetric distributions.

Figure 4 shows an outline of the strategy implemented. In short, the objective was to offer mixed teaching: part of the information was provided to be visited (the student as a spectator) in the theory classes, while another part had to be acquired (the student questioning the world) during the SRP. To complete the study, they produced a final report and shared their results through a slideshow presentation.

A posteriori analysis

Once the course was completed, an analysis of this activity was carried out, taking into account the instructor's and the students' perspectives in order to reveal the restrictions of the strategy followed.

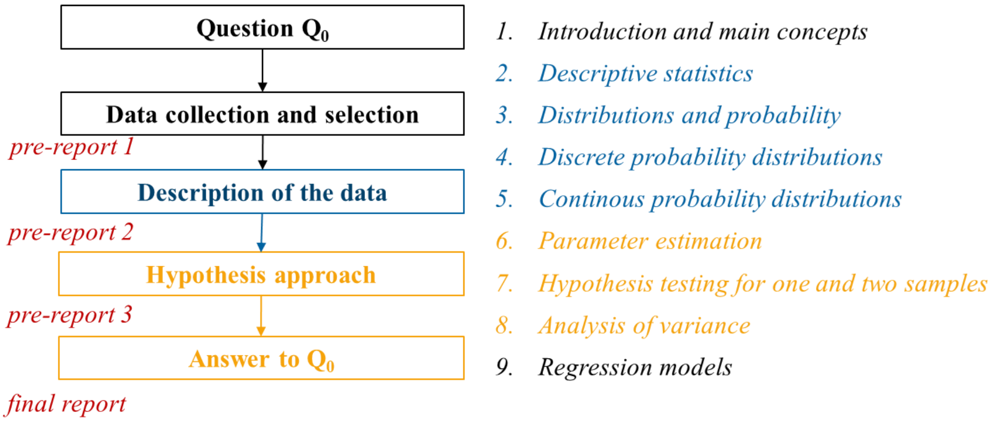


Figure 4: SRP phases and links to the syllabus

Instructor's perspective

The SRP has been a valuable tool to detect the limitations of the current contents of the subject. On one hand, some topics appear as secondary (such as those related to probability) while others show up in the study despite not being part of the syllabus (such as data gathering and processing and some non-parametric approaches). Hence the strategy followed has brought a reflection on the subject contents, which will be modified in the following academic year.

From the point of view of the instructor, the main issue encountered was the amount of time and work required, both on her part and on that of the students. The task was arduous because she had to attend 19 teams and a timely revision of the logbooks was not possible. As a consequence of this lack of feedback and of limited guidance, logbooks were often incorrectly completed. The students had to work outside the classroom because of the extensive syllabus and the limited teaching time. This generated a heavy workload for the students and difficulties for the instructor to provide good feedback about the work done (only about 15% of class time was used to discuss the SRP). It was complicated to find a topic that solved the original educational question and for which enough detailed and relevant data were available. Last, the format instructed for the oral presentation made them repetitive and unappealing.

The instructor mentioned several positive aspects after finishing the SRP and the evaluation of the subject. First, she highlighted the high quality of the projects presented, and the fact that a vast increase in the students' critical thinking was observed when performing statistical analyses. Second, regarding the contents, they acquired better skills regarding the use of R Commander, and they managed to perform a complete statistical inference process: all the teams started with raw data and managed to answer the question proposed. Since the generating question (Q0) was open, and the students formulated their own specific questions, she observed they had a genuine interest in finding the answer. Moreover, the students learnt about the difficulty to collect information (as expected) and about the applicability and usefulness of the subject. Another positive comment was related to being able to evaluate their skills in a much more rigorous manner. The development of skills both for autonomous and team work were also positively assessed.

Students' perspectives through a survey

In order to know the students' perspectives about the applied strategy, a survey was designed and administered to the students. Data were collected anonymously using Google Forms. The first part of the questionnaire included a brief introduction explaining the purpose of the survey as well as the anonymous handling of the answers obtained. The purpose of conducting the survey was to obtain results with the aim of identifying weaknesses and promoting improvement actions for the next SRP implementations.

The questionnaire included 35 questions grouped into six sections: general aspects of the course (Q01 to Q07), general aspects of the project (Q08 to Q12), contents of the project (Q13 to Q19), team work (Q20 to Q23), project management (Q24 to Q31), and final reflections (Q32 to Q35).

A 5-point scale was used for the closed questions of the questionnaire (Q01 to Q32, see Figures 5 to 9). The last section included three open questions that allowed the participants to write down their thoughts and opinions.

The questionnaires were sent to all the students during the last session of the project. The response rate obtained was 88% for the DC/DCE students, and 63% for the DITE students.

The analysis includes an overview of the responses to the closed questions (Q01 to Q32) considering all the participants together, a comparative analysis between the DC/DCE and DITE groups, and the analyses of the open questions in the final reflections section (Q33 to Q35).

Closed questions. Figures 5 to 9 summarise the students' responses to each section of the questionnaire. The responses of both groups (DC/DCE and DITE) were counted together. Q32, despite being part of the final reflections section in the questionnaire, was analysed together with the questions regarding the general aspects of the project.

The respondents tended to agree that the organisation of the course and the learning resources provided helped them learn statistics (Figure 5). They would have preferred not to have the project concentrated at the end of the course, but they evaluated the availability of the course instructors as highly positive.

Their opinion of the project (Figure 6) was positive. They valued it as interesting, related to the subject, and useful for learning. The responses are varied when asked if the project had changed their views on what statistics is (Q12).

When asked about the contents of the project (Figure 7), they considered the project had more content and greater difficulty than desired (Q13 to Q15). They also observed that more in-class time should be dedicated to the project (Q16 and Q18), and that the project took up too much after-class time (Q19). There is a balanced opinion on the weight of the project in the final mark, which means 25% of the total mark seems appropriate.

Team work was not an issue for most of the students (Figure 8). There was a strong agreement about working on the project in teams and the students felt comfortable about it.

As shown in Figure 9, the project helped the students realise the difficulties of data collection and data preparation (Q24), which seemed harder than describing the data (Q25) and analysing them through hypothesis testing (Q26). The respondents did not have the impression they were guided too much (Q27), and perceived a moderate difficulty in adapting to the project (Q28) and in writing the pre-reports (Q29). The pre-reports were useful to learn (Q30), and the project was perceived as properly organised (Q31).

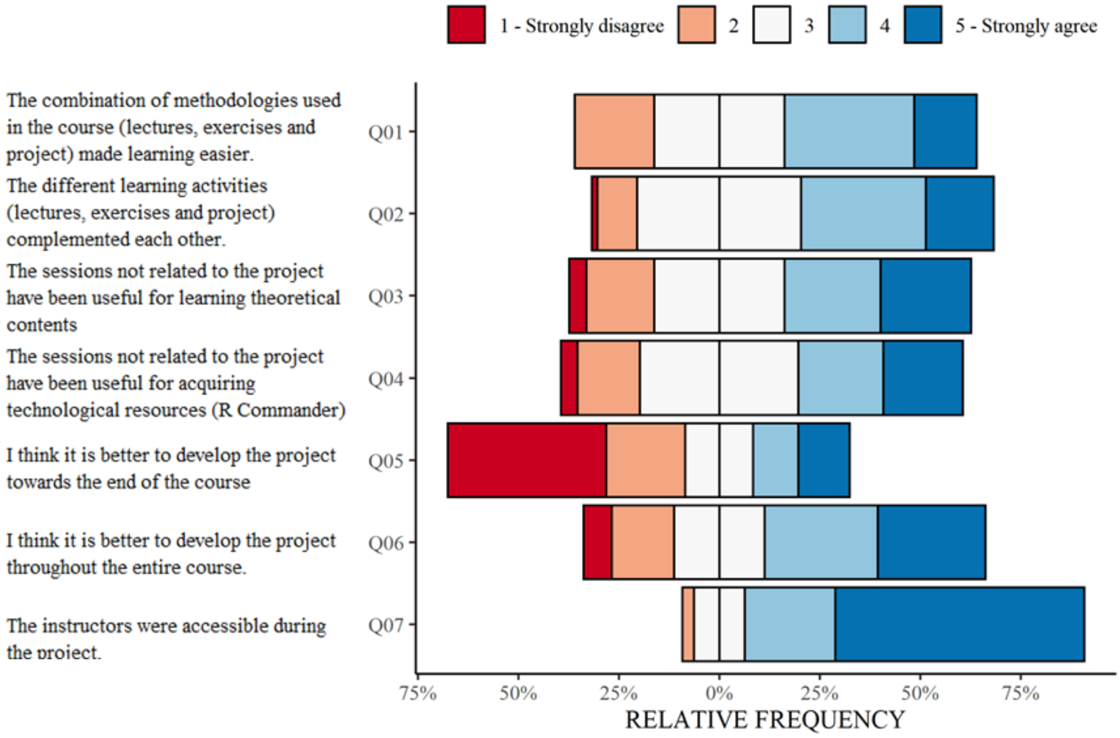


Figure 5: Responses to general aspects of the course (Q01 to Q07)

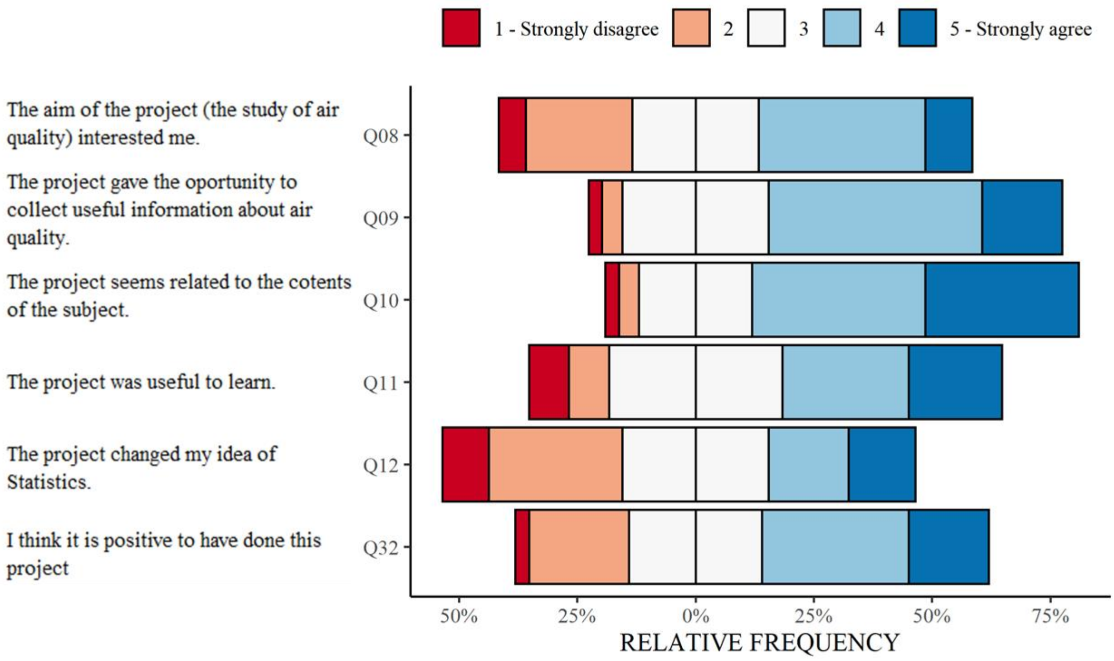


Figure 6: Responses to general aspects of the project (Q08 to Q12) and final reflections (Q32)

The comparative analysis between the two groups was carried out using a test of independence to calculate maximum likelihood. The p-values obtained were calculated using the Real Statistics Resource Pack software (Zaiontz, 2020) in Excel. Figure 10 shows the results corresponding to those questions for which group differences were shown by the acceptance of the alternative hypothesis of non-independence between group and answer.

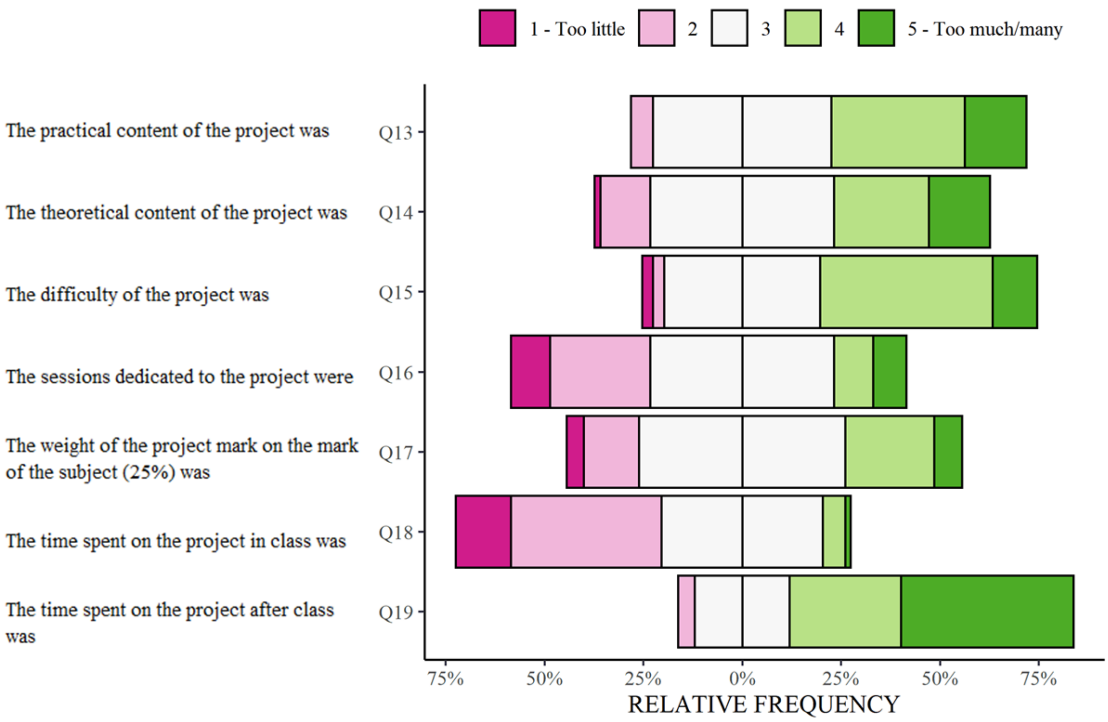


Figure 7: Responses to contents of the project (Q13 to Q19)

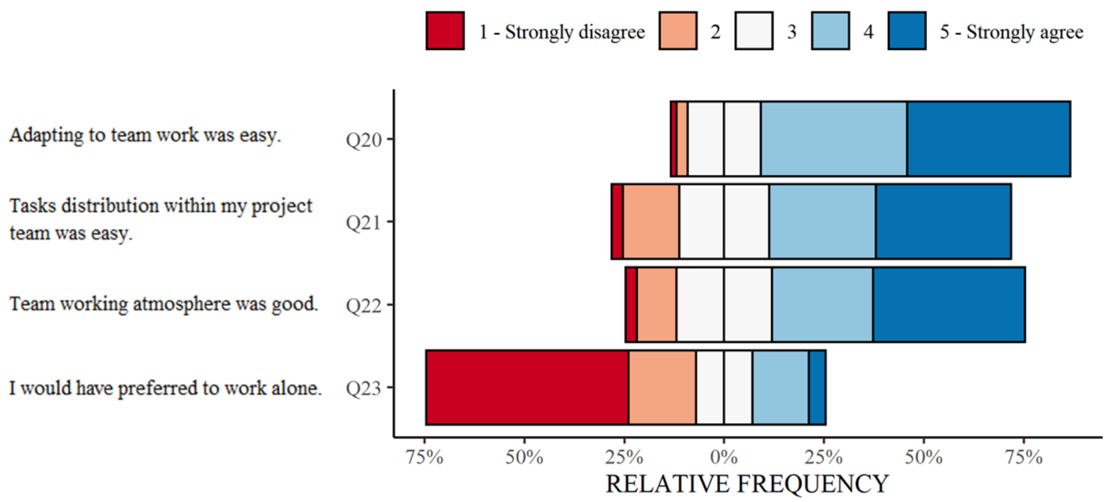


Figure 8: Responses to team work (Q20 to Q23)

DC/DCE students generally disagreed with the statement that the combination of methodologies used made learning easier (Q01). They had a neutral opinion about the idea that the project was useful to learn (Q11), and that the sessions not related to the project (lectures) were useful to learn the theoretical concepts (Q03). The DITE students, in contrast, showed greater agreement with these three statements. The DC/DEC students considered that both the practical and theoretical contents were excessive (Q13, Q14), while the DITE students did not have a clear opinion. The DITE students considered the time spent after class (Q19) somewhat excessive, while, in general, the DC/DEC students were of the opinion it was exaggerated. Moreover, those same students thought that the tasks distribution within their project team (Q21) was harder than that of the other group. Finally, the DC/DCE students considered that it is better to carry out the project throughout the course (Q05, Q06), while the DITE students had no clear opinion about it.

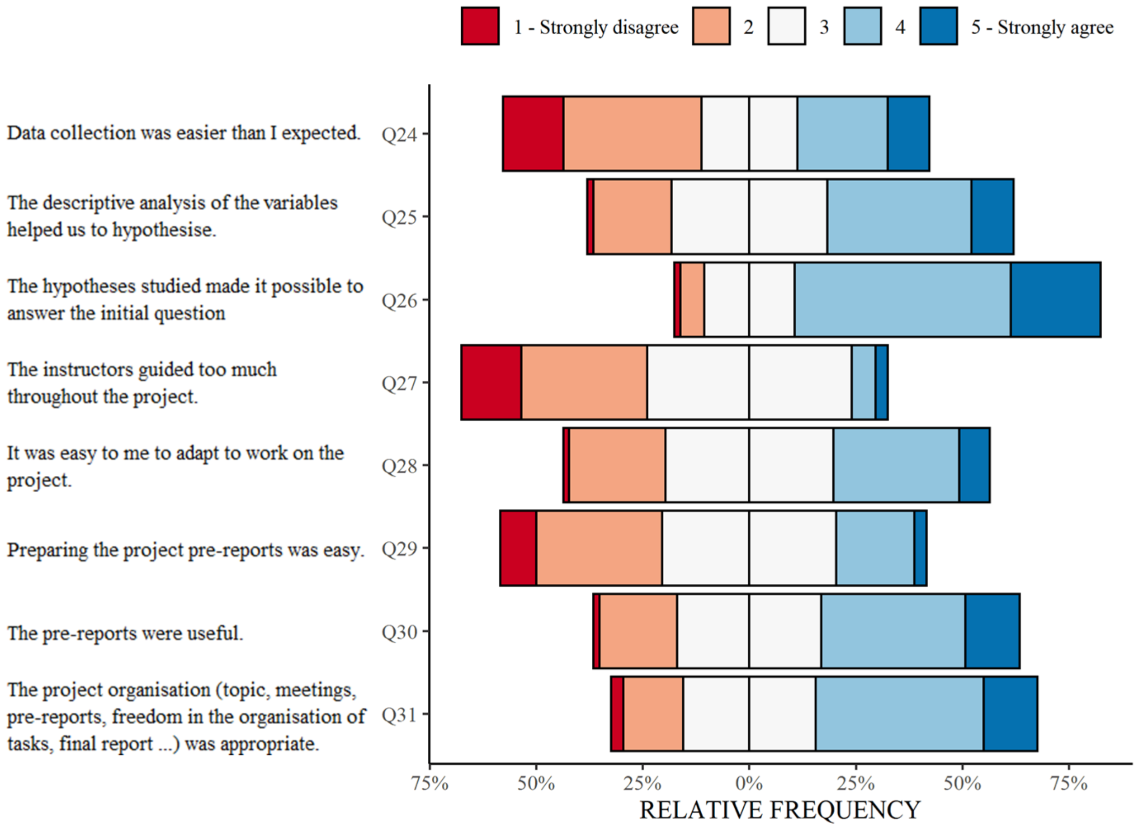


Figure 9: Responses to Project management (Q24 to Q31)

Open questions. Finally, the open questions in the final reflections section (Q33 to Q35) were analysed using text mining techniques. The analysis included (1) data cleaning, removing punctuation marks, numbers, stop words, and words with less than four letters; (2) lemmatisation, identifying the canonical form or lemma of words; and (3) graphical representation using unigram and bigram (a sequence of two adjacent words) word clouds, sketching the most frequent words amongst the students' responses.

A total number of 58 responses were obtained from Q33 (positive aspects of the project), 53 from Q34 (negative aspects of the project), and only 16 from Q35 (additional comments). The results obtained from word clouds were mainly as expected.

As shown in Figure 11, according to the students, the main positive aspects of the project (Q33) were team work ("clase", "trabajo", and "equipo" unigrams or "trabajo.equipo" bigram), the knowledge obtained ("conocimiento", "aprender", and "concepto" unigrams or "conocimiento.adquirido" bigram), and the real-life context ("real" and "vida" unigrams or "vida.real" bigram).

Question		p-value
Q01	The combination of methodologies used in the course (lectures, exercises and project) made learning easier.	0.037 (*)
Q03	The sessions not related to the project have been useful for learning theoretical contents.	0.029 (*)
Q05	I think it is better to develop the project towards the end of the course.	<0.001 (***)
Q06	I think it is better to develop the project throughout the entire course.	0.008 (**)
Q11	The project has been useful to learn.	0.016 (*)
Q13	The practical content of the project has been too little/too much.	0.018 (*)
Q14	The theoretical content of the project has been too little/too much.	0.014 (*)
Q19	The time spent on the project after class was too little/too much.	0.011 (*)
Q21	Tasks distribution within my project team was easy.	0.032 (*)

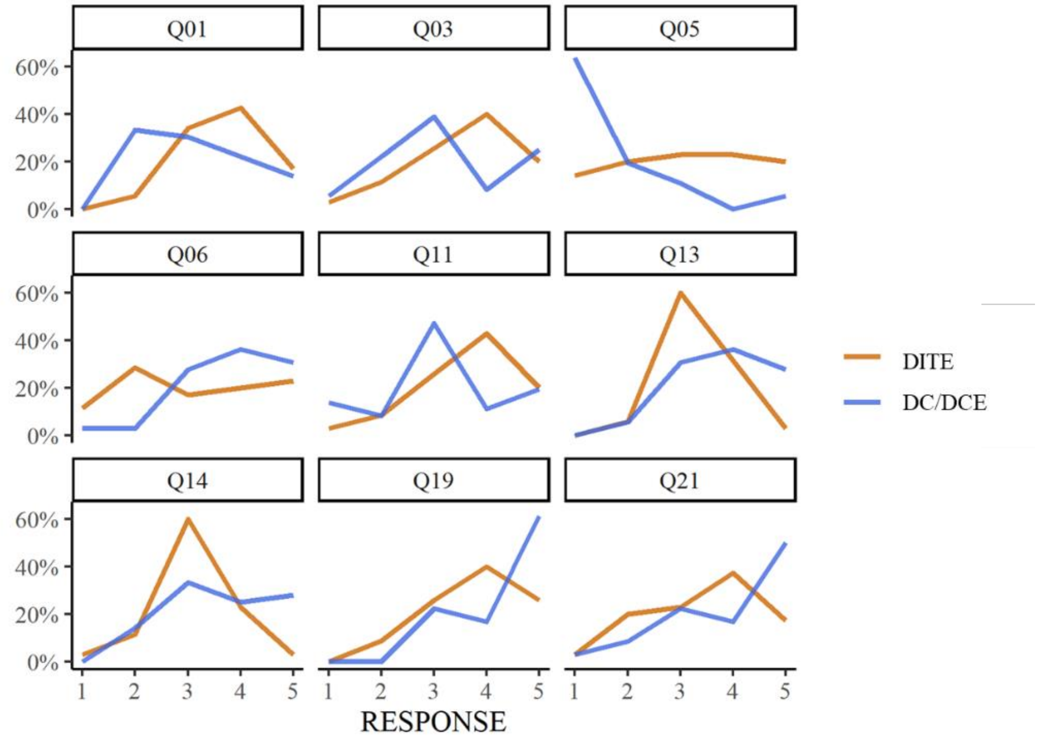


Figure 10: Comparison between the responses of the DC/DCE (in blue) and the DITE (in orange) students to the questions where major differences were observed

(a) (b)

Figure 11: Unigram (a) and bigram (b) word clouds of Q33 (positive aspects)

When analysing the most negative aspects of the project (Q34), what the students mentioned the most was the amount of work needed to develop it ("tiempo", "trabajo", and "demasiado" unigrams or "mucho.tiempo", "demasiado.tiempo", and "carga.trabajo" bigrams), and the calendar objections ("final.curso" and "fuera.horario" bigrams), see Figure 12.

Finally, as can be seen in Figure 13, a less obvious conclusion is reached when analysing the additional comments (Q35). This is probably due to the smaller number of responses, and because negative and positive aspects were mixed. However, it is worth noting that the students were pleased to have carried out the project ("mucho.bien" bigram), but it involved a lot of work and difficulty ("trabajo" and "costar" unigrams, or "costar.mucho", "haber.costar" bigrams).

(a) (b)

Figure 12: Unigram (a) and bigram (b) word clouds of Q34 (negative aspects)

(a) (b)

Figure 13: Unigram (a) and bigram (b) word clouds of Q35 (additional comments)

Students' perspectives through interviews

The second tool used to assess the SPR experimentation was conducting interviews with six students who gave their consent. An interview guide was designed to ensure all items were covered during the interview process. The guide consisted of three sections: (1) an introduction, to set the objectives of the interview; (2) project related-questions including their interest and expectations of the generating question, the reality of the raised case, data collection, and difficulties regarding how to obtain information, data handling and analysis, mixed teaching (SRP and lectures), objectives, usefulness of training and tools/ contents, online teaching, team work, schedule and work organisation, feedback, final presentation, and project duration; and (3) a closing, for any additional interviewee comments, and to estimate their willingness to participate in a similar project in the future.

Although a detailed analysis of the interviews would not fit in this communication, the initial observations corroborate the results obtained through the questionnaire: the project was perceived as useful for learning, but the workload was excessive.

In sum, both the instructor and the students manifest that running this SRP facilitates learning and complements the lectures, although it takes up a significant amount of time and increases workload. Moreover, the contents of the subject are questioned and the need to better adapt the SRP to each degree becomes apparent given the divergent responses to some of the items of the questionnaire.

Conclusions

The experimented SRP was implemented following the strategy of maintaining the course as untouched as possible and running it in parallel to the lectures. However, although the aim was to avoid any interference in the lectures, the SRP ran over the subject contents more than initially expected. The students claim more time devoted to the SRP. The instructor perceives the need for stronger integration of the SRP within the subject, and considers that the contents and instructional practices are questioned and should be revised.

Students value the team work and assumed new roles, asking questions and exploring beyond the topics introduced in class, moving towards the paradigm of questioning the world.

The experience provided substantive feedback to improve the SRP implementation in future statistics courses:

- The SRP should be implemented throughout the course, possibly moving some class hours from theory to face-to-face SRP work;
- Some topics in the syllabus could be replaced by more useful ones;
- The logbooks should be improved by using templates to facilitate prompt feedback;
- The use of an open generating question and of pre-reports could be maintained, but the generating question should be more in accordance with the different degree programs;
- The slideshows could be replaced by posters to make reporting more dynamic.

Acknowledgment

This research has been funded by the RTI2018-101153-B-C21 project of the State R+D+I Programme Oriented to the Challenges of Society (MCIU/AEI/FEDER, UE).

References

Boigues Planes, F. J., Estruch Fuster, V. D., Roig, B. & Vidal Meló, A. (2013). Una propuesta de Recorrido de Estudio e Investigación (REI): Diseño, simulación y decisión de una estrategia de pesca sostenible. *Modelling in Science Education and Learning*, *6*(2), 5–19.

Chevallard, Y. (1992). Fundamental concepts in didactics: perspectives provided by an anthropological approach. *Research in didactique of mathematics: Selected papers*, 131–168.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In *Proceedings of the IV congress of the European society for research in mathematics education* (pp. 21–30).

Dierker, L., Evia, J. R., Singer-Freeman, K., Woods, K., Zupkus, J., Arnholt, A., ... & Rose, J. (2018). Project-based learning in introductory statistics: Comparing course experiences and predicting positive outcomes for students from diverse educational settings. *International Journal of Educational Technology and Learning, 3*(2), 52–64.

Fonseca, C., Pereira, A. & Casas, J. M. (2010). Los Recorridos de Estudio e Investigación como productos de Ingeniería didáctica. In *TICAI 2010: TICs para el Aprendizaje de la Ingeniería* (pp. 107–116). IEEE Sociedad de Educación.

Markulin, K., Bosch, M. & Florensa, I. (2021). Project-based learning in statistics: A critical analysis. *Caminhos da Educação Matemática em Revista* (Online), *11*(1), 200–222.

Santana, A. & Gómez-Blancarte, A. (2019). Opportunities and challenges for project–based learning: the case of statistical projects. In A. Rosas (Ed.), *Research Reports in Mathematics Education: the classroom* (pp. 63–82). Miami, FL: L.D.Books.

Winsløw, C., Matheron, Y. & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics*, *83*(2), 267–284.

Zaiontz, C. (2020) Real Statistics Using Excel (Release 7.6) [Software]. Retrieved from www.real-statistics.com.

A praxeology of the Hough transforms the mathematical education of engineers

César Molina[1] and Cintya Gonzales[2]

[1]Institute of Research for Mathematics Education, Lima, Peru; c.molinaq@pucp.edu.pe

[2]Institute of Research for Mathematics Education, Lima, Peru; cintya.gonzales@pucp.pe

Abstract: Different research focused on the education of future engineers has documented a disconnect between mathematics courses and specialty courses (e.g., circuits, control theory, the strength of materials). To contribute to the creation of relationships between mathematical and engineering knowledge taught in these courses the analysis of engineering courses, especially mixed praxeology, is proposed. That is, praxeology containing both mathematical and engineering elements. The mixed praxeology could within the second stage, be transposed to the teaching of mathematics under didactic devices of mathematical modeling.

Keywords: Anthropological theory of didactic, mixed praxeology, Hough transform.

Introducción

Algunas investigaciones en la formación de futuros ingenieros muestran que la industria demanda egresados con competencias específicas, denominadas del siglo XXI, por ejemplo: pensar y razonar en términos matemáticos, planteo y resolución de problemas matemáticos, realizar modelos matemáticos, representación de entidades matemáticas, manejo de símbolos y del formalismo matemático, comunicarse usando las matemáticas y empleo de herramientas tecnológicas (Alpers, 2013).

Modificar los programas y hacerlos evolucionar hacia las necesidades de la industria, requiere de cambios institucionales de diversa índole, que están asociados a una gran complejidad (Burguignon, 2001) y a factores sociales específicos propios de cada país. En este mismo sentido, Frejd y Bergsten (2018), muestran que los programas de estudio son documentos que están en función de los "contextos políticos, sociales, culturales e históricos en los que se han desarrollado" (p. 117). Estos y otros factores influyen en la transposición externa, como lo han documentado Bosch et al. (2021) en el análisis de programas de cursos de matemáticas en el nivel universitario. Una posibilidad de motivar pequeños cambios en los programas efectivos de estudio, es decir, en la enseñanza que realmente llega al aula y está a cargo del profesor de matemáticas, consiste en el diseño de actividades didácticas específicas, basadas en una transposición interna a cargo del profesor, como se sugiere en Pablo y Romo (en prensa) y en Siero et al. (2022). Estos autores proponen el análisis de una praxeología mixta en la práctica profesional de ingenieros o en los cursos de ingeniería, la cual es objeto de una transposición didáctica que permite transformar la praxeología mixta ingenieril en una praxeología mixta escolar. Con base en la praxeología escolar se genera el diseño de un Recorrido de Estudio y de Investigación (REI) enmarcado en el paradigma del cuestionamiento del mundo (Chevallard, 2013).

En el marco de la Teoría Antropológico de lo Didáctico (TAD), REI están enfocados en los procesos de modelización e indagación y permiten integrar actividades de modelización en la enseñanza (Sala et al., 2020). Cabe resaltar que la modelización es comprendida como un proceso "de reconstrucción y articulación de organizaciones matemáticas de complejidad creciente" (Sala et al., 2020, p. 550) el cual se genera gracias a una cuestión generatriz planteada por la comunidad de estudio y donde las organizaciones matemáticas forman parte de las respuestas a dichas cuestiones.

Por otra parte, algunas investigaciones en las que se han diseñado e implementado REI en cursos de ingeniería (p. ej., Bartolomé et al., 2018; Pablo y Romo, en prensa; Siero et al., 2022), muestran que es posible desarrollar un proyecto de modelización matemática en el aula, con un trabajo autónomo por parte de los estudiantes, quienes investigan, exploran, generan hipótesis y las prueban, construyen y/o adaptan modelos matemáticos para resolver tareas de ingeniería. Es decir, generan relaciones entre saberes de diferente índole: matemáticos, informáticos, ingenieriles y prácticos. Sin embargo, el proceso de diseñar los REI, originados en el estudio de una cuestión de ingeniería, supone un gran reto para los profesores de matemáticas. Por ejemplo, González-Martín et al. (2020) muestran que los profesores de matemáticas con experiencia en ingeniería tienen un mejor desenvolvimiento en incorporar y relacionar aplicaciones propias de la ingeniería con las matemáticas. En tanto, los profesores que solo cuentan con una formación matemática y se desarrollan en la investigación matemática, muestran desventajas para incorporar dichas aplicaciones.

Con base en lo anterior, se considera que una forma de posibilitar que los profesores de matemáticas diseñen e implementen un REI, originado en una cuestión generatriz de la ingeniería, consiste en poner a su disposición un modelo epistemológico praxeológico alternativo, basado en el análisis de praxeologías mixtas que son parte de instituciones de ingeniería. Este modelo permitirá generar transposiciones didácticas para la clase de matemáticas, mediante el diseño de dispositivos didácticos. Por supuesto, esta fase de análisis de instituciones de ingeniería y de la identificación de praxeologías mixtas, constituiría una primera base para el diseño de estos dispositivos. En específico en esta comunicación mostramos un modelo epistemológico praxeológico de la Transformada de Hough que podría constituir una base para diferentes diseños didácticos para la clase de matemáticas

Elementos de la Teoría Antropológico de lo didáctico

La praxeología es la unidad mínima para el análisis de las actividades humanas (Chevallard, 1999). Sus cuatro componentes conforman dos bloques: el técnico-práctico $[T, \tau]$ y el tecnológico-teórico $[\theta, \Theta]$. Donde T es el tipo de tareas (lo que se hace), τ la técnica (cómo se hace), θ la tecnología (que justifica la técnica) y la teoría Θ (que justifica la tecnología). En esta teoría, toda actividad humana tiene lugar en el marco de instituciones, definidas como organizaciones sociales estables que permiten que sus sujetos realicen y hagan posible las actividades humanas gracias a los recursos materiales e intelectuales que están a su disposición y que han sido producidos por las comunidades al enfrentarse a situaciones problemáticas, para resolverlas de manera regular y eficaz (Castela y Romo-Vázquez, 2011).

Modelo praxeológico extendido

Las praxeologías al pasar de una institución a otra sufren transposiciones. Por ejemplo, cuando una praxeología matemática es importada a instituciones de ingeniería es transformada en una praxeología que contiene elementos matemáticos y de ingeniería. Es decir, se convierte en una praxeología mixta (Vázquez, et al. 2016). Castela y Romo-Vázquez (2011) formularon inicialmente el modelo que permite analizar las praxeologías matemáticas en instituciones de ingeniería. Más recientemente, este modelo se ha precisado (Chaachoua, et al., 2009; Castela y Romo, en prensa). En particular, consideramos cuatro instituciones: las matemáticas *M*, la enseñanza de las matemáticas *EM*, las disciplinas de ingeniería *DI* y la enseñanza de las disciplinas de ingeniería *EDI*. A continuación, señalamos el modelo praxeológico extendido:

$$\left[T^{DI}, \begin{matrix} \tau^{M}, \theta^{M}, \Theta^{M} \\ \tau^{DI}, \theta^{DI}, \Theta^{DI} \end{matrix}\right] \begin{matrix} \leftarrow EM \leftarrow M \\ \leftarrow EDI \leftarrow DI \end{matrix}$$

Bajo este modelo es posible identificar una praxeología mixta que involucra a la transformada de Hough. Donde T^{DI} corresponde al tipo de tarea en *DI*; los componentes: $[\tau^{M}, \theta^{M}, \Theta^{M}]$ corresponden a la técnica, la tecnología y la teoría de las matemáticas *M* o de su enseñanza *EM*; y $[\tau^{DI}, \theta^{DI}, \Theta^{DI}]$ corresponde a la técnica, la tecnología y la teoría de la institución de ingeniería de investigación *DI* o de su enseñanza *EDI*. En este caso, la disciplina de la ingeniería es procesamiento digital de imágenes (*PDI*, *EPDI*).

Metodología

Para este trabajo se emplea la metodología originada en Macias y Romo-Vázquez (2014) y ampliada en trabajos de tesis (Guzmán, 2016; Siero, 2017; Vázquez, 2017, Galindo, 2019) Mas recientemente, se ha considerado desde la ingeniería didáctica (Artigue, 2015), como se ilustra en Siero et al. (2022).

Esta metodología se compone de cuatro fases. La fase 1, del análisis preliminar, incluye el análisis de la praxeología mixta en una institución de investigación y/o de enseñanza de ingeniería. La fase 2 se refiere al diseño del REI y análisis a priori, la fase 3 a la experimentación y la fase 4 al análisis a posteriori.

En esta comunicación, presentamos algunos elementos de la fase 1, correspondientes al análisis praxeológico de la transformada de Hough en el contexto del procesamiento digital de imágenes (*PDI*) y de su enseñanza (*EPDI*).

Análisis de la praxeología mixta de la Transformada de Hough en PDI y EPDI

Para este análisis praxeológico se consideraron dos pasos: 1) identificar las instituciones de ingeniería, 2) analizar praxeologías mixtas susceptibles de ser transpuesta al aula. De acuerdo con la literatura considerada en esta investigación, la transformada de Hough se emplea para el diseño de herramientas de la inteligencia artificial, que permiten, por ejemplo, simular la visión humana para detectar líneas rectas en una imagen o en un video digital.

Paso 1. Elección de las instituciones de ingeniería. Para identificar las praxeologías mixtas relacionadas con el uso de la transformada de Hough, se han considerado los diferentes documentos de investigación del *PDI* y de la *EPDI*, que se ilustra en la Tabla 1.

Table 1: Documentos identificados para desarrollar el análisis praxeológico

Autores	Tipo de documento	Institución de referencia	Dirigido a
Hough (1962)	Artículo de investigación	*PDI*	Investigadores, profesores
Duda y Hart (1972)	Artículo de investigación	*PDI*	Investigadores, profesores, ingenieros, estudiantes
Pratt (2007)	Libro	*PDI*	Investigadores
Gonzalez y Woods (2018)	Libro	*PDI, EPDI*	Investigadores, profesores, estudiantes
Mery (17 nov 2020)	Curso virtual	*EPDI*	Estudiante

A partir del análisis de estos documentos encontramos que el objeto de estudio del procesamiento digital de imágenes es la imagen digital, obtenida a partir de las técnicas de muestreo y de cuantización, que dependen del arreglo de sensores del dispositivo electrónico que digitaliza la luz reflejada por el objeto o escena tridimensional, Gonzalez y Woods (2018).

Una imagen se modela como una función bidimensional $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{i}(\boldsymbol{x},\boldsymbol{y}) \cdot \boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})$; con $(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}$. Donde $\boldsymbol{i}(\boldsymbol{x},\boldsymbol{y}) \in [\mathbf{0};+\infty\rangle$ es la cantidad de iluminación incidente sobre la escena (objeto) y $\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y}) \in [\mathbf{0};\mathbf{1}]$ es la cantidad de iluminación reflejada por el objeto, cuando $r = 0$ es absorción total y $r = 1$ es reflectancia total. De aquí que $0 \leq \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) < \infty$, además $\boldsymbol{i}(\boldsymbol{x},\boldsymbol{y})$ queda determinado por la iluminación de la fuente y $\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})$ está relacionado con las características del objeto fotografiado.

Para una imagen digital la función $f(x,y)$ toma valores discretos en el intervalo $\left[\mathbf{0},\mathbf{2}^{\mathbf{k}}-\mathbf{1}\right]$ con k determinando la longitud de un píxel. Por ejemplo, $k = 8$ tenemos un píxel de 8bits cuyos valores oscilan entre 0 a 255 y además son usados para fotos, escaneo e impresión. Este proceso de discretizar la función imagen $f(x,y)$ se le conoce como cuantización, además a este conjunto de valores se les conoce como la escala de grises.

Mientras que el muestreo es la técnica para discretizar las coordenadas espaciales x e y, es decir, x e y toman valores enteros no negativos, digamos i y j, respectivamente. Así $f(i,j)$ es el valor de la imagen digital en el punto (i,j), con $0 \leq i \leq M-1$ y $0 \leq j \leq N-1$. Donde $M \times N$ es el tamaño de dicha imagen. Además, considerando $a_{ij} = f(i,j)$ tenemos que una imagen digital se puede representar como un arreglo matricial $A = \left[a_{ij}\right]$ con M como número de filas y N como el número de columnas.

Paso 2: Análisis de praxeologías mixtas susceptibles de ser transpuesta al aula. Para ejemplificar este paso, detallamos la praxeología mixta de la transformada de Hough (P-TH) encontrada al analizar los documentos de la tabla 1. El tipo de tarea y la técnica P-TH aparecen a continuación:

Tabla 2: Tipo de tareas y técnicas en la praxeología mixta transformada de Hough.

Tipo de tarea: Identificar bordes en una imagen digital **Sub Tipo de Tarea:** T^{PDI}→ Identificar líneas rectas en una imagen digital
Técnica: $(\tau')^{PDI}$→ Algoritmo de transformada de Hough $(\tau'')^{PDI}$→ Algoritmo de transformada de Hough según Duda y Hart (1972)

La técnica $(\tau')^{PDI}$ fue desarrollada por Hough (1962) y está relacionada con la ecuación de la recta $y = mx + c$, considerando los parámetros m y c como variables independiente y dependiente, respectivamente, mientras que x e y son constantes fijas. Sin embargo, esta técnica $(\tau')^{PDI}$ no es efectiva para la tarea que se propone cuando se presenta en la imagen digital rectas verticales, pues el valor de m es indeterminado. Por tal motivo, Duda y Hart (1972) proponen la técnica $(\tau'')^{PDI}$ que se relaciona con la función senoide a partir de la ecuación normal de la recta (ver figura 2a): $\rho = x\cos\theta + y\sin\theta$ y en este caso se considera al parámetro ρ en función de θ y a x, y como constantes fijas.

La representación de la imagen digital se hace en el primer cuadrante del plano cartesiano con origen en la parte superior izquierda (ver Figura 1). A esta representación del plano cartesiano en *PDI* y *EPDI* se le conoce como plano de la imagen.

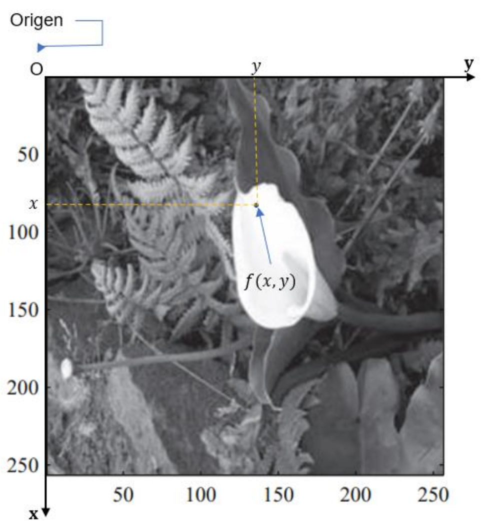


Figure 1: Representación gráfica de una imagen digital[1]

Por otro lado, el plano de parámetros se determina a partir de la ecuación de la recta $\rho = x\cos\theta + y\sin\theta$, con la relación funcional $\rho(\theta)$ y constates fijas x e y. Entonces una recta en el plano de la imagen (dominio) se relaciona con un punto en el espacio de parámetros (contradominio). (Ver Figura 2).

[1] Adaptado de Sundararajan 2017, p.4.

La relación entre el plano de la imagen y el espacio de parámetros está fundamentada en cuatro propiedades que a continuación señalamos:

> Propiedad 1. Un punto en el plano de la imagen corresponde a una curva sinusoidal en el plano de los parámetros.
>
> Propiedad 2. Un punto en el plano de los parámetros corresponde a una línea recta en el plano de la imagen.
>
> Propiedad 3. Los puntos que se encuentran sobre una misma recta en el plano de la imagen corresponden a curvas que pasan por un punto común en el plano de los parámetros.
>
> Propiedad 4. Los puntos que se encuentran en la misma curva en el plano de los parámetros se corresponden con rectas que pasan por el mismo punto en el plano de la imagen. (Duda y Hart, 1972, p. 12)

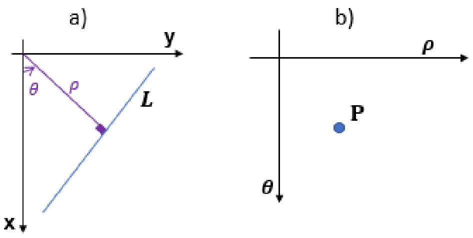


Figure 2: a) Plano de la imagen y b) el espacio de parámetros

A continuación, se presenta en la Tabla 3 las componentes de la técnica $(\boldsymbol{\tau''})^{\boldsymbol{PDI}}$. Expresamos la relación entre la técnica en la disciplina de ingeniería del procesamiento digital de imágenes y la herramienta para concretizarla, MATLAB, mediante la siguiente notación $\tau_i^{PDI} \to \tau_i^{MATLAB}$. Precisamos que MATLAB es un software y un entorno de lenguaje de programación usado tanto en la enseñanza como en la investigación en ingeniería (Mery, 17 nov 2020).

Table 3: Tipos de tareas que componen a la técnica $(\boldsymbol{\tau''})^{\boldsymbol{PDI}}$

Tareas	Técnicas
t_1: Leer la imagen de entrada X t_{12}: Determinar la imagen X en escala de grises t_{13}: Calcular E, la imagen binaria de los bordes de X	$\tau_1^{PDI} \to \tau_1^{MATLAB}$: Realizar la tarea t_1 se ejecuta con la función en MATLAB $\mathrm{imread(filename)}$. Esta función lee la imagen a partir del archivo que puede ser de tipo JPG, entre otros. $\tau_{12}^{PDI} \to \tau_{12}^{MATLAB}$: Realizar la tarea t_{12} se requiere la función en MATLAB $\mathrm{imshow(I)}$, que determina la imagen en escala de grises. $\tau_{13}^{PDI} \to \tau_{13}^{MATLAB}$: Realizar la tarea t_{13} requiere el uso de la función en MATLAB $\mathrm{edge(I, method)}$. Función que detecta los bordes de la imagen. Cuyos argumentos son la imagen y el método para realizar la detección. En particular, en esta función de MATLAB se usa el método CANNY.
t_2: Determinar la matriz H	τ_2^M, τ_2^{PDI}: Según Pratt (2007) para determinar H se considera cuantificar las variables (ρ, θ) considerando las variaciones $-\frac{\rho_{max}}{2} \leq \rho_m \leq \rho_{max};\ -\frac{\pi}{2} \leq$

	$\theta_n \leq \pi$ y donde $\rho_{max} = \sqrt{(x_J)^2 + (y_K)^2}$, además $J \times K$ son las dimensiones de la matriz E. Se cuantifica de tal manera que las dimensiones de H, $M \times N$ queden determinados por números impares.
t_3: Calcular las coordenadas espaciales (x, y) en E donde el valor del píxel sea 1.	τ_3^M, τ_3^{PDI}: $x_j = j + 1/2$ y $y_k = k + 1/2$ donde (j, k) indica los índices del elemento $a_{jk} = f(j, k)$ de la matriz de la imagen binaria, E.
t_4: Determinar la curva senoidal para cada (x, y) en E t_{41}: Calcular la curva (ρ, θ) a través de los valores discretizados de ρ y θ.	τ_4^M : Se toma en cuenta la ecuación normal de la recta $\rho = x \cos\theta + y \sin\theta$ Donde ρ es la distancia del origen a la recta en el plano cartesiano y θ es el ángulo entre el eje X y el radio vector ρ. τ_{41}^M, τ_{41}^{PDI}: Según Pratt (2007), se tiene: $\rho(n) = x_j \cos\theta_n + y_k \operatorname{sen}\theta_n$ y además $\theta_n = \pi - \frac{2\pi(N-n)}{N-1}$ es incrementado sobre el rango $1 \leq n \leq N$ con la restricción $\phi - \frac{\pi}{2} \leq \theta_n \leq \phi + \frac{\pi}{2}$, con $\phi = \arctan\left(\frac{y_k}{x_j}\right)$
t_5: Actualizar la matriz H, sumando +1 en las celdas por donde van pasando las curvas. t_6: Determinar en H los elementos donde hay máximos valores	$\tau_5^{PDI} \rightarrow \tau_5^{MATLAB}$: Se determina una función en MATLAB que tenga como entrada E y tenga como salida la matriz H, vectores ρ y θ. Dentro de esta función se actualiza H a partir de estructuras de control (for, if and else). $\tau_6^{PDI} \rightarrow \tau_6^{MATLAB}$: Se determina una función en MATLAB que tenga como entrada H de la tarea t_5 y que tenga como salida una matriz de los elementos máximos de H.
t_7: determinar la recta con las coordenadas (ρ, θ) del elemento de H correspondiente al máximo valor.	$\tau_7^{PDI} \rightarrow \tau_7^{MATLAB}$: Se determina una función en MATLAB que tenga como entrada la imagen X, una matriz de los máximos en H, los vectores ρ y θ. Además, esta función gráfica sobre la imagen X las líneas rectas para cada $\rho(i)$ y $\theta(j)$ que corresponde a una máximo en H.

Finalmente, en la Tabla 4, presentamos el bloque del logos de la P-TH.

Table 4: Tecnología y Teoría asociada en la P-TH

Tecnologías $\boldsymbol{\theta^{PDI}}$ y $\boldsymbol{\theta^{M}}$	**Teoría**
La tecnología práctica $\boldsymbol{\theta^{PDI}}$ del procesamiento digital de imágenes de las técnicas $\tau_1^{PDI} \rightarrow \tau_1^{MATLAB}$, $\tau_{12}^{PDI} \rightarrow \tau_{12}^{MATLAB}$, $\tau_{13}^{PDI} \rightarrow \tau_{13}^{MATLAB}$ son las funciones propias del entorno de MATLAB.	

La tecnología práctica $\boldsymbol{\theta^{PDI}}$ del procesamiento digital de imágenes de las técnicas τ_{41}^{PDI}, $\tau_5^{PDI} \to \tau_5^{MATLAB}$, $\tau_6^{PDI} \to \tau_6^{MATLAB}$, $\tau_6^{PDI} \to \tau_6^{MATLAB}$ son las cuatro propiedades señaladas en Duda y Hart (1972)	Procesamiento digital de imágenes, Geometría analítica, Trigonometría y álgebra, Lógica proposicional, Vectores y matrices, Teoría de funciones
La tecnología matemática $\boldsymbol{\theta^{M}}$ en el algoritmo de TH comprende: Distancia entre dos puntos, por ejemplo, para determinar ρ_{max} Definición de las coordenadas polares (ρ, θ) Ecuación normal de la recta Definición de función en los códigos fuente de las funciones de MATLAB relacionadas con las técnicas τ_5^{DI}, τ_6^{DI}, τ_7^{DI} Funciones reales dentro de los códigos fuentes de las funciones de MATLAB Definición de matriz y vectores como variables y constantes dentro de las funciones de MATLAB. Lógica proposicional en la implementación de las estructuras de control (if, else, for)	

El análisis praxeológico de la Transformada de Hough nos ha permitido identificar sus elementos relacionados con la enseñanza de la matemática y el procesamiento digital de imágenes. Tomando en cuenta esta relación se planteó una transposición de esta P-TH para la institución de enseñanza de la matemática, el cual lo nombramos praxeología de la transformad de Hough escolar (PTHE) que mostramos a continuación en la Tabla 5.

Table 5: Praxeología de la transformada de Hough escolar

Tipo de Tarea:
$\boldsymbol{T_1^{PTHE}}$: Determinar líneas rectas en un plano cartesiano de ejes entero no negativos, a partir de varios puntos arbitrarios en el plano. $\boldsymbol{T_2^{PTHE}}$: Determinar líneas rectas de una imagen digital desde algoritmos en entornos como MATLAB o Google Colab.
Técnica:
$\boldsymbol{\tau_1^{M}}$: Se calcula rectas a partir de (ρ, θ). Donde (ρ, θ) son parámetros de la ecuación de la recta $x \cos\theta + y \sin\theta = \rho$. Además $(\rho,\ \theta)$ se determina a partir de la intersección de las curvas senoidales, una cantidad de intersecciones mayor a dos. $\boldsymbol{\tau_2^{PDI}}$: Se ejecuta programas implementados en un lenguaje de programación (MATLAB, PYTHON) para determinar las rectas en la imagen digital.
Tecnología:

$\boldsymbol{\theta^{M}}$: Ecuación normal de la recta ($\rho = x\cos\theta + y\sin\theta$), matrices. $\boldsymbol{\theta^{PDI}}$: Algoritmo de la transformada de Hough, propiedades mencionadas en Duda y Hart (1972).
Teoría
Procesamiento digital de imágenes, álgebra y trigonometría, geometría analítica, álgebra Matricial.

Resultados

Castela (2016) señala "cada institución ejerce una actividad propia; aun cuando importa praxeologías de otras instituciones productoras de matemáticas, o no; las desarrolla y las adapta a sus condiciones institucionales específicas" (p. 27). En ese sentido, al analizar la praxeología de la Transformada de Hough considerando los documentos de la tabla 1, se localizaron elementos matemáticos de la componente teórica de forma discursiva, expresados con un lenguaje propio de la disciplina del procesamiento digital de imágenes: el plano de la imagen, espacio de parámetros, arreglo, pixeles, función imagen.

Los elementos tecnológicos de la ingeniería que se encontraron son los siguientes: muestreo, cuantización, segmentación, umbralización, detección de bordes; los cuales se validan principalmente a partir de algoritmos computacionales en entornos como MATLAB, Google Colab.

En la investigación de los referentes epistemológicos de la Transformada de Hough (Hough, 1962) se observa que la componente tecnológica de la ingeniería relativa a la transformada de Hough esta explicada en términos de la electrónica. En tanto, la componente tecnológica matemática se presenta de forma discursiva y está relacionada con la geometría analítica. La relación entre estas componentes está validada por el procesamiento digital de imágenes.

Por otro lado, observamos que en (Duda y Hart, 1972; Pratt, 2007; Gonzalez y Woods, 2018) existen cambios significativos en las tecnologías matemáticas y de ingeniería. Por ejemplo, en Gonzalez y Woods (2018) hay más presencia de elementos de la matemática que justifican a la técnica de ingeniería. De manera general, las técnicas y tecnologías del procesamiento digital de imágenes son más sofisticadas que las que se definen en la geometría analítica y necesitan, en la mayoría de los casos, algoritmos computacionales para ser validadas.

La praxeología de la transformada de Hough ha sufrido cambios para que en las instituciones de investigación y de enseñanza de ingeniería pueda emplearse de manera eficiente. En principio se fundamentó en la ecuación paramétrica pendiente-intercepto, luego se consideró la ecuación normal de la recta en coordenadas polares para salvar la indeterminación que generaba la primera propuesta.

Conclusiones

Consideramos que el trabajo pueda sumar a las investigaciones relacionadas con la formación matemática de los ingenieros. Los resultados obtenidos, presentados aquí brevemente, pueden proporcionar a los profesores de matemática en las escuelas de ingeniería elementos para diseñar actividades didácticas relacionadas con la modelización matemática que involucren problemas de la ingeniería.

Los documentos considerados en la Tabla 1 sirven como referentes epistemológicos para el análisis praxeológico de la transformada de Hough, así como el reconocimiento de elementos propios del procesamiento digital de imágenes. De esta forma, creemos que podemos explicitar la forma en que se utilizan elementos matemáticos, el tipo de lenguaje utilizado y el rol fundamental de los programas computacionales, que pueden ser de interés para los profesores de matemáticas que no son especialistas en ingeniería. Estos nos llevan a cuestionar la forma en que ciertas actividades didácticas pueden ser propuestas para mostrar el uso de los modelos matemáticos a cargo de los programas computacionales y los niveles de control que se pueden tener al explicar tecnologías matemáticas que las sustentan.

Por otro lado, sigue siendo necesario profundizar en las praxeologías del procesamiento digital de imágenes, de diferentes niveles de complejidad praxeológica para explicitar las técnicas, tecnologías y teorías de las matemáticas inmersas en estas actividades de modelización en ingeniería. Finalmente, enfatizamos la necesidad de ingresar en las lógicas de las instituciones de ingeniería, como una vía que permite reconocer relaciones entre conocimientos matemáticos y de ingeniería, así como de posibles relaciones que pueden ser generadas en la enseñanza de las matemáticas

Agradecimientos

Agradecemos a la Dra. Avenilde Romo por compartir sus reflexiones y valiosos aportes para la mejora del artículo. También, a la línea de investigación Epistemología de las matemáticas en la didáctica de las matemáticas: la antropología del conocimiento matemático y el diseño de secuencias didácticas de Pontificia Universidad Católica del Perú.

Referencias

Alpers, B. (Ed.) (2013). *A Framework for Mathematics Curricula in Engineering Education* (A Report of the Mathematics Working Group). European Society for Engineering Education (SEFI). https://www.sefi.be/publication/a-framework-for-mathematics-curricula-in-engineering-education/

Artigue, M. (2015). Perspectives on design research: the case of didactical engineering. In A. Bikner-Ahsbahs, C. Knipping, N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 467–496). Springer.

Bartolomé, E., Florensa, I., Bosch, M. y Gascón, J. (2018). A 'study and research path' enriching the learning of mechanical engineering. *European Journal of Engineering Education*, *44*(3), 330–346. https://doi.org/10.1080/03043797.2018.1490699.

Bosch, M., Hausberger, T., Hochmuth, R. et al. External Didactic Transposition in Undergraduate Mathematics. *Int. J. Res. Undergrad. Math. Ed.* *7*, 140–162 (2021). https://doi.org/10.1007/s40753-020-00132-7.

Duda, R. O. y Hart, P. E. (1972). Use of the Hough Transformation to Detect Lines and Curves in Pictures. *Communication of the ACM*, *15*(1), 11-15. https://doi.org/10.1145/361237.361242

Castela, C. (2016). Cuando las praxeologías viajan de una institución a otra: una aproximación epistemológica del "boundary crossing". *Educación matemática*, *28*(2), 9–12.

Castela y Romo-Vazquez (2011). Des mathématiques à l’automatique: étude des effets de transposition sur la transformée de Laplace dans la formation des ingénieurs. *Recherches en Didactique des Mathématiques, 31*(1), 79–130.

Frejd, P. y Bergsten, C. (2018). Professional modellers’ conceptions of the notion of Mathematical modelling: ideas for Education. *ZDM Mathematics Education*, (50), 117–127. https://doi.org/10.1007/s11858-018-0928-2

Galindo, N. (2019). *Diseño de una unidad de aprendizaje sobre Relaciones Cuantitativas de Estructura-Actividad (QSAR) en la carrera de ingeniería* (Tesis de maestría). https://www.cicata.ipn.mx/assets/files/cicata/ProME/docs/tesis/tesis_maestria/2019/galindo_2019.pdf

Gonzalez, R. C. y Woods, R. E. (2018) *Digital image processing.* (4a ed.). Pearson.

González-Martín y Hernandes-Gomes (2021). Applications in calculus for Engineering. The cases of five teachers with different bacgrounds. En M. Inprasitha, N. Changsri y N. Boonsena (Ed.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education.* (pp. 353–360). https://pme44.kku.ac.th/home/uploads/volumn/pme44_vol2.pdf#page=366

Hough, P. V. C. (1962). Method and Means for Recognizing Complex Patterns (Patente de U.S. No. 30696541962.) https://www.freepatentsonline.com/3069654.html

Mery, D. (17 nov 2020). *27 procesamiento de Imágenes: Transformada de Hough, Watershed, Templates, Movimiento*. YouTube. https://www.youtube.com/watch?v=V94QX4aCglo&list=PLilWJnCHHGl2MECnUo0REYid7ZB7jNiHd&index=27&t=2477s

Pablo-Diaz, E. y Romo-Vázquez, A. (en prensa). Design of a didactic activity on the Hazen-Williams model for engineering education.

Pratt, W. K. (2007). *Digital Image Processing*. (4a ed.). Wiley. https://drive.google.com/file/d/0B30qTepNcnnAOGhlOUQ4Z1o1Sm8/view?resourcekey=0-hSogsioYn1O_qRyq74UVsA

Sala, G., Barquero, B. y Font, V. (2020). Modelización e indagación en la propuesta de un REI codisciplinar de matemáticas e historia**.** *Revista Educação Matemática Pesquisa, 22*(4), 546–562. http://dx.doi.org/10.23925/1983-3156.2020v22i4p546-562

Siero González, L. R., Echavarría Cepeda, L. A., Romo, A., & Navarro Torres, J. (2022). Diseño de un dispositivo de rehabilitación para la trombosis: una actividad de modelización matemática en la formación de ingenieros. *Avances de Investigación en Educación Matemática*, *21*, 107–134. https://doi.org/10.35763/aiem21.4258

Sundararajan, D. (2017) *Digital image processing. A signal processing and algorithmic approach*. Springer. https://www.springer.com/gp/book/9789811061127

Vázquez, R., Romo, A., Romo-Vázquez, R. y Trigueros, M. (2016). La separación ciega de fuentes: un puente entre el álgebra lineal y el análisis de señales. *Educación Matemática*, *28*(2), 31–57.

A study and research path on hyperthermia in children left in parked cars

Vegard Topphol

Department of Mathematical Sciences, Norwegian University of Science and Technology, Norway; vegard.topphol@ntnu.no

This paper presents a study and research path based on a question about hyperthermia deaths in children placed in cars parked in the sun. The text demonstrates the potential of the study and research path for generating questions about modelling, and what sort of mathematics and physics are involved in modelling the human body. It also suggests a broader interdisciplinary potential, and in particular avenues for investigations that include sociological and statistical themes.

Ce document présente un parcours d'étude et de recherche basé sur une question concernant les décès par hyperthermie chez les enfants placés dans des voitures garées au soleil. Le texte démontre le potentiel des parcours d'étude et de recherche pour générer des questions sur la modélisation, et sur les types de mathématiques et de physique qui sont impliqués dans la modélisation du corps humain. Il suggère également un potentiel interdisciplinaire plus large, et en particulier des pistes d'investigation incluant des thèmes sociologiques et statistiques.

Este artículo presenta un recorrido de studio e investigación basados en una pregunta sobre las muertes por hipertermia en niños colocados en los coches al sol. El texto demuestra el potencial del recorrido de estudio e investigación para generar preguntas sobre la modelización, y el tipo de matemáticas y física que intervienen en la modelización del cuerpo humano. También sugiere un potencial interdisciplinario más amplio, y en particular vías de investigación que incluyan temas sociológicos y estadísticos.

Keywords: ATD, study and research path, hyperthermia, modelling

Introduction

Two PhD students (the author and another one), took part in a PhD course about study and research paths, and conducted an SRP individually. The generating question Q_0 = “Why do babies die of heat stroke in cars parked in the sun?” was used as the starting point of the SRP. In a handout describing the SRP, Q_0 was succeeded by the following guidelines:

> Are the possible causes of these deaths studied in the scientific literature? If so, what are the physical and physiological or other factors identified by the researchers? Does the fact that children have a greater ratio of the body’s surface area to its volume than adults play a role according to these research studies? Which one? More broadly, what mathematics is useful or even indispensable to model the relevant factors and their interactions?

Three references were also provided as starting points for the SRP, a fact sheet about heat death (The European Child Safety Alliance, 2013), and two scientific papers (McLaren et al., 2005; Booth et al., 2010). A mid-way seminar was held, where the work was presented, and tips on further investigations were shared. A focus on modelling the human body was suggested at this point.

In this paper, I take as a starting point the report I wrote from the conducted SRP, and use it to answer the following research question: "What sort of mathematical models are used to answer Q_0, and how are they interconnected?"

Theoretical tools

In this study, tools from the ATD are used. These tools will be explained in the following section.

Praxeology

The ATD proposes a *praxeology* as a general model of human activity, including activities related to producing, diffusing and appropriating knowledge (Chevallard, 2020). A praxeology is described in terms of four constituents; type(s) of task(s) (T); a technique or set of techniques (τ) used to solve the given types of tasks; a technology (or discourse, θ) used to describe and explain the techniques; and a theory (Θ) that justifies the technology. These constituents are grouped in the two blocks, called the *praxis* block, consisting of T and τ, and the *logos* block, consisting of θ and Θ. A praxeology $\wp$, can then be described algebraically as $\wp = [T / \tau / \theta / \Theta]$.

Study and research path and the Herbartian schema

In this paper the notion of a study and research path (SRP) and the Herbartian schema are also used. A good description of both can be found in (Chevallard, 2020). In short, a Herbartian schema describes the institutional setting, using the notion of a didactical system S, described algebraically as $S(X;Y;\heartsuit)$, where X is the group of students, Y are the study assistants (e.g. teacher, a librarian…), and ♥ the didactic stake, or the *something* that the X are intended to learn.

The stake ♥ can be a question Q_0, the starting point of an SRP. To describe an SRP, the developed Herbartian schema is used, and can be written symbolically as

$$[S(X;Y;Q_0) \Rightarrow \{A^{\Diamond}_i, W_j, Q_k\}] \Rightarrow A^{\heartsuit}.$$

Here the $A^{\Diamond}_i$ are the pre-existing answers, found in the literature and other sources of information, from which are also extracted the works W_j. These answers and works give rise to derived questions Q_k, and the final answer $A^{\heartsuit}$, which is seen by the participants of the SRP as answering Q_0 to a satisfactory level. The final answer, $A^{\heartsuit}$, is an aggregate of intermediate answers A_k to the derived questions Q_k.

Description of the SRP

The SRP, based on the question Q_0, about heat stroke, with the extra premise that answers should be backed mathematically, and the focus should be on physiological and physical reasons, was conducted over approximately one month in 2019. The background for Q_0 is in the American statistics about heat death and babies left in cars. In 2019, 51 heatstroke deaths among children in parked cars were reported in the U.S., a slight reduction from the 2018 record high of 53 deaths (National Safety Council, n.d.). This issue has gained regular attention from media (e.g. Paybarah, 2019; Kalaichandra, 2019), and is therefore of public interest.

Methodology

The methodology can be divided in two phases. First, the SRP is conducted by the author himself, occupying the role as a student exploring the generating question. And then, from this, a didactical analysis of the resulting data will be done, with the aim of uncovering the didactical potential of this SRP, particularly what sort of mathematical models emerge from the SRP. In the following two sections, these two phases are described in more detail. Note, however, that these phases are not meant to be strictly consecutive. Analysing the SRP *is* part of conducting it.

Method of analysis

The resulting questions that emerge through the SRP are first categorized thematically, according to what sort of answers that are likely to emerge from a careful study of the questions. This includes both partial questions that are directly extracted from the generating question, and questions that are either asked by, or answered by the literature initially provided in the handout. These questions are then categorized according to the nature of the answers an examination of these questions might provide. Most importantly, what sort of questions are likely to give rise to answers with a clear mathematical content, and what sort of mathematics?

Following this a more thorough literature search is conducted, using these initially extracted questions. The literature search is conducted in two steps. First, the references in the literature found so far are examined, to find out how the answers they provide are argued for. This includes examining what theoretical foundations lie behind the answers, which will be an important selection criterion when deciding which branches to follow in the SRP. Due to the particular interest in applying mathematics to answer the questions, after the first step in the literature search, only the branches that seem to harbour a particular potential for mathematically centred analyses will be pursued.

Initial questions and literature search

The starting point of the SRP is the question Q_0, in addition to a fact sheet from the European Child Safety Alliance (The European Child Safety Alliance, 2013), providing the first pre-existing answer ($A_1^\lozenge$) to Q_0.

A reason for starting with this text is that it gives short, easy to understand answers to 5 questions related to Q_0:

1. Why are babies left behind in cars?
2. What is special about cars?
3. What does it mean to be too hot, and how does this affect babies differently from adults?
4. How common is this phenomenon?
5. How can it be prevented?

Q_5, about prevention, is not presented directly as a question, but as tips directed at parents, about how to prevent hyperthermia in the first place. A related issue, which the text does *not* address directly, is the question about how death can be prevented when a baby is already experiencing hyperthermia, with the possible exception “Dial 112 immediately if you see a child alone in a car”. Therefore, Q_5 is divided in two related questions:

a. How can hyperthermia be prevented?
b. How can death be prevented for babies experiencing hyperthermia?

Q_{5a} is directed at parents and society at large, about how we can prevent hyperthermia in the first place. Q_{5b}, on the other hand, addresses the medical question of saving someone who is already experiencing hyperthermia.

These questions can be addressed in different ways, according to how we expect them to be answered. Q_1 is expected to be answered by sociological or psychological reasons. The fact sheet points at both intentionally leaving the baby behind, and unintentionally due to forgetting the baby, or not being aware of the dangers of overheating in a car. Q_2 has to do with the physical properties of a car, and how this affects heat absorption and heat transfer.

In the rest of the paper, I only consider physical and physiological factors. That means, the questions Q_2 and Q_3 mainly. In particular, the differences between children and adults are interesting, since it seems to lend itself to a challenge of modelling, and they relate to the core issue of why this is specific to small children.

These questions have also been answered to greater or lesser degree in the literature which the European Child Safety Alliance base their fact sheet on. I will here present some of the papers and webpages that the fact sheet refers to, what sort of answers they give, and how they arrive at these answers. I will also include some papers and texts found by a limited literature search on the keywords “heat death”, “babies” and “parked cars”.

In the papers referenced by the fact sheet, several different physical and physiological factors are mentioned, and in the next section, I present one example of a physical variable that is discussed.

An example of a physical variable: The colour of the interior of the car

The fact sheet refers to a number of articles and webpages which answers Q_2 and Q_3 experimentally (McLaren et al., 2005; Null 2010) (W_1 and W_2), and by examining 231 lethal hyperthermia cases (Booth et al., 2010) (W_3). In these, the question of interior colour of the car (Q_6), in addition to whether cracking open a window would help in regulating temperature (Q_7). Here, only a lighter interior colour of the car seemed to have any significant effect.

Other papers and webpages also deal with the same question, but the overall trend seems to follow the above-mentioned texts. An experiment from 1995 on two differently coloured cars (Gibbs et al., 1995) (W_4) had the same conclusion as W_1 and W_2. The 1995 paper differed only in adding that the *exterior* colour of the car did not matter significantly. They also mention more clothes, cushioned seats, and a position below window level as reasons for small children being particularly vulnerable.

About allometry

None of the articles and webpages presented so far have dealt directly with modelling the human body to determine the heat response of children, although they do mention its results. Both W_1 and W_3 mentions the surface area to volume ratio as having an effect, and W_3 refers to another paper (Tsuzuki-Hayakawa et al., 1995) (W_5), where they showed that small children had a higher and faster heat increase than their mothers when exposed to moderate heating. They suggested two possible explaining factors. The difference in surface area to mass ratio would result in the bodies of young children being easier to heat, and the thermoregulatory systems, sweating and blood circulation among others, might be less developed in children.

Both these themes, the surface area to volume ratio and the effectiveness of the thermoregulatory system, are important factors in the study of scaling in biology, also called allometry. Therefore, “allometry” and “biological scaling” are also included as keywords in the literature search, in trying to answer the question of what role biological scaling has in answering why children are more vulnerable to overheating (Q_8). In addition, to explain overheating, the question of how heat transfer works also needs to be answered (Q_9).

The first text found by this extension of the search, is the text “Allometry: the study of biological scaling” (Shingleton, 2010) (W_6). This is an educational article presenting the concept of allometry. It introduces the knowledge that different parts of the body scales at different rates in relation to the overall size of the body. Examples from the human body are the heart, which grows at approximately the same rate as the body itself, and the brain, which grows slower than the rest of the body. Of specific mathematical interest, the article also presents the fact that many of the observed scaling relationships turned out to be linear, when plotted on a log-log plot, and follow the equation

$$\log y = \alpha \log x + \log b,$$

which can be written as

$$y = bx^{\alpha}.$$

Here x is body size, y is the organ size, and $\log b$ is the y intercept. The factor α is called the allometric coefficient. Each organ has its specific allometric coefficient, and the size of the coefficient describes how fast a certain body part grows relative to the growth of the rest of the body. A body part having a higher growth rate than the rest of the body, then $\alpha > 1$, and conversely, when $\alpha < 1$ the growth rate of the body part is lower than the rest of the body. Further, the text expands the concept of allometry to include other aspects, such as running speed and metabolic rate.

The last of these two was mentioned in W_5 as one of two dimensions providing explanations to why children are more vulnerable to overheating. The text W_6 does, however, not describe the surface area to volume ratio as one of the allometric variables. By expanding the search to also include the word “surface area” together with allometry, some new texts were found.

The first article showing up was from Britannica (Glitterman, n.d.) (W_7), describing both these dimensions as important measures that displays allometric scaling. Area and body mass are related by area growing by a 2/3 power of the body mass, and metabolic rate grows by a 3/4 power of the body mass. And the second was chapter 4 in an online textbook in biology (Sam Houston State University, n.d.) (W_8), where the relation between surface area and volume is explained through geometrical examples. The first scholarly article showing up in this search (W_9), was an article on how surface area scaling on both microscopic and macroscopic levels are related (Okie, 2013). It explores the different strategies organisms have for dealing with the challenges related to how surface area and volume scales at different rates, and it develops a theory for modelling the effects of these different strategies. The details in this last article go far beyond the scope of this SRP, but the ubiquity of the surface area to volume scaling problem as an explanatory factor in biology which W_9 refers to lends weight to the importance of this dimension.

Heat transfer

The last piece in this puzzle is the question Q_9 about how heat transfer works on bodies. Here the search term "heat transfer", in addition to surface area to volume ratio, was used. The first promising article that was found using this search term was an article describing interdisciplinary teaching in physics and biology, where the heating problem is a major theme (Planinšič & Vollmer, 2008) (W_{10}). Here an example teaching unit is described where they experiment with melting cubes of cheese, and use Newton's law of cooling, in addition to the surface area to volume ratio. Then they apply the physics learned from this to explain the differing metabolic rates of different sized animals. For a more in depth description of Newton's law of cooling, W_{10} references another paper (O'Sullivan, 1990) (W_{11}). Here the law is described in its differential form, which will be used in this SRP. A similar description can also be found at a mathematics teaching web resource (math24.ner, n.d.) (W_{12}). This site also describes the role of heat capacity on the system.

Modelling the human body

From the answers provided by literature, a simple model can be described, based on four assumptions. First, the shape of an ordinary person is largely consistent, and thus a "large" person is just a scaled-up version of a "small" person. Thus, the only dimension important for determining how well a person can stand up to heat exposure is the height. Secondly, all tissues of a person have the same heat conductivity. Thirdly, transfer of energy between a body and the environment is mainly dependent on and proportional to the surface area of the body, while the total temperature is proportional to volume.

In the following calculations, the relations and formulas found in literature are used, particularly W_8 and W_{11}.

A consequence of assuming the growth of a person as purely geometric scaling is that we can use some general geometry true for all bodies in three-dimensional space, following the argumentation shown in W_8. As the dimensions (length, width, and height), scales linearly, the surface area scales quadratically, and volume cubically. Moreover, I will assume that density, heat conductivity and heat capacity is relatively similar for all human bodies. From W_{11}, we also get Newton's law of cooling:

$$\frac{dQ}{dt} = h \cdot A \cdot (T_a - T(t)).$$

Here Q is the thermal energy of a body, A is the surface area, T_a and $T(t)$ are the ambient and body temperatures respectively, and h is a heat transfer coefficient. The energy transfer and the body temperature are both time dependent, meaning that without more information, we are not able to solve this equation. But we can suggest another equation, by the observation that the total thermal energy contained in a body is proportional to the temperature, and dependent on the total heat capacity of the body:

$$\frac{dQ}{dt} = C\frac{dT}{dt}.$$

Here C is the total heat capacity, which is the product of the mass (*m*) and the mass-specific heat capacity of the material (*s*) (Chang, 2008, p. 186). Since the mass is proportional to volume (*V*), we can write this relation as $C = m \cdot s = V \cdot c$, where c is a constant factor. Combined, these two equations give us the equation

$$\frac{dT}{dt} + \frac{A \cdot h}{V \cdot c} T = \frac{A \cdot h}{V \cdot c} T_a,$$

which has the general solution

$$T(t) = T_a + De^{-\frac{A \cdot h}{V \cdot c}t}.$$

Assuming $t = 0$ at the beginning of the heating we can see that $T_0 = T(0) = T_a + D$ implies $D = T_0 - T_a$. D is therefore the temperature difference between the body and the surroundings.

We can interpret the solution as temperature difference decreasing according to $e^{-\frac{A \cdot h}{V \cdot c}t}$. When the body temperature is lower than the ambient temperature, this results in the increase in body temperature. The speed of the temperature change is then dependent upon the coefficient $\frac{A}{V} \cdot \frac{k}{c}$. If both k and c are constants, the only variable parameter is the fraction A/V which is proportional to $L^2/L^3 = 1/L$, where L is height. This fraction decreases as height increases, and consequently, a shorter person, such as a baby, is more prone to heating.

Constructed models and answers to Q_0

From the above argument and calculations, a potential $A^♥$ can be described. Since the body of a child is smaller than the body of an adult, the child has a larger surface area to volume ratio than the adult, and the effect of heating is then much greater on the child than the adult (A_3). In concert with other factors, such as positioning in and thermal characteristics of the car (A_2), and a less developed system for heat regulation (A_8), this makes for a deadly combination. This provides an answer to why children that are left in cars on sunny days are prone to die of heat stroke. The path of the study and research is modelled by a directed graph in Figure 1. The elements of the milieu and intermediate answers are displayed in Table 1.

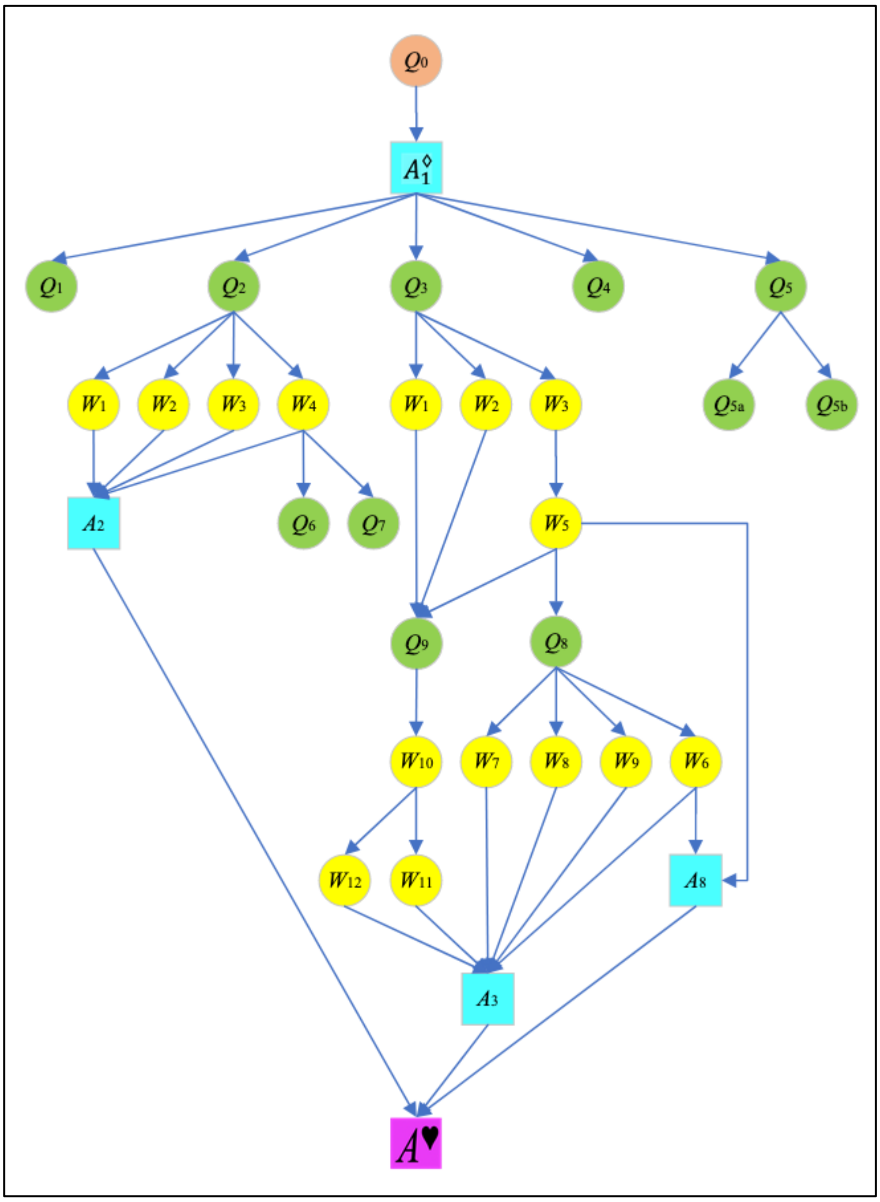


Figure 1: Schematic representation of the SRP

Questions	
Q_0	Why do babies die of heart stroke in cars parked in the sun?
Q_1	Why are babies left behind in cars?
Q_2	What is special about cars?
Q_3	What does it mean to be too hot, and how does it affect babies differently from adults?
Q_4	How common is this phenomenon?
Q_{5a}	How can hyperthermia be prevented?

Q_{5b}	How can death be prevented for babies experiencing hyperthermia?
Q_6	How does the colour (interior/exterior) of the car affect heating?
Q_7	How does opening a window affect heating?
Q_8	Why are children more vulnerable to overheating?
Q_9	How does heat transfer work on a body?
Existing answers to Q_0	
$A_1^{\lozenge}$	European Child Safety Alliance, 2013
Additional works	
W_1	McLaren et al., 2005
W_2	Null, 2010
W_3	Booth et al., 2010
W_4	Gibbs et al., 1995
W_5	Tsuzuki-Hayakawa et al., 1995
W_6	Shingleton, 2010
W_7	Glitterman, n.d.
W_8	Sam Houston State University, n.d.
W_9	Okie, 2013
W_{10}	Planinšič & Vollmer, 2008
W_{11}	O'Sullivan, 1990
W_{12}	math24.ner, n.d.
Intermediate answers	
A_2	Positioning and thermal characteristics of the car
A_3	Effect of the surface area to volume ratio
A_8	Development level of thermo-regulatory system

Table 1: Elements of the SRP

In this SRP, the choice of focusing on physical and physiological factors lead to the development of the model described in the last section, where the role of the surface area of different sized bodies is used to explain differences in how heating affects the bodies. And models explaining the heating of cars have been referenced, but not fully described.

However, other choices could have been made. Several of the questions proposed in the initial exploration can be answered following methods that are different from the ones used to answer Q_3 and Q_2. Notably, several of them seem to lend themselves more to investigations of sociological factors than to physical or physiological. This highlights the interdisciplinary potential of Q_0. Although the focus here is on physical and physiological factors, using geometry and differential equations to model the heating problem, there is also potential for investigating social and sociological factors, in addition to other disciplines within mathematics, such as statistical modelling.

References

Booth, J., Davis, G., Waterbor, J., & McGwinJr., G. (2010). Hyperthermia deaths among children in parked vehicles: anA analysis of 231 fatalities in the United States, 1999 –2007. *Forensic Science, Medicine and Pathology*, 6, 99–105.

Chang, R. (2008). *General chemistry. The essential concepts* (5th ed.). Mc Graw-Hill Higher Education.

Chevallard, Y. (2020). Some sensitive issues in the use and development of the anthropological theory of the didactic. *Educação Matemática Pesquisa, 22*(4), 13–53.

Guard, A., Gallagher, S.S. (2005). Heat related deaths to young children in parked cars: analysis of 171 fatalities in the United States, 1995–2002. *Injury Prevention*, *11*, 33–37.

Gibbs, L., Lawrence, D., & Kohn, M. (1995). Heat exposure in an enclosed automobile. *J La State Med Soc. 6*, 545-546

Glitterman, J. L. (n.d). Allometry. *Encyclopedia Britannica*. https://www.britannica.com/science/allometry

Kalaichandran, A. (2019, August 14). How we can help prevent children from dying in hot cars. *The Washington Post*. https://www.washingtonpost.com/opinions/how-we-can-help-prevent-children-from-dying-in-hot-cars/2019/08/13/b49a016c-be0d-11e9-a5c6-1e74f7ec4a93_story.html

Math24 (n.d.). Newton's law of cooling. *Math24.* https://math24.net/newtons-law-cooling.html

McLaren, C., Null, J., & Quinn, J. (2005). Heat stress from enclosed vehicles, moderate ambient temperatures cause significant temperature rise in enclosed vehicles. *Pediatrics*, *116*(1), 109–112.

National Safety Council (n.d.) Hot car deaths. https://injuryfacts.nsc.org/motor-vehicle/motor-vehicle-safety-issues/hotcars/

Null, J. (2010, June 30). Hyperthermia death of children in vehicles. https://web.archive.org/web/20100701004956/http://ggweather.com/heat/

Okie, J. G. (2012). General models for the spectra of surface area scaling strategies of cells and organisms: Fractality, geometric dissimilitude and internalization. *The American Naturalist, 181*(3), 421–439. https://doi.org/10.1086/669150

O'Sullivan, C. T. (1990). Newton's law of cooling – A critical assessment. *American Journal of Physics 58*(10), 956–960. https://doi.org/10.1119/1.16309

Paybarah, A. (2019, August 1). Painfully common: 52 children died last year in hot cars. *New York Times.* https://www.nytimes.com/2019/08/01/nyregion/newyorktoday/hot-car-deaths-nytoday.html

Planinšič, G., & Vollmer, M. (2008). The surface-to-volume ratio in thermal physics: from cheese cube physics to animal metabolism. *European Journal of Physics, 29,* 369–384.

Sam Houston State University. (n.d.) *Biological design*. https://www.shsu.edu/~bio_mlt/Chapter4.html

Shingleton, A. (2010). Allometry: The study of biological scaling. *Nature Educational Knowledge, 3*(10). https://www.nature.com/scitable/knowledge/library/allometry-the-study-of-biological-scaling-13228439/

The European Child Safety Alliance. (2013, September 26). Children in Hot Cars: Hyperthermia. https://www.safecommunitiesportugal.com/wp-content/uploads/2021/02/Children-in-Hot-Cars-Hyperthermia.pdf

Tsuzuki-Hayakawa, K., Tochihara, Y., & Ohnaka, T. (1995). Thermoregulation during heat exposure of young children compared to their mothers. *European Journal of Applied Physiology 72*, 12-17.

About the Use of QA-Maps in the Development of Lesson Plans by Student Teachers

Reinhard Hochmuth[1] and Jana Peters[2]

[1]Leibniz University Hannover, Faculty of Mathematics and Physics, Hanover, Germany; hochmuth@idmp.uni-hannover.de

[2]Leibniz University Hannover, Faculty of Mathematics and Physics, Hanover, Germany; peters@idmp.uni-hannover.de

In the focus of the reported observations are lesson plans developed by preservice student teachers for inquiry-oriented teaching units on school-related topics. We analyse them with regard to which generating questions Q_0 are considered or whether and to what extent initial questions can be judged to be generating. In particular, we are interested in the QA maps emanating from Q_0 and how these are used in lesson planning. Against the background of the subject-scientific learning theory we explore in some detail the impact of higher levels of codetermination and discover under the surface of an inquiry-oriented lesson planning a de-subjectivised view of learning as a central aspect of the prevailed societal determined school formation of learning processes.

Keywords: Teacher education, inquiry, lesson plans, levels of codetermination.

Introduction

Inquiry-oriented mathematics teaching (IOMT)[1] aims at pupils' deeper understanding of mathematical concepts integrating in particular rationales and uses. IOMT imposes specific requirements for the professional preparation and structuring of the mathematical content to be taught (Artigue & Blomhøj, 2013; Jaworski, Gómez-Chacón & Hochmuth, 2021). Moreover, teaching methods as well as institutional conditions and contexts have to be reflected under which IOMT can be realised. Last but not least, the question of which didactic means are suitable for the a priori preparation and a posteriori reflection of lessons is important. Of course, these issues cannot be answered independently of each other: Tools and designs have to take into account institutional conditions, but, on the other hand, they also have the potential to change those conditions.

In ATD, IOMT is discussed in the realm of the Paradigm of Questioning the World (PQW) in contrast to the Paradigm of Visiting Works (PVW) (Chevallard, 2015). The former includes the establishment of Study and Research Paths (SRPs), generating questions Q_0 and Question-Answer maps (QA maps) (Bosch, 2018; Bosch & Winsløw, 2015; Jessen, 2017). Related ecological issues are discussed with regard to higher levels of codetermination (Barquero, B., Bosch, M., & Gascón, J., 2013). This involves examining key aspects of teaching in terms of various dialectics: the dialectics of the media and the milieu; the individual-collective dialectics; the dialectics of questions and answers; the

[1] We use the notion IOMT to cover the certainly different approaches of traditional Inquiry Based Mathematics Education and the ATD concept of SRPs. We speak here explicitly of teaching and not of learning, in order to mark that no learning whatsoever can be ensured by a certain teaching (cf. the discussion of the technology deficit by Luhmann & Schorr (1982)).

dialectics of the dissemination and reception of answers. Also, adidacticity as well as the design of the didactic contract are pointed out as important aspects of successful SRPs: “Both teacher and students need to accept new roles and responsibilities that take time and effort.” (Barquero, B., Bosch, M., Florensa, I., Ruiz-Munzon, 2020, p. 177). In lesson planning, both would be reflected in adopting a student-centred perspective as opposed to a transmissive teacher-centred perspective.

In this paper we focus on lesson plans of preservice student teachers, who aimed to develop IOMT units on school-based topics as part of their bachelor’s thesis. We analyse them in terms of the extent to which generating questions Q_0 can be judged as actually generating. In particular, we are interested in the QA maps emanating from Q_0, their characteristics and how these are used in lesson planning. Thereby the teaching setting for which the students developed QA maps and lesson plans was not determined in advance. As a rule, however, a traditional teaching setting was chosen by the teacher students, i.e. 90 minutes of teaching in a school class of about 25-30 students, as well as official curricular guidelines were considered. Given these premises, it is certainly fair to ask whether realising the potential of QA maps or IOMT-oriented lesson plans are even possible.

On the other hand, there has been hardly any detailed investigation of whether and how the content-related structuring possibilities of QA maps in lesson planning are (or can be) typically used in the context of a traditional teaching setting, an almost complete adaptation to curricular guidelines and, this is a central point for us, how higher levels of codetermination and corresponding perceptions of student teachers show up in such uses. However, to eventually overcome the dominant PVW and the accompanied transmissive teacher-centred perspective, it is necessary to trace how PVW prevails in arguments and justifications. Among other things, this is of particular importance for the teaching of subject didactics for student teachers, as it enables us to identify starting points for our teaching with regard to IOMT and SRPs.

So far, the role of higher levels of codetermination, like civilisation, in the context of SRPs has essentially been discussed in general terms. Bosch (2018) states:

> It seems clear that the paradigm of questioning the world represents important changes in the organisation of study processes at the different levels of the scale of didactic codeterminacy. Those at the level of civilisations are possibly the most hidden ones, since they correspond to beliefs or assumptions that are difficult to identify, unless we move to another civilisation, through the space or the time. However, changes performed at the higher levels of the scale will remain limited if they do not come with the corresponding modifications at the lower levels. (Bosch, 2018, p. 4051)

Thus, although ATD refers to these higher levels and their importance, to our knowledge there is no specific part of ATD that could be used as a basis for corresponding studies. Regarding higher levels of codetermination, we draw on the subject-scientific learning theory developed in the context of critical psychology (Holzkamp, 1993; for an introduction see also Schraube & Osterkamp, 2013). Already in (Hochmuth & Peters, 2022) it has become apparent that subject-scientific categories are useful to trace certain aspects of higher levels in the institutional enforcement of prevailing praxeologies at lower levels. In the context of this paper, it is in particular relevant that societal insights allow the prevailing school teaching-learning relationships to be characterised as instrumental learning relationships. In this context, we also distinguish between thematic-content and operational-organisational goals of learning actions. Because of the dependence of operational aspects on the superordinate thematic aspects of learning, the latter are also referred to as the primary and the former as the secondary learning objectives. Then an aspect of instrumental learning relationships is that learning and its arrangement is conceived as an operational-goal-based learning in which secondary operational goals dominate and primary-thematic goals tend to be subordinated (Holzkamp, 1993, p. 414).

Against this background we explore aspects of higher levels of codetermination and their role in argumentation and justifications in the choice of questions Q_0, the design of QA maps and their use in developing lesson plans. We emphasise that we do not evaluate issues with the use of QAs as individual deficits of student teachers, but rather as concrete manifestations (symptoms) of the dominant school disciplinary arrangement in its historically determined overall societal mediatedness, as worked out in detail in Holzkamp (1993).

In the next section, we briefly discuss the data and their methodological evaluation. Then we present preliminary observations and results on the following three research questions:

RQ 1 What can be observed regarding the mathematical knowledge that student teachers should have in order to identify a suitable Q_0 and elaborate QA maps?

RQ 2 What are the characteristics of the design and use of generating questions and QA maps?

RQ 3 What societal determined aspects in terms of the highest levels of codetermination can be observed?

Finally, we summarise our findings with respect to the ecological dimensions addressed above and outline some implications for teacher education.

Data

Students for the teaching profession at secondary schools at Leibniz Universität Hannover must attend a bachelor seminar (2 SWS) in which they prepare an exposé for their Bachelor's thesis. At this point, the students have already successfully completed almost all mathematical subject lectures to the extent of approximately a Bachelor's degree in mathematics. In the seminar, the following topics were discussed with the students: (1) forms of explorative-discovering learning (Winter, 1989; Barzel, Holzäpfel, Leuders, Streit, 2012), (2) subject didactics as didactically oriented content-analysis, (3) concepts from ATD (PVW, PQW, praxeologies, SRPs and SRAs, generating questions and QA-maps), and (4) a four-level approach to design research (Hußmann & Prediger, 2016) including a problematisation of semantic and syntactic issues on the basis of basic ideas (in German "Grundvorstellungen"), which are taught in previous modules. After finishing the seminar, the students had to choose an arbitrary school subject and to develop an inquiry-oriented teaching unit focusing in particular on an IOMT-oriented elaboration of the subject matter. They could use QA maps, but did not have to.

In our analyses of 20 theses, we oriented ourselves to steps of a qualitative content analysis (Gläser & Laudel, 2009, pp. 191): After a primary overview, we accordingly developed a consensual structuring of issues which appear in the theses. For this paper we selected topics that appear most regularly and chose representative quotes.

Observations and Results

What can be observed regarding the mathematical knowledge that student teachers should have in order to identify a suitable Q_0 and elaborate QA maps? (RQ 1).

Assuming that the official curricular guidelines are accepted unquestioned in the context of one's own teaching development, the identification of generating questions Q_0 and the elaboration of QA maps is particularly about transforming the curricularly given knowledge into sequences of questions. We can observe in a number of cases that even a comprehensible transformation of given knowledge elements (PVW) into sequences of questions represents an excessive demand. In particular, we found that the professional, e.g., content-related, foundations for determining Q_0 and designing QA maps, as well as linking them to a teaching proposal, are only available to a very varying degree of flexibility and certainty. In particular, weaker theses do not succeed in linking school mathematics with questions that can be interpreted in any way in terms of Q_0 and/or with a view to QA maps. Considering that the authors of the theses are almost at the end of their mathematical studies, it is remarkable that in general the possibly newly acquired mathematical knowledge at university in particular does not contribute to didactic considerations of school mathematics that question its naturalisation. This naturalisation can generally also be seen as an effect of Klein's double discontinuity in the context of the two institutions of school and university. In view of our observations, the more specific research question to be addressed in the future arises, how students could be better prepared for PVW with regard to their basic mathematical knowledge and how helpful mathematical content could be characterised for this.

What are the characteristics of the design and use of generating questions and QA maps? (RQ 2)

With regard to the IOMT character of questions Q_0, QA maps and elaborated lesson plans, we roughly summarise our observations into three characteristic types, regarding the following issues: mode of relation between Q_0 and the curricular standards; the relevance of content-related considerations in the design of lessen plans; the role of learners and teacher's perspective; and properties of the structuring of QA maps. We first describe characteristics of a predominant basic type [1]. Then we specify characteristics of theses in which an IOMT-orientation can be observed to a lesser extent [2], and address finally characteristics of theses that show more IOMT-orientation compared to the basic type [3]. Basically, among all, the theses considered here are on a good to very good level of mathematical knowledge. This means that especially the school mathematics is presented correctly and well-structured and the lesson plans are elaborated understandably and in detail. The rather critical tone of the following characterisations does not reflect grades but focus on the IOMT orientation.

[1] Observations on the basic type *Curriculum dominance:* The initial questions Q_0 arises directly from the curriculum. They read like curricular instructional objectives and competences reformulated into questions. For example, the question “How can the stock of known rates of change and initial

stock be determined as precisely as possible?[2]" addresses the instructional goal of understanding the integral in terms of the concept of reconstruction and accumulation.

Lesson planning is mainly determined by instructional-methodical and secondary operational aspects: This is reflected in formulations such as

> A phase follows in which the pupils are to exchange ideas with their seat neighbours and discuss the context. This is followed by an exchange of ideas about what can be done with the task, taking into account the guiding question. At the same time, this step ensures that any individual misunderstandings of the situation can be uncovered and cleared up through discussion with the partner.

Crucially, neither the choice of group work and its homogeneity nor the misconceptions raised are reflected in relation to mathematical content. In other words, the concrete contents do not play a role for justifications of the chosen social form.

Neither in the QA maps nor in the lesson plans are the learners represented as intentional acting agents: Formulations like

> The pupils reconstruct a quantity of water from the given inflow and outflow rate. For this, they have to form products of time periods and rate of change and add them up for the total effect. This introduces the idea of reconstruction, which is then further developed in the second task.

describe sequences of pupils' actions that the teacher wants to see, but why the learners should act in this way and for what their possible reasons could be is not addressed. Also, *the concept of basic ideas is addressed only from the perspective of the teacher***.** In principle, concepts such as basic ideas can, in fact, be considered from the perspective of the teacher, as a representative of the curriculum, or from the perspective of the learner in lesson planning. Here, despite the claim of IOMT, they are usually treated from the teacher's perspective.

[2] There are theses with a weaker tendency towards IOMT, in which the following aspects can be pointed out beyond what has been mentioned so far: *The questions in the QA maps do not differ much or not at all from structuring of a thematic field along formal-mathematical connections made from the traditional teaching perspective*. Thus, they rather correspond to concept maps, as they are sometimes found in school textbooks for orientation, only the entries are also formulated as questions. The "arrows" do not address potentially intentional learning steps by pupils that are necessary or meaningful in terms of content, but rather mark intended instructional interventions by the teacher. Accordingly, the learning steps planned in relation to the learning objectives are only described in abstract terms, and subject-specific didactic reflection on them remains general. This is also reflected in the fact that alternatives of the content-related-thematic instructional design are not discussed. The little elaborated and reflected content-related connections are compensated by numerous instructional-organisational considerations. These observations lead us to the assessment that lessons

[2] All quotations from student teachers' theses were translated from German into English by the authors.

conducted according to such lesson plans suggest the learning of mathematics in the sense of "insular knowledge".

[3] *Theses with a stronger tendency toward IOMT* show the following characteristics: *The questions in the QA maps are formulated in terms of content in a meaningful learning process context and can be understood, at least from a teaching perspective, in terms of learning paths*. Thereby, primary-thematic reference contexts are addressed beyond curricular specifications, which provide the pedagogical structuring of the subsequent lesson plans with a certain dynamism in terms of content. Interestingly, some theses explicitly state an ambivalence regarding learner intentionality: "Ideally, the QA map questions are posed by the pupils so that the lesson can be constructed based on these questions." and

> However, it should be emphasised that the questions mentioned are only anticipated by the teacher; the actual questions asked depend on the class. In addition, the pupils must be given space and time to ask their questions. In the context, however, questions could be asked that are thematically inappropriate or cause confusion among the pupils.

The avoidance of confusion, for example as a result of a lack of unambiguity, then presents itself as a teaching task, which is solved in particular by means of instructional-methodical decisions. For example, group work is specifically suggested for the purpose of "normalisation":

> Modelling could lead to difficulties for pupils because it is not always clear for them which information is relevant and to be mathematised. For this reason, it is important that the [pupils] not only work independently, but can exchange information with other classmates.

In other words, in lesson plans, primary subject-related learning processes, even to the extent that they are hinted at or explicitly mentioned as possibilities in QA maps, are ultimately "contained" in terms of instructional methodology and thus placed at the disposal of the teacher.

What societal determined aspects in terms of the highest levels of codetermination can be observed? (RQ 3)

Against the background of subject-scientific learning theory, the observations outlined above can be summarized to the effect that the use of the subject-didactic tools Q_0 and QA maps is in the service of establishing instrumental learning relations, which can be seen as a historically specific, societal mediated characteristic of interpersonal relationships (Holzkamp, 1993, pp. 526). In particular the role, which is assigned to them in the context of the ATD towards IOMT is subverted and instead of a transformation to the PQW they serve the reproduction of PVW.

The creation of instrumental learning relation is reflected in a shift from primary-thematic content learning aspects to secondary-operational ones. Secondary-operational aspects address generic goals such as problem-solving skills (as already in Bruner's concept of discovery learning) or process-related competencies (cf. also Gascón, 2011). Also, the concept of basic ideas is used in this way. Overall, lesson planning tends to deprive pupils of the opportunity to make thematic relevant questions to their own problems and then solve their problems through learning. Instead of planning the teaching units from a student-centred perspective, the QA maps are used as a tool for organising teacher centred activities. Initial questions are directly "knowledge questions" of the teacher or the curriculum to which there are correct and incorrect answers. The QA maps merely reproduce the context represented in textbooks and its sequence in a different guise. The pupils are expected to prove themselves as “good” pupils in the sense of PVW.

Thus, under the surface of an inquiry-oriented lesson planning, subject matter potentials are not explored and possible pupils’ learning activities are reduced to secondary operative aspects, i.e., learning activities are “normalised away” in the direction of defensive learning (Holzkamp, 1993, pp. 190). Content-related learning as independent and intentional learning has no place. These observations reflect results of Holzkamp’s analysis of the societal determined school formation of learning processes including as central aspect a de-subjectivised view of learning and indicates the role of higher levels of codetermination for an understanding whether and, if so, which possibilities of the proposed tools are used and how.

Summary and Outlook

With respect to the ecological issues mentioned in the introduction and how these have already been dealt with in ATD, we can summarise our preliminary results as follows:

- Regarding the dialectics of the media and the milieu, we observe teacher dominance. All material is ultimately provided to the pupils by them and their knowledge acquired is assessed by the teacher based on the given curriculum.
- In lesson plans, the pupils are essentially addressed in terms of instructional-methodological decisions. Accordingly, in the individual-collective dialectics, primary thematic-content issues play a subordinate role.
- The dialectics of questions and answers does not function in the sense of an SRP including intentional learners but, as far as a dialectic becomes visible at all, it is addressed from the teacher's perspective and is a priori fixed in lesson plans. Instead of a dialectic of questions and answers, there is usually the traditional organisation of knowledge cast in the form of questions and ready-made answers.
- The dialectics of the dissemination and reception of answers is also not thought of in terms of content, but is reduced to methodological considerations in relation to secondary operational aspects.
- The roles assigned to pupils and teachers are more or less the traditional ones.

With a view to our future teaching the following conclusions can be drawn: The societal suggested dominant forms of thinking and practice, which, as illustrated, limit the use of the tools, should be explicated and problematised with the teacher students in advance. This is at least possible by way of example. A deeper reflection of societal relations would be desirable, but seems only possible with a small number of students under the current study conditions.

Furthermore, the bachelor-theses show the need to work with students on the deep structure of school related mathematics. ATD provides suitable tools for this, such as praxeologies and in terms of lesson planning, the concept of didactic moments. A hurdle for many students will then be that they dispose of mathematical techniques, but to a very different degree of rationales. However, the latter would specifically helpful for working out of QA maps. Although the bachelor’s theses are written at the end of their university subject studies, it seems necessary to discuss with them in detail school subject matter. Finally, against the background of the students' available subject didactic knowledge, such as basic ideas or competencies as concepts, “alternative” uses should be made explicit that more strongly address a learner's perspective.

References

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM, 45*(6), 797-810. http://dx.doi.org/10.1007/s11858-013-0506-6

Barquero, B., Bosch, M., Florensa, I., & Ruiz-Munzon, N. (2020). How to integrate study and research paths into university courses? Teaching formats and ecologies. In T. Hausberger, M. Bosch & F. Chellougui (Eds.), *Proceedings of the Third Conference of INDRUM* (pp. 169-178). Bizerte, Tunisia: University of Carthage and INDRUM.

Barquero, B., Bosch, M., & Gascón, J. (2013). The ecological dimension in the teaching of mathematical modelling at university. *Recherches en didactique des mathématiques, 33*(3), 307-338.

Barzel, B., Holzäpfel, L., Leuders, T., & Streit, C. (2012). *Mathematik unterrichten: Planen, durchführen, reflektieren.* Cornelsen.

Bosch, M. (2018). Study and Research Paths: A model for Inquiry. *Proc. Int. Cong. of Math. 2018, Rio de Janeiro, Vol 4*, 4033-4054.

Bosch, M., & Winsløw, C. (2015). Linking problem solving and learning contents: the challenge of self-sustained study and research processes. *Recherches en didactique des mathématiques*, *35*(2), 357-399.

Chevallard, Y. (2015). Teaching mathematics in tomorrow’s society: A case for an oncoming counter paradigm. In *The proceedings of the 12th international congress on mathematical education*, (pp. 173-187*).* Springer. http://dx.doi.org/10.1007/978-3-319-12688-3_13

Gascón, J. (2011). ¿ Qué problema se plantea el enfoque por competencias? Un análisis desde la teoría antropológica de lo didáctico. *Recherches en didactique des mathématiques, 31*(1), 9-50.

Gläser, J., & Laudel, G. (2009). *Experteninterviews und qualitative Inhaltsanalyse als Instrumente rekonstruierender Untersuchungen*. VS Verlag.

Hochmuth, R., & Peters, J. (2022). About two epistemological related aspects in mathematical practices of empirical sciences. In Y. Chevallard et al. (Eds.), *Advances in the Anthropological Theory of the Didactic* (pp. 327-342). Birkhäuser. http://dx.doi.org/10.1007/978-3-030-76791-4_26

Holzkamp, K. (1993). *Lernen: Subjektwissenschaftliche Grundlegung*. Campus-Verlag.

Hußmann, S., & Prediger, S. (2016). Specifying and structuring mathematical topics. *Journal für Mathematik-Didaktik*, *37*(1), 33-67. http://dx.doi.org/10.1007/s13138-016-0102-8

Jaworski, B., Gómez-Chacón, I. M., & Hochmuth, R. (2021). Conceptual Foundations of the PLATINUM Project. In I. M. Gómez-Chacón et al. (Eds.), *Inquiry in University Mathematics*

Teaching and Learning: The PLATINUM Project (pp. 7-27). Masaryk University Press. http://dx.doi.org/10.5817/CZ.MUNI.M210-9983-2021-2

Jessen, B. E. (2017). How to generate autonomous questioning in secondary mathematics teaching. *Recherches en didactique des mathématiques, 37*(2/3), 217-245.

Luhmann, N., & Schorr, K.E. (1982): Das Technologiedefizit der Erziehung und die Pädagogik. In N. Luhmann et al. (Eds.), *Zwischen Technologie und Selbstreferenz. Fragen an die Pädagogik*, (pp. 11-41). Suhrkamp Verlag.

Schraube, E., & Osterkamp, U. (2013). *Psychology from the Standpoint of the Subject – Selected Writings of Klaus Holzkamp*. Palgrave Macmillan.

Winter, H. (1989). *Entdeckendes Lernen im Mathematikunterricht.* Vieweg.

Enseñanza de ecuaciones diferenciales en el nivel terciario mediante un REI relativo a los contagios de COVID-19

Patricia Sureda[1] y Laura Rossi[2]
[1]Niecyt, Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Universidad Nacional del Centro de la Provincia Buenos Aires (UNICEN), Tandil, Argentina; psureda@exa.unicen.edu.ar
[2]Instituto Superior San Agustín, Misiones, Argentina; lauraro737@gmail.com

Abstract. We design, implement and analyze the perfomance of a Study and Research Path (RSP) relative of COVID-19. The RSP focuses on teaching Differential Equations at the higher level. Here, we present the analysis of the productions of seven students studying to be a math teacher, in the subject: Applied Mathematics. The results show that the students manifested attitudes typical of research pedagogy, which allowed them to study Differential Equations as functional tools to provide answers to problematic questions.

Keywords: Estimation of infections COVID19, pedagogy of research and questioning of the world, higher level.

Resumen. Se diseña, implementa y analiza el funcionamiento de un Recorrido de Estudio e Investigación (REI) relativo a la propagación del COVID-19. El REI hace foco en la enseñanza de ecuaciones diferenciales en el nivel superior. Se analiza el trabajo de siete estudiantes del profesorado de Matemática en la cátedra de Matemática Aplicada. Los resultados muestran que los estudiantes manifestaron actitudes propias de la pedagogía de la investigación, que les permitió estudiar las ecuaciones diferenciales como herramientas funcionales para dar respuestas a cuestiones problemáticas.

Palabras Clave: Estimación de contagios COVID19, pedagogía de la investigación y cuestionamiento del mundo, nivel terciario.

Résumé. Nous concevons, mettons en œuvre et analysons la performance d'un Parcours d'Études et de Recherche (PER) relatif au COVID-19. Le PER se concentre sur l'enseignement des équations différentielles au niveau supérieur. Nous présentons ici l'analyse des productions de sept étudiants en formation de professeur de mathématiques, dans la matière: Mathématiques Appliquées. Les résultats montrent que les étudiants ont manifesté des attitudes typiques de la pédagogie de la recherche, ce qui leur a permis d'étudier les équations différentielles en tant qu'outils fonctionnels pour apporter des réponses à des questions problématiques.

Mots clés: Estimation des infections COVID19, pédagogie de la recherche et questionnement du monde, niveau supérieur.

Introducción

Los cursos de enseñanza de ecuaciones diferenciales (ED) en el nivel terciario y universitario, a menudo consideran que el trabajo "rico" del alumno es resolver ecuaciones diferenciales por diversos métodos a fin de lograr las soluciones explícitas de las mismas. Pero esta forma de enseñanza, reduce el saber a un conjunto desarticulado de conceptos y técnicas que carecen de sentido para el estudiante. Pues la relevancia, esencia y razón de ser de las ED se encuentra en la gran utilidad que poseen, como por ejemplo, en epidemiología, para modelizar la propagación de enfermedades infecto-contagiosas en una población. Por lo tanto, es posible pensar que su estudio escolar emerja de manera funcional como herramienta para dar respuestas a cuestiones problemáticas (Chevallard, 2017). En esta dirección, uno de los objetivos fundamentales de esta propuesta es estudiar las ED como respuesta a la cuestión generatriz Q_0*: ¿Cómo se estima la cantidad de infectados por COVID-19?,* en un intento por introducir en el nivel terciario algunos gestos de una nueva epistemología que permita reemplazar el paradigma de "inventario" de saberes, por una pedagogía de la investigación y cuestionamiento del mundo.

Marco Teórico

Para este trabajo se adoptan los constructos teóricos de Modelización, Recorridos de Estudio e Investigación (REI) y las actitudes de problematización, de ser herbartiano, procognitivo, exotérico y enciclopedista ordinario.

Una de las críticas a la enseñanza tradicional, es la de utilizar la modelización matemática como aplicación de un conocimiento matemático previamente construido a situaciones "reales". Una forma de superar las restricciones transpositivas a las que se enfrenta la modelización matemática para su supervivencia es mediante el diseño y propuesta de un Recorrido de Estudio e Investigación (REI). Para esto, es necesario que la cuestión generatriz prime sobre las respuestas que se les debe aportar y no se convierta, como suele suceder, en el simple hecho de construcciones de saberes que proporcionarán la respuesta (Barquero, Bosch y Gascón; 2011).

El REI es un dispositivo didáctico que organiza el saber en una sucesión de pares de preguntas Q y respuestas R. Las preguntas proveen las razones de ser del estudio y recuperan el sentido que las Organizaciones Matemáticas (OM) perdieron, por mostrarse segmentadas y aisladas de las cuestiones que lo generaron. En particular, en un REI un conjunto de estudiantes X, con ayuda de uno o más profesores Y, estudian una cuestión generatriz Q_0 en búsqueda de una respuesta $R^{\heartsuit}$. Dependiendo de la generatividad de Q_0 y de sus preguntas derivadas, los REI permiten estudiar distintas organizaciones, matemáticas o no, como parte del proceso de construcción de una respuesta válida a Q_0. La relevancia otorgada a las preguntas, es clave para superar el paradigma clásico de "visitar los saberes", e intenta introducir al grupo de clase, en el nuevo paradigma de "interrogar al mundo".

Esta pedagogía descansa en cinco actitudes interrelacionadas (Chevallard, 2013): problematización, de ser herbartiano, procognitivo, exotérico y enciclopedista ordinario.

Herbartiano: Entregarse al estudio de las preguntas, sobre todo matemáticas, en lugar de evitarlas. Procognitivo: La actitud, que en lugar de "remitirse preferentemente y casi exclusivamente al saber ya conocido", se permite abordar preguntas para las cuales no tiene una respuesta. Es decir, conocer siempre hacia adelante. Exotérico: es la actitud de quien no solo estudia para saber, sino también "para verificar lo que cree saber". Actitud de problematización: Se caracteriza por formular preguntas, tal que algunas se conviertan en problemas para al menos un grupo de personas. Enciclopedista ordinario: Se trata de un ciudadano que sabe "poco" de muchos asuntos, pero que está en condiciones de aprender. La contracara de esto es un especialista en matemática que puede ser un enciclopedista ordinario en cocina, música o jardinería.

EL REI

El REI se diseñó en torno a la cuestión generatriz Q_0: ¿Cómo se calcula la propagación de infectados del COVID-19?

El análisis praxeológico y didáctico previo nos permitió inferir que esta cuestión nos permitiría estudiar, por una parte, las Ecuaciones Diferenciales Ordinarias, por ser las que permiten modelizar matemáticamente una enfermedad contagiosa, como es en este caso el COVID-19; y por otra parte, aprender a estimar la cantidad de contagios en el contexto de pandemia que se estaba atravesando durante el estudio de la cuestión.

El diseño del REI se construyó así sobre los modelos matemáticos que utiliza la epidemiología para describir, explicar y predecir los fenómenos y procesos de contagio que se suscitan en enfermedades epidemiológicas. En particular nos decidimos por los modelos determinísticos, debido a que en ellos es posible controlar los factores que intervienen en el estudio del proceso del fenómeno, y predecir con exactitud los resultados. Los modelos determinísticos de contagio que se utilizan son tres: SIR; SEIR y SEAR; y aunque Q_0 nos permitió estudiar en clase los tres modelos, en este trabajo solo describimos lo realizado durante la construcción del modelo epidemiológico matemático SIR. Este modelo, divide a la población de individuos afectados por el virus en tres clases (S, I, R), y se ajusta bastante bien a los primeros resultados de estudios del COVID-19:

S: el grupo de individuos susceptibles de contraer el virus de una enfermedad transmisible, considerando que estas personas no tienen ningún tipo de inmunidad ante el agente infeccioso, lo cual provocaría su contagio ante la exposición del mismo;

I: el grupo de individuos infectados y que son transmisores de la enfermedad al grupo S (susceptible);

R: Es el conjunto de individuos recuperados de la infección, que se convierten en agentes inmunes a la enfermedad y por lo tanto no contagian. Cabe destacar, que en este modelo se considera "recuperados" a los fallecidos debido a que tampoco pueden contagiar.

Así, en el inicio de la pandemia se tiene que la población (N), es la misma que la cantidad de personas Susceptibles de contagiarse el virus (S), esto es: $N = S$ e $I = 1$. Luego la relación $[S \rightarrow I \rightarrow R]$ muestra cómo se produce la variación de Suceptibles a Infectados (I), y finalmente la de Infectados (I) a recuperados (R).

Las ecuaciones diferenciales que determinan las tasas de cambio de los Susceptibles e Infectados son:

$$\frac{dS}{dt} = -\beta \frac{I}{N} S \qquad (1)$$

$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I \qquad (2)$$

$$\frac{dR}{dt} = \gamma I \qquad (3)$$

La primera ecuación diferencial (1), describe la tasa de cambio de las personas susceptibles a las infectadas (S) a medida que transcurre el tiempo (t), que varía solamente cuantos se infectan y pasan de Susceptibles a Infectados (I). La segunda ecuación diferencial (2), muestra la variación de los infectados (I) en función del tiempo (t), que cambian de acuerdo a la diferencia entre los que entran y los que salen de la categoría (Infectados). Por último, la ecuación diferencial (3), muestra la variación de los Recuperados (R), que son las personas que dejan de ser infectadas (I) y entran a ésta categoría (R).

Este modelo establece que se pasa de susceptible a infectado mediante una tasa , que es la tasa de infección por individuos y que representa cuántas nuevas infecciones se producen por unidad de tiempo. Mientras que un infectado se puede recuperar a una tasa . Estos modelos asumen que el tiempo de permanencia en la clase I (el tiempo que permanece infeccioso) está distribuido exponencialmente.

Sin embargo, cuando se estudian epidemias, se debe considerar también el ***número reproductivo básico (*R₀*)***, el cual, técnicamente, representa el número de nuevas infecciones estimadas que puede provocar una persona contagiada en contacto con el grupo de susceptibles (S) durante el tiempo en que el individuo es trasmisor (Diekman, 1990). El R_0 es una variable clave para medir el potencial de transmisión del virus, ya que determina en cada país o región las decisiones sobre las restricciones o flexibilizaciones, que se deberían adoptar. Por ejemplo, en Argentina, cuando se inició la pandemia de COVID-19, el $R_0 = 2{,}3$. Esto es, que cada persona enferma infectaba, en promedio, a 2,3 personas. Luego, como cada una de esas nuevas personas contagiadas, va a contagiar en promedio a otras tantas, en pocas semanas se produciría un crecimiento exponencial de casos.

Si $R_0 = \frac{\beta}{\gamma} > 1$ el crecimiento es exponencial y hay epidemia, en cambio, si $R_0 = \frac{\beta}{\gamma} < 1$ no hay epidemia.

Así, el R_0 determina y proyecta el tamaño de la epidemia.

Finalmente mediante la resolución de ecuaciones diferenciales ordinarias se termina construyendo la expresión exponencial que representa a la cantidad de infectados. Esta es:

$$I(t) = I_0 . e^{k.t}$$

donde la constante $k = \beta - \gamma$. Reemplazando en la ecuación:

$$I(t) = I_0 . e^{(\beta-\gamma).t} \quad (4)$$

La cantidad de infectados por mes depende así de los valores de R_0, β y γ con

$$R_0 = \frac{\beta}{\gamma} \quad (5)$$

Por ejemplo, si se sabe que una tasa de contagio $R_0 = 2{,}5$; hace que al cabo de 30 días se tenga 406 contagiados, se utilizan las expresiones de arriba, para estimar 0.33375 y 0,1335. Finalmente al reemplazar en la expresión algebraica (4) y (5) se tiene que:

$$I(30) = e^{(0{,}20025).30}$$

$$I(30) = 406$$

Que es el resultado al que se quería llegar al finalizar la primera parte del REI.

Durante el diseño del REI, dado que eran las tasas que se manejaban en los medios de comunicación al momento del estudio, sabíamos que les iba a terminar dando un resultado similar.

Metodología

El REI se llevó a cabo en un tercer año del profesorado de Matemática de un Instituto Terciario de gestión semiprivada, en la Provincia de Misiones (Argentina); en la cátedra de Matemática Aplicada. La materia es cuatrimestral y corresponde al tercer año de la carrera. La cursan estudiantes de entre 20 y 30 años.

El REI se implementó en el año 2021 durante dos meses; en ocho clases presenciales de tres horas, y cinco clases virtuales de una hora cada una; con siete estudiantes del profesorado de matemática. Los estudiantes no recibieron información previa para poder contestar la pregunta generatriz propuesta: Q_0*: ¿Cómo se estima la cantidad de infectados del COVID-19?*

Las clases se registraron en audio y se accedió a los registros escritos de los estudiantes.

Análisis de datos

Para dar respuesta a la cuestión generatriz los siete alumnos comenzaron por debatir: la forma en que se producían los contagios, la población factible de contagiarse, la cantidad de personas que podía contagiar cada persona, etc. Agotada la conversación sobre sus saberes y dada la imposibilidad de formular una respuesta decidieron buscar información en internet. Allí, encontraron una noticia de la que extrajeron el siguiente extracto:

> **La universitaria destacó que una persona infectada por este nuevo coronavirus contagia a 2.5 más, así que al cabo de un mes tendríamos 406 nuevas infecciones, "de ahí la importancia de seguir las recomendaciones de permanecer en casa si no es necesario salir, mantener la sana distancia, evitar conglomeraciones y lavarse** las manos con agua y jabón las veces que sea necesario, o utilizar gel-alcohol". Enfatizó que en buena medida, del respeto al autoaislamiento dependerán los resultados finales del paso del COVID-19 por nuestro país.

Figura 1. Noticia sobre cantidad de contagiados tomada de Infobae (28 de marzo 2020)

Al leer la noticia los estudiantes comenzaron a debatir y hacer cuentas que les permitieran explicar esto. Como primera cuestión se preguntan si la noticia indica que cada persona contagia por día 2,5 personas más (Estudiante 2), o si 2,5 es la cantidad de personas que cada persona contagia en promedio, en un mes (Estudiante 5).

Estudiante 2: No entiendo. Una persona contagia 2,5 por día?

Estudiante 5 ¿O el 2,5 es la cantidad de personas que cada persona contagia en un mes? Me parece que es el total de contagios.

La docente conversa con los estudiantes respecto a esta información, y a la necesidad de investigar cómo se justifican los 2,5 contagios por persona. El docente decide no decirles que el 2,5 refiere a la cantidad promedio de personas que contagia una persona durante el tiempo en que es transmisor del virus; y los deja seguir formulando respuestas. Luego, un estudiante calcula numéricamente el aumento de infectados suponiendo que cada día, cada persona contagia 2,5 más.

Estudiante 1: Hoy somos una persona contagiada. Mañana, al segundo día son una persona, más las 2,5 personas que contagian, quiere decir 3,5 personas contagiadas. Al tercer día, 3,5 personas contagian cada una a 2,5 (hace la multiplicación 3,5 por 2,5 y obtiene 8,75); le sumo las 3,25 del día anterior tengo 12,25 personas. Esas contagian a 2,5 personas más. Son 30 más las 12,25. Para el cuarto día hay 42 personas contagiadas; y así sucesivamente, por eso, para mi es una sumatoria pero no sé cómo plantearlo.

Algunos también dicen que el crecimiento no es exponencial (Estudiante 2) e intentan dividir los 406 contagiados en partes iguales a lo largo del mes. Otros proponen la pirámide de Pascal y el binomio de Newton, pero sin llegar a plantear ninguna cuenta (figura 3).

Estudiante 2: Exponencial no es.

Estudiante 3: En dos semanas si hacemos 406 dividido en 30 días, serán 13,5 personas contagiadas por día, se supone, pero no sé si tiene que ser así. No me convence.

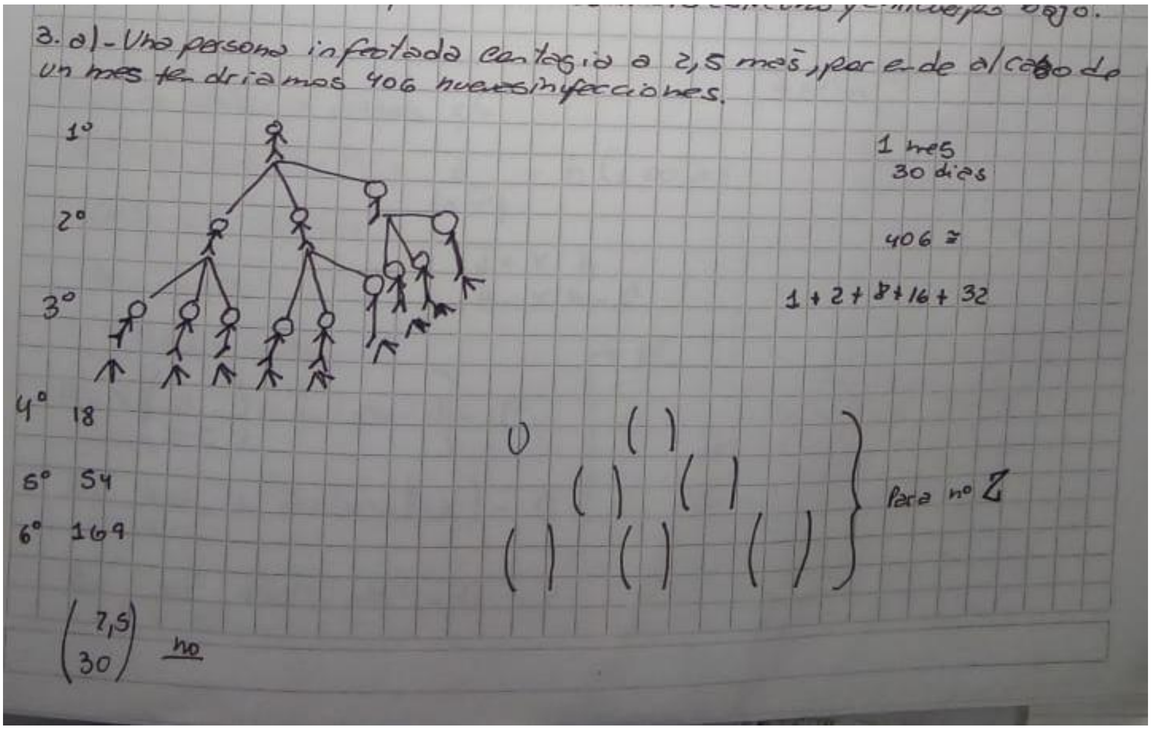


Figura 2. Cálculos sobre la cantidad de contagiados realizada por los estudiantes

Esto muestra, que en un principio, en lugar de ir por obras nuevas los estudiantes recurren a las obras matemáticas que habían estudiado en las distintas materias del profesorado para resolver la situación. Esto se corresponde con las actitudes retrocognitiva, esotérica y pre-herbartiana comunes en la pedagogía monumentalista que predomina en la institución, y que son innatas al intentar dar respuesta a una cuestión a partir de los conocimientos previos, aunque sea un conocimiento que no conduzca a un razonamiento correcto. Aquí el trabajo del profesor era acompañar a los estudiantes a ir por las obras nuevas.

Una vez que el grupo de clase (estudiantes y profesor) concluyen que la respuesta no está en ninguna de las obras matemáticas conocidas, cambian sus actitudes y comienzan a formular nuevas preguntas que emergen como derivadas de la cuestión generatriz:

Q_1*: ¿Cómo obtener la cantidad de contagiados en un mes, si cada una de las persona contagia a 2,5 personas durante el período en que puede contagiar?*

Q_2*: ¿Qué función representa un mes de contagios de COVID-19?*

Con estas nuevas cuestiones, la profesora plantea la necesidad de obtener la cantidad total de contagiados en un mes en Argentina, representar los datos gráficamente e intentar encontrar el modelo que permita calcularlos. Los estudiantes buscan en internet la cantidad de contagios de COVID-19 en el país, para el mismo mes, durante dos años consecutivos (2020-2021), con el propósito de poder encontrar alguna regularidad que les permita dar con el modelo matemático. Para ello escogen los meses de julio y agosto, que son los meses en los cuales se produjo los picos de contagios. Luego, vuelcan los datos en GeoGebra para representarlos gráficamente. Ver en figura 3 la tabla para julio del 2020; y su respectiva gráfica.

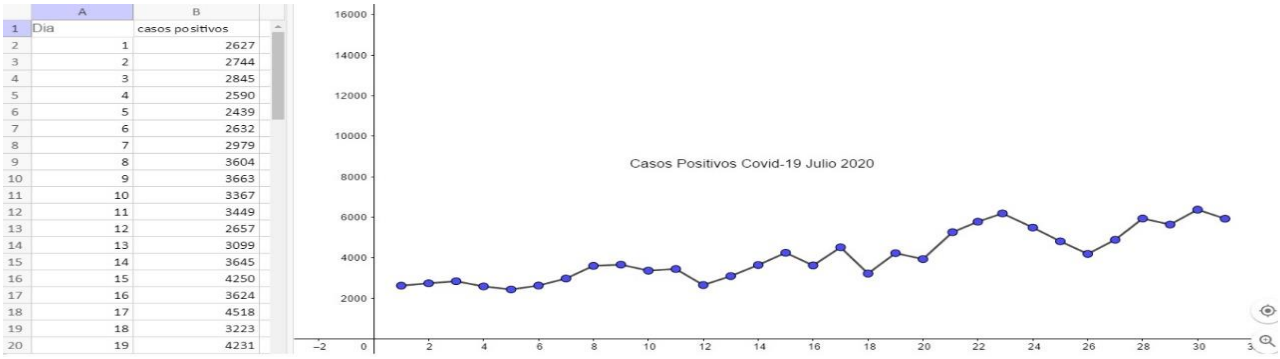


Figura 3. Datos de contagiados de COVID 19 del mes de julio 2020 en GeoGebra

Hecha la tabla de valores y la gráfica, comienzan a buscar una función que les permita aproximar los puntos (figura 4). Con ello obtienen, mediante el uso de GeoGebra, diferentes funciones polinómicas. Por ejemplo, para el mes de julio del 2020 obtienen la expresión:

$$\mathrm{P(x)} = -0{,}15\,x^9 + 3{,}32\,x^8 - 51{,}62x^7 + 544{,}83\,x^6 - 3800{,}93x^5 + 16606{,}67x^4 - 41082{,}77\,x^3 + 46627{,}18\,x^2 - 12236{,}36\,x + 14556{,}18$$

Figura 4. algebraica de contagiados de COVID 19 del mes de julio 2020

Analizando la función polinómica y los datos de partida, concluyen que esta función no se ajusta a ellos. Así, deciden que la situación específica de crecimiento de los contagios no se describe mediante una función polinómica. Pues cuando se habla del crecimiento exponencial de casos, es en un período de tiempo en el cual los casos sólo aumentan. Así formulan las siguientes cuestiones derivadas.

Q_3: ¿Qué función representa un intervalo de tiempo de crecimiento de contagios de COVID-19 en un mes?

Q_4: ¿Qué función representa un intervalo de tiempo de decrecimiento de contagios de COVID-19 en un mes?

En busca de una función que les permita responder estas cuestiones, comienzan a tomar pequeños intervalos de tiempo para los casos en que la cantidad de contagiados crece o decrece, y analizan las funciones que se ajustan a esos datos. Para esto tuvieron que salir del tema, e investigar las utilidades del software GeoGebra , que les permite obtener la ecuación algebraica de una función, a través de un intervalo de puntos. Por ejemplo, en la figura 5 se muestra la aproximación de una gráfica para el intervalo que va entre el 3 y 8 de agosto del año 2021.

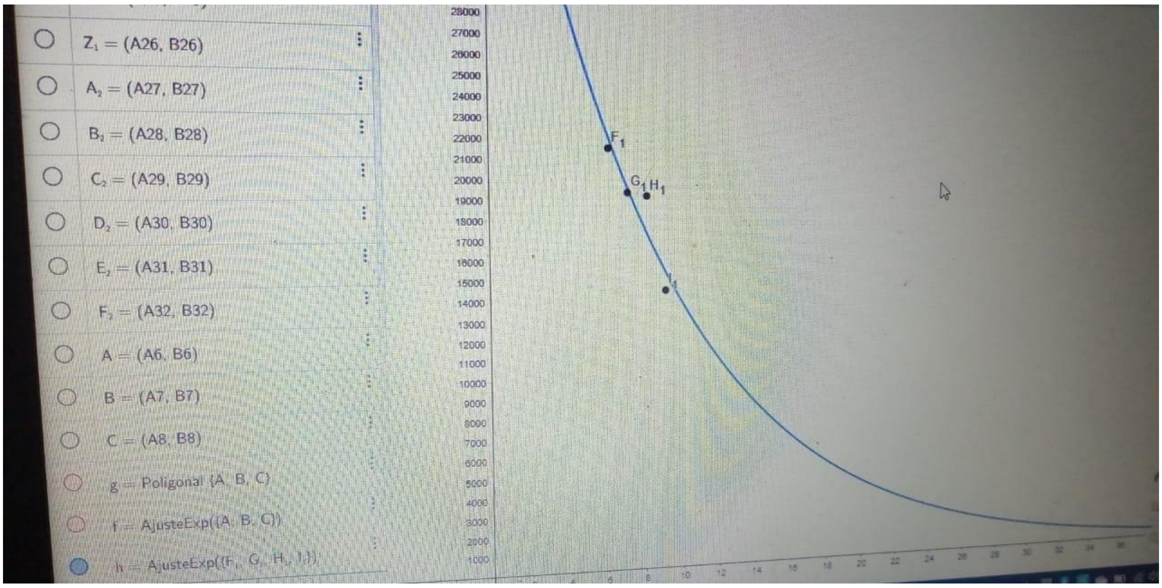


Figura 5. Puntos de decrecimiento de la función entre los días 3 y 8 de agosto del 2021

El análisis realizado mediante el GeoGebra les permite darse cuenta que tanto para los intervalos de decrecimiento como los de crecimiento la función que describe la variación es una exponencial de base e (ver figura 6). Esto da respuesta a las cuestiones Q_3 y Q_4 (*¿Qué función representa un intervalo de tiempo de crecimiento de contagios de COVID-19 en un mes? y, ¿Qué función representa un intervalo de tiempo de decrecimiento de contagios de COVID-19 en un mes?*).

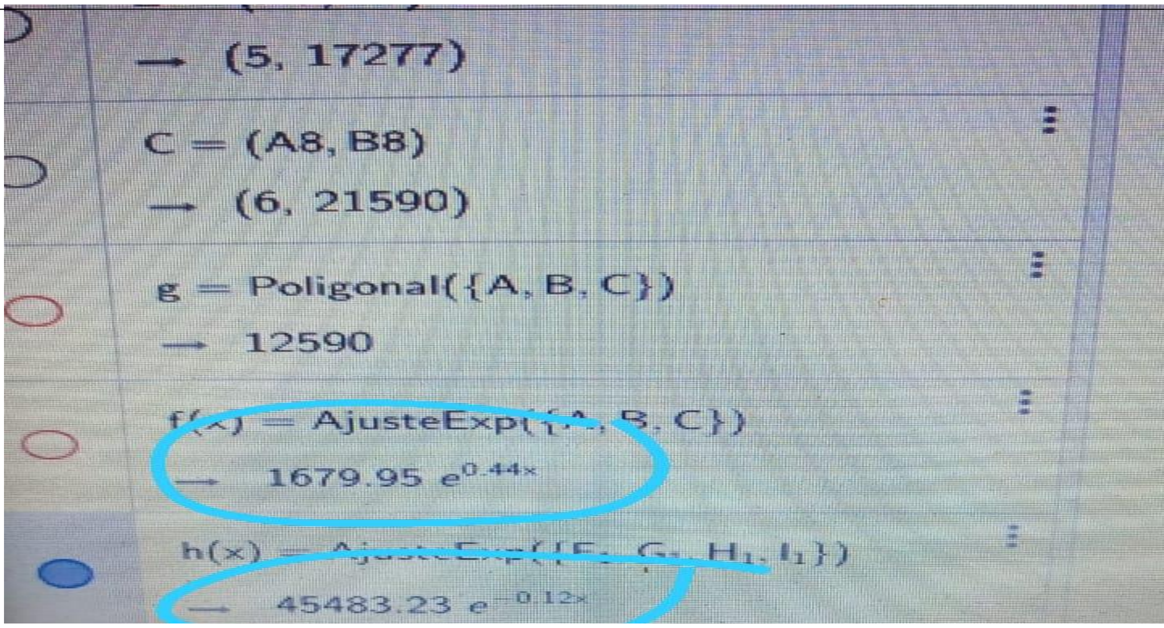


Figura 6. Ajuste de los intervalos de decrecimiento y crecimiento mediante una función

Luego, la clase debate sobre el por qué sucede esto y surge la siguiente cuestión derivada:

Q_5*:¿Cómo se obtienen las funciones de base* e *y qué relación tiene con los infectados de COVID-19?*

Con esta nueva cuestión comienzan a investigar en internet, pero como sus búsquedas son imprecisas e inciertas, interviene el profesor, preguntándoles sobre el comportamiento de los crecimientos poblacionales y modelos epidemiológicos en biología. El Estudiante 1 encuentra el link de un video sobre crecimiento poblacional (https://www.youtube.com/watch?v=MGcIFU8Iw7M), en el cual se explica que los crecimientos poblacionales a partir de una cantidad inicial, se describen a partir de ecuaciones diferenciales, de las cuales se obtiene una función exponencial. La docente les indica que utilicen los libros de análisis matemático donde se encuentra dicho saber para estudiar la idea del por qué se deben utilizar las ecuaciones diferenciales y cómo se obtiene la función exponencial. Luego del análisis del video y el estudio de los libros, se da el siguiente debate en clase:

Estudiante 1: Creo que tiene que ver con ecuaciones diferenciales porque tenemos una cantidad inicial de persona sanas que después se contagian y así se tiene el número de personas infectadas.

Estudiante 2: Pero si es una función exponencial ¿por qué decís que son ecuaciones diferenciales?

Estudiante 1: Porque en el libro de Thomas se desarrolla la función exponencial a partir de ecuaciones diferenciales.

Estudiante 3: Ah! Entonces debemos ver cómo se analizan las ecuaciones diferenciales, porque no las vimos todavía en la materia análisis matemático dos.

Estudiante 4: ¿Pero se puede resolver como una ecuación exponencial?

Estudiante 1: Chicas, sigamos el razonamiento de los datos que tenemos y vemos si nos da.

Los estudiantes reemplazan los valores del problema del video por los datos de la noticia logrando a través de esta ecuación obtener los 406 contagiados en un mes (ver figura 7).

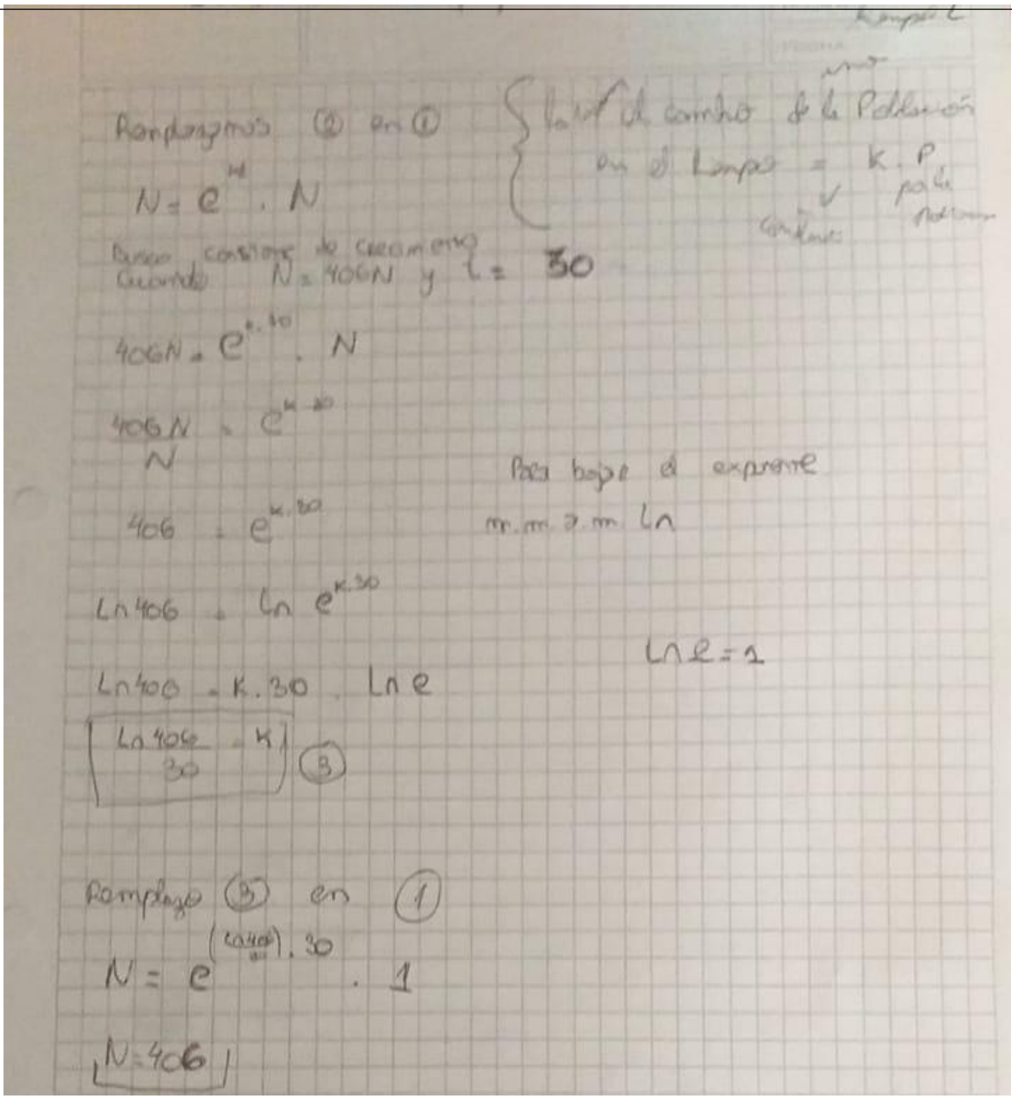


Figura 7. Cálculos relativos a la cantidad de contagiados por COVID-19 en un mes

Cuando los estudiantes le muestran la resolución (Figura 7) a la profesora, esta les hace ver que en esta ecuación no han utilizado el dato 2,5 de la noticia, por lo cual deben seguir investigando esta cuestión. Para ello buscan en internet videos que relacionan modelos matemáticos con COVID-19 y se quedan con el video cuyo link es: https://youtu.be/A1FvgG86rAI que es visto y analizado en clase. En este video se explica que uno de los modelos matemáticos que describe la curva de contagios se obtiene a partir de ecuaciones diferenciales ordinarias, pues son las que más se ajustan al modelo SIR (Susceptible, Infectado, Recuperado). Llegado este punto el grupo de clase profundiza en el estudio de las Ecuaciones Diferenciales Ordinarias y desarrollan el concepto de Ecuaciones Diferenciales Ordinarias vinculados al crecimiento poblacional, como sigue:

> Las ecuaciones diferenciales ordinarias establecen que la razón de crecimiento de la población (estadística) con respecto al tiempo es directamente proporcional a la cantidad de población presente por una constante de crecimiento.
>
> Se utilizan, en general, para calcular el crecimiento o decrecimiento de una cierta población; y además, mostrar la variación de cambio desde el tiempo inicial al tiempo final. (Producción de los estudiantes)

Luego, continúan el desarrollo con el propósito de hallar el modelo matemático epidemiológico a partir de las ecuaciones diferenciales. Para esto, escriben:

P: Población

dP: Diferencial de Población

T: Tiempo

dT: Diferencial de Tiempo

k: Constante

e: Número Euler (2,71828)

c: Constante

Y comienzan con un debate en la pizarra que les permite desarrollar la ecuación de curvas de contagio aplicadas a modelos epidemiológicos

$$P = e^{K\,.t+c}$$

mediante el uso de ecuaciones diferenciales. Para ello toman como referencia el desarrollo y explicación del libro de Thomas (2006) de ecuaciones diferenciales aplicados a problemas de valor inicial a partir de la siguiente ecuación diferencial:

$$\frac{dy}{dt} = ky \qquad (1)$$

Condición Inicial: $y = y_0$, cuando $t = 0$.

Se puede presentar dos situaciones; si y es positiva y creciente, entonces k es positiva, y la ecuación (1) muestra que la razón de crecimiento es proporcional a lo ya acumulado (Thomas, 2006). En caso contrario, si y es positiva y decreciente, k es negativa, y la ecuación (1) muestra que la razón de decaimiento es proporcional a lo que queda aún presente.

Se puede observar que la función constante $y = 0$, Es una solución de la ecuación (1). Para hallar las soluciones distintas de cero, partiendo de la ecuación (1):

$$\frac{dy}{dt} = ky$$

Dividiendo miembro a miembro por y

$$\frac{1}{y}.\frac{dy}{dt} = k$$

Integrando respecto a t; $\int \left(\frac{1}{u}\right) du = \ln|u| + C$

$$\ln|y| = kt + C$$

Por definición de logaritmo natural

$$|y| = e^{kt+C}$$

Por propiedades de la potencia : $e^{a+b} = e^{a} \,.\, e^{b}$

$$y = e^{C}.\,e^{kt}$$

Por definición de módulo, se tiene que, si $|y| = r, \rightarrow y = \pm r$

$$y = \pm e^{C}. e^{kt}$$

Así, $y = Ae^{kt}$, considerando $A = \pm e^{C}$

Si queremos hallar el valor de A para el problema de valor inicial, despejando A, cuando $y = y_0$;$t = 0$; $y_0 = A$

De esta forma, la solución al problema de valor inicial es

$$y = y_0 e^{kt} \qquad (2)$$

La fórmula hallada recibe el nombre de **Ley del cambio exponencial** (Thomas, 2006), en la cual se establece:

Si $k > 0$ se produce crecimiento.

Si $k < 0$ se produce decrecimiento.

El número k es la razón constante de la ecuación.

La derivada de esta ecuación (2) muestra que las únicas funciones que son su propia derivada son múltiplos constantes de la función exponencial, debido justamente a que la derivada de la función exponencial, es la misma función.

Una vez que las alumnas finalizan con este desarrollo, realizan un análisis más profundo de las ecuaciones diferenciales vinculadas al modelo epidemiológico SIR (por cuestiones de reducir el trabajo, no están presentes los modelos previos que conducen a éste), culminando con la siguiente deducción (ver figura 8).

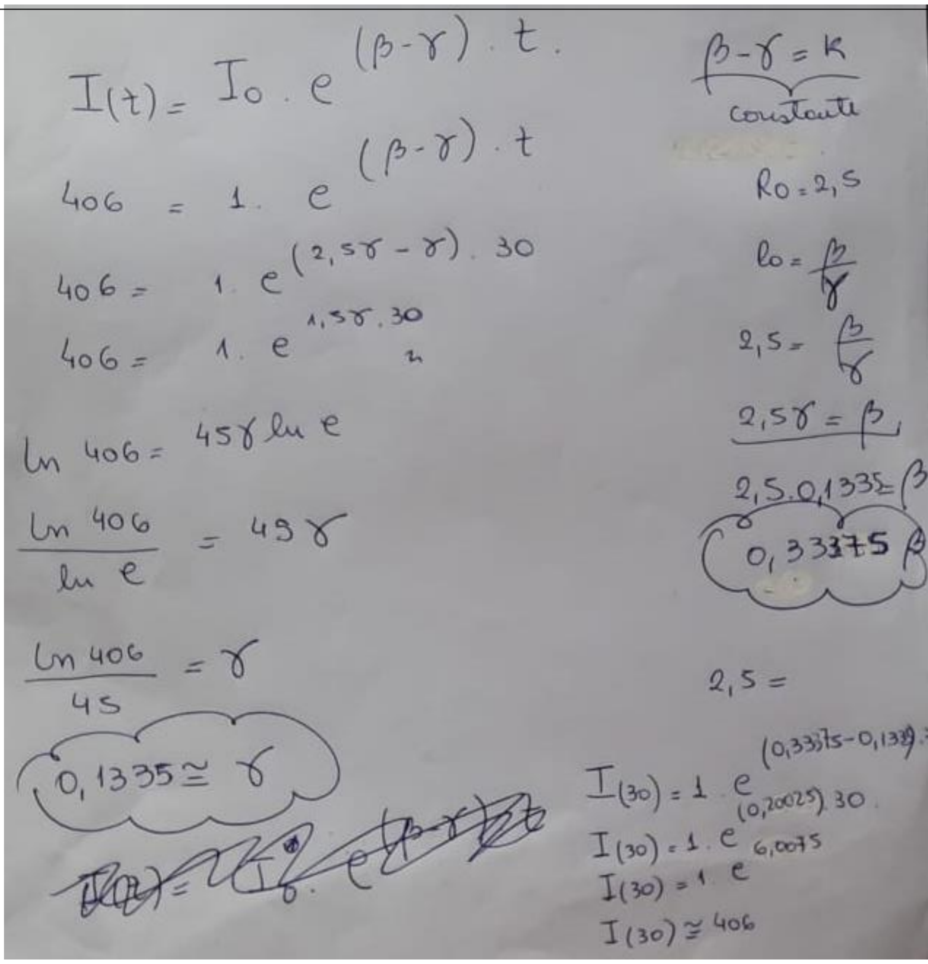


Figura 8. Cálculo de la cantidad de contagiados utilizando el modelo SIR

En la figura 8, los estudiantes utilizaron la fórmula (2) que obtuvieron, para estimar el número de Infectados en un mes, en el modelo SIR:

$$I(t) = I_0 . e^{K.t} \quad (3)$$

Con el estudio del modelo SIR que realizan los estudiantes, deducen que $K = \beta - \gamma$ reemplazando en (3)

$$I(t) = I_0 . e^{(\beta-\gamma).t}$$

Se utiliza el dato de los 2,5 personas, que es el R_0 para despejar .

$$R_0 = \frac{\beta}{\gamma}$$

$$2{,}5 . \gamma = \beta$$

Luego, por sustitución de variables y reemplazando en $I(t)$ se obtiene:

$$I(t) = 1 . e^{(\beta-\gamma).t}$$

donde $I_0 = 1$ debido a que en el tiempo cero se tiene un solo infectado.

$$I(t) = 1. e^{(2,5\gamma-\gamma).t}$$

Reemplazando los valores para $t = 30$

$$406 = 1. e1,5.30$$

Aplicando logaritmo natural miembro a miembro:

$$\ln 406 = 45\gamma$$

$$\gamma = \frac{\ln 406}{45}$$

$$0,1335$$

A partir de dicho valor, los estudiantes realizan una puesta en común, donde establecen que la cantidad de infectados por mes depende de los valores de R_0, β y γ; y coinciden que es por este motivo que la noticia aclara que el $R_0 = 2,5$. Con todas estas conclusiones utilizan esos datos en la función exponencial de contagiados (Figura 7):

$$I(30) = 1. e^{(0,20025).30}$$

$$I(30) = 406$$

Así, los estudiantes logran deducir mediante ecuaciones diferenciales ordinarias, la ecuación general que se utiliza en los modelos epidemiológicos, y que permite representar la curva de contagios de COVID-19. Esto responde la cuestión generatriz Q_0: ¿Cómo se calcula la propagación de infectados del COVID-19?, en el modelo SIR, con las condiciones iniciales indicadas.

Así, no solo lograron responder a la cuestión generatriz y sus derivadas, sino también deducir todas las variables y ecuaciones diferenciales ordinarias que intervienen en el modelo SIR. Luego de esto, los estudiantes pudieron avanzar en el estudio de los modelos SEIR y SEIAR, así como también en la aplicación de ecuaciones diferenciales en otros problemas de la física, como es el decaimiento radiactivo, la datación del carbono 14 y la ley de enfriamiento de Newton.

Conclusiones

Los estudiantes manifestaron al principio gestos característicos de las actitudes *retrocognitiva, esotérica y pre-herbartiana*; sin embargo luego de los primeros fracasos comenzaron a mostrar gestos de las actitudes de *problematización, herbartiana, procognitiva y exotérica*. Esto permitió al REI avanzar hacia la construcción de la respuesta $R^{\heartsuit}$ y del estudio de las organizaciones matemáticas previstas. El medio se construyó con las R_i y las O_j traídas a clase tanto por los alumnos como por el profesor. Internet resultó ser un media fundamental en la construcción del medio.

El REI constituyó un gran desafío tanto para el profesor como para los estudiantes, pero sin embargo, al ser un trabajo que tenía por objetivo poder estimar la cantidad de contagiados por COVID-19, en un momento que se estaba transitando la pandemia, el estudio de la cuestión fue necesariamente codisciplinar, y permitió a los estudiantes a acceder al estudio de las Ecuaciones Diferenciales a partir de su utilidad.

Otra consecuencia de este trabajo fue que los estudiantes comenzaron a cuestionar el saber y la formas de enseñanza de las obras matemáticas, ya que descubrieron una de las “razones de ser” de las ecuaciones diferenciales.

Como síntesis, coincidimos con Chevallard (2017), cuando dice que es necesario una matemática mixta donde se aprecie la razón de ser de las praxeologías matemáticas.

Referencias

Barquero, B.; Bosch, M.; Gascón, J. (2011). Los Recorridos de Estudio e Investigación y la Modelización Matemática en la Enseñanza Universitaria de las Ciencias Experimentales. *Revista Enseñanza de las Ciencias*, Vol. 29(3), 339-352.

Chevallard, Y. (2013). Journal du Seminaire TAD/IDD Théorie Anthropologique du Didactique & Ingénierie Didactique du Développement. Recuperado el 20 de abril del 2022. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/journal-tad-idd-2012-2013-1.pdf

Chevallard, Y. (2017). ¿Por qué enseñar matemáticas en secundaria? Una pregunta vital para los tiempos que se avecinan. *La Gaceta de la RSME*, Vol. 20(1), 159–169.

Zill, D. (2009). Ecuaciones diferenciales con aplicaciones de modelado. Editorial: CENGAGE LARNING. DF- México, pp. 433.

L'analyse *a priori* comme outil de recherche complémentaire pour comprendre le fonctionnement des systèmes d'enseignement

Hamid Chaachoua[1] and Annie Bessot[2]

[1] LIG, Université Grenoble Alpes, France; hamid.chaacoua@imag.fr

[2] LIG, Université Grenoble Alpes, France; annie.bessot@gmail.com

We will try to show how the a priori analysis inspired by the theory of didactic situations (Brousseau, 1998) fits in with the search for an epistemological reference for understanding the functioning of teaching systems. Currently, in these systems, the paradigm of 'la visite des oeuvres' is predominant. This is why we will place ourselves in this case.

Keywords: a priori analysis, reference praxeological model, dominant praxeological model, elementary pinpoint praxeology, scopes of a technique, cost of a praxis.

Nous allons tenter de montrer en quoi l'analyse a priori inspirée de la théorie des situations didactiques (Brousseau, 1998) s'intègre à la recherche d'une référence épistémologique pour comprendre le fonctionnement des systèmes d'enseignement. Actuellement, dans ces systèmes, le paradigme de la visite des œuvres est très largement présent. C'est pourquoi nous nous placerons dans ce cas.

Mots clefs : analyse a priori, modèle praxéologique de référence, modèle praxéologique dominant, praxéologie ponctuelle élémentaire, Portées d'une technique, coût d'une praxis

Introduction

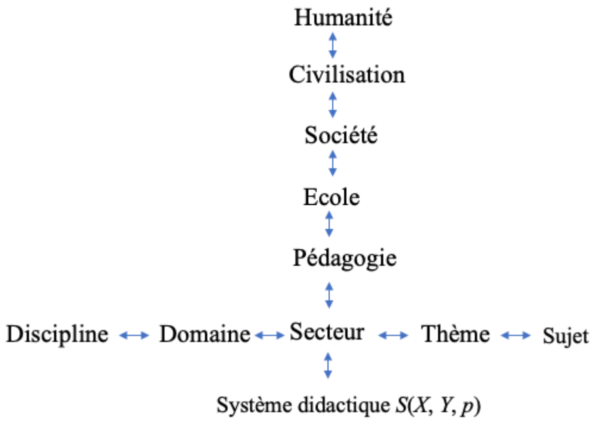
A l'occasion du colloque pour les 70 ans d'Yves Chevallard (2016), dans un diaporama intitulé *Research methods in ATD*, Marianna Bosch déclarait : "We do not use enough the ATD frame in a reflexive way, to explain what we do, how we do it and why."

Nous allons tenter d'emprunter ce chemin réflexif pour montrer en quoi l'analyse *a priori* inspirée de la théorie des situations didactiques (Brousseau, 1998) s'intègre à la recherche d'une référence épistémologique pour comprendre le fonctionnement des systèmes d'enseignement.

L'outil qu'est la notion de transposition didactique (Chevallard, 1985) permet de formuler de nouvelles questions sur les systèmes d'enseignement et de faire émerger des phénomènes qui resteraient sinon invisibles, en particulier l'existence implicite d'un modèle épistémologique dominant.

Ce modèle épistémologique dominant résulte d'un système de conditions et de contraintes qui permet l'existence de certaines pratiques dans l'institution et empêche que d'autres puissent apparaître (Gascon, 1994). On peut rattacher ce système de conditions et de contraintes à l'échelle des niveaux de codétermination didactique introduite par Chevallard (2010). Dans les systèmes d'enseignement actuels, le paradigme de la visite des œuvres est très largement présent. C'est pourquoi nous nous placerons dans ce cas (Figure 1).

Figure 1. L'échelle de codétermination didactique pour le paradigme de visite des œuvres (d'après Bosch 2019)



Le point de vue anthropologique conduit à interpréter les transformations des savoirs du processus de transposition didactique comme des transformations de l'activité humaine liée à l'entrée d'une personne dans différentes institutions *via* la notion de praxéologie.

> La notion de praxéologie est l'une des notions clés de la TAD. Elle fournit un outil neutre et flexible pour décrire tout type de connaissance et aussi tout type d'activité humaine. Elle permet de dépasser les dichotomies classiques entre connaissances procédurales et conceptuelles, entre théorie et pratique, entre pensée et action. (Bosch. 2021, traduction propre)

De ce fait le modèle épistémologique dominant d'une institution d'enseignement sera donné en termes de modèle praxéologique dominant *MPD.*

Rappelons que dans une institution donnée, les praxéologies ponctuelles PMP $[T / \tau / \theta / \Theta]$, centrées sur un unique type de tâches, vivent rarement isolées les unes des autres : d'abord elles se regroupent en praxéologies locales PML $[T_{ij} / \tau_{ij} / \theta_j / \Theta]$ centrées sur une technologie θ_j déterminée ; ensuite en praxéologies régionales PMR, $[T_{ijk} / \tau_{ijk} / \theta_{jk} / \Theta_k]$, formées autour d'une théorie Θ_k. Au-delà, Chevallard (1999) nomme organisation globale le complexe praxéologique obtenu par l'agrégation de plusieurs organisations régionales correspondant à plusieurs théories Θ_k. De ce point de vue, la discipline « mathématiques » peut être considérée comme une organisation globale, amalgamant diverses praxéologies autour de différentes technologies et théories (Chevallard, 2002).

Une analyse *a priori* de praxéologies ponctuelles est pour nous essentielle pour décrire les regroupements existants et en même temps pour envisager des regroupements possibles menant à la construction ascendante du MPD. Comme dans le cadre de la TSD :

> *A priori* ne renvoie pas à une position dans le temps par rapport à l'histoire de la recherche mais à une réflexion d'ordre épistémologique […]. Cette analyse doit mettre en évidence divers phénomènes qui peuvent se produire, en particulier dégager les grandes classes de procédures de résolution et les conditions qui les engendrent par l'étude des variables de la situation. (Bessot, Comiti, 1985, p. 312)

L'objectif de notre article est de mener dans le cadre de la TAD une réflexion sur l'analyse *a priori* des praxéologies ponctuelles ce qui peut permettre une analyse ascendante du MPD d'un système d'enseignement.

Pour exemplifier notre propos nous nous appuyons tout au long de l'article sur l'exemple d'une recherche comparative collective dans le domaine de l'algèbre élémentaire, impliquant le Brésil, la France, l'Espagne, le Japon et le Viêt Nam. La question à l'étude dans cette recherche est de caractériser le MPD de chaque pays à partir d'un modèle de référence construit par le chercheur.

1. Pourquoi construire un modèle épistémologique de référence pour accéder au modèle praxéologique dominant d'une institution ?

Pour comprendre le fonctionnement des systèmes d'enseignement :

> […] le chercheur a besoin d'un « point de vue » particulier, c'est-à-dire d'un modèle alternatif du domaine d'activité mathématique enseigné qui lui serve de cadre de référence pour interpréter le modèle dominant dans l'institution qu'il étudie. (Gascon, 1994, p. 44)

Ce modèle est nommé Modèle Epistémologique de Référence *MER* ou dans la TAD Modèle Praxéologique de Référence *MPR*.

Schneider (2019) écrit à propos du rôle du MER :

> […] et, ajouterais-je volontiers, pour prendre conscience de son propre MER qui lui inspire son analyse didactique et ses propositions. C'est donc, j'insiste, parce qu'il découle d'un pas de côté épistémologique fait par rapport aux pratiques empiriques qu'un MER permet de dénaturaliser des modèles implicites devenus transparents. Dans ces conditions, il peut fonctionner comme une véritable phénoménotechnique pour la recherche.

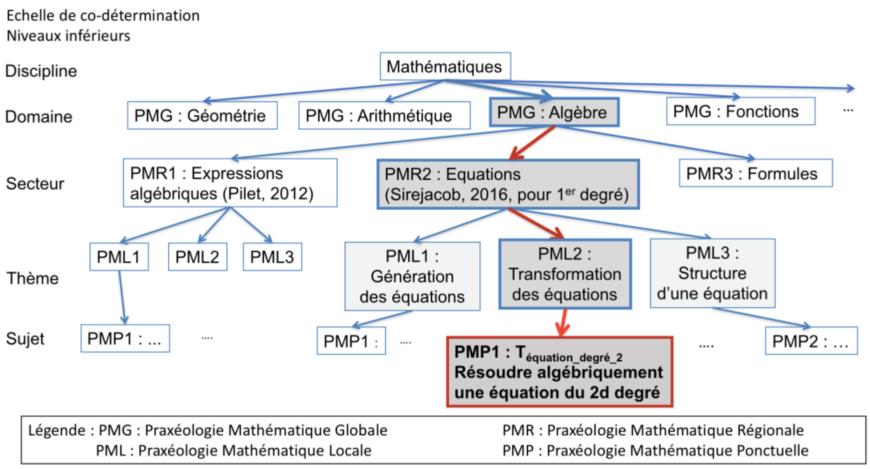
Comme le rappelle Bosch (2019), un MPR se construit comme outil d'étude d'une question de recherche. De plus, le processus d'élaboration et son niveau de granularité relatif à l'échelle des niveaux de codétermination dépendent étroitement de la question de recherche.

Une étude première pour comprendre le fonctionnement des systèmes d'enseignement porte sur la description et la compréhension d'un MPD. Cette étude est souvent un préalable pour d'autres recherches comme l'analyse comparative de plusieurs systèmes d'enseignement (Bessot et Comiti, 2013), la conception d'un environnement informatique d'apprentissage humain ou la construction d'un curriculum alternatif à l'existant mettant en avant un autre paradigme comme celui du questionnement du monde (Chevallard, 2019).

La construction d'un MPR se fait à partir de plusieurs enquêtes. Le point de départ peut être un MPR déjà développé dans d'autres recherches portant sur le même objet de savoir mais qui peut être remanié et enrichi selon la question de recherche de l'étude. Ainsi, un MPR n'est pas « 'le' modèle de tel ou tel objet de savoir, mais 'un' modèle (possible) parmi bien d'autres. » (Bosch, 2019, p. 90). En plus des adaptations d'un MPR existant à une nouvelle question de recherche, un MPR est soumis à l'épreuve empirique des faits et en révision permanente.

Dans notre étude, nous nous sommes appuyés sur plusieurs recherches sur le domaine de l'algèbre (Pilet, 2012 ; Ruiz-Munzón et al., 2020 ; Sirejacob, 2016) pour élaborer un MPR en relation avec notre étude (figure 2). Cette élaboration prend en compte les regroupements locaux, régionaux, globaux, ainsi que la structure hiérarchique de l'échelle des niveaux (inférieurs) de codétermination didactique. Rappelons que la question de recherche à l'étude est la comparaison des modèles praxéologiques dominants des institutions d'enseignement de 5 pays dans le *domaine de l'algèbre élémentaire*.

Figure 2. Représentation partielle d'un MPR de l'algèbre élémentaire



La praxéologie ponctuelle faisant l'objet d'une *analyse a priori* est grisée et encadrée de rouge dans la figure 1.

2. Importance de l'analyse *a priori* des praxéologies ponctuelles dans le cas du paradigme de la visite des œuvres

Redisons que, pour nous, l'étude des praxéologies ponctuelles observées dans un système d'enseignement est particulièrement importante pour envisager les regroupements potentiels et alternatifs en praxéologies locales, régionales et globales et donc pour compléter ou modifier le MPR initial.

Pour cette raison, nous engageons un processus de modélisation des praxéologies (ponctuelles) possibles relatives à un type de tâches T présent dans un système d'enseignement comme sujet d'étude dans le paradigme de la visite des œuvres. Mais en préalable, nous introduisons la notion de praxéologie ponctuelle élémentaire et présentons notre démarche de description d'une technique qui débouche sur la notion de type de tâches pivot.

2.1. Praxéologie ponctuelle élémentaire

Une praxéologie ponctuelle peut concerner plusieurs techniques τ_i pour un même type de tâches. Pour cette raison, nous introduisons la notion de praxéologie ponctuelle élémentaire (Chaachoua et Bessot 2019)

> […] nous définissons pour chaque technique accomplissant un même type de tâches T une praxéologie ponctuelle élémentaire : il y a autant de praxéologies ponctuelles élémentaires que de techniques accomplissant un unique type de tâches T. Les praxéologies ponctuelles élémentaires permettent de « remonter » par regroupement autour de T aux praxéologies ponctuelles au sens de la TAD. (Chaachoua et Bessot, 2019, p. 236)

On trouvera un exemple plus loin, dans le tableau 1, qui présente 5 techniques possibles autour du type de tâches $T_{équation\_degré\_2}$ « résoudre algébriquement une équation du second degré » : chaque ligne de ce tableau correspond à une praxéologie ponctuelle élémentaire autour de $T_{équation\_degré\_2}$.

2.2. Description d'une technique et type de tâches pivot

Nous adoptons le point de vue de T4TEL[1] (Chaachoua 2018) qui décrit une technique par une suite de types de tâches. Pourquoi le choix d'une telle description ? Un premier intérêt est d'uniformiser et de rendre explicite les critères de description d'une technique. Un deuxième intérêt que nous développerons plus loin est de pouvoir calculer le coût d'une technique.

Nous présentons maintenant une troisième raison. Il existe plusieurs façons de décrire une technique pour accomplir un type de tâche T à l'aide de types de tâches. Or plusieurs techniques reposent sur le fait que pour accomplir T, on se ramène à un autre type de tâches T*, ingrédient de la technique, *supposé plus proche d'un élément technologique clé pour la technique*. Nous appellerons ce type de tâches T* *type de tâches pivot* pour l'environnement technologique.

Considérons l'exemple du type de tâches T (Démontrer que 3 droites sont concourantes). L'une des techniques de résolution est de se ramener au type de tâches T* (Démontrer que ces droites sont des droites 'remarquables' d'un triangle). Ce type de tâches ingrédient d'une technique de T est considéré comme type de tâche pivot parce que cette technique convoque un environnement technologique autour des propriétés des droites remarquables d'un triangle (hauteur, médiatrice, médiane, bissectrice).

Maintenant nous vous proposons une analyse *a priori* en trois étapes.

2.3. Étape 1. Analyse *a priori* des praxéologies ponctuelles élémentaires autour de T

Une première enquête portant sur des observables d'une institution d'enseignement vise à construire un répertoire de différentes techniques présentes dans cette institution pour l'accomplissement de T. Chacune de ces techniques possède un environnement technologico-théorique spécifique. Mais cette enquête doit être complétée par l'étude de ces techniques et la recherche d'autres techniques alternatives par des moyens synchroniques (études comparatives entre plusieurs institutions) ou/et diachroniques (étude du passé des institutions).

Exemple du type de tâches $T_{équation\_degré\_2}$ « résoudre algébriquement une équation du second degré »

[1] T4 renvoie au quadruplet praxéologique (Type de tâches, Technique, Technologie, Théorie) et TEL à *Technology Enhanced Learning*. A l'origine, ce qui a motivé le développement de T4TEL est de formaliser les éléments praxéologiques et leurs relations pour une représentation informatique. Par exemple, un type de tâches est représenté par un couple (verbe, complément), une technique comme une suite de types de tâches. Pour plus de développement, se référer à Chaachoua (2018). Soulignons que ce développement trouve son intérêt au sein de la TAD en dehors du contexte particulier des développements informatiques comme le montre cet article.

Pour $T_{\text{équation\_degré\_2}}$ nous avons repéré (enquête menée dans des manuels des 5 pays) 5 techniques différentes pour résoudre ce type de tâches. Le tableau 1 ci-après donne les 5 praxéologies ponctuelles élémentaires qui correspondent à ces 5 techniques et leurs technologies spécifiques.

Technique	Description de la technique comme suite de types de tâches	Technologie de la technique
$\tau_{\text{produit\_nul}}$	- $T_{\text{regrouper\_nul}}$ (Regrouper tous les termes dans un même membre) - $T_{\text{factoriser}}$ (Factoriser une expression algébrique) - $T_{\text{produit\_nul}}$ (Résoudre une équation de la forme $P.Q = 0$)	Propriété du produit nul Propriétés de conservation des égalités Propriété de distributivité Identités remarquables
$\tau_{\text{racine\_carrée}}$	- $T_{\text{regrouper\_cte}}$ (Regrouper tous les termes avec x dans un même membre et les constantes dans l'autre membre) - $T_{\text{factoriser}}$ (Factoriser une expression algébrique) - $T_{\text{carré\_cte}}$ (Résoudre une équation de la forme $(ax+b)^2 = k$)	Règle de la racine carrée Propriétés de conservation des égalités Propriété de distributivité Identités remarquables
$\tau_{\text{sp\_evident\_root}}$	- $T_{\text{trinôme}}$ (Mettre sous la forme $ax^2+bx+c = 0$, $a \neq 0$) - $T_{\text{racine\_évidente\_eq}}$ (Chercher par tâtonnement une solution de l'équation $ax^2+bx+c = 0$) - $T_{\text{sum\_or\_product}}$ (Écrire la relation somme ou produit des racines) - $T_{\text{équation\_degré\_1}}$ (Résoudre une équation du premier degré)	Relation entre les racines et les coefficients du trinôme Propriétés de conservation des égalités Propriétés arithmétiques des nombres
$\tau_{\text{sp\_trial\_root}}$	- $T_{\text{trinôme}}$ (Mettre sous la forme $ax^2+bx+c = 0$, $a \neq 0$) - $T_{\text{sum\_and\_product}}$ (Écrire la relation somme *et* produit des racines) - $T_{\text{racine\_tatonnement\_syst}}$ (Chercher par tâtonnement les solutions d'un système d'équations du premier degré)	Relation entre les racines et les coefficients du trinôme Propriétés de conservation des égalités Propriétés arithmétiques des nombres
$\tau_{\text{discriminant}}$	- $T_{\text{trinôme}}$ (Mettre sous la forme $ax^2+bx+c = 0$, $a \neq 0$) - $T_{\text{discriminant}}$ (Appliquer la formule du discriminant)	Formule du discriminant Propriétés de conservation des égalités Propriété de distributivité Identités remarquables

Tableau 1. Praxéologies ponctuelles élémentaires de $T_{\text{équation\_degré\_2}}$

Du point de vue de l'algèbre, toutes ces technologies peuvent relever d'une même théorie, par exemple celle de l'anneau factoriel R[X] des polynômes à coefficients dans R, comme c'est le cas dans les institutions savantes actuelles ou en 2005 dans l'institution de l'enseignement secondaire au Vietnam (Nguyen, 2005).

La description des cinq techniques dans le tableau 1 fait intervenir trois types de tâches particuliers :

$T_{\text{produit\_nul}}$ (Résoudre une équation de la forme $(ax + b)(cx + d) = 0$, $ac \neq 0$)

$T_{\text{carré\_cte}}$ (Résoudre une équation de la forme $(ax + b)^2 = k$, $a \neq 0$)

$T_{\text{trinôme}}$ (Mettre sous la forme d'un trinôme $ax^2 + bx + c = 0$, $a \neq 0$)

Ces trois types de tâches sont des types de tâches pivots car elles sont liées intimement à des environnements technologiques qui les justifient et les discriminent. Elles ont de plus la propriété d'être des sous-types de tâches générés à partir de $T_{\text{équation\_degré\_2}}$ par une variable (Chaachoua et Bessot, 2019) *forme* de l'équation quadratique.

Cet exemple montre que l'analyse *a priori* des techniques de T peut permettre de formaliser le lien entre types de tâches par une ou plusieurs variables et ainsi de caractériser les types de tâches pivots. Un prolongement est la question des valeurs prises par ces variables dans l'institution étudiée.

2.4. Étape 2. Analyse *a priori* des conditions d'existence et de coexistence des techniques et de technologies de T : portée pragmatique et concurrence des techniques

Cette étape de l'analyse *a priori* va permettre de repérer certaines conditions d'existence et de coexistence des techniques et technologies de ce type de tâches d'un point de vue épistémologique, c'est-à-dire sans prendre en compte ni le point de vue cognitif ni le point de vue des institutions étudiées.

Pour cela nous utilisons les notions de *portée théorique* et de *portée pragmatique des techniques*.

> La portée théorique d'une technique est l'ensemble des tâches où la technique permet d'accomplir une tâche quelconque de cet ensemble en dehors de toute considération des conditions de son exécution. C'est-à-dire qu'on examine cette technique d'un point de vue épistémologique sans prendre en compte le cognitif et donc la maîtrise de sa réalisation par un sujet. Elle sera notée $P_{Th}(\tau)$.
>
> La portée pragmatique d'une technique est l'ensemble des tâches où la technique est fiable dans le sens où elle permet d'accomplir ces tâches avec peu de risque d'échec et à un coût raisonnable. La technique tend à réussir sur cette portée et tend à échouer en dehors. Elle sera notée $P(\tau)$. (Kaspary, Chaachoua et Bessot, 2020, p. 247)

La notion de portée pragmatique correspond à la définition de Chevallard (1999). Et c'est bien cette portée qui est pertinente pour la vie des praxéologies et qui est au cœur des questions didactiques.

Une tâche *t* appartient à la portée théorique d'une technique sous la seule condition que τ accomplisse *t*, quel que soit les risques d'échec ou d'erreurs dans sa réalisation pratique.

La notion de portée pragmatique est délicate à délimiter. Pour mieux comprendre, envisageons théoriquement ce qui peut se passer en cas de coexistence de plusieurs techniques pour accomplir un même type de tâches : en nous inspirant de la théorie des situations didactiques (Brousseau, 1998), nous parlerons alors de concurrence entre techniques et de coût d'une technique. Nous définissons le coût *a priori* d'une technique (c'est-à-dire sans prendre en compte son écologie et son économie dans une institution) par le nombre de types de tâches ingrédients de cette technique[2]. Or, pour une description donnée, le calcul du coût d'une technique fournit des nombres constants pour une technique donnée, indépendamment de la nature d'une tâche *t*. Cette indépendance du coût d'une technique par rapport à *t* est un problème comme le montre l'exemple ci-dessous.

[2] Il existe plusieurs façons de décomposer une technique en sous-type de tâches (voir Chaachoua, 2018).

Prenons l'exemple de la technique $\tau_{produit\_nul}$ pour deux tâches ($x^2 - 7 = 0$) et ($(x - 2)(x-3) = 0$). Il y a plus de gestes à faire pour la première tâche que pour la deuxième pour résoudre algébriquement l'équation avec la technique $\tau_{produit\_nul}$. Le coût ne peut donc pas être invariant.

La question est donc de rechercher un niveau de granularité pour la description d'une technique où le chercheur juge que ce coût est constant sur un ensemble de tâches (sous types de tâches de T). Pour cela le chercheur prend en compte les gestes nécessaires à l'accomplissement d'une technique pour une tâche donnée de T, puis identifie et discrimine les tâches pour lesquels cette technique a le même coût ou non, et enfin regroupe ces tâches dans un même sous-type de tâches T_i de T. Le calcul du coût est donc dépendant de ce sous-type de tâches : c'est pour cela que nous préférons parler du *coût de la praxis* (T_i, τ) plutôt que du coût de la technique τ.

Par exemple, le coût de la technique $\tau_{produit\_nul}$ peut être considéré comme invariant pour toute tâche *t* (comme ($x^2 - 7 = 0$)) du sous-type de tâches $T_{carré\_cte}$ ($x^2 - k = 0$, $k>0$) » : ce coût est celui de la praxis ($T_{carré\_cte}$, $\tau_{produit\_nul}$) ; ce coût diffère du coût de la praxis ($(ax+b)(cx+d) = 0$), $\tau_{produit\_nul}$).

Bien entendu, les critères de ce processus de modélisation sont des éléments de discussion et d'ajustement. Pour mieux comprendre cette démarche, on peut se reporter au paragraphe 3.

La concurrence entre techniques suggère que l'une peut être plus efficace que l'autre sur un certain ensemble de tâches dans le sens où elle est moins coûteuse que sa concurrente.

Par exemple, nous jugeons que $\tau_{racine\_carrée}$ est moins coûteuse que $\tau_{discriminant}$ ou $\tau_{produit\_nul}$ pour le sous-type de tâches $T_{carré\_cte}$. Cette optimalité de $\tau_{racine\_carrée}$ sur ce domaine de concurrence entraîne l'appartenance de *t* à la portée pragmatique de $\tau_{racine\_carrée}$ et l'exclusion de *t* des portées pragmatiques de $\tau_{discriminant}$ et $\tau_{produit\_nul}$.

Ces deux notions de portée pragmatique d'une technique et de concurrence des techniques sont utilisées dans les analyses ultérieures pour caractériser : (i) les tâches appartenant à la portée pragmatique, notée P, (ii) les dynamiques praxéologiques possibles du point de vue épistémologique, c'est-à-dire résultantes de la concurrence des techniques et de leur coût respectif (en termes de praxis), (iii) les dynamiques praxéologiques mise en place dans une institution.

2.5. Étape 3. Analyse *a priori* des praxéologies ponctuelles élémentaires existantes ou pouvant exister dans une institution

En contrepoint aux notions de portées théorique et pragmatique pour prendre en compte l'existant dans une institution, nous introduisons la notion de portée institutionnelle :

> La portée *institutionnelle* d'une technique relative à un type de tâches T est l'ensemble des tâches où cette technique est attendue par une institution. Cette portée est une conséquence des conditions et des contraintes de la vie de τ dans une institution. (Kaspary, Chaachoua et Bessot, 2020, p. 247).

Les portées pragmatique et institutionnelle d'une même technique sont confrontées pour mieux caractériser le fonctionnement d'une institution selon le processus de modélisation suivant.

Dans les premières étapes de l'analyse *a priori*, nous avons étudié des praxéologies ponctuelles élémentaires possibles, ce qui conduit à identifier des sous types de tâches pivots (étape 1) et des outils (portées pragmatiques, concurrence des techniques et coût d'une praxis) pour caractériser les conditions d'existence et de coexistence des techniques et de technologies de T (étape 2).

Une nouvelle enquête cherche à relever les tâches présentes attachées à T à partir d'observables de l'institution (programme, ressources des enseignants, observation de classe, cahier des élèves, etc.). Puis un processus de modélisation impliquant des calculs du coût des praxis cherche à rattacher ces tâches à des sous types de tâches T_i : ces sous types de tâches T_i rendent compte de choix institutionnels. Ce découpage en sous-type de taches T_i est construit par des allers-retours entre le relevé des tâches dans les observables de l'institution et l'analyse *a priori*. La chronogenèse des techniques organisées par l'institution et la portée institutionnelle de ces techniques nourrissent ces allers-retours.

Puis nous poursuivons l'analyse *a priori*, en termes de coût des techniques concurrentes possibles, pour déterminer quels sous types de taches T_i appartiennent à quelles portées pragmatiques de quelle technique. Cette analyse *a priori* permet de caractériser (i) les dynamiques praxéologiques mise en place dans une institution, (ii) les conditions et contraintes institutionnelles de cette mise en place. D'une part, elle peut permettre d'envisager des dynamiques praxéologiques alternatives sous les conditions et contraintes de I. D'autre part, elle peut permettre d'adapter le MPR en tenant compte des regroupements possibles ou impossibles des praxéologies ponctuelles dans une analyse ascendante.

3. Étude du cas du système d'enseignement français à propos de $T_{équation\_degré\_2}$ (Résoudre algébriquement une équation quadratique)

Nous nous restreignons dans cette partie à illustrer les deux premières étapes de l'analyse *a priori*, celle portant sur les praxéologies ponctuelles et celle portant sur certaines conditions de leur coexistence

Nous avons déjà présenté dans le tableau 1, l'analyse *a priori* des praxéologies ponctuelles élémentaires autour de $T_{équation\_degré\_2}$ (étape 1). Le répertoire de 5 techniques - et donc de 5 praxéologies ponctuelles élémentaires, est l'aboutissement de deux enquêtes. Une première enquête a porté sur les observables de l'enseignement français que sont les manuels scolaires français. Nous avons choisi trois manuels par niveaux scolaires - troisième, seconde et première, où $T_{équation\_degré\_2}$ est présent. Nous avons identifié quatre techniques, $\tau_{racine\_carrée}$, $\tau_{produit\_nul}$, $\tau_{discriminan}$t, $\tau_{sp\_evident\_root}$ - cette dernière introduite récemment en première. Puis nous avons complété le répertoire par une seconde enquête menée dans le cadre d'une analyse comparative entre plusieurs institutions de pays différents (analyse synchronique).

Rappelons que, dans ce cas, trois types de tâches pivots ont été mis en évidence par cette analyse *a priori* : $T_{produit\_nul}$, $T_{carré\_cte}$ et $T_{trinôme}$.

Dans le tableau 1, le niveau de description des techniques est suffisant pour caractériser des types de tâches pivots, mais insuffisant pour le calcul du coût. Ce calcul doit prendre en compte la praxis comme présenté dans l'étape 2. Reprenons l'exemple du sous-type de tâches $T_{carré\_cte}$, « $(ax+b)^2 = k$, (k>0) ». Si l'on s'en tient à la description du tableau 1, le coût de de $\tau_{racine\_carrée}$ serait de 3 et celui de $\tau_{discriminant}$ serait de 2. A un niveau de granularité plus bas que celui du tableau 1, le calcul des praxis

($T_{carré\_cte}$, $\tau_{racine\_carrée}$) et ($T_{carré\_cte}$, $\tau_{discriminant}$), donnent un résultat inversé, respectivement 5 et 9 (voir tableau 2).

Résolution algébrique de $T_{carré\_cte}$ « $(ax+b)^2 = k$, (k>0) ».			
$\tau_{racine\_carrée}$		$\tau_{discriminant}$	
Appliquer racine carrée à un nombre	1	Développer une expression sous forme d'un carré	2
Résoudre une équation de degré 1	2[3]	Déplacer un terme ou un facteur d'un membre à un autre	1
Résoudre une équation de degré 1	2	Réduire une expression somme	1
		Identifier les coefficients d'un trinôme pour le calcul de delta	1
		Ecrire Delta	1
		Calculer Delta et conclure sur le signe	1
		Écrire une solution	2

Tableau 2. Calcul du coût des praxis ($T_{carré\_cte}$, $\tau_{racine\_carrée}$) et ($T_{carré\_cte}$, $\tau_{discriminant}$)

Ce calcul de coût permet d'étudier la mise en concurrence des techniques par rapport à un sous-type de tâches de T donné, quand les techniques coexistent dans une institution I. Le découpage de $T_{équation\_degré\_2}$ en sous-types de tâches dans I est une modélisation qui permet au chercheur de caractériser les portées institutionnelles des techniques, et de conduire une analyse *a priori* des praxéologies élémentaires existantes ou pouvant exister dans I (étape 3).

4. Conclusion

La compréhension du fonctionnement d'un système d'enseignement, dans le cadre de la TAD, s'appuie pour nous sur l'explicitation du modèle dominant d'une institution relatif à un objet de savoir. Pour cela, les chercheurs doivent expliciter comme *hypothèse de travail* un modèle praxéologique de référence qui permet de décrire le modèle dominant, de le comprendre et de l'évaluer. Dans le paradigme de la visite des œuvres, un modèle praxéologique de référence peut être proposé à partir d'une analyse descendante selon les niveaux inférieurs de l'échelle de codéterminantion « discipline, domaine, secteur, … » (cf. Figure 2).

[3] On attribue un poids de 2 en prenant en compte des gestes à un niveau de granularité plus bas : par exemple ici (résoudre une équation de degré 1) convoque deux gestes, (soustraire un nombre aux deux membres d'une équation) et (diviser par un nombre les deux membres d'une équation).

Nous avons essayé de montrer ici la place de l'analyse *a priori* dans le questionnement conjoint du modèle de référence et du modèle dominant afin de les adapter et de les enrichir pour les besoins de la recherche selon une analyse ascendante.

L'analyse *a priori* se construit dans des allers-retours constants entre une analyse épistémologique et une analyse institutionnelle, pour repérer des praxéologies ponctuelles institutionnelles et alternatives. Cette démarche a mis en avant le rôle important joué par les outils que sont « portée pragmatique » et « coût d'une praxis ». Nous nous sommes limités aux praxéologies à enseigner ou pouvant être enseignées dans une institution. Nous envisageons de prolonger l'analyse *a priori* du côté des praxéologies enseignées et apprises, pour lesquelles nécessairement interviennent des coûts liés au logos de ces praxéologies.

Notre étude a été envisagée dans le paradigme de la visite des œuvres. Dans le paradigme de questionnement du monde, on est amené à rencontrer des *R*♥ rattachées nécessairement à des praxéologies ponctuelles. Quelle place peut alors occuper une analyse *a priori* de ces praxéologies dans le schéma Herbatien au cœur de ce paradigme (Chevallard, Y. 2007) ?

Références

Bessot A., Comiti C. (1985). Un élargissement du champ de fonctionnement de la numération : étude didactique du processus. *Recherches en Didactique des Mathématiques*. 6(1-2). Grenoble : La Pensée Sauvage. 305-346.

Bessot A., Comiti C. (2013). Apport des recherches comparatives internationales aux recherches en didactique des mathématiques. Le cas de la France et du Viêt Nam. *Recherches en Didactique des Mathématiques.* 33(1). Grenoble : La Pensée Sauvage. 45-77.

Bosch M. (2016). Research methods in ATD. Diaporama d'une communication au colloque pour les 70 ans d'Yves Chevallard. Barcelone : Espagne.

Bosch M. (2019). Les modèles praxéologiques de référence : réflexions méthodologiques en TAD. In J. Pilet & C. Vendeira (Eds.) : *Actes du séminaire de didactique des mathématiques de l'ARDM*. 87-98. HAL Id : hal-03041140, version 1

Bosch M., (2021). Yves Chevallard Unit. Présentation du module 3. ICMI AMOR. https://www.mathunion.org/icmi/awards/amor/awardees-units

Brousseau G. (1998). *Théorie des situations didactiques*. Grenoble : La Pensée Sauvage.

Chaachoua H. (2018). T4TEL un cadre de référence didactique pour la conception des EIAH. In J. Pilet & C. Vendeira (Éd.), *Actes du séminaire national de l'ARDM*. Paris.

Chaachoua H & Bessot A. (2019). La notion de variable dans le modèle praxéologique. *Educação Matemática Pesquisa*: Revista do Programa de Estudos Pós-Graduados em Educação Matemática, v.21. doi: https://doi.org/10.23925/1983-3156.2019v21i4p234-247

Chevallard, Y. (1984). *La transposition didactique : du savoir savant au savoir enseigné*. Grenoble : La Pensée Sauvage.

Chevallard Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en didactique de mathématiques*. 19(2). Grenoble : La Pensée Sauvage. 221-265.

Chevallard Y. (2002.) Organiser l'étude – Ecologie & Régulation. In J.-L. Dorier et al. (eds) : *Actes de la XIe école d'été de didactique des mathématiques* (Corps, 21-30 août 2001). Grenoble : La Pensée Sauvage. 41-56.

Chevallard, Y. (2007). Un concept en émergence : la dialectique des médias et des milieux. In Gueudet G. & Matheron Y. (Eds.) *Actes du séminaire national de didactique des mathématiques*. Année 2007 (pp. 345 – 366). Paris : IREM Paris VII.

Chevallard, Y. (2010). La didactique, dites-vous ? *Éducation & didactique*. 4(1), 139-148.

Chevallard Y. (2019). Des programmes, oui. Mais pour quoi faire ? Vers une réforme fondamentale de l'enseignement. *Recherches en Didactique des Mathématiques*. 39(1). Grenoble : La Pensée Sauvage. 97-115.

Gascón J. (1994). Un nouveau modèle de l'algèbre élémentaire comme alternative à « l'arithmétique généralisée ». *Petit x,* 37. 43-63.

Kaspary, D., Chaachoua, H., Bessot, A. (2020). Qu'apporte la notion de portée d'une technique à l'étude de la dynamique praxéologique *? Annales de didactique et de sciences cognitives*. N° 25. 243-269.

Pilet J. (2012) *Parcours d'enseignement différencié appuyés sur un diagnostic en algèbre élémentaire à la fin de la scolarité obligatoire : modélisation, implémentation dans une plateforme en ligne et évaluation*. Thèse ; Education. Université Paris-Diderot - Paris VII. HAL Id: tel-00784039

Ruiz-Munzón, N., Bosch, M. et Gascón, J. (2020). Un modèle épistémologique de référence pour la recherche sur l'algèbre élémentaire. *Nouveaux cahiers de la recherche en éducation*, 22(1), 123-144.

Schneider M. (2019). Quels usages du concept de modèle épistémologique de référence pour la recherche ? Et pour la formation ? In J.Pilet & C.Vendeira (eds) : *Actes du séminaire de didactique des mathématiques de l'ARDM*. 73-86. HAL Id : hal-03041140, version 1

Sirejacob (2016) Les organisations de savoirs mathématiques à enseigner : les équations au collège. *Petit x*, n°102, pp. 27-55. IREM de Grenoble.

Media and milieu dialectics and new resources available for students

Britta Eyrich Jessen

University of Copenhagen, Department of Science Education, Denmark; britta.jessen@ind.ku.dk

In this paper we present a case study from upper secondary mathematics in Denmark, where students were invited to solve an open problem regarding the design of a facial recognition system. We analyse the activity as a study and research path, where we focus on the media-milieu dialectic. This allows us to identify new potentials regarding resources available to students during mathematics classes but also the need for methodologies on how to design teaching capturing these new resources in order to fully exploit those. pe your abstract here. The purpose of this paragraph is to draw attention to the style for abstracts, which is Normal, italic, and the length is up to 10 lines.

Keywords: Study and Research Paths, media-milieu dialectics, technology

Introduction

When students bring smartphones, tablets, or laptops to the classroom, they also enlarge the number of media available for their study of mathematics regardless of the teachers' planning. The teacher can ask students to leave those tools in the school bag but in this paper, we want to explore what potential emerge, when no limitations are put on students' activities and search for media. Thus, case study of this paper is chosen, as it highlights some unexplored potentials for the design of teaching activities linked to the paradigm of questioning the world. In particular, the dynamics of going forth and back between study and research processes and what resources might support students in the dialectic between media and milieu. The task design was done by upper secondary teachers aiming at developing students 'innovative competence'. It is a new requirement for all disciplines in upper secondary education in Denmark to support the development of this specific competence. In this paper we consider this requirement as an invitation for nurturing more inquiry-based approaches to teaching and learning such as study and research path, SRP (Chevallard, 2004 & 2006, Winsløw, Matheron & Mercier, 2013).

In the curriculum for mathematics, it is stated that: "Students should gain mathematical insights which […] prepare them for citizenship, and the ability to contribute actively, constructively, and innovatively in a democratic society." Moreover, it says that "Through inquiry-based approaches to mathematical topics and problems, the students' mathematical notions and innovative competences should by developed." (Danish Ministry of Education, 2017, p. 3). The general guideline across all disciplines describing the 'innovative competence' defines it as having three competence areas: 'Inquiry of a problem', 'Develop a proposal for a solution' and 'Evaluate the solution'. It is declared that "Innovation concerns the development and evaluation of solutions linking to disciplinary, general, or concrete/authentic problems in relation to the characteristics of the specific [upper secondary] education" (Danish Ministry of Education, 2019). These descriptions align with simple descriptions of the modelling cycle and modelling competence. Kaiser & Schwarz (2006) presents four phased cycle consisting of: idealization [delimit the real world situation to a real world model] mathematisation [describe it mathematically], mathematical considerations [solve the problem mathematically], reinterpretation/validation [assessing the answer against the real world situation] (Kaiser & Schwarz, 2006, p. 197). Thus, in both cases a model is to be constructed in order to solve a real world problem and the validity of the model must be tested against this reality. In our case study the students are asked to model a facial recognition system where all previous learned knowledge from mathematics can be employed to create an innovative solution. We consider this an inquiry-based modelling activity fit for a SRP analysis, though as argued by Barquero and Jessen (2020) the competence and SRP approach represent different lenses for the study of mathematical modelling where the former does not capture a gradually evolving learning environment as SRP and is the case below.

Methodology, SRP and media-milieu dialectic

Analysing the observed teaching we follow the proposal of Winsløw and colleagues (2013) to analyse teaching as a SRP identifying the question and answers, the study and research processes realised during the activity. This means that the problem posed to the student is considered a generating question, Q_0, which is open and has the generating power of nurturing students to pose derived question. Hence, students understand the question posed, but also realise that they need to combine existing knowledge in new ways to answer the question – or to acquire new knowledge. Initially students try to use answers they previously have developed, $A_i^{\lozenge}$, their existing knowledge, to develop answers for the generating question. Often, they will need to combine existing knowledge with new knowledge acquired by studying works, W_j. This can be textbooks, webpages, newspaper articles, youtube videos and all sorts of media produced to disseminate (mathematical) knowledge (Kidron et al., 2014). Or to combine the existing knowledge with new data, D_kThe notion of data is broad and covers both quantitative and qualitative data (Chevallard, 2019). Existing answers, works and data are all media to be studied in SRP's. The study process is characterised as deconstruction of knowledge, where research is considered reconstruction of knowledge as it is when existing knowledge, data and new knowledge are pieced together in terms of partial or more complete answers, $A_l^{\heartsuit}$ to the generating question. The heart signals that the answer is personal to the student or group who developed it. Thus, the reconstruction of knowledge means the development of answers within a milieu, which is "a system of objects acting as a fragment of "nature" for Q, able to produce objective feedback about its possible answers" (Kidron et al, 2014, s. 158). The milieu includes the media and questions addressed.

The Herbartian schema describes the study and research process and indicate the gradually developing milieu. The schema indicates how the didactic system S, which consist of a group of students, X, guided by a group of teachers, Y, study the question, Q_0, while interacting with the milieu developing while an answer $A^{\heartsuit}$. In short:

$$[S(X; Y; Q_0) \rightarrow \{A_i^{\lozenge}, \dots W_j, \dots, D_k\}] \rightarrow A^{\heartsuit}$$

Identifying specific derived questions, partial answers, previously developed answers, data and works studied allow us to identify each step of the dialectic between media and milieu, between study which support students' construction of knowledge or learning. Following the gradually developed answers further allow us to identify the dialectic between theoretical and empirical modelling.

In the following we adopt the methodology suggested by (winsløw et al. (2013) and elaborated by Jessen (2017), where discourse analysis of classroom observations leads to the identification of derived questions, strategies and knowledge employed to identify partial answers. This study is based on observation notes, sound recording and pictures from classroom is analysed. Below we present the result of the analysis in order to discuss how students' answers develop. From this we seek to answer the following research question: When analysing the media-milieu dialectics in the case study, what implications for SRP task design emerge? What measures are needed to fully capture the potential of engaging students in media-milieu dialectics in light of new and further media available?

Context of study

The teaching took place in the late spring semester of grade 10, which is first year of upper secondary school in Denmark. The class followed social science as their study line, which means they have two years of mathematics. They are not particularly interested in mathematics and most of them are struggling to pass the exam. The class had completed working with notions of functions, which means they were familiar with linear functions, exponential functions, power functions and polynomials. They had had a recap of trigonometry from lower secondary and were supposed to initiate the work on vectors in two dimensions, when this activity on innovation were completed. However, this activity were not linked to the following lessons.

The problem posed to the students were the following:

> "Q_0: You are designers of digital solutions to security systems. You have a costumer who needs a facial recognition system. For efficiency the system should only require three measures in order to recognise people. What measures are needed and why?"

It was handed out on paper, together with a picture of a man's face, where grid lines were printed on top of it. Furthermore, certain traits of the face were marked with points. The picture is shown in figure 1. An electronic GeoGebra file, where the picture was imported were available to the students in the learning platform used by the school.

The students worked 2 times 60 mins. on the problem, where two sharing sessions were organised by the end of each lesson. The students worked in their usual groups of 3-5 students. All mathematical domains and aids were allowed including search for further knowledge. The students were told that all strategies were allowed as long as they could explain themselves. The teacher answered questions from the groups and helped with technical issues of working with GeoGebra. The students were encouraged to work autonomously and refused to hint strategies and solutions. Though he highly encouraged the students to be inspired by other groups, pose questions to presenting groups during the plenum sessions and steal the best ideas.

Many groups started by identifying patterns of polynomials in the man's face and experiment with regression. Other groups tried linear regression. Most groups were puzzled by the meaning of measurements, and if coordinates for a point was a measure or if the factors of the polynomials were the measures found.

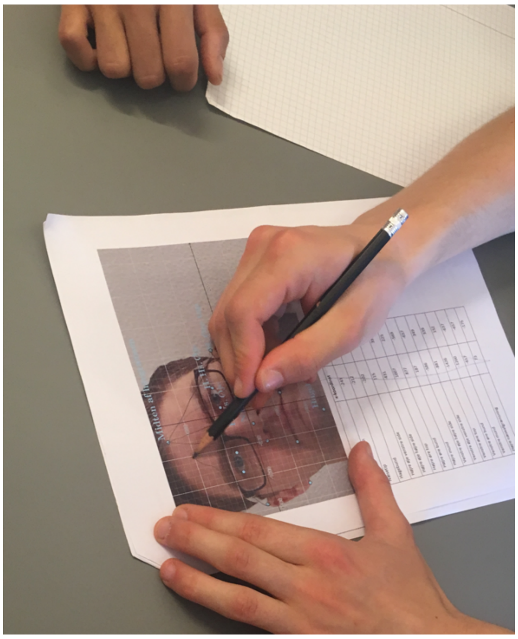


Figure 1: The handout with the picture of a face, gridlines and traits marked with points

In the following we choose to focus on one group which ended up working with trigonometry, as their inquiry highlights new potentials for how students can explore specific problems and assess their own work using other media than usually available.

A posteriori analysis of the lesson

When focusing on one group's work, the didactic system we study consist of $S(X_1, y, Q_0)$, where $X_1 \subseteq X$ is the group of three students we follow (X being the whole class) and y is the teacher. We consider the handout as a document containing the generating question, Q_0, and the initial data D_1 being a man's face with suggested fix points marked in a coordinate system printed on top of the picture. Initially all students studied the paper version of the document, and we can argue that the milieu consisted of $M = \{A_1^\Diamond, A_2^\Diamond, \dots, D_1\}$. Where $\{A_1^\Diamond, A_2^\Diamond, \dots\}$ represent students' existing knowledge. In the following we include y in the diagram considering the teacher being the creator of situations more akin to adidactical situations (Brousseau,1997). The teacher observed the progress of the group we follow in order orchestrate the plenum session but did not interfere. The progress of study was driven by students' engagement with Q_0 and the design of the milieu (being the handout with initial data). The analysis of the two lessons below, show how the student explore the potential of their most recent learning (types of functions) before they focus their inquiry on trigonometry.

Analysis of first lesson

The initial work of the particular group, followed the rest of the class posing derived questions as Q_1: How to fix measures on someone's face? Q_2: Which points to choose. For answering these questions, they studied the handout, D_1, and decided rather quickly that a point with coordinates should count as one measurement. We can consider this a first partial answer, $A_1^\heartsuit$

$$[S(X_1; y; Q_{0,} Q_1, Q_2) \rightarrow \{A_1^\Diamond, A_2^\Diamond, \dots, D_1\}] \rightarrow A_1^\heartsuit$$

This lead the students to consider,Q_3: What kind of mathematical object can be determined from three points in a coordinate system? They drew on a more recently developed answer regarding polynomials and regression, $A_i^{\Diamond}$, to consider, if they could determine a parabola and if it will help them identify a specific person. They noticed that another group, X_2, considered linear regression on three points. They went to the digital version of the handout and opened it in GeoGebra. Using GeoGebra they chose three points and created polynomial regression. They did this a couple of times with fix points chosen as the corner of the eyes and the tip of the nose. Another polynomial was modelled after the tip of ears and the tip of the chin.

Thus, students explore if the facial traits can be modelled with polynomials, $Q_{3,1}$, or as a linear function, $Q_{3,2}$. We consider this a research process done by the students based on existing knowledge on different types of functions, which leads them to construct works in terms of models to be validated against the milieu (existing, knowledge, facial photography in GeoGebra and the conditions for the facial recognition system formulated in Q_0). We consider the empirical models as qualitative data to be studied, D_2. Here the students rejected the polynomials as models suitable for facial recognition as the parabola always pass through the points and they were in doubt how to figure out an acceptable error for the coefficients found through regression. For the same reason they rejected the idea of using other known types of functions.

The next mathematical object recognisable to the students were triangles, drawn between the three points in the coordinate system. Thus, they implicitly pursued the question $Q_{3,3}$: what triangle can be the basis of the facial recognition system?

They experimented in GeoGebra with triangles made from different points in the face. They revisited the notion of points, and that they do not extent in any directions, $A_i^{\Diamond}$. This was important when choosing the traits of the face where the fix point easily could be marked on the picture of a face. They decided to base their solution on points placed at the inner corners of the eyes and the tip of the nose. Figure 2 the left picture show how they have marked the triangle they worked with.

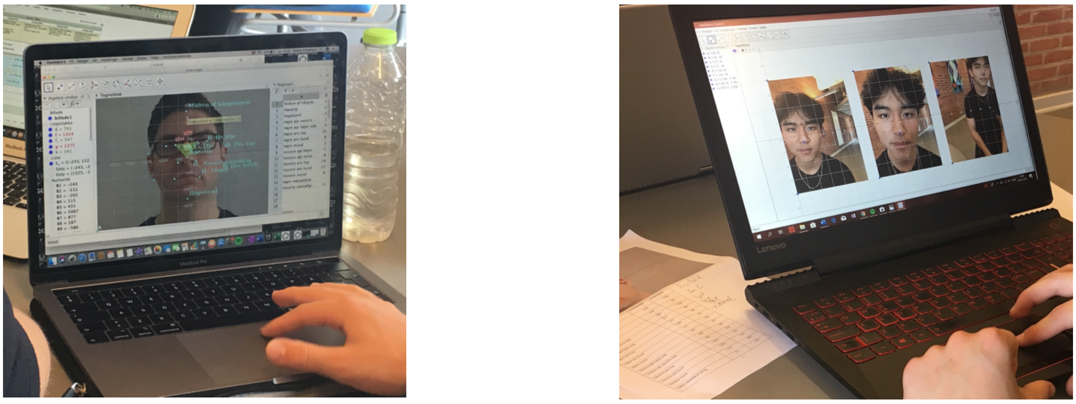


Figure 2: The left picture show students initial work in GeoGebra using the the handout. The right picture show their own pictures when testing hypotheses.

At this point the teacher asked students to present their work in plenum. Since the session was organised as sharing preliminary ideas where groups could 'steal' ideas presented by others, no ideas were discarded completely. Thus, the answers presented by other groups became works to be studied by the rest of the class. The group we follow in this analysis took from the plenum session that others were also looking at triangle, why they kept working on this strategy.

Analysis of second lesson

At the beginning of the second lesson X_1 started discussing $Q_{3,3,1}$: what properties of the triangle could be preserved from one picture to the next one? They had been ‘googling’ facial recognition systems and new that they were based on facial pictures. Companies selling those discussed sensitivity in terms of recognising faces from different angles and distances. As this work was done after school hours we cannot go into details with the study and research done for this path initiated by Q_4: how does facial recognition work in the real world? Though we can conclude from students’ conversation that they constructed an answer $A_4{}^{\heartsuit}$: Most systems are able to recognise faces from different angles and distances, though the faces need to be relatively close to some camera. This became part of the milieu against which their final answer must be validated.

The group agreed the coordinates were insufficient as secure measures and too sensitive to placement of the face in front of a camera. Therefore, they asked $Q_{3,3,1,1}$: are the proportion of side lengths preserved? $Q_{3,3,1,2}$: Is the area of the triangle preserved? $Q_{3,3,1,3}$: Are the angles within the triangle preserved? $Q_{3,3,1,3,1}$: Are these the same for all people? To develop answers for these questions, they took pictures with their smartphones of a group member, D_3, see right side of figure 2. They imported the pictures to GeoGebra and used the functionalities in GeoGebra to find side lengths from coordinates, the area of triangle and angles within the triangle. These instrumented techniques are existing answers students can draw on from previous work within GeoGebra. They further considered if they had to be able to argue how to find the angles within the triangle if they did not have GeoGebra. Thus, they revisited the textbook and a webpage on the law of cosine. We consider this the study of works, W_1 and W_2.

With the new data, D_3, they discovered sufficient difference in the angles found within the tringles between corners of the eyes and tip of the nose, from the man to the student. Thus, validating their model for a facial recognition system in terms of this triangle were a success. So, they went on to exploring if the triangle of the student could be confirmed by pictures from a longer distance and if having a different angle to the camera. Distance proofed to be no problem. The students never realised that they were looking at equilateral triangles and therefore the angles were preserved though side lengths were different when pictures were imported to GeoGebra. However, looking at the right most picture, where the angle to the camera is altered a little, the results were slightly different. The angles within the triangle were varying one degree. Thus, again they started to discuss, how much error margin would be allowed, if they were to differ between different faces. Before they were asked to present their work in plenum by the end of the second lesson, they concluded that they needed to experiment with more data, meaning more pictures of different persons from different distances and different angels to the camera.

If we are to describe the group’s work during the second lesson it looks like this:

$$[S(X_1; y; Q_{0,}Q_3, Q_{3,3,1} \ldots) \Rightarrow \{A_1^{\Diamond}, A_2^{\Diamond},\ A_4^{\heartsuit} \ldots, D_1, D_2, D_3, W_1, W_2, \ldots\}] \Rightarrow A_3{}^{\heartsuit}$$

The second lesson was the last lesson where students worked on the facial recognition system. The quality of answers presented, varied a lot. Other groups had also arrived at the idea of identifying the angles of some triangle in the face, though not tested with real facial pictures. Others still struggled with different types of functions. The plenum presentations were summed up, by the teacher concluding that the mathematics they had learned so far in upper secondary school could be used in various ways to solve concrete problems from the real world. Though some solutions might be more favourable than others with respect to security. From this conclusion some students seemed in doubt when leaving the classroom, if their solution were a good one? If they actually had discovered how such facial recognition systems works? In this sense the openness of the problem with no validation seemed to frustrate some of the students leaving them in doubt if they had learned what was intended.

Discussion of dialectics shown in the case study

From our analysis we see that the problem has a generating power for the particular group of students. They were engaged in exploring the potentials of the most recent knowledge they had constructed (previous learning), they tried to use web searches to understand the technical device they were designing a solution for, they drew on data handed out and produced more in order to validate their own answers. We can also argue that they engaged in the 'inquiry of a problem', they 'developed a proposal for a solution' and they 'evaluated the solution' (Danish Ministry of Education, 2019), which means that they covered the 'innovation competence areas' and developed this competence further.

From the case study we see that students' use of smartphones for taking pictures and web searches enlarged the media drawn upon by students. The searches provided knowledge about facial recognition systems, which became part of the milieu against which the proposed solution had to be evaluated. Furthermore, smartphones enabled the evaluation by providing new pictures imported and analysed in GeoGebra installed on the students' personal laptops.

Though all groups of students had access to smartphones and laptops with GeoGebra, not all groups were this productive in elaborating the milieu and consulting further media. Reasons for this can be the changed didactic contract (Brousseau, 1997), compared to more traditional teaching. For students, these more open problems can be difficult to navigate. Jessen (2017) orchestrate more frequent plenum sessions where preliminary answers were shared and compared. To insist on the openness as in this design and not really unfold or explain how some strategies were more fruitful than others might confuse students rather than teach them the applicability of the mathematics they have learned in school.

To navigate the openness of SRP it might be crucial to complete a priori analyses where potential strategies can be depicted in question-answer maps (Barquero & Jessen, 2020). Still, the maps do not take potential media into account. As seen from our case study a richness emerges when students autonomously question the reality to be modelled, the mathematical knowledge drawn upon as well as seeking and producing data to support their inquiry. However, if we wish more groups of students to pursue such media or create data, we might need to scaffold these processes by proposing repositories students can consult as part of the task design as suggested by Jessen, (2017). Still, this study show a larger creativity compared to other studies. It might be nurtured by the handout, where students copied the data production. If the teacher had foreseen, how the handout and available digital tools invited for this activity, it might have been easier to orchestrate the plenum session of comparing and contrasting different solutions leading to more clarity among the students regarding the intended learning outcomes.

Based on our case study we see a need for including potential media-milieu dialectics in the a priori analysis when designing SRP, if we fully wish to nurture students' engagement with study and research processes and in the de- and reconstruction of knowledge. For such analysis we need to take into consideration more media than handed out by the school system as textbooks and calculators.

Acknowledgements

Lundbeck Foundation Project R284-2017-2997.

References

Barquero Farràs, B., & Jessen, B. E. (2020). Impact of theoretical perspectives on the design of mathematical modelling task. *Avances de investigación en educación matemática, 17*, 98-113.

Brousseau, G. (1997). The Didactic Contract: The Teacher, The student and the Milieu. In N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds.), *Theory of didactical situations in mathematics* (p. 226-249). Dordrecht: Kluwer Academic Publishers.

Chevallard, Y. (2004). Vers une didactique de la codisciplinarité. Notes sur une nouvelle épistémologie scolaire. *Journées de didactique comparée 2004*, Lyon. http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=45

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceedings of the 4th Conference of the European Society for Research in Mathematics Education* (p. 22-30). Barcelona: Fundemi IQS.

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: *An attempt at a principled approach. Hiroshima journal of mathematics education, 12*, 71-114

Danish Ministry of Education (2017). Bilag 112, Matematik B – stx, august 2017. https://www.uvm.dk/-/media/filer/uvm/gym-laereplaner-2017/stx/matematik-b-stx-august-2017-ua.pdf

Danish Ministry of Education (2019*). Elevernes innovative kompetencer Vejledning til lov og bekendtgørelse - Lov om de gymnasiale uddannelser*. https://www.uvm.dk/-/media/filer/uvm/udd/gym/pdf19/aug/190801-vejledning-til-lov-og-bekendtgorelse---elevernes-innovative-kompetencer.pdf

Jessen, B. E. (2017). How to generate autonomous questioning in secondary mathematics teaching? *Recherches en Didactique des Mathematiques, 37*(2/3), 217-245.

Jessen, B. E., & Kjeldsen, T. H. (2021). Mathematical modelling in scientific contexts and in Danish upper secondary education: are there any relations? *Quadrante, 30*(2), 37-57. https://doi.org/10.48489/quadrante.23658

Kidron, I., Artigue, M., Bosch, M., Dreyfus, T., & Haspekian, M.

(2014). Context, Milieu, and Media-Milieus Dialectic: A Case Study on Networking of AiC, TDS, and ATD. In A. Bikner-Ahsbahs, S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (p. 153-177). Springer International Publishing.

Winsløw, C., Matheron, Y. & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies of Mathematics, 83*, 267–284. https://doi.org/10.1007/s10649-012-9453-3

Modelling inquiry in ATD: a framework to capture and analyze the chronogenesis and mesogenesis of an inquiry process

Jana Lackova

University of Geneva, Faculty of Psychology and Educational Sciences, Geneva, Switzerland; jana.lackova@unige.ch

This paper proposes to present a framework based on the tools of the Anthropological theory of the didactic (the Herbartian scheme, the question-answer and media-milieu dialectics) that allows to capture and analyze the chronogenesis and mesogenesis during an inquiry process. Our research questions the place of inquiry in an institutional context that places it in the heart of its educational philosophy, particularly when it becomes an institutionally recognized object. Within the International Baccalaureate (IB), each student must produce a written and individual piece of work entitled "Mathematical Exploration" that should involve an investigation in a field of mathematics.

Keywords: Anthropological theory of the didactic, the Herbartian scheme, dialectics, inquiry, International Baccalaureate.

Introduction

In our thesis, we examine the place of inquiry in mathematics within the International Baccalaureate (IB) because we identified at this institution a good potential for observing inquiry processes. Our research consists of three parts: the institutional analysis of the IB (Lackova & Dorier, 2018), a survey questioning IB mathematics teachers about their attitudes, beliefs and convictions concerning inquiry and finally, an exploratory case study observing two IB mathematics teachers and following the elaboration of an inquiry-based activity of two IB students. The case study took place at the International School of Geneva in Switzerland. In this paper, we are going to briefly introduce the institutional context, the conception of our analysis framework and present some key results concerning an individual inquiry-based research activity.

The institutional context

The IB Diploma Program is intended for students aged 16-19 and its successful achievement leads to a diploma that provides access to higher education. A thorough analysis of the institutional documentation (Lackova & Dorier, 2018) using the scale of didactic co-determinacy (Chevallard, 2005) allowed us to identify elements pointing to the presence of inquiry-based education at each level of this scale. At the level of discipline, problem-solving is considered to be central to learning mathematics and teachers are expected to "provide students with opportunities to learn through mathematical inquiry" (IBO, 2012, p. 11). The IB also requires a summative assessment of inquiry-specific nature in mathematics, compulsory for all students and contributing to 20% of the final grade. This requirement consists of "a piece of written work that involves investigating an area of mathematics" (IBO, 2012, p. 43) called Mathematical exploration (ME). It is an individual, stand-alone research activity conducted on a topic chosen by a student and supervised by the mathematics teacher. The curriculum recommends dedicating 10 hours of in-class time and 10 hours personal work to the exploration. During the ME, the students have access to different sources of information (internet, books, Wolfram alpha, tutor, etc.) and these tools become even primary sources of information by the simple fact of the institutional constraints weighing on the teacher. Indeed, the latter one must supervise about twenty explorations in his class and simply cannot be the first source of information as one would expect in a class governed by a classical didactic contract. As a result, the students' responsibility expands: they find themselves working independently, navigating on the internet, sorting out information and deciding on its relevance and usefulness. Moreover, if they want to use a particular piece of knowledge, they must demonstrate a good understanding of it and be able to apply it correctly. The ME was a good observable example of implementing an inquiry-based activity and the Anthropological theory of the didactic (ATD) provided a theoretical framework to conduct our research.

The ATD tools to model an inquiry process

The appearance of the guided personal projects in French secondary schools as an institutional requirement raised new questions concerning a missing didactical infrastructure, gave birth to the notion of a study and research path (SRP) and resulted in a need for a change in educational paradigms: from visiting works to questioning the world. (Chevallard, 2009). Consequently, the roles and responsibilities of students and teachers need to be redefined. Indeed, a new role of the teacher is defined as that of the director of studies, that is to help students to seek various means of validation rather than behave like a distributor of knowledge.

The Herbartian scheme

In the paradigm of questioning the world, a didactic system (a group of students X and a group of teachers Y) is formed to investigate the question Q: S(X,Y,Q) and develop an answer $A^{♥}$. To produce $A^{♥}$, the didactic system S(X,Y,Q) needs to look for "material" which forms a didactic milieu M. This consists of ready-made answers validated by an institution $A^{◊}$, resources W, which among others are analysis tools, theories or experiments. Depending on the type of inquiry, these main ingredients could be eventually completed by some derived questions Q and data collections D. The Herbartian scheme below is a condensed way that represents this model:

$$S(X, Y, Q) \rightarrow M=\{A_i^{◊}, W_j, Q_k, D_l\} \hookrightarrow A^{♥}.$$

As we can see the Herbartian scheme allows to list the main elements characteristic for an inquiry process in a rather static way, the milieu is however in constant evolution. Its dynamics according to Bosch (2018) "is captured in terms of some dialectics that describe the production, validation and dissemination of $A^{♥}$"(p. 4008).

The role of dialectics in an inquiry process

In reference to the guided personal projects in French secondary schools Chevallard (2002) states that an optimal workspace to conduct a study of a question should be structured certain bipolar tensions that he refers to as dialectics which "are key gestures to achieve a teaching by SRP"((Parra & Otero, 2018, p. 242). Despite their importance in this type of projects, in traditional school systems the dialectical processes remain largely blocked. In order to grasp the dynamics of the processes at stake behind the constitution of the milieu (mesogenesis) and its evolution (chronogenesis) Bosch (2018) suggests using two main dialectics: the media-milieu and the question-answer dialectics.

According to Chevallard (2008), media refers to any system containing information (textbooks, websites, video tutorials, experts etc.) and carries out an intention to inform. The information found in the media (including the teacher) that forms the didactic milieu does not generally provide a definitive answer $A^{\heartsuit}$ to Q. These elements need to be questioned, tested, combined and validated in order to produce a new answer or raise new questions. It is also important to note that in an SRP the teacher is not a privileged source of information and therefore answers coming from the teacher should be object to the same evaluation as from any other media.

Kidron et al. (2014) mention that the Herbartian scheme can include an a-didactic milieu in the sense of the theory of didactical situations whose role is to provide an objective feedback about the possible answers to Q. For Chevallard (2008) the milieu here is also to be regarded in a sense close to the a-didactic milieu from the TDS as "any system that can be considered as devoid of intention in the answer it can give, explicitly or implicitly, to a given question" (p.344). The "continuous interaction between available (partial) answers given by the media and their testing through the interaction with an a-didactic milieu" (Kidron et al., 2014, p. 158) is referred to as the media-milieus dialectic.

For Brousseau (1998) the construction of new knowledge is impossible without an a-didactic milieu and results in an imitation of someone else's answers. Chevallard (2008) states that:

> [...] the existence of a vigorous (and rigorous) dialectic between media and milieu is a crucial condition for a process of study and research not to be reduced to the uncritical copying of scattered elements of response in the institutions of the society (p. 345)

For Kidron and al. (2014) "the production and organization of an appropriate a-didactic milieu for Q is an essential aspect of the study process" but "finding and creating an appropriate experimental milieu [...] can be one of the most challenging issues to tackle (p.158).

To avoid any confusion, we will denote M_D the didactic milieu from the Herbatian scheme and M_A the a-didactic milieu in sense of TDS.

The question-answer dialectic helps to unfold and describe the different paths followed during the inquiry process (including dead ends and abandoned attempts). When approaching the question Q at the beginning of the inquiry, one starts by searching for available answers $A^{\diamondsuit}$ and studies them. This study usually generates new derived questions that help to move on with the inquiry. According to Bosch (2018) the question-answer dialectics "provides visible proof of the progress of the inquiry and contributes to what is called the chronogenesis of the process" (p. 4008).

Using the above-mentioned theoretical elements from the ATD, we put in place a methodology and developed a framework that enables us to capture the chronogenesis ant the mesogenesis of the ME.

Data collection and data treatment

We followed two volunteer students (17 years old) in the same Mathematics Standard Level class during their explorations from April to June 2019. We conducted two interviews to obtain information about their mathematical profile (pre-exploration) and their experience with the exploration (post-exploration). Since an important part of the work is done at home, we asked these students to keep a logbook and to record their work using an action camera. This was also the main difficulty of collecting the data because the students did not always think to turn on the action camera and this caused some gaps in the chronology. We however obtained reasonable data (notes in logbook, recordings of the computer screen) to reconstruct the chronology of the exploration, identify the accessed media and the research carried out by the students. We then transcribed the data and created a chronology of the exploration listing each action of the student and categorizing each media and its content that appeared in the recordings based on the elements from the Herbartian scheme as shown in Table 1. Obviously, the questions were formulated a posteriori based on the visible data, particularly on the google search input line.

Action		**Media**		**Elements in M_D**		**Production**	
Q	question	V	video	A♦	institutional answer	A♥	produced answer
GS	google search	Doc	document	GD	works-google doc		
GSi	image search	WS	website	Img	works-image		
YTS	youtube search	R	google result	Fgr	works-figure		
Std	student's reflection	PPT	power point	DGr	desmos graphing		
		T	teacher	W	various works		
		Exp	expert	Dt	various data		

Table 1 : Categorization of the obtained data

This enabled us to create a chronological list (Figure 1) in respect to the order of appearance of the different identified elements that we call *occurrences*. Finally, we split the chronology into sections, the beginning of a new section being marked when a question was identified. For example, the label V 2.2 refers to a second video from section 2.

Chronology	Occurance	Description
1	***Q 1.1***	***Unknown***
2	GS 1.1	Unknown
3	PPT 1.1	contains useful data about the bridge + annotated picture
4	GD 1.1	Math sites
5	W 1.1	The link of PPT 1.1 is copied to GD 1.1
6	***Q 2.1***	***how to model a bridge using functions?***
7	GS 2.1	how to model a bridge using functions
8	V 2.1	Khan Academy: Modelling with composite functions
9	V 2.2	Model Parabolic Bridge as quadratic equation
10	A♦ 2.1	"translating" natural language problem (from V 2.2)
11	A♦ 2.2	intercept form of a quadratic equation (from V 2.2)
12	V 2.3	Bridge modelling with Dynabridge Dynamo package
13	V 2.4	Modelling with function combination
14	Doc 2.1	How do functions shape the Sydney Harbour Bridge?
15	A♦ 2.3	Quadratic model of a bridge – solved example
16	Doc 2.1	How do functions shape the Sydney Harbour Bridge?
17	A♦ 2.3	Step 1: Find a structure which has a clear photograph taken from the side (from Doc 2.1)
18	A♦ 1.4	Step 2: Research the height/width/length of your structure (from Doc 2.1)
19	***Q 3.1***	***How will I proceed with my exploration?***
20	Std 3.1	Student's reflection
21	A♥ 3.1	I need to find a clear picture of the GGB taken from the side
22	A♥ 3.2	I need to find the dimensions of the bridge

Figure 1: Extract from the Chronology of the Exploration

Construction of an analysis framework

Even though the Herbartian schema provided a good model to categorize the different "ingredients" of inquiry, it proved rather limited when trying to capture the dynamics of an inquiry process. We wanted to identify the interactions of the student with the media, understand how the information from the media enters the didactic milieu M_D and contributes to the production of $A^♥$, and finally if and how an a-didactic milieu M_A is organized to validate $A^♥$. These questions lead us to the conception of an analysis framework at three different levels as summarized in the scheme on Figure 2.

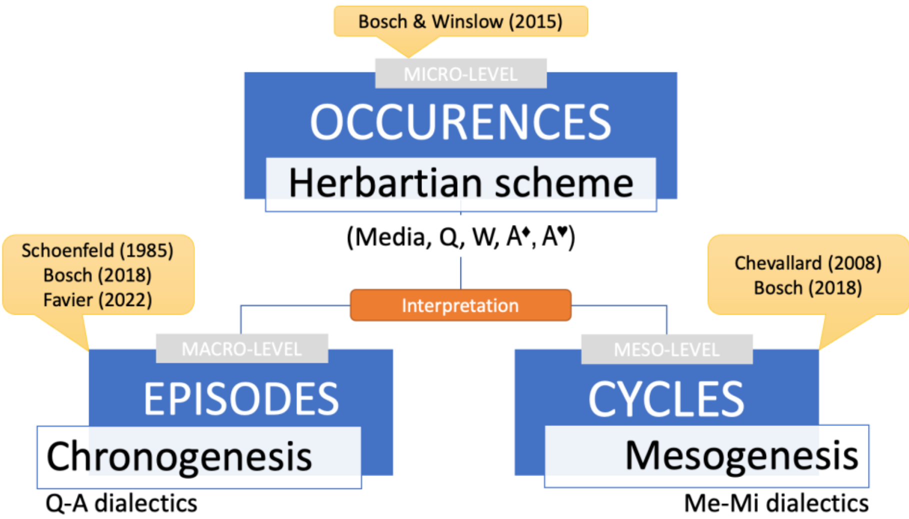
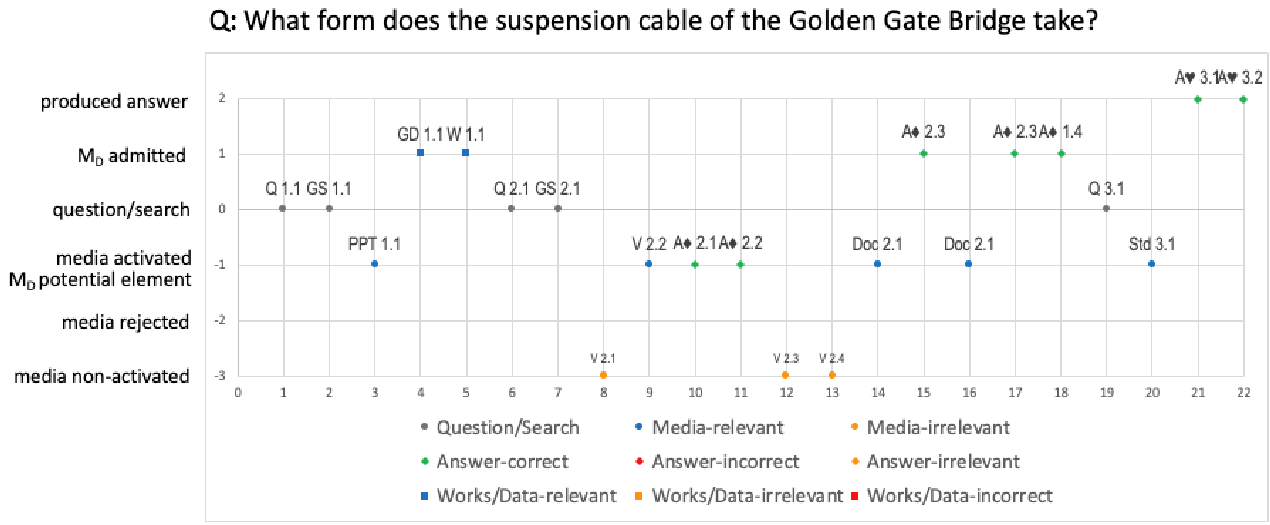


Figure 2: Analysis framework scheme

Micro level

At the micro-level, we labelled each media as relevant or irrelevant according to its potential to contribute to the exploration and activated or non-activated according to whether the student activated it or not. We consider that a media was activated when there was observable evidence such as reading, taking notes, highlighting, watching a video, etc. The content of the media was labelled as admitted, rejected or potential according to whether it was admitted or not in M_D, and correct/incorrect/relevant/irrelevant according to the "quality" of the obtained answers or works. We expressed this complex reality graphically as shown on Graphics 1.

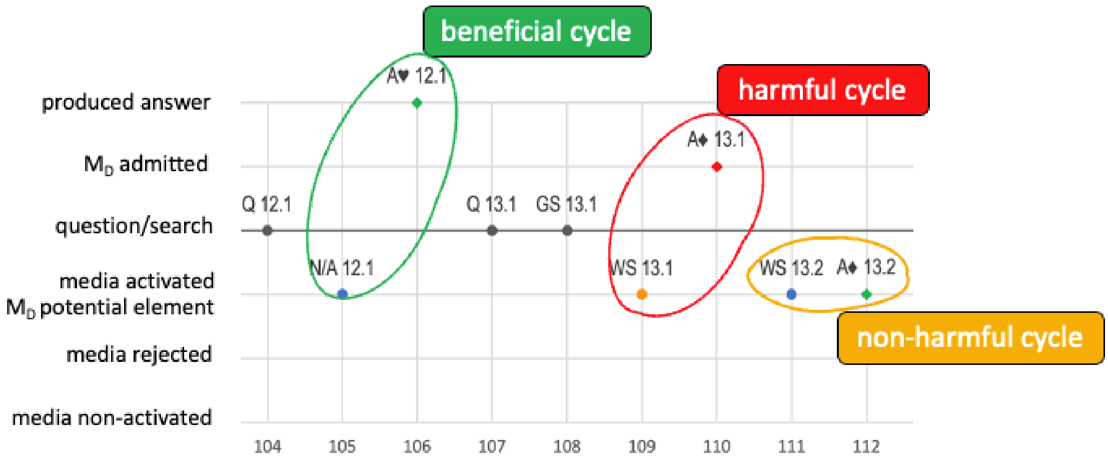


Graphics 1 : Example of graphic representation of an episode

This graphical representation of the chronology of the exploration enables us to make visible the interactions with media and capture the moments when different elements were admitted to the milieu M_D and answers were produced. It is important to note that the chronology is not expressed in terms of time but in terms of the appearance of occurrences (an event unit = 1) in chronological order.

Meso-level

At the meso-level, we split the chronology into cycles. The beginning of a cycle is marked when a new media occurs and expresses the interaction of the student with a media. An example of coding into cycles is provided on Graphics 2.



Graphics 2: Example of coding into Cycles

Based on the type of the interaction with the media, we were able to identify beneficial, non-harmful and harmful cycles and defined them as follows.

Beneficial cycles

- Efficient non-activation: Media is recognized immediately as irrelevant because out of subject and therefore not activated.
- Pertinent elimination: Media is activated because at first sight it seems potentially useful but is rejected as irrelevant.
- Pertinent admission: Relevant media is successfully activated, and new elements are admitted into the milieu, which is thus enriched but not yet productive.
- Pertinent production: Relevant media successfully activated, relevant information retrieved, and a correct partial answer is produced and/or and a new question is generated.

Non-harmful cycles

- Non-activated occasion: Media not recognized as relevant and thus not activated.
- Missed occasion: Relevant media successfully activated however answers remain potential and don’t enter the milieu.
- Irrelevant admission: Irrelevant media is activated, and irrelevant information is admitted to the milieu.
- Irrelevant production: Irrelevant media activated, and irrelevant information is admitted to the milieu. Irrelevant answer is produced.

Harmful cycles

- Incorrect admission: A media containing an incorrect information is activated and this information admitted into M_D.
- Incorrect production: A media containing an incorrect information is activated and this information admitted into M_D. Incorrect answer is produced.

At the meso-level but we expect to be able to characterize the conditions in which a media-milieu dialectics can exist.

Macro level

At the macro-level, we split the chronology into episodes. On the one hand, Bosch (2018) reminds us that the chronogenesis of an inquiry process can be captured through the question-answer dialectic, which allowed us to make an initial division into sections; each new question marking the beginning of a section. On the other hand, we adapted Schoenfeld's (1985, pp. 297–301) analysis tools and divided the exploration into episodes based on the definition of an episode as:

> A period of time during which an individual or a problem-solving group is engaged in one large task [...] or a closely related body of tasks in the service of the same goal [...] (p. 292).

In relation to the student's main activity, we identified the following episodes:

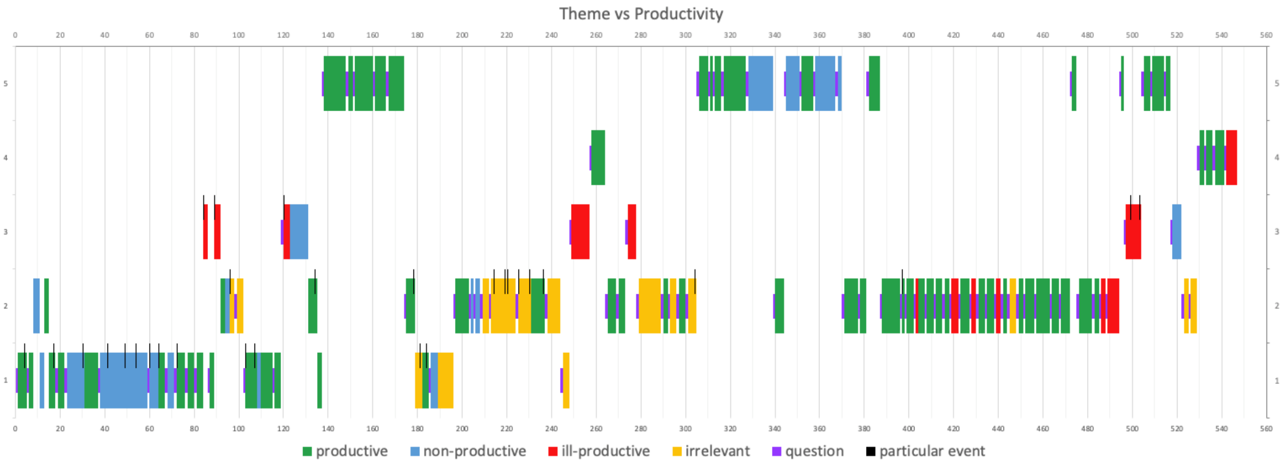
- Exploration: this episode is characterized by a rather unstructured search for information, preceded by a rather general question.
- Study: this episode begins when a media item catches the inquirer's attention and the latter stops and studies the content of this media item. New information may be admitted into the milieu M_D.
- Planning: during this episode, the inquirer sets up a structured plan for proceeding; this phase may be implicit.
- Implementation: the inquirer carries out a plan or a procedure.
- Analysis: the inquirer tries to better understand the problem, and analyzes it in light of new information
- Verification: refers to episodes where the inquirer verifies responses obtained from the media or his or her own generated responses.
- Regulation: this episode considers interventions initiated by the teacher, tutor or another expert in the field.

On top of that Schoenfeld (1985) suggests to mark new-information points when “a previously unnoticed piece of information is obtained or recognized” (p.299). Based on this idea we created a mark that we called particular event.

- Particular event: new information, keeping track of sources, adjusting search, difficulty

We then characterized each episode according to the mathematical or non-mathematical theme addressed (these categories depend obviously on the main topic of the ME) and its productivity (productive, non-productive, ill-productive and irrelevant). We refer to a productive episode when, a relevant media was activated, institutional answers were accepted in the M_D and a correct response was eventually produced. A non-productive episode occurs when, after a search, a relevant media is activated but nothing is admitted to the milieu and no action is taken or answer is produced. An ill-productive episode is identified when, after a search, incorrect information is allowed into M_D or an incorrect response is produced. It seemed important to us to consider separately the irrelevant episodes, because it has much less harmful effects on exploration than an ill-productive episode but will still be less neutral than a non-productive episode. An irrelevant episode results either in an irrelevant path or an irrelevant production. This happens when irrelevant information is admitted into the M_D or eventually an irrelevant answer is produced.

This coding allowed us to create graphical representations of the exploration chronology such as Theme vs Type of Episode or Theme versus Productivity. An example is illustrated in Graphics 3. Note that the size of each episode is expressed in terms of the number of occurrences.



Graphics 3 : Productivity vs Theme

We will now proceed to a brief introduction of an IB student's ME and present some preliminary results from the analysis at the macro-level using the above framework.

Preliminary results

A brief introduction of the Golden Gate Bridge exploration

This student chose to investigate the Golden Gate Bridge in San Francisco (Figure 3). She decided to analyze the shape of the curve formed by the bridge's suspension cable and find out if it forms a parabola or not.

Figure 3: The photo of the San Francisco Bridge used by the student[1]

At first glance, two curves that have a similar shape could be potential candidates for modeling: a parabola and a catenary. A parabola is the graphical representation of a quadratic function, and a catenary can be modeled by a hyperbolic cosine and is obtained by suspending a cable or a chain between two points. The suspension cables are largely believed to be catenaries, but the answer is far from being straightforward. In general, when the horizontal deck of the bridge is suspended on vertical rods attached to the main suspension cable, different forces act on the suspension cable, and

[1] https://en.wikipedia.org/wiki/Golden_Gate_Bridge#/media/File:Golden_Gate_Bridge_Dec_15_2015_by_D_Ramey_Logan.jpg

it will take the shape of a parabola. This is rather a simplified explanation, but sufficient to demonstrate the difficulties that the student encountered during her exploration. It is important to state that solid knowledge on vector analysis and calculus and their applications in physics would be necessary to justify a quadratic model of a suspension bridge.

Prior knowledge of the class and the student's profile

During the exploration assignment, the students in the observed class had available knowledge on quadratic functions, trigonometric functions, as well as the basics of differential calculus, including the derivative of some known functions and their applications for optimization. We also point out that the program does not cover hyperbolic functions, so modeling using a catenary would necessarily require the use of media. The particularity of the student whose exploration we have analyzed here is that she missed the introductory courses on functions, especially on the quadratic function. Overall, she is a rather scholar and hard-working student, coming from a culture where teaching and learning mathematics is associated with repetitive exercises and calculations. She describes her level of math as follows: “So basically, I don't know if I am good at it or is it just because I know the basics, I don't know if I am logical”. She herself prefers the calculative side of math because: it's not easy for me to visualize and apply math in the real world. As for choosing the topic of her exploration, she says:

> I’ll do modelling of the San Francisco bridge using functions and because I have never done functions, doing this might help me you know…

We will now examine what the analysis at the macro-level allows us to understand about how the student conducted her exploration and what she really learned while taking into account her profile and prior knowledge.

Macro-level analysis: preliminary results

The generating question Q is: What form does the suspension cable of the Golden Gate Bridge take? The overall chronology (547 occurrences) was split into 89 sections (which is also the number of derived questions) that were further refined into 122 episodes. Within this ME five main themes were identified:

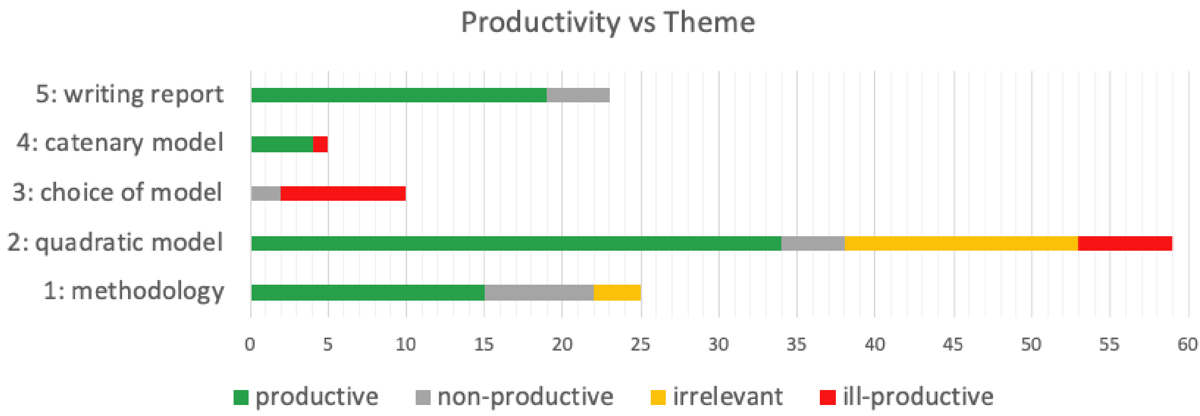
1. Methodology (25 episodes)
2. Quadratic model (59 episodes)
3. Choice of model (10 episodes)
4. Catenary model (5 episodes)
5. Writing the report (23 episodes)

We can notice that the majority of the student's work concerns the theme 1: Quadratic model which represents 48% of the episodes. We then have about the same proportion of episodes devoted to Methodology (20%) and Writing the report (19%), finally 8% to the Choice of the model and only 4% to the theme concerning the Catenary model. Indeed, the quadratic model is characterized by a rich activity and represents a certain comfort zone for the student. We also noticed an inversely proportional relationship between the number of episodes and the progressive distance from the student’s comfort zone. To illustrate this, we propose to examine the productivity of the episodes (Table 2).

	productive	non-productive	irrelevant	ill-productive	Grand Total
1: methodology	15	7	3	0	25
2: quadratic model	34	4	15	6	59
3: choice of model	0	2	0	8	10
4: catenary model	4	0	0	1	5
5: writing report	19	4	0	0	23
Grand Total	**72**	**17**	**18**	**15**	**122**

Table 2: Productivity vs Theme

We can see that there are no ill-productive episodes in methodology and writing report, the proportion of ill-productive episodes amounts to about 5% when the student operates in the quadratic model and 20% for the catenary model, on the other hand it is 80% in the theme choice of model. The Graphics 4 provides a clear visualization of what we just described.

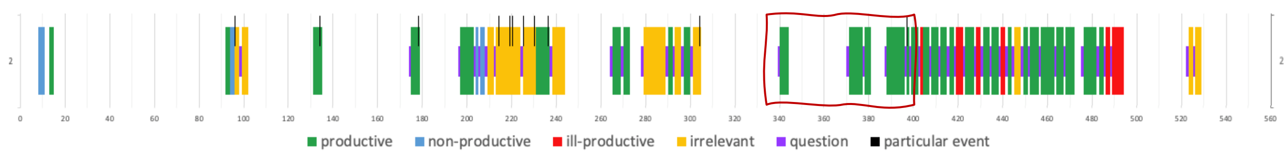


Graphics 4: Productivity vs Theme (condensed)

We can interpret this using the concept of the student's cognitive universe (CU) which is a set of objects towards which an individual has developed a stable personal relationship. (Chevallard, 2003). Some of the objects from the theme Quadratic model were already in the student’s CU while others became part of it as a result of the work done on the ME. There is a clear tendency that the more the nature of the exploration leads the student to deal with object at a more complex level than the ones in her CU, the less activity there is and the proportion of ill-productive episodes increases.

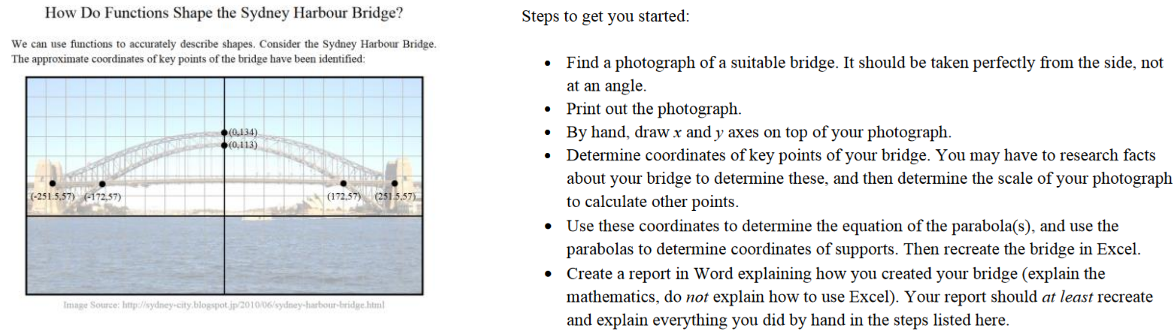
An example of a sequence of productive episodes in the quadratic model

In the quadratic model, of the 59 episodes, there were 34 productive, 4 non-productive, 15 irrelevant and 6 ill-productive episodes. The example below is situated as indicates the wrapped part of the chronology of Graphics 5.



Graphics 5: Chronology of episodes in the Quadratic model

In particular, we identified two media that contributed greatly to the success of this part. A document from the beginning of the ME (Section 2 Episode 5) giving a protocol of how to model a bridge with a quadratic function (see Figure 4) and a tutorial in the form of a video (Section 20 Episode 27) that proposes a quadratic modeling of a suspension cable of the Lion's Gate Bridge (Figure 5).


How Do Functions Shape the Sydney Harbour Bridge?

We can use functions to accurately describe shapes. Consider the Sydney Harbour Bridge. The approximate coordinates of key points of the bridge have been identified:

Steps to get you started:

- Find a photograph of a suitable bridge. It should be taken perfectly from the side, not at an angle.
- Print out the photograph.
- By hand, draw *x* and *y* axes on top of your photograph.
- Determine coordinates of key points of your bridge. You may have to research facts about your bridge to determine these, and then determine the scale of your photograph to calculate other points.
- Use these coordinates to determine the equation of the parabola(s), and use the parabolas to determine coordinates of supports. Then recreate the bridge in Excel.
- Create a report in Word explaining how you created your bridge (explain the mathematics, do *not* explain how to use Excel). Your report should *at least* recreate and explain everything you did by hand in the steps listed here.

Figure 4 : Extract of the document D 2.1[2]

The student was able to extract the methodology of this document, personalize and apply it to her work.

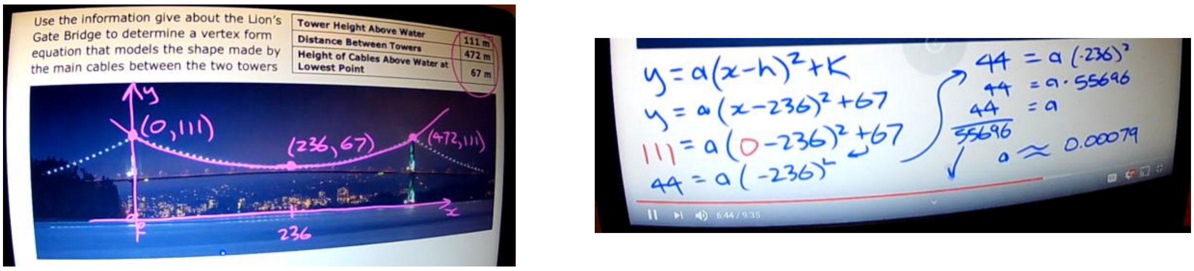


Figure 5: Screenshot of the video V20.1[3]

If we compare these documents with the productions found in the student's logbook (Figure 6 et Figure 7) we can see that she succeeded in extracting relevant information from these two media, integrating it into M_D, adapting it and applying it to the problem of her exploration.

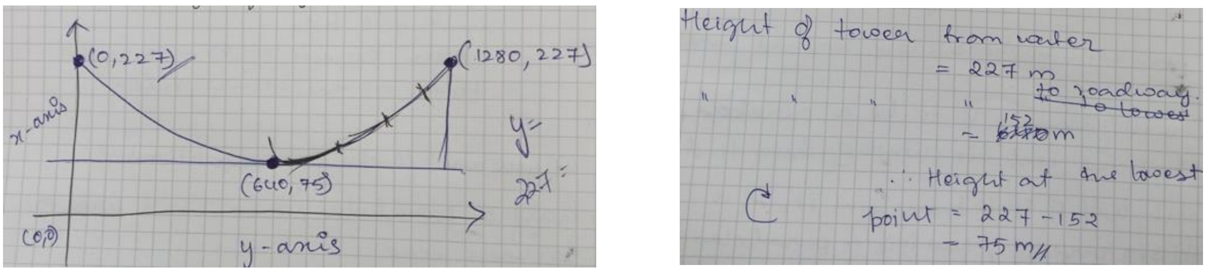


Figure 6: Quadratic model of the Golden Gate Bridge

---

[2] https://mrbertman.com/projects/bridge.pdf

[3] https://www.youtube.com/watch?v=CAu3LdK8xS8

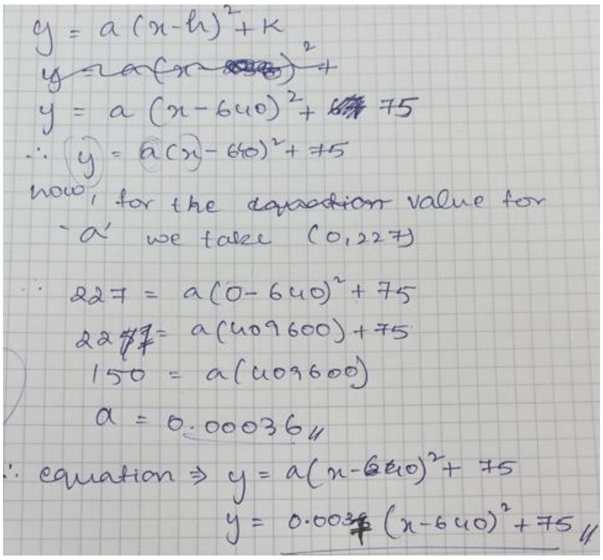


Figure 7 : Finding the equation of the quadratic function

This production, taken from the logbook, illustrates the research-study and media-milieu dialectics that lead to the production of a correct answer and to building of knowledge related to quadratic functions. This example could give the impression that the learning process was quick and straightforward, but in reality, it led to a rich mathematical activity characterized by research, encounters with different mathematical objects, moments of reflection, but also by blockages, dead ends or irrelevant paths.

Conclusion and perspectives

The analysis at the macro-level in terms of episodes led us to identify four cases with respect to productivity. The didactical milieu M_D is enriched through a series of exploration-study episodes when the encountered objects are at a similar level of complexity as those contained in the student's CU. In this case the student is able to activate a relevant and reject an irrelevant media, create an a-didactical milieu that provides the necessary feedback, the media-milieu dialectic is set until a satisfactory answer to the inquiry is obtained and validated. These are the so-called productive episodes. However, it happens that the student misses important information and relevant media is rejected. This can happen when the student "zaps" between the different media and is not sufficiently assiduous in studying the answers or when the information in the media is "too far" from his or her CU. In the latter case, irrelevant or erroneous information is admitted into the M_D and contaminates it. In this situation the student rejects relevant media, while he will be inclined to pursue irrelevant or erroneous paths. These are irrelevant respectively ill-productive episodes. The exploration will inevitably lead the students to go outside of their cognitive universe and to build new skills and new knowledge. This learning seems to be closely linked to the "distance" from the student's CU. Indeed, when the student encounters mathematical objects close to his/her cognitive universe, even if he/she has not yet developed stable personal and/or institutional relationships with them, he/she seems to be able to "manage" on his/her own with the media to which he/she has access and to conduct a relevant and fruitful inquiry.

References

Bosch, M. (2018). Study and research paths: A model for inquiry. In *Proceedings of the International Congress of Mathematicians (ICM 2018)* (Vol. 1–0, pp. 4015–4035). WORLD SCIENTIFIC. https://doi.org/10.1142/9789813272880_0210

Brousseau, G. (1998). *La théorie des situations didactiques*. La pensée sauvage.

Chevallard, Y. (2002). Les TPE comme un problème didactique. In T. Assude & B. Grugeon (Eds.), *Actes du séminaire national de didactique des mathématiques. Année 2001* (pp. 177–188). IREM de Paris 7 et ARDM. http://www.irem.univ-paris-diderot.fr/up/publications/AAR02005.pdf

Chevallard, Y. (2003). Approche anthropologique du rapport au savoir et didactique des mathématiques. In S. Maury & M. Caillot (Eds.), *Rapport au savoir et didactiques* (pp. 81–104). Fabert. http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=42

Chevallard, Y. (2005). La place des mathématiques vivantes dans l'éducation secondaire: Transposition didactique des mathématiques et nouvelle épistémologie scolaire. *La Place Des Mathématiques Vivantes Dans l'enseignement Secondaire*. 3e Université d'été Animath, (Saint-Flour, 22-27 août 2004), Paris. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/La_place_des_mathematiques_vivantes_au_secondaire.pdf

Chevallard, Y. (2008). Un concept en émergence: La dialectique des média et des milieux. In G. Gueudet & Y. Matheron (Eds.), *Actes du Séminaire national de didactique des mathématiques*. IREM.

Chevallard, Y. (2009, April). *La notion de PER: problèmes et avancées* [Communication]. IUFM, Toulouse. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/La_notion_de_PER___problemes_et_avancees.pdf

IBO. (2012). *Diploma Programme Mathematics SL guide*. International Baccalaureate Organization.

Kidron, I., Artigue, M., Bosch, M., Dreyfus, T., & Haspekian, M. (2014). Context, Milieu, and Media-Milieus Dialectic: A Case Study on Networking of AiC, TDS, and ATD. In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of Theories as a Research Practice in Mathematics Education* (pp. 153–177). Springer International Publishing. https://doi.org/10.1007/978-3-319-05389-9_10

Lackova, J., & Dorier, J.-L. (2018). La démarche d'investigation dans le cadre du Baccalaureat International. *Actes Du Colloque EMF 2018*. Mathématique en scène - des ponts entre les disciplines, Paris (Gennevilliers). Prochainement disponible en ligne.

Parra, V., & Otero, M. R. (2018). Study and Research path: Indicators of the development of the dialectics. *Pré-Actes CITAD6*, 241–253. https://citad6.sciencesconf.org/data/pages/Pre_proceedings_citad_7.pdf

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic Press Inc.

Pour une nouvelle approche de l'équivalence quantitative

Anne-Marie Rinaldi[1]

[1]LIRDEF, Université Paul-Valéry Montpellier 3, France; anne-marie.rinaldi@univ-montp3.fr

Abstract: The contribution is based on the results of research conducted in an IRES group in Montpellier (September 2019 – June 2021). This collaborative research was conducted to design and experiment with a system for teaching arithmetic in elementary first year for 7 to 8 years old attending school in France. It made it possible, among other things, to show how the proposed tasks based on actual manipulations and followed by calculation sessions lead students to establish an equivalence relationship between two numerical expressions. Thus, in response to a "classic" question about the teaching of arithmetic, appears a question on the articulation between arithmetic and algebra.

Keywords: elementary arithmetic, early algebra, number fact, learning

Résumé : La contribution s'appuie sur les résultats d'une recherche conduite dans un groupe IRES de Montpellier (septembre 2019 – juin 2021). Cette recherche collaborative a été menée pour concevoir et expérimenter un dispositif d'enseignement du calcul sous vingt en cours élémentaire première année pour des enfants de 7 à 8 ans scolarisés en France. Elle a permis, entre autres, de montrer en quoi les tâches proposées à partir de manipulations effectives et suivies de séance de calcul amènent les élèves à établir une relation d'équivalence entre deux expressions numériques. C'est ainsi, qu'en réponse à une question « classique » autour de l'enseignement du calcul, apparait une question vive sur l'articulation entre arithmétique et algèbre.

Mots clefs : calcul élémentaire, pré-algèbre, faits numériques, apprentissages

Introduction

Notre contribution relève de l'axe 2 dans le sens où nous nous appuyons sur les résultats d'une recherche collaborative conduite dans un groupe IRES de Montpellier1 (septembre 2019 – juin 2021), pour montrer en quoi une recherche conduite dans le but de construire, consolider et renforcer le répertoire2 additif sous vingt (R≤ 20) à l'école élémentaire Française pour des enfants de 7 à 8 ans

[1] Le groupe IRES de Montpellier est un des groupes de l'Institut pour la Recherche sur l'Enseignement des Sciences implanté dans l'académie de Montpellier. Sa spécificité est de regrouper des enseignants, des formateurs qui enseignent dans le premier, le second degré ou à l'université. Les membres du groupe qui ont participé à la recherche collaborative citée dans la contribution sont : Sonia Bayle, PEMF; Sophie Gastal, PEMF; Alain Moreau, PEMF; Anne-Marie Rinaldi, MCF didactique des mathématiques.

[2] Ce répertoire englobe les tables d'addition des nombres inférieurs à dix et toutes les décompositions et recompositions des nombres inférieurs à vingt.

nous amène à soulever des questions propres au domaine de *l'early algebra* (EA) et à avancer ainsi la nécessité de *mettre à jour*, certains contenus hérités du domaine de l'arithmétique selon Chevallard (1997).

> Ensuite, les *contenus de l'étude scolaire* – qui sont aujourd'hui, pour l'essentiel, des *savoirs*, c'est-à-dire des œuvres permettant *en principe* d'accéder à beaucoup d'autres œuvres – apparaissent tout au contraire, fréquemment, fermés sur eux-mêmes, frappés d'autisme épistémologique, et en particulier devenus muets sur leurs *raisons d'être* – en tant qu'œuvres, et en tant qu'œuvres inscrites au répertoire de l'École. Leur mise à jour nécessitera un effort spécifique, immense, pour créer à des œuvres nouvelles un rapport nouveau, selon une didactique libérée de ce que quelques-uns prennent pour un destin obligé – le rapport rituel-fétichiste à des œuvres moribondes. (Chevallard, 1997, pp.2)

Afin d'étayer notre propos, nous présentons dans une première partie, les éléments théoriques en lien avec la notion d'équivalence quantitative que nous retenons pour *créer* un *rapport nouveau* au calcul. Puis, nous explicitons les raisons sur le plan de la progressivité des apprentissages et sur le plan mathématique qui nous ont poussé à introduire en CE1 la technique de calcul en appui sur dix[3] - à partir de manipulations et en opérant directement sur les nombres - pour amener les élèves à remplacer une expression numérique par une expression numérique équivalente. Pour finir, nous nous fondons sur les résultats de l'expérimentation du dispositif d'enseignement, pour débattre et conclure sur les biens fondés de cette nouvelle approche de l'équivalence quantitative au début de l'école élémentaire.

L'équivalence quantitative : essai de définition

Dans l'introduction d'un ouvrage paru en 2020 regroupant des contributions de chercheurs membres de l'Observatoire International de la pensée algébrique (OIPA), Squalli (2020) fait remonter aux années 1970 la volonté de comprendre comment l'enseignement de l'algèbre pourrait s'articuler à l'enseignement de l'arithmétique tout en sachant que le passage pour les élèves d'un mode de pensée à l'autre est loin d'être facile à réaliser et pose problème. Toujours d'après Squalli (2020) citant Carraher et Schliemann (2007), bon nombre de difficultés sont documentées par la recherche. Nous retenons trois d'entre elles :

> « - Les élèves voient le signe d'égalité comme un signe d'annonce de résultats (Booth, 1984 ; Kieran, 1981 ; Vergnaud, 1985 ; Vergnaud, Cortes et Favre-Artigue, 1988) ;
>
> - Ils ont tendance à rechercher une valeur numérique simple (Booth, 1984). Le refus de laisser les opérations en suspens conduit à des erreurs de concaténation (Bednarz et Janvier, 1996) ;
>
> - Ils ne reconnaissent pas les propriétés de commutativité et de distributivité (Boulton-Lewis et al., 2001 ; Demana et Leitzel, 1988 ; MacGregor, 1996). » (Squalli, 2020, p. 6)

Dans une précédente étude (Rinaldi, 2021), nous faisions l'hypothèse que les causes de ces difficultés seraient dues, entre autres, au fait de développer, dans les premières années de l'école élémentaire

[3] Les éléments qui définissent avec précision la technique en appui sur dix sont développés dans la suite de l'article mais d'ores et déjà pour calculer avec cette technique $7 + 5$, on calcule $7 + 3 + 2$ ou $5 + 5 + 2$.

essentiellement la valence pragmatique du calcul (trouver le résultat d'un calcul) sans développer en même temps la valence épistémique du calcul (apprendre des propriétés mathématiques en calculant). Or en cherchant par exemple à expliquer pourquoi et comment on peut remplacer le calcul $53 - 27$ afin de rendre celui-ci plus facile, les élèves peuvent être amenés à trouver puis écrire : $53 - 27 = 56 - 30$. Dans cette égalité, le symbole égal « = » indique que les deux expressions numériques $53 - 27$ et $56 - 30$ sont équivalentes « quantitativement ». L'intention est surtout de prouver, en l'occurrence grâce à la propriété de conservation des écarts, la similitude des expressions situées des deux côtés du signe égal, sans avoir besoin au préalable, d'effectuer le calcul correspondant à chaque expression. Auquel cas, le symbole d'égalité est employé comme un symbole relationnel entre $53 - 27$ et $56 - 30$ et non pas comme un symbole opérationnel, qui manifeste toujours quant à lui, le fait qu'on indique à droite du signe égal, le résultat d'un calcul.

Cependant, avant d'arriver à ce symbolisme conventionnel qui lie deux expressions numériques, les élèves ont manipulé. Afin de mesurer un même segment, ils ont utilisé différentes règles, les unes commençant par exemple à la graduation 2 ou à la graduation 3, les autres commençant à la graduation 8 ou à la graduation 10 (Rinaldi, 2013). Ces règles dites « cassées », car ayant toutes la particularité de ne pas commencer à la graduation 0, ne permettent pas de trouver grâce à une lecture directe la longueur du segment. Leur utilisation successive puis conjointe permet ainsi d'appréhender la notion d'écart entre deux nombres et la propriété de conservation des écarts dans le domaine de la mesure.

C'est pourquoi, dans le but de développer la pensée algébrique des jeunes enfants (CE1), dans la continuité des travaux de recherche de Rinaldi (2013, 2021), Squalli (2002) et de ceux d'Anwandter Cuellar, Lessard, Boily et Mailhot (2008), nous avons recherché des activités qui permettaient d'opérer directement sur des collections de cardinal inférieur à vingt et de préciser en quoi certaines d'entre-elles peuvent être équivalentes « quantitativement ». L'enjeu étant par exemple d'établir une équivalence entre une collection de cardinal $7 + 6$ et une autre collection de cardinal $10 + 3$. Nous justifions ce choix en donnant quelques repères de progressivité dans les apprentissages dont nous avons tenu compte.

Le calcul sous vingt

Le calcul sous vingt s'applique à des nombres entiers inférieurs ou égaux à dix. La plus grande somme que l'on puisse obtenir est dix plus dix soit vingt. Une des caractéristiques des nombres entiers inférieurs à vingt, est qu'il est relativement facile de les représenter grâce à des configurations de doigts, des dessins, des schémas ; sur ces dessins, ces schémas, on peut éventuellement « montrer » que onze égal dix plus un et que chaque nombre compris entre dix et dix-neuf correspond à dix plus « … » avec « … » compris entre zéro et neuf.

Par ailleurs, si on se réfère au programme de l'école maternelle en vigueur à la rentrée 2020 et à l'annexe de ces programmes parue en 2021, il est indiqué que la construction des quantités est essentielle et que cette construction passe d'abord entre deux et quatre ans par la connaissance des petits nombres, c'est-à-dire des nombres inférieurs à cinq pour s'étendre jusqu'à dix voir plus. Nous en déduisons donc qu'en appui sur des manipulations effectives puis mentales pour décomposer et recomposer les premiers nombres, les élèves vont peu à peu connaître les faits numériques sous dix, et théoriquement les maîtriser à la fin du cours préparatoire comme indiqué dans le guide[4]pour enseigner les nombres, le calcul et la résolution de problèmes au cours préparatoire (2021).

Nous pensons que le processus de compréhension des nombres au-delà de dix ne se déroule pas de la même manière que pour les nombres sous dix, car le nombre dix joue un rôle majeur dans nos deux systèmes de numération, parlée et écrite. En effet ces deux systèmes sont tous deux basés sur le groupement par dix. Par la suite, nous développons d'autres arguments, liés au calcul, qui incitent à s'appuyer sur les faits numériques sous dix pour construire le répertoire sous vingt avant d'appréhender les faits numériques sous vingt.

Techniques de calcul sous vingt

Si nous considérons la tâche qui consiste à chercher le résultat de a plus b avec a et b nombres entiers inférieurs à dix, plusieurs techniques sont envisageables. Or chacune de ces techniques, en se référant à la théorie anthropologique du didactique, développée par Chevallard (2002) est justifiée par une technologie qui permet en même temps de la penser, voire de la produire. Par conséquent, chaque technique fait appel et mobilise des connaissances mathématiques spécifiques. Cette spécificité permet d'envisager le classement suivant :

- Le comptage

Pour effectuer cinq plus trois, il s'agit de compter : « un, deux, trois, quatre, …huit ». Cette technique, pour être appliquée correctement, suppose que certains principes explicités dans Bideaud J. et al.éd. (1991) soient acquis. Elle devient difficile à mettre en œuvre dès que la somme des deux termes est supérieure à dix.

- Le sur-comptage

Pour effectuer cinq plus trois, il s'agit de compter à partir de cinq, le nombre de fois indiqué par le nombre trois : « six, sept, huit ». Cette technique, si elle est maîtrisée, peut s'avérer économique, c'est-à-dire peu couteuse en temps et sans trop de risque d'erreurs à partir du moment où le sur-comptage s'opère à partir du plus grand nombre et que le nombre indiqué par le plus petit nombre est inférieur ou égal à trois ou quatre.

- La récupération en mémoire des faits numériques connus

Cette technique permet d'obtenir le résultat immédiatement et s'avère peu couteuse cognitivement.

- Les techniques utilisant la décomposition et recomposition des nombres, les presque doubles et l'appui sur dix

Ces dernières techniques ont en commun de s'appuyer sur une bonne connaissance des nombres inférieurs à 20, de faire appel à des faits numériques connus et d'utiliser les propriétés de l'addition. Nous explicitons dans le paragraphe suivant la technologie propre à la technique en appui sur dix.

Technique en appui sur dix : éléments de technologie

Avec la technique en appui sur dix, pour calculer $7 + 5$, on calcule $7 + 3 + 2$ ou $5 + 5 + 2$. Les éléments que nous proposons par la suite, permettent d'identifier le domaine d'application, le principe et les connaissances mathématiques nécessaires pour mettre en œuvre et valider la technique en appui

[4] Le guide est disponible en suivant ce lien : https://eduscol.education.fr/177/mathematiques-cycle-2

sur dix. Ils vont également permettre de préciser le mode d'emploi et de faciliter la mise en œuvre de la technique.

- *Domaine d'application de la technique*

La technique en appui sur dix s'applique pour calculer $a + b$ avec a et b entiers naturels compris entre 0 et 9 et si la somme $a + b$ est supérieure à 10 et inférieure à 20. Cette technique est donc spécifique au calcul sous vingt. Quand le domaine numérique est étendu au-delà de 20, la technique ne s'appuie plus sur dix mais sur un multiple de dix. C'est ainsi que pour calculer $27 + 8$, on calcule $27 + 3 + 5$. Le multiple de dix sur lequel on s'appuie est alors trente.

- *Principe de la technique*

La technique en appui sur dix consiste à décomposer un des termes du calcul en fonction de l'autre terme du calcul, afin de se ramener à un calcul de la forme dix plus « … », avec « … » entier naturel compris entre zéro et neuf.

- *Connaissances mathématiques relatives à la technique*

- décomposer additivement un nombre compris entre zéro et neuf
- mobiliser directement les faits numériques connus ou retrouver les résultats du répertoire additif sous dix
- recomposer un nombre de la forme dix plus « … », avec « … » entier naturel compris entre zéro et neuf
- utiliser l'associativité de l'addition et éventuellement la commutativité de l'addition sur l'ensemble des entiers naturels.

- *Mode d'emploi de la technique*

Si nous reprenons l'exemple du calcul $7 + 5$ et que nous cherchons à expliquer comment nous nous y prenons pour effectuer le calcul demandé en partant du nombre 7, nous pouvons avoir éventuellement le discours suivant : « Pour effectuer $7 + 5$, j'ajoute à 7 le complément de 7 à 10 afin d'obtenir 10 puis le complément à 3 de 5. » Ce discours met en avant la nécessité de convoquer deux faits numériques, $7 + 3 = 10$ et $3 + 2 = 5$ pour remplacer le calcul initial par un calcul équivalent : $7 + 5 = 7 + 3 + 2$.

Le schéma de la **figure 1** construit à partir de bandes de longueurs respectives 7, 5, 7, 3, 2, 10, 2 et 12 permet d'expliquer[5] les différentes étapes de la mise en œuvre de la technique en appui sur dix. En effet, la seconde ligne du schéma illustre le fait que la bande 5 soit échangée contre deux bandes 3 et 2 mises dans le prolongement l'une de l'autre et bout à bout. Les trois bandes ainsi obtenues 7, 3 et 2, ont pour longueur, la longueur des deux bandes initiales 7 et 5 mises bout à bout et dans le prolongement l'une de l'autre. Les expressions numériques $7 + 5$ et $7 + 3 + 2$ sont équivalentes. De la même manière, la seconde ligne et la troisième de **la figure 1** illustrent le fait que les expressions $7 + 3 + 2$ et $10 + 2$ sont équivalentes. De ligne en ligne, on obtient donc une suite d'égalités : $7 + 5 = 7 + 3 + 2 = 10 + 2 = 12$.

[5] Une autre manière par exemple d'expliquer la technique serait de dire : « Pour effectuer $7 + 5$, j'ajoute à 7, le complément de 7 à 10 et je compense, en soustrayant à 5, le complément à 5 de 7 ».

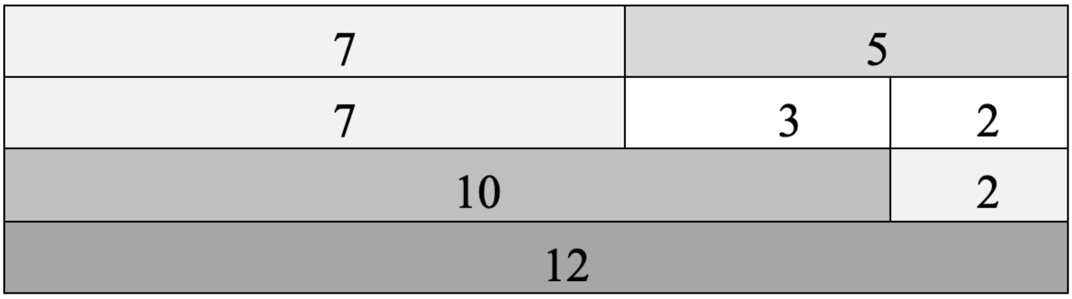


Figure 1 : exemple de schéma associé à la mise en œuvre de la technique en appui sur dix

Pour conclure, le discours qui vient d’être explicité, associé de surcroit à la manipulation effective « de bandes » (figure1) permet à notre sens d’illustrer sur un exemple pourquoi une expression numérique peut être remplacée par une autre expression numérique équivalente. Nous présentons certains résultats d’analyses associés à l’introduction de ce discours dans un dispositif d’enseignement en CE1.

Éléments d’analyse propres à l’introduction de la technique en appui sur dix

En lien avec les éléments théoriques présentés précédemment, nous avons conçu un dispositif d’enseignement qui s’appuie sur trois types de tâches : celles qui consistent à manipuler des collections discrètes (configuration de doigts) ou continues (bandes de longueurs données) ; à produire une expression numérique de la forme $a + b$ ou $a - b$; à trouver le résultat d’un calcul en mobilisant la technique en appui sur dix. En annexe 1, nous présentons un tableau de programmation qui montre comment les tâches sont imbriquées les unes aux autres et qui donne l’objectif notionnel poursuivi d’une séquence[6] à l’autre. Pour étayer notre analyse, nous revenons sur les effets de la manipulation « des bandes » au tout début de l’expérimentation et sur les effets de la technique en appui sur dix en particulier dans une séance de calcul avant de préciser les limites de l’analyse.

Méthodologie de recueil et d’analyse des données

L’expérimentation du dispositif d’enseignement s’est déroulée de septembre 2020 à décembre 2020 dans les deux classes de CE1[7] de deux membres du groupe IRES. La classe A est une classe de CP/CE1 située au centre-ville qui a pour effectif 35 élèves dont 11 sont des élèves de CE1 et la classe B est une classe de CE1 dédoublée située en REP+ qui a pour effectif 9 élèves. Pour chaque classe, nous avons recueilli comme données, les réponses des élèves aux évaluations, les feuilles de calculs renseignées individuellement, les enregistrements vidéo (avec une caméra orientée vers le tableau) des séances pendant les phases de lancement, de synthèse et d’institutionnalisation locale et, avec des caméras orientées sur un, deux, quatre, voire l’ensemble des élèves pendant les phases de recherche.

[6] L’ensemble des séquences du dispositif est disponible en suivant le lien suivant : https://irem.edu.umontpellier.fr/ressources-et-publications/ressources-cycle-2/

[7] En 2019-2020, nous avions conçu, observé et analysé un dispositif proche de celui- qui en fait a servi de base à celui - dans trois classes de CE1 : la classe A, la classe B et une autre classe de CE1 située en REP+.

Notre analyse des données est menée en deux temps. Un premier temps où, nous repérons à travers les manipulations, les discours, les calculs effectués par les élèves, les techniques qu'ils utilisent pour mettre en évidence une relation d'équivalence entre deux quantités, respectivement deux expressions numériques. Nous explicitons les savoirs mathématiques sur lesquels les élèves s'appuient pour communiquer, expliquer et justifier leurs techniques. Un second temps où nous repérons, en lien avec les interventions de l'enseignant – pendant les phases de restitution qui suivent les phases de recherche – si la notion d'équivalence est présente, semi ébauchée ou absente.

Les effets de la manipulation des bandes sur le sens de l'équivalence quantitative

La première séquence a permis à chaque élève de réaliser son propre jeu de bandes constitué de deux bandes de longueur dix et de neuf bandes de longueur allant de un à neuf afin de représenter tous les nombres de un à vingt. C'est grâce à ce matériel, que les élèves en binômes, ont calculé et validé le résultat de la somme de deux nombres inférieurs à dix et rechercher des compléments. Une situation évoquée et analysée dans Rinaldi (2022) montre que certains élèves, alors qu'ils connaissent à chaque fois la réponse du calcul, par exemple grâce au sur comptage (9+2) ou à l'utilisation des presque double (5+6) sont amenés à rechercher, une expression équivalente à $9+2$ et à 5+6 car la bande onze n'existe pas dans leur jeu. Pour chacun des calculs, une écriture arithmétique est proposée et c'est justement cette écriture qui fait l'objet d'une institutionnalisation locale : $9+2=10+1$ et $6+5=10+1$. A la droite du signe égal (=), l'enseignant n'a pas écrit le résultat du calcul mais une expression numérique qui se lit directement dix plus un et qui correspond donc au nombre onze.

Les effets de l'utilisation de la technique en appui sur dix sur le sens de l'équivalence quantitative

Alors que la technique en appui sur dix a été découverte à partir de manipulations effectives et utilisée pour effectuer la somme de deux nombres inférieurs à dix (cf. annexe 1), nous faisons le choix de centrer notre analyse sur la première séance de la quatrième séquence qui consiste à calculer une somme de trois termes. Le fait qu'il y ait trois termes amène à choisir parmi ces trois termes deux d'entre eux pour s'engager dans le calcul. Ce choix dépend de la connaissance des faits numériques et de l'adaptabilité dont fait preuve chaque calculateur[8].

Nous présentons dans un diagramme en bâtons (figure 2) le nombre de réponses correctes pour chaque calcul (couleur noire), de calculs faux (couleur grise), de calculs non faits (bandes verticales blanches sur fond noir) sur les 13 réponses recueillies dans la classe B. Nous avons regroupé les calculs en fonction de leurs caractéristiques sans respecter l'ordre dans lequel ils étaient posés afin de faciliter la lecture et l'analyse des données.

[8] Exemple : Pour calculer 8+3+2 avec la technique en appui sur 10, on peut

- Choisir 8 et 2 ou 2 et 8 ce qui permet d'obtenir directement la somme sous la forme demandée : $\boxed{8+2}$+3.

- Choisir 8 et 3 ou 3 et 8 ce qui oblige à calculer cette somme sous la forme $\boxed{8+2}$+1 et à effectuer 1+2 pour trouver le résultat final sous la forme demandée : 8+3+2= $\boxed{8+2}$+3

- Choisir 3 et 2 ou 2 et 3 amène à calculer 5+8 ou 8+5sous la forme $\boxed{5+5}$+3 ou $\boxed{8+2}$+3.

Il est à noter que 2 élèves sur les 13 présents n'ont cherché que les deux premiers calculs qui sont $8 + 4 + 1$ et $5 + 3 + 5$. Tous les autres élèves ont cherché les réponses à tous les calculs en utilisant la technique en appui sur dix.

Comme attendu, les calculs les mieux réussis sont les trois premiers ($5 + 3 + 5$, $2 + 4 + 8$, $9 + 3 + 1$). Les dix élèves sur treize qui les ont faits justes ont directement regroupé les deux termes dont la somme égale 10 (5 et 5, 2 et 8, 9 et 1).

Le dernier calcul $4 + 7 + 0$ est aussi très bien réussi (9 réponses justes sur 11). Sur les 9 réponses proposées, une seule correspond à 4+6+1.Les 8 autres sont de la forme 7+3+1.

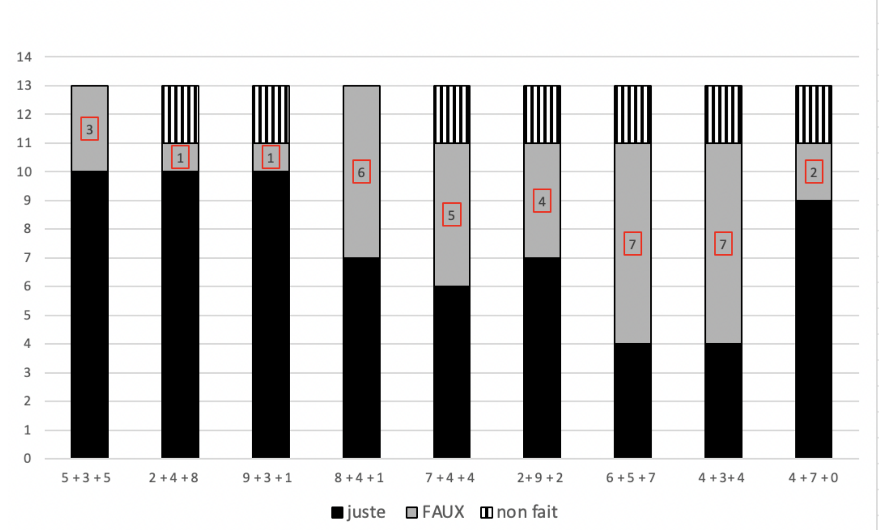


Figure 2 : nombre de résultats justes et faux par calcul correspondant à la somme de trois termes

Les calculs $8 + 4 + 1$ et $7 + 4 + 4$ se ressemblent car le premier chiffre du calcul est le chiffre le plus grand des trois et pour obtenir 10, il faut ensuite décomposer le second terme. Les 9 élèves ont d'ailleurs commencé par obtenir $8 + 2$ et $7 + 3$. Quand leurs réponses sont fausses, c'est qu'ils n'ont pas effectué le calcul de $2 + 1$ et de $4 + 1$. Ils n'ont pas reporté sur le troisième terme le complément à 2 de 10 ou le complément à 3 de 4.

$2 + 9 + 2$ a la particularité d'avoir le plus grand chiffre 9 au milieu. 10 élèves sur les 11 se ramènent d'ailleurs à 9+1+…. . Une seule élève se ramène à 2+8+3. Les réponses fausses s'expliquent par le non report du nombre 1 qu'il reste à ajouter à 2.

Parmi les quatre réponses justes à $6 + 5 + 7$, calcul où le dernier chiffre est celui qui est le plus grand, on trouve 6+5+7=6+4+8 ou 6+5+7=5+5+8 ou 6+5+7=7+3+8.

Parmi les quatre réponses justes à $4 + 3 + 4$, on trouve autant de 7+3+1 que de 4+6+1. Or il semblerait que le premier calcul soit moins source d'erreurs que le second.

Pour conclure, les résultats obtenus dans la classe B sont encourageants car la plupart des élèves arrivent à utiliser leur connaissance des faits numériques sous dix et leur connaissance des décompositions de dix pour remplacer une expression numérique par une expression numérique de la forme dix plus « … ». De plus, il est à noter, en étudiant une à une les productions, qu'ils ont accordé, de l'importance au choix du premier nombre. En effet, ils ont lu dans son intégralité toute l’expression numérique située à gauche du signe égal notamment pour regrouper les deux termes dont la somme faisait directement dix et gagner en efficacité pour effectuer le calcul demandé.

Les limites liées aux données analysées

Les données utilisées dans cette contribution, empruntées à deux séquences ne sont pas suffisantes pour montrer, comment à terme, la technique en appui sur dix est maîtrisée pour plus de la moitié des élèves des classes A et B. Par ailleurs, les enseignants des classes A et B ont constaté que les élèves se servent spontanément et assez fréquemment dans l’ensemble, de la technique en appui sur dix pour effectuer un calcul additif posé en colonne. Dans les deux classes, les élèves utilisent beaucoup moins leurs doigts pour compter.

Une seconde limite du recueil de données est due au fait que les effectifs des deux classes réunies s’élèvent à 20 élèves. Il est donc difficile de prévoir, dans ces conditions, les résultats d’une étude similaire menée à grande échelle.

Un dernier point très important mérite d’être soulevé. Les enseignants des classes A et B ont contribué à la conception du dispositif d’enseignement. Ils savent donc que ces séances demandent en amont de bien connaître les repères de progressivité de la maternelle au CE1, d’avoir conscience des difficultés que les élèves éprouvent « en général » à maîtriser les faits numériques pour les combiner et de connaître les éléments de technologie se référant à la technique en appui sur dix. Les séances, nécessitent également, tout en cadrant le déroulement, de laisser les élèves chercher, de les observer, de s’appuyer sur leurs erreurs, de les aider éventuellement à formuler leurs pensées. Elles demandent d’avoir du recul car ne l’oublions pas, la maîtrise de la technique en appui sur dix n’est pas un objectif en soi. Nous ne savons pas si avec des enseignants de classe ordinaire qui n’auraient pas contribué à concevoir le dispositif d’enseignement, les résultats seraient aussi probants.

Conclusion

L’étude nous a permis de montrer, qu’un des enjeux de l’apprentissage de la technique en appui sur dix est d’arriver, au cours élémentaire 1(CE1), en suivant le mode d’emploi de cette technique, à remplacer le calcul initial d’une somme par le calcul d’une « autre » somme, qui elle « contient » dix. Le travail préalable à l’introduction de la technique, associé à la manipulation des « doigts » et des « bandes » a mis en évidence les résistances de certains élèves à effectuer des « groupements » ou « des distributions » afin d’obtenir une collection équivalente. En revanche, le matériel utilisé, les bandes, s’est avéré efficace pour valider ou invalider la relation d’équivalence entre deux expressions numériques. Nul besoin d’associer à chacune des expressions numériques situées de part et d’autre du signe égal un résultat pour les comparer. Par ailleurs, le fait de poursuivre ce travail en opérant directement sur les nombres et en utilisant la commutativité et l’associativité de l’addition inscrit bien, comme annoncé, l’étude dans le domaine *early algebra* (EA).

Cependant, en CE1, dans le domaine de l’arithmétique élémentaire, la maîtrise de la technique en appui sur dix n’est pas un objectif en soi. Beaucoup de calculs additifs sous vingt, se résolvent très simplement grâce au sur-comptage, à l’utilisation des doubles et des presque doubles. L’utilisation de la technique en appui sur dix, est avant tout un prétexte pour travailler la décomposition et recomposition des nombres sous vingt et mobiliser les faits numériques sous dix. En ce sens, le côté innovant de la recherche : intégrer l'étude de la technique en appui sur dix dans un dispositif d'enseignement du calcul sous vingt, revêt un caractère un peu formel qui ne doit pas pour autant masquer le véritable enjeu de l'étude : réfléchir à une nouvelle approche de la relation d'équivalence quantitative et lui apposer au cycle 2, la recherche d'expressions numériques équivalentes.

References

Anwandter Cuellar, N., Lessard, G., Boily, M. & Mailhot, D. (2018). L'émergence de la pensée algébrique au préscolaire : les stratégies des élèves concernant la notion d'équivalence mathématique. *McGill Journal of Education / Revue des sciences de l'éducation de McGill*, *53*(1), 146–168.

Bideaud J.Meljac C. et Fischer J.-P. (éds.) *Les Chemins du nombre*, Lille, Presses universitaires du Septentrion, 1991.

Chevallard, Y. (1997). Questions vives, savoirs moribonds : le problème curriculaire aujourd'hui. Colloque Défendre et transformer l'école pour tous, Marseille, 3,4 et 5 octobre 1997.

Chevallard, Y. (2002). Organiser l'étude. 1. Structures et fonctions. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot & R. Floris (Eds.), *Actes de la XI^{e} Ecole d'été de didactique des mathématiques*, (pp. 3-33). Grenoble : La Pensée sauvage.

Gersten, R., Jordan, N.-C. and Flojo, J.-R. (2005**).** Early identification and interventions for students with mathematics difficulties. *Journal of learning disabilities, 38* (4), 293-304.

Rinaldi, A.-M. (2013). Mesurer avec une règle cassée pour comprendre la technique usuelle de la soustraction posée. *Grand N*, *91*, 93-119.

Rinaldi, A.-M. (2021). Habiletés calculatoires : les enjeux de la réécriture de calculs soustractifs. *Recherches En Didactique Des Mathématiques*, *41*(2), 217–259.

Rinaldi, A.-M. (à paraitre 2022). Le calcul sous vingt : une possibilité de travailler la notion d'équivalence à l'école élémentaire. *Repères IREM, (126).*

Squalli, H., Oliveira, I., Bronner, A. et Larguier, M. (2020). *Le développement de la pensée algébrique à l'école primaire et au début du secondaire. Recherches et perspectives curriculaires*. Québec : Livres en ligne du CRIRES. En ligne : https ://lel.crires.ulaval.ca/oeuvre/le-developpement-de-la-pensee- algebrique-lecole-primaire-et-au-debut-du-secondaire-recherches.

Squalli, H.(2002). Le développement de la pensée algébrique à l'école primaire : un exemple de raisonnements à l'aide de concepts mathématiques. *Instantanés mathématiques,* automne 2002, 1-10.

Programme du cycle 1 en vigueur à la rentrée 2020. Bulletin officiel n°31 du 30-07-2020

Annexes du programme d'enseignement de l'école maternelle Bulletin officiel n°25 du 24-06-2021

Le guide pour enseigner les nombres, le calcul et la résolution de problèmes au cours préparatoire (2020). https://eduscol.education.fr/1486/apprentissages-au-cp-et-au-ce1

Consulté le 20 juin 2021

Annexe 1 : programmation des séquences d'enseignement

Séquences du dispositif	Objectif notionnel	Tâches proposées
Séquence 1 : 5 séances Semaines 2 et 3 de septembre	Maîtrise des faits numériques et passage aux écritures arithmétiques : a+b=? et a+?=b	- Manipulation de *bandes* -Manipulation *de configurations de doigts*
Séquence 2 : 2 séances Semaine 4 de septembre	Maîtrise des décompositions et recompositions de 10 (F=10) et passage aux écritures arithmétiques : 10 = ? +? et a+? = 10	-Manipulation de *bandes* -Manipulation de *configurations de doigts* -Utilisation de *cartes à jouer*
Séquence 3 : 3 séances Semaine 1 d'octobre	Mise en œuvre de la technique de calcul en appui sur dix et passage à l'écriture arithmétique : a+b=10+⋯	-Manipulation de *bandes* -Manipulation de *configurations de doigts* -Utilisation de nopmbres
Séquence 4 : 3 séances Semaine 2 d'octobre	Calcul en ligne avec appui sur dix de plusieurs termes inférieurs à dix	- Utilisation des nombres
Tout au long de l'année de CE1	Adaptabilité aux calculs : utilisation de la technique de l'appui sur dix lorsqu'elle paraît la plus adaptée au calcul proposé Calculs en ligne et en colonne : utilisation possible de la technique en appui sur dix pour réaliser des calculs en ligne ou posés	

Propuesta de un proceso de estudio para la formación matemática de ingenieros

Pedro Vivas Caballero[1], Mercedes Hidalgo Herrero[2] y Tomás Ángel Sierra Delgado[3]

[1]Universidad Francisco de Vitoria, Escuela Politécnica Superior, Madrid, España; pedro.vivas@ufv.es

[2]Universidad Complutense de Madrid, Facultad de Educación, Madrid, España; mhidalgo@ucm.es

[3]Universidad Complutense de Madrid, Facultad de Educación, Madrid, España; tomass@edu.ucm.es

Abstract: *In this paper, after a brief analysis of the current epistemological model of mathematical training in engineering degrees in Madrid, we have detected the phenomenon of applicationism. To deal with this phenomenon we have designed and implemented a study process around the flow of vector fields for students to give meaning to surface integrals. The didactic organization designed, an activity of the study and research paths proposed by the anthropological theory of the didactic, has been experimented with two groups of students of the 2nd year of the degree of Industrial Systems Engineers. The discussion of the results obtained has allowed us to analyze the quality of the generative question, origin of the study process, and consider some of the restrictions which have been presented in its development to take them into account in the future design of new study processes.*

Keywords*: Surface integrals, flow of vector fields, study and research activity, Anthropological Theory of the Didactic and engineering.*

Resumen: *En este trabajo, después de un breve análisis del modelo epistemológico vigente de la formación matemática en los grados de ingeniería en Madrid, hemos detectado el fenómeno del aplicacionismo. Para hacer frente a dicho fenómeno hemos diseñado e implementado un proceso de estudio en torno al flujo de campos vectoriales con el fin de que los estudiantes doten de significado a las integrales de superficie. La organización didáctica diseñada, una actividad de estudio e investigación que propone la teoría antropológica de lo didáctico, ha sido experimentada con dos grupos de estudiantes de 2º curso del grado de Ingenieros de Sistemas Industriales. La discusión de resultados obtenidos nos ha permitido analizar la calidad de la cuestión generatriz, origen del proceso de estudio, y considerar alguna de las restricciones que se han presentado en su desarrollo para tenerlas en cuenta en el futuro diseño de nuevos procesos de estudio.*

Palabras clave*: Integrales de superficie, flujo de campos vectoriales, actividad de estudio e investigación, teoría antropológica de lo didáctico e ingeniería.*

Introducción

En este trabajo pretendemos realizar un breve análisis sobre la formación matemática que reciben actualmente los estudiantes de los Grados en Ingeniería. Uno de nuestros objetivos es indagar sobre el tipo de actividad matemática que resulta más habitual en dicha enseñanza.

Así como Barquero, Bosch y Gascón (2014) muestran la existencia del fenómeno del *aplicacionismo* en la formación matemática de los grados de Ciencias Experimentales analizando la existencia de una serie de indicadores (Figura 1), nuestra experiencia en la formación matemática en los grados de Ingeniería nos permite afirmar que dicho fenómeno está vigente en dicha formación, fruto de una enseñanza muy ligada al paradigma del "monumentalismo".

- I_1: *Las matemáticas se mantienen independientes de otras disciplinas*
- I_2: *Las herramientas matemáticas básicas son comunes para todos los científicos*
- I_3: *La organización de los contenidos matemáticos sigue la lógica de los conceptos*
- I_4: *Las aplicaciones siempre van después de la formación matemática básica*
- I_5: *Muchos sistemas extra-matemáticos pueden ser construidos sin ninguna referencia matemática.*

Figura 1. Indicadores del aplicacionismo según Barquero, Bosch y Gascón (2014, p. 7)

Trejo, Camarena & Trejo (2013) para modificar la enseñanza tradicional proponen desarrollar una estrategia metodológica que integre el bloque del saber-hacer con el saber proponiendo un estudio de las matemáticas ligado a las Ciencias y creando hábitos de estudio individual y en equipo para la resolución de problemas.

Romo-Vázquez (2014) afirma que el paradigma didáctico dominante *teoría-aplicación* no permite satisfacer las necesidades prácticas del trabajo del ingeniero. Propone el paradigma de la modelización matemática como modelo alternativo. E introduce herramientas teóricas y metodológicas, dentro del marco de la teoría antropológica de lo didáctico (TAD), que utiliza en el diseño y experimentación de actividades didácticas con futuros ingenieros biomédicos.

Otras investigaciones desarrolladas dentro de la TAD en el ámbito de la ingeniería (Florensa, Bosch, Gascón, & Winslow, 2018; Bartolomé, Florensa, Bosch, & Gascón, 2019) han mostrado que, a través de los recorridos de estudio e investigación (REI), los estudiantes de ingeniería han conseguido dotar de significado a conceptos matemáticos relacionados con la Elasticidad Lineal General.

Nosotros queremos hacer frente al fenómeno del *aplicacionismo* en la formación matemática de ingenieros desarrollando un primer trabajo que forma parte de un plan investigación de tesis doctoral.

Nuestra propuesta pretende un cambio de paradigma respaldado por investigaciones previas realizadas en el marco de la TAD. Presentaremos un primer diseño e implementación de una organización didáctica (OD) denominada Actividad de Estudio e Investigación (AEI) para estudiantes de 2º curso de ingeniería industrial. Pero, primero analizaremos algunos rasgos del modelo epistemológico dominante (MED) en la formación matemática de ingenieros.

Aproximación al modelo epistemológico vigente en la formación matemática de ingenieros

El concepto de *ingeniería*, según la Real Academia Española (RAE), se define actualmente como *"Conjunto de conocimientos y técnicas científicas y empíricas aplicadas a la invención, el diseño, el desarrollo, la construcción, el mantenimiento y el perfeccionamiento de tecnologías, estructuras, máquinas, herramientas, sistemas, materiales y procesos para la resolución de problemas prácticos"* (RAE, s.f.). Dentro de ese conjunto de conocimientos, podemos encontrar uno muy habitual y necesario: las matemáticas. Por tanto, las matemáticas son una herramienta para resolver muchos de los tipos de tareas que se le plantean al ingeniero que es uno de los profesionales que aporta soluciones útiles para la sociedad (Vázquez, 2012).

El estudio de las matemáticas otorga, según Curbeira, Bravo, & Bravo (2013), a un ingeniero industrial en formación la capacidad de modelizar y analizar los procesos a los que se enfrentará en su vida laboral, así como un pensamiento lógico, algorítmico y heurístico que nace de dichos procesos.

Sin embargo, algunos programas de matemáticas de dicha formación señalan explícitamente que su objetivo es que las matemáticas aprendidas sean aplicadas a posteriori en su vida académica y profesional (Figuras 1 y 2), es decir, de algún modo, fomentan el *aplicacionismo* donde se pueden apreciar los indicadores $\boldsymbol{I}_1, \boldsymbol{I}_3$ e $\boldsymbol{I}_4$.

OBJETIVO

El curso de Matemáticas II, proporciona al alumno las herramientas matemáticas para conocer cómo resolver distintos tipos de ecuaciones diferenciales, que describen numerosos fenómenos físicos. Así el alumno podrá analizar en profundidad, cualquiera de los modelos que aparecen en el mundo de la Ingeniería Industrial a lo largo de su carrera académica y profesional.

CONTENIDOS

BLOQUE A: CÁLCULO DE VARIAS VARIABLES

Tema 1: Superficies y Sólidos

oEspacio tridimensional. Coordenadas cartesianas
oFunciones escalares de dos variables: Dominio y su representación gráfica
oSuperficies y curvas en tres dimensiones
oSuperficies cuadráticas
oSólidos simples y su proyección ortogonal

Tema 2: Cálculo diferencial en varias variables

oDerivadas parciales de primer orden y de órdenes superiores
oFunción diferenciable. Diferencial total
oParametrización de una curva
oGradiente. Curvas y superficies de nivel
oPlano tangente y vector normal

Tema 3: Integrales múltiples

oIntegral doble
oCambio de variable: Coordenadas polares
oIntegral triple
oCambio de variable: Coordenadas cilíndricas y esféricas

Figura 2. Objetivo y parte de los contenidos de la asignatura de Matemáticas II UFV (Guía Docente, p. 2) http://notas.ufv.es/documentos/gd/5722_p.pdf

Asimismo, tenemos los programas docentes de universidades como la Rey Juan Carlos o la Politécnica de Madrid cuyo objetivo explícito es aprender matemáticas independientemente de otras disciplinas ($\boldsymbol{I}_1$)(Figuras 3 y 4).

II.-Presentación
El objetivo de la asignatura es que los alumnos adquieran los conocimientos básicos de cálculo diferencial e integral de varias variables reales, campos vectoriales, ecuaciones diferenciales y aproximación numérica. Las habilidades y técnicas adquiridas les permitirán un mejor seguimiento y comprensión de otras asignaturas del grado. Es muy recomendable haber cursado la asignatura de Matemáticas de la modalidad de Ciencias y Tecnología de Bachillerato, así como la asignatura Matemáticas I del primer cuatrimestre de la titulación.

Figura 3. Objetivo de la asignatura de Matemáticas II URJC (Guía Docente, p. 3)
https://gestion3.urjc.es/guiasdocentes/mostrarGuias.jsp#

5.1. Descripción de la asignatura

Este curso se dedica al estudio del Cálculo Vectorial: teoría de campos, integrales de línea y superficie y los teoremas integrales de Green, Gauss y Stokes.

El objeto de esta asignatura es dotar a los estudiantes de aquellas herramientas matemáticas que subyacen en problemas técnicos que abordan en otras asignaturas del grado y que están relacionados con el cálculo integral sobre curvas y superficies, tales como el cálculo del flujo o del trabajo. El conocimiento de dichas herramientas (las aplicaciones de la integral múltiple, los teoremas integrales) es de sumo interés, por ejemplo, en el estudio de la teoría de campos como el electromagnético y el gravitatorio; campos conservativos que admiten potencial escalar o campos que admiten potencial vector. Se aborda con rigor pero sin demostraciones excesivamente teóricas el estudio de condiciones para la resolución de dichos problemas.

Figura 4. Objetivo de la asignatura de Ampliación de Cálculo UPM (Guía Docente, p. 6)
https://www.upm.es/comun_gauss/publico/guias/2021-22/2S/GA_05TI_55000021_2S_2021-22.pdf

Además, aunque en algunas guías docentes de matemáticas para la formación de ingenieros el objetivo propuesto es el análisis de *problemas prácticos* y la *modelización*, la metodología utilizada está influida por el aplicacionismo presentándose además de los indicadores $\boldsymbol{I_1}, \boldsymbol{I_2}$ e $\boldsymbol{I}_3$, el $\boldsymbol{I_4}$ tal como aparece en las siguientes guías docentes (Figuras 5−7).

Metodología Presencial: Actividades

1. Clase magistral y presentaciones generales (52 horas; 100% presencial): El profesor explicará los conceptos fundamentales de cada tema, incidiendo en lo más importante y resolviendo a continuación una serie de problemas tipo, con los que el alumno aprenderá a identificar los elementos esenciales del planteamiento y se iniciará, adquiriendo habilidad y soltura, en la resolución de problemas del tema.

2. Resolución en clase de problemas propuestos (60 horas; 100% presencial, incluidas las horas dedicadas a pruebas cortas de evaluación continua e intercuatrimestrales). En estas sesiones se explicarán, corregirán y analizarán problemas de cada tema análogos a los resueltos en las lecciones expositivas y también otros de mayor complejidad, previamente propuestos por el profesor y trabajados por el alumno.

3. Prácticas con ordenador (8 horas; 100% presencial, incluidas las horas dedicadas a los controles de prácticas). Se realizarán en grupos reducidos. En ellas los alumnos ejercitarán los conceptos y técnicas estudiadas, resolviendo problemas prácticos con ayuda del software MATLAB.

Figura 5. Metodología de la asignatura de Cálculo ICAI (Guía Docente, p. 7)
https://repositorio.comillas.edu/xmlui/bitstream/handle/11531/58882/Gu%c3%ada%20Docente.pdf?sequence=-1&isAllowed=y

• Sesiones teórico-prácticas: en ellas se expondrán, con la ayuda de materiales audiovisuales, los conceptos clave de la asignatura. Estas clases se desarrollarán en un ambiente dinámico, centrado en la interacción profesor-alumno y alumno-alumno.

• Clases prácticas: pretenden el refuerzo, manipulación y dominio de los conceptos teóricos. Predominará la metodología del aprendizaje basado en problemas, casos prácticos, proyectos y medios digitales. Se favorecerá un entorno colaborativo y constructivo de aprendizaje mediante la interacción alumno-alumno como eje de la resolución de los problemas propuestos.

Figura 6. Metodología de la asignatura de Matemáticas II UFV (Guía Docente, p. 4)
http://notas.ufv.es/documentos/gd/5722_p.pdf

IV.B.-Actividades formativas	
Tipo	**Descripción**
Lecturas	Clases magistrales
Prácticas / Resolución de ejercicios	Clases prácticas de resolución de problemas
Otras	Tutorías individuales o en grupos
Otras	Preparación individual de las clases teóricas, prácticas y de pruebas

Figura 7. Metodología de la asignatura de Matemáticas II URJC (Guía docente, p. 4)
https://gestion3.urjc.es/guiasdocentes/mostrarGuias.jsp#

Esta pequeña muestra de guías docentes pone de manifiesto cómo el aplicacionismo es un rasgo esencial de la epistemología dominante de las matemáticas impartidas en los primeros cursos de los grados de Ingeniería.

También mostramos brevemente dos ejemplos de libros de texto de cálculo en varias variables recomendados en la bibliografía de Matemáticas II del Grado de Ingeniería de Sistemas Industriales

(GISI) de la Universidad Francisco de Vitoria (UFV): Mora Flores (2019) y Apostol (2002). Dichos textos, que contienen parte de los conocimientos que pretendemos desarrollar en nuestro proceso de estudio, no presentan casos prácticos en los que sea necesario utilizar dichos conocimientos (***I***1). Tampoco contienen un bloque de ejercicios contextualizados al final de cada tema. Se limitan a la exposición de teoremas, definiciones, etc. (Figuras 8 y 10) (***I***3). para después aplicarlos en una serie de ejemplos y ejercicios abstractos (Figuras 9 y 11) (***I***4) que, para dotarles de cierta relevancia ante los estudiantes, son marcados como problemas modelos de examen.

12.7 Integrales de superficie

En muchos aspectos, las integrales de superficie son análogas a las integrales de línea. Definimos las integrales de línea mediante una representación paramétrica de la curva. Análogamente, definiremos las integrales de superficie en función de una representación paramétrica de la superficie. Demostraremos luego que en ciertas condiciones generales el valor de la integral es independiente de la representación.

DEFINICIÓN DE INTEGRAL DE SUPERFICIE. *Sea $S = \boldsymbol{r}(T)$ una superficie paramétrica descrita por una función diferenciable $\boldsymbol{r}$ definida en una región T del plano uv, y sea f un campo escalar definido y acotado en S. La integral de superficie de f sobre S se representa con el símbolo $\iint_{\boldsymbol{r}(T)} f\,dS$ [o por $\iint_S f(x,y,z)\,dS$], y está definida por la ecuación*

$$\iint_{\boldsymbol{r}(T)} f\,dS = \iint_T f[\boldsymbol{r}(u,v)] \left\| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right\| du\,dv \qquad (12.14)$$

siempre que exista la integral doble del segundo miembro.

Figura 8. Definición de Integral de Superficie (Apostol, 2002, p. 525)

12.10 Ejercicios

1. Sea S la semiesfera $x^2+y^2+z^2=1$, $z\geq 0$, y $\boldsymbol{F}(x,y,z)=x\boldsymbol{i}+y\boldsymbol{j}$. Sea $\boldsymbol{n}$ el vector normal unitario exterior a S. Calcular el valor de la integral de superficie $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS$, empleando:
   a) la representación vectorial $\boldsymbol{r}(u,v) = \operatorname{sen} u \cos v\,\boldsymbol{i} + \operatorname{sen} u \operatorname{sen} v\,\boldsymbol{j} + \cos u\,\boldsymbol{k}$,
   b) la representación explícita $z = \sqrt{1-x^2-y^2}$.

Figura 9. Ejemplo de ejercicios propuestos sobre "Integrales de superficie" (Apostol, 2002, p. 532)

8.5 Flujo través de una superficie S

Campos escalares y campos vectoriales.

Definición 8.5
Sea $U \subseteq \mathbb{R}^n$ un conjunto abierto. Una aplicación $f: U \longrightarrow \mathbb{R}$ se denomina *campo escalar* o función escalar. Una función $f: U \longrightarrow \mathbb{R}^n$ se denomina *campo vectorial*.

$$\iint_S F\cdot N\,dS = \iint_D F(r(u,v))\cdot \frac{\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}}{\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\|}\left\|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right\| dA = \iint_D F(r(u,v))\cdot\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\,dA$$

Figura 10. Definición de Flujo (Mora Flores, 2019, p. 373-374)

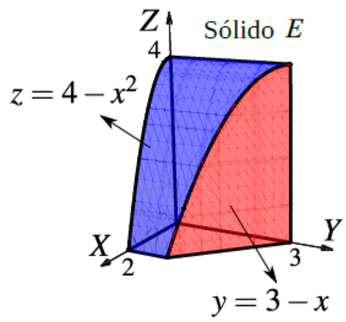
Se puede observar que los modelos matemáticos que aparecen en los tipos de tareas planteados siempre vienen proporcionados de antemano en la propia tarea.

Ejercicios

8.8.1 Consideremos el campo de fuerzas **F** con

$\mathbf{F}(x, y, z) = (x+\text{sen}(y))\ \hat{\imath}+(\ln(xz)-y)\ \hat{\jmath}+(2z+\arctan(xy))\ \hat{k}$

Calcule la integral de superficie $\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$ donde S es la frontera del sólido E, el cual se muestra en la figura a la derecha y **N** es el vector normal unitario siempre exterior a E.



8.8.2 Use el teorema de la divergencia para calcular $\iint_S \mathbf{F}\cdot\mathbf{N}\,dS$ donde S es la frontera del sólido E, limitado por la superficie $z = x^2 + y^2 + 5$ y el plano $z = 10$, tal y como se muestra en la figura a la derecha, $\mathbf{F}(x, y, z) = 2x\ \hat{\imath} + y\ \hat{\jmath} + z\ \hat{k}$ y N es el vector normal unitario exterior a E.

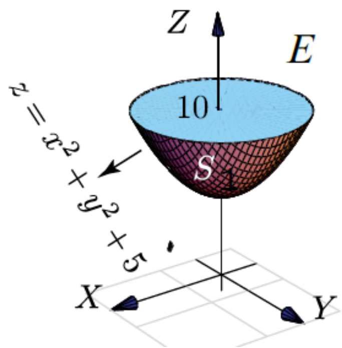


Figura 11. Ejercicios propuestos sobre "Integrales de superficie" (Mora Flores, 2019, p. 386)

Así, para que la modelización matemática se normalice dentro de las instituciones, consideramos necesario introducir nuevos dispositivos didácticos dentro de la enseñanza de las matemáticas que transformen la actividad científica escolar, ya que actuar sobre las propias instituciones para incidir sobre el *aplicacionismo*, modelo muy anclado en la sociedad (Barquero, Bosch y Gascón, 2014; Plaza & Villa-Ochoa, 2019), se torna una tarea ardua. Por ello, a continuación, nos hemos planteado una hipótesis de trabajo y hemos empezado a diseñar una OD para hacer frente al fenómeno del *aplicacionismo* detectado en la formación matemáticas en los GISI, que posteriormente hemos experimentado.

Planteamiento de la hipótesis de trabajo

La hipótesis ***H*** de trabajo que nos planteamos para realizar un primer intento de abordar dicho fenómeno es la siguiente:

> ***H***: *La implementación de una **AEI** que parta de la cuestión **Q** (fig. 12), que plantea una posible necesidad real basada en su futuro profesional, va a permitir que los estudiantes de ingeniería doten de significado a las integrales de superficie.*

Q: La nueva escudería Fórmula UFV Racing quiere diseñar el vehículo que competirá en los próximos años. Para ello, necesita saber ciertos parámetros de algunas de las piezas aerodinámicas que conforman la carrocería y así poder diseñar los motores.

*Una de las piezas aerodinámicas en las que necesitan tu ayuda son las DOS tomas de aire que refrigeran el motor. Para poder sacarle el máximo partido posible a la potencia del motor **es imprescindible averiguar el flujo de aire que atraviesa dichas piezas**. Por limitaciones en la geometría del monoplaza, el diseño de la pieza es fijo y su modelo es de la forma:*

$$x = y^2 + z^2 \quad con \quad \begin{cases} y \in \left[-\frac{3}{10}, \frac{3}{10}\right] \\ z \in \left[-\frac{1}{10}, \frac{1}{10}\right] \end{cases}$$

El viento a lo largo del circuito con respecto al monoplaza se puede describir según el campo vectorial:

$$\vec{F}(x, y, z) = -100x \cdot \vec{\imath}$$

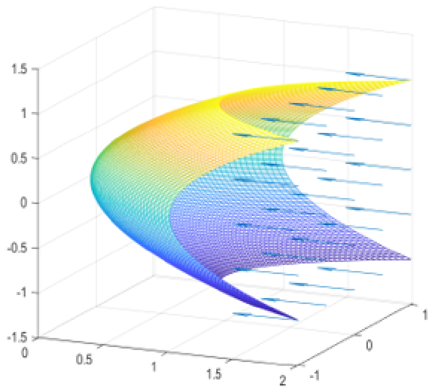


Figura 12. Cuestión generatriz *Q*

La metodología que seguimos a lo largo de la ***AEI*** es la siguiente:

- Trabajo de investigación por parte de los alumnos se realizará en grupos de 3 o 4 personas asignados previamente de forma aleatoria.
- Cada grupo tiene acceso a un documento en el Aula Virtual en el que anotarán diariamente los descubrimientos asociados a cada sesión de trabajo.
- El profesor actúa como guía de estudio de las cuestiones planteadas en las sesiones, orientando y coordinando la búsqueda de las respuestas posibles en función del avance de dicha investigación.
- Al finalizar cada sesión el profesor y los estudiantes, es decir, toda la comunidad de estudio valida el camino buscado en la investigación o, en su defecto, planteará las cuestiones de confirmación, es decir, aquellas cuya respuesta permite a los estudiantes decidir el camino a seguir.
- Finalizada la ***OD***, cada grupo elabora un informe evaluable especificando las respuestas a ***Q*** obtenidas y el proceso para obtenerlas. Después, toda la comunidad de estudio las validará para una mejor comprensión por parte de todos.

La organización didáctica por diseñar

La organización matemática (***OM***) que pretendemos sea elaborada por los estudiantes en dicha ***AEI*** son las *integrales de superficie* dentro del cálculo multivariable, contenido impartido en 2º curso del GISI de la UFV de Madrid.

Basaremos la ***AEI*** en el modelo del diseño de tareas dentro del marco de la TAD (García, Barquero, Florensa, & Bosch, 2019). Se parte de una cuestión generatriz $\boldsymbol{Q}$ que es el motor del proceso, generando una serie de cuestiones derivadas $\boldsymbol{Q_i}$**.** El estudio de estas cuestiones se realiza de forma compartida y colaborativa para encontrar posibles respuestas $\boldsymbol{R}$ a $\boldsymbol{Q}$ y $\boldsymbol{R_i}$ a $\boldsymbol{Q_i}$, culminando el proceso una vez validadas las respuestas por toda la comunidad de estudio.

Una parte importante a la hora de diseñar una ***AEI*** es ser conscientes de las restricciones que existen a la hora de su aplicación, que trataremos en la última sección.

Nuestro objetivo a largo plazo sería desarrollar toda la disciplina *Matemáticas II* a partir de cuestiones o tipos de tareas que permitan dar significado a las nociones que propone la guía docente. La ***AEI*** que hemos desarrollado partirá de la cuestión generatriz $\boldsymbol{Q}$, ya explicitada en la sección 3. La ***OM*** que se pretende estudiar corresponde a uno de los cinco temas que componen el *cálculo multivariable* a impartir en un cuatrimestre. Debido a la restricción institucional de la temporalización, nos hemos visto obligados a diseñar la ***AEI*** con 5 sesiones de 1,5 h (2 sesiones por semana). Y nos hemos apoyado en un *mapa de cuestiones* (Florensa, Bosch, Gascón, & Winslow, 2018) (Figura 14).

Los estudiantes fuera del aula deben buscar y recopilar la información que permita encontrar una respuesta a las cuestiones planteadas en la sesión anterior para tratarlas en la siguiente sesión.

Las características de la institución donde implementar una ***OD*** juegan un papel fundamental. El proyecto formativo, que promueve la UFV, denominado *“Formar para transformar”*[1], no solo hace totalmente compatible la implementación de una ***AEI***, sino que considera explícitamente que la forma de llevar a cabo el estudio está muy relacionada con la metodología didáctica de las ***AEI*** (Chevallard, 2009). No obstante, el aplicacionismo sigue siendo el MED, tal y como hemos observado en la sección 2.

Una organización didáctica para el estudio de las integrales de superficie

Como vimos en la sección 3, $\boldsymbol{Q}$ tiene un enfoque claramente automovilístico debido a las características del centro universitario donde queremos implementar la ***OD***, instalado en el Motor Sport Institute. En el mismo edificio se realiza como actividad principal el diseño y puesta a punto de vehículos de automoción para la alta competición.

La ***OM*** que se pretende estudiar se enmarca en el *tema 4 sobre Integrales de superficie*. La cuestión $\boldsymbol{Q}$ se planteará una vez finalizado el *tema 3 sobre Integrales múltiples* (Figura 13).

Figura 13. Esquema curricular en el que se ubica nuestra *OD*

[1] https://www.ufv.es/la-universidad/sobre-ufv/

Nuestra pretensión es que la búsqueda de respuestas a la cuestión generatriz $\boldsymbol{Q}$ conduzca al uso de la noción de *integrales de superficie* para poder calcular el *flujo de campos vectoriales*, noción que se define:

$$\boldsymbol{\Phi} = \iint \vec{\boldsymbol{F}} \cdot \overrightarrow{\boldsymbol{ds}} = \iint \vec{\boldsymbol{F}}[\vec{\boldsymbol{r}}(\boldsymbol{u},\boldsymbol{v})] \cdot \vec{\boldsymbol{N}}(\boldsymbol{u},\boldsymbol{v})\ \boldsymbol{dS}$$

Siendo $\vec{\boldsymbol{F}}[\vec{\boldsymbol{r}}(\boldsymbol{u},\boldsymbol{v})]$ el campo vectorial en la superficie $\vec{\boldsymbol{r}}(\boldsymbol{u},\boldsymbol{v})$ parametrizada en las variables $\boldsymbol{u}$ y $\boldsymbol{v}$, y $\vec{\boldsymbol{N}}(\boldsymbol{u},\boldsymbol{v})$ el vector normal en cada porción de superficie $\boldsymbol{dS}$ (Mora Flores, 2019).

El diseño de $\boldsymbol{Q}$ se ha realizado teniendo en cuenta la restricción temporal para su estudio, siendo esta el motivo principal por el que $\boldsymbol{Q}$ es cerrada y única. En la figura 14 presentamos un *mapa de cuestiones* posibles, derivadas de $\boldsymbol{Q}$, que pueden ser propuestas por los estudiantes a lo largo de la ***AEI***.

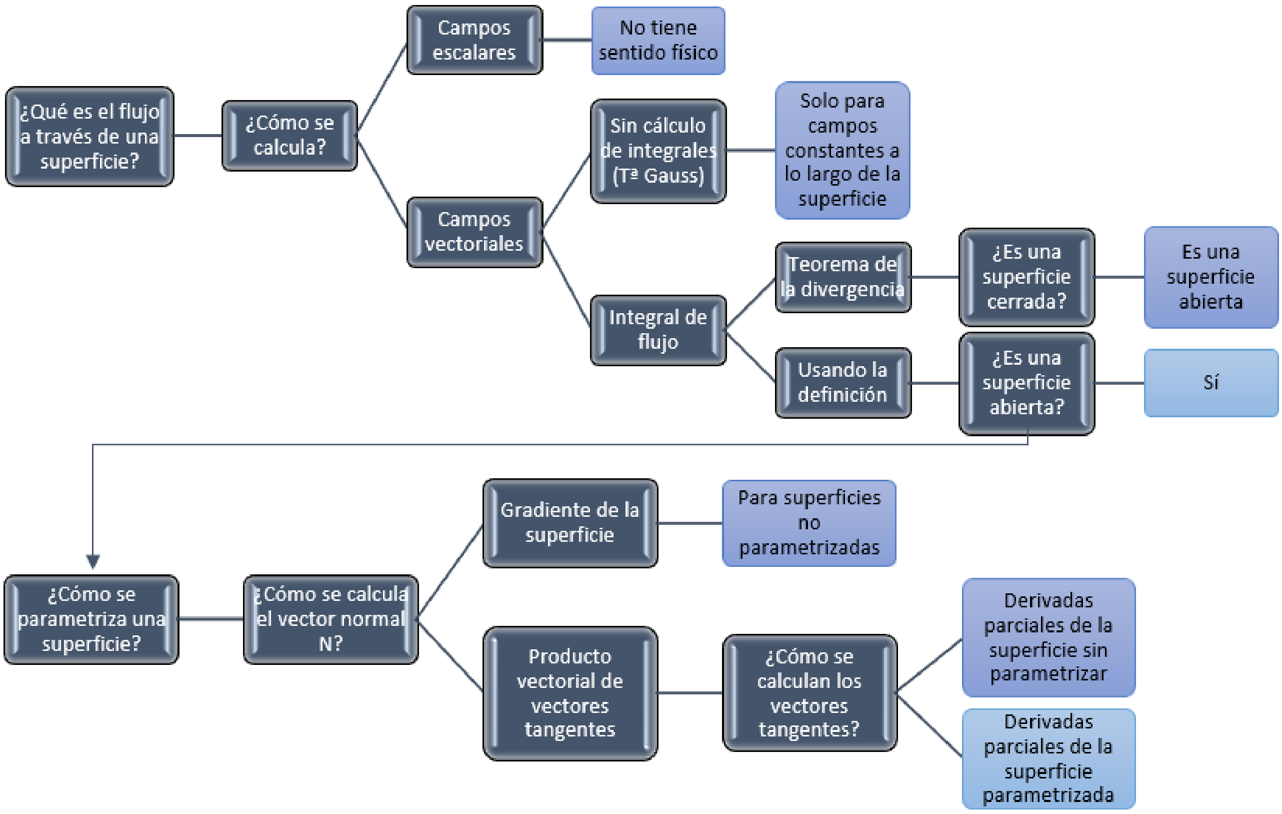


Figura 14. Mapa de cuestiones

Teniendo en cuenta el mapa de cuestiones anterior, se pretende que las sesiones de la ***OD*** se desarrollen de la siguiente manera (Figura 15).

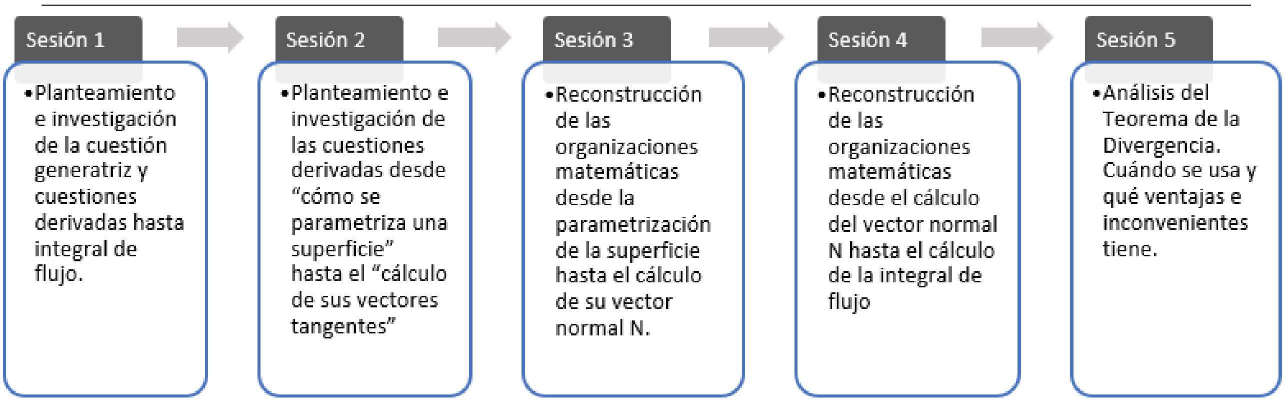


Figura 15. Esquema provisional del desarrollo de las sesiones

Implementación del proceso de estudio

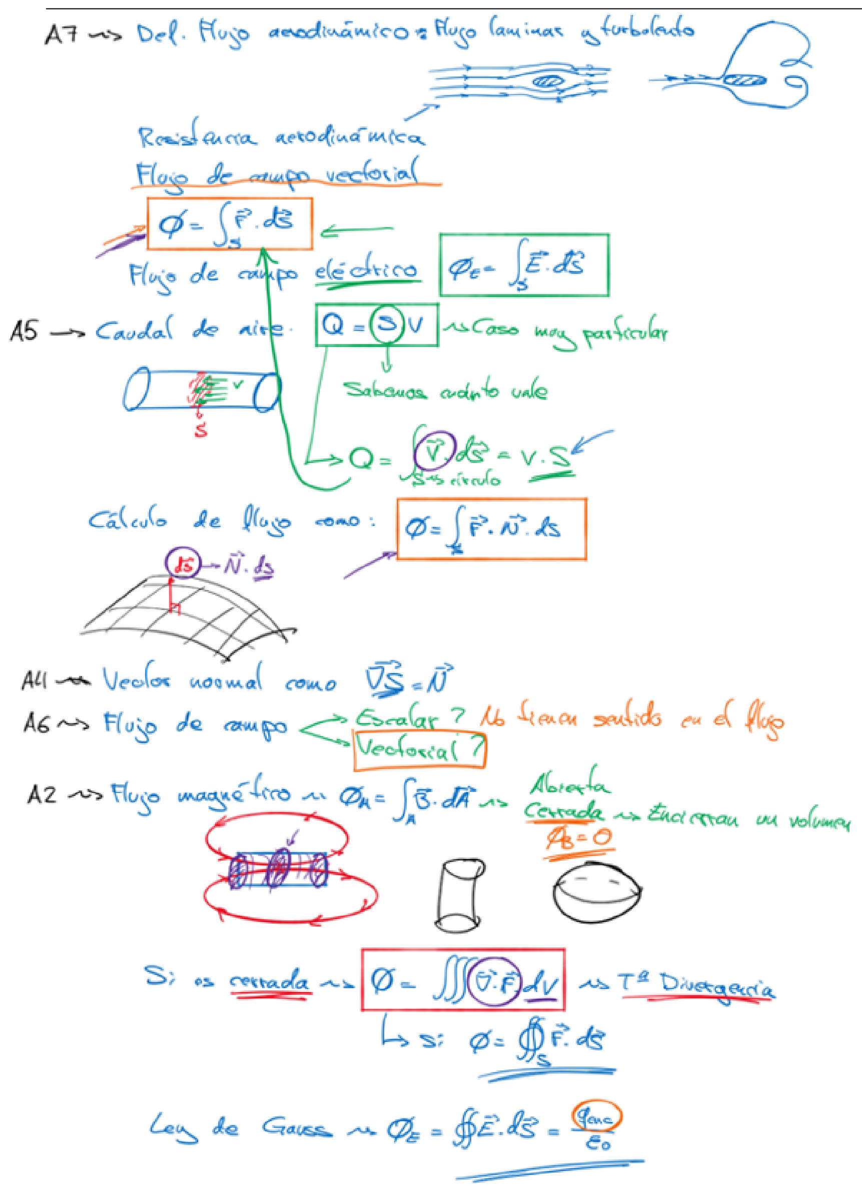
Al principio de cada curso, en la asignatura *Matemáticas II* se plantea a los estudiantes un proyecto automovilístico que abarca gran parte de los contenidos de la asignatura, incluyendo la parte de *integrales de superficie* para el cálculo del *flujo de campos vectoriales*. Dicho proyecto deben realizarlo por grupos de 3 o 4 personas que realizan conforme avanza la asignatura y de forma paralela, pero siguiendo el modelo docente aplicacionista.

Durante el curso 2021/22 hemos podido empezar a experimentar una pequeña parte del programa, intentando seguir la metodología de las ***AEI***, para poder confirmar o rechazar las hipótesis de partida.

En la ***OD*** han participado dos grupos (A y B) formados por un total 59 estudiantes. Se han generado 16 equipos reducidos de estudiantes (A1, A2, …, A7 y B1, B2, …, B9).

En la primera sesión, se les planteó la cuestión ***Q*** y se les explicó la dinámica de trabajo, que tuvo buena aceptación. Todos los alumnos disponían de dispositivos electrónicos para la búsqueda de información, más allá de la bibliografía recomendada, a recopilar en los archivos compartidos.

Todas las sesiones, exceptuando la última, han seguido la siguiente estructura: durante la primera mitad de la clase, los alumnos buscaban y elaboraban información sobre las cuestiones planteadas, mientras que, en la segunda mitad, los representantes de cada grupo exponían por turnos lo recogido. Dicha información era debatida por la comunidad de estudio. Mediante el uso de las pertinentes cuestiones de confirmación, que comentaremos más adelante, hemos provocado que los propios estudiantes decidieran el camino que realmente se buscaba. La información obtenida se iba recopilando *in situ* a la vez que se iba exponiendo para tener un esquema con los avances realizados en cada sesión (Figura 16).



Cada grupo de alumnos tenía asignado una notación (A1, A2, A3, …) y, de forma aleatoria, iban exponiendo los conceptos que habían encontrado en el proceso de investigación, para ser debatidos tanto por el docente como por el resto de los grupos. Dichos conceptos se iban anotando en el esquema mostrado.

El grupo A7 expuso los tipos de flujo que se pueden dar en un fluido (laminar o turbulento) y la definición integral de flujo de campo vectorial.

El grupo A5 planteó el concepto de *Caudal Q*, siendo éste un caso particular, y la definición de flujo a través del vector normal a la superficie.

El resto de los grupos exponían otros conceptos como el vector normal a partir del gradiente (A4), la posibilidad de flujo de campos escalares (A6) o flujo magnético (A2) además de los que se habían comentado anteriormente por otros grupos.

Figura 16. Esquema de conceptos elaborados *in situ* por el profesor

En el esquema de la Figura 16, realizado tras la primera sesión con el grupo A, se observa cómo algunos grupos han localizado algunos conceptos considerados en el mapa de cuestiones (A7 localiza la "Integral de Flujo", A6 se pregunta si pueden ser campos escalares o vectoriales, A2 expone el Tª de la Divergencia). En general, aunque su contenido es similar al informe diario de cada grupo, ha resultado útil de cara a desarrollar las versiones posteriores del mapa de cuestiones. No obstante, dichos informes diarios permiten profundizar más detalladamente en la información recogida.

Al final de cada sesión se dejaban planteadas cuestiones cuya respuesta debía buscar cada grupo y presentarla en la siguiente sesión.

El análisis de los archivos compartidos, donde anotaban la información proporcionada al final de cada sesión junto con los esquemas realizados, permitió observar que, en algunos casos, la información recogida respondía a cuestiones distintas a la que se planteaba, que no se recogían en el mapa inicial. Por ejemplo, el concepto de *caudal* $\boldsymbol{Q}$ *de un fluido a través de un conducto* cuya expresión es:

$$\boldsymbol{Q} = \boldsymbol{V} \cdot \boldsymbol{S}$$

Donde $\boldsymbol{V}$ es la velocidad del fluido y $\boldsymbol{S}$ el área plana de la sección transversal del conducto. Aquí al plantear las cuestiones de confirmación siguientes: *¿El caudal sirve para calcular el flujo en una superficie curvada? ¿El caudal sirve para calcular el flujo para cualquier campo de velocidades o tiene que ser constante a lo largo de la superficie?*, se descartó el uso de dicha noción pues en la cuestión $\boldsymbol{Q}$ la superficie es curvada y el campo vectorial depende de la coordenada $\boldsymbol{x}$.

También apareció el *flujo de campo eléctrico* dado por el *teorema de Gauss*. Pero, fue descartado después de plantear la cuestión de confirmación: *¿El Teorema de Gauss sirve para calcular el flujo a través de superficies abiertas o cerradas?* Pues solo sirve para superficies cerradas. Este concepto de *flujo* se consideró en el mapa de cuestiones porque, aunque no daba respuesta a la cuestión principal, formaba parte del temario y era uno de los objetivos secundarios de la ***OD***. Es decir, se perseguía que dieran significado a dicho concepto descubriendo por qué no contestaba a la cuestión planteada y decidiendo desecharlo. Un estudiante no solo comprende los conocimientos que decide idóneos para dar respuesta a la cuestión, sino también aquellos que no son adecuados.

Durante la primera mitad de la última sesión, se reconstruyeron los elementos de la ***OM*** encontrados en las sesiones anteriores profundizando en aquellos cuya comprensión resultó más difícil. En la segunda mitad de la sesión, los alumnos trabajaron las técnicas descubiertas (Figura 17) para dar respuesta a $\boldsymbol{Q}$.

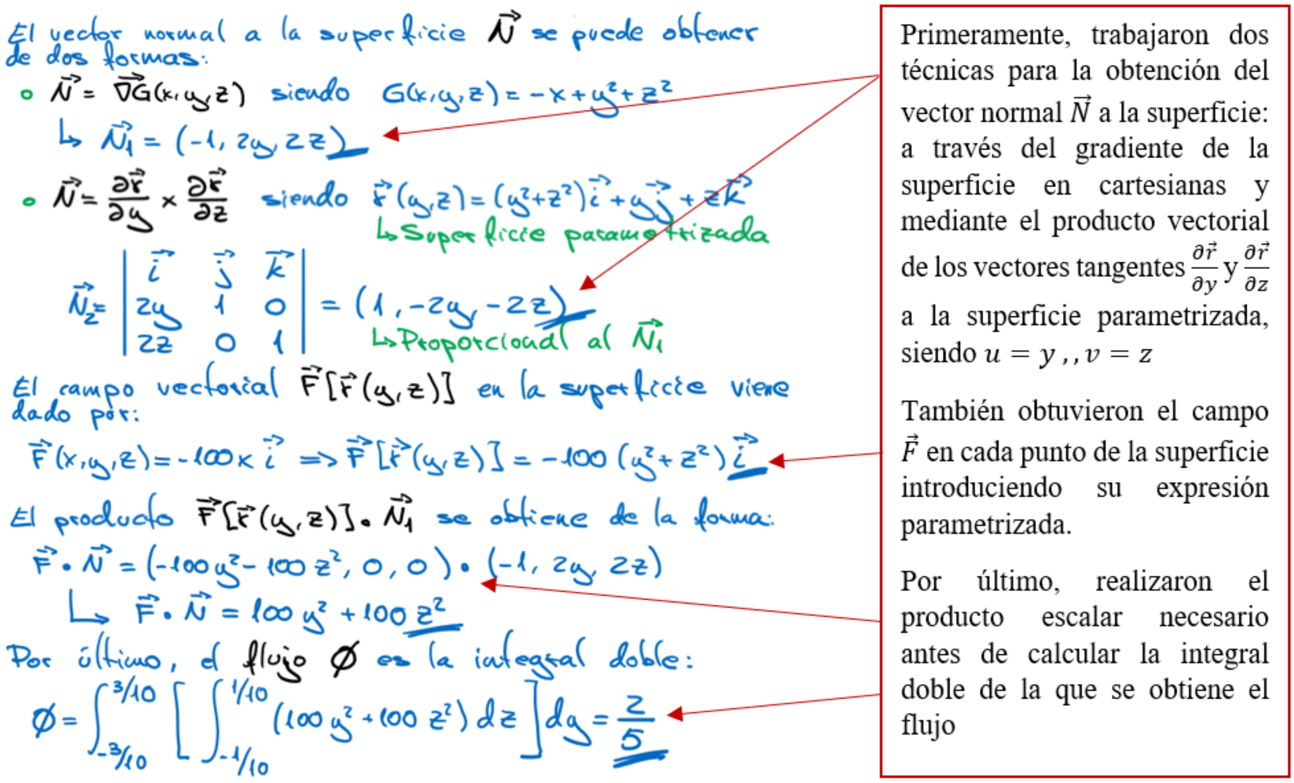


Figura 17. Técnicas descubiertas por los estudiantes

Finalmente, cada grupo entregó un informe para ser evaluado por el profesor con dos partes diferenciadas: una explicando el marco teórico en el que se apoyaban los cálculos realizados para dar respuesta a $\boldsymbol{Q}$ y, otra exponiendo las respuestas que resolvían la cuestión. La calidad de dichos informes no fue la esperada en cuanto a la estructura y la redacción (escasez de tecnicismos y de calidad narrativa en las explicaciones), pero, en general, sí respondieron correctamente, lo que nos permite considerar que nuestra hipótesis de partida puede resultar plausible. En la Figura 18, mostramos de forma breve un ejemplo de informe realizado por el grupo B7 con la respuesta correcta a $\boldsymbol{Q}$.

Calculamos el flujo, que viene determinado por la siguiente fórmula:

$$\Phi = \int \vec{F}\,\vec{N}\,ds$$

La región de integración se trata de un rectángulo, por lo que tendremos que operar en coordenadas cartesianas.

$$\vec{N} = VG \rightarrow G(x, y, z) = (-1{,}2y, 2z)$$

$$\vec{F}\,[x(y, z)] = (-100y^2 - 100z^2, 0{,}0)$$

Una vez obtenida la fórmula del campo, sustituimos en la fórmula anterior para obtener el flujo de aire que atraviesa las piezas a estudiar:

$$\Phi = \int_{-3/10}^{3/10} dy \int_{-1/10}^{1/10} dz\ (100y^2 + 100z^2 = \frac{\mathbf{2}}{\mathbf{5}}$$

Figura 18. Ejemplo parcial del informe del grupo B7 con la respuesta a $\boldsymbol{Q}$

Otros grupos entregaron una respuesta a la cuestión principal con algún error de aritmética como, por ejemplo, en el cálculo del producto escalar (Figura 19), en los límites de integración (Figura 20) o en las variables de integración (Figura 21), pero inicialmente la técnica empleada para el cálculo de flujo sí la han desarrollado correctamente.

RESULTADOS Y DISCUSIÓN

Diseño de la pieza y geometría:

$$x = y^2 + z^2 con \begin{cases} y \in [-\frac{3}{10}, \frac{3}{10}] \\ z \in [-\frac{1}{10}, \frac{1}{10}] \end{cases}$$

Campo vectorial del viento:

$$\vec{F}(x, y, z) = 100x\,\vec{\imath}$$

Calculo del flujo de aire que atraviesa dicha pieza:

$$Flujo\ de\ aire = \iint \vec{F}\,\vec{N}\,dS$$

La superficie es: $S(x, y, z) = x - y^2 - z^2$

Calculamos el gradiente, que será la normal de la superficie: $\nabla S(x, y, z) = (1, -2y, -2z)$

$$\boxed{\vec{F} * \vec{N} = 1 + 2y^3 + 2z^3}$$

$$\emptyset = \iint \vec{F}\,\vec{N}\,dS;\ \iint 1 + 2y^3 + 2z^3 = \int_{-\frac{3}{10}}^{\frac{3}{10}} dy \int_{-\frac{1}{10}}^{\frac{1}{10}} dz\ (1 + 2y^3 + 2z^3)$$

$$\int_{-\frac{3}{10}}^{\frac{3}{10}} dy\ z + 2y^3 + \frac{6z^4}{4}\begin{pmatrix} \frac{1}{10} \\ -\frac{1}{10} \end{pmatrix} = \int_{-\frac{3}{10}}^{\frac{3}{10}} 2y^3 + 0{,}2$$

Finalmente realizando la integral, el flujo de aire queda:

$$\emptyset = 0{,}12$$

En este caso, el cálculo del producto escalar es erróneo, ya que debería haber realizado:

$(100x, 0{,}0) \cdot (1, -2y, -2z)$

Siendo $x = y^2 + z^2$

Figura 19. Error en el cálculo del producto escalar

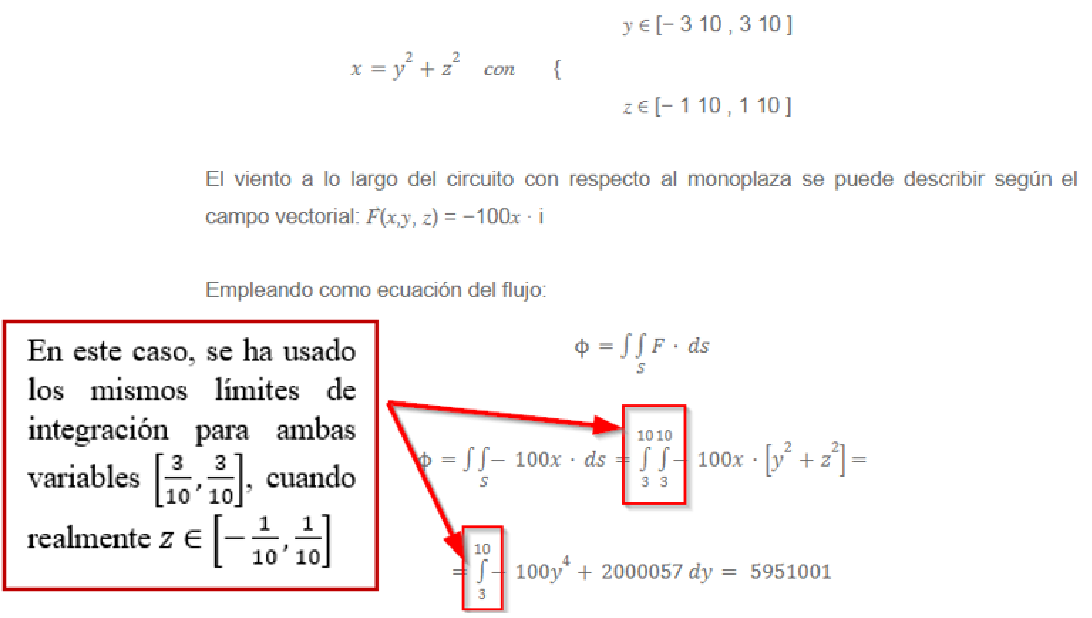


Figura 20. Error en los límites de integración

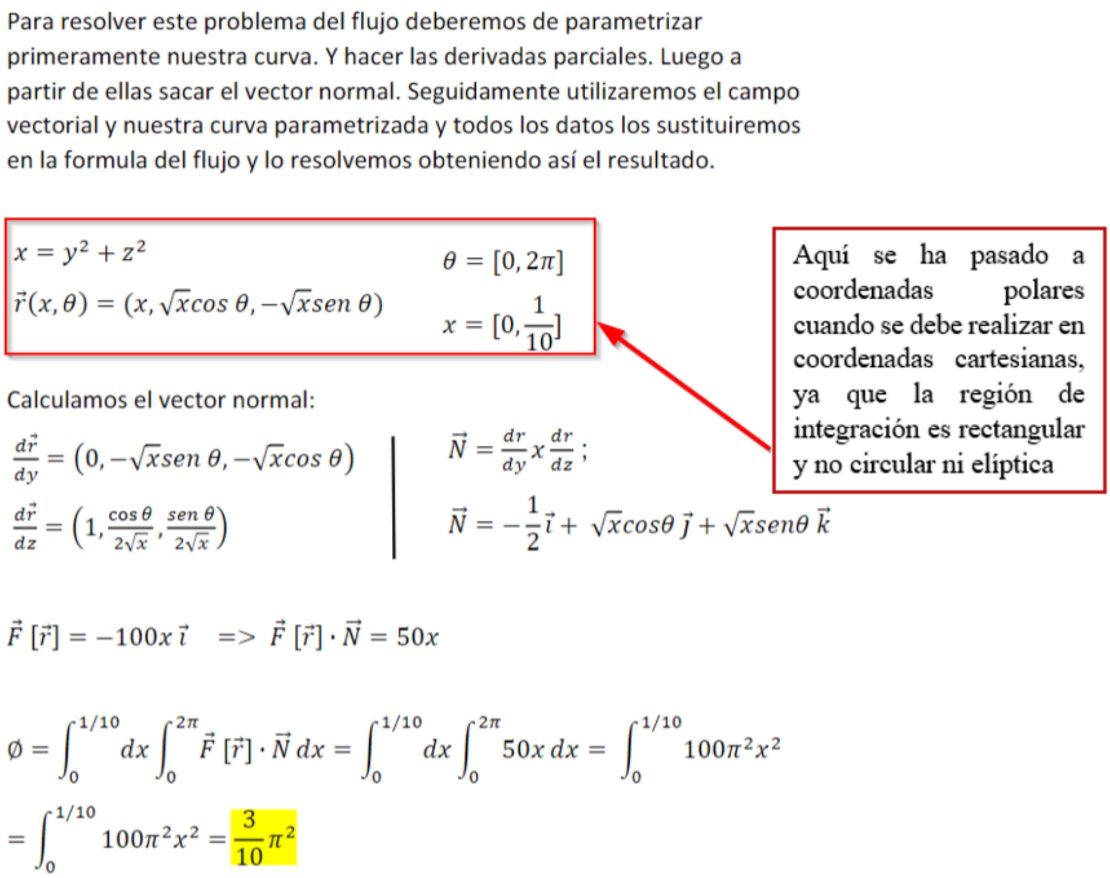


Figura 21. Error en las variables de integración (el grupo quiso resaltar la respuesta subrayándola en amarillo)

Breve discusión de los resultados y conclusiones

En base a las investigaciones y a las guías docentes de distintas universidades que tratamos en la introducción y en el MED, resulta que el aplicacionismo es una práctica bastante habitual en la formación matemática de los grados de Ingeniería, tal y como nos muestra la presencia de los indicadores asociados a dicha práctica. No hemos observado la presencia del I_5 ya que tratamos con una disciplina de matemáticas donde no se presentan cuestiones sobre sistemas extramatemáticos.

En el desarrollo de nuestra ***AEI*** con los estudiantes de segundo de GISI se detectaron ciertas restricciones similares a las expuestas por Barquero, Bosch y Gascón (2014). Inicialmente, hubo dificultades para que el profesor, actuando como director del proceso de estudio, consiguiera que los estudiantes fueran más autónomos y tomasen la iniciativa en la búsqueda de respuestas. Sin embargo, la nueva metodología, que incluye el trabajo en equipo, el planteamiento de cuestiones, la búsqueda de respuestas, etc., fue poco a poco aceptada por los estudiantes debido, entre otras razones, a la mayor motivación que les produjo el uso de las nuevas tecnologías para el estudio de las matemáticas.

En cuanto al mapa de cuestiones planteado (Figura 14), se podría considerar, más bien, mapa de conceptos ya que lo que se plantea, en su mayoría, son una secuenciación de los conceptos que presumiblemente se van a desarrollar en la ***AEI***. Dicho mapa se debería revisar en este sentido para futuras experimentaciones.

No obstante, en el trabajo realizado por los estudiantes, destacamos la gran agilidad que demostraron para la búsqueda de las respuestas a lo planteado, ya que recorrieron dicho mapa en menos tiempo de lo esperado. Este hecho propició que se pudieran debatir con cierta profundidad algunos conceptos que encontraron fuera del mapa de cuestiones como los tipos de *flujo* (laminar o turbulento) o el *caudal* en un conducto.

La ***OD*** experimentada ha estado muy condicionada por la restricción del tiempo del que disponíamos para el estudio de la ***OM***, obtenida como respuesta a la cuestión ***Q***. Dicha restricción ha motivado que eligiéramos una cuestión ***Q*** bastante cerrada, ya que el profesor ha aportado el modelo de superficie, y, en consecuencia, la respuesta también ha sido cerrada y única. Sin embargo, a pesar de ello, han aparecido distintos conceptos como el caudal o los tipos de flujo.

También la validación del camino seguido, mediante las cuestiones de confirmación, ha sido planteaba por el profesor en lugar de dejar que surgiera de los estudiantes. Pensamos que la propuesta de una cuestión ***Q*** más abierta donde los estudiantes deban construir el modelo de superficie a partir de las características de las piezas proporcionadas por los planos de diseño puede hacer que el proceso de estudio desarrollado sea mucho más enriquecedor y productivo, pues ello provocará que surjan otros conocimientos matemáticos.

Como propuesta de futuro dentro de un trabajo de tesis doctoral, queremos diseñar e implementar ***OD*** siguiendo la metodología de los REI donde la actividad matemática desarrollada en la formación matemática de los futuros ingenieros sea una actividad de modelización. De este modo, intentaremos que las matemáticas que se proponen estudiar en la guía docente surjan como respuesta a las cuestiones que les han dado origen.

Referencias

Apostol, T. M. (2002). *Calculus (Vol. II): Cálculo con funciones de varias variables y álgebra lineal, con aplicaciones a las ecuaciones diferenciales y a las probabilidades* (7.ª ed.). Reverte. Barcelona, España.

Barquero, B., Bosch, M., & Gascón, J. (2014). Incidencia del «aplicacionismo» en la integración de la modelización matemática en la enseñanza universitaria de las ciencias experimentales. *Enseñanza de las ciencias*, *32,* 83–100.

Bartolomé, E., Florensa, I., Bosch, M., & Gascón, J. (2019). Un 'camino de estudio e investigación' que enriquece el aprendizaje de la ingeniería mecánica. *Revista europea de educación en ingeniería, 14,* 330–346.

Curbeira, D., Bravo, M. d., & Bravo, G. (2013). La formación inicial de habilidades profesionales del ingeniero industrial desde el contexto de la matemática. *Ciencia y Sociedad*, *38,* 377–403.

Chevallard Y. (2009) *Remarques sur la notion d'infrastructure didactique et sur le rôle des PER*. Conférence prononcée aux Journées Ampère tenues à l'INRP, Lyon. http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=155&var_recherche=infrastructure+didactique

Florensa, I., Bosch, M., Gascón, J., & Winslow, C. (2018). Itinerarios de estudio e investigación: una nueva herramienta para el diseño y la gestión del aprendizaje basado en proyectos en ingeniería. *Revista internacional de educación en ingeniería*, *34 (*6*),* 1848–1862.

García, F. J., Barquero, B., Florensa, I., & Bosch, M. (2019). Diseño de tareas en el marco de la Teoría Antropológica de lo Didáctico. *AIEM - Avances de Investigación en Educación Matemática*, *15,* 75–94.

Mora Flores, W. (2019). *Cálculo en varias variables.* Costa Rica: Revista digital Matemática, Educación e Internet.

Plaza, L. F., & Villa-Ochoa, J. A. (2019). Obstáculos detectados en la formación matemática de ingenieros. Una revisión de literatura. *Revista Virtual Universidad Católica del Norte*, *58,* 223–241.

RAE. (s.f.). *Real Academia Española*. Obtenido de RAE: https://dpej.rae.es/lema/ingenier%C3%ADa

Romo-Vázquez, A. (2014). La modelización matemática en la formación de ingenieros. *Educación matemática, 26* (especial 25 años), 314–338.

Trejo, E., Camarena, P., & Trejo, N. (2013). Las matemáticas en la formación de un ingeniero: la matemática en contexto como propuesta metodológica. *Revista de Docencia Universitaria*, *11 (*1*),* 397–424.

Vázquez. (2012). ¿Qué ingenieros necesita México? *Innovación Educativa*, *12 (*60*)* 125–136.

REI codisciplinar en matemática y microeconomía: resultados preliminares de su implementación en el nivel universitario argentino

Velma Marina Benítez [1] and Verónica Parra[2]

[1]Universidad Nacional de Misiones (UNaM), Argentina; velmabenitez63@gmail.com

[2]Universidad Nacional del Centro de la Provincia de Buenos Aires (UNCPBA), Núcleo de Investigación en Educación en Ciencia y Tecnología (NIECyT), Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina; vparra@niecyt.exa.unicen.edu.ar

Abstract. This paper presents the first results of an SRP implemented at the Argentine university level. SRP relates Economics and Mathematics and its generating question is Q_0: How much benefit do producers and consumers receive by the existence of a competitive market? The implementation took place from March to August 2021 (16 weeks), with a group of approximately 35 students of the Contador Público, Licenciatura en Administración de Empresas and Licenciatura en Economía of a Faculty of Economic Sciences of an Argentine National University, within the course "Mathematical Analysis". The SRP was developed entirely through the Chamilo platform and with synchronous meetings, through Google meet. We present here the derived questions formulated by each study group starting from Q_0.

Palabras clave: Recorrido de estudio en investigación, nivel universitario argentino, economía, matemática.

Resumen. En este trabajo se presentan los primeros resultados de un REI implementado en el nivel universitario argentino. El REI vincula la Economía y la Matemática y su pregunta generatriz es Q_0: ¿Cuánto beneficio reciben los productores y los consumidores por la existencia de un mercado competitivo? La implementación se realizó durante todo el primer cuatrimestre del año 2021 (16 semanas), con un grupo de 35 estudiantes de primer año de las carreras Contador Público, Licenciatura en Administración de Empresas y Licenciatura en Economía de una Facultad de Ciencias Económicas de una Universidad Nacional argentina, dentro del curso "Análisis Matemático". El REI se desarrolló en su totalidad a través de la plataforma Chamilo y con encuentros sincrónicos, por Google meet. Presentamos aquí las preguntas derivadas formuladas por cada grupo de estudio a partir de Q_0.

Keywords: Research and study path, Argentine university level, economics, mathematics.

Introducción.

Los resultados de diversas investigaciones (Barquero, 2009; Serrano, Bosch y Gascón, 2007; Gómez-Chacón, 2009; Fonseca, 2004; Fonseca, 2010, 2011; Salgado, 2019; Serrano, Bosch y Gascón, 2010, entre otros) ponen de manifiesto que en la universidad se enseña de forma descontextualizada, como un conjunto ordenado y finito de operaciones que permite resolver un problema. El aprendizaje de los estudiantes se convierte así en un proceso ligado a una serie de reglas, axiomas, postulados y teoremas para resolver ejercicios tipos y aplicarlos a modelos preestablecidos, que trae como consecuencia el considerar a la matemática como un conjunto de reglas aplicadas en forma sistemática para dar respuestas y luego reproducir la obra presentada.

En cuanto a la relación entre la matemática y microeconomía, los profesionales de esta última han expresado que la matemática les permite mejorar su productividad, de manera que la cooperación activa entre matemáticos y economistas ha contribuido a que la matemática, utilizada como un recurso, tenga una presencia significativa en la enseñanza de la economía. Sin embargo, en la realidad esto no sucede, las áreas son enseñadas sin conexión o, en el mejor de los casos, con vínculos muy débiles. No sólo es importante que en las carreras de ciencias económicas el estudiante pueda "usar" la matemática y la economía al resolver un problema, sino también, interpretar el resultado económico obtenido. En este último aspecto se presentan grandes dificultades (García, Azcarate, Moreno, 2006).

Examinando diferentes aspectos de la relación cultural con los saberes, Chevallard (2004) pone en evidencia el "enfoque monumentalista" en la enseñanza de la matemática. La enseñanza se reduce al estudio de respuestas en lugar de preguntas, consideradas por Chevallard como carentes de legitimidad y sentido. Ante la necesidad de un cambio en la pedagogía tradicional se requiere de una modificación sustancial del modelo pedagógico reinante que sustituya la pedagogía monumentalista por una pedagogía que permita recuperar el sentido y las razones de ser de las praxeologías matemáticas, y poder así cambiar el modelo de enseñanza actual. Dentro del marco de la TAD se propone la pedagogía de la investigación y del cuestionamiento del mundo (Chevallard, 2004, 2007, 2013b) cuyo dispositivo vital son los recorridos de estudio e investigación (REI).

Resulta interesante cuestionarse sobre las formas de concretar este cambio de pedagogía a nivel disciplinar y/o codisciplinar puesto que la modelización adopta aquí un papel central. La modelización matemática aboga por el estudio de todo tipo de sistemas, en nuestro caso, microeconómicos y los REI, en este sentido, tienen un doble objetivo: por un lado, mostrar las condiciones (y limitaciones) que se requieren para comenzar a superar el monumentalismo y, por el otro, indagar en las condiciones (y limitaciones) que permitan (y no) hacer vivir un tipo de modelización matemática.

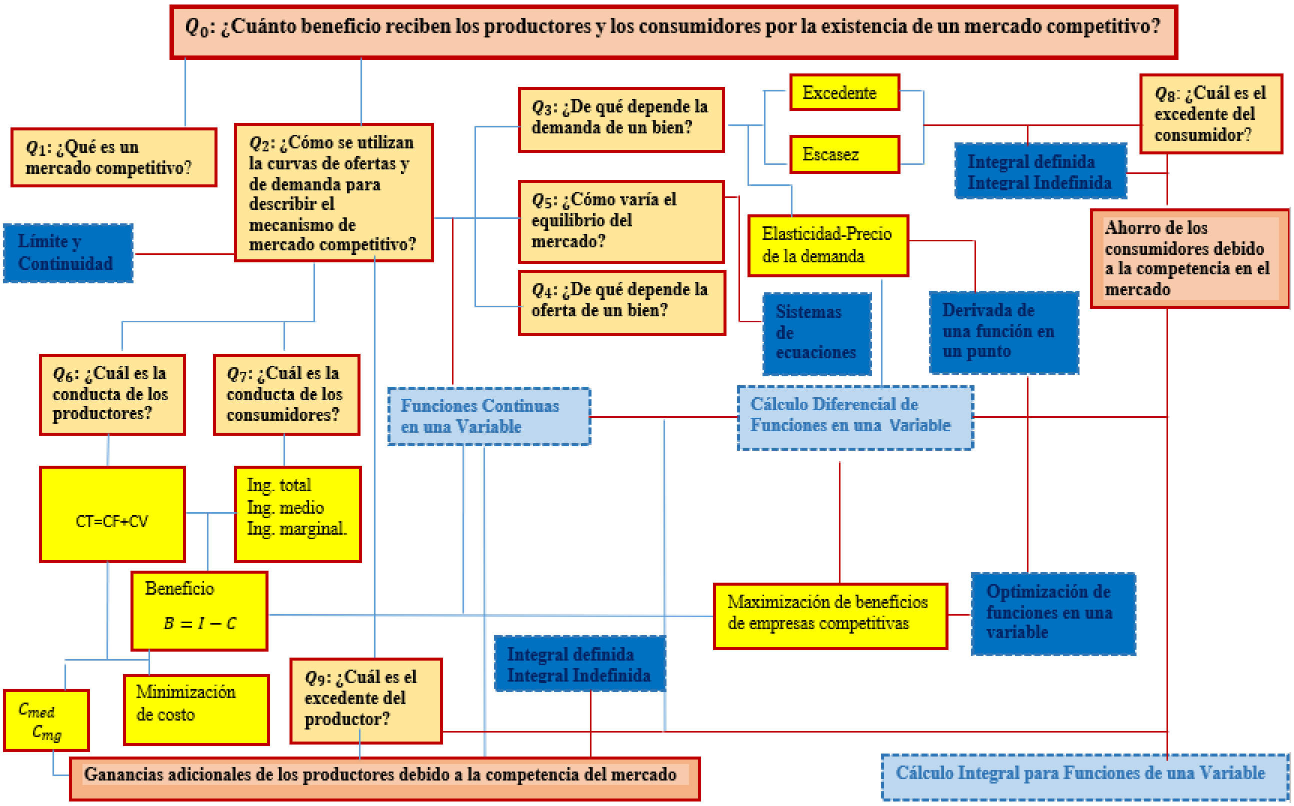
En la actualidad, en la universidad hay quienes intentan apropiarse de nuevos paradigmas y hacen serios cuestionamientos a las formas rígidas del acceso al conocimiento, sin embargo, hay otros sectores que se alejan de cualquier cuestionamiento del mundo (Chevallard, 1998) que pueda encauzar la enseñanza y que provoque en los estudiantes a tener una actitud autónoma e independiente en la construcción del conocimiento. Nuestro trabajo, que es parte de una tesis doctoral, aún en una etapa de análisis incipiente, pretende apuntar en este sentido.

La tesis tiene por objetivo evaluar y analizar un REI co-disciplinar en Matemática y Microeconomía diseñado en 2020 e implementado durante el ciclo académico 2021 en un curso de Análisis Matemático del primer año de una Facultad de Economía de una Universidad Argentina. Es importante explicitar que este recorrido tiene un fin educativo y es abordar el programa de estudios (o al menos, la mayor parte de él) del curso de Análisis Matemático y del cuso de Microeconomía. La evaluación y análisis del REI en sí mismo contempla: la descripción y análisis de las posibles praxeologías involucradas en el REI a partir de un posible modelo praxeológico de referencia (MPR), la descripción y análisis de las praxeologías efectivamente reconstruidas por la comunidad de estudio para responder a la cuestión generatriz y un análisis detallado de la evolución del REI en términos de dialécticas y de actitudes. En este último sentido nos proponemos, no sólo hacer uso del conjunto de indicadores para cada una de las dialécticas (Parra y Otero, 2017, 2020) y actitudes (Donvito, Otero y Sureda, 2014) sino también establecer el grado de vinculación entre el funcionamiento de cada dialéctica entre sí y con las actitudes. Por cuestiones de avances de la investigación, presentamos aquí algunos resultados preliminares: listaremos ciertas preguntas derivadas formuladas por cada grupo de estudio y algunos de los saberes matemáticos que, a partir de ellas, pudieron estudiarse. Insistimos en que se trata de resultados incipientes y por el momento, parciales.

Metodología de la investigación

El REI parte de la pregunta generatriz *Q_0: ¿Cuánto beneficio reciben los productores y los consumidores por la existencia de un mercado competitivo?* Previo a la implementación del REI se elaboró un primer potencial modelo praxeológico de referencia (MPR) donde se contemplan posibles preguntas derivadas, organizaciones matemáticas y organizaciones económicas que podrían ponerse en juego. Algunas de ellas son organizaciones matemáticas relativas al Cálculo Integral de Funciones en una Variable y praxeologías económicas referentes del Cálculo del Excedente del Consumidor y del Productor, por ejemplo. Uno de los objetivos de este MPR es analizar la potencialidad de la pregunta generatriz a partir de las posibles preguntas derivadas que, en la posición de investigadores/didactas, se pueden proponer, formular y anticipar. Este análisis de preguntas-praxeologías permitirá, además, pensar en una posible gestión del REI por parte del profesor-investigador puesto que es muy probable que varias de estas preguntas surjan por parte de los estudiantes. Finalmente, este MPR podría, incluso, sacar a la luz y dar cuenta de algún fenómeno didáctico emergente a lo largo del proceso de estudio.

A partir de este posible MPR, se podrían estudiar aspectos de 5 de las 6 unidades del programa de estudios del curso “Análisis Matemático” y conceptos de 6 de las 8 unidades del programa del curso Microeconomía. Presentamos a continuación solo un esquema del mismo:



Esquema 1: Esquema de un posible MPR

La implementación se realizó desde marzo a agosto del año 2021, con un grupo de aproximadamente 35 estudiantes de primer año de las carreras Contador Público, Licenciatura en Administración de Empresas y Licenciatura en Economía de la Facultad de Ciencias Económicas de la Universidad Nacional de Misiones, en la cátedra de Análisis Matemático. Este REI se desarrolló en su totalidad a través de la plataforma Chamilo (encuentros asincrónicos) y con encuentros sincrónicos, por Google meet. La implementación abarcó todo el primer cuatrimestre, cuya duración fue de 16 semanas. Se realizaron un total de 48 encuentros, distribuidos en 3 encuentros semanales de 2 horas cada uno. Estos horarios están planificados por la institución para el desarrollo "usual" del curso Análisis Matemático. Los estudiantes se organizaron en grupos de 4 o 5 integrantes según sus propias decisiones de conformación. Al inicio del recorrido se coordinó con los estudiantes cual sería la metodología de trabajo, poco convencional en un desarrollo usual del curso "Análisis Matemático", y se les explicó que formaría parte de un proyecto de investigación. El grupo de docentes a cargo del curso de Análisis Matemático estaba compuesto por el profesor responsable (que es el investigador), dos ayudantes de cátedra que son profesores de matemática y un licenciado en economía.

Se realizó observación participante, se hicieron registros de las clases haciendo uso de la opción Grabar disponible en Google meet. Se tomaron notas de campo, se solicitaron informes y se recogieron todas las producciones de los grupos formados por los alumnos, incluso, las presentaciones por diapositivas realizadas por ellos.

Algunos resultados preliminares: pregunta generatriz y sus derivadas.

Como ya se mencionó, los estudiantes se dividieron en 7 grupos de trabajo, por elección propia, y la profesora-investigadora introdujo la pregunta generatriz *Q_0: ¿Cuánto beneficio reciben los productores y los consumidores por la existencia de un mercado competitivo?*, lo que al principio provocó ansiedad porque no se trataba de resolver un problema específico, donde estuviera involucrada un saber matemático o económico previamente analizado. Los estudiantes presentaron buena predisposición en participar del desarrollo del curso y asumieron un compromiso que se mantuvo a lo largo de todo el cuatrimestre.

A partir de Q_0, cada grupo formuló nuevas preguntas que trataban, al menos parcialmente, de responder recurriendo a apuntes de cátedra de Microeconomía (curso que se desarrollaba en paralelo al de Análisis Matemático), apuntes de Matemática y cursos afines, a páginas web, libros de ambas áreas, al equipo de profesores, entre otros.

Una vez formuladas las diferentes preguntas derivadas de Q0, justificando la razón de ser de cada una, cada grupo las presentó al resto del curso a través de una presentación por diapositivas cuyo diseño y forma de difusión no tenía restricciones. Luego de esta puesta en común se acordó comenzar por las preguntas relacionadas con la oferta, la demanda y el punto de equilibrio, y a partir de allí empezar a trabajar las preguntas acordadas por toda la clase. A continuación, listaremos las preguntas explicitadas por cada uno de los siete grupos. Notaremos QGi, con i desde 1 a 7, para indicar que esa pregunta fue formulada por el grupo Gi.

Grupo 1.

$QG1_{0.1}$: Cuando hablamos de mercado ¿Qué fuerzas interactúan? ¿De qué lado del mostrador se posicionan los Consumidores y Productores?

$QG1_{0.2}$: ¿Cómo serían llamados los Productores y Consumidores? ¿Se puede representar gráficamente? ¿Cómo se verán?

$QG1_{0.3}$: Sabemos que las empresas al vender sus bienes y servicios reciben un beneficio, pero, ¿Qué es un beneficio? ¿Cómo se calcula el beneficio? ¿Habrá beneficios que sean negativos, nulos, normales?

$QG1_{0.4}$: ¿A qué se llama mercados competitivos? ¿Se da en la realidad o es un modelo en que nos basamos para estudiar los mercados de competencia imperfeta?

$QG1_{0.5}$: ¿Cuándo se dice que un mercado competitivo se encuentra en equilibrio?

$QG1_{0.6}$: ¿Cuál es la relación entre productores y consumidores?

Grupo 2.

$QG2_{0.1}$: ¿Que tipos de mercado existen?

$QG2_{0.2}$: ¿En qué consiste el mercado competitivo?

$QG2_{0.3}$: ¿Cómo se logra un mercado competitivo?

$QG2_{0.4}$: ¿Cuándo un mercado deja de ser competitivo?

$QG2_{0.5}$: ¿Qué es un productor? ¿Y un consumidor?

$QG2_{0.6}$: ¿Cómo se relacionan los productores y consumidores dentro de un mercado?

$QG2_{0.7}$: ¿De qué manera interviene el Estado?

$QG2_{0.8}$: ¿Qué es un modelo?

$QG2_{0.9}$: ¿Cuáles son los supuestos de la competencia perfecta?

Grupo 3.

$QG3_{0.1}$: ¿Cuándo un mercado es competitivo, Cuando no, ¿qué nos lleva a saber?

$QG3_{0.2}$: ¿Cuáles son los elementos básicos de la oferta y la demanda?

$QG3_{0.3}$: ¿Qué tipo de bienes formarían parte de un mercado competitivo?

$QG3_{0.4}$: ¿Cuáles son las características de un mercado competitivo?

$QG3_{0.5}$: ¿Cómo controlan los vendedores el grado del precio?

$QG3_{0.6}$: ¿Cómo se daría el punto de equilibrio en un mercado competitivo?

Grupo 4.

$QG4_{0.2}$: ¿Quiénes son los productores? $QG4_{0.3}$: ¿Quiénes son los consumidores?

$QG4_{0.4}$: ¿Qué es el Mercado? $QG4_{0.5}$: ¿Cuándo se dice que un mercado es competitivo?

$QG4_{0.6}$: ¿Qué es la oferta? $QG4_{0.7}$: ¿Qué es la Demanda?

Grupo 5.

$QG5_{0.1}$: ¿Qué es un mercado?

$QG5_{0.2}$: ¿A qué denominamos "Mercado Competitivo"? ¿Cuáles son sus características?

$QG5_{0.3}$: ¿Qué es beneficio cuándo hablamos acerca del mercado?

$QG5_{0.4}$: ¿Quiénes se benefician en un mercado competitivo? ¿Cómo se mide?

$QG5_{0.5}$: ¿A qué nos referimos cuando hablamos de un mercado flexible?

$QG5_{0.6}$: ¿Los plazos forman parte de un elemento fundamental a la hora de trabajar en un mercado competitivo?

$QG5_{0.7}$: ¿Qué es la demanda? $QG5_{0.8}$: ¿A qué nos referimos con cantidad demandada?

$QG5_{0.9}$: ¿Cuál es la ley de demanda? $QG5_{0.10}$: ¿Qué gráfica la curva de Demanda?, y $QG5_{0.11}$: ¿Cuáles son sus desplazamientos?

$QG5_{0.12}$: ¿Qué es la oferta? $QG5_{0.13}$: ¿A qué nos referimos con Cantidad ofrecida?

$QG5_{0.14}$: ¿Cuál es la ley de oferta? $QG5_{0.15}$: ¿Qué gráfica la curva de Oferta?, y $QG5_{0.16}$: ¿Cuáles son sus desplazamientos?

$QG5_{0.17}$: ¿Cómo se comportan los Consumidores y los Productores?

Grupo 6.

$QG6_{0.1}$: ¿Qué es un mercado?

$QG6_{0.2}$: ¿Qué es un mercado competitivo? ¿Cómo se caracteriza?

$QG6_{0.3}$: ¿Cuáles son las ventajas de un mercado competitivo?

$QG6_{0.4}$: ¿Qué elementos tiene un mercado competitivo?

$QG6_{0.5}$: ¿Cuándo podemos decir que el mercado se encuentra en equilibrio?

$QG6_{0.6}$: ¿Cómo se calcula el punto de equilibrio de un mercado?

$QG6_{0.7}$: ¿Qué es la ley de oferta y la ley de demanda? $QG6_{0.8}$: ¿Cuáles son sus determinantes?

$QG6_{0.9}$: ¿Qué factores inciden en un mercado competitivo? ¿Cómo hallamos el excedente del consumidor y del productor?

$QG6_{0.10}$: ¿Qué papel cumplen los productores en el mercado competitivo?

$QG6_{0.11}$: ¿Cómo se benefician los productores en un mercado competitivo?

$QG6_{0.12}$: ¿Qué papel cumplen los consumidores en un mercado competitivo?

$QG6_{0.13}$: ¿Cómo se benefician los consumidores en un mercado competitivo?

$QG6_{0.14}$: ¿Cómo se calcula el beneficio obtenido por los productores?

$QG6_{0.15}$: ¿Cómo se calcula el beneficio obtenido por los consumidores?

$QG6_{0.16}$: ¿En qué nivel de producción se maximizan los beneficios en un mercado competitivo?

$QG6_{0.17}$: ¿Qué significa que un mercado sea atomizado?

$QG6_{0.18}$: ¿Que son las fallas del mercado?

$QG6_{0.19}$: ¿Qué tipos de barreras de entrada al mercado no deben existir para que un mercado sea competitivo?

$QG6_{0.20}$: ¿Cuándo se produce la eficiencia de mercado y a que hace referencia?

$QG6_{0.21}$: ¿Cuáles son las diferencias entre un mercado competitivo perfecto he imperfecto?

$QG6_{0.22}$: ¿Cómo influye el mercado en la economía?

Grupo 7.

Este grupo, indicó realizar un análisis de Q_0 a partir de las palabras clave y por otro lado sobre el significado de cada una y su implicación en el conjunto de la pregunta.

Palabra clave “Beneficio”: ¿Qué entendemos por beneficio? ¿Cómo se aumenta el beneficio en una empresa? ¿Qué beneficio obtiene el consumidor en contraste a un mercado no competitivo? ¿Cómo se maximiza el beneficio de una empresa? ¿Qué variables hay que tener en cuenta para el estudio de la maximización del beneficio en una empresa?

Palabra clave “Productor”: ¿A quién denominamos productor? ¿Cómo influye el productor al mercado competitivo? ¿De qué variable depende la producción total? ¿Cuáles son las etapas de la producción? ¿Puede un productor operar si no obtiene beneficio? ¿Qué ocurre en un mercado competitivo con un número de productores fijos?

Palabra clave “Consumidor”: ¿Qué entendemos por “Consumidor”? ¿Cómo afecta la cantidad de consumidores en un mercado competitivo? ¿Cómo se analiza la influencia que aportan en el mercado competitivo?

Palabra clave “Mercado competitivo”: ¿A qué denominamos “Mercado competitivo”? ¿Existe un mercado perfectamente competitivo? ¿Cómo se clasifican los mercados competitivos? ¿Cómo afecta el aumento o disminución de consumidores a un mercado competitivo?

Estas preguntas han permitido el estudio de organizaciones económicas (microeconomía) que, tal como lo organizó el G7, podríamos clasificarlas en función de “palabras clave”, y organizaciones matemáticas, ligadas al cálculo diferencial e integral en una variable como ser, límites de funciones en una variable, funciones continuas en una variable, derivada de una función en un punto, ecuación de la recta tangente a la curva en un punto, diferencial, optimización de funciones en una variable, integral definida, y especialmente las relacionadas al excedente del consumidor y del productor. Este estudio, como ya lo indicamos, abarcó las 16 semanas y las preguntas fueron analizadas en cada grupo de alumnos y puestas en común con los demás integrantes de la comunidad de estudio.

Por ejemplo, el grupo 7, respondiendo a la pregunta: *¿Cómo se maximiza el beneficio de una empresa? ¿Qué variables hay que tener en cuenta para el estudio de la maximización del beneficio en una empresa?*, formulan un modelo: identifican las variables, reconstruyen saberes matemáticos para resolverlo, y realizan un análisis gráfico, impulsado por preguntas tales como *¿En qué intervalo el* $I(x) > C(x)$*? ¿En qué intervalo el* $I(x) < C(x)$*? ¿Para qué valores de “x”* $I'(x) = 0$ *y* $C'(x) = 0$*? ¿Cuándo el beneficio es máximo?* (Figura 1).

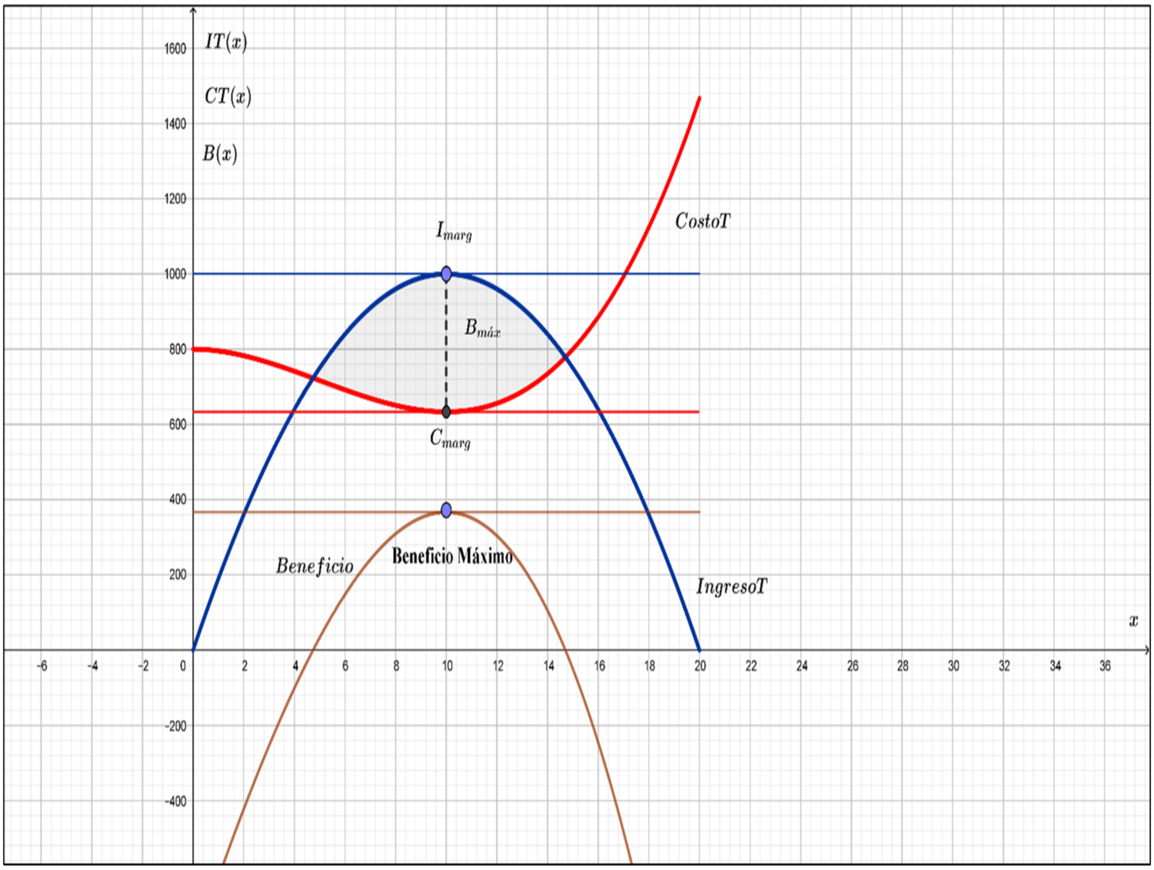


Figura 1: Análisis a partir de GeoGebra de uno de G7

Aportan así respuestas del tipo*: los beneficios son máximos cuando se alcanza la máxima diferencia entre los ingresos totales y los costos totales. Como se puede ver en la figura, la diferencia positiva máxima entre los ingresos totales y los costos totales se alcanza en el punto en el que la pendiente de la recta tangente a la curva de ingresos totales es igual a la pendiente de la recta tangente a la curva de costos totales. Se puede mostrar gráficamente que el costo marginal debe ser igual al ingreso marginal para que los beneficios sean máximos*.

Esto permitió estudiar praxeologías matemáticas: ecuaciones, inecuaciones, derivada de una función en un punto, crecimiento y decrecimiento de una función en un intervalo, máximos y mínimos absolutos y relativos, ecuación de la recta tangente a la curva en un punto e integrales definidas.

El grupo 6, por su parte, respondiendo a la pregunta $QG6_{0.9}$: *¿Qué factores inciden en un mercado competitivo? y ¿cómo hallamos el excedente del consumidor y del productor?*, calculan el excedente del consumidor y del productor en un mercado competitivo, y presentan un gráfico como se muestra en la Figura 2.

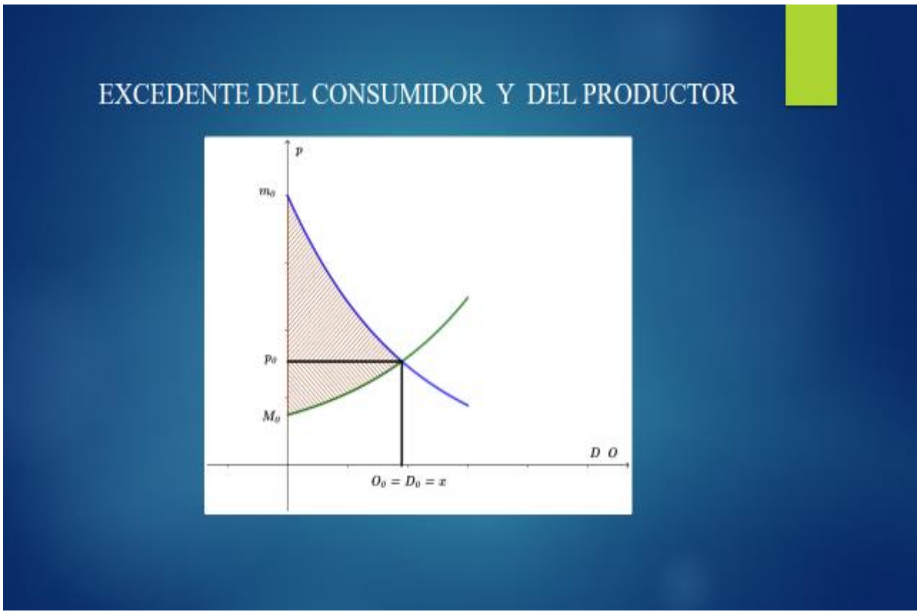


Figura 2: Presentación del grupo 6

Este grupo, busca información en sus apuntes de Microeconomía y en Internet para indicar que : si $p = f(O)$,donde p es el precio y O la cantidad ofrecida, gráficamente el excedente del productor es el área entre la curva de oferta y la recta $p = p_0$ siendo p_0 el precio de mercado $EP = O_0.p_0$ - $\int_0^{O_0} f(O).dO$; $p = f(D)$, la ganancia total del consumidor gráficamente es el área situada entre la curva de demanda y la recta $p = p_0$ y se denomina excedente del consumidor $EC = \int_0^{D_0} f(D).dD - D_0.\,p_0$.

Esto permitió estudiar aspectos de OM: funciones continuas de una variable, crecimiento de una función, decrecimiento de una función, sistema de ecuaciones e integrales definidas (regla de Barrow).

Hemos presentado aquí muy brevemente sólo dos ejemplos del estudio realizado a partir de dos preguntas derivadas. Insistimos que el análisis está en proceso y apunta al relevamiento de los indicadores del funcionamiento de las dialécticas, como así también de las actitudes, para luego establecer el grado de vinculación de cada dialéctica entre sí y con las actitudes.

Consideraciones finales.

El análisis realizado hasta el momento resulta ser alentador. Haber mantenido vivo el REI durante el desarrollo de todo el curso pareciera ser uno de los resultados más prometedores. Nos queda un camino por recorrer en el análisis de los datos, pero hasta aquí, concluimos en que se han generado una enorme cantidad de preguntas derivadas que han conducido a estudiar obras matemáticas, al menos, con un sentido.

References

Barquero, B. (2009). Ecología de la modelización matemática en la enseñanza universitaria de las matemáticas (Tesis doctoral). Universitat Autónoma de Barcelona, España.

Barquero Farràs, B., Ruiz-Munzón, N., Monreal Galán, J. I. & Barajas Frutos, M. (2016). Un Recorrido de Estudio e Investigación sobre la comparación de realidad y previsión de la evolución de los usuarios de Facebook. Su diseño como c-unidad e implementación a nivel universitario. *Revista CIDUI*, 1-11.

Chevallard, Y. (1998). *Análisis de prácticas de enseñanza y didáctica de las matemáticas: una aproximación antropológica*. Université de la Rochèlle.

Chevallard, Y. (2004). Vers une didactique de la codisciplinarité. Notes sur une nouvelle épistémologie scolaire. *Journées de Didactique Comparée,* Lyon. http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=45

Chevallard, Y. (2007). Passé et présent de la théorie anthropologique du didactique. En L. Ruiz-Higueras, A. Estepa, & F. Javier García (Éd.). *Sociedad, Escuela y Mathemáticas. Aportaciones de la Teoría Antropológica de la Didáctica*, Universidad de Jaén, 2007, 705-746. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Passe_et_present_de_la_TAD-2.pdf

Chevallard, Y. (2013a). Théorie Anthropologique du Didactique & Ingénierie Didactique du Développement. *Journal du Seminaire TAD/IDD 2013*. Aix-Marseille. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/journal-tad-idd-2012-2013-5.pdf

Chevallard, Y. (2013b). *La matemática en la escuela: Por una revolución epistemológica y didáctica*. Buenos Aires, Argentina: Libros del Zorzal.

Donvito, A., Otero, M. & Sureda, P. (2014). Actitudes de la Pedagogía de la investigación en el marco de la TAD: un análisis en tres escuelas secundarias. *Ikastorratza, e-Revista de didáctica*,12, 1-27.

Florensa, I., Bosch, M. & Gascón, J. (2020). Reference epistemological model: what form and function in school institutions? *Educação Matemática Pesquisa*, 22(4), 240-249.

Fonseca, C. (2004). *Discontinuidades matemáticas y didácticas entre la secundaria y la universidad* (Tesis doctoral). Universidad de Vigo, España.

Fonseca, C. (2010). Una posible "razón de ser" de la diagonalizacion en ciencias económicas y empresariales. En: A. Bronner, M. Larguier, M. Artaud, M., Bosch, Y. Chevallard, G. Cirade y C. Ladage (eds.), *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action* (pp.595-614). Montpellier: IUFM de l'Académie de Montpellier.

Fonseca, C. (2011). Recorridos de Estudio e Investigación: una propuesta dentro de la teoría antropológica de lo didáctico para la creación de secuencias de enseñanza y aprendizaje. *Paradigma*, 32(1), 55-70.

García, L., Azcarate, C. & Moreno, M. (2006). Creencias, concepciones y conocimiento profesional de profesores que enseñan cálculo diferencial a estudiantes de ciencias económicas. *Relime*, 9(1), 85-116.

García, F., Bosch, M., Gascón, J. & Ruiz, L. (2006). Integración de la proporcionalidad escolar en una organización matemática regional en torno a la modelización funcional: los planes de ahorro. En L. Ruiz Higueras, A. Estepa & F. J. García (Eds.), *Sociedad, escuela y matemáticas:*

aportaciones de la teoría antropológica de lo didáctico (pp. 439-460). Jaén, España: Publicaciones de la Universidad de Jaén.

Gómez-Chacón, I. Ma. (2009). Actitudes matemáticas: propuestas para la transición del bachillerato a la universidad. *Educación Matemática*, 21(3), 5-32.

Parra, V. & Otero, M. R. (2017). Enseñanza de la matemática por recorridos de estudio e investigación: indicadores didáctico-matemáticos de las "dialécticas". *Educación Matemática*. 29(3), 9-50.

Parra, V. & Otero, M. R. (2020). Research and study paths: Indicators of the development of the dialectics. *Educação Matemática Pesquisa*, 22(4), 856-869.

Salgado, D. (2019). *Diseño, implementación, análisis y evaluación de un Recorrido de Estudio e Investigación en el nivel universitario relativo al Cálculo en dos variables* (Tesis doctoral). Universidad Nacional de Centro de la Provincia de Buenos Aires, Argentina.

Serrano, L., Bosch, M. y Gascón, J. (2007). Diseño de organizaciones didácticas para la articulación del bachillerato con el primer ciclo universitario. En L. Ruiz-Higueras, A. Estepa y F. J. García (Eds.) *Sociedad, escuela y matemáticas. Aportaciones de la Teoría Antropológica de lo Didáctico* (pp.757-766). Jaén: Publicaciones de la Universidad de Jaén.

Serrano, L., Bosch, M. y Gascón, J. (2010). Cómo hacer una previsión de ventas: propuesta de recorrido de estudio e investigación en un primer curso universitario de administración y dirección de empresas. En: A. Bronner, M. Larguier, M. Artaud, M., Bosch, Y. Chevallard, G. Cirade y C. Ladage (eds.), *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action* (pp.835-857). Montpellier: IUFM de l'Académie de Montpellier.

Study of asymptotes in Calculus in mathematics textbooks for general upper secondary education in Croatia

Ana Katalenić[1], Aleksandra Čižmešija[2] and Željka Milin Šipuš[3]

[1]Faculty of Education, J. J. Strossmayer University of Osijek, Croatia; akatalenic@foozos.hr

[2]Faculty of Science, University of Zagreb, Croatia; cizmesij@math.hr

[2]Faculty of Science, University of Zagreb, Croatia; milin@math.hr

Abstract: Asymptotes in the upper secondary mathematics curriculum are related to graphs and limits of elementary functions. We analyzed fourth-grade mathematics textbooks for general upper secondary education in Croatia concerning the study of asymptotes in Calculus. Asymptotes occurred in several praxeologies confined to two units; one related to function limit and the other to applying derivatives in function graphing. The study of asymptotes was organized around two didactic moments: the institutionalization of formulas and working on determining asymptotes. We propose exploring the asymptotic behaviour of a function to motivate the construction of formulas with a limit for the coefficients of slant asymptote and generalized asymptotic relations.

Résumé: Les asymptotes du programme de mathématiques du secondaire supérieur sont liées aux graphes et aux limites des fonctions élémentaires. Nous avons analysé des manuels de mathématiques de quatrième année pour l'enseignement secondaire supérieur général en Croatie concernant l'étude des asymptotes en calcul. Des asymptotes se sont produites dans plusieurs praxéologies confinées à deux unités; l'un lié à la limite de fonction et l'autre à l'application de dérivées dans la représentation graphique de fonctions. L'étude des asymptotes s'est organisée autour de deux moments didactiques: l'institutionnalisation des formules et le travail sur la détermination des asymptotes. Nous proposons d'explorer le comportement asymptotique d'une fonction pour motiver la construction de formules avec une limite pour les coefficients des relations asymptotique oblique et asymptotique généralisée.

Keywords: asymptotes and asymptotic behaviour, limit of a function, textbook analysis, upper secondary education.

Introduction

Asymptotes are included in the upper secondary mathematics curriculum as a notion related to conics (hyperbola), graphs and limits of elementary functions. Research shows that upper secondary and university students describe an asymptote of a function as a line to which a function graph approaches in a monotonic way without intersecting it, which corresponds to their first encounter with asymptotes as a property of rational and exponential functions (Katalenić et al., 2020; Mudaly & Mpofu, 2019). Further studying of asymptotes requires knowledge of their relationship with function limiting behaviour at a point or at infinity. Studies implicate that exploring the behaviour of a function graph at infinity contributes to student construction of the definition of a horizontal asymptote by using the limit (Dahl, 2017; Kidron, 2011; Swinyard & Larsen, 2012), as well as deducing techniques of finding more general, polynomial asymptotes of rational functions (Yerushalmy, 1997).

Asymptotic or limiting behaviour of functions is a rich mathematical body of knowledge which connects graphic and different symbolic representations of a function and relies on knowledge of limits. Studying asymptotes at the high school level with a focus on asymptotic behaviour also provides insight into important advanced mathematical concepts, such as the approximation of a function by a simpler function for large magnitude arguments. For example, common formulas with a limit for equations of asymptotic lines of a function can be interpreted in terms of asymptotic equivalence, with two functions considered asymptotic in the analytic sense if their quotient tends to one as their argument tends to infinity, and asymptotic in the geometric sense if their difference tends to zero at infinity (Dobbs, 2011).

Previous research on upper secondary mathematics textbooks in Croatia showed that the content about asymptotes was fragmented and unconnected, with an emphasis on practical work (Čižmešija et al., 2017). Additionally, university mathematics students' knowledge of asymptotes was related to the upper secondary content, with vague indications of further development of the notions of asymptotes and asymptotic behaviour (Katalenić et al., 2020, 2022). Following the previous research on asymptotes, and taking into account the significant impact of a textbook on teaching practice, the motivation for our study was twofold. We want to examine whether the new, recently implemented problem-solving oriented mathematics curriculum brought any changes in textbook content regarding asymptotes and asymptotic behaviour, and how the organization of content in textbooks contributes to developing these notions.

Methodology

According to the Croatian national curriculum, the basic Calculus is taught in the fourth grade of the general upper secondary education. We analysed the corresponding textbook units in textbooks of four Croatian commercial publishers by scanning for occurrences of asymptotes or limiting behaviour of functions that contribute to the study of asymptotes. The analysed textbooks are the only textbooks currently in use in the fourth grade.

In the textbook analysis, we used four elements [$T/\tau/\theta/\Theta$] of *praxeology* (Bergé, 2008; González-Martín et al., 2013) considering that separate content blocks in textbooks contain practical or discursive elements or both. The type of task T is identified from the question or problem posed, and a technique τ from the content presented or implied in solving the task. Content that describes, supports, complements, or evaluates a practical activity or a particular mathematical notion, identifies the technology θ as a discursive activity within the corresponding theory Θ. Praxeological analysis results in a representation of activities and knowledge which construct a particular body of knowledge in the textbook.

On the other hand, the process of the study describes the means of creating (or recreating) mathematical knowledge (Barbé et al., 2005; García et al., 2006). Chevallard (1999) described six moments in the process of study: the first encounter (with a notion), exploration of the type of tasks, work on a technique, construction of the discourse, institutionalization, and evaluation. For a complete, connected and coherent knowledge, the process of study should be organized around all six moments (Barbé et al., 2005; Corica & Otero, 2012; Serrano Martínez et al., 2020), although it is typically organized around the institutionalization of knowledge and work on applying it (Barbé et al., 2005; González-Martín et al., 2013; Serrano Martínez et al., 2020).

Teaching and learning certain mathematical content in classrooms can take different paths, depending on the teacher. Also, by organizing the mathematical content in the textbook, its authors present their vision of the study process, implicitly suggesting it to teachers. Since teachers in Croatia rely heavily on textbooks when organizing the study (Domović et al., 2012), textbook analyses provide significant insight into the study process. Taking this into account, we state the following research question:

RQ: How is the process of study of asymptotes in Calculus organized in new upper secondary textbooks in Croatia regarding praxeologies and study moments?

Reference epistemological model

We propose a reference epistemological model (REM) for the study of asymptotes in basic Calculus (see *Figure 1*) which builds on the reference model from Barbé et al. (2005) for studying limits in upper secondary education. Asymptotes and asymptotic behaviour are related to two particular tasks, T_{L1} and T_{L2}, from MO_1 from their model.

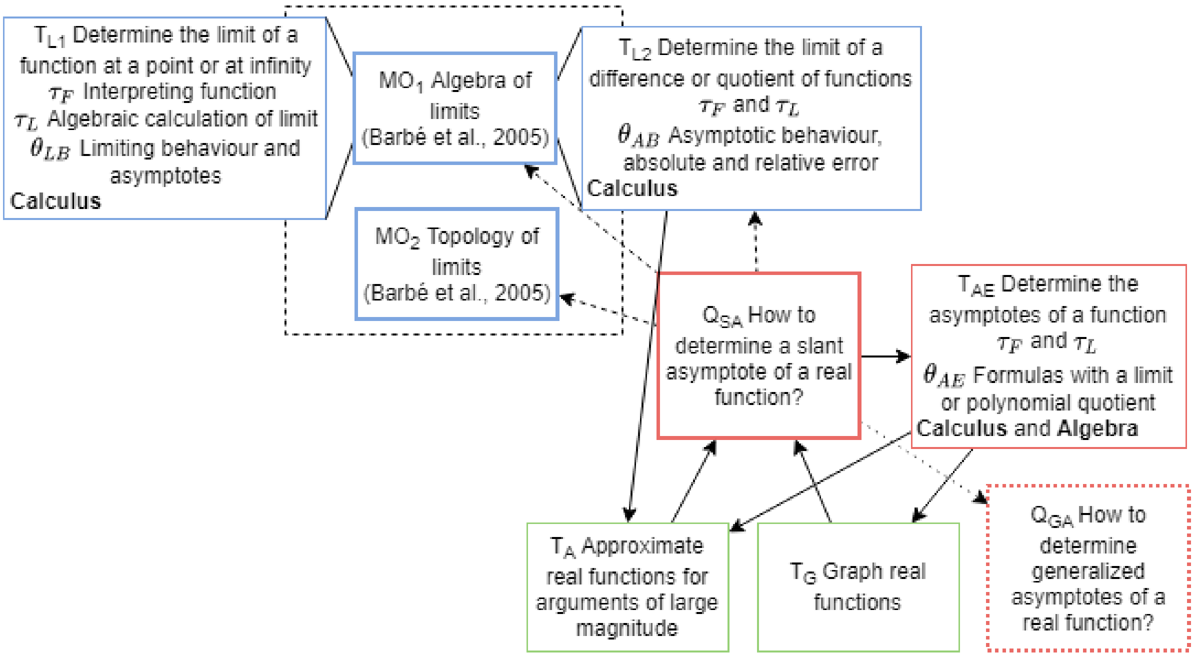


Figure 1: Reference model (REM) for the study of asymptotes[1]

The first encounter with asymptotes in the upper secondary Calculus is in the discursive block when determining the limit of a function as $x \to a$ or $x \to \infty$ (see the example $[T_{L1}/\tau_L,\tau_F/\theta_{LB}/C]$ in *Figure 2*). Complementing Barbé et al. (2005), the task to determine the limit of a quotient or difference of two functions (T_{L2}) contributes to discourse about asymptotic behaviour and approximation of a function with a simpler one for arguments of large magnitude (θ_{AB}).

Example 1. Asymptotic behaviour from the limit of difference of functions.

T_{L2}: Calculate $\lim_{x\to\infty}(f(x) - g(x))$, for $f(x) = \sqrt{x + x^2}$ and $g(x) = x + \frac{1}{2}$.

τ_L: Rationalization and evaluation.

τ_F: Interpretation of the graphs and values of *f* and *g*.

θ_{AB}: Approximation of $f(x)$ with $g(x)$ for large values *x* gives absolute error tending to 0 at infinity. Functions *f* and *g* are asymptotic as $x \to \infty$.

Example 2. Asymptotic behaviour from the limit of quotient of functions.

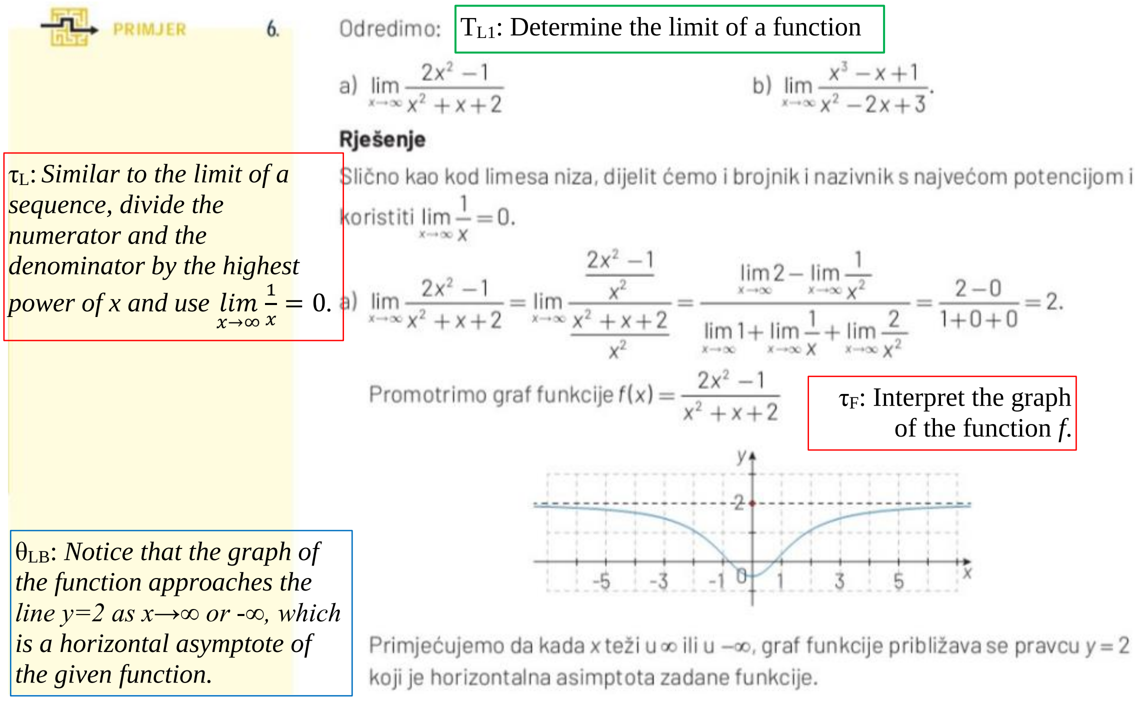
[1] We use the label C for the domain (theory) Calculus and A for Algebra.

T_{L2}: Calculate $\lim\limits_{x\to\infty}\frac{\sqrt{x^2+1}}{x+1}$, for $f(x)=\sqrt{x^2+1}$ and $g(x)=\ x+1$.

τ_L: Division by highest power of x and evaluation.

τ_F: Interpretation of the graphs and values of f and g.

θ_{AB}: Approximation of $f(x)$ with $g(x)$ for large values x gives relative error tending to 0 at infinity. Functions f and g are not asymptotic as $x\to\infty$ in the "approaching" way.



PRIMJER 6. Odredimo: T_{L1}: Determine the limit of a function

a) $\lim\limits_{x\to\infty}\frac{2x^2-1}{x^2+x+2}$ b) $\lim\limits_{x\to\infty}\frac{x^3-x+1}{x^2-2x+3}$.

Rješenje

τ_L: *Similar to the limit of a sequence, divide the numerator and the denominator by the highest power of x and use* $\lim\limits_{x\to\infty}\frac{1}{x}=0$.

Slično kao kod limesa niza, dijelit ćemo i brojnik i nazivnik s najvećom potencijom i koristiti $\lim\limits_{x\to\infty}\frac{1}{x}=0$.

a) $\lim\limits_{x\to\infty}\frac{2x^2-1}{x^2+x+2}=\lim\limits_{x\to\infty}\frac{\frac{2x^2-1}{x^2}}{\frac{x^2+x+2}{x^2}}=\frac{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}\frac{2}{x^2}}=\frac{2-0}{1+0+0}=2.$

Promotrimo graf funkcije $f(x)=\frac{2x^2-1}{x^2+x+2}$ τ_F: Interpret the graph of the function f.

θ_{LB}: *Notice that the graph of the function approaches the line y=2 as x→∞ or -∞, which is a horizontal asymptote of the given function.*

Primjećujemo da kada x teži u ∞ ili u $-\infty$, graf funkcije približava se pravcu $y=2$ koji je horizontalna asimptota zadane funkcije.

Figure 2: First encounter of asymptote when calculating limit from Textbook 3

Graphing (T_G) and approximating functions (T_A) are two tasks which motivate exploring and constructing the discourse of asymptotes. Building on previous discourses about limiting (θ_{LB}) and asymptotic behaviour (θ_{AB}), a more or less formal definition of asymptote of a function can be produced (discourse θ_D in *Figure 3*). The formulas with a limit for the vertical and horizontal asymptote[2] (part of the discourse θ_{AE}) are known from the discourse about limiting behaviour (θ_{LB}):

- the line $x=a$ is a vertical asymptote of function f if $\lim\limits_{x\to a}f(x)=\infty$,
- the line $y=l$ is a horizontal asymptote of function f if $\lim\limits_{x\to\infty}f(x)=l$.

[2] For simplicity, the formulas below define the two-sided asymptotes. If the limits are one-sided, that is, $x\to a^+$ or $x\to a^-$, or the limits are taken at +∞ or -∞, then the corresponding asymptotes are respectively regarded as right or left.

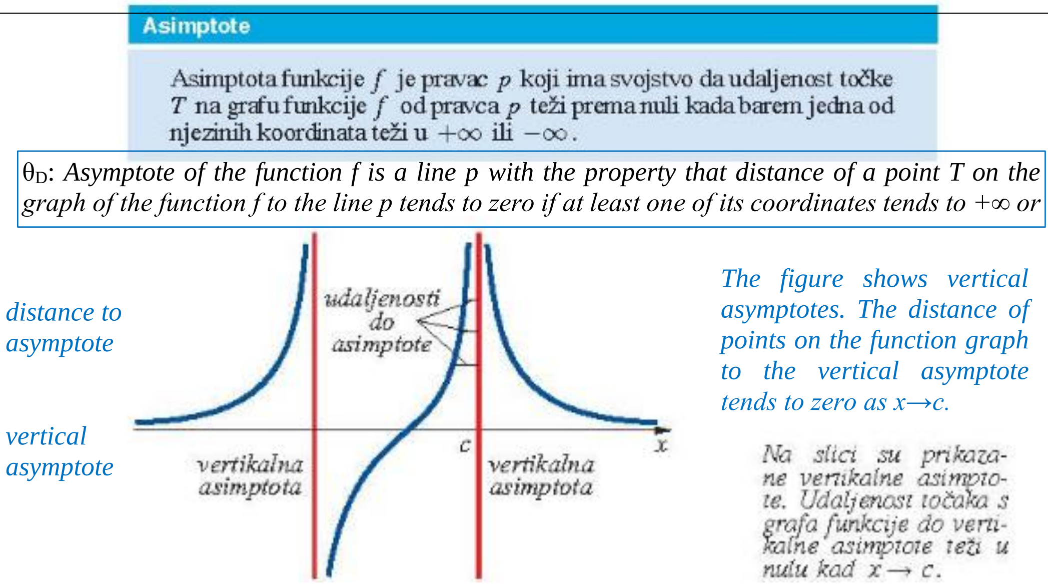
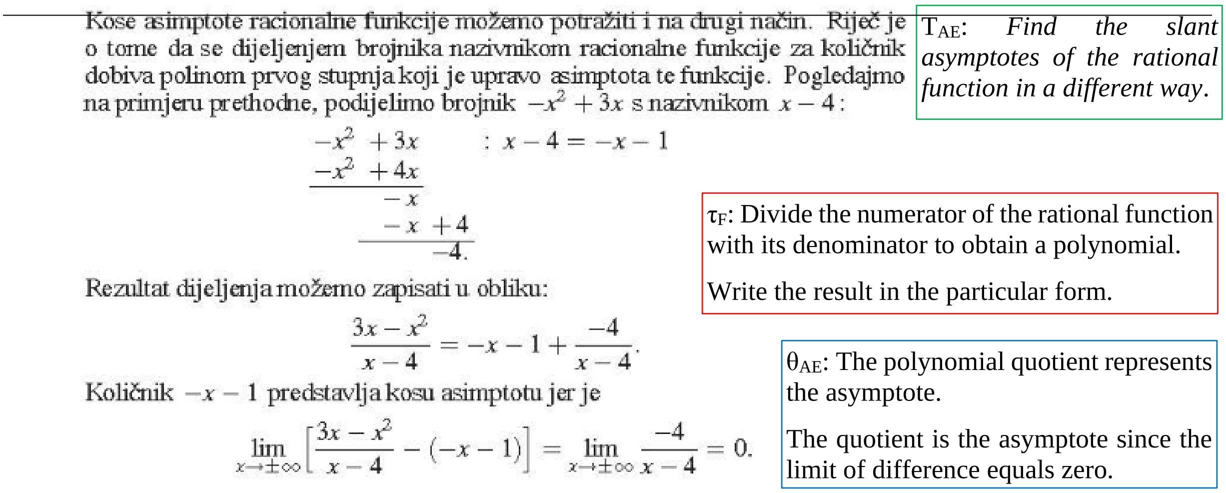


Figure 3: Distance definition of asymptote from Textbook 1

The question Q_{SA} "*How to determine the slant asymptote of a real function*?" arises naturally. It is essential for a coherent study of asymptotes for two reasons. Resolving it: (i) includes different praxeologies and (ii) provides an opportunity for increasing in complexity toward generalized (e.g. polynomial) asymptotes. Answering this question requires using both limit praxeologies MO_1 and MO_2, as well as the distance definition of asymptote (θ_D), as it produces the formulas with a limit for the coefficients of the slant asymptote of a function (part of the discourse θ_{AE}).

Example 3. Deducing the formula with limit for the slant asymptote.
According to the distance definition of asymptote, ... θ_D
the line $y = kx + l$ is a slant asymptote of the function f if the limit $\lim\limits_{x\to\infty}(f(x) - kx - l) = 0$ exists. .... θ_{AB}
Let $g(x) = f(x) - kx - l$. Then $\lim\limits_{x\to\infty} g(x) = 0$ and $\frac{f(x)}{x} - k = \frac{g(x)+l}{x}$ holds. .... MO_2
Calculate $\lim\limits_{x\to\infty}\left(\frac{f(x)}{x} - k\right) = \lim\limits_{x\to\infty}\frac{g(x)+l}{x} = 0$. .... MO_1
Hence $0 = \lim\limits_{x\to\infty}\frac{f(x)}{x} - \lim\limits_{x\to\infty} k = \lim\limits_{x\to\infty}\frac{f(x)}{x} - k$, so $k = \lim\limits_{x\to\infty}\frac{f(x)}{x}$ and $l = \lim\limits_{x\to\infty}(f(x) - kx)$. .... MO_1
Formulas for the coefficients k and l of the slant asymptote $y = kx + l$ of the function f are $k = \lim\limits_{x\to\infty}\frac{f(x)}{x}$ and $l = \lim\limits_{x\to\infty}(f(x) - kx)$. .... θ_{AE}

Yerushalmy (1997) presented an ICT supported inquiry about linear and polynomial asymptotes of rational functions justified with an informal infinitesimal discourse. Asymptotes of a rational function are determined by interpreting its formula regarding the zeros of the denominator for the vertical asymptote and the polynomial quotient of the numerator and denominator for the slant asymptote (see the example [$T_{AE}/\tau_F/\theta_{AE}$/C,A] in *Figure 4*). Calculating corresponding limits justifies the praxeology.



Kose asimptote racionalne funkcije možemo potražiti i na drugi način. Riječ je o tome da se dijeljenjem brojnika nazivnikom racionalne funkcije za količnik dobiva polinom prvog stupnja koji je upravo asimptota te funkcije. Pogledajmo na primjeru prethodne, podijelimo brojnik $-x^2+3x$ s nazivnikom $x-4$:

$$\begin{array}{l} -x^2 + 3x \quad : x-4 = -x-1 \\ \underline{-x^2 + 4x} \\ \quad -x \\ \quad \underline{-x + 4} \\ \qquad -4. \end{array}$$

Rezultat dijeljenja možemo zapisati u obliku:

$$\frac{3x-x^2}{x-4} = -x-1+\frac{-4}{x-4}.$$

Količnik $-x-1$ predstavlja kosu asimptotu jer je

$$\lim_{x\to\pm\infty}\left[\frac{3x-x^2}{x-4}-(-x-1)\right] = \lim_{x\to\pm\infty}\frac{-4}{x-4} = 0.$$

Figure 4: Determining slant asymptote of a rational function from the polynomial quotient from Textbook 1

The praxeology for finding asymptotes ([$T_{AE}/\tau_F,\tau_L/\theta_{AE}$/A,C]) is an integral part of the common graphing procedure in Calculus and the related approximation method.

Results

The textbook analysis revealed that, in Calculus, asymptotes were mentioned within two separate units: the unit about the limit of a function, and the unit about applying derivatives to graph functions. We found tasks corresponding to T_{L1}, T_{AE}, and T_G from our REM, and two additional tasks:

T_B Describe the behaviour of a (rational) function as $x\to a$ or $x\to\infty$ given its Cartesian graph.

T_{B1} Determine different types of asymptotes of a function given its Cartesian graph.

Techniques used to solve the tasks were aligned with the REM: interpretation of a function graph (τ_F) for T_{L1}, T_{AE}, T_B, and T_{B1}, algebraic manipulation and interpretation of a function formula (τ_F) for T_{AE}, algebraic calculation of limit (τ_L) for T_{L1} and T_{AE}. All analysed textbooks systematized the usual, traditional step-by-step graphing procedure (τ_G) for solving the task on graphing real functions (T_G) in the following way:

1. Examine the function – domain, breakpoints, zeros, parity, periodicity, limiting behaviour and asymptotes;
2. Examine the first derivative – stationary points, intervals of monotonicity, local extremes;
3. Examine the second derivative – intervals of convexity or concavity, inflexion points.

The discourses reported both in our REM and in all textbooks were:

θ_{LB} The connection between the equation of asymptote, *the limiting behaviour* and the limit of a function at a point or at infinity.

θ_D The *distance definition* of an asymptote, as in *Figure 3*.

θ_{AE} The *formulas with a limit* for different types of asymptotes, and the connection between the *formula of rational functions* and equations of different types of its asymptotes.

Additionally, we observed that the analysed textbooks included the following discourses:

θ_A *Recognition* of an equation of asymptote from a function graph.

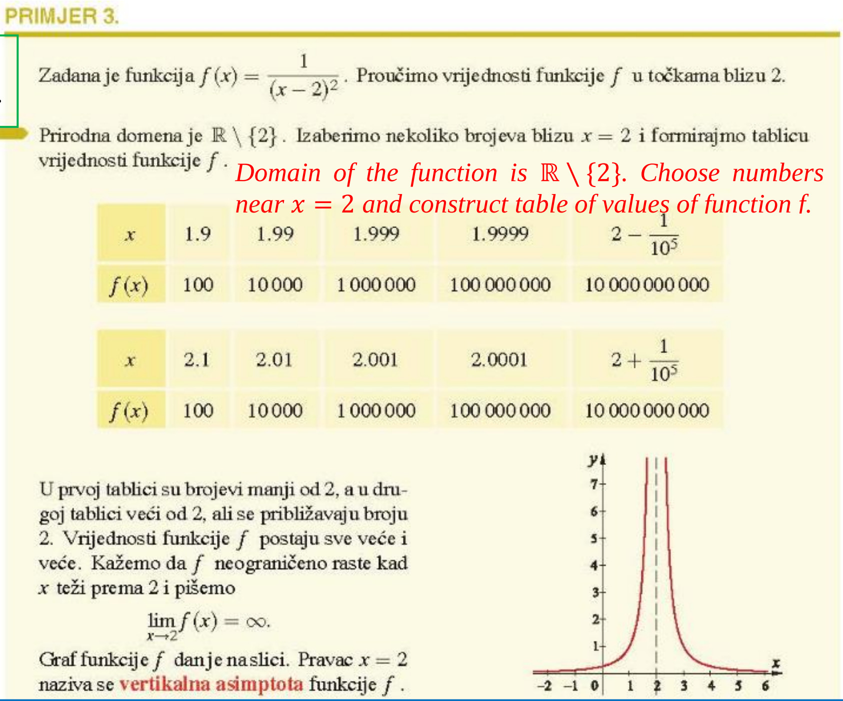
θ_D *Relative position* of a function graph and its asymptote.

Praxeologies related to asymptotes in the textbooks

The first encounter of asymptotes appeared in the section about the limit of a function, when describing function behaviour in Textbooks 2 and 4 (exploration of [$T_B/\tau_F/\theta_{LB}/C$] in *Figure 5*), and when calculating the limit of a function in Textbook 2 (working on [$T_{L1}/\tau_L,\tau_F/\theta_{LB}/C$] in *Figure 2*). Textbook 1 included the task T_B but vertical and horizontal asymptotes were introduced when separately framing[3] discourse about limiting behaviour (institutionalization of θ_{LB}), and asymptotes were recognized when calculating the limit of a function (working on [$T_{L1}/\tau_L/\theta_A/C$]).

T_B Describe the behaviour of $f(x) = \frac{1}{(x-2)^2}$ at points near 2.

τ_F Interpreting function graph and numeric values.



PRIMJER 3.

Zadana je funkcija $f(x) = \frac{1}{(x-2)^2}$. Proučimo vrijednosti funkcije f u točkama blizu 2.

Prirodna domena je $\mathbb{R} \setminus \{2\}$. Izaberimo nekoliko brojeva blizu $x = 2$ i formirajmo tablicu vrijednosti funkcije f.

x	1.9	1.99	1.999	1.9999	$2 - \frac{1}{10^5}$
$f(x)$	100	10 000	1 000 000	100 000 000	10 000 000 000

x	2.1	2.01	2.001	2.0001	$2 + \frac{1}{10^5}$
$f(x)$	100	10 000	1 000 000	100 000 000	10 000 000 000

U prvoj tablici su brojevi manji od 2, a u drugoj tablici veći od 2, ali se približavaju broju 2. Vrijednosti funkcije f postaju sve veće i veće. Kažemo da f neograničeno raste kad x teži prema 2 i pišemo

$$\lim_{x \to 2} f(x) = \infty.$$

Graf funkcije f dan je na slici. Pravac $x = 2$ naziva se vertikalna asimptota funkcije f.

Domain of the function is $\mathbb{R} \setminus \{2\}$. Choose numbers near $x = 2$ and construct table of values of function f.

In the first table, numbers are less than 2, and in the second table, numbers are greater than 2, but they are approaching number 2.

Function graph is given in the figure.

θ_{LB} The function f increases unboundedly, $\lim_{x \to 2} f(x) = \infty$, the line $x = 2$ is vertical asymptote of function f.

Figure 5: First encounter of a vertical asymptote when describing function behaviour from Textbook 4

As already mentioned, the study of asymptotes was a part of a textbook unit about using derivatives to graph functions. Textbooks 1 and 3 presented the step-by-step graphing procedure (τ_G) first, graphed functions without determining their asymptotes, and only then recognized the asymptote from the function graph (working on [$T_G/\tau_G/\theta_A/C$] in *Figure 6*). In Textbooks 2 and 4, the graphing procedure (τ_G) followed after introducing the asymptotes.

[3] We used this word to note that the content was emphasised in a visually coloured frame.

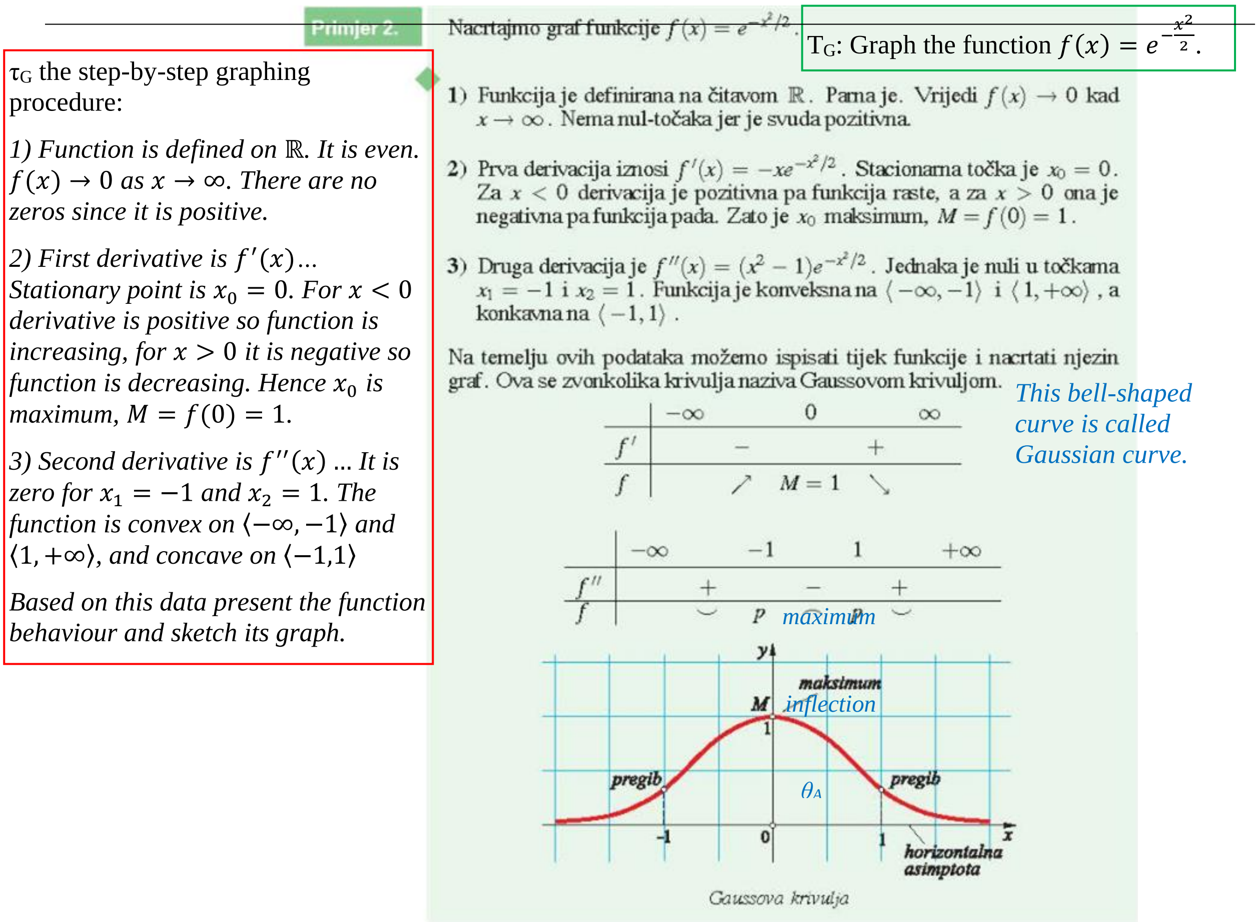


Figure 6: Recognizing a horizontal asymptote when graphing function from Textbook 1

Organization of study of asymptotes in the textbooks

The section on asymptotes started in all textbooks by acknowledging that, when graphing a function, it is useful to determine if the function has asymptotes and their equations. Recalling the types of asymptotes from the graphs of elementary functions followed, without reference to the limit of the function (exploration of [T_{B1}/τ_F/θ_A/A]).

Organization in Textbook 1.

After the aforementioned common introduction, Textbook 1 proceeded with framing the distance definition of asymptote (institutionalization of θ_D in *Figure 2*) and presenting that asymptote is vertical as $x \to a$ and slant as $x \to \infty$. The subsection *Slant asymptotes* started with deducing the formulas with a limit for the coefficients of slant asymptote, similar as in *Figure 7*. Here, $\lim_{x\to\infty}\left(\frac{f(x)}{x} - k - \frac{l}{x}\right) = 0$ follows from $\lim_{x\to\infty}(f(x) - kx - l) = 0$ by simple division with x, differently from the work in *Example 3*. The formulas were then framed (institutionalization of θ_{AE}), adding that “a horizontal asymptote is a special case of a slant asymptote”. Afterwards, asymptotes of a function were determined by calculating the appropriate limit (working on [T_{AE}/τ_L/θ_{AE}/C]).

~~Funkcija može imati i kosu asimptotu. Kosa asimptota funkcije je pravac p čija je~~ jednadžba $y = kx + l$ i čija udaljenost od grafa funkcije teži 0 kada x teži u ∞. To znači da je

A function can also have a slant asymptote. The slant asymptote of a function is a line p with equation $y = kx + l$ such that its distance from the function graph tends to 0 as x tends to ∞. That means

$$\lim_{x\to\infty}(f(x)-y)=0$$

$$\lim_{x\to\infty}(f(x)-kx-l)=0.$$

Odredimo k tako da gornju jednakost podijelimo s x.

Let us find k by dividing the above equation by x.

$$\lim_{x\to\infty}\frac{f(x)-kx-l}{x}=\lim_{x\to\infty}\frac{f(x)}{x}-\lim_{x\to\infty}\frac{kx}{x}-\lim_{x\to\infty}\frac{l}{x}=\lim_{x\to\infty}\frac{f(x)}{x}-k-0=0$$

It follows that Slijedi da je $k=\lim_{x\to\infty}\frac{f(x)}{x}$.

Now, it is Sada je $l=\lim_{x\to\infty}(f(x)-kx)$.

Figure 7: Deducing formulas with limit for the coefficients of the slant asymptote from Textbook 2

Next subsection entitled *Graphing rational functions* started with presenting discourse about the vertical asymptote of a rational function "in the zeros of the denominator (which are not zeros of the numerator)" (θ_{AE}). Recognizing asymptotes of a rational function given its Cartesian graph was complemented with the relative position discourse (θ_P), e.g. "zero of the denominator is simple … the graph comes from different sides of the vertical asymptote" (exploration of [$T_{AE}/\tau_F/\theta_A,\theta_P$/A]). The following discourses about the horizontal asymptote of a rational function were given and exemplified by calculating the limit at infinity of functions ([T_{L1}/τ_L]):

- If the degree of the numerator is less than the degree of the denominator, then the equation of horizontal asymptote is $y = 0$;
- If the degree of the numerator equals the degree of the denominator with the leading coefficients of the respective polynomials a_n and b_n, then the equation of horizontal asymptote is $y = \frac{a_n}{b_n}$.

Rational functions were graphed using the step-by-step graphing procedure including asymptotes (working on [T_G/τ_G] and [$T_{AE}/\tau_F,\tau_L/\theta_{AE}$/A,C]). Additionally, determining the slant asymptote of a rational function from the polynomial quotient was presented (*Figure 4*).

Organization in Textbook 2

After the introduction as above, Textbook 2 proceeded with framing the distance definition of asymptote (institutionalization of θ_D) and presenting the discourse that if $x \to a$ and $f(x) \to \infty$, then the asymptote is vertical and if $x \to \infty$ and $f(x) \to a$, then the asymptote is horizontal. The formulas with a limit for the vertical asymptote were framed (institutionalization of θ_{AE}), adding that vertical asymptotes can exist only in endpoints of its domain. Determining the vertical asymptotes of a rational function by calculating the appropriate limit was complemented with the discourse about its connection to a rational function formula (θ_{AE}) and the relative position discourse (θ_P) (exploration of [$T_{AE}/\tau_L/\theta_{AE},\theta_P$/A,C]). Analogous organization for the case of horizontal asymptote of a rational function followed. This textbook posed two questions. The first one stated here was Q_{AE1} "When the equation of a horizontal asymptote is given by $y = 0$ and when is it $y = c, c \neq 0$? Can you determine the value of c without calculating the limit?". Textbook continued with deducing formulas with a limit for the coefficients of slant asymptote (*Figure 7*), framing the formulas (institutionalization of θ_{AE}) and determining asymptotes of rational functions by calculating appropriate limit and from the zeros of the denominator (working on [$T_{AE}/\tau_F,\tau_L/\theta_{AE}$/A,C]). It was noted that, according to the examples, "the horizontal asymptote needs not to be calculated … its equation follows from the procedure of calculating slant asymptote if $k = 0$". The second posed question was Q_{AE2} "When will the rational function have a slant asymptote? Describe how it depends on the degree of the polynomials in the numerator and the denominator."

In this textbook, the section about using differential calculus (and determining asymptotes) for graphing function followed.

Organization in Textbook 3

Textbook 3 started with framing the distance definition of asymptote (institutionalization of θ_D) and proceeded with describing function behaviour near a breakpoint and at infinity (exploration of $[T_B/\tau_F/\theta_{LB}/C]$). The formulas with a limit for the vertical and horizontal asymptote were framed (institutionalization of θ_{AE}), adding the discourse about vertical asymptotes in a breakpoint of a function and in a zero of the denominator of a rational function. Two tasks followed – determining vertical asymptotes of a rational function from the zeros of the denominator of its formula and recognizing the vertical and slant asymptote of a function given its Cartesian graph. The latter task motivated the "more complex calculation of slant asymptotes". The formulas with a limit for the coefficients of slant asymptote were framed (institutionalization of θ_{AE}), adding that a horizontal asymptote is a special case of a slant asymptote. Afterwards, asymptotes of a function were determined by calculating the appropriate limit and from the zeros of the denominator (working on $[T_{AE}/\tau_F,\tau_L/\theta_{AE}/A,C]$). As a side note, determining the slant asymptote of a rational function from the polynomial quotient and a particularly written equality was presented (exploration of $[T_{AE}/\tau_F,\tau_L/\theta_{AE}/A,C]$ in *Figure 8*).

In this textbook, the previously presented step-by-step graphing procedure was now complemented with determining asymptotes, and tasks were solved (working on $[T_G/\tau_G]$ and $[T_{AE}/\tau_F,\tau_L/\theta_{AE}/A,C]$).

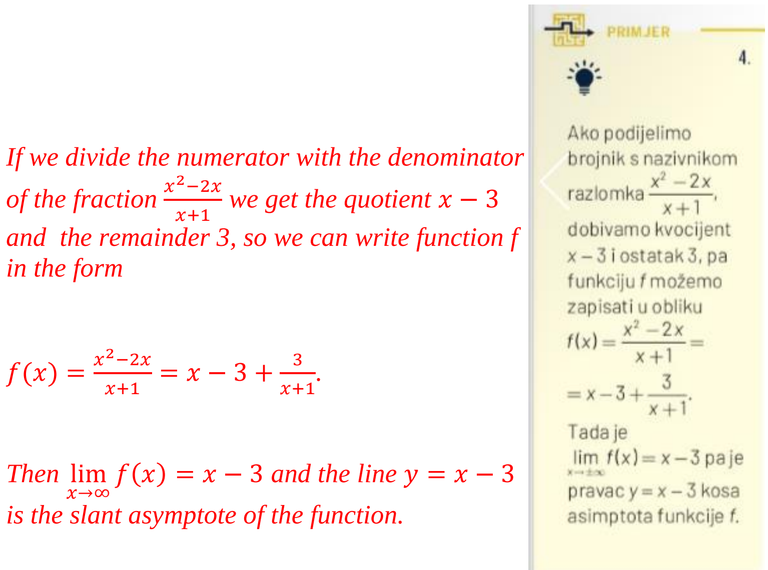


Figure 8: Determining the slant asymptote of a rational function from the polynomial quotient with particularly written equality from Textbook 3

Organization in Textbook 4

After the above-mentioned introduction common to all four textbooks, Textbook 4 proceeded with framing the formulas with a limit for vertical asymptote (institutionalization of θ_{AE}) and determining vertical asymptotes of a function by calculating the appropriate limit (working on [$T_{AE}/\tau_L/\theta_{AE}/C$]). The distance definition of the right-hand and the left-hand slant asymptote $y = kx + l$ and that $k = 0$ results with a horizontal asymptote, was framed (institutionalization of θ_D). The discourse was supported with a figure similar to *Figure 3*, and formulas with a limit for the coefficients of slant asymptotes were given. Determining the slant asymptote of a function by calculating the appropriate limit was complemented with relative position discourse: "rational function always has the same right-hand and left-hand slant asymptote" (working on [$T_{AE}/\tau_L/\theta_{AE}/C$]).

In this textbook, the section about function behaviour and its use for graphing function followed.

Comparison of the analysed textbooks

The analysed textbooks implemented the praxeologies in a different way (Table 1). Exploring asymptotes when describing function limiting behaviour or calculating limit, was coherent with our REM. Our analysis further showed that determining asymptotes was an integral part of working on graphing function, though the chronology of studying asymptotes and presenting the step-by-step graphing procedure varied in textbooks. All analysed textbooks explored different types of asymptotes when recalling the graphs of elementary functions but did not relate their asymptotes and limiting behaviour. The textbooks included working on determining slant asymptotes by calculating formulas with a limit for the coefficients of a slant asymptote − Textbooks 1 and 2 deduced the formulas using the algebra of limits and Textbooks 3 and 4 presented them ready.

REM – study moments	*Textbook 1*	*Textbook 2*	*Textbook 3*	*Textbook 4*
1st encounter in [$T_{L1}/\tau/\theta_{LB}/C$]	θ_{LB} isolated from [T_B/τ_F]	[$T_B/\tau_F/\theta_{LB}/C$]	[$T_{L1}/\tau/\theta_{LB}/C$]	[$T_B/\tau_F/\theta_{LB}/C$]
Exploring [$T_{L2}/\tau_L/\theta_{AB}/C$]	[$T_{L1}/\tau_L/\theta_A/C$] [$T_{L1},T_{L2}/\tau_L$]	[$T_{L1},T_{L2}/\tau_L$]	[$T_{L1},T_{L2}/\tau_L$]	[$T_{L1},T_{L2}/\tau_L$]
Raison d'étre in T_A and T_G	Useful in graphing; Exploration of [$T_{B1}/\tau_F/\theta_A/A$]	Useful in graphing; Exploration of [$T_{B1}/\tau_F/\theta_A/A$]	Useful in graphing; Exploration of [$T_{B1}/\tau_F/\theta_A/A$]	Useful in graphing; Exploration of [$T_{B1}/\tau_F/\theta_A/A$]
Definition θ_D for Q_{SA}	Institutionalize θ_D	Institutionalize θ_D	Institutionalize θ_D	Institutionalize θ_D
Construction of formulas with limit	Deducing θ_{AE}; Working on [$T_{AE}/\tau_L/\theta_{AE}/C$]	Deducing θ_{AE}; Working on [$T_{AE}/\tau_L/\theta_{AE},\theta_P/A,C$]	Exploration of [$T_B/\tau_F/\theta_{LB}/C$]; Institutionalize θ_{AE}; Working on [$T_{AE}/\tau_F,\tau_L/\theta_{AE}/A,C$]	Institutionalize θ_{AE}; Working on [$T_{AE}/\tau_L/\theta_{AE},\theta_P/C$]
Construction of polynomial quotient	Construction of θ_{AE}; Exploration of [$T_{AE}/\tau_F,\tau_L/\theta_{AE},\theta_P/A$]	Stating questions Q_{AE1} and Q_{AE2}	Exploration of [$T_{AE}/\tau_F,\tau_L/\theta_{AE}/A,C$]	None
Working on T_G with T_{AE}	1. T_G without T_{AE}; 2. Study T_{AE}; 3. T_{AE} is part of τ_G	1. Study T_{AE}; 2. T_{AE} is part of τ_G	1. T_G without T_{AE}; 2. Study T_{AE}; 3. T_{AE} is part of τ_G	1. Study T_{AE}; 2. T_{AE} is part of τ_G
Working on T_A with T_{AE}	Logistic function	None	None	None

Table 1: Summary of organization of study of asymptotes in REM and textbooks

Regarding the asymptotes of rational functions, Textbook 1 included an extensive discussion, with exploration, of all types of asymptotes, Textbook 2 discussed vertical asymptotes from the zeros of the denominator of a function formula and posed questions on investigating two other types of asymptotes, and Textbook 3 discussed vertical asymptote and explored the slant asymptote. Given a rational function $f(x) = p(x) + \frac{r(x)}{q(x)}$, Textbook 1 evaluated $\lim_{x\to\infty} f(x) - p(x) = 0$ and Textbook 3 wrote $\lim_{x\to\infty} f(x) = p(x)$, which is an imprecise way of describing the asymptotic relation of two functions as equal at infinity.

We found no occurrence of exploring the asymptotic behaviour of two functions and approximating one of them with the other one for arguments of large magnitude, even though the limits of difference and quotient were calculated. Instead, the focus was on the relative position of the function graph and its asymptotes. Discourses of asymptotes were mainly institutionalized − presented ready in visually emphasised frames. They were applied to work on determining asymptotes. Textbook 1 was the only textbook that included an example of modelling with logistic function with a question potentially related to approximation (*Figure 9*).

Zadatak 2. Novi virus širi se bez kontrole u pogonu u kojem radi 500 osoba. Broj zaraženih ravna se po formuli

$$f(t) = \frac{500}{1 + 499e^{-0.8t}},$$

vrijeme t mjeri se u danima. Dakle, u trenutku $t = 0$ zaražena je samo jedna osoba.

1) Koliki će broj zaraženih biti nakon 5 dana?

2) U kojem će danu biti najveći broj novozaraženih osoba? Koliki je taj broj?

3) U kojem će trenutku biti zaražene sve osobe?

A new virus is spreading without control in the facility with 500 workers. The number of once infected persons is calculated from the formula

$$f(t) = \frac{500}{1 + 499e^{-0.8t}}$$

time t is measured in days.

Hence, in the moment $t = 0$ only one person is infected.

1) What is the number of once infected person after 5 days?

2) Which day will the number of once infected persons be the largest?

3) At which moment will all the persons be once infected?

Figure 9: Example of modelling with logistic function from Textbook 1

Discussion

The analysis showed that the organization of analysed textbooks corresponds to the paradigm of visiting monuments and praxeologies were confined within two separate textbook units. Study of asymptotes revolved around three didactic moments: an exploration of tasks, institutionalization of discourse and working on technique.

First confinement appeared regarding the encounter with horizontal and vertical asymptotes in the exploration of the limit of a function graphically. Previous editions of the textbooks did not acknowledge the connection between the equation of asymptotes and the limiting behaviour of function (Čižmešija et al., 2017). In the new textbooks, there was an intention to connect two pieces of knowledge − asymptotes of function graph and limit of a function at the breakpoint and at infinity. However, this connection was omitted when recalling the asymptotes of elementary functions.

Second confinement is related to the procedure of graphing functions, which was institutionalized and worked in the textbook unit about applying derivatives. The *raison d'être* of asymptotes was announcing their use in graphing functions. The definition of asymptote, formulas with a limit for different types of asymptotes and the connection of their equations with the formula of a rational function were institutionalized and worked as a part of the graphing procedure. Students were not encouraged to analyse functions and their asymptotes using ICT. Hence, there were no opportunities to coherently study asymptotic and limiting behaviour.

Conclusion

Following traditional study in upper secondary mathematics education in Croatia, the study of asymptotes is restricted to graphing real functions using content from Calculus. Though using ICT is encouraged in the curriculum, working on the step-by-step paper and pencil graphing procedure subsists. Its *raison d'être* is a straightforward application of derivatives. Thus, the formulas with a limit for different types of asymptotes are mainly institutionalized and worked on as a given.

Some of the analysed textbooks explore functions relevant to modelling, but these have only horizontal and/or vertical asymptote, and their asymptotic behaviour remained unattended. Questions that provoke inquiring about the asymptotic behaviour of functions, including those functions with a different behaviour at $+\infty$ and $-\infty$, or at different sides of a vertical asymptote, and interpreting asymptotes as easy-to-calculate approximations of functions for arguments of large magnitude of relevant functions, should be considered. Exploring the asymptotic behaviour of a function with a slant asymptote while working on evaluating limit or graphing functions would motivate the construction of formulas with a limit for the coefficient of slant asymptote and generalized (e.g. polynomial) asymptotic relations. Such a study organized around different didactic moments could emphasize *raison d'être* and contribute to the completion of praxeologies related to asymptotes.

References

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. *Educational Studies in Mathematics*, *59*(1–3), 235–268. https://doi.org/10.1007/s10649-005-5889-z

Bergé, A. (2008). The completeness property of the set of real numbers in the transition from calculus to analysis. *Educational Studies in Mathematics, 67*(3), 217–235. https://doi.org/10.1007/s10649-007-9101-5

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique [Analysis of teaching practices in anthropological theory of didactics]. *Recherches en didactique des mathématiques, 19*(2), 221–266.

Čižmešija, A., Katalenić, A., & Milin Šipuš, Ž. (2017). Asymptote as a body of knowledge to be taught in textbooks for Croatian secondary education. In Z. Kolar-Begović, R. Kolar-Šuper, & L. Jukić-Matić (Eds.), *Mathematics education as a science and a profession* (pp. 127–147). Element.

Corica, A. R., & Otero, M. R. (2012). Estudio sobre las praxeologías que se proponen estudiaren un curso universitario de cálculo [Study of the praxeologies proposed to study in a university calculus course]. *Bolema: Boletim de Educação Matemática, 26*(42b), 459–482. https://doi.org/10.1590/S0103-636X2012000200004

Dahl, B. (2017). First-year non-STEM majors' use of definitions to solve calculus tasks: Benefits of using concept image over concept definition? *International Journal of Science and Mathematics Education*, *15*(7), 1303–1322. https://doi.org/10.1007/s10763-016-9751-9

Dobbs, D. E. (2011). Polynomial asymptotes of the second kind. *International Journal of Mathematical Education in Science and Technology*, *42*(2), 276–282.

Domović, V., Glasnović Gracin, D., & Jurčec, L. (2012). Uporaba matematičkih udžbenika s obzirom na inicijalno obrazovanje učitelja [Use of mathematics textbooks with regard to initial teacher education]. *Sociologija i prostor : časopis za istraživanje prostornoga i sociokulturnog razvoja*, *50*(2 (193)), 237–256. https://doi.org/10.5673/sip.50.2.6

García, F. J., Pérez, J. G., Higueras, L. R., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *ZDM Mathematics Education*, *38*(3), 226–246. https://doi.org/10.1007/BF02652807

González-Martín, A. S., Giraldo, V., & Souto, A. M. (2013). The introduction of real numbers in secondary education: An institutional analysis of textbooks. *Research in Mathematics Education*, *15*(3), 230–248. https://doi.org/10.1080/14794802.2013.803778

Katalenić, A., Čižmešija, A., & Milin Šipuš, Ž. (2022). Prospective mathematics teachers' knowledge of asymptotes and asymptotic behaviour in calculus. *International Journal of Science and Mathematics Education*, 1–28. https://doi.org/10.1007/s10763-022-10247-9

Katalenić, A., Milin Šipuš, Ž., & Čižmešija, A. (2020). Asymptotes and asymptotic behaviour in graphing functions and curves: An analysis of the Croatian upper secondary education within the anthropological theory of the didactic. *International Journal of Science and Mathematics Education*, *18*(6), 1185–1205. https://doi.org/10.1007/s10763-019-10020-5

Kidron, I. (2011). Constructing knowledge about the notion of limit in the definition of the horizontal asymptote. *International Journal of Science and Mathematics Education*, *9*(6), 1261–1279. https://doi.org/10.1007/s10763-010-9258-8

Mudaly, V., & Mpofu, S. (2019). Learners' views on asymptotes of a hyperbola and exponential function: A commognitive approach. *Problems of Education in the 21st Century*, *77*(6), 734–744. https://doi.org/10.33225/pec/19.77.734

Serrano Martínez, L., Bosch, M., & Gascón, J. (2020). An overview of "bridging courses" from the ATD perspective. *Educação Matemática Pesquisa : Revista do Programa de Estudos Pós-Graduados em Educação Matemática*, *22*(4), 579–592. https://doi.org/10.23925/1983-3156.2020v22i4p579-592

Swinyard, C., & Larsen, S. (2012). Coming to understand the formal definition of limit: Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education*, *43*(4), 465–493. https://doi.org/10.5951/jresematheduc.43.4.0465

Yerushalmy, M. (1997). Reaching the unreachable: Technology and the semantics of asymptotes. *International Journal of Computers for Mathematical Learning*, *2*(1), 1–25. https://doi.org/10.1023/A:1009769330476

Teachers’ perspective of mathematical analysis with Q&A maps

Matija Bašić[1] and Željka Milin Šipuš[1]

[1]University of Zagreb, Faculty of Science, Department of Mathematics, Zagreb, Croatia; mbasic@math.hr and milin@math.hr

Abstract. We study the perspective of teachers with experience in design and implementation of inquiry based mathematics teaching. We focus on the ways the selected teachers consider mathematical analysis and the role that tools from ATD might have in supporting them in their own inquiry on teaching. In particular, we study the potential of the Q&A maps and the various dialectics. Our findings show that after involvement in inquiry oriented projects, the teachers value collaboration with colleagues and researchers, and shift their style of teaching towards the paradigm of questioning the world.

Keywords: Question and answer maps, Anthropological theory of didactic, Inquiry based teaching.

Introduction

In the recent two decades there has been a growing number of initiatives promoting inquiry based teaching, in science and in mathematics, around the world. In Europe, these initiatives have taken place also in form of international (EU) projects, such as Primas, Fibonacci, Mascil and other. The projects bring together teachers and researchers in mathematics education in a common goal to find and articulate new way of collaboration and create teaching materials that support “active learning”. We focus on the case of two connected projects: Mathematics Education – Relevant, Interesting and Applicable (MERIA, 2016-2019) and Teachers' Inquiry in Mathematics Education (TIME, 2019-2022). In these projects teachers from Croatia, Denmark, the Netherlands and Slovenia have been exposed to tenets of Theory of Didactic Situations (TDS) and Realistic Mathematics Education (RME), as frameworks for designing and implementing inquiry-based teaching, as well as Lesson Study (LS), as a collection of principles guiding teachers to improve their practice through planning, observing and reflecting on lessons in small communities.

As researchers on these projects, we have closely followed groups of Croatian teachers and the changes in their practice over the course of five years. This experience has shown us that even the very experienced and enthusiastic teachers with firm mathematical and didactical knowledge find it challenging to read research papers on mathematics education and to design inquiry based lesson as evidenced in the literature (Florensa et al., 2015; Bosch, 2018). Similar findings can be found in research, therefore contributing to the thesis that teaching is a semi-profession needing “less specialised and less highly developed body of knowledge and skills, and less emphasis on theoretical and conceptual bases for practice” (Guerriero, 2017, p.23). This is related to our observation that, in design teachers rely on their experience and even when they use tools, they use them on an intuitive level, often without carrying out their actions with scientific precision and purpose.

In this paper our intention is to use Anthropological Theory of Didactics (ATD) as a framework that provides the language to study these phenomena and the tools that might be suitable to support teachers in their own inquiry on teaching. ATD concretizes the idea of inquiry based teaching through the teaching paradigm of “questioning the world”, contrasted with the more traditional approach of "visiting the monuments". Instead of showing "monumental" theorems of mathematics to students, teachers are invited to guide the students in answering more open questions that motivate emergence of mathematical concepts. These activities are organized under the name of Study and Research Paths (SRPs) introduced by Chevallard (2006, 2015) and also documented in a series of papers (e.g. Winsløw, et al., 2013; Barquero et al., 2016; Florensa, et al., 2019). Following up on this research, we aim to gather the attitudes on lesson design for inquiry based mathematics teaching (IBMT) developed by experienced teachers and explore which tools from ATD might be suitable to further support them in the design process, and by this, also in their profession.

Theoretical framework and research question

According to ATD, mathematical knowledge and works are modelled by praxeologies (mathematical organizations) which consist of a practice block ("know how") that includes types of tasks and techniques to carry them out, and a logos block ("know why") of technologies and theories that describe and justify the techniques.

With respect to mathematical instruction, the teaching format of Study and Research Paths (SRPs) strives to "dialectically combine 'inquiry' with 'transmission', and thus move mathematical instruction from the paradigm of 'visiting the monuments' towards 'questioning the world" (Barquero et al., 2016, p. 341). SRP-based teaching is organized on students' queries of open questions that starts with a generating question, and further, in search of a definite answer, encourages new derived questions and their answers. The process goes through phases of "research" in which students inquire or solve related (sub)problems, and "studies" where students seek and refer to relevant existing knowledge. The "study" component brings rational and research-like perspective towards existing knowledge, as Winsløw et al. (2013) state: "Human societies cannot afford to discard the knowledge accumulated by previous generations, or to require that every generation reconstruct it anew." (p. 269).

Thus, posing questions is essential in the construction of knowledge, and ATD further proposes an explicit tool for teachers and students to manage the process and describe the knowledge in form of question-answer maps (Q&A maps), as a tree-like diagrams that depict a possible, usually non-linear, trajectory of the process (Winsløw et al., 2013). Its use may have different intertwined purposes: to present the epistemological considerations, to model the inquiry process and to support "collaboration and communication with and among teachers" (ibid.).

Q&A maps can be understood in terms of dialectics which are another tool for analysing the development of SRPs offered by ATD. The dialectic refers to "two poles of opposing action within the studying process which are not dual, on the other hand: one action calls the other and vice versa" (Gazzola et al., 2019, p. 492). Chevallard (2013) describes nine dialectics that can occur in the development of SRP: question and answer dialectic (also called dialectic of the study and the research), dialectic of media-milieu, of the individual and the collective, of the skydiver and the truffles, of subject and out-of-subject, of black boxes and clear boxes, of reading and writing, of the analysis-synthesis, praxeological and didactic, and finally of the dissemination and the reception.

A crucial underlying construct of Q&A maps is the inclusion of explicit epistemological questioning in them, which makes SRP activities different that other problem-based learning approaches

(Florensa et al., 2019). It comprises praxeological (mathematical) analysis of the underlying knowledge as part of the SRP development. This is the focus of our study – to analyse work of teachers involved in the project TIME regarding this part of the lesson design. We pose the following questions: *How do experienced teachers (with a few or more years of experience with IBMT) approach mathematical analysis? How do they understand "mathematical analysis" and which procedures they follow? Which tools of ATD can be observed to be used in their practice? What potential for the use of the tool Q&A map they appreciate after a first encounter?*

Context of the study

In this study we focus on the experience and attitudes of six teachers involved in the project TIME. Project TIME is organized into two intertwined parts. One part is dedicated to establishing teams of mathematics teachers in the partnering school and engaging them in Lesson study activities. Second part is dedicated to design principles and supports teachers in design of lessons. To engage teams in LS, some teachers were selected as team leaders and introduced to principles of LS through international workshops led by experts on LS in the project. After that, subsequent local workshops were held for other members of the team.

The project relies on the previous project MERIA. In MERIA, three of the six teachers were involved in all aspects of the project, which means they were attending many meetings to discuss the theoretical frameworks and new designs, they have implemented and improved inquiry-based lessons and led workshops for other teachers. The other three teachers attended a sequence of workshops at the end of the project in which the methodology and designed scenarios were presented.

During the project TIME, the teams have designed new scenarios and hence experienced the design process as well. They were supported through (online) workshops on design principles based on the principles of TDS and RME as chosen frameworks for designing and implementing inquiry-based teaching in the projects MERIA and TIME. Through these workshops teachers gained more focus on the formulation of goals of the lesson and the use of language of design principles. Examples of the activities were constructing an iceberg model for concept formalization (from RME), formulating students' difficulties, or formulating new design principles (as a piece of advice). The design process was described in terms of the phases of LS (study, planning, observation, reflection), which can be considered as the structural support to teachers in organizing the design and its implementation. In this way, we may view teachers as experienced in inquiry based teaching design, although their designs have not strictly followed the theoretical principles and we can only say that the designs were inspired by the principles to a certain, personal, extent. None of the six teachers that participated in this study have had prior experience with ATD.

Methodology

We gather data in two ways: by semi-structured interviews and by organizing a workshop with a semi-structured reflection session. All activities were led by the authors.

Teachers are interviewed before the workshop in the form of a semi-structured interview (Cohen et al., 2011). Each of the six teachers was interviewed separately and the interview lasted for about 15 minutes. The teachers are asked the following questions:

1. How was your teaching practice changed by project TIME (and project MERIA)? What did you learn?
2. Can you say more about the design in the project?
3. How do you perform mathematical analysis when you want to design a new lesson? Do you follow any procedures?

If some of the questions are not clear, the authors would rephrase the question keeping in mind the order of the questions, e.g. mathematical analysis was not mentioned before the question about the design and the design was not mentioned before the first question.

The interviews were transcribed and natural units of meaning were recognized. For the first question, we related the answers to dialectics. For the second question we listed the adjectives which teachers used to describe design. For the third question, we coded the procedures by short descriptions and singled out one quote that that both authors interpreted as relevant to the research questions. Findings are shown summarized and commented. Based on the interviews, we were aware of the very strong influence of the collaborative aspect emphasized by the teachers and have decided to observe the team dynamics during the workshop more carefully.

After the interviews, teachers were invited to a two-hour workshop. The goal of the workshop was to test a new tool (Q&A map) and reflect on its use in design. The workshop started with a brief introduction of the paradigm of questioning the world from ATD linked immediately to the question-and-answer map. After presenting a few examples from the literature (Winsløw et al., 2013), participants were given asked to construct a Q&A map that would support them in designing a lesson based on the topic of linear programming. A typical textbook task presented in Figure 1 served was shown as a source of inspiration for the discussion, but the teachers were instructed that they have the freedom to design the Q&A map in any way they find suitable.

The company produces ice cream with two flavours: vanilla flavour and coffee flavour. To produce one box of half-litre ice cream, you need:

One box of half-litre ice cream	Number of eggs	Decilitres (dl) of cream
Vanilla flavour	2	3
Coffee flavour	1	3

There are 500 eggs and 900 dl of cream in the warehouse. The profit from the sale of vanilla ice cream is 3 kunas per a box, and from coffee ice cream 2 kunas per a box. How many boxes of each type do you need to produce for maximum profit?

Figure 1: A typical textbook task for linear programming

The participants were divided into two groups of three. Participants from each group work in the same school and have collaborated intensively over the years. The participants were explicitly allowed to use any media they find useful. The group discussion was observed by the authors and lasted for about 60 minutes. After working on the map, teachers were invited to a semi-structured discussion led by the authors based on the following questions:

1. What did you consider in your analysis of the problem?
2. Which mathematical aspects of the solution you found relevant and interesting?
3. What did you rely on in your study?
4. Have you constructed the Q&A map?
5. Do you feel that you have reached a final form of the Q&A map, one that you are satisfied with?
6. What is your experience with the Q&A map? Did this workshop support you in the challenging aspect of the design?
7. Would your analysis be significantly different if you did not work in groups, but on your own?

The reflection session was recorded and transcribed. We present the results in the form of short quotations. This is followed by our discussion and interpretation of the results as relating to our main research questions.

Results of the interview

For the first question, the answers of all participants have been quite similar. We have identified five main topics, which we relate to dialectics. Figure 2 shows which teacher has provided answers related to which topic, abbreviated by the first letters of the dialectic.

The individual and the collaborative (IC). The participants value the collaboration in their team as a big change for their school and them personally. One participant emphasizes the importance of sharing responsibility for the design and implementation of the lesson. Two participants emphasize the quality of conversations. Exchange of ideas is mentioned, especially meeting colleagues from other countries and observing lessons in a different educational system.

The skydiver and the truffles (ST). The participants value the collaboration with researchers working at university and the influence of research on their teaching because of the wider perspective. One teacher mentions that this is not new to her from the project, but that research gives confirmation of the ideas that are intuitively used.

Questions and answers (QA). The participants value the quality of questions being raised. Now they question their teaching from more aspects, thinking about the students, the goals and why they are doing what they do.

The media and the milieu (MM). All participants engage in the study of various *resources*. They use official textbooks, foreign and old textbooks, the internet, lecture notes and other materials they used as students of didactics of mathematics. Some of them mention looking into the *history* of mathematics and reading professional and/or *research papers*. Almost all participants express that they rely on their *experience* (intuition). One participant mention *colleagues* as a resource.

The analysis-synthesis, praxeological and didactic (AS). The participants start their design by setting the goal first and then thinking about the activities. This is an important change because they used to start with activity and/or content. Now they think about the students' difficulties and needs more. The participants give examples of tools (e.g. template based on phases from TDS) making their orchestration of the inquiry-based lessons more structured in order to make students' inquiries more open.

	Teacher 1	Teacher 2	Teacher 3	Teacher 4	Teacher 5	Teacher 6
PROJECT	IC, MM Focus on argumentation	IC, ST, MM, AS Long-term change	IC, QA, MM, AS Different viewpoint	IC, ST, MM, AS Shared responsibility	IC, AS Structure makes study easier	IC, ST, QA Conversation is important, education is complex
DESIGN	New, time-consuming, hard to follow the given form	Too theoretical, intuitive principles and new terminology	Takes time, need help Overcame resistance to theory	Complex, many aspects, still learning	Easier in collaboration, change of paradigm	Long process of learning, intuitive principles and new terminology
ANALYSIS	Resources, textbook criticism, own experience, standard structure "I start from the goals and possible student difficulties"	Resources, textbook criticism, own experience "No conscious structure: I start with the internet and old notes"	Resources (colleagues), textbook criticism, standard structure, iceberg model "I try out and improve the design"	Resources, own experience, TDS structure "We need a bridge to research and practical advice"	Resources, textbook criticism, own experience, TDS structure "Students change, so we adapt and change, too"	Resources (foreign textbooks, history), iceberg model "I ask myself what my students need to learn, what do I need to guide them"

Figure 2: Summary of the interviews with the teachers

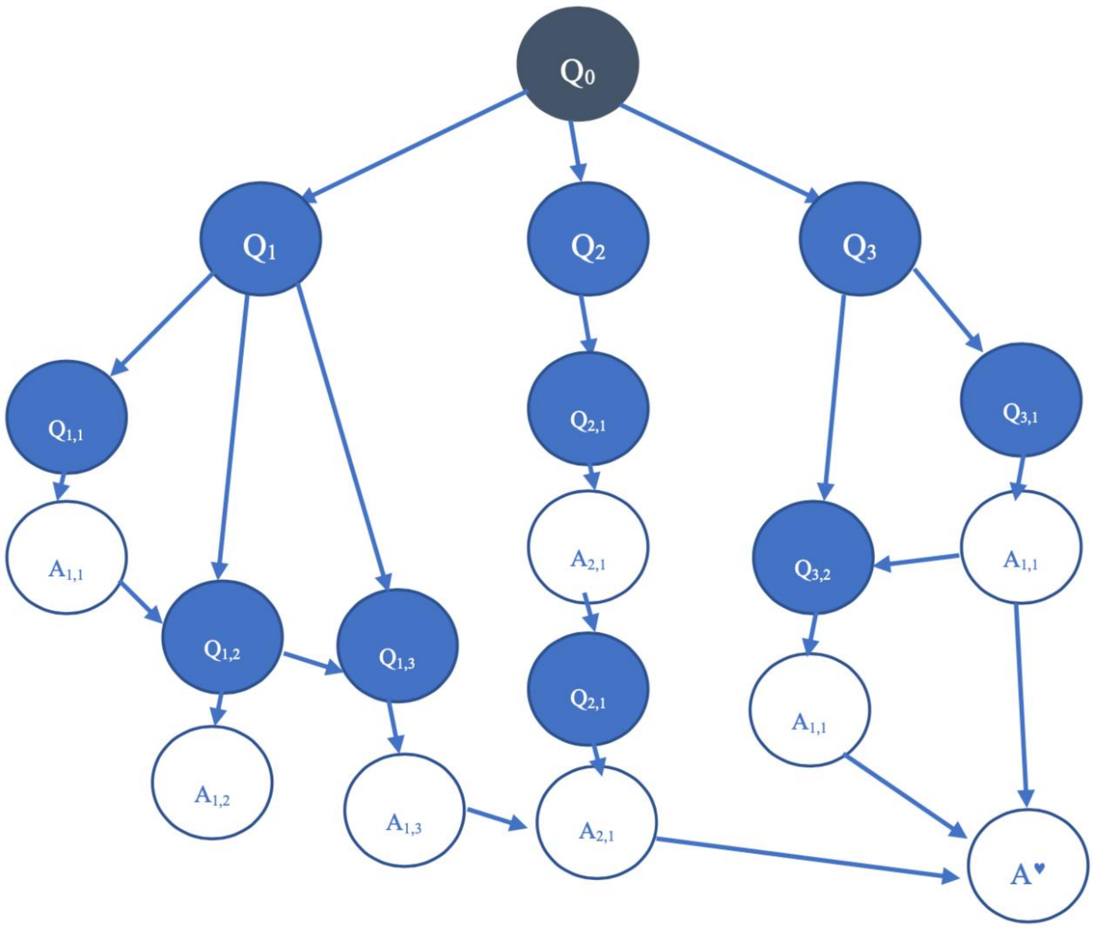
The main conclusion of the interpretation of answers to second question is that *design is challenging*. The participants express that design is difficult for them, this kind of activity is new in their practice, and they are *still learning*. Design is *time-consuming* when done properly. The theoretical underpinnings and use of principles based on research seems to be in line with their intuition about teaching, but they find it *hard to use the terminology* used by researchers. One participant says, "we need a bridge". Another participant first recalls the resistance that she had towards using theory and suggested templates, but now wants to support other teachers in accepting them.

As an answer to the third question all participant discussed the use of resources. Without any prompts many expressed *textbook criticism*. The participants express their dissatisfaction with available textbooks. Sometimes they do not agree with the style of presenting the content, but they express severe criticism when the order of the presented material is not logical. They say "if I do not understand immediately when reading, my students will also have a problem using the textbook".

When asked about the mathematical analysis itself or "what do you do after consulting the resources", all participants describe the process of writing the didactic scenario. Some participants follow a certain *"standard" structure* for their lessons. It is the structure used also in training the prospective teachers. The lesson starts with a motivational example, after which there are more activities with increasing complexity, depending on the type of a lesson (introducing new concepts or deepening the knowledge). One participant emphasizes the structure introduced in the project *based on TDS*. Two participants explicitly mention the *iceberg model* (from RME**)** as a tool they have learned to use. They have learned how to use this tool in the project workshop and see it as supporting the study of the progressive formalization of a concept.

A Q&A map

In this section we present one possible way to obtain a Q&A map which we use as our epistemological reference. The map has been constructed by the authors and presents steps in the solution to the task from Figure 1.



Q_0. What is the maximal possible profit with the given quantity constraints?

Q_1. How to describe/represent the quantity constraints?

$Q_{1,1}$. Which quantities are possible?

$A_{1,1}$. We can write pairs of numbers, in a table.

$Q_{1,2}$. Which relations between possible quantities in general?

$A_{1,2}$. With introduced variable, we have inequalities $2x + y \leq 500$, $3x + 3y \leq 900$.

$Q_{1,3}$. How to represent quantity constraints graphically?

$A_{1,3}$. We can draw the xy-plot representing all allowed pairs of quantities. Graph.

Q_2. Can we guess the solution?

$Q_{2,1}$. How to justify the solution?

$A_{2,1}$. We find the pair (200, 100) a optimal either by trial and error or by solving the system of equations $2x + y = 500, 3x + 3y = 900$.

$Q_{2,1,1}$. Why is the profit maximal for the pair in which all quantities are used?

$A_{2,1,1}$. We consider the family of parallel lines $P = 3x + 2y$ and P is maximal for the line passing through the vertex of the allowed area.

Q_3. How to describe/represent the profit?

$Q_{3,1}$. How to maximize a function of two variables?

$A_{3,1}$. More information needed from resources. The function is linear, has no global minimum or maximum, so the constraints are important. The extrema are obtained at 'boundary points'.

$Q_{3,2}$. How to relate the profit function with the constraints?

$A_{3,2}$. Using the graphical representation, the equation for the profit is $P = 3x + 2y$.

$A^{\blacktriangledown}$. The optimal profit is attained in the vertex (200,100) of the allowed area. At this point the profit is maximal because among all the lines of the form $P = 3x + 2y$ and intersecting the allowed area, this one has the largest P.

Figure 3: Steps in the solution presented as a Q&A map

While it does not show the full potential of the given task (for students in the imaginary lesson nor the teachers in the workshop) for developing the inquiry process, it enabled authors to follow the work of the teachers during the workshop and to provide a baseline for understanding teachers' reasoning.

Results of the workshop

During workshop teachers received explicit training for the use and development of the tool of Q&A maps. Here we present the main results of their group work and the reflection session. Teachers were given the instruction to use the problem in Figure 1 and perform the mathematical analysis for the lesson including that problem. It was allowed to modify the problem and the only requirement in the group work was to formulate the analysis through the development of a Q&A map.

In one group teachers had difficulties to formulate the Q&A map from scratch, so we directed them to write down different questions that the students might ask themselves. The group reported that they found it easier to first write a list of activities (instead of questions) and during that process "arrows started to appear" as they observed the need to use different information. Their activities are: (1) table with possibilities, (2) introduction of variables, (3) inequalities for a certain ingredient, (4) profit function, (5) representation in the coordinate system, (6) finding the intersection. The arrows pointed from (3) and (4) to (5).

After this, the group made a draft of the Q&A map with a few very specific questions: (Q1) What is the profit with 200 vanilla ice-creams and 150 coffee ice-creams? (Q2) How much ingredients is needed for that amount of ice-cream? (Q3) What is the maximal possible quantity of coffee/vanilla ice-cream? (Q4) Which combination of amounts of ice-creams will gain the maximal profit?

The second group started they work with the dilemma should they produce a Q&A map from the perspective of the students or the teachers. We find this interesting in two ways. First, the teachers have shown great attention to the students' perspective, which is aligned with the change of the teaching paradigm (in ATD and the projects MERIA and TIME). Second, both possibilities were planned for the workshop, depending on the time. The design of the Q&A map from the perspective of students serves as an example of the mathematical analysis characteristic for implementation of SRP (or other inquiry-based activities), while the Q&A map from the perspective of teachers serves as an example of a didactical analysis characteristic for implementation of SRP-TE (with the initial question "how to teach…"). The group has produced drafts of both Q&A maps. First one (student perspective, Figure 3) is very similar to the one described in Figure 2 as it has three related branches.

(Q1) What is profit?
- (Q1,1) How to calculate profit?
- (Q1,2) What are variables?
- (Q1, 3) How to determine optimal values?

(Q2) What is the meaning of data?
- (Q2,1) What to do with quantities of eggs and cream?
  - (Q2,1,1) Do we need to use all of eggs and cream?
- (Q2,2) How do we use data from the table?
- (Q2,3) How can we connect data form the table?
  - (Q2,3,1) How many vanilla ice-cream can we produce?
  - (Q2,3,2) How to write this using mathematics?

(Q3) What does maximum mean?
- (Q3,1) Why cannot we produce and earn as much as we like?
- (Q3,2) Where do the constraints come from?
- (Q3,3) Could we only produce vanilla ice-cream since it is more expensive?

Figure 4: Questions produced by the teachers in the workshop

In the draft of a Q&A map from the perspective of teachers there are a few main questions without further development:

- Which problem to start with?
- Does the problem provide the inductive approach (from concrete values to introduction of variables)?

- Does the milieu provide opportunities for validation?
- How many conditions should be used?
- What if the vertex of the allowed area does not have integer coordinates?

Teachers were motivated to analyse the mathematical solution path by the provided tool. They already had experience teaching this piece of mathematical knowledge, although it is not part of the standard curriculum in Croatia. They found the task rich since “there were many data (ice cream flavours, ingredients, warehouse condition, request to maximize profit), and it was not clear how to relate them”. Furthermore, they agreed that for students it would not be immediate that one possible solution path could be graphical. One teacher argued that the construction of mathematical solution path does not go linearly: “choose the variables, write down the equation and solve”.

Teachers found the Q&A map to be a new tool, although not completely unknown. They stated that they needed more experience, since the tool structure limited them, and that “the branching was hard”. Generally, they considered that it was very important for students to ask questions in such a way, and that was new to them. Therefore, they thought that the Q&A map had potential. One teacher said that she realized that she always tried to prepare what she wanted to say to students, how to support them or how to explain something, or what students would ask her but not themselves.

Teachers also got engaged in a discussion of didactical elements – of students’ possible reasoning. It led them to conclude that the task requires modifications, that it should be changed in a way that there would be “more than one vertex in the domain”, which means that “more ingredients should be involved, with more conditions on them”. Otherwise, students would not be motivated to justify their solution (why the profit is maximal, since there is only one point obtained which “must be the solution”). They also argued that “vertices are not needed to be given as integers, since they do not require additional reasoning about the solution.” One teacher further argued that there is a difference – as a student she would ask herself “do I need to consume all ingredients”, but as a teacher I ask myself “what kind of a task should I choose?”.

At the end of the reflection session, teachers concluded that “there is no way they could do it alone” because different members of the group have brought different considerations and experience even though they know each other very well (personally and professionally).

Discussion and conclusion

From the gathered data we may conclude that even the experienced teachers find the mathematical analysis and design of inquiry-based lessons very complex and challenging. In terms of the media-milieu dialectic, the interviews showed that experienced teachers are aware of many different resources when designing milieus for new lessons. They approach textbooks with criticism, rely on the materials they used as university students and value history of mathematics as a source of inspiration. Although they appreciate theory of didactics and research, they feel the need to collaborate with university professors and researchers in mathematics education as a “bridge”. This may be interpreted also in terms of the skydiver and the truffles dialectics, in which the researchers are considered as the “knowledgeable others” providing a wider perspective of the subject.

Furthermore, the teachers emphasize the collaboration with their colleagues as the main change in their practice. In the workshop with the given problem involved teachers did not use other resources and they have relied on their experience and used colleagues as media. This is in line with the interviews, where the collaboration and use of own experience has been highlighted by all teachers. In terms of dialectics, we may thus see the prominence of the dialectic of questions and answers and the dialectic of the individual and the collective.

We conclude that projects MERIA and TIME have significantly oriented involved teachers towards the paradigm of questioning the world. This is confirmed in this study as teachers emphasized their shift towards thinking not only about students' difficulties and responses during the lesson, but also about the importance of posing questions. In this respect, Q&A maps proved as one more tool that is natural to read and close to teachers' understanding of mathematical analysis, although its production was difficult in terms of organization of branching. Nonetheless, even without the tree-like structure of the map, posing questions and the art of posing the "right questions" is a driving force of the mathematical analysis that interested teachers are trying to master. These findings thus seem encouraging in the light that ATD bring not only tools for research in mathematics education, but the tools for the lesson design and implementation of questions into contemporary mathematics teaching profession.

References

Barquero, B., Serrano, L., Ruiz-Munzon, N. (2016). A bridge between inquiry and transmission: The study and research paths at university level. In E. Nardi, C. Winsløw & T. Hausberger (Eds.), *Proceedings of the First Conference of International Network for Didactic Research in University Mathematics*, *(INDRUM 2016, 31 March-2 April 2016)* (pp. 340-349).

Bosch, M. (2018). Study and Research Paths: a model for inquiry. In B. Sirakov, P. N. de Souza, & M. Viana (Eds.), *International congress of Mathematicians* (Vol. 3, pp. 4001–4022). World Scientific Publishing Co. Pte. Ltd.

Bosch, M., Gascón, J., and Nicolás, P. (2018). Questioning Mathematical Knowledge in Different Didactic Paradigms: the Case of Group Theory, *International Journal of Research in Undergraduate Mathematics Education 4*, 23-37.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In Bosch, M. (Ed.) *Proceedings of the 4th Conference of the European Society for Research in Mathematics Education* (CERME 4). (pp. 21-30) FUNDEMI-IQS.

Chevallard, Y. (2013). Éléments de théorie anthropologique du didactique (TAD) Une initiation à la didactique fondamentale. *Journée de didactique*.

Chevallard, Y. (2015). Teaching Mathematics in Tomorrow's Society: A Case for an Oncoming Counter Paradigm. In S.J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education* (pp. 173-187). Dordrecht: Springer.

Cohen, L., Manion, L., & Morrison, K. R. B. (2011). *Research Methods in Education*, Routledge.

Florensa, I., Bosch, M., & Gascón, J. (2015). The epistemological dimension in didactics: Two problematic issues. In K. Krainer & N. Vondrova (Eds.), *Proceedings of CERME9* (pp. 2635-2641).

Florensa, I., Barquero, B., Bosch, M., Gascón, J. (2019). Study and research paths at university level: managing, analysing and institutionalizing knowledge. In Jankvist, U.T, van den Heuvel-Panhuizen, M., Veldhuis, M. (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education*, (pp. 2484-2491).

Gazzola, M. P., Otero, M. R., Llanos, V. C. (2019). The characteristics didactic gestures of a study and research path involving mathematics and physics at secondary school, *European Journal of Education Studies 6*(7), 491-502.

Guerriero, S. (ed.) (2017), *Pedagogical Knowledge and the Changing Nature of the Teaching Profession*, Educational Research and Innovation, OECD Publishing.

Winsløw, C., Matheron, Y., & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics*, *83*(2), 267-284.

Teaching Circular Motion according Antropological Theory of Didactic

Alexandre Jurandir Aparecido[1] and Débora Coimbra[2]

[1]Federal University of Uberlândia, Science and Mathematics Teaching Master Program, Brazil; alexandre.aparecido150@gmail.com

[2]Federal University of Uberlândia, Exact and Natural Sciences Institute of Pontal, Brazil; debora.coimbra@ufu.br

Abstract

According to Anthropological Theory of Didactics, whose paradigm of questioning the world was implemented through the analysis of bicycle lanes as an option of urban transport, and considering students cannot passively receive the knowledge transmitted by teachers but rather actively build this knowledge by their actions and interactions this article aims to present a didactic proposal for circular motion teaching to first year of High School level. We used digital technologies to implement initial problematization to compare linear and angular physical quantities, to study historical and artistic contexts, to visit virtual exposition of Leonardo da Vinci's artifacts. Moreover, the bicycle and rack of gates studies worked out to apply the institutionalized knowledge in everyday life situations.

Keywords*: Angular speed; Anthropological Theory of Didactics; urban transport options.*

Résumé

Selon Théorie Antropologique du Didactique, dont le paradigme du questionnement du monde a été mis en œuvre à travers l'analyse des pistes cyclables comme option de transport urbain, et, considérant que les élèves ne peuvent pas recevoir passivement les connaissances transmises par les enseignants mais plutôt construire activement ces connaissances par leurs actions et interactions cet article a pour objectif de présenter une proposition didactique pour l'enseignement du mouvement circulaire aux élèves de première année de lycée. Nous avons utilisé les technologies numériques pour mettre en œuvre une problématisation initiale pour comparer des quantités physiques linéaires et angulaires, pour étudier des contextes historiques et artistiques, pour visiter une exposition virtuelle des artefacts de Léonard de Vinci. De plus, les études sur les bicyclettes et les crémaillères ont été élaborées pour appliquer les connaissances institutionnalisées dans des situations de la vie quotidienne.

Resumen

De acuerdo con la Teoría Antropológica de la Didáctica, cuyo paradigma de cuestionamiento del mundo se implementó a través del análisis de las ciclovías como una opción de transporte urbano, y *considerando que los estudiantes no pueden recibir pasivamente los conocimientos transmitidos por los docentes, sino que construyen activamente estos conocimientos con sus acciones e interacciones. Este artículo tiene como objetivo presentar una propuesta didáctica para la enseñanza del movimiento circular en el primer año del nivel secundario. Utilizamos tecnologías digitales para implementar la problematización inicial para comparar cantidades físicas lineales y angulares, para estudiar contextos históricos y artísticos, para visitar exposiciones virtuales de artefactos de Leonardo da Vinci. Además, los estudios de bicicletas y rejillas sirvieron para aplicar los conocimientos institucionalizados en situaciones de la vida cotidiana.*

1. Introduction

Nowadays, questions about the importance and usefulness of the contents addressed by science teaching in basic contemporary school are growing. Many authors recognize a worrying dissociation between school and professional practice and daily life. Particularly concerning to Physics teaching and despite this, Newtonian Mechanics is considerably accurate when applied to everyday situations and, its conceptual structuring, abstraction and formulation in mathematical language ensure its non-incorporation inside popular culture. In other words, even old-fashioned contents as Kinematics still does not part of common sense of 21st century individuals. From this perspective, themes from the previous three centuries such as Kinematics, Thermodynamics and Electricity, figure magnificently in the current curricula of several countries because they have been appropriate to the school domain and also because they have been operationalized in the form of exercises and problems suitable for evaluative activities in these contexts and beyond them, as national exams (Pietrocola, 2008).

The Anthropological Theory of Didactic (ATD) refers to the dominant paradigm in teaching, its fragmentation in stagnant disciplines, such as epistemological "monumentalism", in which knowledge travels in pieces or packages sanctified by tradition, in which its supposed inviting beauties demand students to visit them so that they can enjoy, have fun and even fall in love. This intellectual pilgrimage of contestable purpose requires strong devotion from teachers and educators in their narrative exhibitions and extraordinary receptivity of students to accept them as their mentors. Nevertheless, students are practically reduced to mere spectators, even when educators passionately urge them to enjoy the spectacle that is science.

In this movement of teaching transformation, efforts to stimulate students' critical sense and curiosity in face of solutions to a certain kind of problems emerging from everyday reality are urgent. In this paradigm of questioning the world, we proposed the study of bike lanes as an option for urban transport. We proposed as a generating question: "To arrive at this school, is a bike a good option of transport?" The answer demands to understand bicycle functioning as coupled circular motion. To design the study and research path (SRP), we had looking for data about bike lanes in our city in last thirty years. We also applied a preconception survey test, inspired by Silva's work (1990), to explore the comparison between linear and angular speed in simple everyday situations. We organized the didactical sequence according to Theory of Didactical Situations of Guy Brousseau as shown in next section.

Survey of Prior Conception

The first step in our didactic sequence consisted in the application of a survey of previous conceptions that stimulated in students a conflict when they try to use speed concept in simple and everyday situations of bodies in circular movements. These situations, after, would use as adidactic situations in first classroom. As linear speed concept had already been worked out previously in study of rectilinear movements, there was an expected conflict in situations that involved the impression of simultaneity and that was expected to be the gap for the understanding of the angular speed concept. The subsequent and complementary purpose of this survey would be to trigger the institutionalization of angular speed concept from the didactic situations analyzed, as shown in Table 1.

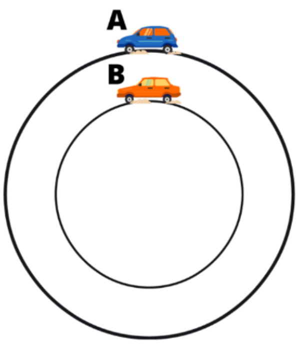
***Table 1.* Five phases of first didactical situation** (Jessen and Wislow, 2017)

Phases	**Roles of Teacher**	**Roles of Students**	**Milieu**
Devolution	Introduces, hands over the milieu	Access the milieu, try to take on a problem	Questions organized in *Microsoft Forms*
Action	Observes and reflects	Act and reflect	Questions being explored
Formulation	Organizes, if needed restart through questions	Formulate as specifically, as possible	Open Discussion
Validation	Listens and evaluates if needed, comparing different groups answers	Argue, try to follow others' arguments	Guided discussion
Institucionali-zation	Presents and explains angular speed definition	Listen, reflects and associates	Institucionalized knowledge

Despite we know ATD transcends Brousseau's theory of didactic situations (TDS), we found TDS a useful tool for planning classroom interactions.

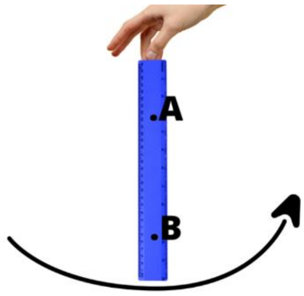
The first question posed the following situation: what could be said about the speed of two cars that describe concentric circular trajectories, as in Figure 1, taking the same time to complete a lap, that is, leaving and arriving simultaneously? Among three alternatives, the student could claim greater speed of A car, B car or that both had the same speed. Subsequently, a justification for the chosen statement was asked.

***Figure 1.* Situation illustrated in the first question of survey.**



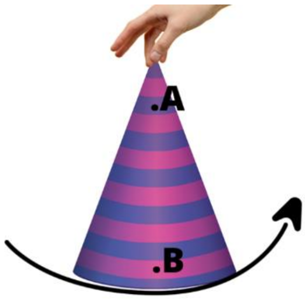
Likewise, in next question, the student must compare speeds of two points located distinctly along a ruler that oscillates when held by one end.

***Figure 2*. Situation illustrated in the second question of survey.**



In the third question, the two points are located along a cone.

***Figure 3*. Situation illustrated in the third question.**

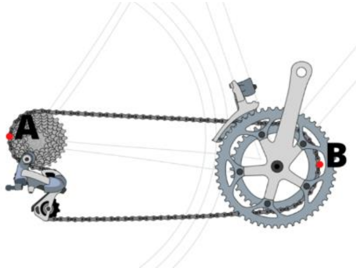


Intentionally, the kind of speed referred in questions is not specified. As expected, in the first question, most of the students (83%) understand that car A has a speed greater than car B, because it describes a longer trajectory: "*the diameter of car A trajectory is greater than that of car B, so in order to reach car A together would have to run more while B goes slower*".

However, this perception of a difference in the extension of the trajectories described by the bodies or points analyzed is not so evident for the students in the other situations: "*Who moves the ruler is the hand that is making a movement with a single speed*"; "*because it is a straight and rigid ruler, the two points have the same speed because both sides are equal*". Most of the students, fifty-three percent, indicated in the questionnaire that points A and B had the same speed, while thirty-nine percent indicated that point B, the one further from the axis of rotation, was faster.

In the fourth and last question of the questionnaire, we present a situation of motion transmission in a ratchet and crown system of a bicycle. Here there is an inversion, because, unlike the previous situations, now the linear velocity is conserved while the angular speed varies.

***Figure 6*. Situation illustrated in the fourth question of survey**



The data collected, alternatives chosen, and justifications presented, indicated that the students understood that there was something to be elucidated: “*because they are two points of the same chain, so they will have equal values*”; “*when pedaling, the crown goes slower because it is bigger, and the ratchet goes faster because it is smaller*”. Fifty-five percent of the students stated that points A and B had the same speed, while twenty-seven percent stated that at point A, located at the ratchet of the bicycle, the speed was higher.

The test was initially applied for individual resolution, followed by a moment for group discussion of four students, comprising the time interval of two classes. In the two following classes, the didactic situations were resumed for the moment of the intended institutionalization. It was explained to the students that there were incorrect answers because in all the situations analyzed there was a magnitude that was conserved while another varied in relation to the comparison of the movement of the two bodies or points indicated. This finding was built with the insertion of the standard scientific model, in the definition of angular speed and distinction with linear one. Therefore, inserted in this standard scientific model after didactic transposition, the students were able to reassess the situations proposed in the questionnaire and understand that it was necessary to define the magnitude, linear or angular, so that it was possible to compare and conclude which body or point had greater speed.

Other activities

Following the survey of previous conceptions and systematizations as described, we proposed a visit activity to a virtual exhibition celebrating 500 years of Leonardo da Vinci held by São Paulo State Museum of Image and Sound (MIS), Brazil. The exhibition is considered Leonardo da Vinci's work the most complete and detailed investigation touring the world and featured 18 thematic areas that showed the trajectory of the great Renaissance genius. With experiences in augmented reality, it was possible to walk through all exhibition areas in an immersive way and explore the works on display in detail.

The proposed activity was to explore in Da Vinci's work the inventions that presented angular movement mechanisms, more specifically the bicycle and the aerial screw, precursor of the helicopter. The activity was intended to complement the establishment of relationships between circular motion and Art. In the aid texts of the exhibition, students were able to explore the mystery regarding the authorship of the bicycle project, falsely attributed to Da Vinci. Forensic investigations indicate that the project was carefully implemented by some explorer who came into contact with his notes a few centuries after his death. Students were invited to analyze and reflect on the evidence presented that supports this story. Comparative elements to the real objects (bicycle and helicopter) were established by the teacher in the institutionalization phase. Table 2 shows the virtual tour organized according TSD phases.

***Table 2.* Five phases of second didactical situation** (Jessen and Wislow, 2017)

Phases	Roles of Teacher	Roles of Students	Milieu
Devolution	Introduces, hands over the milieu, focusing the special artifacts of interest	Access the exposition and try to take on a problem	MIS virtual exposition

Action	Observes and reflects	Visit and reflect	Virtual exposition and artifacts
Formulation	Organizes and if needed restarted through questions	Formulate as specifically as Possible	Open discussion
Validation	Listens and evaluates, if needed	Argue, try to follow others' arguments	Guided discussion
Institucionali-zation	Comparing artifacts with bicycle and helicopter	Listen and reflect	Institutionalized knowledge

After virtual tour, we studied coupled movements using the bicycle as generating equipment. With the concept of angular speed systematized in previous classes, we detail the mechanisms involved in the gears and the movement transmission. For this activity, in addition to using a real bike in the classroom, we designed a Geogebra computational simulation exploring these relationships.

Considering the relevant scales, we discuss the viability of the bicycle as a urban transport in Franca city, Brazil, where the school where this didactic sequence was applied is located. Inspired by Bertasi's work (2018), students were asked about the availability and conditions of bike lanes in the city to answer the question of whether the bike would be a suitable alternative transport for the journey to school. Other practical situation was treated, as rank of gates, with number of teeth as complementary studies.

Final Remarks

Next step of this project is to study how physical and mathematics knowledges are imbricated to didactical ones, in situations proposed to approach circular motion, through praxeological analysis. We intent to design again the SRP, changing order of didactic situations to refine validation.

References

Bertasi, R. (2018) Personal Communication.

Chevallard Y. (2015) Teaching Mathematics in Tomorrow's Society: A Case for an Oncoming Counter Paradigm. In: Cho S. (eds) *The Proceedings of the 12th International Congress on Mathematical Education.* Springer, Cham. https://doi.org/10.1007/978-3-319-12688-3_13

Jessen, B. & Winsløw, C. (2017). The theory of Didactical Situations. In Winsløw, C. (Ed.), *Practical Guide to Inquiry Based Mathematics* Ed., pp. 29-42.

Pietrocola, M. (2008). Inovação curricular em Física: transposição didática e a sobrevivência dos saberes. *Encontro de Pesquisa em Ensino de Física*. Curitiba.

Silva, D. (1990). *O ensino construtivista da velocidade angular*. Textos. Investigações para o ensino de Ciências. Número 4. Coordenadoria de Comunicação Social. Universidade de São Paulo. Faculdade de Educação. 59p.

Textualization and Educational Reforms: analysis of a praxeology in the last eighty years

Viviana Carolina Llanos[1] and Maria Rita Otero[2]

Type the author(s) institution(s) here, country; email@google.com

[1]Núcleo de Investigación en Educación en Ciencia y Tecnología (NIECyT). Universidad Nacional del Centro de la Provincia de Buenos Aires (UNICEN). Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina; vcllanos@niecyt.exa.unicen.edu.ar

[2] Núcleo de Investigación en Educación en Ciencia y Tecnología (NIECyT). Universidad Nacional del Centro de la Provincia de Buenos Aires (UNICEN). Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina; rotero@niecyt.exa.unicen.edu.ar.

188 math textbooks for the secondary school printed during the last 80 years are analyzed. This analysis is based on the praxeologies related to the parabolas identifying the transformation of this piece of knowledge in the different books over time. The Anthropological Theory of Didactics (ATD)is adopted as well as the notions of didactic transposition, praxeology and the scale of didactic codeterminacy levels. This analysis shows a transformation in books that go from information systems to activity folders. Thus, we notice an impoverishment of knowledge over time.

Keywords: Textbooks, Polynomial function of second degree, Secondary school.

Introduction.

Textualization is one of the main vehicles through which transpositive effects are produced (Chevallard, 1985) in particular, when educational reforms occur, transpositive phenomena are incremented because such reforms are spread through the “new” school books. In the successive educational reforms in Argentina, beginning with the one in 1994 and the one that took place between 2007 and 2012, the changes in textualization considerably affected the knowledge to be taught. In previous works, we analyzed various characteristics of books related to the modifications on mathematical knowledge, images, and arguing (Otero and Llanos, 2019; Llanos and Otero, 2018). In this research, math textbooks for secondary school edited in the last 80 years are analyzed from an intentional sample of 188 books. The analysis over time focuses on the praxeologies related to the parabolas whether or not the -predominant- functional point of view is adopted, depending on the periodization. Anthropological Theory of Didactics (ATD) (Chevallard, 1985, 1999, 2001, 2013) seems to be a particularly appropriate tool for carrying out the aforementioned analysis based on the notions of didactic transposition, praxeology and the scale of didactic codeterminacy levels.

Here, the questions that guide the research are: which are the transformations of knowledge in the selected topic when considering the textbooks that correspond to the educational reforms in Argentina? How could these transformations be related to the levels of the scale of didactic codeterminacy?

Methodology

188 math textbooks for the secondary level published between 1940 and 2020 are intentionally selected. The books are classified according to the criteria that contemplates the period of their first edition; and four periods related to substantial changes in curricular designs and educational reforms in Argentina are generated:

- Period 1, 20 texts edited between 1940 and 1973;
- Period 2, 34 texts edited between 1974 and 1994,
- Period 3, 84 texts edited between 1995 and 2007,
- Period 4, 49 texts edited between 2008 and 2020.

In the first two periods, the selected books almost correspond to the available universe, given that in the first 50 years considered, the number of existing books was limited by the characteristics of the publishing industry, which are changing considerably with the development of the information and communication technologies. The reforms that began to take place in the country in 1990 and later in 2007 produced a massive edition of textbooks. From the last periods, only those that best describe the selection are considered to avoid theoretical saturation, i.e. texts that do not add information are not analyzed.

In search of a praxeology that, to a greater or lesser extent, or at least nominally speaking, is timeless for the selected period, "the polynomial function of second degree" was chosen. In the 80 years considered, the polynomial function of second degree is included in the curricula and particularly in the last two periods, in which it is integrated in the so-called "Algebra and the study of functions block", content that is transversal in 4 of the 6 years of secondary school. In 3rd year (14/15-year-old students), it appears as an example of functions to study from formulas, tables and graphs, and what is considered "the study of functions" includes analysis of domain, image, positivity and negativity sets, zeros, maximum or minimum from "observing" a given graphic representation, as proposed as a model in the same program. In 4th year (15/16-year-old students) it is appropriate to study: second degree equations, quadratic functions (reading graphs and domain, equivalent algebraic expressions of the quadratic function). In 5th year (16/17-year-old students) it is time to study polynomial functions: zeros, graphs; which also include those of grade two. In 6th year (17/18-year-old students) there is an item within the same block called "complete study of simple functions" and the polynomial function of grade two appears again. That is to say, that factually this is one of the most repeated pieces of knowledge in high school that lives on after all the reforms, this is why we analyze how it is proposed and transformed its study in the textbooks over the last eighty years.

Scheme of praxeological analysis in textbooks.

Table 1 lists the components of the practical-technical and technological-theoretical block. This table allows us to observe the differences between the textbooks of the four periods considered.

Period of edition				
Type of book				
Name of praxeology	Gender of tasks	Type of tasks	Techniques	Technological-theoretical elements

Table 1: Analysis for each period

Initially, the publication period of the books and the "type of text" are identified, which allows differentiating the books to study -books that inform-, from those we call here "activity folders". Column one explains the knowledge related to the praxeology in question: polynomial functions of degree two, parabola, quadratic equations, equivalent analytical representations. Column two informs about the gender of tasks proposed by the chapters of the selected textbooks, for example: calculate, graph, prove. Column three considers the types of tasks linked to the genders of the previous column. Column four includes the proposed techniques to solve the tasks mentioned above. The last column refers to the technologies and theories that would determine the justification for the practical-technical block.

Analysis of textbooks.

Table 2 summarizes the general characteristics of each period considered. The type of text is relevant because it distinguishes between those that are genuine and relatively complete information systems, which would be the usual texts, from those that are actually exercise folders, which do not contain the information to solve these tasks. The table is completed for each period and the components that are not present are specified.

Period 1 (1940-1973)				
Type of book: information system				
Name of praxeology	**Gender of tasks**	**Type of tasks**	**Techniques**	**Technological-theoretical elements**
Parabola	Build Test (geometrically)	T_1: mechanical construction of a parabola.	τ: using ruler, compass and string.	Θ: Definition of geometrical locus. Name of the fundamental elements. Cases and properties for the construction of the parabola.
		T_2: construction of a parabola by points	τ: draw a perpendicular to *d*. τ: use the notion of distance to determine F. τ: parallel lines, radius of circle centered at F to obtain points *p*.	
Equation of the parabola	Build	T_3: Analyze the equation of the	τ: identify F y *d*.	Θ: Generalized definition of the equation of a parabola for V=

	Analyze Determine Calculate	parabola $y^2 = 2px$ y $x^2 = 2py$.	τ: measure of the distance between two points τ: convert the difference of squares to the product. τ: build the parabola	(0;0) y $F = \left(\frac{p}{2}; 0\right)$ y $F = \left(0; \frac{p}{2}\right)$ respectively. Generalized definition of the equation of a parabola for V= (h;k) and $F = \left(\frac{p}{2} + h; k\right)$ Characteristics of the curve.
		T4: Build and analyze the equation of the parabolas: $(y-k)^2 = 2p(x-h)$	τ: build the parabola $y^2 = 2px$ τ: translation of the axes with origin at the vertex of the parabola	
Polynomial function of second degree	Build Analyze Calculate Solve	T5: Represent the functions of the form $y = ax^2 + bx + c$, with $a \neq 0$.	τ: get common factor a. τ: complete squares of the binomial $x + \frac{b}{2a}$, $y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$ τ: binomial sum square: $y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$ τ: simplify and distributive property of a. $y + \frac{b^2 - 4ac}{4a} = a(x + \frac{b}{2a})^2$	Θ: Generalized definition of polynomial function of second degree. Validity conditions of the formula. Determinations of its fundamental elements: Vertex $V = \left(-\frac{b}{2a}; -\frac{b^2-4ac}{4a}\right)$ Axis of symmetry $x = -\frac{b}{2a}$ Parameter $p = \frac{1}{2a}$ Focus $F\left(-\frac{b}{2a}; \frac{1}{4a} - \frac{b^2-4ac}{4a}\right)$ Directrix d, line $y = -\frac{1}{4a} - \frac{b^2-4ac}{4a}$ x-intercept: $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$
		T6: Calculate the coordinates of the vertex, equation of the axis, value of the parameters F y d.		
		T7: Analyze transformations for: (i) $a \neq 0$ y b=c=0, (ii) $a, c \neq 0, b$=0 y (iii) $a, b \neq 0$, c=0		
		T8: Solve the equations graphically: $ax^2 + bx + c = 0$		
Period 2 (1974-1994)				
Type of book: information system				
Graph of the polynomial function of second degree	Analyze Calculate Graph Observe	T1: Being $f: x \to ax^2 + bx + c$; with $a \neq 0, x \in R$. Graph f.	τ: generate a table of values. τ: graph points on a system of cartesian axes	Θ: Generalized definition of polynomial function of second degree. Validity conditions of the formula. Analysis of the variation of the graphs for each parameter. Canonical form of polynomial function of second degree.
		T2: Analyze the variations of the matrix parabola $y = ax^2$, in relation to the parameters a, b and c.	τ: change the parameters one by one τ: observe the graph.	

Second degree polynomial equations: analytical solution. Equivalent analytical representations	Analyze Determine Find	T_3: Given a function in canonical form, obtain the polynomial form.	τ: square binomial. τ: common factor by groups.	Θ: Definition of polynomial, canonical and factored forms.
		T_4: Given a function in polynomial form, obtain the canonical and factored forms.	τ: complete the trinomial. τ: factor the perfect square trinomial. τ: solve x, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ For the canonical form: τ: replace $h = -\frac{b}{2a}$ y $k = -\frac{b^2}{4a} + c$	Θ: Definition for each technique τ. Definition of a second-degree polynomial equation. Demonstration of the formula to solve the quadratic equation. Definition of the relationships that link the axis of symmetry, V, zeros with the coefficients of the canonical and factored form.
Polynomial equations of second degree. Graph solution.	Graph Analyze Determine	T_5: Graph the function	τ: value table τ: $f(x_a) = 0$ τ: identify the intersection between the parabola with the axis x.	Θ: Definition of zero. Properties of zeros. Demonstration of the addition and multiplication properties of zeros.
Period 3 (1995-2007)				
Type of book: activity folders				
Quadratic functions	Identify Observe Indicate Graph	T_1: Being $f: x \rightarrow ax^2 + bx + c$; identify parameters and graph.	τ: generate a table of values. τ: graph points on a system of cartesian axes	Θ: **It is replaced by:** *"The graphs of quadratic functions always have a vertical axis of symmetry. The vertex is the point where the parabola intersects the axis of symmetry"*
		T_2: Analyze the vertical and horizontal displacement of $f(x) = x^2$	τ: look at the graphs. τ: identify displacement units.	Θ: Shifted parabola formula $g(x) = (x - p)^2 + k$; (p indicate the horizontal displacement and k the vertical one).
Quadratic equations	Find	T_3: Find the zeros of the function $f(x) = ax^2 + bx + c$ con *a=1*	τ: Solve x if it is possible. τ: replace the letters in the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ τ: If b=0, isolate x directly.	Θ: **It is replaced by:** They can all be solved using the solving formula

			τ: If c=0, take out common factor x.	
Discriminant	Identify	T_4: Identify the type of zeros of function f	τ: $\Delta = b^2 - 4ac$	Δ>0: the equation has two different real zeros. Δ<0: the equation has no real roots Δ=0: the equation has a single double real zero.
Canonic form	Find	T_5: Find the canonical formula, given the vertex and a point	τ: Substitute the coordinates of the vertex into the formula. τ: verify with the point that *a*=1.	Θ: **It is replaced by** the formula
			τ: **not found for** *a*≠1	
Factored form	Find	T_6: Find the factored form using Bhaskara	τ: calculate the *x* values: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ τ: replace in formula	Θ: **It is replaced by** the formula
Period 4 (2008-2020)				
Type of book: activity folders				
Quadratic functions	Identify Observe Indicate graph	T_1: Graph the function $f(x) = ax^2 + bx + c$,	τ: generate a table of values. τ: graph points on a system of cartesian axes	Θ: **It is replaced by:** *"Each parabola has a vertical symmetry axis and on it, a point called vertex in which the curve goes from being increasing to decreasing and vice versa"*
		T_2: Mark the vertex and the axis of symmetry of a quadratic function		
Canonic form	Find	T_3: Find the canonic form, given the vertex and a point.	τ: Substitute vertex coordinates into formula.	Θ: **It is replaced by:** *"If the vertex of a quadratic function is known, we can obtain its formula in the canonical form"*
			τ: **not found** for *a*≠1	
Quadratics equations	Find	T_4: Find the zeros and the vertex of the function $f(x) = ax^2 + bx + c$	τ: substitute the values of *a*, *b* and *c* in the formula $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ τ: calculate $x_v = -\frac{b}{2a}$ e $y_v = \frac{c}{a}$	Θ: **It is replaced by** the formula
			τ: identify the ordinate to the origin $f(x) = c$ τ: Find the symmetrical point:	Θ: **It is replaced by:** *"If the function is expressed in polynomial form, we look for the symmetry of the ordinate to the*

			$x^2 + bx + c = c$ $x(x + b) = 0$ $x = 0$ y $x = -b$ $x_v = \frac{0 + (-b)}{2}$ $f(x_v) = y_v$ τ: equalizing to zero the canonical form: $(x - x_v)^2 + y_v = 0$	*origin and we obtain the zeros and the vertex"*
Factored form	Find	T_6: Find factored form using Bhaskara formula	τ: replace the zeros in the formula of factored form (for a=1) τ: **not found** for $a≠1$	Θ: **It is replaced by** the formula

Table 2: praxeological analysis in each period.

From table 2, it is possible to identify that there are elements in the practical-technical or technological-theoretical block that disappear from textbooks, which were indicated as "not found" or "replaced by".

Analysis and discussion

In table 3 there is a (+) for the characteristics and elements added from one period to the next one, and a (-) to identify what is missing.

	Period 1	Period 2	Period 3	Period 4
Praxeology	**The geometric system prevails.** Parabola as a geometrical locus (definition, fundamental elements, equation of the parabola, polynomial function of degree two)	(-) geometrical system. (+) **The functional system prevails**. Polynomial functions of second degree. Parabola and analysis of its components. Conditions of validity of the formulas. Equivalent algebraic representations. Resolution of second-degree equations. Deduction and justification of Bhaskara formula. Discriminant analysis.	(-) mathematical notation system. (+) **Alphabetic -not mathematical-writing prevails.** Construction of the graph by points. Quadratic equations: incomplete, b=0 or c=0 and a=1, or Bhaskara formula (-) • Analysis of equivalent algebraic representations.	(-) They generally disappear: • The canonical and factored forms of the function. If they are for a=1. • Bhaskara Formula. (+) Quadratic equations: solution from the canonical form.

			• Proof of Bhaskara formula
Techniques	Geometric and algebraic techniques. Questioning and scope of techniques.	(-) geometric techniques. (+) techniques in the graphic, algebraic and functional representation system.	(-) justification of algebraic techniques. Study of the complete form with $a \neq 1$. (+) a single technique, without justification (get some highlights and graph).
Technological-theoretical elements	Generalized definitions. Theorems and their respective proofs. Conditions of validity of constructions, parameters, formulas.	(-) Theorems, demonstrations. (+) Definitions and properties of the coefficients of the equations and the points	**(-)** There is not. Or it is reduced to isolated verbal statements.

Table 3: changes in time

In periods 1 and 2 the books are information systems that allow the fairly complete studying of the polynomial function of second degree, both geometrical-analytical or functional. The general definitions, the validity analysis of formulas, the characteristics of parabolas and the analysis of its notable points, be it on the geometrical or functional framework, are always present as elements of the technological-theoretical environment. Algebraic and mutually equivalent representations known as polynomial, canonical and factored forms are obtained and justified using the technique for completing the perfect-square trinomial. Relationships are established between the parameters in one way or another with any value of *a*. The books of these periods contain the information that allows studying this praxeology at different levels of depth.

In period 3, a different type of text begins to emerge, which we call here “activity folders”, whose characteristic is an increase in “practical” tasks and a progressive disappearance of technological and theoretical elements. We interpret this fact as an impoverishment of the school book as an information system, since it appears increasingly reduced and disjointed, in the margins, to give rise to the so-called “activities”. There is also a decline in mathematical writing (relatively simple) and its replacement by verbal statements, for example, expressions such as:

- *“The graphs of quadratic functions always have an axis of vertical symmetry. Mark in red the point where the parabola intersects the axis of symmetry. That point is the vertex”.*
- *“All quadratic equations can be worked out with the solving formula”.*
- *“To obtain the canonical form, replace the coordinates of the vertex in the formula. Substitute a point from the graph into the formula and obtain a”*

Other examples can be added, relative to the factored form: *“If I solve Bhaskara and find the roots, substitute in the formula and search for the value of a.”*

In general, in the books corresponding to the third period, Bhaskara formula, also called “solver”, is proposed as an exclusive and self-evident technique to obtain the zeros and the vertex. When *b*=0, it is prescribed: *"if the equation does not have a linear term, the x is cleared directly*"; or if c=0, it is indicated: "extract common factor x". In these cases, there is no justification or questioning of techniques, as these are unique.

In Period 4, the disappearance of the technological-theoretical environment is more noticeable. Most of the books only contain "activities". The study begins with a "definition" of the polynomial function of second degree, and the books draw a distinction between the complete and incomplete form. Then, it is observed that in general they will only treat functions with a=1 and incomplete. The reason for this distinction would be to avoid the use of Bhaskara formula, so as to be able to "clear" the x in the equation and obtain the roots. The justifications about the notable points of the curve are proposed in written verbal form, for example: *"each parabola has a vertical axis of symmetry, and on it, a point called vertex in which the curve goes from being increasing to decreasing or vice versa. The zeros are the abscissas of the points of contact between its graph and the x axis"*.

In some books, as it has been mentioned, Bhaskara formula is avoided while it continues to appear in others. Thus, as shown in Table 2, some books obtain the roots by setting the canonical form equal to zero with, they only use the expressions in which this is possible, without making it explicit. But others also eliminate the canonical form and only deal with the incomplete polynomial form and the factored form when c=0.

That is to say that the most recent books do not have the necessary information to show the equivalence of all the possible algebraic representations of the same parabola. So much so that several books directly exclude the canonical and factored forms.

This reduction in algebraic writing and algebraic calculation even in the sixth year seems to be due to reasons originated at the level of pedagogy, which aims to teach a more "friendly" mathematics and avoiding mathematical writing would be the way to achieve it.

On the other hand, the substitution of books for exercise folders could also be understood as an economic issue for our social standards, or as a way -wrong as far as we are concerned- of facing criticism of the pedagogical tandem: definition, exemplification, exercise, typical of traditional teaching.

This transformation of books as a relatively complete information system, where praxeologies are reasonably proposed with all their components, towards a folder of practical activities, means that the information will have to be found elsewhere. We could think that the emergence of the Internet and its mass use would justify this, as long as teaching promotes the use and questioning of all existing media, which unfortunately does not seem to be happening at the moment, because the dominant teaching paradigm does not promote questioning.

In all the periods considered, the analysis of the parameters in each of the possible ways was based on the visualization of the graph. What varies between one and the other, especially in the last two periods, is related to the emergence of ICTs, and their supposedly massive use. In this last case, due to the way the software is used, the ostensiveness increases and the analysis seems to be impoverished, if there is no questioning.

This work does not seek to sustain the relevance of studying or not the polynomial function of second degree in high school or any other piece of knowledge, but rather, to make clear the "impoverishment of knowledge" due to the transformations of textualization. In the books of the last period, knowledge is distorted, and increasingly removed from its mathematical origins, its uses, its reason for being. Although all the texts respond to the educational reforms originated in the noosphere, those of the last period are endorsed by the Ministry of Education in Argentina and distributed free of charge.

Conclusions

This paper analyzed the way in which the praxeology of the parabola is treated in textualization, for eighty years, as a result of changes in school programs and educational reforms. The praxeology mentioned is always present in the texts, but in an increasingly reduced way. Thus, in the books of the last period, the elements of the theoretical technological environment are practically not found. This could be admitted if teaching were carried out in the paradigm of research and questioning, where all information systems must be questioned.

References

Chevallard, Y. (1985) *La transposition didactique: du savoir savant au savoir enseigné*, Grenoble: La Pensée Sauvage (2e éd.1991). [Traducida al español: Chevallard Y. (1997) La transposición didáctica. Del saber sabio al saber enseñado. AIQUE, Buenos Aires]

Chevallard, Y. (1999) El análisis de las practicas docentes en la teoría antropológica de lo didáctico. *Recherches en Didactique des Mathématiques, 19* (2), 221- 266.

Chevallard, Y. (2001). *Les TPE comme problème didactique. Séminaire national de didactique des mathématiques*. Disponible en http://yves.chevallard.free.fr/spip/spip/article.php3?id_article=14

Chevallard, Y. (2013). Enseñar Matemáticas en la Sociedad de Mañana: Alegato a Favor de un Contraparadigma Emergente. *Journal of Research in Mathematics Education, 2* (2), 161-182. doi: 10.4471/redimat.2013.26.

Llanos, V. C.; Otero, M. R. (2018). Characteristics of the images and the arguing from the mathematics textbooks for the secondary school in Argentina: analysis of the changes along 67 years. *International Journal of Research in Education and Science (IJRES), 4* (1), pp. 98-105.

Otero, M. R.; Llanos, V. C. (2019). Los libros escolares de matemática y física en Argentina entre 1961 y 2009: el papel de las imágenes. IARTEM e-journal, 11(1), pp. 1-21.

An example of the questions-answers dialectic in accounting teaching

Monica Martinez-Blasco[1], Kristina Markulin[2], Francesc Prior Sanz[3] and Marianna Bosch Casabò[4]

[1]IQS School of Management, Universitat Ramon Llull, Spain; monica.martinez@iqs.url.edu

[2]IQS School of Management, Universitat Ramon Llull, Spain; kristina.markulin@iqs.url.edu

[3]IQS School of Management, Universitat Ramon Llull, Spain; francesc.prior@iqs.url.edu

[4]Facultat d'Educació, Universitat de Barcelona, Spain; marianna.bosch@ub.edu

Abstract. *This paper shows how an inquiry activity framed by a Study and Research Path (SRP) can be implemented in a traditional university course of accountancy. We describe the functioning of the SRP, especially focusing on the questions-answers dialectic. We also analyse the consequences of the SRP implementation for both the students and the teacher. Our research shows that the proposed path allows enriching what is pre-established by the curriculum content, as well as developing skills such as teamwork, self-criticism, and the urge to inquire in students.*

Resumen. *Este trabajo muestra cómo una actividad en el marco del Recorrido de Estudio e Investigación (REI) puede ser implementada en un curso universitario de contabilidad. Describimos el funcionamiento del REI centrándonos especialmente en la dialéctica de preguntas y respuestas. También analizamos las consecuencias de su implementación tanto en los estudiantes como en el profesor. Nuestra investigación muestra que el recorrido propuesto permite enriquecer lo preestablecido por los contenidos curriculares, así como desarrollar otras habilidades como el trabajo en equipo, la autocrítica y el afán de indagación.*

Introduction

Study and Research Paths (SRPs) are an inquiry-based teaching format including an associated methodology for its design and analysis. By considering an open generating question an SRP is launched, which leads to moments of searching and studying information in numerous sources, as well as moments of research and creation of new solutions, including the adaptation of the information acquired in response to specific requests. This paper focuses on the design and implementation of an SRP in a university course of Fundamentals of Accounting for first-year students of Business Administration and Management. The SRP is generated by the problem of applying value-added tax (hereafter VAT) to the purchase and sale transactions of goods. It aims to illustrate the threefold use of the questions-answers dialectic (QA dialectic): in the design of the SRP; as a tool to manage the inquiry process for both the students and the teacher; and in the a posteriori analysis to examine and interpret the decisions made and the results obtained. We will also see in what aspects the implementation of the SRP has become a productive tool for the teacher (and first author of this communication) to locally transform the current transmissive pedagogy prevailing at the university and, above all, to question the knowledge to be taught in university accountancy courses.

Traditionally, accounting courses in higher education take place in instructor-centred learning environments where traditional learning activities such as lectures and textbook exercises are common. Different learning methodologies have been introduced into accounting courses in recent years as a response to the demand for not only technical skills, but also soft skills in business and accounting graduates. For undergraduate accounting students, inquiry-based learning and team-based learning are two teaching models that have been widely promoted in academic institutions. Both are active and structured teaching approaches that put the student at the centre of the learning process by providing groups of students with scenarios and challenges to investigate and for them to present appropriate solutions. Stanley and Marsden (2012) saw the deployment of inquiry-based learning (IBL) in accounting as one viable solution to this call. According to Christensen et al. (2018), these instructional proposals also seem to improve students' attitudes towards teamwork, a key component of their professional competence.

Nowadays, academic educators question traditional passive accounting education (e.g., lectures, problem-solving) and traditional goals (professional certification examinations) as a method of preparing accounting students for successful professional employment (Eskola, 2011). To generate more qualified and adaptive accountants, active approaches (such as team building and learning-to-learn methodologies) are widely promoted. The differences between active and passive education tactics lie mainly in whether teaching is primarily focused on imparting facts and practices, or on enabling students to build their own accounting concepts and possibly even change their minds in the process (Leveson, 2004).

Incorporating active teaching strategies into the accounting curriculum, such as student-centred, problem-based, and project-based assignments has resulted in positive educational outcomes (Lindquist, 1997). Curriculum improvements that involve students in learning experiences fulfil the acknowledged needs of university graduates for increased communication, reflective assessment, and analytical and critical thinking skills. However, this type of innovative educational proposal tends to emphasise the organisation of the students' work in class and encourages more productive teacher-student interactions hardly intervening in—or questioning—the type of knowledge of accountancy that is to be taught. IBL is proposed as an alternative way to teach a given organisation of content without really interfering in its standardised structure.

The implementation of an SRP in an introductory accounting course for a business degree is based on earlier experiences using the Anthropological Theory of the Didactic (ATD) in the teaching of engineering and statistics subjects (Bartolomé et al., 2018; Florensa et al., 2018; Markulin et al., 2021, 2022). These investigations use the didactic engineering (DE) and QA dialectic to prepare, experience and analyse the SRP. In particular, QA maps show interesting results when used as an epistemological analytical tool, since it facilitates the description of curricular contents from a more dynamic and alternative perspective. It makes us rethink the knowledge that is to be learnt—or taught—which might otherwise be taken for granted.

The DE technique is based on the Theory of Didactical Situations (Brousseau, 1997), and was reformulated within the ATD to provide a methodological framework to work with SRPs (Barquero & Bosch, 2015). The four steps of the DE methodology employed in this paper are as follows. In the first step, preliminary analysis, we figure out which didactic phenomenon we would like to address. An a priori analysis is the second step, and it affects the choice and analysis of the generating character of the question proposed to launch the SRP. The third step consist in an in vivo analysis based on class observations and regular pauses to review the in-process inquiry. The a posteriori analysis is the fourth and final step. It includes the analysis of all the data gathered, its comparison with the a priori analysis and its development towards the initially formulated didactic issue.

The first contribution of this paper is to show how an inquiry activity framed by an SRP can be implemented in a traditional university course of accountancy. We will describe the implementation of the SRP, paying special attention to the QA dialectic managed by the teacher and the students. We will examine some of the teaching strategies and resources elaborated for this purpose, their functioning, potential and limitations. The second contribution is to analyse how the SRP implementation affects both the students and the teacher in their usual didactic practice. The next section presents the details of the a priori analysis of the SRP: the description of the question proposed to the students and its relation to the course objectives. In section three, prior to the conclusions, we examine the dynamics established during the SRP's implementation through the questions raised by the students in the form of "session demands" and the results of the survey they answered at the end of the SRP.

Task description and class dynamics

Task description

This task is designed for students in the context of the "Fundamentals of Financial Accounting" course to introduce them to the purchase and sale of goods subject to VAT. This is a fundamental part of the course content, and the experimental teaching dynamic was applied to a group of 25 undergraduate students aged 18 to 20 years old (19 males, 6 females). This course was taught in English, and, for the SRP, the students were organised in teams of five members. The course lasted from September 2020 to April 2021, and included two two-hour sessions per week. The SRP was completed during February and March in a total of 16 sessions.

Up until the moment the task was presented to them, the students had learnt the double-entry accounting technique and the steps to follow in the accounting cycle, but not the valuation and accounting of purchases and sales of goods. They did not know how to deal with indirect taxes either. The generating question of the SRP started with reading a letter that a company based in Spain, PanPot SL, received from the tax agency. In it, the tax agency requested the company to rectify the amount paid for the VAT settlement of a quarter. According to the tax agency, the amount was wrongly calculated, although the financial director assured that no error had been made. Under this circumstance, the company manager, Miss Chen, requested a report to find out if the Spanish tax authority was right, and therefore, established generating question Q_0 for the students' inquiry. The main objective was for the students to reach a conclusion regarding Miss Chen's dilemma, and prepare a report and a short video for her. This letter the students received as part of the documentation to be analysed can be found in Annex 1. The amounts of Input VAT and Output VAT declared by the company, as well as the amount for the VAT settlement are mentioned in the letter. It also shows the amount of VAT calculated by the tax agency, which is different from the VAT declared by the company.

The students also received the initial balance sheet of the company, including a description of the economic transactions during the accounting period. Accounting information and economic transactions can be found in Annex 2. Each transaction required basic accounting knowledge related to the double-entry accounting system.

The economic events described are the following:

- Economic event 1 is a transaction where goods are sold at a discount. This transaction is partially paid in advance, in cash and on credit. The description of the transaction in class also stated that transport costs were to be borne by the buyer.
- Economic event 2 is an international purchase transaction to be paid on credit. This transaction had been prepaid and, as a result, the amount of the transaction to be paid results from subtracting the advance to suppliers from the balance sheet already paid.
- Economic event 3 is related to a return on the sale presented in transaction 1. The return is agreed at a certain price, which consequently decreases the amount of accounts receivable.
- Economic event 4 is a transaction where purchases are returned to a Chinese supplier due to low quality at an agreed price. This transaction reduces the debt that PanPot owes the supplier in the accounts payable, as well as the value of the purchase without VAT.
- Economic event 5 is a sale transaction to be paid in cash at delivery. Delivery expenses were paid by PanPot also in cash.
- Economic event 6 is an additional discount applied to PanPot that reduces accounts payable.

The accounting information did not only include the economic events, but also additional data on depreciation rates and on calculating the value of ending inventory. This information was included to make the students consider whether or not it could affect the resolution of the task. Therefore, not only information that was useful to them for the resolution of the task, but also totally irrelevant data were provided.

Figure 1 shows the a priori analysis performed by the teacher based on previous courses implemented adopting a traditional class structure. It shows the topics related to the task that were expected to appear throughout the sessions. This figure is based on a traditional type of teaching where the teacher is basically the content provider. As can be seen, in traditional teaching the explanation to solve Q_0 would be sequential and in the following order: first, we would explain how to valuate and prepare the journal entry (JE) for a purchase and a sale of goods without taking into consideration VAT. This includes a review of the rules for measurement standards (valuation) that affect these two types of transactions, as well as their practical application in several examples that illustrate the theory. Secondly, we would introduce the concept of VAT (Q_2); what type of products it is charged on in Spain, in which percentages, and how we journalise transactions taking VAT into account (Q_3). Finally, we would explain how to settle VAT and how often (Q_4). The final answer to Q_0 would be given by the resolution of Q_4, which would be obtained by combining the answer to Q_1 with those to Q_2 and Q_3.

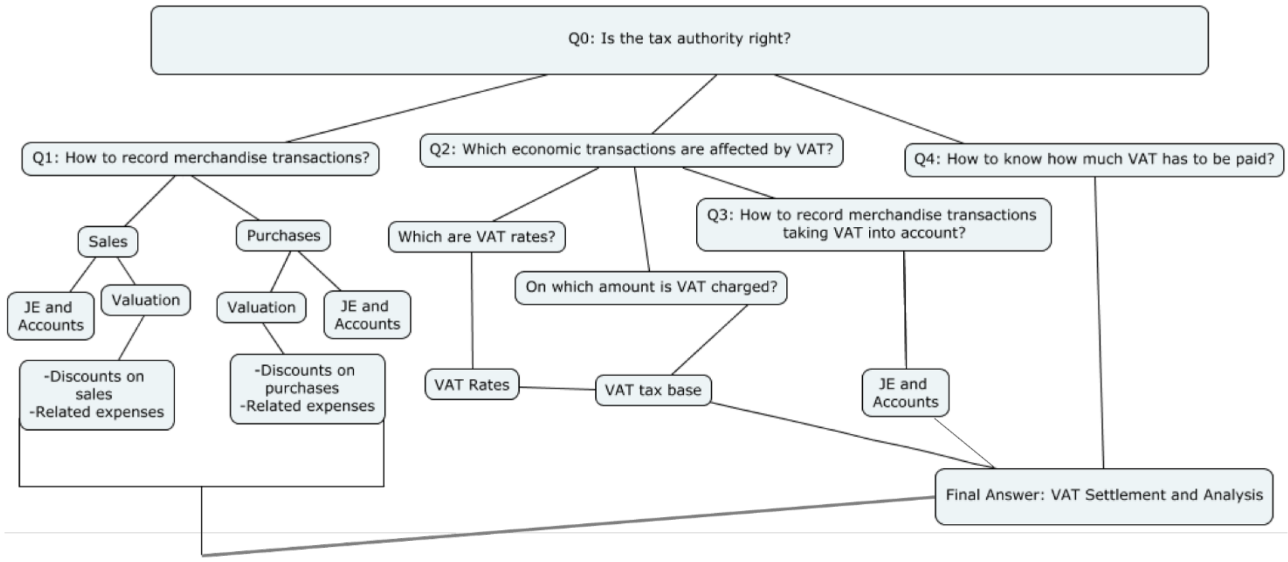


Figure 1: A priori analysis, focused on the vision of the theoretical answers to the problem

Class dynamics

Two two-hour sessions per week were devoted to the SRP for two months. Approximately every two sessions, each student teams had to submit a Team Logbook including the questions raised, the answers provided and some "demands" for the class discussion. Therefore, the topics discussed in class were on-demand by the students and were not predefined by the teacher. Demands were pooled and the teachers provided the students with the necessary tools to find the answers to the questions. This methodology allowed the class to advance as a whole and not only as individual teams. The number of sessions to be devoted to the task was not planned in advance. The task started without knowing how much time the research sessions and the students' demands would take up. Curiously, the number of sessions dedicated to the task was the same as the number of sessions that would have been dedicated to it in a traditional classroom. The difference, as will be shown in the following section, lies in the fact that many more topics were discussed than the ones established in the a priori analysis.

The Team Logbook was the main instrument used to organise the QA dialectic. The document (see Annex 3) was structured into three sections. The first one, which listed the questions addressed, included a record of the questions that had been discussed in class during the session. The second section of the Logbook contained the results obtained by the students applying the questions previously addressed in class (listed in the first section of the logbook). It reflects the individual work of each team. In this section, the students wrote down how the questions discussed in the classroom allowed them to make further progress in solving Q_0. The third section listed the new questions and demands that needed to be addressed and would be discussed in the following session.

The class sessions were organised as follows: The first session introduced the problem and the documentation. The teacher also explained how to complete the Team Logbook and how to take advantage of all its sections. The students analysed the documentation and presented the first questions and demands of information by uploading the Team Logbook on Moodle. From submission 2 to submission 5 and for submission 7, the teams' demands regarding the previous sessions uploaded on the Logbook were addressed and discussed with the whole class, and the teacher guided the students to find the answers through providing response tools (such as web pages, specific theory content prepared to answer Logbooks demands, etc.). The teacher organised the teams' demands by topic, and provided resources, but did not answer the demands directly. Some sessions, like the ones for submission 6, were fully dedicated to teamwork, since all the demands had been discussed previously. The final session before submission 8 was the last session where the teacher addressed the Logbook demands.

Demands and class experience

Demands and topics covered

Figure 2 chronologically shows, using a scale of greens, how the questions proposed in the a priori analysis (Figure 1) arose in accordance with each submission. Figure 2 is based on the students' demands per submission obtained from each Team Logbook. Digital Annex 4 shows all the demands made by the teams per submission.

As can be seen, the order in which the questions appeared in the students' Logbooks was the opposite of the order that would have been developed in a traditional classroom. In the first round of demands, the students were already asking about how to find out how much VAT they should pay, a topic that in traditional sessions would not have been addressed until the end of the sessions.

After that, the students' questions revolved around finding out information about VAT, its rates and how it works. After working on how to find out the amount of VAT to pay (Q_4) and on how VAT works (Q_2), the remaining questions were related to its application to specific transactions (Q_3 and Q_1).

There was no submission of new demands in session 6, which entirely focused on developing the results of the previous session. It is also worthy of note that the colour of submission 7 does not appear in any topic. This is because the questions in submission 7 had already appeared in previous sessions, and were discussed again in submission 8. From then on, there were no new questions, and all the expected topics from the a priori analysis (as well as additional ones) were discussed. At this point, the teacher, with the students' consent, established there would be no more rounds of demands, and settled the date for the submission of the final report.

Thanks to the research developed by the students in this SRP, we did not only deal with the topic of VAT in Spain, but also discussed the different rates in European countries, and how tax rates vary over time. In the traditional sessions, this discussion would not have taken place, since they focus only on Spanish legislation. Moreover, the students also asked themselves whether they should perform a complete accounting cycle to find the answer to the question or not. In a traditional class, they would not have considered the option of not going through a complete accounting cycle, since the standardised task is to perform it.

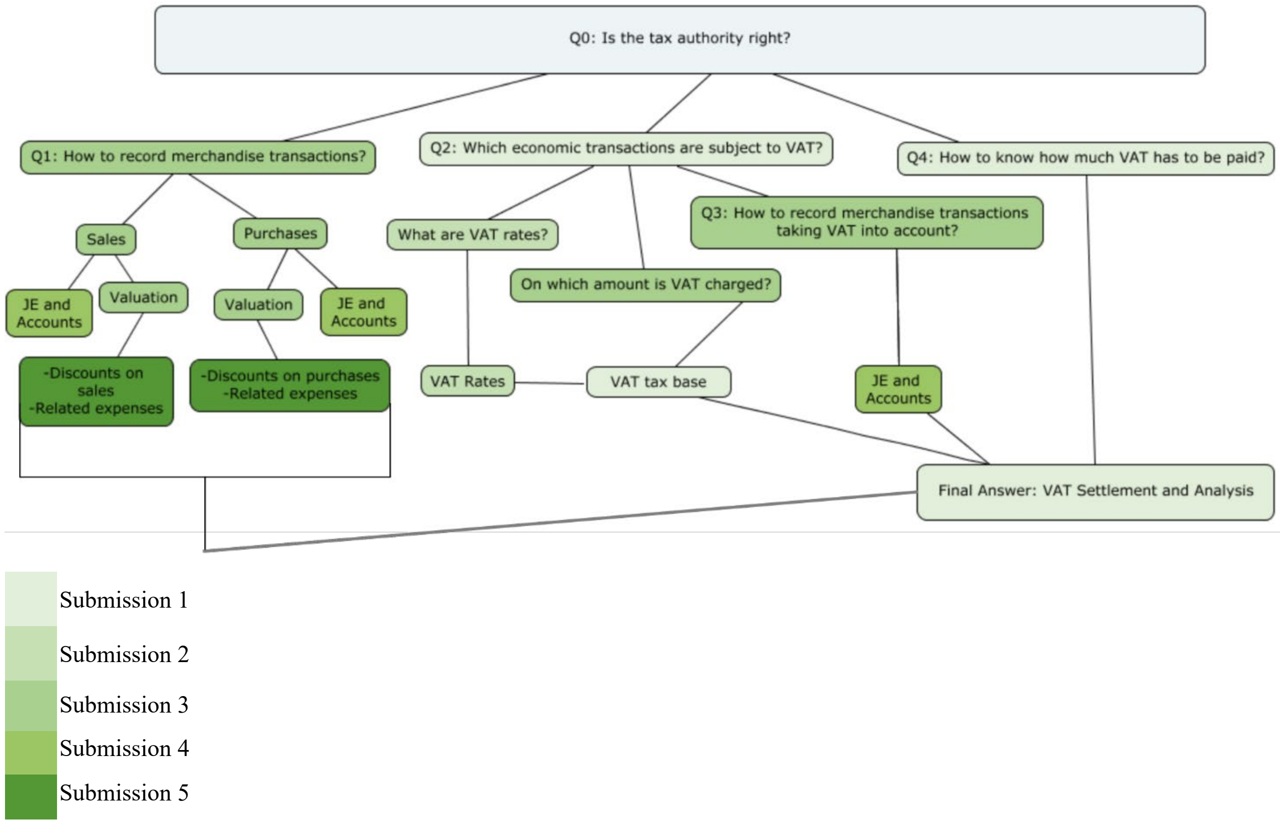


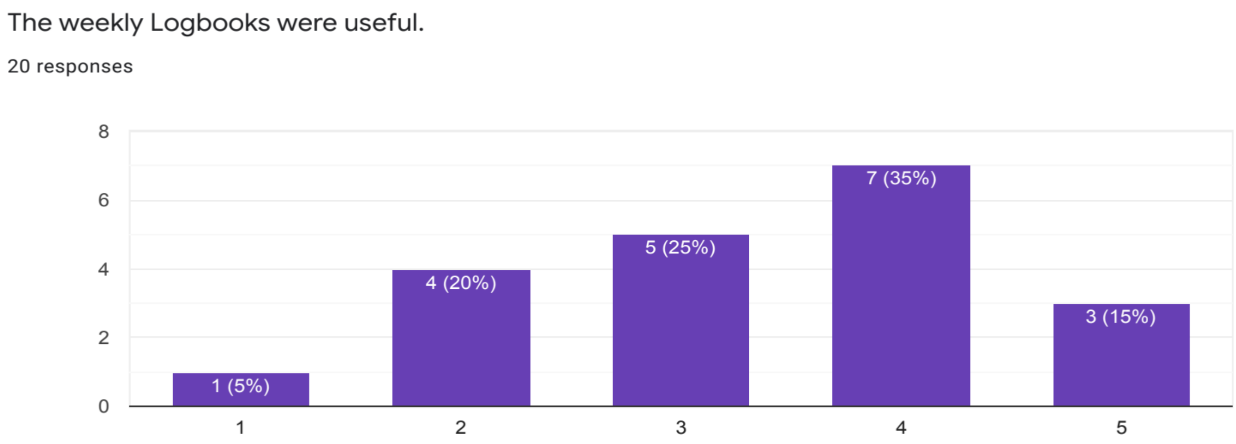
Figure 2: Chronology of the demands

Another issue we would like to highlight is that some of the students' demands appear repeatedly in different sessions. We believe the repeated questions indicate that the teacher facilitated the search for the answer without providing the answer directly. Those students who did not find the answer to their question(s) using the research tools provided asked the question(s) again.

Class experience

One of the first issues we would like to point out was the difficulty presented by the introduction of the Team Logbook into the classroom dynamics. The operability of the Team Logbook was not easy to introduce to the students and the teacher had to repeat its objective and functionality several times. However, we believe that the effort required to introduce this tool was worthwhile. Figure 3 shows the results of the student surveys conducted at the end of the project. The survey consisted of 37 questions, which were grouped into six blocks of information to be collected. These blocks assessed general aspects of the course, general aspects of the project, project content, teamwork, project organisation and final reflections. The majority of the students (75%) valued the introduction of this teaching tool positively (score 3-5 on a scale from 1 to 5). We only obtained 20 answers, as 5 students had dropped the course by the end of the project.

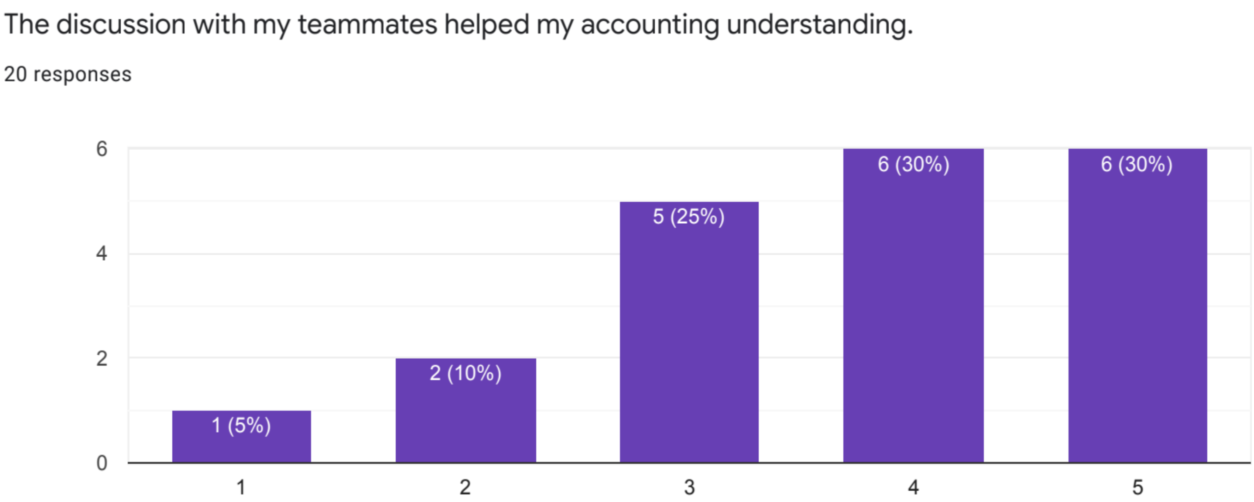
We hope that the three Heading Styles will suffice to structure your paper. Please avoid numbering sections (as opposed to lists and footnotes) 1, 1.2, etc.



1-strongly disagree; 5-strongly agree

Figure 3: Student survey results of the usefulness of the Team Logbooks

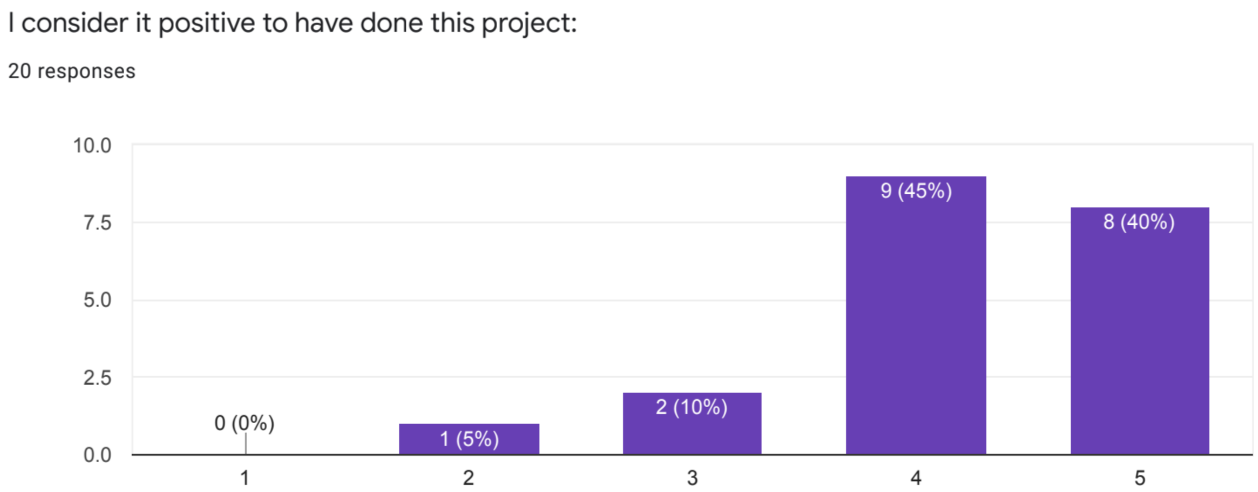
We would also like to point out that perhaps the hardest aspect highlighted by the students was having to work in groups. Although this is considered an important aspect to develop competencies, in practice, students are used to working alone. Figure 4 shows that the majority of the students considered the discussion with their teammates helped them to better understand accounting.



1-strongly disagree; 5-strongly agree

Figure 4: Student survey results of the usefulness of teamwork with respect to the understanding of accounting

Finally, figure 5 shows that most of the students valued the project positively, as 95% of the students agreed or strongly agreed on the positive impact of having performed the task.



1-strongly disagree; 5-strongly agree

Figure 5. Student survey results of the overall evaluation of the task

Conclusions

In this paper, we analysed the implementation of an SRP as an alternative to traditional accounting classes. Regarding the development of the a priori QA map (Figure 1), the SRP's generating question was powerful enough to allow the teaching of a basic part of the fundamental accounting course, such as accounting for the purchase and sale of goods subject to indirect taxes (VAT in this case). This task, implemented as an SRP, made it possible not only to discuss traditional topics, but also others that would not be dealt with in a conventional accounting class. The students' curiosity for knowledge went beyond what is covered in an academic syllabus.

Real time management of the SRP depended on the Team Logbook. Therefore, the Team Logbook was a very useful tool not only for the teacher to know how the students were progressing in their learning process, but also for the students themselves. Hence, the Team Logbook was a resource that students could use to review all the contents discussed in class and see how they were progressing in their inquiry.

An a posteriori reflection and analysis show three findings on how this SRP implementation enriches the traditional accounting teaching experience. Through the SRP initially proposed, the students did not only work on pre-established content, but also developed skills such as teamwork, reflection and self-criticism, and the urge to inquire, amongst others. The students left behind the usual position of wait-and-see and had to consider some questions to move on with resolving the task, which was considered as positive. Concerning the content itself, a shift in the teaching structure led to a different organisation of the topics. Up to now, the traditional way of delivering content in a class was to first teach the accounting of the purchase and sale of goods, and then to add indirect tax to the transactions. The SRP implementation shows that, in the students' logic, the process is the opposite. Using an SRP in class allowed the teacher to be more aware of teaching and learning difficulties presented by certain topics. Those topics are the ones that were repeatedly asked about in the students' Logbooks.

Finally, after having obtained the students' feedback on the experience, as well as the application of the DE methodology to analyse the implementation of the SRP, further development of its teaching is necessary. This is nothing but the DE preliminary analysis: continuously questioning knowledge and its current academic curriculum organisation.

Acknowledgment

Funded by project RTI2018-101153-B-C21, Programa Estatal de I+D+i orientado a los Retos de la Sociedad (MCIU/AEI/FEDER, UE).

References

Barquero, B., & Bosch, M. (2015). Didactic engineering as a research methodology: From fundamental situations to study and research paths. In A. Watson, M. Ohtani, & J. Ainley (Eds.), *Task design in mathematics education* (pp. 249-272). Springer.

Bartolomé, E., Florensa, I., Bosch, M., & Gascón, J. (2018). A 'study and research path' enriching the learning of mechanical engineering. *European Journal of Engineering Education*, *44*, 1-17.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. N. Ballacheff, M. Cooper, R. Sutherland, & V. Warfield (Eds. & Trans.). Kluwer.

Christensen, J., Harrison, J., Hollindale, J., & Wood, K. (2018). Implementing team-based learning (TBL) in accounting courses. *Accounting Education*, *28*, 1-25.

Eskola, A. (2011). Good learning in accounting: Phenomenographic study on experiences of Finnish higher education students. *Jyväskylä studies in business and economics*, *101*. University of Jyväskylä.

Florensa, I., Bosch, M., Gascón, J., & Winsløw, C. (2018). Study and Research Paths: A New tool for Design and Management of Project Based Learning in Engineering. *International Journal of Engineering Education*, *34*, 1848-1862.

Leveson, L. (2004). Encouraging better learning through better teaching: A study of approaches to teaching in accounting. *Accounting Education*, *13*, 529-548.

Lindquist, T.M. (1997). An experimental test on cooperative learning with faculty members as subjects. *Journal of Education for Business*, *72*(3), 157-163.

Markulin, K., Bosch, M., & Florensa, I. (2021). Un recorrido de estudio e investigación para la enseñanza universitaria de la estadística [A study and research path for the university teaching of statistics]. In P. D. Diago, D. F. Yáñez, M. T. González-Astudillo, & D. Carillo (Eds.) *Investigación en Educación Matemática XXIV* (pp. 417-424). SEIEM. https://www.seiem.es/docs/actas/24/ActasXXIVSEIEM.pdf

Markulin, K., Bosch, M., Florensa, I., & Montañola, C. (2022). The evolution of a study and research path in Statistics. *ÉpiDEMES*.

Stanley, T., & Marsden, S. (2012). Problem-based learning: Does accounting education need it? *Journal of Accounting Education*, *30*, 267–289.

Annex 1: Letter from Taxation Authority

https://drive.google.com/file/d/14jFOR46OJJVXkj8s9KscrUZYyKIndRMo/view?usp=sharing

Annex 2: Accounting information given

https://drive.google.com/file/d/1CIJOxQdsCFQ-m3pLgEKelJMePu7KQYay/view?usp=sharing

Annex 3: Team Logbook

https://drive.google.com/file/d/1XSf62ME8EhyeM04PqQl8rSgr5ehrPoBL/view?usp=sharing

Annex 4: Demands per submission

https://docs.google.com/spreadsheets/d/1Imm3Dndhvs1VLWAfbIBGXJTfQVcm15z1/edit?usp=sharing&ouid=106599841629929888905&rtpof=true&sd=true

Transpositive phenomena at the interface between mathematics and physics: the case of quantum mechanics

Nathan Lombard[1] and Thomas Hausberger[2]

[1]University of Montpellier, Montpellier, France; nathan.lombard@umontpellier.fr

[2]University of Montpellier, Montpellier, France; thomas.hausberger@umontpellier.fr

Mathematics as a service course for other disciplines is a vivid issue in mathematics education. Yet, existing studies mainly focus on mathematics as being used, whereas in advanced physics their status is rather on an equal footing with the domain of application. In this article, we focus on the case of quantum mechanics. By presenting praxeological organisations from an introductory quantum mechanics course at the University of Montpellier (France) we wish to describe the linkage between mathematics and physics it shows, with a focus on linear algebra and Hilbert spaces. Our main concern is to investigate the transpositive phenomena that may occur when mathematics and physics are as intimately intertwined as it is in quantum mechanics.

Introduction

Mathematics as a service course for other disciplines is a vivid issue in mathematics education. Within the broader issue of the relationship between math and physics, mathematics as a service course for engineering education is a focus of current research (Pepin et al., 2021). Many educational studies concerned with mathematics for engineers use the perspective of the Anthropological Theory of the Didactic (ATD; Chevallard & Bosch (2020)) in order to situate the interdisciplinary context in the setting of different institutions (professional or dedicated to teaching) that influence each other. In a pioneering study, Castela and Romo (2011) described how the practice of mathematics in a context where it is applied (to automatics) may consequently be modified, due to the particular ecology of the host institution that uses mathematics. Forms of validation of the observed techniques endorsed by the institution were cautiously studied (while restricting the foundation of the discourse to mathematical arguments) and the pragmatist dimension of the activity (related to the application of mathematics) was put to the fore. More recently, Hochmuth and Peters (2020) focused on the "mixture" of discourses when mathematics is used in signal processing. They found out that mathematical practices may be also enriched in the context of its application to give birth to new mathematical practices: the introduction and justification of the Dirac impulse is an example.

However the previous works mainly focus on mathematical objects and concepts as mere tools, whereas in advanced physics their status is more on an equal footing with their physical counterparts. In such a context indeed, though one certainly cannot formulate physics without utilising mathematical objects, the history of the mathematisation of physics also led to the development of new mathematical structures which were then studied for themselves by mathematicians (Schlote & Schneider, 2011).

This study focuses on the case of quantum mechanics. As a starting point, an epistemological study was carried out (in the form of a synthesis of works conducted by historians) to show quantum mechanics is indeed a field lying at the interface between mathematics and physics. Throughout its history, it repeatedly gave rise to periods of coevolution and mutual influence between physics and mathematics. As a consequence, we may expect the aforementioned institutional influences to be more complex.

Wawro et al. (2017) showed that bringing into play various mathematical frameworks in the context of quantum mechanics requires “meta-representational competence” for students to be able to reason about and symbolise concepts related to eigenvalue theory in quantum physics, for instance to navigate between Dirac notation or matrix notation. This issue may be formulated as the difficulty to connect mathematical practices to a physical setting where they could fit and live. This being said, the authors presented it as a cognitive issue.

In contrast, in the present work we wish to show another aspect of this problem by presenting it under the institutional perspective offered by the ATD. By studying an introductory quantum mechanics course at the University of Montpellier, we wish to describe the linkage between mathematics and physics it shows, with a focus on linear algebra and Hilbert spaces. Our main concern is to investigate the new didactic phenomena that may occur when mathematics and physics are as intimately intertwined as it is in quantum mechanics. We will begin by presenting the context of the study, its methodology and our research questions, once the theoretical framework is introduced. The fourth section is dedicated to the analysis of the data. Results are discussed in the fifth section, before concluding with further perspectives.

Context of the study

The course *Quantum Mechanics I* is an introductory course to quantum mechanics. It takes place in *Licence 3*, that is during the third year of bachelor’s degree (fifth semester). It is taught by two distinct teachers, one for the lectures (1.5 hours per week) and one for the tutorials (1.5 hours). The lecturer wrote the exercise sheets and drafted solutions to the attention of her colleague. She also wrote the content of her course in a notebook and made these documents available to us. Students specialise in physics from the third semester. In addition to introductory courses in Calculus and Linear Algebra during the first year (*Algebra and Analysis I & II)*, they are taught mathematical prerequisites by physicists in the teaching units *Mathematics for physics I & II.* Afterwards, most of them will take more advanced quantum mechanics classes at the master’s level.

The course *Quantum Mechanics I* starts with qualitative approaches to the topic. This material is presented in the first chapter of the course, entitled *The Quantum World*, and is featured in the first two exercise sheets. The third exercise sheet covers Bohr's model of the atom. After a brief presentation of historical experiments, the second chapter (*Wave function and Schrödinger equation*) begins with the introduction of the wave function together with the equation describing its dynamics: the Schrödinger equation. Born’s probabilistic interpretation of the wave function is given: its squared modulus represents the probability density to find the particle it represents. We consider all that regards wave functions as a sector in the sense of ATD (see next section), dedicated to wave mechanics. It includes the fourth and fifth exercise sheets, each corresponding to a theme and respectively entitled *Normalisation of wave function and probabilistic interpretation* and *Potential step and potential well*. The latter theme consists in the study of the resolution of the Time-Independent Schrödinger Equation (TISE in the following) in various contexts. Observables, in the first place related to physical quantities, are ultimately defined as hermitian operators and presented in the framework of wave mechanics at the end of the chapter, but without being the topic of any exercise. Then, in the third chapter (*Notion of quantum state. Dirac notation and the formalism of quantum mechanics*), the linearity of the space of wave functions as well as Born's probabilistic interpretation motivate the introduction of the notion of Hilbert space as the mathematical structure of quantum states — described as kets (vectors) in Dirac notation. Many notions from the previous chapter are presented again using Dirac notation. New concepts are also introduced, such as the position and impulsion representations, as well as the Fourier transform relating them. Several postulates of quantum mechanics are given in the fourth, eponymous chapter, and illustrated in the context of two-level systems as an important example in chapter five (*Two-level systems*). In particular, it is the only one to be treated in the tutorials, in the sixth (and last) exercise sheet, titled *Operators and Dirac formalism*. An exam topic covers a three-level system, which led us to consider *n*-level systems as a whole theme. In our model, this last part of the course (chapters 3-5) makes up another sector, which we call *Quantum states and postulates in Dirac notation*. Moreover, we only pointed out themes relative to linear algebra and Hilbert spaces.

Theoretical framework and methodology

The main concept we rely on is that of *praxeology* from the Anthropological Theory of the Didactic (Chevallard & Bosch, 2020). We undertook a praxeological analysis of the fourth, fifth and sixth exercise sheets of the course *Quantum Mechanics I* in order to build a *praxeological model of reference* (Florensa et al., 2015) for this introductory course in quantum mechanics with a focus on basic Hilbert analysis and linear algebra. This case study presents the situation at the University of Montpellier. Praxeologies will be presented according to their praxis and logos blocks, that is, their four components: type of task T, technique τ, technology θ and theory Θ. Tables will be given to highlight how *punctual praxeologies* may be unified through their technologies to make up *local praxeological organisations* (LPOs), themselves unified through an overarching theory in a *regional praxeological organisation* (RPO).

In the ATD model, RPOs correspond in general to sectors and LPOs to themes. Thanks to preliminary analyses of our data, we identified a course content that should correspond to two main RPOs and three LPOs. For each of them, we will present their $[T,\tau,\theta,\Theta]$-content based on an analysis of the exercise sheets. The number of exercises being rather low, one finds only a few instances of each type of task. However, we could infer their generality through the consultation of several textbooks of reference (Cohen-Tannoudji et al., 1991; Aslangul, 2007).

In our analysis, we pay a close attention to ostensives (Bosch, 1994; Bosch & Chevallard, 1999). Indeed, semiotic analysis of praxeologies, based on the analysis of ostensives, allows to draw connections between classroom practice and the related institutional setting. For instance, we discuss whether a given praxeological element (praxis or logos) refers to mathematics, physics, or may be considered "mixed", meaning the origin of its ostensives can be traced back to praxeologies from both fields. Also, the examination of the courses of linear algebra (first year) and mathematics for physics (second year) gives pieces of information about the praxeological equipment the professor teaching the course may anticipate.

The *didactic transposition* and other transposition processes allow to determine the institutional sources of praxeologies and account for their circulation between the institutions in charge of the development of math and physics as research and teaching disciplines. Conditions imposed on the *ecology* of the institution importing a praxeology imply its modification, so it may be transferred; hence the necessity of a *transposition* (Chevallard 1999, p. 231). The ecological study of conditions and constraints goes beyond the scope of this paper. Our goal is to describe the praxeologies that may be observed in an introductory quantum mechanics course and account for the transposition phenomena that may be observed. Especially, we wish to enlighten new phenomena in comparison to the case of a pragmatist application of mathematics in an engineering course (such as the case study of Castela and Romo, 2011).

We wish to address the following research questions: How to characterise the praxeological organisations that mobilise linear algebra and Hilbert spaces in an introductory quantum mechanics course? What transpositive phenomena can be observed, within and between mathematics and physics institutions?

Praxeological analyses

The analysis of lecture notes together with the titles of exercise sheets allowed us to identify two main sectors: *Probabilistic interpretation and dynamics of stationary wave functions*; *Quantum states and postulates in Dirac notation*. With a focus on Hilbert spaces and Linear Algebra, our analysis will be restricted to three themes: *Probability and wave functions*; *The TISE in function space*; *Quantum mechanics of two-level systems*. In this section of the paper, we will provide a detailed analysis of praxeologies that are unified by these themes and sectors.

A first RPO: probabilistic interpretation and dynamics of stationary wave functions

The types of task $T_{1.1}$, $T_{1.2}$ and $T_{1.3}$ are related to Born's probabilistic interpretation of wave functions and appear in the fourth exercise sheet (see table 1). Their techniques consist in applying formulae introduced during the lectures. The type of tasks $T_{1.1}$ – *Normalise a wave function* is particular in so far as it appears throughout the three exercise sheets we study here (see next section). Importantly, here, a normalised wave function is simply a wave function whose squared modulus equals one. There is no notion of functions as vectors, nor any geometrical interpretation of normalised functions.

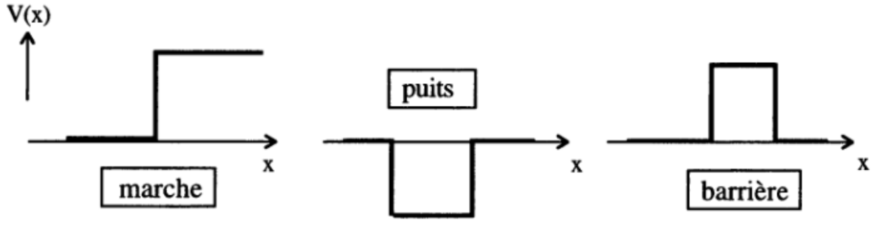


Figure 1: Several possible "shapes" for the one-dimensional potential *V* (from left to right: step, well, barrier ; Aslangul, 2007)

Wave functions are solutions of the TISE. The fifth homework set is mostly devoted to finding the possible states of a particle in a one-dimensional potential (a step, a barrier, and an "infinite well", see Figure 1). Mathematically, it consists in solving the TISE for a given potential function V:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

This formula may be interpreted as describing the state $\psi(x)$ of a particle evolving in a potential $V(x)$. $\hbar$ and m are physical parameters of the problem; E is an unknown scalar interpreted as the energy of the particle.

The resolution of the TISE is presented in the praxeology $P_{2.2}$. It consists in focusing on intervals where the potential is constant, thus applying $P_{2.1}$ (see table 1). At this point, students are to employ a praxeology coming from the math class *Algebra and Analysis II*. It is a rather common practice at this level of studies (see Figure 3). Indeed, the TISE with V constant is a second-order ODE with constant coefficients. Once $P_{2.1}$ is put at play, students have to merge the solutions on the various intervals using "continuity relations", which are a crucial technological element of the praxeology $P_{2.2}$. The class of functions at stake, however, is never specified.

In the last exercise of this homework sheet, an additional praxeology is put at play: $P_{2.3}$. Indeed, the boundary conditions now avail the students to infer solutions exist only for discrete energy values E_n (indexed by $n \in \mathbb{N}$). The technology which allows them to make sense of this technique is given in the written notes of the professor. The TISE is presented as an eigenvalue problem, even though the vector space in question is still not specified, as it is simply called "the function space" (*l'espace des fonctions*, see Figure 2). There, the teacher is making what may be considered a crucial didactical gesture in pointing out the mathematical status of the objects involved in the TISE: an "operator", a "scalar", the "vector" as an element of the "function space". Whereas dimensional analysis is a common practice in classical physics to gain control on the formulae, a new form of analysis of semiotic representations is introduced here in quantum mechanics: a mathematical analysis in terms of mathematical concepts. Importantly, when they reach this course, students have only encountered eigenvalue problems in second year and in finite dimension (see below). At the University of Montpellier, the theory of eigenvalue problems in infinite dimension, called spectral analysis, is part of functional analysis and is taught at the master's level in mathematics. Besides, no connection to Schrödinger equation is shown, as the theory is mainly developed in the setting of bounded operators.

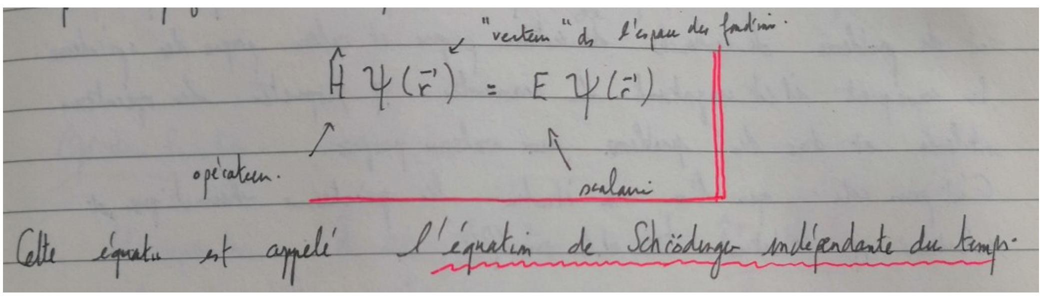


Figure 2: Excerpt of the course notes: presentation of the TISE

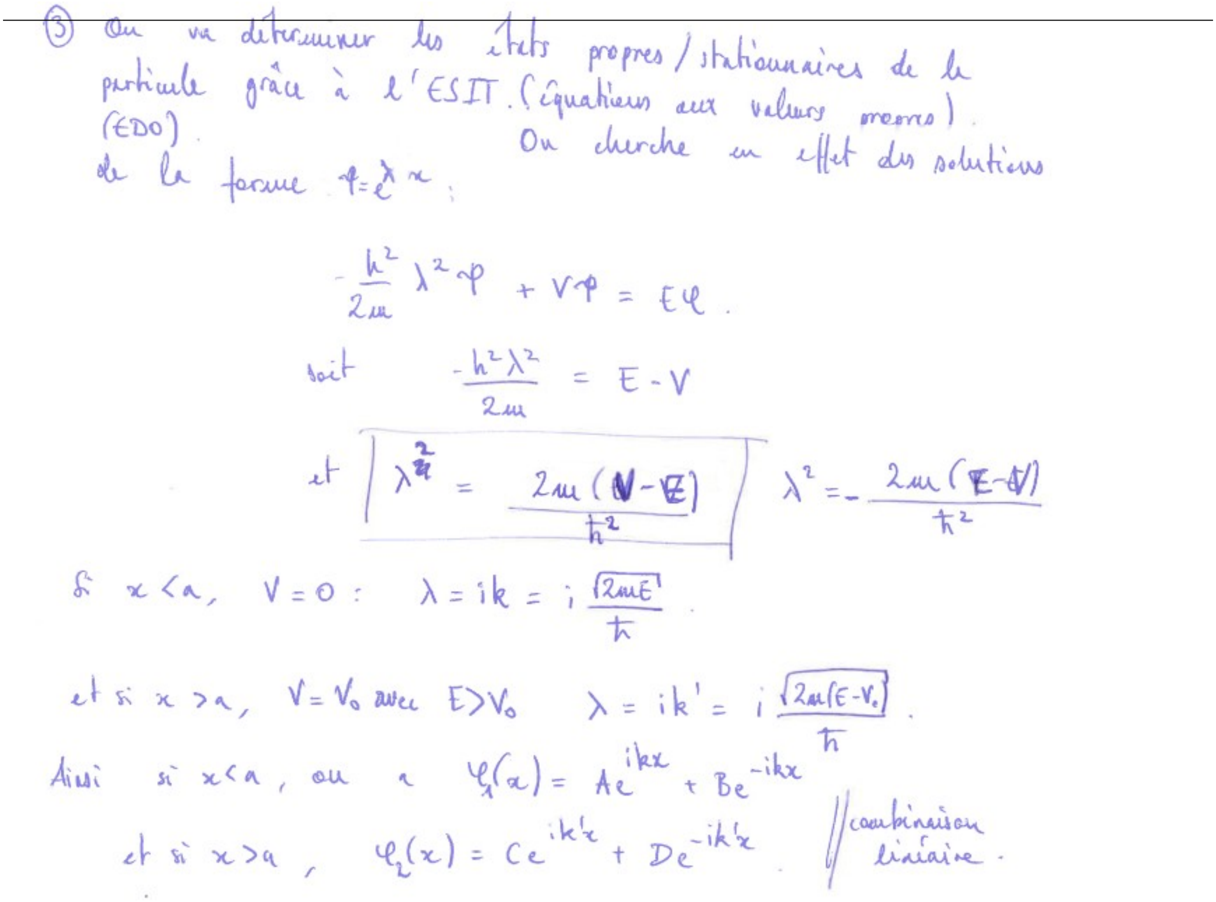


Figure 3: The first step in the resolution of the TISE in the case of a step potential

So, in this part of the course, the TISE has a dual nature (analysis and algebra), which is exemplified by the formulation of the previous exercise we considered, as well as its solution (see Figure 3). There, the TISE is both presented as an ordinary differential equation (*EDO* in French) and an eigenvalue equation (*équation aux valeurs propres*).

Hence, the two LPOs we put forth in the first RPO correspond to the fourth and fifth homework sets, respectively. They are brought together by pieces of theory of varied nature: results from pure physics grounded in optics or electromagnetism; mathematical theorems about differential equations, integrals or elementary probability theory; the mixed theory around Born's probabilistic interpretation of the wave function. Even though there is a rather wide conceptual spectrum involved, the unity of this practice holds thanks to a common set of ostensives, allowing links between mathematical symbols and physical phenomena.

In the tables 1 and 2, we highlighted this plurality by pointing out the institutional source of praxeological elements with an index: φ for physics, μ for mathematics, and M for mixed elements.

<table>
<tr><th></th><th>Type of tasks T</th><th>Technique τ</th><th>Technology θ</th><th>Theory Θ</th></tr>
<tr><td>$P_{1.1}$</td><td>$T_{1.1}$: Normalise a wave function</td><td>$\tau_{1.1}$: Use formula $\theta_{1.2}$ to compute the wave function parameter so that its squared modulus equals one</td><td rowspan="3">θ_1 –
– The probability p to observe a particle described by the wave function in a volume V_0 is given by (formula $\theta_{1.1}$):
$p(V_0) = \int_{V_0} \psi(r,t)\psi^*(r,t)d^3v$
– In particular, a wave function is normalised when $p(V) = 1$, with (formula $\theta_{1.2}$):
$p(V) = \int_{V} \psi(r,t)\psi^*(r,t)d^3v$
– Given an observable X, the average value of the measure on X is given by (formula $\theta_{1.3}$):
$\langle X \rangle = \int_{V} \psi(r,t)X\psi^*(r,t)d^3v$</td><td rowspan="6">$\Theta_{1\text{-}2}$ –
$\Theta_{\varphi 1}$: Schrödinger equation as a wave equation
Θ_{M1} : Schrödinger equation as an eigenvalue problem
Θ_{M2} :Born's probabilistic interpretation of the wave function
$\Theta_{\mu 1}$: ODE theory
$\Theta_{\mu 2}$: Linear Algebra
$\Theta_{\mu 3}$: Integration theory
$\Theta_{\mu 4}$: Probability Theory</td></tr>
<tr><td>$P_{1.2}$</td><td>$T_{1.2}$: Compute the average value of the position of a particle described by the wave function</td><td>$\tau_{1.2}$: Use formula $\theta_{1.3}$ with $X = r$</td></tr>
<tr><td>$P_{1.3}$</td><td>$T_{1.3}$: Compute the probability p to observe a particle described by the wave function in a volume V_0</td><td>$\tau_{1.3}$: Use formula $\theta_{1.1}$</td></tr>
<tr><td>$P_{2.1}$</td><td>$T_{2.1}$: Solve the TISE as an ODE for V constant</td><td>$\tau_{2.1}$: Employ the mathematical praxeology to solve a second order ODE with constant coefficients</td><td rowspan="3">θ_2 –
– The TISE with V constant is a second order ODE with constant coefficients
– The solutions of the TISE are wave functions, which implies a certain level of regularity, though unspecified
– Continuity relations and boundary conditions to ensure the regularity of the overall solution
– The TISE is an eigenvalue problem in an unspecified function space</td></tr>
<tr><td>$P_{2.2}$</td><td>$T_{2.2}$: Solve the TISE</td><td>$\tau_{2.2}$: Consider intervals on which the potential function V is constant, apply $P_{2.1}$, apply "continuity relations" to merge the solutions obtained on the various intervals</td></tr>
<tr><td>$P_{2.3}$</td><td>$T_{2.3}$: Calculate the allowed values of the energy E</td><td>$\tau_{2.3}$: Given solutions of the TISE and the boundary conditions, deduce a set of allowed energy values E_n (indexed by $n \in \mathbb{N}$)</td></tr>
</table>

Table 1: The LPOs that are related to the fourth and fifth exercise sheet (within the sector *Probabilistic interpretation and dynamics of stationary wave functions*)

A second RPO: Quantum states and postulates in Dirac notation

In the sixth and last exercise sheet, the ostensives differ significantly. On the one hand, the physical context is that of a double-well potential, which is dealt with, mathematically, in the framework of the two-dimensional complex vector space. On the other hand, the formalism relies on the Dirac notation. The "states" (*états*) are thus now written using kets in the Dirac notation $|\psi_G\rangle$. Ostensives such as "eigenstates", "diagonalise" (*diagonaliser*) or "eigenvectors" (*vecteurs propres*) have appeared. As a matter of fact the resolution of the TISE is still at stake (see table 2) but, compared to the first RPO, the statements of tasks differ significantly (Figures 4 and 5) to point out a clearer connection with Linear Algebra.

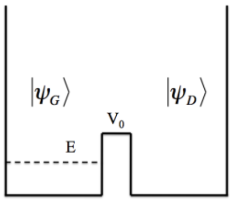
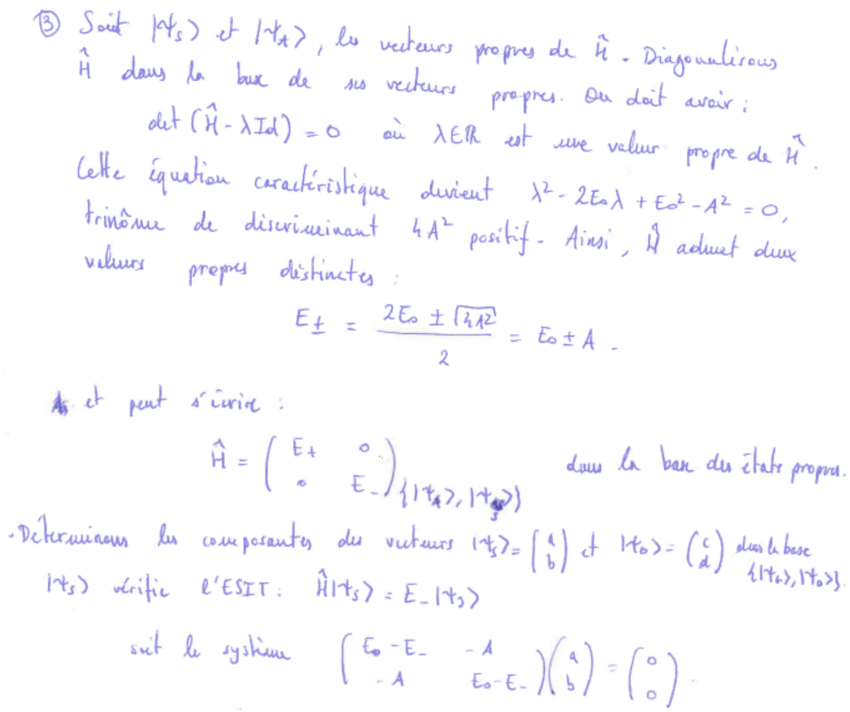


Figure 1: puits de potentiel reliés par une barrière de hauteur V_0. On distingue deux états quantiques, l'état gauche $|\psi_G\rangle$ qui décrit une particule dans le puits de gauche et l'état droit $|\psi_D\rangle$ qui décrit une particule dans le puits de droite. La particule a une énergie E.

2. Les états droite $|\psi_D\rangle$ et gauche $|\psi_G\rangle$ sont-ils des états propres du hamiltonien H?
3. Diagonaliser le hamiltonien. Déterminer les vecteurs propres et les normaliser. L'un d'eux sera noté $|\psi_S\rangle$ pour symétrique et l'autre $|\psi_A\rangle$ pour antisymétrique.

Figure 4: Excerpt of instructions relative to the last exercise of the homework sheet no. 6



③ Soit $|\psi_S\rangle$ et $|\psi_A\rangle$, les vecteurs propres de $\hat{H}$. Diagonalisons $\hat{H}$ dans la base de ses vecteurs propres. On doit avoir :

$\det(\hat{H} - \lambda Id) = 0$ où $\lambda \in \mathbb{R}$ est une valeur propre de $\hat{H}$.

Cette équation caractéristique devient $\lambda^2 - 2E_0\lambda + E_0^2 - A^2 = 0$, trinôme de discriminant $4A^2$ positif. Ainsi, $\hat{H}$ admet deux valeurs propres distinctes :

$$E_\pm = \frac{2E_0 \pm \sqrt{4A^2}}{2} = E_0 \pm A.$$

$\hat{H}$ peut s'écrire :

$$\hat{H} = \begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix}_{\{|\psi_A\rangle, |\psi_S\rangle\}}$$ dans la base des états propres.

· Déterminons les composantes des vecteurs $|\psi_S\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ et $|\psi_A\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$ dans la base $\{|\psi_G\rangle, |\psi_D\rangle\}$.

$|\psi_S\rangle$ vérifie l'ESIT : $\hat{H}|\psi_S\rangle = E_-|\psi_S\rangle$

soit le système $\begin{pmatrix} E_0 - E_- & -A \\ -A & E_0 - E_- \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Figure 5: Diagonalization of the Hamiltonian operator, followed by the computation of its eigenvectors

Firstly, the statements of the tasks subsumed under the praxeologies $P_{3.1}$ and $P_{3.2}$ involve ostensives referring to low-dimensional linear algebra as it is taught in mathematics or mathematical physics (Figure 4). Techniques to solve the tasks consist in finding the roots of the characteristic polynomial of an n-dimensional matrix M, and then, for each root (or eigenvalue) λ, in solving the system $M - \lambda I_n = 0$. Such techniques are reminded as well in the course written by the professor, this time using Dirac notation. The eigenvalue equation obeyed by the eigenvectors is called the TISE in this context (Figure 5).

At this point of the course, the technological discourse grows with the statements that the eigenvalues of a Hermitian operator are real and their eigenvectors are orthogonal. The demonstrations are given in Dirac notation during the lectures and also assigned to students in the tutorials. Regarding the type of tasks $T_{3.3}$, *Compute the probability to observe a particle [in a given state]* (see table 2), the technique involves the computation of a scalar product in Dirac notation. It relies on two aspects: on the one hand, the orthogonality of eigenvectors, and, on the other hand, the probabilistic interpretation of the squared modulus of the scalar product between a bra and a ket (Born's interpretation in Dirac notation). The latter is rather briefly mentioned in the chapter entitled *The Postulates of Quantum Mechanics*: the probability to measure a given eigenvalue is expressed in terms of the scalar product with the corresponding eigenstate. In the preceding chapters, the link is only made implicitly through equalities of the kind $\psi(a) = \langle a | \psi \rangle$which point to the praxeology $P_{1.3}$. Finally, the type of tasks *to normalise* appears again in the new context and the technique puts the Dirac notation at play. The normalisation condition now reads $\langle \psi | \psi \rangle = 1$.

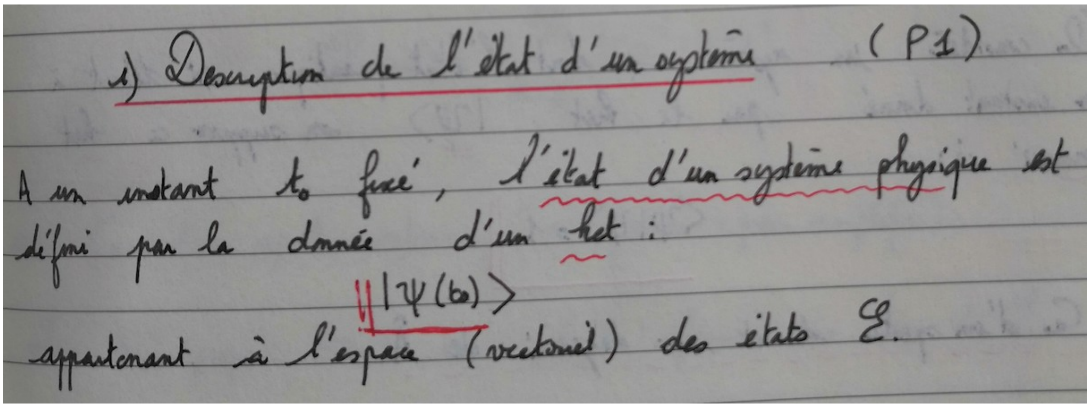


Figure 6: The first postulate of quantum mechanics in the written notes of the professor

Compared to the previous one, this LPO is much more mixed: the ostensives from both math and physics intertwine at every level of the praxis and logos. For instance, both phrases *eigenvectors* and *eigenstates* are used, depending on the context (the former being employed to describe the Hamiltonian matrix, whereas the latter refers to the physical system). In table 2, Θ_{M3} refers to this mixed logos. Statements of quantum postulates give other striking examples of intertwining of concepts from mathematics and physics (Figure 6).

<table>
<tr><th></th><th>Type of tasks T</th><th>Technique τ</th><th>Technology θ</th><th>Theory Θ</th></tr>
<tr><td>$P_{3.1}$</td><td>$T_{3.1}$: Diagonalise the Hamiltonian of a n-level system</td><td>$\tau_{3.1}$: Put to use the mathematical praxeology to diagonalise a low-dimensional matrix (P3’)</td><td rowspan="4">θ_3 :

The matrix representing the Hamiltonian of an n-level system is Hermitian and n-dimensional;
its eigenvectors obey the TISE:
$\hat{H}\,|\psi\rangle = E\,|\psi\rangle$

$\mathscr{P}(a_n) = |\langle u_n|\psi\rangle|^2$

$|\psi\rangle$ is normalised when $\langle\psi\,|\psi\rangle = 1$

θ_μ : Hermitian matrices are diagonalisable with real eigenvalues and eigenvectors are orthogonal</td><td rowspan="4">Θ_3 :
Θ_{M3} : A selection of postulates of quantum mechanics:
– States are represented by kets $|\psi\rangle$
– Observables are represented by Hamiltonian operators
– Measures of physical values are eigenvalues of the corresponding observables
– TISE: $\hat{H}\,|\psi\rangle = E\,|\psi\rangle$
– The coordinate of a state vector $|\psi\rangle$ in the basis of eigenvectors $|a\rangle$ of an observable A is given by the equation $\psi(a) = \langle a\,|\psi\rangle$
– Born’s probability interpretation
$\Theta_{\mu 2}$: Linear Algebra
$\Theta_{\mu 4}$: Probability Theory</td></tr>
<tr><td>$P_{3.2}$</td><td>$T_{3.2}$: Find the eigenstates of the Hamiltonian of a n-level system</td><td>$\tau_{3.2}$: Given its eigenvalues, put to use the mathematical praxeology to find the eigenvectors of a low-dimensional matrix (P3”)</td></tr>
<tr><td>$P_{3.3}$</td><td>$T_{3.3}$: Compute the probability $\mathscr{P}$ to observe a particle $|\psi\rangle$ in a given state $|u_i\rangle$ of a n-level system (associated to the eigenvalue a_i of an observable)</td><td>$\tau_{3.3}$: Write the decomposition of the state in terms of the eigenvectors:
Apply the formula:
$\mathscr{P}(a_i) = |\langle u_i|\psi\rangle|^2$</td></tr>
<tr><td>$P_{3.4}$</td><td>$T_{3.4}$: Normalise the state vector $|\psi\rangle$ of a n-level system</td><td>$\tau_{3.4}$: The normalised state vector is</td></tr>
</table>

Table 2: The LPO that is related to the sixth exercise sheet (within the RPO *Quantum states and postulates in Dirac notation*)

As a matter of fact, the logos on Hilbert space theory highlighted by our analyses remains quite limited. During the lectures on the postulates, several formulae linking the two RPOs (that is, wave functions and quantum states in *n*-level systems) are given. Most formulae in Dirac notation written in the course about finite systems are indeed valid in infinite dimension. Dirac notation thus serves to carry out the abstract point of view of vector spaces (actually Hilbert spaces, since the Dirac bracket is identified with the scalar product) which has a unifying potential for praxeological organisations. Nevertheless, a mathematical analysis is not undertaken in the course to relate the semiotic representations with their corresponding mathematical status (in the infinite dimensional case). A complete elucidation would require much mathematical technicality (duality, distributions, spectral theory) associated with Hilbert spaces and Hilbert analysis. It may be hypothesised that the lecturer was therefore reluctant to make steps in this direction and consider more advanced elements regarding Hilbert spaces.

Discussion

The praxeological organisation we presented is characterised by the presence of two RPOs unifying similar types of tasks related to quantum systems: the resolution of the time-independent Schrödinger equation and the calculation of probabilities of physical measurements. However, the first RPO serves as an introduction to a first formalism (that of wave functions) allowing to link quantum principles to physical experiments, while the second one is more in line with the unifying framework of the postulates of quantum mechanics. It is based on the Dirac formalism, and at the same time it particularises the study devolved to the students to *n*-level systems. This sequence seems to be common in the teaching of quantum mechanics in France, as numerous textbooks attest (e.g. Cohen-Tannoudji et al., 1991). The transition from the first RPO to the second one has a high cost in terms of mathematical knowledge, which some of these textbooks try to assume by devoting a whole chapter to the mathematical formalism.

The mathematical praxeologies (that is those that can be claimed by an institution whose teaching or research discipline falls within the field of mathematics) are of different types. On the one hand, we have identified a praxeology of analysis related to ODEs and transposed directly from mathematics. It is activated via the identification of the Schrödinger equation as an ODE. The result is a disconnection between the praxis and the logos of the quantum mechanical praxeology that aims to be developed: while the logos points to a problem of eigenvectors and eigenvalues, anticipating the second RPO, the praxis falls back on elementary analysis. The mathematical praxeology which carries the general abstract point of view (linked to the spectral theory of operators in a Hilbert space) is not available. The case of FUGS (formalising-unifying-generalising-simplifying concepts) in didactics of linear or abstract algebra (Dorier, 2000; Hausberger, 2012) has shown that a work of unification of different examples, over a long time, is to be undertaken to motivate the abstract point of view. In this respect, these elements of the logos seem to be given too early.

A second set of mathematical praxeologies is transposed from mathematics courses, this time in linear algebra. It is related to the diagonalization of matrices applied to the case of *n*-level systems. We call the corresponding praxeologies P_3' and P_3'' (relating to finding eigenvalues and eigenvectors respectively, of *n*-dimensional matrices; they appear in any textbook on linear algebra). Ostensives such as "eigenstates" point to a practice at the interface between two fields.

This question is not about finding the eigenvalues of a matrix, but rather the eigenstates of the Hamiltonian. We can hypothesise that these ostensives play a crucial role in coordinating the original mathematical praxeologies P_3' and P_3'' and the closely related quantum mechanical praxeologies $P_{3.1}$ and $P_{3.2}$. This may be interpreted in several ways: either considering their union as a transposition of P_3' and P_3'', or considering $P_{3.1}$ and $P_{3.2}$ as new praxeologies that put P_3' and P_3'' at play as techniques. Our analyses of the data led us to the second option. However, a forthcoming analysis of the dialectics of systems of models (Garcia et al., 2006) at play should allow us to refine our model. Otherwise, following Castela et al. (2011), we could consider that new technological elements have come to enrich the properly mathematical logos of P_3' and P_3''. However, unlike this model, we notice the titles of the types of task have been modified. Furthermore, a mixed logos, specific of the user institution, is present. Its mixed nature makes it cannot be overlooked as disconnected from any mathematical practice. The existence of this mixed logos, which testifies to the theoretical development of quantum mechanics in an interface between mathematics and physics, is indeed the new phenomenon we wished to study.

Then, further praxeological elements related to mathematics do not come from the mathematical praxeologies that are available in the equipment of physics students. They have to do with the new aspects of linear algebra related to Hilbert spaces (such as θ_μ, see table 2). In this instance, their function is to clarify the (mixed) notion of observable. Indeed, the unification between the case of spaces of wave functions (that is, of infinite dimension with a possibly continuous spectrum) and that of n-level systems (that is, finite-dimensional) is essentially conveyed by Dirac operational formalism. The links between scalar product and duality, the projections in orthonormal bases and changes of bases, are hidden behind the manipulations of Dirac's ostensives and the necessity to support these practices by mathematical proofs does not appear in the discourse of the institution in charge of teaching quantum mechanics (in the framework of this course, in Montpellier).

So, the situation can be compared to that described by Castela and Romo (2011). There, various choices of didactic transposition are opted for when teaching a mathematical praxeology in a professional training context brings o play a theory, which would certainly validate the techniques involved, but whose reconstruction is very costly. In the case of quantum mechanics, if such a reconstruction conformed to mathematical practice, its cost would put its teaching at a master's level, as numerous books show (see for instance Le Bellac (2013)). Several books are devoted to developing a more extended mixed logos, but they rather focus in this case on the issues raised by the physical context, for instance trying to link the mathematical formalism and its physical interpretation much more precisely. It involves advanced contents such as the theory of C^*-algebras or representations of Lie groups, and these references may be addressed to a mixed public (see for instance Mackey (2004)) or directly to mathematicians (such as Strocchi (2008)). The phenomenon of circulation of praxeologies is thus not limited to a movement from mathematics, but also towards mathematics.

We conducted interviews with quantum mechanics and mathematical physics teachers at the University of Montpellier. The following excerpt sheds light on the praxeological organisations we presented. It depicts a state of equilibrium between conditions and constraints coming from both the teaching and the professional institution, and in mathematics as well as in physics:

I mean, one cannot start with two years of spectral theory just to solve $u'' = \lambda u$, it would make no sense. [...]

yes, one deals with differential operators in infinite dimensions, but [...] most of the time we actually consider 2×2 or 3×3 matrices [...]

there is a line here, very narrow I mean, in order to convey a message without being too formal. I believe this is the main difficulty.

Conclusion and perspectives

In this study, we have given an overview of the current practice in an introductory course on quantum mechanics by analysing the praxeologies it puts at play. We may hypothesise it provides some insights about the *dominant praxeological model* that currently exists in France regarding such courses. By analysing ostensives and the way they relate to institutional praxeologies, we shed light on some of the transposition processes that are at play, which confirmed the complexity of circulation of knowledge between math and physics, and we pointed to some of the tensions that apply to didactic transposition processes.

We wish to continue our work in two directions. By analysing the interviews we performed, we wish to better understand the network of conditions and constraints determining the praxeological organisations we presented in this study. Then, we wish to conduct an experiment using the framework of *study and research paths* (SRP, Winsløw et al. (2013)). It will involve students from both mathematics and physics, whose cooperation throughout the SRP should foster a better articulation between the two institutional parlances. Furthermore, we have focused our SRP on the topic of quantum computers to prompt a reordering between the introduction of wave functions and two-level systems.

References

Aslangul, C. (2007). *Mécanique quantique 1, Fondements et premières applications.* de Boeck.

Bosch, M. (1994). *La dimensión ostensiva en la actividad matemática, el caso de la proporcionalidad.* Tesis doctoral, Universitat Autònoma de Barcelona.

Bosch, M., & Chevallard, Y. (1999). La Sensibilité de l’activité mathématique aux ostensifs. *Recherches en didactique des mathématiques*, *19*(1), 77-124.

Castela, C., & Romo, A. (2011). Des mathématiques à l’automatique : Étude des effets de transposition sur la transformée de Laplace dans la formation des ingénieurs. *Recherches en Didactique des Mathematiques*, *31*(1), 79-130.

Chevallard, Y. (1999). L’analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathematiques*, *19*(2), 221-266.

Chevallard, Y., & Bosch, M. (2020). Anthropological Theory of the Didactic (ATD). In Lerman, S. (Ed.) *Encyclopedia of Mathematics Education* (2nd ed.), 53-61. Springer.

Cohen-Tannoudji, C., Diu, B., & Laloe, F. (1991). *Quantum Mechanics, Vol. 1* (1st edition). Wiley.

Dorier, J.-L. (2000). *On the Teaching of Linear Algebra*. Kluwer Academic Publishers.

Florensa, I., Bosch, M., & Gascón, J. (2015). The epistemological dimension in didactics : Two problematic issues. In Krainer, K. & Vondrová, N. (Eds.) *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education*, 2635-2641. Charles University in Prague, Faculty of Education and ERME.

Garcia, F. J., Pérez, J. G., Higueras, L. R., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *ZDM*, *38*(3), 226-246.

Hausberger, T. (2012). Le challenge de la pensée structuraliste dans l'apprentissage de l'algèbre abstraite : une approche épistémologique. In Dorier, J.-L., Coutat, & S. (Eds.) *EMF 2012*, Feb 2012, Genève, Suisse. (p. 425-434).

Hochmuth, R., & Peters, J. (2020). About the "Mixture" of Discourses in the Use of Mathematics in Signal Theory. *Educação Matemática Pesquisa : Revista Do Programa de Estudos Pós-Graduados Em Educação Matemática*, *22*(4), 454-471.

Le Bellac, M. (2013). Physique quantique, Tome I : Fondements. EDP Sciences/CNRS Éditions.

Mackey, G. W. (2004). Mathematical foundations of quantum mechanics. Dover Publications.

Pepin, B., Biehler, R., & Gueudet, G. (Eds.). (2021). International Journal of Research in Undergraduate Mathematics Education, Special Issue : Mathematics in/for Engineering Education, 7(2). Springer.

Schlote, K.-H., & Schneider, M. (Eds.). (2011). Mathematics meets physics, A contribution to their interaction in the 19th and the first half of the 20th century. Verlag Harri Deutsch.

Strocchi, F. (2008). An introduction to the mathematical structure of quantum mechanics : A short course for mathematicians (2nd ed). World Scientific.

Wawro, M., Watson, K., & Christensen, W. (2017). Meta-representational competence with linear algebra in quantum mechanics. In Dooley, T., & Gueudet, G. (Eds.) *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education*, 2282-2289. DCU Institute of Education and ERME.

Winsløw, C., Matheron, Y., & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics*, *83*(2), 267-284.

Un modèle praxéologique de référence pour des praxéologies mixtes dans des tâches de programmation

Hamid Chaachoua, Rosamaria Crisci and Pierre Tchounikine
Laboratoire Informatique de Grenoble, Université Grenoble Alpes, France
hamid.chaachoua@univ-grenoble-alpes.fr, rosamaria.crisci@gmail.com, Pierre.Tchounikine@imag.fr

Abstract. The introduction of algorithms in education is often linked to the teaching of mathematics. The question then arises: how to model the knowledge related to a mixed mathematical organisation in which praxeologies specific to computer science allow the mobilisation of mathematical praxeologies? We have addressed this question in the case of primary education around the notion of Euclidean division.

Resumen. La introducción de algoritmos en la educación suele estar vinculada a la enseñanza de las matemáticas. Se plantea entonces la cuestión de cómo modelizar los conocimientos relacionados con una organización matemática mixta en la que las praxeologías propias de la informática permiten la movilización de las praxeologías matemáticas. Hemos abordado esta cuestión en el caso de la educación primaria en torno a la noción de división euclidian.

Résumé. L'introduction de l'algorithmique à l'école est souvent articulée avec l'enseignement des mathématiques. Se pose alors la question : comment modéliser les savoirs relatifs à une organisation mathématique mixte dans laquelle des praxéologies propres à l'informatique permettent de mobiliser des praxéologies mathématiques ? Nous avons abordé cette question dans le cas de l'enseignement primaire autour de la notion de division euclidienne.

Keywords: Algorithms, Euclidean division, Mixed mathematical organisation.

Contexte et problématique

En France, l'algorithmique et la programmation font partie des programmes du lycée (15 - 18 ans) depuis 2010 ainsi que des programmes scolaires du primaire - cycle 3 (9 - 12 ans) depuis 2016. L'habitat le plus naturel pour l'introduction de ces objets dans l'enseignement secondaire en France est constitué par les disciplines des mathématiques et celui de technologie. Depuis 2018, une discipline Numérique et Sciences Informatiques (NSI) a vu le jour dans les programmes de première et terminale[1]. Cependant, pour le cycle 3, il n'y a pas d'indication sur un habitat potentiel pour la pratique de l'algorithmique et la programmation. C'est dans ce contexte que nous avons conduit le projet EXPIRE[2], où nous avons mis en œuvre un ensemble d'actions qui visent à coupler une initiation

[1] https://eduscol.education.fr/2068/programmes-et-ressources-en-numerique-et-sciences-informatiques-voie-g

[2] EXPIRE (EXpérimenter la Pensée Informatique pour la Réussite des Élèves ; cf. http://lig-membres.imag.fr/tchounikine/projetEXPIRE.html) est une opération soutenue par l'État dans le cadre du volet e-FRAN (Espace de formation, de recherche et d'animation numérique) du Programme d'Investissement d'Avenir, opéré par la

à l'algorithmique et l'enseignement des mathématiques. Plus précisément, le projet vise à utiliser l'introduction en cycle 3 de l'algorithmique et de la programmation comme vecteurs d'apprentissage de notions de mathématiques. Cet objectif s'inscrit dans une problématique écologique (Artaud, 1998) pour faire place à une nouvelle notion dans un curriculum existant, mais aussi dans une problématique de la transposition didactique (Chevallard, 1985) de la notion d'algorithmique.

Dans ce projet, plusieurs séquences ont été construites, mises en œuvre et observées (Crisci, 2020). Pour cela, nous avons identifié un environnement de programmation adéquat, ainsi que des principes pour la conception des séquences.

Choix du logiciel

Un algorithme peut être défini comme un enchaînement d'actions, dans un certain ordre, dont chacune a un effet, et dont l'exécution complète permet de résoudre un problème et plus précisément :

> Un algorithme est une procédure de résolution de problème, s'appliquant à une famille d'instances du problème et produisant, en un nombre fini d'étapes constructives, effectives, non-ambigües et organisées, la réponse au problème pour toute instance de cette famille. (Modeste, 2012)

Dans ce sens, l'élaboration d'un algorithme consiste à décomposer un problème en sous-problèmes. Un algorithme se décrit généralement en langage naturel. Il peut faire l'objet d'une programmation, c'est-à-dire d'une traduction dans un langage interprétable et exécutable par un ordinateur.

La différence entre algorithme et programme est l'objet de plusieurs réflexions épistémologiques et en se référant à Dowek (2011), on peut dire qu'un programme informatique peut être défini comme « un algorithme écrit dans un langage de programmation » et peut donc être directement exécuté sur une machine. Il existe une multitude de langages de programmation, mais ceux qui mieux s'adaptent à des enfants sont les environnements comme Scratch, qui permettent de faire de la programmation sans la complexité d'un langage de programmation, en proposant des structures visuelles (des blocs d'instruction) et des aides (manipulation des blocs par glissé-déposé, contrôle syntaxique automatique, etc.) qui font qu'il est possible de superposer la phase d'algorithmique et de programmation : les élèves écrivent leur algorithme en agençant les blocs Scratch, ce qui permet de l'exécuter directement. Le choix de ce type de logiciel de superposer l'algorithmique et la programmation peut être vu et considéré comme un choix didactique mérite d'être questionné d'un point de vue épistémologique et didactique, mais qui ne sera pas développé dans cette contribution. Enfin, une autre raison du choix de cet environnement est qui a l'avantage d'être de plus en plus diffusé et utilisé au collège et à l'école primaire.

Séquences : principes d'élaboration

La construction des tâches de chaque séquence respecte les principes suivants (Chaachoua et al., 2018) :

Caisse des Dépôts. Il implique l'Université de Grenoble Alpes, la ville de Grenoble, le CCSTI La Casemate, l'Espé et le Rectorat de l'Académie de Grenoble.

- L'algorithme/programme que l'élève doit écrire définit le comportement d'un objet ou d'un personnage (par exemple, son déplacement) à l'écran ;

- La tâche de l'élève consiste à construire un algorithme qui permet d'obtenir un comportement cible, qui est stipulé par l'énoncé de la tâche ;

- l'algorithme attendu est congruent avec le une explicitation de la procédure de calcul attendue ;

- L'exécution de l'algorithme permet de visualiser la procédure proposée par l'élève.

En suivant ces principes, nous avons élaboré des séquences sur des notions mathématiques du cycle 3 : la division euclidienne, la décomposition additive, la résolution de problèmes et les fractions. La conception de ces séquences, ainsi que l'étude de leur mise en œuvre dans des conditions écologiques a nécessité l'élaboration d'un modèle praxéologique de référence (Bosch, M. & Gascàn, J. 2005) objet central de cet article.

Problématique

Notre séquence intègre d'un côté des activités mathématiques et d'un autre côté des activités algorithmiques. Pour cette raison, nous allons reprendre la notion de praxéologie mathématique mixte, que nous notons OMM, présentée par (Chevallard, 2001) : une organisation mathématique mixte est un ensemble de praxéologies qui articulent les mathématiques à d'autres disciplines scolaires ou plus largement à d'autres savoirs ne relevant pas des disciplines scolaires et donc au « contact du monde ».

> Penser en termes de mathématiques mixtes, c'était surtout aller au contact du monde, ne pas craindre de se mêler à lui, de rechercher le métissage. « Mixte » est un mot qui nous vient du latin miscere, « mêler, mélanger ». (La chose s'entend mieux peut-être en anglais : « mixed mathematics»). Ce que j'appellerai une organisation mathématique mixte (OMM) est le fruit d'une telle hybridation. Or ce sont de telles OMM que nous avons progressivement expulsées des mathématiques enseignées alors qu'elles furent si longtemps au cœur de la culture mathématique dispensée au secondaire. (Chevallard, 2001, p.10)

Dans cet article nous nous demandons comment modéliser les savoirs relatifs à une organisation mathématique mixte dans laquelle des praxéologies propres de l'informatique permettent de mobiliser des praxéologies mathématiques ?

Un modèle praxéologique de référence pour l'algorithmique

Dans le contexte de notre recherche, nous cherchons à modéliser les tâches qui demandent de façon explicite de produire un algorithme qui permet de résoudre une situation qui mobilise des connaissances mathématiques.

Michèle Couderette (2016) a étudié des praxéologies d'ordre algorithmique dans l'articulation entre deux disciplines : les mathématiques et l'informatique (voir Figure 1).

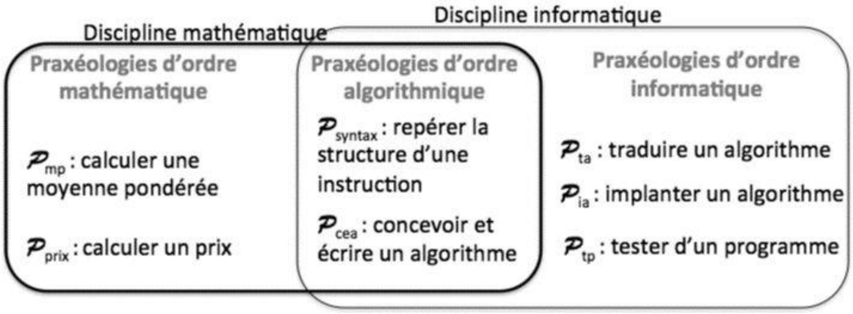


Figure 1. Exemple de praxéologies d'ordres mathématique, algorithmique et informatique identifiées par Couderette (2016)

Elle distingue trois catégories de praxéologies : mathématique, algorithmique et informatique. Ces praxéologies sont imbriquées les unes dans les autres. Par exemple, la praxéologie Pcea convoque des praxéologies mathématiques P_{mp} et P_{prix}. L'étude des dépendances entre des praxéologies différentes est importante. En particulier, nous pensons qu'un modèle explicitant les articulations entre les types tâches d'une organisation mathématique intégrant une dimension algorithmique puisse être opérationnel pour caractériser les conditions de son introduction dans le curriculum.

Dans une autre recherche, Strock et Artaud (2019) étudient les difficultés à faire émerger les éléments technologiques et théoriques propres de l'algorithmique et essaient d'en identifier les causes. Les auteurs trouvent sept types de tâches susceptibles d'être identifiés et travaillés, et qui sont en lien les uns les autres donnant lieu au réseau de types de tâches présenté dans la Figure 2. Ils précisent :

> Obtenir un programme réalisant l'exécution d'un type de tâches donné T peut demander l'écriture d'un algorithme réalisant T, le test de l'algorithme obtenu, la programmation de cet algorithme et le test du programme obtenu. Pour écrire un algorithme réalisant T, on peut se procurer un algorithme réalisant un type de tâches T' (qui peut être égal à T) et éventuellement le modifier pour qu'il exécute T, ce qui suppose de comprendre et d'analyser un algorithme, puis le tester. (Strock et Artaud, 2019).

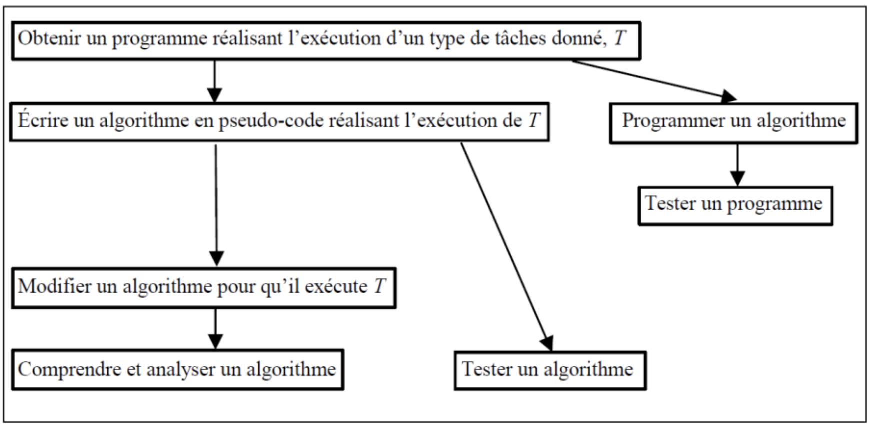


Figure 2. Réseau de types de tâches liés à l'algorithmique (Strock et Artaud, 2019)

Observons que Strock et Artaud indiquent la relation « flèche » présente dans la Figure 2 comme la relation « être sous-type de tâches de ». Puisque nous adopterons des définitions issues de T4TEL[3] (Chaachoua 2018), nous identifierons plutôt cette relation comme « être ingrédient de la technique de ». En effet, dans T4TEL une technique est décrite par une suite de types de tâches. Un des intérêts d'une telle description est d'uniformiser et de rendre explicite les critères de description d'une technique (Chaachoua et Bessot, soumis à CITAD 7).

Nous nous sommes appuyés sur les travaux de Couderette et de Strock et Artaud pour construire un modèle praxéologique de référence fonctionnel pour la description et l'analyse de savoirs relevant des mathématiques et de l'algorithmique. En particulier, pour notre étude, on s'intéresse aux praxéologies des types de tâches qui sont : T_{prog}[T*] (Créer un programme dans Scratch qui réalise un type de tâches T*). Par exemple, T* peut-être (Déplacer le lutin pour atteindre une cible C sous des conditions), nous y reviendrons plus loin.

Les ingrédients des techniques de ce type de tâches T_{prog}[T*] vont relever des domaines de l'algorithmique pour T_{prog} et du domaine dont relève le type de tâches T*. Cette formalisation du modèle praxéologique nous donne des outils puissants pour la modélisation des savoirs en jeu dans une organisation mathématique mixte : c'est un autre intérêt de la formalisation des techniques comme suite de types de tâches.

Observons que le type de tâches T_{prog}[T*] présente dans son complément un renvoi explicite au type de tâches T*. Cette propriété des types de tâches T_{prog}[T*] et T* peut être aussi retrouvée au niveau de leurs techniques. Autrement dit, si T est un ingrédient d'une technique T*, il peut intervenir également dans une technique τ_{prog} de T_{prog}[T*].

Nous allons poursuivre cette modélisation à travers une étude cas sur une des séquences étudiées dans le projet Expire : « division euclidienne ».

Etude de cas : Séquence didactique « division euclidienne »

Présentation de la séquence

La séquence didactique sur la division euclidienne, a été inspirée de certaines activités existantes et éprouvées dans la communauté de professeurs d'écoles français et des didacticiens de mathématiques. L'activité de base peut être modélisée par le type de tâches $T_{Déplacer}$ (Déplacer un objet situé sur la case 0 d'une bande numérique pour qu'il se rapproche au maximum d'une case cible a avec des sauts constants de longueur b). Ce type de tâches[4] relève de la division-quotition, pour l'accomplir il faut

[3] T4 renvoie au quadruplet praxéologique (Type de tâches, Technique, Technologie, Théorie) et TEL à Technology Enhanced Learning. A l'origine, ce qui a motivé le développement de T4TEL est de formaliser les éléments praxéologiques et leurs relations pour une représentation informatique. Cependant, ce développement trouve son intérêt au sein de la TAD en dehors du contexte particulier des développements informatiques comme le montre cet article.

[4] Nous avons pu repérer ce type de tâches dans les manuels Euro Maths [Peltier et al., 2009] et dans Cap Maths [Charnay et al., 2018

déterminer le quotient de la division euclidienne de a par b. L'énoncé peut, de plus, demander également l'explicitation du reste.

La séquence[5] vise un enjeu d'étude autour du type de tâches mathématique T_{DE} (Déterminer un ou deux termes inconnus de l'expression $a = bq + r$, étant le terme a connu et avec $0 \leq r < b$ ») à partir de la recherche de nombre de parts dans une situation de division-quotition.

Nous avons créé un dispositif dans l'environnement de programmation visuelle Scratch qui est une transposition d'un dispositif tangible utilisé dans l'enseignement traditionnel. Celui-ci n'a pas été intégré dans la séquence, seule la version scratch a été utilisée.

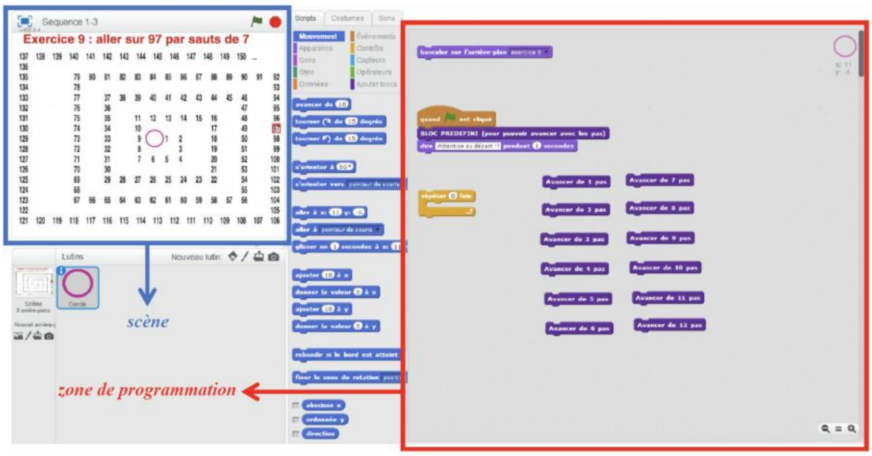


Figure 3. Interface de l'environnement préparé sur Scratch.

Les tâches proposées aux élèves relèvent de plusieurs types de tâches comme : (i) $T_{prog}[T_{DE.1}]$ où $T_{DE.1}$ (Déplacer le lutin pour se rapprocher au maximum de la cible C, sans la dépasser et en utilisant un même saut S) ; (ii) $T_{prog}[T_{DE.2}]$ où $T_{DE.2}$: (Déplacer le lutin pour atteindre la cible C en utilisant un même saut S (autant de fois que vous souhaitez) et une seule fois un joker au choix (des sauts de 1 à S - 1))

Par exemple, une tâche de $T_{prog}[T_{DE.2}]$ est « Écrivez un programme dans Scratch pour permettre au cercle-repère d'arriver exactement sur le nombre encadré (i.e., la cible, qui est mise en évidence sur la bande numérique), en utilisant le saut indiqué (autant de fois que vous souhaitez), et en utilisant une seule fois un joker au choix (des sauts de 1 à 6) ». Dans cette tâche, C = 97 et S = 7. Plusieurs algorithmes / programmes sont possibles ou attendus par l'institution (Figures 4a et 4b). L'objectif de la séquence est de faire évoluer les programmes du type (Figure 4a) vers des programmes du type (Figure 4b). En effet, dans la séquence de la division euclidienne nous visons une formule cible ($C = S \times q + r, 0 \leq r < s$) qui correspond à une structure cible du programme de la figue (4.a). C'est dans les termes de cette correspondance entre écriture mathématique visée et structure cible que nous reprenons le concept de programme de type congruent défini par Briant (2013).

[5] Nous présentons ici qu'un aspect de la séquence. Pour accéder à la situation complète se référer à Crisci (2020)

Pour cela, nous avons joué sur le choix de certaines valeurs des variables didactiques et l'ordre des tâches pour favoriser cette dynamique praxéologique.

Figure 4a . Un programme possible **Figure 4b . Un programme attendu**

Un modèle praxéologique de référence pour les praxéologies de la séquence

La séquence proposée a été construite autour de l'étude des praxéologies autour du type de tâches T_{DE}. On cherche à ce que ce type de tâches soit mobilisé comme ingrédient d'une technique des types de tâches $T_{DE.1}$ et $T_{DE.2}$ (on désignera $T_{DE.X}$ l'un des deux types de tâches). Le premier type de tâches vise à faire comprendre la notion du reste et le second type de tâches vise établir la définition de la division euclidienne. En fait, ce qui est proposé aux élèves ce sont des tâches de $T_{prog}[T_{DE.X}]$ et on vise à ce que la production d'un programme attendu (comme celui de la Figure 4a) mobilise des techniques qui sont visées pour accomplir $T_{DE.X}$ reposent sur l'environnement technologique :

θ : Étant donnée une cible C et un saut S inférieur ou égal à C, il existe un couple d'entiers uniques (q , r) tels que $C=S\times q+r\ 0\leq r<s$.

Concentrons-nous dans la suite au type de tâches $T_{DE.2}$ La séquence repose sur un jeu de plusieurs variables (Chaachoua et al. 2018), dont les valeurs de C et S, pour faire évoluer les techniques de $T_{prog}[T_{DE.2}]$ vers la création d'un programme attendu qui a une structure congruente à l'écriture mathématique visée par la séquence.

Les ingrédients de technique de $T_{prog}[T_{DE.2}]$ relevant de l'algorithmique et de la programmation nécessaires pour la réalisation des tâches proposées sont les suivantes :

— associer des blocs d'instructions simples ;

— utiliser la boucle « répéter » ;

— lancer le programme.

La façon dont ces ingrédients seront mobilisés et combinés dépend des praxéologies du type de tâches $T_{DE.2}$. En effet, une technique possible est d'associer quelques blocs « Avancer de 7 pas » et « lancer le programme », comme dans l'exemple de la figure 4, ensuite on évalue le nombre de cases manquantes et on ajuste. Une autre technique qui optimise la précédente, consiste à faire une boucle avec l'instruction « répéter », puis ajouter les instructions « Avancer de 7 pas » : cela aboutit à un programme comme celui de la Figure 4a. Une autre technique plus optimale à la précédente, consiste à mobiliser le type de tâches T_{DE} (Déterminer un ou deux termes inconnus de l'expression a = bq +r, étant le terme a connu et avec $0 \leq r < b$ »). A ce type de tâches on peut mobiliser une technique de la division euclidienne justifiée par l'enjeu didactique θ. Nous voyons que le type de tâches T_{DE} est ce qui est visé comme ingrédient dans la technique de $T_{DE.2}$ et qui va participer donc à la technique de $T_{prog}[T_{DE.2}]$.

Dans le processus de notre étude nous avons construit au préalable un modèle praxéologique de référence du type de tâches mathématique T_{DE} (Déterminer un ou deux termes inconnus de l'expression $a = bq + r$, étant le terme a connu et avec $0 \leq r < b$ »). Son élaboration s'est appuyée sur plusieurs enquêtes et selon une méthodologie décrite par Chaachoua H., Pilet J., Bessot A. (2022). Elle combine des allers-retours entre plusieurs enquêtes. D'une part, des études épistémologique et didactique pour caractériser le savoir et identifier les conditions et contraintes pour son enseignement et l'apprentissage des élèves, notamment à partir des travaux de (Banwitiya, 1993), (Brousseau, 1972a), (Fischbein et al. 1985), (Squire et Bryant, 2002), (Vergnaud, 1994). D'autre part, des enquêtes institutionnelles avec des analyses d'un contingent à partir des programmes scolaires (MEN, 2015), les manuels et d'autres ressources institutionnelles, mais également des travaux qui ont analysé eux même le savoir à enseigner comme (Gautier et al., 2008).

Le modèle praxéologique de référence de T_{DE} construit s'articule autour des organisations mathématiques ponctuelles (OMP) relatives aux types de tâches suivants :

- $T_{décomposer}$ « Décomposer un nombre naturel a sous forme de produit de deux nombres naturels » ; Ce type de tâches a un rôle très important dans l'OM. Il fait travailler les concepts de multiples et diviseurs d'un nombre naturel, nécessaires pour la compréhension de la division euclidienne.

- $T_{quotient}$ « Déterminer le quotient de la division euclidienne de deux nombres naturels a et b » ;

- T_{reste} « Déterminer le reste de la division euclidienne de deux nombres naturels a et b » ;

Par ailleurs, nous incluons dans l'organisation mathématique de référence le type de tâches $T_{représenter}$ « représenter la division euclidienne de deux nombres naturels a et b ». Ce type de tâches fait référence à la représentation de l'expression mathématique liée à la division euclidienne. Il peut être accompli avec des ostensifs différents et, par conséquent, peut donner lieu à des productions différentes.

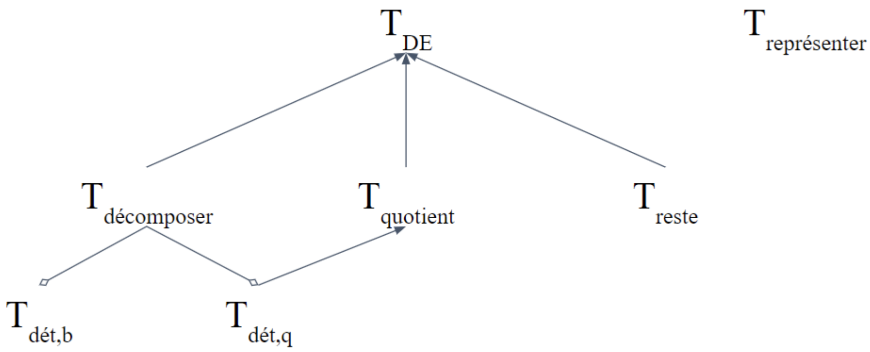


Figure 5. Structuration des types de tâches. Les notations → et ◇— indiquent respectivement « est un sous-type de tâche de » et « est ingrédient de ».

Chacun des types de tâches de la Figure 5 a son propre développement de sa ou ses praxéologies que nous ne présentons pas ici.

La construction d'un modèle praxéologique de référence de $T_{prog}[T_{DE.2}]$ va combiner des praxéologies relatives au type de tâches T_{DE} avec des éléments praxéologiques de relevant de l'algorithmique. Il y a donc un lien entre les techniques de $T_{prog}[T_{DE.2}]$ et T_{DE}.

Au niveau praxéologique relevant de la programmation nous pouvons retrouver les ingrédients de techniques suivants :

- $T_{choix, b}$ « Choisir le type de saut avancer de b »
- $T_{choix, q}$ « Choisir le nombre de sauts q »
- $T_{choix, r}$ « Choisir le type de saut avancer de r »
- $T_{répéter}$ « Utiliser le bloc répéter dans le programme Scratch
- $T_{dupliquer}$ « Dupliquer un bloc d'instruction »
- $T_{associer}$ « Associer deux blocs d'instruction entre eux »
- T_{tapper} « Taper le nombre q dans le bloc répéter »
- T_{lancer} « Lancer le programme crée sur Scratch »
- $T_{valider}$ « Valider le programme »

La combinaison de ces ingrédients donne lieu à des techniques différentes et donc des programmes Scratch différents. Nous considérons les ingrédients T_{tapper}, $T_{associer}$, $T_{dupliquer}$ et T_{lancer} comme des types de tâches élémentaires (Chaachoua, 2018). Il n'est donc pas nécessaire de détailler leurs techniques. Les types de tâches $T_{choix, b}$, $T_{choix, q}$, $T_{choix, r}$ peuvent renvoyer dans leurs techniques respectivement aux ingrédients qui sont des types de tâches rattachés au T_{DE} (Figure 5) : $T_{det, b}$, $T_{det, q}$ ou $T_{quotient}$ et T_{reste}.

Les types de tâches T_{lancer} et $T_{valider}$ ont un statut différent par rapport aux autres, car ils dépassent la consigne relative à la création d'un programme Scratch. Dans le dispositif Scratch l'accomplissement de T_{lancer} se fait avec un clic sur le drapeau vert qui est au-dessus de la scène, ou avec un double clic sur le programme. On le considère comme type de tâches élémentaire. L'accomplissement du type de tâches $T_{valider}$ consiste à vérifier, après l'exécution du programme prise en charge par le dispositif, que toutes les contraintes déterminées par le type de tâches de contexte que le programme doit réaliser soient respectées. Ce type de tâches relève du topos de l'élève et est complètement à sa charge.

Pour modéliser les savoirs issus d'une organisation mathématique mixte nous avons construit une OM de référence basée sur un ensemble d'organisations mathématiques ponctuelles relevant du type de tâches T_{DE} (Déterminer un ou deux termes inconnus de l'expression $a = bq + r$, étant le terme a connu et avec $0 \leq r < b$). Pour représenter le bloc praxis de cette organisation, nous nous sommes servis surtout de la notion d'ingrédient d'une technique et nous avons décrit les ingrédients de techniques pour les types de tâches étudiés, jusqu'à un certain niveau de granularité.

Ensuite, nous avons défini le type de tâches $T_{prog}[T_{DE.2}]$ (Créer un programme dans Scratch qui réalise $T_{DE.2}$) en définissant des variables pour le type de tâches $T_{prog}[T_{DE.2}]$ issues du dispositif conçu pour l'OMM dans Scratch, nous avons pu construire un autre modèle de référence $T_{prog}[T_{DE.2}]$. En particulier, les ingrédients des techniques de T_{DE} se présentent à un deuxième niveau de granularité, le premier étant constitué par des ingrédients algorithmiques. L'ensemble de ces constructions nous a amenés à représenter les blocs praxis de l'OMM de la séquence à l'aide d'un graphe mettant en lumière diverses relations entre les types de tâches pouvant émerger lors de la réalisation de l'OMM.

Une modélisation de ce type nous a permis de concevoir un dispositif d'évaluation des praxéologies mathématiques acquises grâce au travail de l'OMM et concevoir une grille d'analyse pour observer, décrire et analyser les praxéologies didactiques mises en œuvre par les enseignants (Crisci, 2020). Ces mises en œuvre ont montré la pertinence du modèle de référence et contribué de fait à sa validation.

Conclusion

Un modèle praxéologique de référence sur la notion mathématique visée, ainsi que la structuration des praxéologies sur plusieurs niveaux de granularité, issue du cadre T4TEL, a permis de concevoir une modélisation des savoirs d'une organisation mathématique mixte dans laquelle les praxéologies propres à l'informatique permettant d'évoquer des praxéologies mathématiques.

Bien que cette modélisation ait été faite dans un cas d'étude particulier, celui de travailler la division euclidienne dans un environnement Scratch, la démarche de conception du modèle praxéologique de référence peut se généraliser aux types de tâches $T_{prog}[T]$ où T est un type de tâches mathématiques.

Le niveau scolaire et les séquences construites autour du type de tâches T_{DE} étudiés, convoquent des praxéologies algorithmiques qui ne présentent pas de complexité. Il serait bien de poursuivre ce travail, par un choix convenable de T, pour engager davantage le travail sur la dimension algorithmique.

Références

Artaud, M. (1998). Introduction à l'approche écologique du didactique. Actes de la IXème école d'été de didactique des mathématiques, pages 101–139.

Banwitiya, Y. (1993). L'ingénierie du sens en mathématique : la division dans N, Q et D à l'école primaire. Thèse de doctorat, Université Bordeaux I.

Bosch, M., Gascon, J. (2005). La praxéologie comme unité d'analyse des processus didactiques. Dans Mercier A. & Margolinas C. (Dir) *Balises pour la didactique des mathématiques*. pp. 197 – 122. Grenoble : La Pensée Sauvage.

Briant, N. (2013). Etude didactique de la reprise de l'algèbre par l'introduction de l'algorithmique au niveau de la classe de seconde du lycée français. Thèse de doctorat, Université Montpellier 2.

Brousseau, G. (1972). La division euclidienne aux cours élémentaire et cours moyen. Les mathématiques de l'école élémentaire, pages 267–278.

Chaachoua H. (2018). T4TEL un cadre de référence didactique pour la conception des EIAH. In J. Pilet & C. Vendeira (Éd.), Actes du séminaire national de l'ARDM. Paris.

Chaachoua H., Pilet J., Bessot A. (2022) The Analysis of Dominant Praxeological Models with a Reference Praxeological Model: A Case Study on Quadratic Equations. In: Chevallard Y. et al. (eds) Advances in the Anthropological Theory of the Didactic. Birkhäuser, Cham.

Chaachoua, H. (2020). T4TEL : Un cadre de référence pour la formalisation et l'extension du modèle praxéologique. Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática, 22(4), 103-118.

Chaachoua, H., Tchounikine, P. et Crisci, R. (2018). L'algorithmique et la programmation pour la construction du sens de la division euclidienne.

Charnay, R., Combier, G., Dussuc, M.-P., Anselmo, B. Et Madier, D. (2018). CAPMaths CM1 Éd.2018 - Mon cahier de recherches. Hatier. Library Catalog : www.editionshatier.fr

Chevallard, Y. (1985). La transposition didactique : du savoir savant au savoir enseigné. La Pensée Sauvage. Grenoble.

Chevallard, Y. (2001). Les mathématiques et le monde : dépasser l'horreur instrumentale. Quadrature, 41:25–40.

Couderette, M. (2016). Enseignement de l'algorithmique en classe de seconde. une introduction curriculaire problématique. Annales de de didactique et de sciences cognitives, 21:267–296

Crisci, R. (2020). Etude des conditions de viabilité d'une approche basée sur l'algorithmique et la programmation pour l'apprentissage de la division euclidienne l'école primaire. Thèse de doctorat. Université Grenoble-Alpes.

des processus didactiques. In A. Mercier & C. Margolinas (dir.), Balises en didactique des mathématiques (pp. 107-122), Grenoble, La Pensée sauvage.

Dowek, G. (2011). Les quatre concepts de l'informatique. In Sciences et technologies de l'information et de la communication en milieu éducatif : Analyse de pratiques et enjeux didactiques. Actes du quatrième colloque international DIDAPRO 4 - Dida&Stic

Fischbein, E., Deri, M., Nello, M. S. Et Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. Journal for research in mathematics education, pages 3–17.

Gautier, F., Goupil-Lebot, M.-T., Le Lous, A., Le Poche, G., Prophète, Y., Reichert, F. Et Roux, V. (2008). Les divisions. Liaison cycle 3–6ème. 28, 29, 209.

MEN (2015). Programme d'enseignement du cycle de consolidation (cycle 3).

Modeste, S. (2012). Enseigner l'algorithme pour quoi? Quelles nouvelles questions pour les mathématiques ? Quels apports pour l'apprentissage de la preuve ? Thèse de doctorat, Université de Grenoble.

Peltier, M.-L., Briand, J., Ngono, B. Et Vergnes, D. (2009). Euro Maths CM1 éd. 2009 - Manuel de l'élève + Aide mémoire. Hatier. 61.

Squire, S. Et Bryant, P. (2002). The influence of sharing on children's initial concept of division. Journal of experimental child psychology, 81(1):1–43.

Strock, J.-M. et Artaud, M. (2019). Du logos des organisations algorithmiques dans l'enseignement secondaire. Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática, 21.

Vergnaud, G. (1994). Multiplicative conceptual field: what and why. The development of multiplicative reasoning in the learning of mathematics, pages 41–59.

Análisis de un REI-FP sobre los conocimientos lógicos en la formación matemático-didáctica inicial de maestros

A.M. Lerma[1], B. Barquero[2], F.J. García[3], M. Hidalgo-Herrero[4], A. Ruiz-Olarría[5] and T.A. Sierra[6]

[1]Universidad de Jaén, España; alerma@ujaen.es

[2]Universidad de Barcelona, España; bbarquero@ub.edu

[3]Universidad de Jaén, España; fjgarcia@ujaen.es

[4]Complutense de Madrid, España; mhidalgo@ucm.es

[5]Autónoma de Madrid, España; arolarri@gmail.com

[6]Computense de Madrid, España; tomass@edu.ucm.es

In this paper, we address a mathematical-didactic problem in early childhood teacher education, concerning the teaching of logical knowledge. As an answer to this problem, we have analysed a study and research paths for teacher education (SRP-TE). It aims at provoking prospective teachers into providing answers to the professional questions about what logical knowledge is and how to provide a meaning to their teaching in early childhood education.

Keywords: Teacher Education, Early Childhood Education, Logical Knowledge, Anthropological Theory of the Didactic, Study and Research Paths.

Dans cet article, nous abordons un problème mathématique et didactique dans la formation des professeurs des écoles, en relation avec l'enseignement du développement de la pensée logique. En réponse à ce problème, nous analysons un parcours d'étude et de recherche pour la formation des professeurs (PER-FP), dans ce cas des enseignants de l'école maternelle, à travers lequel l'objectif est que les futurs enseignant abordent et apportent des réponses aux questions professionnelles sur ce que sont les connaissances dites logiques et comment organiser leur enseignement fonctionnel à l'école maternelle.

Mots-clés: Formation des Enseignants, L'école Maternelle, La Pensée Logique, Théorie Anthropologique de la Didactique, Parcours d'Étude et de Recherche.

En este trabajo abordamos un problema matemático-didáctico de la formación de maestros de Educación Infantil en torno a la enseñanza de los conocimientos lógicos. Como respuesta a dicho problema se analiza un recorrido de estudio e investigación para la formación del profesorado (REI-FP), en este caso maestros de Educación Infantil, mediante el que se persigue que los maestros-estudiantes traten y aporten respuestas a las cuestiones profesionales sobre qué son los llamados conocimientos lógicos y cómo organizar su enseñanza con sentido en Educación Infantil.

Palabras clave: Formación de Maestros, Educación Infantil, Conocimientos Lógicos, Teoría Antropológica de lo Didáctico, Recorrido de Estudio e Investigación.

Introducción

Uno de los temas específicos de la formación matemático-didáctica de los maestros de Educación Infantil (EI, en adelante) es el de los conocimientos lógicos. Postulamos que la causa de que estos conocimientos formen parte solo del currículo actual de EI se debe a la influencia de los trabajos de psicología cognitiva (Piaget & Szeminska, 1967) donde se plantea que para que el alumno pueda llegar a construir los saberes en torno al número natural es necesario haber trabajado la clasificación y la ordenación. En España dichos conocimientos lógicos aparecen por primera vez en los currículos de Educación Infantil, Primaria y Secundaria en 1970 con la reforma educativa de las "matemáticas modernas”. Sin embargo, a finales de los años 80 se realiza una contrarreforma de dichos currículos que elimina, en general, los conocimientos lógicos salvo en la EI. En consecuencia, dichos temas se siguen estudiando en la formación de maestros de EI. Así, por ejemplo, en Muñoz-Catalán y Carrillo (2018, p. 21) se plantea que para que el niño pueda comprender el número debe hacerse “partiendo de la seriación y la clasificación”. Sin embargo, Hans Freudenthal afirma:

> In no way does the child constitute the single number as a class of mutually equivalent sets, not even unconsciously. It is a self-deceit of adult optics to stress the invariance under one-to-one mappings; this is the mental attitude of an adult mathematician who cannot forget his own theory of natural numbers. The child learns this one invariance in a much broader context, not that the number is invariant under one-to-one mappings, but that even if I count again tomorrow I still get five fingers, that all men have the same number of certain things, and that the number of marbles under the handkerchief does not change if I say “abracadabra”. The invariance under one-to-one mappings is a cosy corner in this large complex, the hobby of adults who sell it as the numerosity aspect. (Freudenthal, 1973, p.192)

Igualmente, otros autores (Hughes, 1986; Ermel, 1990; Brissiaud, 1993) consideran que tratar las actividades lógicas como la clasificación y la ordenación principalmente como actividades que preparan para la construcción del número hace que estas pierdan su razón fundamental. En la misma línea, Lacasta et al. (2012, p. 372) proponen que “Las actividades de tipo lógico y relacional deben ser apreciadas por sus finalidades propias y no por su supuesto carácter prenumérico”.

Así, en este trabajo consideramos que los conocimientos lógicos permiten resolver una problemática específica. El ciudadano medio, cuando tiene que enfrentarse a los diferentes problemas que se presentan en el mundo donde vive, casi seguro que tendrá que recurrir a designar, clasificar, ordenar y organizar objetos. Por tanto, no debe asumirse necesariamente que sean conocimientos prenuméricos y quedar restringidos a la EI. Nuestra propuesta debe entenderse en el marco de un proyecto amplio cuyo objetivo es, por un lado, explicar y aclarar qué entendemos por conocimientos lógicos y, por otro, analizar si es posible articularlos con otras áreas de las matemáticas, estableciendo las bases de la actividad científica y lógica, que viene condicionada por el desarrollo de la capacidad de identificar semejanzas y diferencias, de comparar, de clasificar y ordenar, de designar y simbolizar, de identificar y utilizar algoritmos y de realizar deducciones iniciales. De acuerdo con Orús (1992), consideramos que los conocimientos lógicos deberían seguir siendo objeto de estudio, al menos a lo largo de toda la Educación Primaria (EP, en adelante).

Nuestra propuesta tiene como objetivo dar respuesta al siguiente problema docente del formador de maestros, enunciado en el marco de la teoría antropológica de lo didáctico:

> ¿*Qué praxeologías puede utilizar el formador de maestros para que el futuro maestro sea capaz de elaborar las praxeologías matemáticas y didácticas necesarias para gestionar el estudio de los conocimientos lógicos en la Educación Infantil de modo que aflore su razón de ser*?

La búsqueda de posibles respuestas a esta cuestión nos conduce a considerar dos problemas interrelacionados e inseparables:

A. Un problema de *ingeniería matemática*, que consiste en construir algunos de los elementos de la praxeología matemática de referencia sobre los conocimientos lógicos, tanto en la institución de EI como en la formación de maestros.

B. Un problema de *ingeniería didáctica* en el que se diseñen praxeologías didácticas que contengan las cuestiones a las que responden los conocimientos lógicos, tanto en la EI como en la formación de maestros.

Diseño de un REI-FP y contexto de experimentación

Esta investigación se desarrolla en el marco de la teoría antropológica de lo didáctico (TAD), donde los *recorridos de estudio e investigación* (REI) (Bosch, 2018) se proponen como dispositivos didácticos para transitar hacia el denominado paradigma del "cuestionamiento del mundo" (Chevallard, 2015). Los REI se fundamentan en el estudio y la investigación de preguntas abiertas para las que las matemáticas son una herramienta que proporciona respuestas, lo que lo aleja del paradigma de la "visita de las obras", centrado en la presentación de contenidos matemáticos para ser mostrados a los estudiantes. En lo que afecta a la formación docente, es necesario plantearla desde el paradigma de "cuestionar el mundo de la profesión docente" (García et al., 2020). En este contexto, los *recorridos de estudio e investigación* para la formación del profesorado (REI-FP) conducen a poner en el punto de partida cuestiones sobre la profesión y a trabajar con los docentes los conocimientos matemáticos y didácticos necesarios para abordar estas cuestiones.

El REI-FP en el que aquí nos centramos se inicia con la cuestión de la profesión ($Q_{0\text{-FP}}$): *¿A qué llamamos conocimientos lógicos*? *¿Cómo se puede caracterizar y organizar su estudio en las etapas de educación infantil y primaria*? Debemos subrayar tres aspectos importantes sobre este REI-FP.

En primer lugar, la cuestión generatriz de partida es amplia y compleja y no pretendemos, ya de partida, abarcar la extensa tipología de saberes que tradicionalmente se han asignado a este dominio de la matemática. Las actividades diseñadas y, en particular, el REI a ser "vivido" por los maestros en formación, va a suponer, inevitablemente, una acotación del vasto territorio que conlleva dicho dominio. Las actividades lógicas asociadas al REI-FP que presentaremos llevan a considerar tipos de tareas cuya respuesta pasa por utilizar técnicas de clasificación, categorización, ordenación y designación, así como su justificación. En segundo lugar, debemos destacar la dificultad en el diseño de tipos de tareas que supongan un reto para estos maestros en formación, donde los saberes matemáticos a estudiar surjan como una buena respuesta de los tipos de tareas propuestos. Esto nos ha llevado a diseñar un REI sobre la "Sala de objetos perdidos" que no está pensado, de partida, para implementarse directamente en EI o EP, sino que ha sido especialmente diseñado para implementarse en la formación de maestros. En tercer lugar, este REI-FP ha sido experimentado en la formación matemático-didáctica del grado de Maestros de EI de las universidades de Jaén, Autónoma de Madrid y Complutense de Madrid y del doble grado de Maestros de EI y EP de la Universidad de Barcelona. No ha sido objetivo del trabajo comparar las adaptaciones particulares que se han realizado en los distintos contextos de formación, foco de futuros trabajos.

Contexto para el diseño, implementación y análisis del REI-FP

La propuesta de formación fue diseñada, y adaptada a cada contexto de formación, por el equipo de investigadores. En esta comunicación, nos centraremos en la implementación desarrollada en la Universidad de Barcelona, durante el primer semestre de los cursos 2020-21 y 2021-22, con los maestros en formación del doble grado de Maestros de EI y EP, de la asignatura de "Didáctica de la Matemática" (asignatura anual, 3er curso, 9 créditos ECTS). Trabajar con el doble grado nos permitirá abordar la transición entre infantil y primaria. Por ello, a lo largo del texto, haremos mención a ambas etapas educativas.

Solo se han experimentado los tres primeros módulos (módulos 0, 1 y 2) del REI-FP. Es decir, los estudiantes en formación no han llegado a convertirse en diseñadores, docentes y analistas de adaptaciones de actividades en la EI. La puesta en funcionamiento de los módulos 0, 1 y 2 se ha realizado con 42 estudiantes del grado, durante un total de 6 semanas con 4 horas de docencia presencial por semana, en dos sesiones de 2 horas. A lo largo de toda la implementación, los maestros en formación trabajan en los mismos grupos de trabajo, 10 grupos en total en el aula, con entre 3 y 5 miembros. Los estudiantes trabajan con autonomía en su grupo, bajo la guía de la formadora, y se pide que realicen la entrega de informes al finalizar cada módulo y, en ocasiones, la presentación de respuestas parciales para facilitar la puesta en común, la emergencia y comparación de propuestas por parte de los grupos.

Diseño y análisis de un REI-FP sobre los conocimientos lógicos

Módulo 0: exploración de la cuestión generatriz

El módulo 0 comienza presentando la cuestión generatriz $Q_{0\text{-FP}}$ (introducida en la sección anterior) sobre a qué se llama conocimientos lógicos y cómo se organiza su enseñanza y aprendizaje en la etapa de EI y EP. Se pide a los futuros maestros (estudiantes, en adelante) que exploren algunas respuestas existentes en documentos y recursos, seleccionados previamente por los diseñadores, que permitan aportar una primera aproximación a los modelos epistemológicos y didácticos dominantes relativos a estos conocimientos. Los diseñadores organizan esta selección en dos grandes líneas de exploración: [Línea 1] cómo se propone en los currículos (u otros documentos oficiales) la enseñanza de los conocimientos lógicos (qué estatus tiene, qué terminología se emplea, qué relación tiene con otros dominios de la matemática, etc.); y, [Línea 2] analizar una selección de actividades propuestas en estas etapas educativas (tradicionalmente bajo el título de “lógica”) para describir qué tipo de tareas se propone y qué actividad matemática conlleva. Cada grupo se encarga solo de una de estas líneas, pudiendo consultar otros documentos al margen de los ofrecidos.

Después de este primer análisis, se hace una puesta en común, que permite poner de manifiesto algunas necesidades importantes de los estudiantes, incluso vislumbrar fenómenos didácticos de relevancia relativos a los conocimientos lógicos. En primer lugar, se abre la necesidad de compartir y definir una *terminología* concreta para referirse a los conocimientos lógicos, así como dotarlos de significado. En otras palabras, a partir de la exploración de ambas líneas, empieza a aparecer terminología específica para referirse al conocimiento lógico (por ejemplo: atributos, clasificación, identificación, selección, ordenación, comparación, secuenciación, listas, entre otros), aunque no se asocian estos términos a definiciones claras más allá de las usuales en el lenguaje natural. Los estudiantes junto con la formadora consensúan crear un *glosario* que se irá completando a lo largo del desarrollo del resto de módulos.

En segundo lugar, los estudiantes destacan el hecho de que no encuentran grandes diferencias entre los currículos de EI y EP, sin que en ninguno de ellos se aporte una descripción específica. Destacan que, cuando se hace referencia a alguna de las “acciones” o “tareas” lógicas, se asocian a competencias o dimensiones generales del quehacer matemático como, por ejemplo: *razonar*, *validar*, *pensar críticamente*, *pensar matemáticamente*, *resolver problemas*, *probar*, etc. Dada la poca explicitación de los “contenidos” que puede abarcar este “bloque” curricular, los estudiantes destacan la falta de conexión con otros dominios de la matemática escolar que tienen estructuras más explícitas como, por ejemplo, los bloques curriculares de números y operaciones, estadística y geometría. En resumen, los estudiantes señalan la transparencia de este cuerpo del saber escolar, tanto en la etapa de EI como de EP, y la falta de descriptores claros para referirse a estos saberes.

Módulo 1: vivencia y experimentación del REI la “Sala de objetos perdidos”

En este módulo, se pide a los futuros docentes que adopten un rol de estudiante para vivir una propuesta didáctica. En este caso, dicha experiencia se basa en el REI “Sala de objetos perdidos” diseñado para que emerjan los conocimientos lógicos que aparecen en la figura 1. El objetivo de este módulo es crear un medio didáctico compartido entre los estudiantes y los formadores que les permita posteriormente realizar el análisis del conocimiento matemático que ha aparecido en la actividad vivida y evidenciar la necesidad de ampliar las herramientas para su análisis.

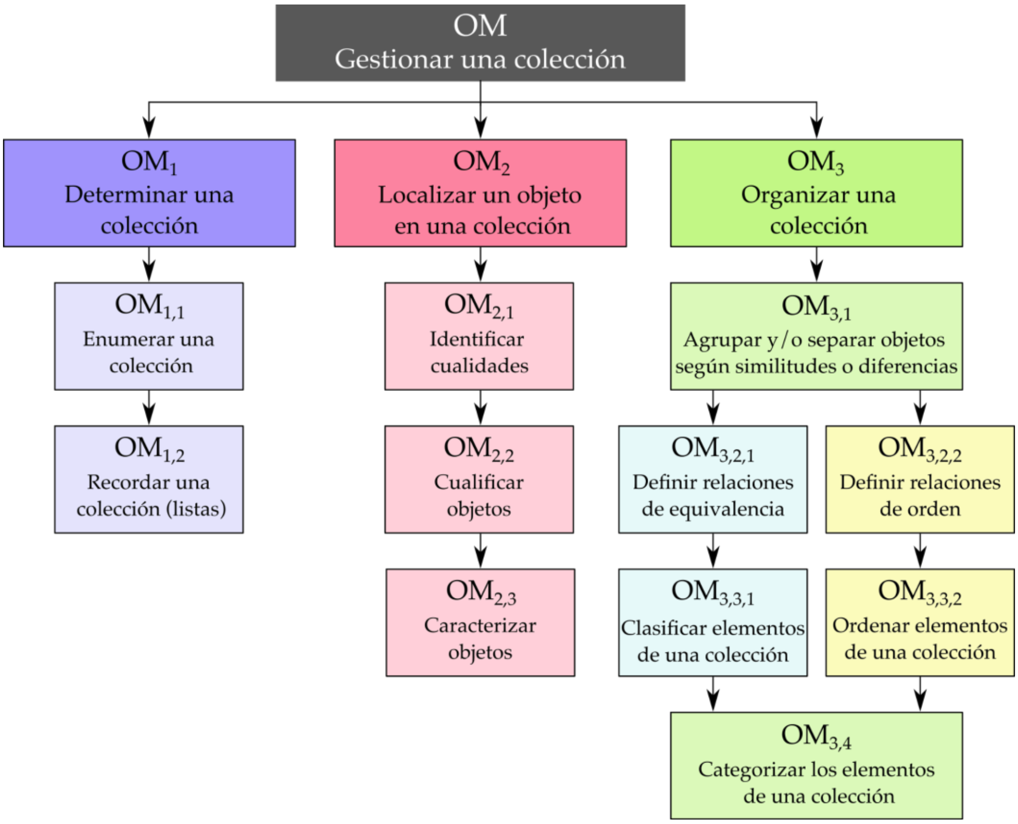


Figura 1: Organizaciones matemáticas (OM) que emergen durante la vivencia del REI (análisis tras una experimentación previa durante el curso 2019/20)

El REI "Sala de objetos perdidos" comienza con la presentación de la situación de un colegio que pretende destinar una pequeña sala para organizar la gran cantidad de objetos perdidos que se van encontrando a lo largo del curso escolar. Este colegio siempre ha utilizado cajas para guardar los diferentes objetos perdidos, sin ningún criterio sobre cómo organizarlos. La cuestión inicial Q_0, que inicia este REI, es "*¿cómo organizar y gestionar la* nueva *sala de objetos perdidos de la escuela*?" Más concretamente, se pide a los estudiantes que trabajen en propuestas para organizar y gestionar esta nueva sala, que tendrán que ser expuestas y debatidas, para ser presentadas al colegio.

Los estudiantes, organizados en grupos, reciben el enlace a un documento en línea (enlace a una *presentación de Google*) con muchos objetos organizados en 9 cajas, que simulan los que se han perdido en el colegio, junto a la cuestión Q_0 expuesta anteriormente. Como respuesta final, los estudiantes diseñarán una propuesta conjunta que será la que se presente a la escuela. Se espera que los grupos, guiados por los formadores, pasen por las siguientes fases:

Fase 1. Localización de objetos en cajas. En esta fase se realiza la primera exploración de los objetos perdidos. Los estudiantes tendrán que dar respuesta a la pregunta Q_1: "*Dada una colección de objetos, ¿cómo podemos localizar un objeto concreto?*". Para provocar algunos conflictos, se pide a los estudiantes que estudien distintas situaciones que se pueden presentar cuando las familias de los niños del colegio reclamen algún objeto perdido.

Centrándonos en el grupo de estudiantes analizado en esta comunicación, puede verse cómo en la propuesta que este hace de las distintas situaciones se indican como necesarios ciertos atributos para poder caracterizar los objetos ($OM_{2,1}$, $OM_{2,2}$ y $OM_{2,3}$). El grupo explicita que *necesita más detalles para poder determinar si los objetos que se reclaman están en las cajas de los objetos perdidos*. En algunos casos incluso indica *qué características son las necesarias para determinar qué objeto es el que se está reclamando* (características que diferencian al objeto de otros que comparten los atributos principales que lo caracterizan). Tras este análisis, manifiestan, como se puede ver en la figura 2, la necesidad de clasificar los objetos de las cajas para que sea más sencilla su localización ($OM_{3,1}$, $OM_{3,2,1}$ y $OM_{3,3,1}$).

```
Hemos decidido que sería más sencillo localizar los objetos si estuvieran
clasificados. Comenzamos haciendo diferentes grupos de objetos y terminamos
haciendo 5 clases.
```

Figura 2: Extracto de la conclusión obtenida tras el análisis de las situaciones, donde el grupo hace referencia explícita a la clasificación e incluso a las clases de equivalencia que se obtienen tras esta

Fase 2. Diseño de la propuesta. Una vez que, en la Fase 1, los estudiantes encuentran la necesidad de organizar la colección de objetos, tendrán que abordar la cuestión Q_2: "*¿Cómo organizar y gestionar los objetos perdidos de la escuela para que nuestra propuesta sea la elegida*?".

En la fase anterior, el grupo indica en su informe que "*empieza a hacer diferentes grupos de objetos y termina con 5 clases*" (ver figura 2), es decir, inicia el proceso de clasificación explorando las características de los objetos ($OM_{3,1}$), hasta que los clasifica atendiendo al criterio *tipo "general" de objeto* (accesorio personal, objeto electrónico, material escolar, juguete y libro). Durante esta fase constatan la necesidad de establecer nuevas clases/relaciones de equivalencia dentro de las ya construidas, indicando que *necesitan nuevos subgrupos y que para crearlos tendrán que tener en cuenta las cualidades de los objetos* ($OM_{2,1}$ y $OM_{2,2}$). Para cada una de las clases realizadas, estudian distintos criterios ($OM_{3,1}$ y $OM_{3,2,1}$). Por ejemplo, se plantean clasificar los objetos que se encuentran en la clase *material escolar* usando como criterio el *color*, aunque lo abandonan pronto, *ya que este criterio considerará equivalentes objetos que son muy diferentes* (ver figura 3). Finalmente, el grupo decide considerar como criterio el *tipo de objeto y su funcionalidad* y dejar una caja de *otros* que aglutine todos los elementos que formarían clases unitarias ($OM_{3,3,1}$) bajo la relación de equivalencia definida ($OM_{3,2,1}$). Por ejemplo, definen las clases *mochilas y estuches* porque son contenedores; *bolígrafos y correctores tipo Tipp-Ex* porque el segundo corrige al primero; *afiladores y gomas* porque están vinculados con lápices.

```
Con respecto al "material escolar", hemos decidido crear algunos subgrupos
dependiendo del color de los objetos, pero tendríamos objetos muy diferentes
dentro de estos grupos. Por ejemplo, en el subgrupo "azul" encontraríamos
lápices de colores, marcadores, bolígrafos, etc.
```

Figura 3: Extracto de las reflexiones del grupo durante la fase 1

Después de definir las clases de equivalencia ($OM_{3,2,1}$), atendiendo a los distintos criterios ($OM_{3,1}$), deciden colocar cada clase de equivalencia en cajas diferentes ($OM_{3,3,1}$) que irán en las baldas de unas estanterías. Además, en cada caja pondrán una *etiqueta de color con un código QR y el nombre* que representa los objetos que se encuentran en su interior. El nombre es la característica definitoria de los objetos de la clase, pero no seleccionan un representante canónico de la misma. A la hora de colocar las cajas en las baldas toman como criterio el peso de estas ($OM_{3,1}$), dejando las más pesadas en las baldas inferiores ($OM_{3,2,2}$ y $OM_{3,3,2}$). Aplicando esta ordenación en la clasificación anteriormente definida (ordenación de las clases del conjunto cociente obtenido por la relación de equivalencia), los estudiantes categorizan los objetos perdidos ($OM_{3,4}$).

Una vez que han decidido cómo organizar los objetos y cómo colocarlos en las estanterías de la sala, el grupo toma la decisión de diseñar una página *web* donde se recoja la clasificación realizada. En esta página, que estará conectada con los códigos *QR* de las cajas, se recogerán las fotos de todos los objetos perdidos de la escuela ($OM_{1,1}$ y $OM_{1,2}$). De esta forma, no será necesario estar delante de los objetos para poder saber si un objeto concreto está en la sala, es decir, la página *web* permitirá controlar la colección de objetos perdidos y llevar a cabo anticipaciones (ver figura 4). Además, incluyen un formulario de *Google* para reclamar los objetos perdidos, en el que los familiares tendrán que responder preguntas sobre los detalles del objeto que se ha perdido ($OM_{3,2}$).

> Consideramos que esto [página *web* y formulario de *Google*] es una idea innovadora, económica y útil de cara a que el conserje no pierda tiempo buscando en el interior de cada caja.

Figura 4: Extracto de las reflexiones finales de la fase 1

Fase 3. Evaluación de las propuestas. En esta fase los estudiantes tendrán que evaluar las propuestas. Para ello, darán respuesta a la cuestión Q_3: "*¿Cómo comparar y evaluar las diferentes propuestas?*". Esto implica responder a si *todas las propuestas mejoran la localización de los objetos perdidos* y a si *hay propuestas más convenientes que otras.*

Una vez que todos los grupos exponen las propuestas, cada uno decide qué características tenía que tener una buena clasificación o categorización y organización de los objetos perdidos. Después de este trabajo, entre todos los grupos, se decide qué debe cumplir dicha organización (ver figura 5).

> 1. Se tienen que crear clases/categorías generales y, dentro de estas, nuevas clases/categorías (es decir, crear subclases/subcategorías).
> 2. Claridad y facilidad para entender las clases/categorías (subclases/subcategorías).
> 3. Las clases/categorías (subclases/subcategorías) tienen que ser disjuntas. Es decir, no puede haber ambigüedad a la hora de ubicar un objeto.
> 4. Cualquier objeto que se pierda tiene que pertenecer a una de las clases/categorías creadas previamente. Por tanto, se debe incluir en cada una de ellas la subclase/subcategoría "otros".
> 5. Flexibilidad.

Figura 5: Criterios que tiene que cumplir una buena clasificación/categorización y organización de los objetos perdidos

Finalmente, cada grupo analiza sus propuestas teniendo en cuenta estos criterios. En este sentido, el grupo aquí analizado modifica la clasificación realizada, ya que *solo se había centrado en los objetos que se le había proporcionado*. Por ejemplo, en su propuesta no habían considerado que se pudiera perder ropa. Tras este análisis conjunto, estudian los posibles objetos que pueden perderse en el colegio y los incluyen en su clasificación. Otro aspecto que comentan que mejoran son *los datos que recogen de los objetos perdidos*, para que se ajusten mejor los datos con los criterios según los cuales proponen la clasificación de los objetos.

Módulo 2: análisis del REI vivido

En este módulo, se propone a los estudiantes adoptar el rol de "analistas" y abordar la tarea profesional de analizar la actividad matemática que se desarrolla en el REI "Sala de objetos perdidos" y, en particular, el análisis del conocimiento lógico que ha entrado en juego.

Se pide a los grupos que tomen como material de partida, para ser analizado, sus informes grupales y las puestas en común realizadas a lo largo de la vivencia del REI para describir qué conocimientos lógicos han aparecido en la experiencia, así como darles nombre y una descripción. A modo de síntesis mostramos, en la figura 6, el análisis realizado por el grupo seleccionado.

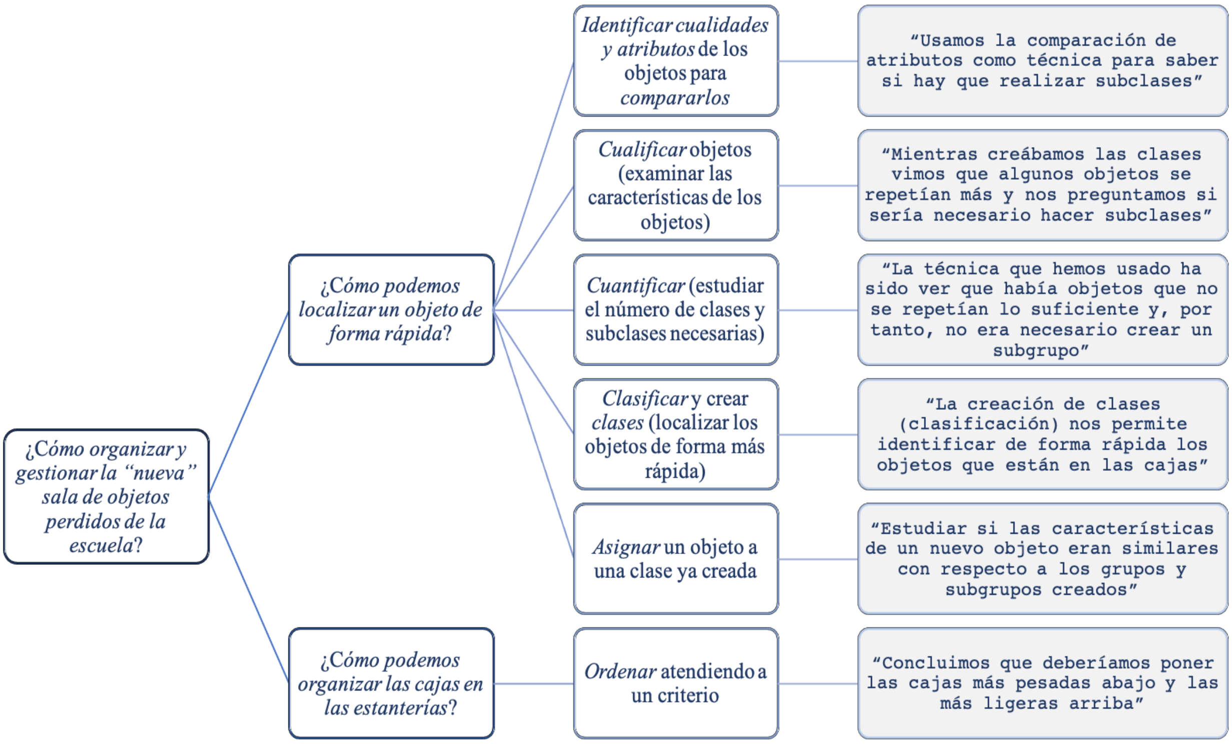


Figura 6: Esquema síntesis del análisis realizado durante el módulo 2 (acompañado de algunas reflexiones de los estudiantes del grupo)

Teniendo en cuenta todos los conocimientos que este grupo pone en funcionamiento (ver módulo 1), siendo bastante rico el análisis que propone, solo se centra en la actividad matemática relacionada con la localización de un objeto en una colección (OM_2) y la organización de una colección (OM_3), dejando de lado lo referente a la determinación de una colección (OM_1), a pesar de que aparezca en sus informes. Además, queremos subrayar también cómo, aunque no se recoge en el esquema de la figura 1, también emergen organizaciones matemáticas relacionadas con el número y la numeración, en el sentido cardinal (identificadas por los estudiantes del grupo) y en el sentido ordinal que (aunque no ha sido identificado por el grupo) aparece, por ejemplo, en el momento de comunicar a otra persona dónde se encuentra una caja concreta dentro de la estantería.

Una vez terminado este primer análisis grupal, la formadora lo completa poniendo en común los distintos conocimientos descritos y analizados por los distintos grupos, dándoles además una definición más concreta, junto con ejemplificaciones, fruto de la experiencia.

A partir de este trabajo, se enriquece el *glosario*, permitiendo a los estudiantes ir más allá de la primera interpretación espontánea-natural iniciada en el módulo 0. Este trabajo consigue delimitar y compartir con los estudiantes un marco común epistemológico de referencia (bastante más amplio que el que aparece en los módulos 0 y 1 anteriores) sobre los conocimientos lógicos que permite institucionalizar gran parte de las tareas o cuestiones, técnicas y justificaciones relativas a las organizaciones matemáticas que han entrado en juego. Incluso logra mostrar componentes de ellas que son más difíciles de ser explicitadas como ocurre, por ejemplo, con las *relaciones de equivalencia* y *relaciones de orden*, ambas usadas implícitamente (si bien difícilmente detectadas) por los estudiantes.

Conclusiones

En esta contribución hemos esbozado una propuesta de REI-FP sobre el conocimiento lógico, que lleva a los estudiantes a cuestionarse qué es este conocimiento en la EI y EP y cómo describir, analizar y planificar su enseñanza y aprendizaje en estas etapas escolares. Aunque no se han analizado todos los módulos del REI-FP, consideramos que este análisis preliminar nos ha llevado a evidenciar que las cuestiones problemáticas abordadas a lo largo de este recorrido de formación han ayudado a los docentes y educadores a construir progresivamente el equipamiento praxeológico necesario para referirse, analizar y diseñar actividades que involucren conocimientos lógicos.

Los procesos desarrollados en estas experimentaciones han mostrado la necesidad de poner a los estudiantes ante un profundo cuestionamiento epistemológico y didáctico sobre cómo se interpreta este saber lógico desde la TAD y su vinculación con otros dominios matemáticos (y extramatemáticos). El cuestionamiento propuesto podría ir acompañado de la construcción de modelos epistemológicos robustos que puedan ser utilizados como modelos de referencia para el diseño de actividades *ad hoc*. Se espera que dichos modelos brinden herramientas a los futuros maestros para que puedan analizar el conocimiento lógico que aparece naturalmente en distintas situaciones de enseñanza, sus relaciones con otros dominios matemáticos y su razón de ser para ser reintroducido en la escuela.

Agradecimientos

Esta investigación se ha financiado en el marco del proyecto I+D+i: RTI2018-101153-A-C22 (MCIU/AEI/FEDER, UE).

Referencias

Barquero, B., Bosch, M., & Romo, A. (2018). Mathematical modelling in teacher education: Dealing with institutional constraints. *ZDM Mathematics Education, 50,* 31-43. https://doi.org/10.1007/s11858-017-0907-z

Bosch, M. (2018). Study and Research Paths: a model for inquiry. In B. Sirakov, P. N. de Souza, & M. Viana (Eds.), *International Congress of Mathematicians* (vol. 3, pp. 4001–4022). World Scientific Publishing Co. Pte. Ltd.

Brissiaud, R. (1993). *El aprendizaje del cálculo. Más allá de Piaget y de la teoría de conjuntos*. Aprendizaje Visor.

Chevallard, Y. (2015). Teaching mathematics in tomorrow's society: a case for an oncoming counter paradigm. In S. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 173–187). Springer International Publishing.

Ermel (1990). *Apprentissages numériques*. (Grande section de maternelle). Hatier.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* D. Reidel Publishing Company.

García, F.J., Sierra, T.A., Hidalgo, M. y Rodríguez, E. (2020). The education of prospective early childhood teachers within the paradigm of questioning the world. In M. Bosch, Y. Chevallard, F.J. García, & J. Monaghan (Eds.), *Working with the Anthropological Theory of the Didactic in Mathematics Education: A comprehensive casebook* (pp. 169-188). Routledge.

Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number.* Harvard University Press.

Hughes, M. (1986). *Children and number: Difficulties in learning mathematics.* Basil Blackwell.

Orús, P. (1992). *Le raisonnement des élèves dans la relation didactique; effets d'une initiation à l'analyse classificatoire dans la scolarité obligatoire*. [Tesis doctoral, Université de Bordeaux I]. Ed. IREM de Bordeaux.

National Council of Teachers of Mathematics [NCTM] (2000). *Principles and Standards for school mathematics.* NCTM.

Piaget, J., & Szeminska, A. (1941). *La genèse du nombre chez l'enfant.* [The development of the number concept in the child]. Delachaux, Niestle.

Ruiz-Olarría, A. (2015). *La formación matemático-didáctica del profesorado de secundaria. De las matemáticas por enseñar a las matemáticas para la enseñanza* [Tesis doctoral, Universidad Autónoma de Madrid]. Repositorio institucional UAM. http://hdl.handle.net/10486/665889

Sarama, J., & Clements, D.H. (2009). *Early childhood mathematics education research: learning trajectories for young children.* Routledge.

Creando Recorridos de Estudio e Investigación para la Formación de Profesorado Universitario en Ciencias Sociales: Dispositivo ProfessorLab Marketing

Noemí Ruiz-Munzón[1], Alexandra Masó-Llorente[2] and Alexandra Samper-Martínez[3]

[1]Universitat Pompeu Fabra, Escola Superior de Ciències Soicals i de l'Empresa-Tecnocampus, Barcelona, España; nruiz@tecnocampus.cat

[2]Universitat Pompeu Fabra, Escola Superior de Ciències Soicals i de l'Empresa-Tecnocampus, Barcelona, España; amaso@tecnocampus.cat

[2]Universitat Pompeu Fabra, Escola Superior Politècnica-Tecnocampus, Barcelona, España; asamper@tecnocampus.cat

Abstract.

Each university develops its own programs to homogenise their teaching practices in light of the absence of a systematised continuous training on general pedagogical content. This paper presents the results of a modular application of Study and Research Paths for Teacher Training (SRP-TT) framework. The paper shows the case of a university teachers' team, from Tecnocampus university centre (attached to University Pompeu Fabra), with different levels of affiliation and experience in the field of marketing. The SRP-TT allowed the generation of a conceptual and methodological framework for the systematisation of the continuous training for the university teachers' team. Thus, the SRP-TT acted as a catalytic device for dialogue between teachers from the same teaching corpus, raising the conceptual epistemological consensus and the production of related teaching materials.

Resumen.

Cada universidad desarrolla programas propios para homogeneizar la práctica de su corpus docente ante la falta de formación continua sistematizada para éste en relación a contenidos pedagógicos generales. El trabajo presenta los resultados de la aplicación modular del marco de Recorridos de Estudio e Investigación para Formación de Profesorado (REI-FP) para el caso de un equipo de profesores universitarios en el centro universitario Tecnocampus (adscrito a la Universidad Pompeu Fabra) con distintos índices de afiliación y experiencia, procedentes del ámbito del marketing. Como resultado, los REI-FP permiten generar un marco conceptual y metodológico para la sistematización de la formación continua del profesorado universitario, ejerciendo como dispositivo catalizador del diálogo entre profesores del mismo corpus docente, alzando el consenso epistemológico conceptual y la producción de materiales docentes vinculados.

Résumé.

Chaque université développe ses propres programmes pour homogénéiser la pratique de son corpus d'enseignement en l'absence d'une formation continue systématisée de ce corpus par rapport aux contenus pédagogiques généraux. Cet article présente les résultats de l'application modulaire du cadre PER-FE (Parcours d'Études et de Recherche pour la Formation des Enseignants) pour le cas d'une équipe d'enseignants universitaires du centre universitaire Tecnocampus (rattaché à l'Université Pompeu Fabra) ayant différents niveaux d'affiliation et d'expérience dans le domaine du marketing. En conséquence, le PER-FE permet la génération d'un cadre conceptuel et méthodologique pour la systématisation de la formation continue des enseignants universitaires, agissant comme un dispositif catalyseur pour le dialogue entre les enseignants du même corpus d'enseignement, élevant le consensus épistémologique conceptuel et la production de matériel didactique connexe.

Keywords: Marketing, Universidad, Formación continua de profesorado, Recorrido de Estudio e Investigación.

1. La formación para el profesorado universitario

Actualmente, en España no existe una formación inicial unificada para el profesor universitario (Rodríguez-Espinar, 2020). En los últimos años, cada universidad ha desarrollado programas propios para dotar de competencias docentes a aquellos investigadores principiantes, en general, doctorandos con becas de Formación del Profesorado Universitario o de Formación del Personal Investigador. Sin embargo, dicha formación se caracteriza por: (1) estar fragmentada y poco sistematizada (Aramburuzabala et al., 2013), (2) encontrarse centrada en contenidos pedagógicos generales que no tienen en cuenta la naturaleza del conocimiento involucrado en el proceso de enseñanza y aprendizaje (Barquero, Florensa & Ruiz-Olarría, 2019), y (3) encontrarse centrada solo en los procesos de formación inicial de la identidad profesional del profesor (Cuadra-Martínez et al., 2021). Esta tendencia no solo se manifiesta a nivel nacional, sino que también se hace patente a nivel europeo (Rodríguez Espinar, 2020).

La implementación de reformas educativas que inducen a una elevación de la calidad del proceso educativo (Gómez et al., 2021) junto con la aceleración de una cultura derivada del crecimiento de la información en la era digital (Hernández-Campillo et al., 2020), derivan en una mayor necesidad de una formación inicial y una formación continua estructurada para garantizar un mejor desempeño de los profesores universitarios y una elevada calidad de enseñanza.

Una de las dificultades con las que se encuentra la problemática de la formación del profesorado universitario es la figura del docente a tiempo parcial (PT en adelante), del inglés *part-time faculty position* o *adjunct labour* (Colby & Colby, 2017), existente en la mayoría de universidades. Esta figura nace con la vocación de mejorar la calidad de la enseñanza por un lado, pero también para descargar de carga docente (*teaching load)* a los profesores que ostentan posiciones a tiempo completo (FT en adelante), del inglés *full-time* o *tenure-track positions*, para que éstos últimos se puedan dedicar a investigación a fin de mejorar la transferencia de conocimiento (Meixner, Kruck &

Madden, 2010). Sin embargo, la figura del PT, también debe generar una identidad profesional y mantener una formación continua teniendo en cuenta que sus índices de afiliación y experiencia en enseñanza universitaria distan de los de los profesores FT (Jolley, Cross & Bryant, 2014), lo que presenta un reto para el sistema universitario.

Dado que en el contexto universitario español la formación del profesorado universitario se caracteriza por la ausencia de un marco de referencia nacional (Aramburuzabala et al., 2013), en el presente artículo se plantea un dispositivo de formación continua ajustado también al perfil de profesorado universitario PT y, al mismo tiempo, que considera e incluye las particularidades del centro universitario. El dispositivo ha sido denominado ProfessorLab, y pretende ser el punto de inicio para dotar de herramientas didáctico-pedagógicas que emanan de las necesidades de la práctica de los profesores universitarios.

2. Los Recorridos de Estudio e Investigación para Formación de Profesorado (REI-FP)

En los trabajos de Ruiz-Olarría (2015); Barquero, Bosch & Romo (2018); y Florensa, Bosch y Gascón (2019), entre otros, se desarrolla el marco conceptual y metodológico de los Recorridos de Estudio e Investigación (REI) para la Formación de Profesorado (FP). En esta línea, Lerma, Barquero, García, Hidalgo, Ruiz-Olarría & Sierra (2021) presentan una estructura modular del modelo de REI-FP (Figura 1):

Figura 1. Estructura Modular de un REI-FP (Lerma et al. 2021)

Según estos autores, un REI-FP parte de una cuestión o situación problemática en torno a la profesión del profesor universitario. Para responder a esta cuestión, la comunidad de estudio realiza un proceso de investigación orientado a la producción de posibles respuestas a través de cinco módulos. La puesta en marcha de uno o varios módulos viene limitada por las posibilidades de experimentación, así como la intencionalidad de la formación que se quiere desarrollar. Además la secuencia de los módulos puede ser alterada cronológicamente, por ejemplo, según la problemática puede ser más apropiado iniciar el estudio en el módulo 1 (tomando un problema docente espontáneo, por ejemplo) y posteriormente plantear el módulo 0 (reformulando el problema en términos didáctico-matemáticos).

En el módulo 0 se ubica el primer encuentro y su familiarización con la cuestión profesional (contiene tanto el contexto como la pregunta). También se desarrolla una primera búsqueda de respuestas existentes a través de diferentes materiales o recursos. El objetivo de este módulo no es tanto el de elaborar una respuesta, sino cuestionar los hallazgos en términos de posibles fenómenos didácticos que tales respuestas generan. Una vez establecida la problemática sobre la que se pretende trabajar, el objetivo del módulo 1 es realizar un proceso de estudio, en forma de REI, que hace que la comunidad de estudio experimente un tipo de actividad, en cierta forma poco familiar o habitual, asumiendo el rol de estudiante.

El módulo 2, consiste en analizar el REI vivido o un REI del que se disponga material (vídeos, diario de sesiones, etc.). Consiste, por lo tanto, en analizar desde el punto de vista de la didacta el proceso seguido y las respuestas elaboradas, tanto a nivel de contenido como de las acciones didácticas que se han realizado. Esta actividad permite dotar al profesorado de nuevas competencias docentes y herramientas de análisis para su actividad.

En el módulo 3 se sitúa a los participantes (profesorado) en el rol de “ingeniero didáctico”. Se trata de que los profesores diseñen o readapten un proceso de estudio para los alumnos, tomando en consideración las restricciones institucionales bajo los que se desarrollará su futura experimentación, identificando las variables didácticas más relevantes para el desarrollo del proceso de estudio, etc.

Finalmente, el módulo 4 consiste propiamente en la experimentación o puesta en marcha del proceso de estudio propuesto en el módulo 3, así como en el análisis tanto en vivo como a posteriori del proceso y de los resultados obtenidos de la implementación.

3. El dispositivo ProfessorLab: aplicación del REI-FP para la formación continua

Existe un amplio consenso en la necesidad de incluir en la formación del profesorado (formación inicial o continua) conocimientos disciplinares más allá de los contenidos que el profesor debe enseñar (Barquero, Florensa & Ruiz-Olarría, 2019). La concepción del dispositivo ProfessorLab se sustenta, por lo tanto, en la asunción de que la formación profesional específica del profesorado requiere de un equipamiento praxeológico (o conjunto de praxeologías disponible) cuya construcción y desarrollo es responsabilidad de la comunidad de investigadores en didáctica en estrecha colaboración con la profesión docente (Chevallard & Cirade, 2009).

Por otra parte, en el proceso de enseñanza, en este caso de ciencias sociales, “el saber enseñar” se nutre tanto de un *logos* derivado de la propia disciplina, como de la experiencia adquirida en el mundo profesional. Además, en el caso del profesor FT también se sugiere una distancia epistemológica entre los saberes académicos del propio profesor respecto a los saberes de las disciplinas (Perafán-Echeverri, 2013). No obstante, en las propuestas de formación (Rodríguez Espinar, 2020; Guasch, Alvarez & Espasa, 2010, Postareff, Lindblom-Ylänne & Nevgi, 2007, Fry, Ketteridge & Marshall, 2009), no se aborda la dificultad transpositiva que supone el paso entre el “saber sabio” al “saber por enseñar” (Chevallard, 1997).

ProfessorLab es un dispositivo que propicia un espacio de diálogo entre los diferentes agentes creadores y difusores del conocimiento a nivel universitario. Este dispositivo consiste en la realización de diferentes sesiones con el objetivo, por un lado, de evidenciar la necesaria reconstrucción del conocimiento, creando un modelo epistemológico propio de la comunidad de profesores que lo pone en práctica (Gascón, 2014). Permite poner de manifiesto la razón de ser de las técnicas y contenidos docentes, a qué preguntas permiten dar respuesta, bajo qué premisas son válidas las técnicas, etc. Por otro lado, durante el proceso, se introducen herramientas didáctico-pedagógicas que ayudan al profesor universitario a cuestionar su propia práctica docente. De esta manera, se pasa de una mera actividad de investigación o reflexión grupal, a un proceso de formación del profesorado universitario.

El dispositivo ProfessorLab se estructura en cuatro fases que derivan de los módulos 0, 1 y 3 del REI-FP que hemos mostrado en la sección anterior. El grupo de profesores universitarios participante puede variar siendo creando subgrupos de trabajo en cada una de las fases. Además, existe un/a formador/a (con conocimientos en didáctica) que asume un rol de guía de las sesiones.

La fase 0 y fase 1 se desarrollan en el marco del módulo 0 del REI-FP con la explicitación de la cuestión profesional (Q_0) en torno a los modelos epistemológicos de la disciplina. Una vez que el grupo de participantes ha acordado un marco general sobre el modelo epistemológico y habiendo reformulado la cuestión inicial, se inicia la fase 2 en la que se realiza un trabajo de investigación (ubicándose así en el marco del módulo 1 del REI-FP) basado en la dialéctica de preguntas y respuestas. Una de las características más destacables del proceso es la no existencia de una respuesta predefinida sino que la respuesta ($A^{♥}$) será construida por la comunidad de estudio (X, los profesores universitarios e Y el formador/a). Esto constituye el sistema didáctico $S(X; Y; Q_0)$. El proceso seguido puede ser descrito en términos del esquema herbartiano (Bosch, 2018; Barquero et al., 2021):

$$[S(X; Y; Q_0) \Rightarrow \{A_1^{\Diamond}, A_2^{\Diamond},\ldots, A_i^{\Diamond}, W_{j+1}, W_{j+2},\ldots,W_m, Q_{k+1}, Q_{k+2},\ldots,Q_n \}] \Rightarrow A^{♥}$$

Donde Q_k son las preguntas derivadas, W_j es cualquier tipo de trabajo (conocimiento o material, por ejemplo, una figura o un dibujo, pero también un laboratorio) y $A_i^{\Diamond}$ corresponde a trabajos consensuados de la comunidad científica como el "modelo de las 4P de Kotler", la "regresión lineal”, la "media móvil", etc. Ambas se combinan para obtener una respuesta final al problema planteado.

Finalmente, a partir de la respuesta construida se desarrolla la fase 3 con la elaboración de propuestas didácticas basadas en $A^{♥}$. Llegando así a desarrollarse, por lo tanto, en el marco del módulo 3 del REI-FP. La implementación de dichas propuestas no se contempla, a priori en una primera edición del dispositivo ProfessorLab, por las limitaciones y restricciones que comportaría la elección de los docentes para la viabilidad de su ejecución.

4. Descripción del caso particular: ProfessorLab Marketing

En este apartado describimos la aplicación práctica del dispositivo ProfessorLab, en el caso particular de un grupo de profesorado universitario en el ámbito del marketing, de ahí su nombre: “ProfessorLab Marketing”. El caso se desarrolló en el centro universitario TecnoCampus perteneciente al Grupo Universidad Pompeu Fabra, el curso 2019-20, en el marco de un proyecto de innovación docente con la participación de 14 profesores universitarios. El 28% eran profesores universitarios FT y un 72% PT, donde su formación en didáctica o pedagogía era heterogénea y poco estructurada. Los detalles de la experimentación se pueden consultar en Ruiz-Munzón, Samper-Martínez & Masó-Lorente (2021).

El dispositivo “ProfessoLab Marketing”, siguiendo el modelo del REI-FP, tomó como cuestión de la profesión (fase 0):

¿Qué se entiende por Marketing? ¿Qué debe saber un estudiante sobre Marketing para ser un buen profesional en el futuro?

A partir del análisis de las respuestas a un cuestionario inicial (Ruiz-Munzón, Samper-Martínez & Masó-Lorente, 2021), que se lanzó a todo el profesorado (PT y FT) que imparte docencia en el área de marketing del centro universitario TecnoCampus, se confirma que existe la percepción, entre los 10 participantes de esta fase, de que hay una brecha entre la vertiente teórica del marketing y su práctica. Se elige, así, el campo de estudio y los potenciales participantes que intervendrán en las siguientes fases.

Para el desarrollo de las fases 1, 2 y 3, se empleó el modelo cualitativo de grupos focales, del inglés Focus Groups (FG) (Wilson, 1997). La fase 1 se caracterizó por la apertura de la introspección dialógica (Ash, 2014) y la exploración de la brecha teórico-práctica a nivel sustantivo entre profesores PT y FT. La fase 2 conllevó la organización conceptual y estructuración teórica con tal de acordar cuál es el saber que se propone para ser enseñado (Chevallard & Bosch, 2014). Esto ayudaría a diferenciar qué partes se corresponden entre sí, además de emerger lo que se puede interpretar como el *logos* de la actividad (teorías, conceptos propios, conceptos de otras áreas) y qué partes se corresponden con la *praxis* (herramientas, uso de datos o recursos).

Para garantizar una rica recopilación de datos representativa de toda la muestra, las fases 1 y 2 se desarrollaron aplicando la técnica de Grupos Nominales[1] (GN) en cada uno de los FG. El desarrollo de los GN se realiza mediante entrevistas a los participantes que trabajan las ideas simultáneamente e independientemente antes de verbalizarlas. Así, se consigue respuesta de todos los participantes y no solamente de los miembros vocales (Delbecq et al., 1976).

El modelo de GF empleado fue híbrido dadas las restricciones que se derivan de la situación pandémica COVID19, que tuvieron lugar a mitad del proyecto. Inicialmente, los grupos focales se desarrollaron en formato presencial, trabajando en la fase 1 con un modelo *face to face* y en fase 2, se adoptó un modelo virtual.

En la última etapa, la fase 3, se diseñaron y editaron contenidos específicos, como los mapas pregunta-respuesta (*Q-A maps*), propuestos en Winsløw, Mahteron & Mercier (2013). En los *Q-A maps* se plasma la respuesta final ($A^\heartsuit$) que responde a las preguntas formuladas por la propia comunidad de estudio Q_k del tipo: ¿cómo empieza el marketing operativo?, ¿cómo determinar el precio del producto o servicio?, ¿qué mecanismos tiene la empresa para conseguir ingresos?, ¿cómo los precios ayudan a posicionar a la empresa y a explicar su posicionamiento?, ¿cómo estar presente en el momento de compra (*placement*)?, ¿cómo hacemos llegar el producto?, ¿qué es un canal de distribución?, ¿cuándo nos preguntamos cómo conectar con el cliente?, ¿qué es un plan de comunicación?, ¿cómo se elabora?, etc. La Figura 2 muestra en amarillo las preguntas que se formularon, en verde las $A_i^\Diamond$ del ámbito del marketing que aportan respuestas, en azul las $A_i^\Diamond$ de otras disciplinas y en naranja un

[1] La técnica de debate por grupos nominales es una técnica creativa empleada para la generación de ideas y el análisis de problemas. Se trata de un modelo de análisis completamente estructurado con intervenciones marcadas para permitir alcanzar un número elevado de conclusiones sobre las cuestiones planteadas (Delbecq et al., 1976)

trabajo W_j a realizar a partir de fuentes externas (datos, informes de otros departamentos, etc.). Por lo tanto, de la Figura 2 se pretende destacar la distribución de colores y no tanto el contenido concreto del esquema obtenido de una implementación.

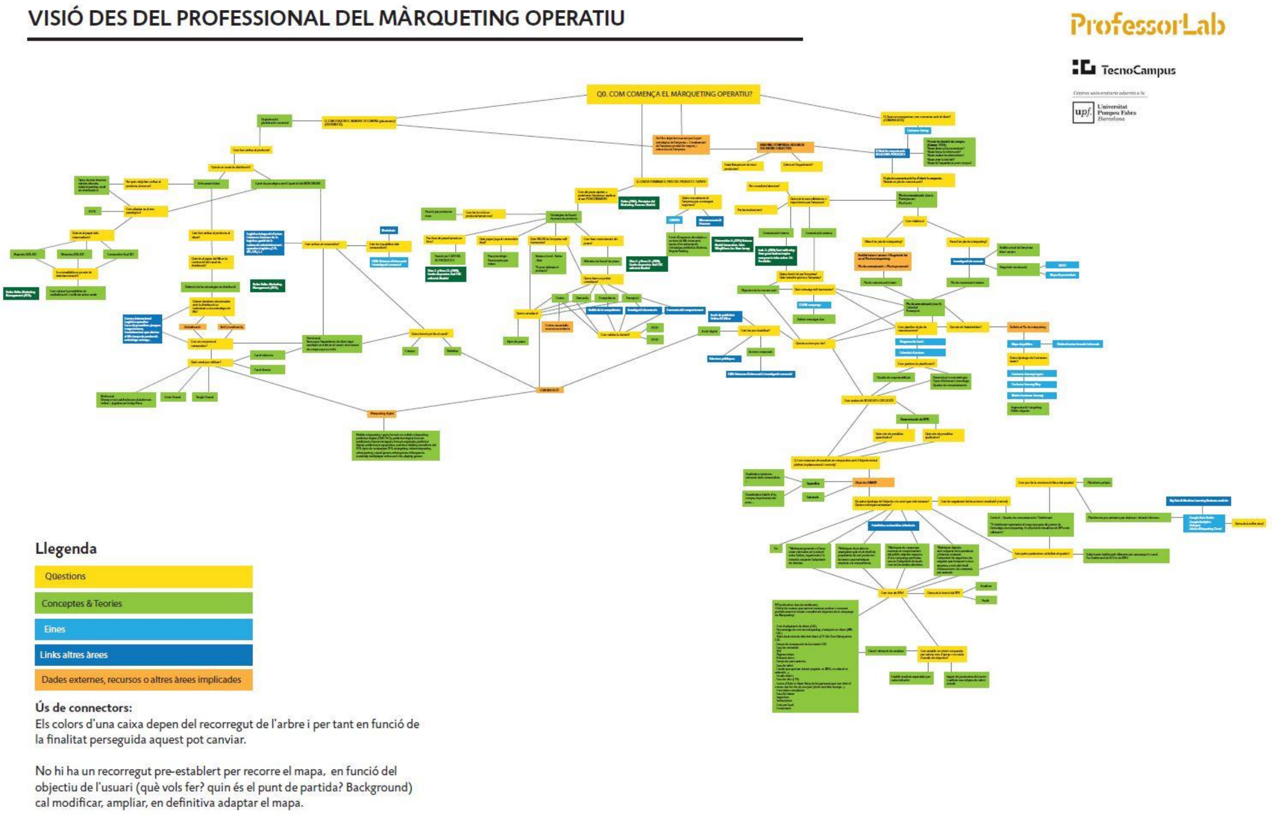


Figura 2. *Q-A map* original (en catalán) del marketing operativo

También se desarrollaron, a partir del consenso epistemológico alcanzado, otros materiales audiovisuales útiles para unificar la labor docente tanto en lo que respecta a las asignaturas de marketing como a la hora de tutorizar trabajos de final de grado (p.e. vídeos sobre metodología de aplicación de herramientas de análisis propias del marketing o videotutoriales para el desarrollo de aspectos del plan de marketing).

Se completó la respuesta final $A^♥$ con acciones de difusión interna a través de una jornada online para el profesorado (37 participantes) donde se expusieron los materiales creados por el equipo de profesores universitarios ProfessorLab Marketing a fin de recoger su validación e ideas para posibles mejoras futuras. De esta manera, se creó un nuevo espacio de debate sobre los modelos epistemológicos, para los participantes del FG y para una parte significativa del cuerpo de profesorado del TecnoCampus, aunque esta vez partiendo de una visión predefinida.

5. Resultados

La experiencia de la puesta en marcha del dispositivo ProfessorLab en el ámbito del Marketing apunta a que podría ser útil para la formación del profesorado universitario en ciencias sociales. La aplicación del REI-FP, permite abordar algunos de los fenómenos didácticos que han sido puestos de manifiesto en otras investigaciones de la Teoría Antropológica de la Didáctica, identificamos a continuación qué ha aportado cada fase a los diferentes módulos del REI-FP:

Módulo 0 del REI-FP: permite realizar una confrontación de opiniones sobre una cuestión legítima de la profesión por la comunidad de profesores universitarios, lo cual posibilita visualizar fenómenos didáctico-epistemológicos de la profesión docente (y no del profesor/a a título individual). En el caso ProfessorLab Marketing la articulación de las dos visiones, a través de las experiencias particulares del grupo de profesores universitarios, facilitó un momento de catarsis a través de las dificultades de articular una visión del Marketing como disciplina ("saber sabio") y a la vez asumir un punto de vista de semi-profesión donde "se aprende haciendo" (lo que podemos denominar por "saber profesional"). Conllevó un inicio del diálogo para repensar la epistemología sobre el marketing tomando su faceta instrumental; en lugar de una visión más euclidiana como cuerpo de conocimiento coherente en sí mismo, como la que se encuentra en la mayoría de los libros de texto. En definitiva, el método permitió caracterizar la visión más tradicional del Marketing, que asume en definitiva para su enseñanza un paradigma monumentalista (Chevallard, 2015), y contraponerla a una visión más pragmática desarrollada desde la práctica del marketing (articulando las experiencias de los profesores universitarios en el mundo laboral). Mostrando así la razón de ser del conocimiento, es decir, a qué preguntas profesionales y bajo qué supuestos da respuesta el conocimiento de marketing, y permitiendo repensar el paradigma epistemológico actual. Por ende, se trata de hacer emerger y hacer explícita la línea de pensamiento de la propia profesión en su proceso de transposición didáctica del "saber profesional" al "saber sabio".

Módulo 1 del REI-FP: permite definir una estructura inicial de bloques de conocimiento y el análisis profundo de alguno de dichos bloques. En el caso ProfessorLab Marketing esto se dio a través de responder a: ¿qué preguntas dan sentido al uso o construcción del conocimiento propio del marketing?, ¿qué otras disciplinas se requieren?, ¿qué recursos se necesitan (p.e. información de la empresa, datos o definición de indicadores)?, etc. De esta manera, la aplicación del módulo 1 permitió visualizar la estructura suyamente de los contenidos de marketing para los profesores universitarios (identificar el *logos* y la *praxis*), emergiendo así los diferentes "saberes sabios" de referencia que nunca son cuestionados o puestos en duda, y mostrando el alcance y limitaciones o dificultades de cada técnica de análisis relacionada con la práctica del marketing ("saber profesional").

En este proceso, los profesores universitarios participantes en ProfessorLab Marketing, asumieron de forma natural su rol de estudiantes investigadores, y no se produjo ninguna resistencia a abandonar su rol de experto en la materia. Se creó una colaboración entre iguales, donde la *dialéctica de preguntas y respuestas* permitía mantener en cierto modo su estatus de expertos. Además, no había una respuesta prestablecida y se desarrolló el REI sin muchas de las restricciones entorno a la *topogénesis* reportadas en otras implementaciones (Barquero et al., 2021).

Módulo 3 del REI-FP: su aplicación permite la actividad de edición del material de análisis, en este caso los *Q-A maps*, y el desarrollo de materiales visuales para la docencia (vídeos explicativos de técnicas específicas del área del marketing). Todo ello se corresponde con el momento de institucionalización y validación de los REI y, en definitiva, de la $A^{\heartsuit}$ elaborada, por parte de la comunidad universitaria. El rol de los profesores universitarios en esta fase fue de diseñadores, y se puede identificar como el connato hacía construir o analizar el "saber a enseñar".

En la sesión final de exposición de resultados a la comunidad de profesores universitarios del centro universitario en cuestión (Ruiz-Munzón, Samper-Martínez & Masó-Lorente, 2021), se mostró la potencialidad del *Q-A map* como instrumento para analizar la interrelación entre asignaturas de los diferentes grados que se imparten en el centro, como lo recogen algunos de los comentarios de los profesores participantes en la jornada de presentación: *"[...] la creación de los mapas, la interrelación de unas cosas con otras, ...¡Esto da [una] visión global!", "[...] el enfoque holístico del mapa (mapas), y cómo, sin la aspiración de convertirse en una taxonomía, da una visión de mucha utilidad.*", etc. La actividad auto conclusiva permitió introducir explícitamente algunas nociones de didáctica, como: la transposición didáctica, la escala de co-determinación didáctica, el concepto de *logos* y *praxis* en particular.

6. Conclusiones y futuras líneas de investigación

La aplicación del método REI-FP permite, por una parte, alcanzar altos niveles de diálogo en relación a la metodología pedagógica que aplican los propios profesores universitarios dentro de un mismo ámbito de conocimiento. Por otro lado, el diálogo inducido a través de la aplicación del método, permite llegar a ciertos grados de consenso epistemológico conceptual sobre diferentes áreas de estudio, lo cual posibilita que profesores universitarios de un mismo centro universitario (ya sean PT o FT) impartan una misma visión conceptual contundente y clara. Finalmente, el método REI-FP facilita la producción conjunta de materiales para la enseñanza que permiten homogeneizar la labor docente en el seno de cada centro universitario. En esta línea sería apropiado definir nuevas fases del dispositivo que permitan desarrollar el diseño, ejecución y análisis de propuestas de enseñanza basadas en el modelo epistemológico, en cabiéndose estás en los módulos 2 y 4.

No obstante, una de las limitaciones de la aplicación del dispositivo ProfessorLab a destacar, y por tanto de REI-FP para la formación continua del profesorado universitario, se encuentra vinculado a la infraestructura institucional necesaria para darle justamente una continuidad o ampliación del proyecto y, por consiguiente, a la formación del cuerpo docente (tanto de profesorado PT, como FT).

Otra posible futura línea de investigación a abordar sobre el dispositivo ProfessorLab es analizar su potencialidad si se toma una disciplina más madura o cohesionada, como puede ser el ámbito de la economía. Cabría abordar, en este caso, cómo esto modifica la topogénesis en las diferentes fases de aplicación del método, y la identificación de dificultades en la búsqueda del consenso o de la construcción de los *Q-A maps*. Además, y de forma más general, cabría seguir indagando en qué herramientas didácticas son más útiles para los profesores universitarios y qué características o competencias deben pedirse al formador de formadores.

Agradecimientos

Las autoras reconocen y agradecen el apoyo financiero de la Fundación Tecnocampus Mataró-Maresme.

Bibliografía

Aramburuzabala, P., Hernández-Castilla, R., & Ángel-Uríbe, I. C. (2013). Modelos y tendencias de la formación docente universitaria. *Revista de Curriculum y Formación de Profesorado O*, 17(3), 345–357.

Ash, M. (2014). *Knowledge That Counts: An Examination of the Theory Practice Gap Between Business and Marketing Academics and Business Practitioners Examined in Terms of Their Respective Epistemic Stances*. PhD thesis. Derby: University of Derby. Accedido desde https://tinyurl.com/y4dq3pxt

Barquero, B., Bosch, M., & Romo, A. (2018). Mathematical modelling in teacher education: dealing with institutional constraints. *Zdm*, *0*(0), 0. https://doi.org/10.1007/s11858-017-0907-z

Barquero, B., Bosch M., Florensa, I. & Ruiz-Munzón, N. (2021) Study and research paths in the frontier between paradigms, *International Journal of Mathematical Education in Science and Technology*, DOI: 10.1080/0020739X.2021.1988166

Barquero, B., Florensa, I., & Ruiz-Olarría, A. (2019). The education of school and university teachers within the paradigm of questioning the world. In *Working. with the Anthropological Theory of the Didactic in Mathematics Education* (pp. 189-212). Routledge.

Benavides León, C. A., y López Rodríguez, N. M. (2020). Retos contemporáneos para la formación permanente del profesorado universitario. *Educación y Educadores*, 23(1), 71-88.

Bosch, M. (2018). Study and Research Paths: a model for inquiry. In *B. Sirakov, P. N. de Souza, & M. Viana (Eds.), International congress of Mathematicians* (Vol. 3, pp. 4001–4022). Rio de Janeiro: World Scientific Publishing Co. Pte. Ltd.

Chevallard, Y. (1997). *La transposición didáctica. Del saber sabio al saber enseñado*. Buenos Aires: Aiqué.

Chevallard, Y. & Bosch, M. (2014). Didactic transposition in Mathematics Education. In *S. Lerman (Ed.), ACM Encyclopedia of Mathematics Education* (pp. 170–174). Dordrecht: Springer. doi: 10.1145/1869746.1869759

Chevallard, Y. & Cirade, G. (2009). Pour une formation professionnelle d'université. Éléments d'une problématique de rupture. *Recherche et formation, (60)*, 51-62.

Colby, R., & Colby, R. S. (2017). Real faculty but not: the full-time, non-tenure-track position as contingent labor. *Contingency, exploitation, and solidarity: labor and action in english composition*, 57-70.

Cuadra-Martínez, D., Castro-Carrasco, P. J., Oyanadel, C. & González-Palta, I. N. (2021). Identidad profesional docente en la formación universitaria: una revisión sistemática de estudios cualitativos. *Formación Universitaria,* 14(4), 79–92. https://doi.org/10.4067/s0718-50062021000400079

Delbecq, A., Van de Ven, A., & Gustafson, D. (1976). Group Techniques for Program Planning: A Guide to Nominal Group and Delphi Processes. In *The Journal of Applied Behavioral Science*. Green Briar Press. https://doi.org/10.1177/002188637601200414

Florensa, I., Bosch, M. & Gascón, J. (2019). Análisis a posteriori de un REI-FP como herramienta de formación del profesorado. *Educaçao Matemática Pesquisai. v.21 (4),* 382-394.

Fry, H., Ketteridge, S., & Marshall, S. (2009). A handbook for teaching and learning in higher education. In *H. Fry, S. Ketteridge and S. Marshall (eds), A handbook for teaching and learning in higher education* (3rd edition). Oxford, UK: Routledge.

Gascón, J. (2014). Los modelos epistemológicos de referencia como instrumentos de emancipación de la didáctica y la historia de las matemáticas. Educación Matemática, Special Issue: XXV years, 99–123. https://www.redalyc.org/pdf/405/40540854006.pdf

Gómez, G. Á., Moya, J. V., Ricardo, J. E. & Sánchez, C. V. (2021). La formación continua de los docentes de la educación superior como sustento del modelo pedagógico. Revista Conrado, 17(S1), 431–439.

Guasch, T., Alvarez, I., & Espasa, A. (2010). University teacher competencies in a virtual teaching/learning environment: Analysis of a teacher training experience. *Teaching and Teacher Education*, *26*(2), 199–206.

Hernández Campillo, T. R., Carvajal Hernández, B. M. & Legañoa Ferrá, M. de los Á. (2020). Análisis a las competencias informacionales en la formación continua de los docentes universitarios. Bibliotecas. Anales de Investigación, XVI(1), 61–69.

Jolley, M. R., Cross, E., & Bryant, M. (2014). A Critical Challenge: The Engagement and Assessment of Contingent, Part-Time Adjunct Faculty Professors in United States Community Colleges. *Community College Journal of Research and Practice, 38*(2–3), 218–230. https://doi.org/10.1080/10668926.2014.851969

Lerma, A. M., Barquero, B., García, F. J., Hidalgo-Herrero, M., Ruiz-Olarría, A. & Sierra, T. (2021). Los conocimientos lógicos en la formación matemático-didáctica de maestros. En *Diago, P. D., Yáñez D. F., González-Astudillo, M. T. y Carrillo, D. (Eds.), Investigación en Educación Matemática XXIV* (pp. 385 – 392).

Meixner, C., Kruck, S. E., & Madden, L. T. (2010). Inclusion of part-time faculty for the benefit of faculty and students. *College teaching, 58*(4), 141-147.

Perafán Echeverri, G. A. (2013). La transposición didáctica como estatuto epistemológico fundante de los saberes académicos del profesor. *Revista Folios, 37,* pp. 83-93. ISSN: 0123-4870.

Postareff, L., Lindblom-Ylänne, S., & Nevgi, A. (2007). The effect of pedagogical training on teaching in higher education. *Teaching and Teacher Education, 23*(5), 557–571.

Rodríguez Espinar, S. (2020). Luces y sombras en la formación pedagógica del profesorado universitario en España. *REDU. Revista de Docencia Universitaria*, *18*(1), 143. https://doi.org/10.4995/redu.2020.13183

Ruiz-Munzón, N.; Samper-Martínez, A. & Masó-Lorente, A. (2021). "Think Global, Act Local": Un Método de Diálogo Epistemológico para Reducir la Brecha entre Universidad-Empresa en el Ámbito del Marketing. *En Buzón-García, O., Romero-García, C. y Verdú-Vázquez, A. (Eds). Innovaciones Metodológicas con TIC en Educación* (pp1802-1821). Dykinson.

Ruiz-Olarría, A. (2015). *La formación matemático-didáctica del profesorado de secundaria. De las matemáticas por enseñar a las matemáticas para la enseñanza*. (*Doctoral* Dissertation). Universidad Autónoma de Madrid.

Wilson, V. (1997). Focus groups: A useful qualitative method for educational research? *British Educational Research Journal*, *23*(2), 209–224. https://doi.org/10.1080/0141192970230207

Winsløw, C., Matheron, Y. & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics*, *83*(2), 267-284. doi: 10.1007/s10649-012-9453-3

How did the Croatian noosphere respond to the problem of the shortage of mathematics teachers?

Jelena Pleština[1], Matija Bašić[2], Željka Milin Šipuš[2] and Nikola Koceić-Bilan[1]

[1]University of Split, Faculty of Science, Department of Mathematics, Split, Croatia; jplestina@pmfst.hr koceic@pmfst.hr

[2]University of Zagreb, Faculty of Science, Department of Mathematics, Zagreb, Croatia; mbasic@math.hr zeljka.milin.sipus@math.hr

This paper deals with the problem of professionalization of the teaching concerning the new policy on the necessary qualifications of mathematics teachers in Croatian primary schools created due to the lack of candidates for that position. Using tools of the Anthropological Theory of the Didactics, the problem is analysed from three perspectives: the Ministry of Science and Education, which is trying to meet the needs of the labour market; the mathematical academic community representing the postulates of the profession; and finally, the perspectives of students and candidates.

Keywords: Anthropological Theory of the Didactics, mathematics teacher education, noosphere, praxeological equipment.

Introduction

In this paper we paint the landscape of the Croatian educational system which faces a shortage of Science, Technology, Engineering, and Mathematics (STEM) teachers. Since mathematics and mathematics education is an area of interest of authors, this paper is dedicated to mathematics teachers in the upper grades of primary schools (students aged 11-14), where the shortage is greatest. The paper tries to investigate whether this social problem affects the issue of the professionalization of mathematics teaching.

Who can be a mathematics teacher in Croatian primary schools?

According to the current regulations by Ministry of Science and Education (MSE), only the next five groups[1] of candidates[2] , marked from A to E in the order in which they are listed, may teach mathematics in Croatian primary schools.

A. Candidates who graduated from one of the following master study programs with specialization in education: Mathematics, Mathematics and Informatics, Mathematics and Physics.

B. Candidates who graduated from non-teaching master study programs of Mathematics (pure mathematics, statistics, applied mathematics, computer science, …).

[1] Given in the order of those who have the highest priority in employment, according to the lower one.

[2] Candidates from all groups except the last, completed either a five-year study under the Bologna program or a four-year study before the Bologna implementation process.

C. Lower primary school teachers with an enhanced program in the subject of mathematics.[3]

D. Graduate engineers, regardless of academic title, with at least 55 ECTS credits in mathematics.

E. University bachelors of the one of the following programs: Mathematics (regardless of study program), Mathematics and Informatics and Mathematics and Physics.

Current policy also stipulates that those candidates who have not acquired pedagogical-psychological knowledge during their regular university education must complete additional pedagogical-psychological education in the total amount of at least 55 ECTS credits. However, the MSE has given primary school principals the right to employ a person who does not meet the requirements of any of the A-E groups for the post of mathematics teacher for a limited period until finding an appropriate candidate. The MSE included group D in the previous policy to compensate for the shortage of mathematics teachers in upper primary schools. Thus, the network of formal education pathways leading to the workplace of a mathematics teacher in upper primary school has further branched out, and it needs to be seen from three perspectives: the MSE, which is trying to meet the needs of the labor market; the mathematical academic community representing the postulates of the profession; and finally, the perspectives of students and candidates who are going through the prescribed paths.

Croatia is not the only European country facing this problem. In England, for example, the shortage of mathematics teachers has been tried to make up for by retraining teachers of other subjects (Sani & Burghes 2021). Comparing the Danish (under Anglo-Saxon influence) and French (comparable to German) traditions of didactics of mathematics, Winsløw and Durand-Guerrier (2007) conclude that the idea that only mathematical knowledge “represents true knowledge” dominates among teachers and mathematicians, regardless of the state. On the other hand, general pedagogical knowledge and didactic knowledge are seen as "(possibly highly developed) 'craft'". Does this also apply to Croatia? Is the decision to introduce group D based on such an evaluation of the categories of knowledge needed for the position of mathematics teacher?

Theoretical framework and research questions

Anthropological Theory of the Didactics (ATD) observes mathematical and didactic activities at the complexity of the social institutions (Bosch & Gascón, 2006) and therefore it provides suitable tools for analysis of the problems discussed in this article.

According to ATD, every human activity, including mathematical one, can be described in terms of praxeology (Bosch & Gascón, 2006). Mathematical praxeology Π is represented by an ordered quadruple $[T, \tau, \theta, \Theta]$, where: T is a type of problem, τ is a technique that can solve a task t which is type T, θ a technology that explains and justifies τ, and Θ a theory that is a formal argument and thus explains θ. The ordered pair $[T, \tau]$ is called the praxis block of mathematical praxeology Π and refers to the question of *knowing how*; while the ordered pair $[\theta, \Theta]$ is called the *logos block* and answers the question of *knowing why*. Mathematical praxeology Π can be considered as *point praxeology* if it

[3] The program completed by group C candidates existed only from 1998 to 2006, so this paper will not take them into account.

is generated by one type of problem in the observed institution. Point praxeologies that have a common technology make up a *local praxeology*, and local praxeologies that have a common theory make up a *regional praxeology* (Barbé, Bosch, Espinoza, & Gascón, 2005).

Didactic activity is also a praxeology, called *didactic praxeology*, that aims to expand some praxeology (in our case, mathematical praxeology) among a group of people. An important topic of research in ATD is the issue of teacher education, which "raises the question of the praxeological equipment of teachers" (Cirade & Crumière, 2019, p.159). The term scale of levels of didactic co-determinacy (Chevallard, 2002) denotes a hierarchy of determination levels between mathematical and didactic praxeologies: Civilization↔Society↔School↔Pedagogy↔Discipline↔Domain↔Sector↔Theme↔Subject. Noosphere of the educational system or “sphere of those who think about education” (Bosch & Gascón, 2006, p. 52) “consists of all those persons who share an interest in the teaching system, and who “act out” their impulses in some way or another” (Chevallard, 2013, p. 2). The noosphere can be observed as a level on the scale of didactic co-determinancy levels located between the Society and the School or as an integrated part of the Society (Bosch & Chevallard, 2019).

Given that the results of changes in education are difficult to see at first glance and often are far-reaching, this paper seeks to explore the following questions relevant to the professionalization of the mathematics teaching profession.

RQ1: How can praxeological equipment relevant to the position of a mathematics teacher in primary school of candidates from different groups (A, B, and D) be compared?

RQ2: What conditions and constraints, and on which levels of the didactic co-determinacy, can be identified in solving the issue of lack of mathematics teachers?

The relationship and interaction of the MSE, the mathematical academic community, and candidates for the position of a mathematics teacher in primary schools can be described by the conditions and constraints that come from different levels of didactic co-determinacy. Thus, in our case, the MSE and the mathematical academic community are the noosphere of the Croatian education system.

Methodology

The perspective of the MSE, which is trying to solve the problem of the shortage of mathematics teachers in primary schools, is given through the analysis of the adopted policies, which, as far as the conditions allowed, were influenced by the mathematical academic community.

Praxeological equipment assessment for candidates for the position of mathematics teacher was made through the analysis of the approved programs. The analysis of the course content shattered the *illusion of a unique mathematical knowledge* (Bosch, 2006), given with the fact that candidates come from different institutions in which mathematics has a different status. Due to the diversity of study programs in group D, we singled out specific data for four candidates who completed different study programs and applied to the Faculty of Science in Split (FSS) for further evaluation of requirements for mathematics education certificate. The analysis was deepened by semi-structured interviews with mathematics university teachers who assessed mathematical knowledge of candidates. Furthermore, among candidates from group D, a questionnaire was conducted, with the purpose to get insight into candidates' didactic praxeologies. To examine some specific mathematical and didactic knowledge, an additional interview was conducted with one candidate from each of groups A, B, and D. The transcripts of the answers were subjected to praxeological analysis.

Results and discussion

Analysis of regulations and events around it

Encouraged by the lack of candidates who would like to work as mathematics teachers in primary school, which is constraint on Society level, the MSE in early 2019 included group D in the existing regulations on employment in primary school. Until the middle of 2020 the faculty at which the candidate graduated issued a certificate on the acquired number of ECTS credits in mathematics at the request of the candidate. Due to the persistence of the Croatian mathematical community, in the middle of 2020, the policy was amended and supplemented: only those faculties that conduct study programs for group A may issue a certificate on ECTS credits in mathematics. The MSE is part of the noosphere that wants to solve the problem but does not pay attention to mathematics as a discipline. On the other hand, the mathematical academic community is part of the noosphere that wants to respect the conditions and limitations of the Discipline level when solving problems. Due to conditions and restrictions at different institutions, the certification process is not uniform. At the other mathematical departments in Croatia, the procedure is as follows: at the request of the candidate, the number of ECTS credits in mathematics acquired by the candidate during university education is determined; if that number is at least 55, then the candidate receives a certificate. Practice shows that apart from a few candidates who have graduated from one of the study programs in physics, no non-mathematical study program awards students with enough ECTS credits. On the other hand, only at the FSS, candidates who have completed some of the study programs in the STEM field are additionally prescribed differential exams if it is determined that the number of ECTS credits in mathematics is less than 55.

The mathematical academic community demanded a more precise definition of the program of supplementary pedagogical-psychological education, which is mandatory for all candidates who have not completed teacher training program. The request was denied, which means that the policy allows the candidate to complete additional pedagogical-psychological education at the faculty that does not conduct a mathematics study program with a specialization in education, but only a general program in education, resulting in the fact that candidates do not need to take courses on didactics of mathematics. Thus, if the candidates from groups B and D have completed the general program of supplementary pedagogical education, then it is the responsibility of the candidate to connect mathematical and didactic praxeologies.

Since the described events are placed on the scale of didactic co-determinacy at the level of Society and School, the question arises how the decisions of the noosphere will affect the lower levels of the observed scale?

Analysis of programs allowing a position of a mathematics teacher in primary school

Group A

University education of a candidate from group A is usually performed in 3 years of undergraduate study and 2 years of graduate study. The undergraduate study program consists of mathematics courses, with only a few courses dedicated to the future teaching profession (application of technology to mathematics teaching, elementary mathematics in the curriculum, mathematics competitions). The first year of undergraduate study consists of general mathematics courses, and later more advanced and specialized courses. Students are introduced to the deductive structure of mathematics, so proofs are an indispensable part of every course discourse. Although graduate students take some advanced mathematics courses, most courses at this level of study are devoted to mathematics teaching. A minor portion of ECTS credits goes to general didactic, psychological, and pedagogical courses, while a major of ECTS credits is devoted to courses in mathematics methods of teaching and mentored methodological practice. Thus, candidates from group A have developed primarily mathematical praxeologies during their undergraduate studies, but from the second year of their studies, they also began to develop didactic praxeologies.

Group B

The difference between teaching and non-teaching mathematics programs at the undergraduate level is reflected in some advanced and specialized mathematical courses, which in the case of non-teaching programs continue at the graduate level of study. Thus, candidates from group B during their studies are equipped exclusively with mathematical praxeologies, many of which are too advanced for teaching in primary school.

Group D

When the different status of mathematics as a discipline for mathematical study programs (core status) and the other study programs in the STEM field (auxiliary status) are taken into account (Bosch & Winsløw, 2020), group D distinguishes from the other groups. Although there are large differences in mathematics courses among study programs completed by group D candidates (for example, a physics study program and a technical college program), these study programs have in common the need for mathematics at the tool level to solve problems relevant to a particular profession. The deductive nature of mathematics is not emphasized in the courses of these study programs. Mathematics courses at technical faculties are mostly about calculus through which *knowing how* is taught, and *knowing why* is given less attention, or explanations are sought in applications relevant to a particular profession. Although exams for these courses can be technically complicated, they only check whether a *technique* has been adopted for a particular type of task. Students learn many *local praxeologies* that are difficult to integrate into *regional praxeologies* due to the lack of mathematical theory. This phenomenon is known as the *thematic autism* (Barbé, 2005) and is noticed in both secondary education (ibid.) and university education. Especially for courses taken by many students, such as calculus (Winsløw, 2012).

The education of four candidates, who applied to the FSS for a group D certificate, was analyzed. For this article, candidates will be noted with D1, D2, D3 and D4. Table 1 provides basic information about these candidates: what degree they have, how many ECTS credits in mathematics they acquired during their university education, and what additional exams the committee prescribed. The content of some prescribed exams will be explained in the following few sentences. Course Introduction to

Mathematics is a typical bridging course for a first-year student through which elementary concepts from mathematics, already known in primary and secondary school, are viewed from a formal mathematical perspective. In this course, the student gets acquainted with proving techniques and building of mathematical theory. Considering the content of mathematics courses at the technical faculties, the committee's opinion is that the students from group D had not acquired the knowledge from this course. Also, technical faculties do not cover themes in geometry domain or number theory domain in their programs.

Through interviews with university teachers who held exams, we learned that they did not ask candidates to prove any statement because they concluded that candidates could not master it within a reasonable time. We quote part of the instruction of a teacher who examines the course Introduction to Mathematics:

> My idea is to deepen your knowledge of the content of the course, I will not emphasize the same things that I put to students for whom this is the basis for continuing their studies. Also, I will not ask you proofs. It will be important that you understand the terms well and that you demonstrate your understanding with examples.[4]

The university teachers saw the need to change the usual *didactic contract* (Bosch, 2006), according to which candidates from groups A and B passed the courses. Thus, the candidates from group D are not equipped with praxeologies that use proof as a technique, i.e., the logos block of praxeologies with which they are equipped is impoverished.

	D1	D2	D3	D4
Master's Diploma	electrical engineer	shipbuilding engineer	traffic engineer	physics teacher
ECTS credits	35	24	28	58
Prescribed courses by the FSS	IM, EG, INT, HM	IM, EG, INT, HM, IMLST, C	IM, EG, INT, HM, IMLST	None

Table 1. Some basic information about interviewed candidates from group D. In the third row are ECTS credits in mathematics achieved during university education. Prescribed courses: Introduction to Mathematics (IM), Elementary Geometry (EG), Introduction to Number Theory (INT), History of Mathematics (HM), Introduction to Mathematical Logic and Set Theory (IMLST), Combinatorics (C)

Analysis of interviews with several mathematics teachers

Teachers from group D

From the interviews with the candidates, we present only some data that seem to be the most indicative. Due to the lack of mathematics teachers, all four interviewed candidates from group D

[4] Translated from Croatian.

had worked as mathematics teachers without a license (D1 over 10 years, D2 over two years, D3 over one year, and D5 over 5 years). The candidates were asked about knowledge on which they rely in the teaching. D1 and D2 said they rely entirely on the mathematics knowledge they acquired during their primary education, D3 only partially relies on that knowledge, and D4 does not rely on it. The situation is polarized in terms of relying on the knowledge of mathematics acquired at the faculty from which they graduated. D1 and D4 rely entirely on that knowledge, and D2 and D4 do not rely on it at all. All candidates rely heavily on the experience gained working as a mathematics teacher. Candidates D1, D2, and D3 who are prescribed additional exams generally agree that this material is partly relevant for the position of a mathematics teacher in primary school and that it only partly affect their teaching. All candidates met the requirements of psychological-pedagogical education, but only candidate D4 passed the courses about methods of teaching mathematics and had mentored mathematical methodological practice. To the question to share impressions on what they consider important, D4 stated: "The courses that helped me the most were courses about methods of teaching mathematics. Knowledge of mathematics alone is not enough."[5]

Teachers from different groups

One candidate from each group A, B, and D (labeled as a, b, d) was interviewed to compare mathematical and didactic knowledge. All candidates have experience in teaching mathematics at the primary school level. Candidates b and d have completed additional psychological-pedagogical education, but courses of teaching mathematics were not represented in programs.

This paper presents the results of praxeological analysis whose object of knowledge is a circumcircle of a triangle as the sector of Euclidean geometry, required for teaching in primary school, which was assessed as particularly interesting for the comparison of candidates from different groups. At the university level, candidates from group D learned about it only through the prescribed course Elementary geometry as sector of Euclidean Geometry is not included in any aspect of the study programs they graduated from. Furthermore, in interviews, candidates from group D confirm that their teaching practice is not based on the prescribed courses but relies mainly on their knowledge gained during primary education and prior teaching experience.

Let T denote the type of problem being observed: construct the circumcircle of a given triangle. Furthermore, by τ_i we denote the technique that the candidate used in solving the tasks T, by θ_i the technology, by Θ_i the theory that the candidate showed in the discourse; where $i \in \{a, b, d\}$.

i	τ_i	θ_i	Θ_i
a	By constructing a perpendicular bisector of at least two sides of a triangle.	Definition of a perpendicular line segment bisector.	„Circumcircle of the triangle is a circle passing through all three vertices of the triangle."

[5] Translated from Croatian.

			Proof that all three perpendicular bisectors of the sides of a triangle intersect at one point equidistant from the vertices of that triangle.
b	By constructing a perpendicular bisector of at least two sides of a triangle.	Definition of a perpendicular line segment bisector. „A circumcircle of a triangle is the circle whose center is the intersection of all three perpendicular bisectors of the sides of the triangle, and the radius is the distance of the center to any vertex."	Proof that all three perpendicular bisectors of the sides of a triangle intersect at one point equidistant from the vertices of that triangle.
d	By constructing a perpendicular bisector of at least two sides of a triangle.	„A circumcircle of the triangle is the circle whose center is the intersection of all three perpendicular bisectors of the triangle's sides. "	Does not exist.

Table 2. The result of praxeological analysis.

All candidates solved the task using the same technique: by constructing perpendicular bisectors of the sides of a triangle, they obtained a unique point which is an element of all three perpendicular bisectors, that is, equidistant from the vertices of the triangle. Candidate *d* knew that the perpendicular bisectors of the sides of a triangle intersect at one point which is the center of the circumcircle of a triangle, but could not explain why, even after the examiner's sub-questions about the definition and properties of a circumcircle of the triangle and a perpendicular line segment bisector. Candidate *d* did not see a difference between the definition and properties of the observed mathematical objects. Candidate *b* proved, using the definition of a perpendicular line segment bisector, that all three bisectors of the sides of a triangle intersect at the same point and that this point is equidistant from the vertices of the triangle. Although b defined a circumcircle of the triangle as "a circle whose center is the intersection of all three perpendicular bisectors of the sides of the triangle, and radius is the distance of the center to any vertex"; he knew that all triangles have a circumcircle, but not all quadrilaterals do. Candidate a knew everything that candidate b did, but assisted by the examiner's sub-questions, he defined a circumcircle of the triangle as a circle passing through all three vertices of the triangle.

Candidates were also asked to list the most important steps in the development of a teaching unit about a circumcircle of the triangle, which can be perceived as a type of task T' of didactic praxeology. Candidate d's technique for solving the T' problem relied only on the techniques of relevant mathematical praxeologies: 1) how to construct a perpendicular line segment bisector; 2) how to use 1) to construct a circumcircle of the triangle. On the other hand, the technique of candidates a i b or solving problem T' included elements of theoretical blocks $[\theta_a/\Theta_a]$, i.e., $[\theta_b/\Theta_b]$ of the problem T. These candidates saw the need to define terms relevant to that teaching unit, and they provided some explanations of the techniques of relevant mathematical praxeologies.

Conclusion

The social problem of the shortage of mathematics teachers has prompted negotiations within the noosphere, i.e., negotiations between the MSE and the mathematical academic community, which have resulted in expanding the range of ways (with group D) to become a mathematics teacher in Croatian primary schools. Concerning the initial conditions for group D set by the MSE, the mathematical academic community, representing the conditions from the level of the Discipline, managed to set somewhat more precise requirements for mathematical praxeologies that candidates must be equipped with but failed to set requirements for didactic knowledge in a mathematical context. However, the mathematical praxeologies with which the candidates from group D are equipped through additional prescribed courses differ from the praxeologies that the candidates from groups A and B acquire through the same courses. Namely, due to the urgent situation in the labour market, university teachers abandoned the standard criteria, which resulted in a far weaker logos block than usual.

This not so clear reasoning on the choices of which qualifications are relaxed and which are required leads to firming the position of teaching as a semi-profession. In conclusion, we may hope that the future will bring more communication and collaboration between the stakeholders and more informed decisions, considering the relevance of mathematical and didactic knowledge.

References

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic Restrictions on the Teacher's Practice: The Case of Limits of Functions in Spanish High Schools. *Educ Stud Math* 59, 235–268 (2005). https://doi.org/10.1007/s10649-005-5889-z

Bosch, M., & Gascón, J. (2006). Twenty five years of the didactic transposition. *ICMI Bulletin*, 58, 51–65, June 2006.

Bosch, M. & Chevallard, Y. (2019). A short (and somewhat subjective) glossary of the ATD. In M. Bosch, Y. Chevallard, F. J. García & J. Monaghan (Eds.), Working with the Anthropological Theory of the Didactic in Mathematics Education, (pp. xviii-xxxvii). Routledge.

Bosch, M., & Winsløw, C. (2020). The external didactic transposition of mathematics at the university level: dilemmas and challengesLa transposition didactique externe des mathématiques au niveau universitaire: dilemmes et défis. Educação Matemática Pesquisa : Revista Do Programa De Estudos Pós-Graduados Em Educação Matemática, 22(4), 373–386. https://doi.org/10.23925/1983-3156.2020V22I4P373-386

Chevallard, Y. (2002). Organiser l'étude : 3. Écologie & régulation. In J. L. Dorier et al. (Eds.), *Actes de la 11e école de didactique des mathématiques*. (pp. 41-56). Grenoble : La Pensée Sauvage.

Chevallard, Y. (2013). A Theoretical Approach to Curricula. *Journal für Mathematik-Didaktik*. 13. 215-230. (1992). https://doi.org/10.1007/BF03338779

Cirade, G., & Crumière., A. (2019.) The study of teachers' mathematical and didactic praxeologies as a tool for teacher education. In M. Bosch, Y. Chevallard, F. J. García & J. Monaghan (Eds.), *Working with the Anthropological Theory of the Didactic in Mathematics Education*, (pp.159-168). Routledge.

Sani, N., & Burghes, D. (2021). Longitudinal study of 'retraining' non-maths specialist teachers to become capable, confident teachers of mathematics. *International Journal of Mathematical Education in Science and Technology.* 1-27. https://doi.org/10.1080/0020739X.2021.1890248

Winsløw, C., & Durand-Guerrier, V. (2007). Education of lower secondary mathematics teachers in Denmark and France: A comparative study of characteristics of the systems and their products. *Nordisk Matematikkdidaktikk*. 12(2), 5-32.

Winsløw, C. (2015). Mathematics at University: The Anthropological Approach. *Selected Regular Lectures from the 12th International Congress on Mathematical Education*, 859–875. https://doi.org/10.1007/978-3-319-17187-6_48

Identifying the logos block that composes para-didactic praxeology in mathematics lesson design: Case studies from pre-service teachers in Japan

Toru Hayata[1], Yusuke Shinno[2], Akio Matsuzaki[3], and Tatsuya Mizoguchi[4]

[1]Department of Education, Naruto University of Education, Japan, thayata@naruto-u.ac.jp

[2]School of Education, Hiroshima University, Japan, shinno@hiroshima-u.ac.jp

[3]Department of Education, Saitama University, Japan, makio@mail.saitama-u.ac.jp

[4]Faculty of Regional Sciences, Tottori University, Japan, mizoguci@tottori-u.ac.jp

Abstract

Our cross-cultural study consisted of two stages: the first stage investigated cultural specificities and commonalities within a country (Japan), and the second stage compared such characteristics between different countries (Japan and Thailand). In this paper, we aim to report the results of the first stage of the project, which utilised case studies to characterise Japanese pre-service teachers' para-didactic praxeologies in mathematics lesson design. In particular, we analysed the lessons designed by four teams of teachers and focused on differences in their logos blocks. The analysis of the four teams showed that their mathematical praxeologies were almost the same, but their didactic praxeologies were very different. Although the praxis of their specific praxeology differed, their theory was based on semi-structured problem-solving.

Résumé.

Notre étude interculturelle s'est déroulée en deux étapes : la première étape a permis d'étudier les spécificités et les points communs culturels au sein d'un pays (le Japon), et la seconde étape a permis de comparer ces caractéristiques entre différents pays (le Japon et la Thaïlande). Dans cet article, nous souhaitons rapporter les résultats de la première étape du projet, qui a utilisé des études de cas pour caractériser les praxéologies para-didactiques des enseignants en formation initiale japonais dans la conception des leçons de mathématiques. En particulier, nous avons analysé les leçons conçues par quatre équipes d'enseignants et nous nous sommes concentrés sur les différences dans leurs blocs de logos. L'analyse des quatre équipes a montré que leurs praxéologies mathématiques étaient presque les mêmes, mais que leurs praxéologies didactiques étaient très différentes. Bien que leur praxéologie spécifique soit différente, leur théorie était basée sur la résolution de problèmes semi-structurés.

Resumen

Nuestro estudio transcultural constó de dos fases: en la primera se investigaron las especificidades culturales y los aspectos comunes dentro de un país (Japón), y en la segunda se compararon dichas *características entre distintos países (Japón y Tailandia). El objetivo de este artículo es presentar los resultados de la primera fase del proyecto, que utilizó estudios de casos para caracterizar las praxeologías para-didácticas de los profesores japoneses en el diseño de las clases de matemáticas. En concreto, analizamos las lecciones diseñadas por cuatro equipos de profesores y nos centramos en las diferencias de sus bloques de logos. El análisis de los cuatro equipos mostró que sus praxeologías matemáticas eran casi iguales, pero sus praxeologías didácticas eran muy diferentes. Aunque la praxis de su praxeología específica difería, su teoría se basaba en la resolución de problemas semiestructurados.*

Introduction

Lesson study is well-established in professional teaching communities in Japan (e.g. Fernandez & Yoshida, 2004; Stigler & Hiebert, 1999) and is used for teacher preparation at universities (e.g. Nakamura, 2019; Peterson, 2005). It has been incorporated into pre-service teacher education at universities or schools in various forms. For example, Nakamura (2019) reported student teachers experience some aspects of lesson study (lesson planning, research lesson, and post-lesson discussion) through practicum, whereas the full-cycle model (Fujii, 2016) entails two additional stages (goal setting and reflection). Lesson planning is a crucial phase in the lesson study cycle because teachers' mathematical and didactical knowledge guides their preparation to design a lesson during this phase. In the Japanese context, pre-service teachers often undergo this phase in order to develop their knowledge and skills for *kyouzai kenkyuu* (教材研究), which refers to the 'study of [a] topic, curriculum, learning, learning progression and related teaching materials' (Bahn, 2018, p. 167). Although different aspects and resources are involved and used in the process of *kyouzai kenkyuu* (e.g. Fujii, 2016; Melville & Corey, 2021; Watanabe et al., 2008), one can understand that a lesson plan is its main product (e.g. Shinno & Mizoguchi, 2021).

The present study is part of a wider research project that offers an international approach to *kyouzai kenkyuu*. Although lesson study is well-known globally as a useful approach to facilitate teachers' professional development, how teachers in different countries engage in *kyouzai kenkyuu* during lesson planning is relatively unknown. To better understand this, we conducted a cross-cultural study on lesson study and particularly focused on lesson planning (Hayata et al., 2018; Mizoguchi et al., 2020; Mizoghichi et al., 2021). Our cross-cultural study consisted of two stages. The first stage investigated cultural specificities and commonalities within a country (Japan), and the second stage compared such characteristics between different countries (Japan and Thailand). In this paper, we aim to report the results of the first stage of the project, which utilised case studies to characterise Japanese pre-service teachers' (para)didactic praxeologies in mathematics lesson design.

Theoretical perspective

Within the anthropological theory of the didactic, the notion of *praxeology* is used to characterise any human activity in an institution (Chevallard, 2019; Chevallard & Bosch, 2020). The four components of praxeology are defined as follows. In the praxis (practical) block, *task type* indicates the problems of a given task, whereas *technique* is the way the task is performed. In the logos (theoretical) block, *technology* is the manner in which the technique is explained and justified, whereas *theory* is employed to explain or justify the technology. One can consider the distinction between the mathematical and didactic organisation of knowledge when analysing praxeology. Mathematical praxeology (MP) refers to mathematical practice and knowledge, whereas didactic praxeology (DP) refers to teachers' pedagogies of MP (for example, procedure to introduce today's lesson problem).

When examining the logos block of mathematical and didactic praxeology, it appears that MP is easier to characterise than DP, because technology and/or theory are often regarded as mathematical concepts or theorems. However, it is more difficult for researchers to identify the logos block of didactic praxeology because the theoretical elements of DP are often implicit. Moreover, the distinction between the technology and theory of DP was unclear in a previous praxeological analysis (cf. Winsløw et al., 2018).

Another crucial notion is para-didactic praxeology, which refers to teachers' practices and knowledge outside the classroom (Miyakawa & Winsløw, 2013, 2019; Winsløw, 2012). Lesson planning is an activity that is typically viewed as a para-didactic praxeology. It is therefore challenging for us to consider how to characterise the logos block of the para-didactic praxeology of *kyouzai kenkyuu* (Shinno & Mizoguichi, 2021).

In this paper, we offer case studies that allow us to interpret and identify the observable parts of the praxis block and the underlying parts of the logos were determined by investigating the lesson planning activities of three groups of pre-service teachers who belonged to different universities in Japan. We then discuss some differences and similarities among the groups, which may lead to a better understanding of Japan's cultural specificity.

Thus, our research questions are as follows:

RQ1: What logos block of para-didactic praxeologies can be identified in Japanese pre-service teachers' lesson planning activities?

RQ2: To what extent can the logos block explain the similarities and/or differences between mathematics lesson plans produced by the groups of pre-service teachers?

Context and method

Method

Participants in this study are undergraduate students, who are preservice teachers, from three universities (A, B, and C) located in in different regions of Japan. We asked three students (pre-service teachers) in each university to participate this study voluntarily. All participants are mathematics major students in the faculty of education and had completed their teaching practice (practicum). Regarding the university C, the group was divided into two sub-groups, so we labelled them C-1 and C-2. Three students as one group (or sub-group) from each university formed a team and were asked to design a lesson based on the problem described below. All other aspects were left to their choice, including their designations (e.g., grade, domain). Additionally, tuning the problem was allowed (e.g., changing a numerical value, changing Alice's name); however, these specific examples were not presented to the participants.

We collected data from video recording of their lesson planning processes for all groups of preservice teachers, as well as the lesson plans produced by them. During the planning activities, one of the authors observed their group activities without any intervention. We also prepared a questionnaire which was conducted after the experiments, asking each group about their lesson designs; 1) how did you consider a given problem in relation to the curriculum? 2) what is the aim of the designed lesson? 3) how do you assess students' performances in the lesson? In this paper, we mainly used the data from their lesson plans, although other data were also considered to better understand their productions.

Concerning the future international comparison, it was necessary to maintain consistency in the experimental settings. With this in mind, the problem chosen for lesson design was 'Alice was born in November 2002. How old will Alice be in September 2017?' This problem (called "Alice's problem") was selected from the International Open Bank of Mathematical Problems (http://mathopenbank.org/). For the selection of the problem, we concerned a cultural bias to avoid in the international comparison. For example, if we chose a problem from a textbook in Japan, Japanese participants will be more familiar with this problem, but participants from other countries will not be so familiar.

The problem was presented to participants in English—a language that was not their native tongue. This choice was based on our belief that the translation process itself was one of the important parts of our study and, therefore, we did not want the authors to translate the materials as their own knowledge and background could affect the translation. With these assumptions established, participants were furnished with the following basic tools: textbooks and teachers' manuals used in the region, the national curriculum, and a personal computer with no internet access. They were then asked to create a lesson plan. While they performed this activity, the facilitators did not interfere in any way. However, Team A asked to see the commonly used format for lesson plans in their region. Facilitators allowed this and provided the requested documents.

A priori analysis of Alice's problem

This is a short a priori analysis of Alice's problem. The task consisted of finding Alice's age in September 2017. Several techniques were developed for estimating age. The first one is expressed as counting 1 years old in September 2013, 2 years old in September 2014…(τ_{cnt}). The second one is expressed as 2017-2002-1=14(τ_{sup}). In these techniques, one technology is contextual explanation of the calculation, such that Alice would have aged by a year in November 2017, and since her birthday has not yet arrived as of September 2017. Another technology is more mathematical discourse to explain the reason why two subtractions appear, which is based on the ordinal structure and the relationship between the natural number. The theory uses the properties of natural number to explain the ordinalities.

The third technique involves focusing on the number of months in a given period; that is, nine months in 2017, two months in 2002, and 12 months × 14 years in between. Consequently, the solution of $(9+2+12\times14)\div12$ results in Alice's age (after rounding down; $14.916\cdots\rightarrow14$) ($\tau_{cal}$). In this technique, the technology can also be a contextual explanation of the calculation, considering the number of months and years. However, there are several possible variations of these techniques, such as using (decimal) fractions. In such case, the theory referred to must include one in which the corresponding mathematical concept is relevant (e.g. the theory of rational numbers).

Result of each experiment

The pre-service teachers were divided into three teams, and they collaboratively worked to create a lesson plan to describe the lesson. In this experiment, although we did not direct them to do so, the teachers wrote a simplified lesson plan. This plan is generally called *ryakuan*, and we assumed that writing *ryakuan* is something that the pre-service teachers had experienced in their teaching practice. In addition to the flows and details of the lesson, it is common for a regular lesson plan in Japan to include, for example, perspectives on evaluating students, the mathematical background of contents, and interpretations of the curriculum. In *ryakuan*, some of them are omitted depending on their situations and/or purpose. Although *ryakuan*, several teams wrote a short perspective on (virtual) students, and other things. For the purposes of our RQs, we will only deal with the design part of the one-hour lesson (details, lesson objectives, blackboard management plan). In what follows, we refer to these only as the lesson plan. However, as for Team A and B, no MP is written in their lesson plan. So, just to identify the MPs, only Team A and B referred to the notes they had written during the design process.

Their lesson plans are presented in Figures 1, 2, and 3. The lesson plans were actually in Japanese but have been translated by the authors. All teams did not describe all of the lessons they designed in their lesson plans (for example, B did not describe any of the MPs they designed). This suggested that we needed to go beyond the analysis of their MP and DP, and instead examine their para-didactical praxeology.

2. Purpose of this lesson

Students can become able to interpret meaning of the problem, to devise methodology, and to feel closer to students themselves.

3. Progress of this lesson

Time	Learning Activity	Point to note for instruction	Assessment
5 min	1. Look at the problem and realize that they can solve the problem by using subtraction.	・Through hiding one element of the problem students can be engaged in the exposition.	・ Are students using previous learning to understand the problem?
	Aim Let's use a 4-digit sum to think about the problem.		
15min	2. Look at the previously covered months and think individually about what type of calculation is required.	・By having students focus on the month, they will find the key to the calculation and get ideas about how to solve it.	・Are they progressing to engage with the problem?? (Student's note, and observation)
15 min	3. Individual thoughts are announced to the class and discussed.	・ By having them present, they will be able to explain it in their own words and take on board the opinions of their friends.	・Are they able to explain their thoughts in words or with the use of graphs, tables, etc. ? (speech)
10 min	4. Think about those close to them (teachers or parents, etc.)	・ By thinking about those close to them they can solidify their competence and extend their abilities to use such calculations.	・Are they trying to apply what they have learned in practical activities? (copybooks, speech)

4. Plan of blackboard

1. What kind of calculation is it?

Alice was born in November 2002. How old will be Alice in September 2017?

Devising a plan to solve by using 4-digit subtraction.

2017-2002=15

Answer 15 years old

Figure 1. **Team A's lesson plan**

2. Purpose of this contents

The students are asked to think about the meaning of the problem and whether it is possible to solve it using what they have already learned.

To understand that there is not only one way to solve a problem, but also many other ways to think about it.

To develop an attitude of communicating one's own ideas to others in an easy-to-understand manner by using diagrams and graphs.

	Students' activities	Points to consider in support and instruction, and assessment/evaluation.
Introduction	Students will know the learning tasks for this class. Kenta was born in November 2002. How old will Alice be in September 2017?	
Development	Students solve the problem individually.	If some students can't solve a problem, the teacher gives them a hint on how to think about it. To encourage students to generate many ways of thinking, when a student solves a problem using one method, the teacher asks them to think of another one. Solving problems by using diagrams and graphs to help students elucidate their thought process.
	Groups present their ideas.	Students listen with the following in mind: do they think similar to or different from your thinking.
	Students decide on one idea to present to the group. They summarise their ideas on the whiteboard and think about how to present them.	The teacher adjusts the groups' opinion so that they have different opinions of each other.
	Each group will present their ideas	The teacher should instruct the group to teach each other so that everyone in the group understands the ideas being presented.
Matome	Students summarise their presented ideas.	Teacher should make the students aware that there are many ways to solve a problem

Figure 2. **Team B's lesson plan**

1. Purpose of this lesson

Students investigate about different units with consideration of real-situation.

	Learning activities	Points to consider in support and instruction (assessment/evaluation).
Introduction (5 min)	1. To grasp the problem Alice was born in November 2002. How old will Alice be in September 2017? 2. To present today's objective Let students think of a method to ascertain Alice's age	
Development (35 min)	3. Students solve the problem individually C: Students use a number line.	Review their focus on units such as 2017, November, etc. Then, have them review their previous calculations in metres, litres, etc. Have them reflect on the use of number lines and fractions.
	C: Students use mixed fractions ($2017\frac{9}{12} - 2002\frac{11}{12} = 2016\frac{21}{12} - 2002\frac{11}{12} = 14\frac{10}{12}$) C: Subtract 2012 from 2017 and you get 15. It is not possible to draw from September to November, so the answer is 14 years old.	Teacher should encourage the students to relate the relationship between the year and the month to the idea of carrying-over or mixed fractions.
	4. Students indulge in classroom discussion	If you cannot easily use a number line, it is more convenient to use mixed fractions.
Matome (10 min)	5. Matome Able to solve problems by subtracting between mixed fractions	

Figure3. Team C-1's lesson plan

(The problem in pre-translated version is also English)

1. Purpose of this lesson

Students can set variables according to the problem.

Realize that by using letter formulas, one can quickly obtain answers.

	Learning activities	Points to consider in support and instruction (assessment/evaluation).
Introduction (5min)	1. To grasp the problem Alice was born in November 2002. How old will Alice be in September 2017? T: Consider Alice's age.	
Development (35min)	Today's objective In what way can you quickly ascertain anyone's age? T: What are the possible solutions? ・counting (years) ・construct a mathematical sentence ・counting (months) e.g. teacher XX born July, 1964	We stress that if the person you're dealing with is younger in age, like Alice, you can count their age. However, age becomes harder to count when the person is older, such as the principal
	T: If it is November, how many months are there? Let it be expressed in mathematical sentence. ・$12 \times (2017 - 2002)$ months T: But it's September. What do you do? ・$12 \times (2017 - 2002) - 2$months	1 Year = 12 Months Have the teacher focus on the birth month and the current month, and figure out how to find -2.
	T: Let's use letters to express the sentence so that we can ascertain the ages of other people. Year of birth:x Year of month: y $12 \times (2017 - 2002) - (11 - 9)$ $12 \times (2017 - x) - (y - 9)$ months	Finally, divide by 12, since a year is 12 months.
Matome (10 min)	5. Matome Using letters, you can easily ascertain anyone's age. $+\alpha$ Application: How old will be Alice in December 2065?	

Blackboard management plan

Problem: Task: Prospects:	・If it is November… ・But it's September… Year of birth:x Year of month: y Mathematical Sentences: Matome:

Figure4. Team C-2's lesson plan

(The problem in pre-translated version is also English)

Analysis

Mathematical praxeology in designed lesson

We did not forbid them to modify Alice's problem, but there was very little difference in the design of the task type in the MP of each team. Therefore, all groups share the same task. Teams C-1 and C-2 retained the English framing of the problem, whereas the other teams translated it into their native language (Japanese). The following Table.1 represents τ in their designed MP.

Team	Technique
A	τ_{sup}: After calculating 2017–2012, correct for the fact that November cannot be subtracted from September so 2017-2012-1.
B	τ_{cnt} : Draw a number line and count using it. τ_{sup} : After calculating the equation 2017–2012, correct for the fact that November cannot be subtracted from September so 2017-2012-1. $\tau_{cal\text{-}frac}$: Calculate the mixed fraction equation $2017\frac{9}{12} - 2002\frac{11}{12} = 14\frac{8}{12}$
C-1	τ_{cnt} : Draw a number line and count it out. τ_{sup} : After calculating the equation 2017–2012, correct for the fact that November cannot be subtracted from September. $\tau_{cal\text{-}frac}$: Calculate the mixed fraction equation $2017\frac{9}{12} - 2002\frac{11}{12} = 14\frac{8}{12}$
C-2	τ_{cnt} : Draw a number line and count it out. $\tau_{sup\text{-}mon}$: Calculate the equation12 × (2017–2012) and the correct it by adding -2 so 12 × (2017–2012) – 2, and devide by 12 and round down it. τ_{cal} : Calculate the equation 12 × (2017–2012) - (11 - 9), and devide by 12 and round down it. τ_{gen} : Set Alice's year of birth as *x* and birth month as *y,* and calculate the equation.12×(2017 – x) – (y – 9) and devide by 12 and round down it.

***Table 1*. The predicated technique of the MP related to lesson designs of each team.**

Almost all of the designed τ was identified in our a priori analysis. τ_{cal} has two variations, that is, calculating by fraction (Teams B and C-1), or converting the year to months (Team C-2). Team C-2 also designed $\tau_{sup\text{-}mon}$, which is a variation of τ_{sup}, because only months were used in their designed lesson. These differences were based on their chosen grades and field of mathematics, that is, below "field" levels of decision is conditions and constraints for it.

τ_{gen} was not assumed in our a priori analysis. It was designed as a general algebraic formula for Alice's birth year and month as technique. Team C-2 designed Alice's problem in algebra. In Japan, due to mathematical properties, the (national) curriculum, and the (tacit) agreement among teachers, being able to express generalities in terms of mathematical characters is considered an important educational aim. The chosen field of mathematics may condition and constrain the designing of τ_{gen}.

As described above, although minor differences were observed, the designed MP was generally consistent with those identified in our a priori analysis, and there were no major differences among the teams.

Didactic praxeologies in designed lesson

All teams designed and organised their DPs to let students grasp the problem, solve problems by themselves, discuss, and do the *matome*. The *matome* is an act of teaching that summarises the content of the lesson and emphasises its important points (Fujii et al., 1998). Team C-2 did not explicitly design discussions, but it did design specific conversational cues that encouraged students to think. We presumed that this was collaborative discussion that involved both teachers and students. Generally, this type of lesson flow when teaching mathematics is called ‘structured problem solving’ (Stigler and Hiebert, 1999). The first part may include a review and reflection of what was learned in the previous hour (Stigler and Hiebert, 1999), but not all teams followed this step. This was probably because we asked them to design only one lesson. All the designed DPs presented significant similarities and differences.

For instance, Team A sought to present the problem to the students by designing the technique to intentionally show wrong answers (i.e. 2017-2002=15). Team C-2 designed a technique for the same type of task, emphasising that it is difficult to count older people’s ages (such as the principal of their school). What they are specifically doing (praxis) differs, but the technique is presumed to be the same. It is implicit, but their technique could reflect that the teacher should not just present the problem-text, the teacher has to make the students want to ask something about problem-text, or that it is important for students to have interests and concerns.

The teams that chose elementary schools (except C-2) designed a DP that involved tasks in which children solve problems on their own with as little help as possible from the teacher and techniques in which diagrams/graphs/number lines are presented. Team C-2 designed the same type of task, but the technique employed involved providing the students with a specific example. Their technique could reflect that students should be able to solve problems without any help from the teacher or that the teacher should not give the answer to the problem directly but should provide clues for thinking.

With the exception of the DP of Team C-2, the DPs were designed to encourage discussions (we labelled DP_{SD}). However, there was a significant difference in the praxis of the DP_{SD} that organises the DP. The following table presents the same.

	Type of task	Technique
$DP_{SD\text{-}A\text{-}1}$	The teacher encourages students to explain (their solution) in their own words.	The teacher provides time to make a presentation.
$DP_{SD\text{-}A\text{-}2}$	The teacher encourages students to incorporate the opinions of other students into their own solutions.	The teacher provides time to make a presentation.

$DP_{SD\text{-}A\text{-}insuf}$	The teacher stimulates students' interest so that they want to discuss.	The teacher proposes an insufficient answer (2017-2002=15).
$DP_{SD\text{-}B\text{-}1}$	The teacher encourages students to compare their own solutions with those of other students and consider their similarities and differences.	The teacher sets up a group of several students and encourages discussion.
$DP_{SD\text{-}B\text{-}2}$	The teacher encourages students in each group to construct a single group opinion and present it.	The teacher sets up a group of several students and encourages discussion.
$DP_{SD\text{-}B\text{-}3}$	The teacher encourages students teach each other so that the group's opinions are understood by all group members.	The teacher sets up a group of several students and encourages discussion.
$DP_{SD\text{-}B\text{-}4}$	The teacher adjusts the groups' opinions so that they are as different as possible.	Teacher will request individual groups [it is not designed to be obvious].
$DP_{SD\text{-}C1\text{-}1}$	The teacher helps students make connection between mathematical content through group discussions.	The teacher addresses students' solutions, emphasises the important content to be taught, and encourages consensus building [it is not designed to be obvious].
$DP_{SD\text{-}C1\text{-}2}$	The teacher helps students recognise the values of mixed fraction.	The teacher addresses students' solutions, emphasises the important content to be taught, and encourages consensus building [it is not designed to be obvious].
$DP_{SD\text{-}C2\text{-}1}$	The teacher leads discussions with students about the mathematical value of mathematical sentences and letters [it is not designed to be obvious].	The teacher addresses students' solutions, emphasises the important content to be taught, and encourages consensus building [it is not designed to be obvious].
$DP_{SD\text{-}C2\text{-}insuf}$	The teacher creates a situation in which the students can easily express in the equation and use it as a cue for discussion.	The teacher proposes an insufficient answer: 12 × (2017–2012).

***Table 2*. DP and praxis block related to each team's lesson design**

The logos that justify the DPs of these teams also differ. Team A's technology is based on the notion that when students explain their ideas and/or solutions in their own words, their mathematical understanding deepens and their competencies improve. It can also be inferred that when students solve problems in different ways, share their methods, and reflect on their differences and similarities, they deepen their mathematical understanding and enhance their competencies. Team C-1's technology is based on the notion that students' mathematical understanding can be deepened through discussion. This is evident from the fact that they always associate their DP with their MP.

C-2 did not design their DP to be related to students' discussions. However, the teachers designed specific discussions to be had with the students. Team C-1's technology can be inferred to be the same as that of Team C-2 (i.e. it deepens students' mathematical understanding through discussion). The different approaches to realising the same logos may possibly result from the difference in the schools they chose.

The same tendency can be observed in the design of the DP for the *matome*. The DP designed by B-1 for the *matome* is based on the technology that there is educational value in not deciding that a single way of solving a problem is best and knowing that there are many ways to solve a problem. Furthermore, the DP of C-1 and C-2 is based on the technology that there is educational value in deepening students' recognition of the value of certain mathematical content. The *matome* was not included in Team A's lesson plan. However, they designed their DP to emphasise potential ways to solve this problem. For example, one explanation was 'Alice is 15 years old in November 2017. It is September, so she is 14 years old'. Their technology was based on the notion that there is educational value in explaining students' words.

Another interesting tendency is that Teams A and C-2 designed to treat insufficient answers as τ (we labelled them as $DP_{SD\text{-}A\text{-}insuf}$ and $DP_{SD\text{-}C2\text{-}insuf}$). We can interpret insufficient answers as part of (and/or in the middle of) τ_{sup}. By dealing with it intentionally, the type of tasks itself is different, but as mentioned above, they were designed to encourage discussions. Team B did not design it; however, they predicted students who could not solve it or develop ideas for finding Alice's age. $DP_{SD\text{-}B\text{-}insuf}$ may be included in such failures and DPs for them (if then, it is not explicitly designed).

	Technology	Theory
$DP_{SD\text{-}A\text{-}1}$	When students explain their ideas and/or solutions in their own words, their mathematical understanding deepens, and their competences improve.	Semi-structured problem solving
$DP_{SD\text{-}A\text{-}2}$		
$DP_{SD\text{-}B\text{-}1}$	When students solve problems in different ways, share their methods, and reflect on their differences and similarities, they deepen their mathematical understanding and enhance their competences.	Semi-structured problem solving
$DP_{SD\text{-}B\text{-}2}$		
$DP_{SD\text{-}B\text{-}3}$		
$DP_{SD\text{-}B\text{-}4}$		
$DP_{SD\text{-}C1\text{-}1}$	Discussion between students facilitates a deeper understanding of mathematics.	Semi-structured problem solving
$DP_{SD\text{-}C1\text{-}2}$		

$DP_{SD\text{-}C2\text{-}1}$	Discussion facilitates a deeper understanding of mathematics	Semi-structured problem solving

***Table 3*. DP and logos block related to each team's lesson design**

The technologies in Table 3 are the ones to explain or justify the corresponding techniques in different ways. However, all the technologies are more or less related to the Japanese pedagogical notion called "*neriage*", which refers to collective activities aiming at elaborating and refining students' different ideas (Mizoguchi, 2013). The "neriage" is one of the key elements of the Japanese teaching pattern which is today well-known as 'structured problem solving' (Stigler & Hiebert, 1999), although what we identified in their teaching plans is an informal version of what Stigler and Hiebert (1999) described as 'structured problem-solving'. Therefore, we term it as "semi-structured problem solving" as the theory that underpins these technologies. Generally, 'structured problem-solving' consists of a sequence of five phases: 1) reviewing the previous lesson, 2) presenting the problem for the day, 3) students working individually or in groups, 4) discussing solution methods, and 5) highlighting and summarising the major points (Stigler & Hiebert, pp. 80-81). Although the pre-service teachers' lesson plans did not faithfully follow the five phases, their plans did share some characteristic aspects related to structured problem-solving in an informal manner (for example, all lesson plans include a discussion/*matome* that corresponds to 4) in the second half of their class), which may explain the technological discourse of their DPs.

Only Team A explicitly designed a DP for assessment and evaluation. However, they did not voluntarily design their assessment and evaluation DP. They only did so because the format of their lesson plans included a section for assessment/evaluation. In the video, one of Team A's members said, 'I cannot remember another topic to include in their lesson plan', to which another student replied, 'Let me think.... evaluation…is it enough?' It is suggested that their logos block of the praxeology that related with assessment is the structure of the semi-structured problem solving, dictate by the format, rather than the significance or educational role of the evaluation.

Only teams A and C-2 designed DPs in which the teacher asks students to solve a different but similar problem after the *matome*. However, their DPs were very different. Team C-2 designed a DP in which students were asked to solve a problem in which the year and month variables of Alice's problem had been changed. In contrast, Team A designed a DP that required the students to ascertain the age of a real, familiar person (like their parents) rather than that of an imaginary person like Alice. These DPs are based on the common technique that there is value in understanding the content taught by solving analogous problems. In addition to this, it is presumed that Team A's technique has educational value as thinking about familiar things enables students to understand mathematical content and students should be able to solve familiar problems through what they learn.

Another interesting difference is that only teams A and C-2 designed a DP that involved the blackboard (in their blackboard management plan). Generally, most mathematics classes in Japan use the blackboard (Baldry et al., 2022), so it would be odd for teams B and C-1 to not include the blackboard, and we can assume they also utilised it in their plan. In other words, they may have designed a DP that involved the blackboard without mentioning it explicitly. This DP may be based on the logos related to the function of the blackboard in Japan. It is generally understood that the blackboard not only presents information to students but also offers a summary of the class in chronological order, which enables the students to relive the class just by reading it. Teams B and C-1 may have thought that since they had designed the outline of the class, what would be written on the blackboard would follow the same. In fact, the blackboard management plans designed by teams A and C-2 are consistent with their overall lesson DP.

Discussion

The Japanese teaching pattern, which is called structured problem-solving (Stigler & Hiebert, 1999), could be experienced by pre-service teachers in different situations; for example, teaching practicum, observing in-service teachers' lessons, or method course classes at university. It seems that their understanding of the teaching pattern affects their lesson planning activities. This notion is very polysemous, but it usually aims to emphasise that students solve the problem, promote discovery and (deeper) understanding of mathematical concepts, and foster competencies such as communication skills. The DPs of all teams were largely organised to realise these goals, which is why they can be considered semi-structured problem solving.

The exact DP designed by each team varied considerably, especially in terms of their praxis. However, almost all their logos, especially in terms of theory, were common. It implies that all the variations in praxis could be linked with the induvial groups. The technology employed by each team differed significantly. For instance, teams C-1 and C-2 focused on technology for understanding mathematical content, Team B placed little emphasis on such technology, and Team A adopted an in-between approach. Despite these differences, the theory part is consistently interpreted as semi-structured problem solving. Hence, their praxeological organisations were locally- and regionally-embedded (Chevallard, 2019, p.94).

Concluding remarks

These differences appeared in the DP related to encouraging students to discuss in groups and the *matome*. Considering that Teams C-1 and C-2 also share similarities in these aspects, these differences may result from certain conditions and constraints. Given the role of the *matome*, it is in effect the difference between praxis and logos in the praxeologies of the teacher encouraging students to discuss in groups that characterise each team's DP. It can also be argued that every team to sought to achieve their goal by encouraging the students to indulge in discussion. It is not clear enough what produces these differences. There may be more detailed differences in the semi-structured problem solving for each team, or it may be due to differences in the para-didactical infrastructure. In addition, whether these differences are due to individual or regional differences will require further experimentation and analysis.

As mentioned earlier, the praxeology of writing lesson plans has an impact on the lesson design process. For instance, Team A may have designed the DP on assessment/evaluation as they considered it important to include within the lesson plan because their format have evaluation section.

Conversely, the format commonly used in other teams' regions does not have it. Influenced by this, they did not explicitly design and describe the evaluation. The design of this DP is implicitly or explicitly influenced by their para-didactical infrastructure (e.g., the format and function of their lesson plan) and para-didactical praxeology (i.e. praxeology of the writing lesson plan) (Miyakawa and Winsløw, 2019), although we in this study did not investigate into the para-didactical infrastructures in each region. A lesson plan is not solely written to outline the entire lesson but may also serve to organise important issues within the lesson study (Corey and Ninomiya, 2019). It is highly likely that these factors influenced the teams' lesson designs. However, the dynamism between them was not the subject of this study. This is a limitation of this study and of our methodology.

As mentioned above, this study is part of an international comparison project with Thailand. Our experimental methodology is potentially applied to Thai cases to clarify commonalities and specificities of designing lesson process, compared with the Japanese cases in this paper. However, it remains to be seen whether this method can be used for international comparison. It is also a future task to consider the effect of didactical infrastructure, which is responsible for these differences.

References

Bahn, J. (2018). Japanese vocabulary—A proposal for standard transcriptions. In M. Quaresma et al. (Eds.), Mathematics lesson study around the world (pp. 165–169). Springer International Publishing.

Baldry, F., Mann, J., Horsman, R., Koiwa, D., & Foster, C. (2022). The use of carefully planned board work to support the productive discussion of multiple student responses in a Japanese problem-solving lesson. Journal of Mathematics Teacher Education. https://doi.org/10.1007/s10857-021-09511-6

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. Hiroshima Journal of Mathematics Education, 12, 71–114.

Chevallard, Y., & Bosch, M. (2020a). Anthropological theory of the didactic (ATD). In S. Lerman (Ed.), Encyclopedia of mathematics education (pp. 53–61). Springer.

Corey D.L., & Ninomiya H. (2019). Values of the Japanese Mathematics Teacher Community. In: P. Clarkson, W. Seah, J. Pang (Eds.), Values and valuing in mathematics education. Springer. https://doi.org/10.1007/978-3-030-16892-6_4

Fernandez, C., & Yoshida, M. (2004). Lesson study: A Japanese way to improving mathematics teaching and learning. Routledge.

Fujii, T. (2016). Designing and adapting tasks lesson planning: A critical process of lesson study. ZDM Mathematics Education, 48, 411–423. https://doi.org/10.1007/s11858-016-0770-3

Fujii, T., Kumagai, K., Shimizu, S., & Sugiyama, S. (1998). A cross-cultural study of classroom practices based on a common topic. Tsukuba Journal of Educational Study in Mathematics, 17, 185–194.

Hayata, T., Mizoguchi, T., Matsuzaki, A., Shinno, Y., Inprasitha, M., Changsri, N., & Kunseeda, P. (2018). A comparative research of mathematics lesson design by pre-service teachers: A case of Japan and Thailand [Conference presentation]. The 11th International Conference on Educational Research: Innovations for Capacity Building and Networking, Khon Kaen University, Thailand.

Melville, M. D., & Corey, D. L. (2021). Kyouzaikenkyuu: An exploration of Japanese mathematics teachers' daily planning practices. Journal of Mathematics Teacher Education, 18, 1–3. https://doi.org/10.1007/s10857-021-09493-5

Miyakawa, T., & Winsløw, C. (2013). Developing mathematics teacher knowledge: The paradidactic infrastructure of "open lesson" in Japan. Journal of Mathematics Teacher Education, 16, 185–209. https://doi.org/10.1007/s10857-013-9236-5

Miyakawa, T & Winsløw, C. (2019). Paradidactic infrastructure for sharing and documenting mathematics teacher knowledge: a case study of "practice research" in Japan. Journal of Mathematics Teacher Education, 22, 281–303. https://doi.org/10.1007/s10857-017-9394-y

Mizoguchi, T. (2013). Design of problem solving lesson and teacher's assistance: Based on refining and elaborating mathematical activities, Proceedings of the 6th East Asia Regional Conference on Mathematics Education (EARCOME6), vol.2, 194-283.

Mizoguchi, T., Inprasitha, M., Changsri, N. & Shinno, Y. (2020). Describing researchers' ways of seeing a lesson: As the first work of the cross-cultural study on lesson study between Japan and Thailand. Educação Matemática Pesquisa: Revista do Programa de Estudos Pós-Graduados em Educação Matemática, 22(4) - Congresso Internacional sobre a TAD - CITAD 6, 836–844.

Mizoguchi, T., Shinno, Y. & Hayata, T. (2021, July 17). How can we classify the teachers' paradidactic praxeologies in different institutional settings? [Conference presentation]. The 14th International Congress on Mathematical Education, Shanghai.

Nakamura, K. (2019). How lesson study helps student teachers learn how to teach mathematics trough problem-solving: Case study of a student teacher in Japan. In R. Huang, A. Takahashi, & J. da Ponte (Eds.), Theory and practice of lesson study in mathematics (pp. 507–525). Springer.

Peterson, B. (2005). Student teaching in Japan: The lesson. Journal of Mathematics Teacher Education, 8, 61–74.

Shinno, Y., & Mizoguchi, T. (2021). Theoretical approaches to teachers' lesson designs involving the adaptation of mathematics textbooks: Two cases from kyouzai kenkyuu in Japan. ZDM Mathematics Education, 53(6), 1387–1402. https://doi.org/10.1007/s11858-021-01269-8

Stigler, J. W., & Hiebert, J. (1999). The Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom. Free Press.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective Lesson Study and beyond. In F. Arbaugh & P. M. Taylor (Eds.), AMTE Monograph 5: Inquiry into mathematics teacher education (pp. 131–142). Information Age Publishing.

Winsløw, C. (2012). A comparative perspective on teacher collaboration: The cases of lesson study in Japan and of multidisciplinary teaching in Denmark. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), From text to "lived" resources. Mathematics curriculum materials and teacher development (pp. 291–304). Springer.

Winsløw, C., Bahn, J., & Rasmussen, K. (2018). Theorizing lesson study: Two related frameworks and two Danish case studies. In M. Quaresma, C. Winsløw, S. Clivaz, J. da Ponte, A. Ní Shúilleabháin, & A. Takahashi (Eds.), Mathematics lesson study around the world (pp. 123–142). Springer.

La recherche collaborative comme institution abritant des systèmes de recherche
Le cas d'un projet en résolution de problèmes mathématiques

Cécile Redondo
ADEF, Aix-Marseille Université, France

Michèle Artaud
ADEF, Aix-Marseille Université, France

Karine Bernad
LIRDEF, Université de Montpellier, France

Abstract. Our paper presents a way of modelling the factors, working methods and processes at play within an educational research system involving teachers and researchers. This modelling takes as a touchstone a work conducted in the academy of Aix-Marseille for two years. Within the framework of the anthropological theory of didactics, the proposed model makes it possible to characterise the emerging research system and its functioning as well as the conditions and constraints of the evolution of the relationships to the object studied: mathematical problem solving (MPR).

Résumé. Notre communication présente une manière de modéliser les facteurs, modalités de travail et processus en jeu au sein d'un dispositif de recherche en éducation faisant collaborer des enseignants et des chercheurs. Cette modélisation prend comme pierre de touche un travail conduit dans l'académie d'Aix-Marseille durant deux années. Dans le cadre de la théorie anthropologique du didactique, le modèle proposé permet de caractériser le système de recherche émergeant et son fonctionnement ainsi que les conditions et contraintes de l'évolution des rapports à l'objet étudié : la résolution de problèmes mathématiques (RPM).

Liste des editeurs (Eds)
Advances and challenges in the anthropological theory of the diactic (pp. xx-yy)
VII congrès international de la TAD (Bellaterra, 19-23 juin 2022)
Axe 3. *La TAD et la professionnalisation du métier d'enseignant*
Editorial, año

1. Introduction

Depuis 2017 une dynamique locale initiée par une fédération de recherches en éducation (SFERE-Provence) et le rectorat d'Aix-Marseille vise à accompagner les enseignants des réseaux d'éducation prioritaire dans un processus de professionnalisation face aux besoins déclarés par les établissements. Il s'agit de développer des travaux coopératifs entre divers acteurs de ces réseaux (enseignants, personnels de direction, voire d'autres membres de la communauté éducative : personnels d'encadrement, animateurs, éducateurs, parents, élèves, etc.), et des chercheurs s'intéressant à des problématiques relevant du champ de l'éducation et de la formation. Cette politique volontariste développée à l'échelle locale, a vu naître depuis 2017 quatre « vagues » de projets, chacun sur une durée de deux ans.

Dans ce cadre, nous avons été responsables et co-porteurs[1] d'un projet de septembre 2019 à juin 2021, mené en collaboration avec trois établissements scolaires : un collège et deux écoles publiques d'un même secteur scolaire dans une zone urbaine de la région du sud de la France. Ce secteur scolaire fait face à de nombreuses difficultés sociales et scolaires des élèves qui se traduisent en partie par de faibles scores obtenus par les élèves aux évaluations nationales passées à l'entrée en sixième (élèves de 11-12 ans). C'est ce problème qui déclenche la demande d'intervention de chercheurs à propos de la résolution de problèmes en mathématiques (RPM) par le chef d'établissement du collège. Ainsi, le dispositif de travail conjoint avec les chercheurs est mis en place en juin 2019 à partir d'une analyse et d'un ajustement de la commande initiale justifié par la faisabilité pratique et scientifique du projet.

Quatre axes principaux caractérisent notre dispositif : un temps et une présence prolongés dans les établissements scolaires (deux ans) ; une visée de changement et d'amélioration des pratiques professionnelles (en lien avec l'amélioration des résultats des élèves) ; un engagement collectif des acteurs ; une planification concertée de la démarche d'intervention « avec » les acteurs (et non pas « sur » ces derniers).

[1] Sont désignés comme co-porteurs du projet quatre enseignants-chercheurs intervenant directement dans le dispositif dont les auteurs de ce texte, ainsi que plusieurs personnels de direction et d'encadrement de l'Éducation nationale : un principal de collège, une inspectrice et deux conseillères pédagogiques de circonscription (premier degré), ainsi qu'un inspecteur académique rattaché au secteur.

La collaboration s'organise ainsi entre chercheurs et praticiens. La démarche générale consiste en un « accompagnement » des équipes d'enseignants sur le terrain, traduisant aussi une position singulière du chercheur dans ses modalités d'intervention : « celle d'un accompagnateur, d'un facilitateur qui encourage la réflexion et l'échange entre les acteurs concernés par le problème ; il organise le contexte, préside les rencontres, fournit différentes ressources tels des cadres conceptuels pour soutenir la réflexion des participants, etc. » (Morrissette, 2013, p. 45). Il ne s'agit donc pas d'administrer des contenus, mais d'organiser une structure, un mode de fonctionnement, et de promouvoir des possibilités pour permettre aux acteurs de décider de certaines voies à prendre pour faire face à des situations de travail spécifiques (et les améliorer). L'implication dans le dispositif engage aussi les participants (enseignants comme chercheurs) à transformer ensemble leurs praxéologies (Chevallard, 1998), voire à en créer de nouvelles de toutes pièces, dans la perspective de leur développement professionnel. À l'aune de ce double objectif, produire des connaissances et permettre le développement professionnel des praticiens engagés dans la recherche, le dispositif relève des recherches dites collaboratives (Desgagné, 1997).

2. Cadre théorique

Le cadre théorique qui nous permet de modéliser la collaboration et ce qui est à l'œuvre dans ce dispositif est issu de la théorie anthropologique du didactique (TAD, Chevallard, 1992, 2007, 2019). Le qualificatif anthropologique témoigne de l'intention de cette théorie du didactique d'être attentive à toutes les dimensions du réel social qui influent sur le didactique. Pour le dire autrement, un phénomène didactique ne peut être expliqué en ne considérant que l'institution qui l'abrite. Ce qualificatif témoigne aussi de l'intention d'analyser le didactique partout présent autour de nous, passé, présent et à venir, dans n'importe quelle situation portant une visée didactique (une situation vécue dans la rue, dans une entreprise, à l'université, etc.). Il s'agit d'étudier l'émergence, la vie et la diffusion des connaissances auprès des personnes et des institutions, dans tout domaine de la société où quelque chose est étudié, enseigné ou appris.

La notion d'institution est ainsi une notion fondamentale de la TAD. Yves Chevallard l'explicite comme suit :

> Une institution *I* est un dispositif social « total », qui peut certes n'avoir qu'une extension très réduite dans l'espace social (il existe des « micro-institutions »),

> mais qui permet – et impose – à ses sujets, c'est-à-dire aux personnes *x* qui viennent y occuper les différentes positions *p* offertes dans *I*, la mise en jeu de manières de faire et de penser propres. Ainsi la classe est-elle une institution (dont les deux positions essentielles sont celles de professeur et d'élève), de même que l'établissement (où d'autres positions apparaissent : celles de CPE, d'infirmière conseillère de santé, etc.), de même encore que cette institution qui englobe classes et établissements et qui foisonne en positions de toutes sortes, le système éducatif. (Chevallard, 2003, p. 2)

Une même institution peut abriter plusieurs sous-institutions elles-mêmes garantes d'un fonctionnement permis et/ou imposé aux personnes qui y occupent une position *p*.

En TAD, l'étude d'une question est modélisée par le *schéma herbartien* : $[S(X\,;Y\,;Q) \rightarrow M] \hookrightarrow R^{\heartsuit}$.

Celui-ci fait l'énoncé du « bilan du travail du système didactique » (Chevallard, 2011, p. 21). Dans sa forme développée, reproduite ci-dessous, le schéma herbartien intègre les composants du milieu *M* de l'étude de la question *Q*.

$[S(X\,;Y\,;Q) \rightarrow M = \{R_1, R_2, \ldots, R_m, O_{m+1}, O_{m+2}, \ldots, O_n, Q_{n+1}, Q_{n+2}, \ldots, Q_p, D_{p+1}, D_{p+2}, \ldots, D_q\}] \hookrightarrow R^{\heartsuit}$ (Chevallard, 2019)

Dans ce formalisme, $S(X\,;Y\,;Q)$ désigne le système didactique formé autour de l'étude de la question *Q*, *X* représentant l'instance des étudiants, *Y* l'instance des aides à l'étude. Ce système se donne un milieu didactique *M* en vue de fabriquer une réponse $R^{\heartsuit}$ à la question *Q*. Celui-ci est composé de plusieurs réponses $R^{\lozenge}$ déjà existantes, dans la culture ou la littérature ; de plusieurs œuvres *O* de différentes sortes, comme des théories, des expériences, des récits, etc. permettant le recueil des $R^{\lozenge}$ ou leur analyse pour la fabrication de $R^{\heartsuit}$; de données *D* ; de questions Q_i produites par l'étude de la question initiale *Q* et qui peuvent être relatives aux réponses $R^{\lozenge}$, aux œuvres *O* ou aux données *D*.

Lorsque les étudiants du système didactique sont des chercheurs, on parlera de système de recherche et on le notera $S(\Xi, Z, Q)$, où Ξ désigne les chercheurs et *Z* les aides à la recherche (Chevallard, 2013). Le système de recherche s'organise alors autour d'un objet de travail commun qui peut être coconstruit à partir duquel décident de travailler les acteurs de la nouvelle institution : il s'agit de la question *Q* dans la modélisation ci-dessus. Cet objet commun de travail peut être choisi ou imposé, mais en tous les cas, il pose problème. Il va susciter l'action et la collaboration des acteurs afin d'essayer de produire une réponse $R^♥$ à *Q*. L'étude d'une question par un système didactique ou un système de recherche modifie les rapports de la position d'étudiant comme de celle d'aide à l'étude à un certain nombre d'objets.

La question du rapport à un objet a été développée par Y. Chevallard (1989 ; 2003 ; 2019) pour modéliser la relativité institutionnelle de la connaissance. Nous ne nous intéresserons pas ici aux personnes, mais aux positions institutionnelles, c'est-à-dire aux positions *p* que des personnes *x* occupent dans des institutions *I*.

Faire évoluer le rapport de la position d'élève à un objet pourrait supposer de faire évoluer le rapport de la position de professeur à un certain nombre d'objets : il est nécessaire d'identifier ces objets ainsi que de déterminer le rapport à ces objets qu'il est utile, voire indispensable, de faire exister dans la position de professeur. Cette détermination prend appui sur une observation de l'activité institutionnelle selon différentes modalités (questionnaires, traces écrites de l'activité des instances, comptes rendus d'observation de classes, etc.) et sur une analyse didactique de cette activité à l'aide de la notion de praxéologie.

En TAD, en effet, l'activité humaine est modélisée par la notion de praxéologie (Chevallard, 1998, 2007). Lorsque l'on analyse l'activité d'un système didactique ou d'un système de recherche, l'enjeu de l'étude, soit la réponse $R^♥$ produite par cette activité, est une praxéologie ou, parfois, une partie d'une praxéologie. Il faut la distinguer des praxéologies qui permettent sa production, que ces praxéologies fassent partie du milieu ou soient parties prenantes de la réalisation du processus d'étude de la question *Q* (Artaud, 2019 ; Artaud & Cirade, 2021). Ce point est particulièrement crucial lorsque le processus d'étude, qui s'analyse en termes de praxéologies didactiques (d'étude ou de directions d'étude), produit une praxéologie de direction d'étude, qui est donc à nouveau une praxéologie didactique. Cela peut notamment être le cas dans la formation des professeurs. Mais il est également délicat lorsque l'enjeu de l'étude n'est pas didactique. En effet, le milieu comprend généralement des praxéologies de même nature que l'enjeu de l'étude mais qui font partie du processus d'étude, parce qu'elles en permettent la réalisation sans s'inscrire pour autant dans la réponse $R^♥$.

3. Analyse

3.1. Le système de recherche et son fonctionnement

La recherche collaborative considérée se présente comme étant abritée par une institution, celle que l'on nommera « Projets SFERE-Provence/DAFIP », qui définit certaines conditions et contraintes. On peut citer par exemple : la demande émane d'établissements, et pas de la position de professeur, même si celle-ci peut la soutenir ; la durée est de deux ans ; les chercheurs doivent « accompagner », et « accompagner n'est pas former ».

Au sein de cette institution « Projets SFERE-Provence/DAFIP » se crée un système de recherche $S(\Xi, Z, Q)$ que l'on peut considérer comme une sous-institution de l'institution mandante « Projets SFERE-Provence/DAFIP ». Cette institution mandante peut elle-même être considérée comme une sous-institution de l'institution « recherche collaborative ». En ce qui nous concerne, la question Q autour de laquelle s'est constitué le système de recherche est : Comment améliorer les résultats des élèves de cycle 3 en RPM ?

L'ensemble des aides à l'étude est a priori vide, et les chercheurs de ce système de recherche, les ξ appartenant à Ξ, sont issus de deux autres institutions : l'institution de recherche en sciences de l'éducation et en didactique des mathématiques, par le biais du laboratoire ADEF (Apprentissage Didactique Évaluation Formation) ; l'institution d'enseignement des mathématiques, à travers trois de ses établissements : deux écoles primaires et un collège. On peut éventuellement scinder, pour les besoins de l'analyse, la position de chercheur, p_ξ de ce système de recherche en deux sous-positions : l'une de chercheur-accompagnateur et l'autre de chercheur-enseignant. La création de cette institution pour mener une enquête sur la RPM suppose des objets communs, et notamment l'objet « RPM au cycle 3 »[2]. Ce sont alors de nouveaux rapports à ces objets qu'il s'agit de créer à partir des rapports des individus occupant les deux positions institutionnelles externes au système de recherche, chercheur en didactique des mathématiques et en sciences de l'éducation d'une part, enseignant du cycle 3 d'autre part.

Le fonctionnement du système de recherche va ainsi créer un rapport de l'instance p_ξ à l'objet « RPM au cycle 3 », rapport que l'on pourra analyser en

[2] Le cycle 3 comprend les eux dernières années de l'école primaire et la première année du collège (élèves de 9 à 12 ans).

termes de praxéologies. C’est ainsi l’assujettissement à cette institution $S(\Xi, Z, Q)$ qui fera évoluer les rapports de chaque ξ à cet objet, et qui permettra donc notamment qu’évolue le rapport de la position de professeur à ce même objet pour les établissements concernés. Le rapport de la position de chercheur-accompagnateur a également été modifié mais nous n’examinerons pas ce point ici. Le processus de création de ces nouveaux rapports s’inscrit dans une certaine spatialité (celle de l’établissement dans notre cas), dans une temporalité longue (deux ans) et dans une dynamique de continuité école-collège. Tout un lot de conditions et de contraintes pèse sur la constitution et l’évolution de ces rapports. Parmi ces conditions et ces contraintes, on peut citer par exemple le rapport à la recherche de la position de professeur, qui peut gêner ou au contraire favoriser la collaboration. Dans notre système de recherche, les chercheurs-accompagnateurs ne se sont pas mis en position d’aide à la recherche pendant une période. Les chercheurs-professeurs se voyant comme expérimentateurs mais non comme coproducteurs, cela a gêné l’avancée du temps de la recherche jusqu’à ce que le rapport des chercheurs-professeurs change et qu’ils s’autorisent à produire des ingrédients au sein du système de recherche.

Pour produire une réponse à la question Q et, dans ce même processus d’étude, créer un rapport à la RPM propre au système de recherche, celui-ci va convoquer un milieu M, assez riche. Sans prétendre être exhaustifs, nous présentons ci-dessous des éléments ayant été intégrés dans le milieu en nous référant au schéma herbartien développé : $[S(\Xi, Z, Q) \rightarrow M = \{R_1, R_2, \ldots, R_m, O_{m+1}, O_{m+2}, \ldots, O_n, Q_{n+1}, Q_{n+2}, \ldots, Q_p, D_{p+1}, D_{p+2}, \ldots, D_q\}] \hookrightarrow R^{\heartsuit}$.

Questions Q_i : Quel est le rapport à la RPM existant dans les établissements concernés en position de professeur et en position d'élève ? Quel est celui que l’institution voudrait voir exister ? Quel est celui qui serait nécessaire pour que les résultats des élèves des établissements concernés s’améliorent ? Comment modifier le rapport des élèves aux problèmes de proportionnalité ?

Éléments de réponses $R^{\Diamond}$: La RPM n’est pas un domaine des mathématiques pour l’institution mais on apprend des mathématiques pour résoudre des problèmes et à travers la RPM. Les problèmes proposés aux évaluations nationales relèvent de types de problèmes « classiques » (proportionnalité sur représentée, traitement et gestion de données, détermination de l’heure de la fin d’un événement, principalement). Les difficultés des élèves sont davantage liées aux techniques mathématiques pour traiter les types de problèmes proposés et à leur justification qu'à des aspects très transversaux. Il y a un déficit de mise en forme écrite des techniques, et de leur amalgamation.

Données : Réponses à deux questionnaires élaborés pour faire un diagnostic/état des lieux approfondi des préoccupations des acteurs, des pratiques en cours et des difficultés ; mesures des évolutions éventuelles. Traces de l'activité des enseignants et des élèves, de l'institution ; textes officiels (programmes, documents-ressources notamment), résultats des évaluations nationales, observation de la passation 2019, comptes-rendus de séances de travail et d'observation de classe, enregistrements, travaux des élèves sur l'évaluation nationale 2020 et sur trois problèmes proposés en amont, problèmes proposés par les enseignants sur la proportionnalité et leur exploitation, etc.

Œuvres : Conditions et contraintes influant sur les rapports en position d'élève ; la notion d'explicitation ; les notions de praxéologie mathématique et didactique, ainsi que leur mobilisation pour analyser et développer ; la notion de grandeur et une organisation mathématique pertinente relative à cette notion ; une praxéologie mathématique autour de la proportionnalité.

Une première partie du processus d'étude de la question Q a abouti à découper une question secondaire Q_p « Comment améliorer les résultats des élèves relatifs à la résolution de problèmes de proportionnalité ? » C'est l'étude de cette question que nous examinerons plus particulièrement dans ce qui suit.

3.2. Les éléments de réponse apportés en termes d'évolution des rapports de la position de professeur ($R^{♥}$)

Le travail mené dans la première partie du projet, notamment à partir de l'analyse des évaluations nationales de mathématiques à l'entrée en sixième (rentrées 2019 et 2020) et d'une évaluation complémentaire coconstruite, a mené le collectif de chercheurs Ξ au constat que la proportionnalité était source de difficultés. D'un côté, la proportionnalité porte sur les grandeurs et cette notion de grandeur doit être apparente dans les praxéologies mises en place ce qui n'est pas le cas dans le rapport à la proportionnalité prévalant en cycle 3 ; d'un autre côté, il y a un problème de juxtaposition de techniques qui gêne la RPM de proportionnalité : le tableau de proportionnalité, la technique du retour à l'unité et celle du produit en croix. La décision a ainsi été prise de préparer une praxéologie mathématique qui articule les techniques en une seule en amalgamant les différentes techniques selon leurs occasions d'emploi (Artaud, 2010 ; 2019) et qui fasse apparaître et manipuler plus nettement la notion de grandeur (Artaud, 2021). Puis, une fois cette praxéologie mathématique mise au point, le système de recherche a travaillé sur une praxéologie didactique, et notamment sur l'institutionnalisation en classe de la praxéologie mathématique produite. Ce sont donc les rapports à plusieurs objets qui vont être sollicités et modifiés : on peut notamment citer les objets proportionnalité, grandeur, institutionnalisation. Nous examinerons ci-dessous ce que la réponse produite par le système de recherche a permis de faire évoluer dans le rapport en position de professeur des établissements concernés par le projet.

La restitution du projet organisée au printemps-été 2021 a permis d'accéder aux déclarations des enseignants portant sur le changement de rapport à la proportionnalité et à l'institutionnalisation, mais aussi à des productions soutenant ces déclarations. Le nouveau rapport à la proportionnalité intègre désormais l'emploi des grandeurs par l'intermédiaire de l'inscription systématique des unités et le fait que le coefficient de proportionnalité est une grandeur quotient. Cela a amené à inscrire le travail avec le coefficient de proportionnalité principalement avec des grandeurs de même espèce. Ce travail a été complété par l'étude d'un petit nombre de situations issues de la vie courante qui ont du sens pour les élèves ; on écrit alors le coefficient de proportionnalité « en extension » : par exemple, « 3 euros par kilo » pour la grandeur quotient prix/masse. Plusieurs éléments systématiquement présents dans l'ancien rapport à la proportionnalité, qui occasionnaient souvent la manipulation de nombres au détriment des grandeurs, aboutissant à des égalités fautives, retrouvent une juste place : le tableau de proportionnalité, la technique du retour à l'unité et celle du produit en croix. La modification la plus significative concerne le tableau de proportionnalité : utilisé maintenant avec parcimonie, principalement avec une fonction de synthèse, il contient les grandeurs, et non leurs seules mesures.

Le nouveau rapport à l'institutionnalisation intègre quant à lui une mise en forme discursive des techniques et, solidairement, des discours justificatifs des techniques, avec un passage systématique à l'écrit. Le texte est davantage présent et les différents ingrédients de technique sont amalgamés selon leurs occasions d'emploi. Les notions de modélisation et de mise en forme mathématique sont également présentes dans le nouveau rapport. Elles viennent remplacer de façon plus fonctionnelle des éléments qui étaient présents plus structurellement dans le rapport initial (comme le fait de « rédiger une phrase réponse » ou de « faire une hypothèse »).

9

Cette modification du rapport à la résolution de problèmes de proportionnalité dans la position de professeur des établissements concernés va au-delà de ce thème mathématique. Les enseignants verbalisent le fait que ce sont leurs praxéologies professorales qui ont évolué, soit les savoir-faire mais aussi les justifications technologico-théoriques des *praxis*. Ils ont en effet un discours dans lequel ils ne prennent plus la RPM comme un seul bloc. Ils intègrent l'idée que le thème mathématique – comme celui de la proportionnalité – est un axe pertinent de travail ; ils considèrent que l'écriture et la mise au point préalable par le professeur de la praxéologie mathématique relative au thème est un aspect important du travail du professeur, de nature à améliorer l'aide qu'il peut apporter aux élèves pour qu'ils réussissent en RPM.

4. Conclusion et perspectives

Chaque recherche collaborative est une enquête sur une question *Q* qui a avantage à être analysée comme sous-institution d'une institution mandante, qu'elle soit institutionnellement repérée comme ici avec les projets SFERE-Provence/DAFIP ou qu'elle relève plus largement du « collège invisible » qu'est la recherche collaborative. Le fait de voir ce système comme élément d'une institution permet d'analyser des rapports en position de chercheur qui se créent, se modifient, mais aussi les conditions de création de ces systèmes de recherche. On pourrait par exemple examiner en quoi le fait qu'un système de recherche relève clairement d'une institution mandante ou d'un collège invisible favorise, permet ou au contraire gêne, empêche la création de rapports institutionnels à certains objets. Par exemple, ici, on peut penser que les systèmes de recherche créés dans l'institution « Projets SFERE-Provence/DAFIP » ont peu de chances de faire évoluer les rapports en position de professeur de l'institution scolaire compte tenu du petit nombre de ces projets et de leur labilité - l'institution mandante va en effet cesser d'exister en 2023, remplacée par une institution d'une autre nature « Ampiric ».

C'est l'étude sur un thème mathématique précis qui a permis que se dégagent des éléments de réponses à la question étudiée par le système de recherche. Mais c'est le travail antérieur qui a permis de créer les conditions pour que se dégage ce thème ou encore pour que les rapports des enseignants impliqués dans le projet évoluent suffisamment pour s'engager dans l'étude de la RPM sur un thème précis. Naturellement, le contexte spécifique de l'établissement, la fracture entre temps court de l'année scolaire (enseignants aux prises par exemple avec les questions d'évaluations nationales et des compétences) et temps long de la recherche impactent le déroulement du projet, comme l'injonction du travail collectif à distance en lien avec la gestion sanitaire de la crise du Covid-19.

Cette modélisation du processus de collaboration permet ainsi d'identifier les ingrédients du processus, de caractériser les rôles/responsabilités/interactions des acteurs et d'inférer des éléments de compréhension qui donnent du sens au travail effectué. La formalisation *via* le modèle de recherche proposé mériterait toutefois d'être éprouvée dans d'autres contextes de recherches collaboratives pour étendre le domaine de validité du modèle produit. Un des enjeux serait de parvenir à identifier des conditions et des contraintes écologiques qui caractérisent chaque contexte empirique de recherche collaborative pour envisager la mesure dans laquelle la constitution d'un système de recherche est productive, permet de faire évoluer les rapports à quels types d'objets, etc. La modélisation proposée a également vocation à évoluer. Elle pourrait être enrichie de façon à mieux comprendre comment les actions individuelles s'agglomèrent en activités collectives, comment se développent les relations/interactions entre chercheurs et acteurs, comment la dynamique de collaboration se maintient ou encore comment se règlent les désaccords et les différents qui peuvent survenir dans la collaboration.

Références

Artaud, M. (2010). Conditions de diffusion de la TAD dans le continent didactique. Les techniques d'analyse de praxéologies comme pierre de touche. Dans A. Bronner, M. Larguier, M. Artaud, M. Bosch, Y. Chevallard, G. Cirade & C. Ladage (Éds), *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d'action* (pp. 233-253). Montpellier : IUFM.

Artaud, M. (2019). Praxeologies to be taught and praxeologies for teaching: a delicate frontier. In Marianna Bosch, Yves Chevallard, Francisco Javier García, John Monaghan (Eds), *Working with the Anthropological Theory of the Didactic in Mathematics Education* (pp. 33-40). Londres, Royaume Uni : Routledge.

Artaud, M. (2021). Des grandeurs et de leur mesure : Besoins praxéologiques de la position de professeur et leur satisfaction. In H. Chaachoua, A. Bessot et al. (Éds), *Nouvelles perspectives en didactique : point de vue de l'élève,* *questions curriculaires, grandeurs et mesure Vol. 1* (pp. 197-226). Grenoble, France : La pensée sauvage.

Artaud, M. & Cirade, G. (2021). La TAD comme milieu pour l'étude de l'activité des institutions didactiques. *Caminhos da educação matemática em revista (online)*, *11*(1), 388-411. https://aplicacoes.ifs.edu.br/periodicos/index.php/caminhos_da_educacao_matematica/article/view/762/638

Chevallard, Y. (1989). Le concept de rapport au savoir : Rapport personnel, rapport institutionnel, rapport officiel. In *Actes du séminaire de didactique des mathématiques et de l'informatique* (pp. 211-235). Grenoble, France : DidaTech, UJF.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique : Perspectives apportées par une approche anthropologique. *Recherches en didactique des mathématiques*, *12*(1), 73-112.

Chevallard, Y. (1998). Analyse des pratiques enseignantes et didactique des mathématiques : L'approche anthropologique. In *Actes de l'université d'été de La Rochelle, 4-11 juillet 1998* (pp. 91-120). Clermont-Ferrand, France : IREM.

Chevallard, Y. (2003). Approche anthropologique du rapport au savoir et didactique des mathématiques. In S. Maury & M. Caillot (Éds.), *Rapport au savoir et didactiques* (pp. 81-104). Paris, France : Fabert. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/Approche_anthropologique_rapport_au_savoir.pdf

Chevallard, Y. (2007). Passé et présent de la théorie anthropologique du didactique. In L. Ruiz-Higueras, A. Estepa & F. J. García (Eds), *Sociedad, Escuela y matemáticas. Aportaciones de la teoría antropológica de lo didáctico (TAD)* (pp. 705-746). Jaén, Espagne : Publicaciones de la Universidad de Jaén.

Chevallard, Y. (2011). *Didactique fondamentale*. Module 1 : Leçons de didactique. Aix-Marseille Université. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/DFM_2011-2012_Module_1_LD_.pdf

Chevallard, Y. (2013). *Journal du séminaire TAD/IDD. Année 2012-2013*. http://yves.chevallard.free.fr/spip/spip/IMG/pdf/journal-tad-idd-2012-2013-3.pdf

Chevallard, Y. (2019). On using the ATD: Some clarifications and comments. *Educação Matemática Pesquisa*, *21(*4), 1-17. https://doi.org/10.23925/1983-3156.2019v21i4p001-017

Desgagné, S. (1997). Le concept de recherche collaborative : L'idée d'un rapprochement entre chercheurs universitaires et praticiens enseignants. *Revue des sciences de l'éducation, 23*(2), 371-393.

Morrissette J. (2013). Recherche-action et recherche collaborative : Quel rapport aux savoirs et à la production de savoirs ? *Nouvelles pratiques sociales*, *2*(25), 35-49. https://www.erudit.org/fr/revues/nps/2013-v25-n2-nps01030/1020820ar.pdf

Mathematical knowledge for the profession of upper secondary mathematics teaching

Carl Winsløw

University of Copenhagen, Faculty of Science, Denmark; winslow@ind.ku.dk

In this paper, we first outline existing answers to the question about the mathematical knowledge that is required for the profession of teaching mathematics at the upper secondary level: what it is, and how it can be developed and especially assessed. We then present the idea of study and research lines as a model that could be used for the development and assessment purposes to conclude a classical teacher education programme in which most mathematics is learned in standard undergraduate courses..

Keywords: Study and research lines; Klein's second discontinuity.

1. Some teacher education problems.

Let us consider the following broad question:

(1) *What mathematical knowledge should future teachers have, and what is required to enable and ensure that they have it?*

The problem is one of considerable age, involving societal, institutional and academic interest. Concerning age, it has progressively reached its current form and models along with the introduction of universal school education in most Western countries during the 19th century. For society, the question of preparing teachers forms part of the regulation of school systems, which is in general highly political. It involves, for instance, norms about the mathematics that "all citizens" must know or be able to do, and norms related to perceived mathematical needs in the labour market and in higher education. The discussion of how to prepare teachers is naturally dependent on such norms, expressed as goals and requirements set for school institutions. When it comes to institutions, both those in which the teachers exercise, and those entrusted with preparing teachers, have evident interests in the problem; even within institutions, interests and positions may differ and be in conflict. Finally, the problem (in more delimited forms) has led to both theoretical and empirical research, which is of considerable intellectual interest and depth.

An anthropological approach to these questions must produce studies and analyses of actual teacher preparation practices, while explicitly addressing the institutional and societal contexts as variable factors that codetermine the practices and their outcome. This, in our time and age, clearly calls for a strongly international perspective. Researchers working within the ATD (Anthropological Theory of the Didactic) paradigm cannot consider specific institutional, national or even Western contexts as a kind of nature that can be assumed to be known and present everywhere. Even in studies reporting on just one context (like the present study), an explication of how it resembles and differs from similar contexts is needed in order for the study to be of substantial interest to scholars outside of that context. Teacher education institutions, and regulation of their practice, is no exception. In fact, as contexts they are inseparable from the school institutions they serve, so that analysing answers to (1) involves taking into account a large complex of institutional conditions.

In this paper, we will mainly consider a special case of (1), namely:

(2) *What kinds of questions can be used to characterize, develop and test relevant mathematical knowledge among future upper secondary school teachers?*

More specifically, we will discuss how one may single out, and assess, mathematical knowledge deemed to be relevant for future teachers in upper secondary schools (with further characteristics, naturally). Our own data come from a somewhat classical university context (to be further specified), with many variables apparently fixed. We will examine these data critically in the light of studies carried out in other national contexts as well as international, quantitative studies of mathematics teacher knowledge. We first outline results from such other studies (section 2), then present our institutional and didactical context (section 3), some concrete proposals and results on (2) (section 4-5), and finally some overall conclusions related to the international field (section 6).

2. A Herbartian view on (1), (2): what do we know, if anything?

There are certainly many strong viewpoints and opinions when it comes to (1), or just its special cases related to mathematics, such as (2). Providing even a summary of some of the most common ones would be the topic of a book series rather than of a conference paper; what we will do here is to present a selection of answers that we deem relevant to the questions. In doing so, we shall assume and open-minded and critical “Herbartian” perspective, which involves indeed a “receptive attitude towards yet unanswered questions and unsolved problems” (Chevallard, 2015, p. 178).

Even mapping out the present contents and structure of teacher education in general, with its variations across institutions and societies, seems to be a daunting task. The Organisation for Economic Co-operation and Development (OECD) regularly issues reports providing partial overviews; for instance, concerning content focus in teacher education within the so-called TALIS (Teaching and learning international survey) countries, we learn that

> For pre-primary school teachers, academic subjects are mandatory in 20 of the 33 countries with available data for 2013; however, as expected, mandatory academic subjects are more common for prospective teachers of general subjects at the upper secondary level (in 28 of 34 countries). In addition, courses in academic subjects are specific to prospective teachers at the pre-primary level in around two-thirds of countries and in around three-quarters of countries at the primary level; but only in one-third of countries at the upper secondary level. In around two out of three countries, there are common courses for all prospective teachers, regardless of the level of education they will teach. This may make it easier for teachers to move among the different levels of education. (OECD, 2014)

In particular, we note that upper secondary school teacher education (virtually everywhere) involves mandatory academic courses in mathematics, but only in 1/3 of the participating countries are at least some of them specifically directed towards the teaching profession. This means that the postsecondary mathematical knowledge of the teacher students is mostly acquired in “general” academic mathematics courses, such as the undergraduate sequences in domains like analysis and algebra. But in some countries, including the United States, Germany and France, students will also encounter mathematics courses specifically designed for future teachers, as well as subject specific and general pedagogy. These courses appear often, but not always, towards the end of the students' university degree.

These and similar facts do not answer (1), but merely inform us of the current and general state of affairs. However, there is a long tradition in American social science to conduct quantitative, correlative studies on (1). Early research (e.g. Eisenberg, 1977) suggested that teachers' knowledge of academic mathematics, like abstract algebra, does not correlate significantly with their students' learning (measured as gain in knowledge over about one year). Eisenberg formulates some reservations though, including that there could be a minimal level of academic mathematics background, largely held by all participants in the study, which does actually correlate with students' learning. Monk (1994) studied correlations between the number of academic mathematics courses taken by secondary level mathematics teachers, and their students' performance gains, and in fact did find positive correlations with having taken up to about 5 courses. This suggests – with multiple caveats - that a minimal undergraduate mathematics background of up to a year of full time academic mathematics study does have a positive effect on the teachers' efficiency, but that anything beyond that has little or no effect. Subsequent international studies – in particular the Teacher Education and Development Study in Mathematics (TEDS-M) – seem to confirm, with much more nuance, that content knowledge up to some point is one significant factor in teachers' efficiency, but also that much stronger correlations are found when measuring various forms of "mathematical knowledge for teaching" (see e.g. Hill, Rowan & Ball, 2005; Mesa & Leckrone, 2020).

The recent TEDS-M project investigated mathematics teacher education and teacher knowledge using this theoretical framework. It is international and involved 17 countries, including Spain, Germany, Norway and Switzerland. The study investigates, in particular, how successful teacher education programmes in these countries are, in terms of their graduates' mathematical knowledge for teaching. Based on data from the best performing programmes, researchers even derived policy principles for "world class standards" for mathematics teacher education (Schmidt, Burroughs & Cogan, 2013). Among their conclusions (p. 12) is that

> analysis of the course taking patterns of the top performing teacher preparation programs across 17 countries revealed a reasonable consensus about what courses are appropriate for preparing future teachers. For example, more rigorous mathematics content, such as university level mathematics at the middle school level, characterized the best programs.

The TEDS-M project does not include upper secondary school teacher education, but it is plausible that most if not all of courses identified as "consensual" in top lower secondary teacher education programmes, would also be found in the education of teachers for this higher level (if any differences exist at all). They identified six course topics in "university mathematics", three topics in "mathematics education", and one topic from "school mathematics" (namely, "functions and equations").

Despite the evident methodological criticisms that could bear on this and almost any empirical research, it will not do for a Herbartian researcher to simply dismiss large-scale studies like those cited above as irrelevant or coming from another intellectual planet. First, there is a long way from ignoring a study to questioning it. Secondly, studies of the type introduced are anthropological facts themselves, part of the factors that contribute to determine policies – for schools, for teacher education, and for research. They lead to schemes for defining and measuring various elements of teacher knowledge. They deserve study and questioning.

ATD researchers have mostly taken a more experimental approach to the problem (1). The vision of a "paradigm of questioning the world" suggests that teachers should above all be leaders of inquiry, having themselves a substantial experience as inquirers. Certainly, it would be a gross mistake to discard answers as entirely secondary to questions, or knowledge as accidental products of diligent googling. ATD maintains that there exists a vital *dialectics* between questions and answers, or (in mathematics) between problems and syntheses (Bosch & Winsløw, 2017). This would also be at the heart of our inquiry into the problems (1) and (2): identifying and examining answers that exist, whether produced within our beyond ATD; and deriving new questions from them.

Within ATD, several researchers have recently experimented so-called study and research paths for teacher education (SRP-TE, see e.g. Otero et al., 2017; Barquero et al., 2018). These first experiences with SRP-TE indicate that they could provide a valuable experience for future or current teachers, especially when it comes to discover conditions and constraints for teaching in their own institutions, and to explore challenging questions that lead far beyond the boundaries of what school mathematics is in their current context. For initial teacher education, Otero et al. (2017) also identified constraints, such as the students' lack of familiarity with mathematical modelling and inquiry. They ascribe this to

> the fact that although they have experienced four years of "hard" university studies, the utility of the science they aim at teaching had never been visible. The epistemological conception about the mathematics produced by the traditional paradigm is so ingrained, that it is complex to reverse it. (…) It is unlikely that a teacher whose training has been answers-based teaching can teach by means of questions. Therefore, our final message is that the training of teachers must change profoundly (Otero et al., p. 236).

The challenge is well known: the gap between synthetic university mathematics education, and emerging requirements for teachers to present mathematics as problem solving. This can be seen as a modern version what Klein called the *second discontinuity* (Winsløw & Grønbæk, 2014), which could arise already at universities, to the extent these (as in the case studied by the authors) introduce problem driven activities related to, and transcending, school mathematics. It is not clear what role study and research paths could occupy in mathematics teacher education as such. The requirements they present to students seem indeed far from requirements found in the general university mathematics courses which, as we have seen, occupy a good part of the time of most students who prepare for teaching mathematics at upper secondary school. While waiting, preparing or researching for profound change, and also to smoothen the discontinuities that would remain, we consider that less radical proposals for answering (2) are also called for.

3. A context to address (2): a capstone course in Copenhagen

Based on the international surveys of mathematics teacher education for the secondary level, the situation in Denmark seems to be relatively usual. As in some other countries, it is common to qualify for teaching in two subjects: a major subject, and a minor. Upper secondary teachers need to complete a master degree lasting at least 5 years, followed by practical and theoretical training in pedagogy (about ½ year). The five years are split between the two subjects roughly in the ratio 3:2. Thus, a minor subject in mathematics will involve course credits corresponding to about 2 years. Of these, about 1½ years are spent on mandatory mathematics courses (roughly: calculus and analysis up to Fourier Analysis and Hilbert space; linear and abstract algebra up to rings and fields; discrete mathematics; probability and statistics; differential geometry), while the remaining credits include mandatory courses on Didactics and History of Mathematics, as well as two optional mathematics courses.

One of these optional courses is “Mathematics in a teaching context” (UvMat). It is, however, taken by almost all minor students and even some major students. The course focuses on “elementary mathematics from an advanced standpoint”, following Klein’s (2016) ideas, as outlined by Winsløw and Grønbæk (2014). It should suffice to note here the themes currently covered (based on undergraduate mathematics methods): recursion and induction proofs, polynomials, the real number system, exponential and logarithmic functions, functions and sequences as theoretical and empirical models, probability theory up to normal and binomial distributions. The last theme was not included in previous versions of the course but was introduced from 2018, following a major reform of Danish high school, in which probability theory was given more emphasis. This new theme replaced “calculus” (which, based on the earlier syllabus, was focused on by Winsløw et al., 2014).

Formally, UvMat is organised as many other mathematics courses at this university: with lectures, exercise sessions, and a written exam to conclude. It differs from the standard courses by the contents, which is largely known to students in advance; what is new is in the ways the content is to be mastered, especially in terms of solving “challenging problems on elementary contents” based on the experience and knowledge the students have from undergraduate courses. This is both emphasised in the course and at the exam. We believe that some of the principles and examples of the task design that has been carried out in the course is relevant to provide what we called above “less radical proposals for answering (2)”. We outline these in the next section.

4. Study and Research Lines

Study and Research Paths (SRP) can be described as non-linear sequences of questions and answers that branch out from an initial question with high generating power; these are often represented by QA-graphs (Matheron, Mercier & Winsløw, 2013). Many if not most of the examples that appear in experimental research with SRP involve both questions and answers that go substantially beyond the usual domains of school mathematics. This is also consistent with the origin of SRP, as these were first proposed to give sense to a new work format in French high school, aimed at having students work autonomously and beyond the borders of single disciplines, based on challenging questions (Chevallard & Matheron, 2002; Chevallard, 2015). This is also true for examples provided in the emerging literature on SRP within teacher education (SRP-TE, exemplified above).

The arborescence of an SRP comes essentially from the idea that students could and should themselves develop the initial question, and could go in multiple directions not only in terms of methods for answering, but also in the derivation of new questions. This is usually problematic in contexts where specific “answers” are to be taught. While a SRP can certainly start from a purely mathematical question and never go beyond mathematical questions and answers, one could suspect that it would be difficult for many students to go directly from undergraduate mathematics courses (all focused on answers) to autonomous study and research of an open mathematical question. Another challenge with that idea in the case of a course like UvMat is the need to declare and assess students' learning with respect to a relatively broad range of secondary school contents.

For these and other reasons, we have developed two forms of work for the course, namely *short and long sequences of questions*, which satisfy all or most of the following *criteria for relevance to future teachers*:

- The question involves mathematical objects that are central to (Danish) upper secondary school mathematics
- The question requires use of undergraduate mathematics methods (or at least more mathematical maturity than high school students normally have)
- The question asks for explaining, arguing or justifying in mathematical ways that could be relevant for an upper secondary teacher
- The questions are not "standard type of task" but requires some problem solving, like combining knowledge from different areas
- If possible, the students formulate results by themselves – so we try to replace "Prove that…" by "Is it the case that… justify your answer".

In the course, both short and long sequences of questions appear – the short ones (2-3 questions) for exercise sessions and the long ones (6 questions) for mandatory weekly assignments. We call them *short* and *long study and research lines* (SRL) since the questions are all given, but are connected to form what could, possibly, form a small or large branch of a study and research path. For the exam, five short study and research lines are proposed, and the students get individual grades based on their answers.

One case of a long SRL is presented and analysed by Huo (in preparation). Here are two examples of short SRL:

State, with justification, whether these equations have rational solutions:

a) $x^2 - \sqrt{3} = 0$

b) $2^x = x^3$ (Exam, June 2021, exercise 2)

a) Find the decimal representations of $\frac{14}{99}$, $\frac{784}{999}$ and $\frac{5716}{9999}$. Formulate a general hypothesis from what you see.

b) Prove your hypothesis from a).

c) Provide an informal justification for your hypothesis from a), which addresses one of the three given examples. (Exam, June 2021, exercise 3)

A computer algebra system (Maple) is used throughout the course, and can be used as a milieu even when this is not explicitly required. At the exam, students may use Maple and bring along any books, notes and files they wish (but internet use is not possible, to avoid communication). Students can thus engage in study and research, in the sense of ATD, while attempting to develop answers. Still, a major difference with SRP-TE (and the study and research activities that it may include) is that the mathematical questions to be developed are given directly, and are constructed with (the didactical) question (2) in mind: to test and develop mathematical knowledge deemed relevant to future high school teachers. Two other main differences are equally important to note: unlike most SRP-TE, the proposed SRL require the use of undergraduate mathematics, and ulike almost all SRP-TE, the professional environment in which the SRL could be relevant is not explicitly given. This makes SRL potentially more viable in a context such as UvMat, which is indeed an undergraduate course on mathematics, given to students who may have no professional experience.

We shall include a more detailed analysis of the examples given above in the next section. But note that while the questions are certainly within the themes of the course, none of them is a standard task for the students, although the amount of challenge varies, as we shall now see. Indeed, the SRL differs from standard exercises in undergraduate courses in that they are concerned with high school level mathematics, but still require – or potentially develop – a more sophisticated relationship to it, or, what Klein (2016) calls a “higher standpoint” – that, indeed, of the teacher. For a more detailed discussion of this, we refer to Winsløw (2020).

5. Student answers.

Both of the exercises are related to a major theme in the course, namely the mathematical properties and representations of real numbers. In the course, several mathematical results are studied which can be used in this regards, such as:

The rational root theorem. Consider the polynomial $p(x) = a_n x^n + \cdots + a_1 x + a_0$ where the coefficients a_k are all integers and $a_n \neq 0$, $a_0 \neq 0$. If $\frac{M}{N}$ is an irreducible fraction of integers and $p\left(\frac{M}{N}\right) = 0$, then $M|a_0$ and $N|a_n$.

Decimal representations of real numbers. Every real number x can be written uniquely on the form

$$(*) \qquad x = \pm\left(N + \sum_{k=1}^{\infty} c_k 10^{-k}\right)$$

where: N is a natural number or 0, $c_k \in \{0,1,\dots,9\}$ for all $k \in \mathbb{N}$, and there is no $m \in \mathbb{N}$ such that $c_{k+m} = 9$ for all $k \in \mathbb{N}$. Moreover, x is rational if and only if (c_k) is periodical from some step.

The first of the two SRL given above include two questions that look similar but are in practice quite different. The first equation is easily solved (both manually and with a Computer Algebra System , abbreviated CAS), while the second one can only be solved by CAS and yields a (to all students) weird result, involving the Lambert W-function which none seem to have ever seen. In the end, some kind of indirect proof is needed in both cases (as neither has rational solutions). But seeing the familiar solutions ($\pm 3^{1/4}$) in a), and using familiar techniques from the course, especially the rational root theorem above, make more than 80% of the students succeed here; in fact, those who do not manage that generally fail or get the lowest passing grade.

In b), less than 20% produce a correct proof that no solution can be rational; those who succeed are all students who end up with top grades. Here it is necessary to work with an abstract (hypothetical) rational solution, and derive a contradiction. So, even if the essence of the latter argument is not very different (one can, for instance, use the rational root theorem), the lack of a familiar (real) solution seems to be decisive. Reasoning about solutions one cannot recognize or find, and whose existence would even take some justification, leads b) to have an altogether higher level of abstraction.

Similar observations can be made on the second SRL. Here, Maple immediately let the students see the decimals, like

$$\frac{784}{999} = 0.784784784784784478$$

and form the hypothesis that the period of the decimal is simply the nominator of the fraction. Using (*) with $x = \frac{784}{999}$ it is clear that $N = 0$ and that $784 = 999x = 1000x - x = c_1 10^2 + c_2 10^1 + c_3 + \sum_{k=1}^{\infty}(c_{k+4} - c_k)10^{-k}$. The result follows from the uniqueness of decimal representation (of 784).

Indeed, working from concrete numbers (in a) and c)) makes the majority succeed (86% and 73% respectively), while only 32% arrive at producing a correct proof of the general statement. The latter students all get middle or top grades overall. Note that we do not pose the questions b) and c) in what would seem to be the natural order: first a "heuristic" argument, then the more formal one based on series. What we presented above can be regarded as a "midway" approach since we work with a concrete example but do not use the more suggestive notation $x = 0.c_1c_2c_3c_4 \ldots$ The reason for this order is that students are expected to organise their inquiry into b) by themselves, and if they succeed, that would most likely would make c) an easy addition to b). On the other hand, the success rates show that many students can do a) and c), but not b). This, then, corresponds also to a relevant assessment of the status of their knowledge concerning decimal representations of real numbers.

More generally, abstraction and formal reasoning – which is here, as already stated, never standard – form crucial obstacles in these SRL. This can to some extent surprise, given that formal reasoning (with much more abstract objects) form an essential ingredient in the general undergraduate diet. In fact, trials with random undergraduate students who did not attend UvMat showed even less success with questions that required students' autonomous reasoning about non-explicit or general objects (Winsløw et al., 2014). Thus, it could be a main goal for SRL for future teachers to develop their autonomous capacity with formal reasoning about upper secondary level mathematical objects.

6. Conclusions.

Much of recent, large-scale research on (1) suggests that future teachers need substantial work – and success with – challenging questions that are closely related to the mathematics they will actually teach. Undergraduate courses may support this, at least up to about a year of such (Monk, 1994), but does not guarantee that such work succeeds or even takes place (Winsløw et al., 2015).

As for (2), SRL focusing on developing a more targeted experience with study on research on questions related to secondary level mathematical objects, seem to be a promising avenue to bridge the gap between what students can achieve in general mathematics courses and what will make them successful and creative mathematics teachers under current circumstances. This is in itself an important task, given most secondary teachers' preparation, and under the assumption that advanced mathematical knowledge can be made a resource, rather than an obstacle, to teachers' work with secondary mathematics.

It seems plausible that work with more ambitious, open SRP-TE would be essential to succeed in teaching under circumstances where the profession involves the management of students work with SRP. Even so, the assessment part of (2) – focusing on diagnosing specific mathematical outcomes, and in relation to common measures of "mathematical knowledge for teaching" (cf. section 2) – may still require use of instruments like SRL.

More importantly, it is an interesting hypothesis to explore if preliminary work with SRL type challenges, to develop teachers' autonomy and creativity with challenging (yet, school-relevant) types of mathematical questions, could indeed form a worthwhile preparation for engaging in the more complex and professional questions that are usually involved in SRP-TE. Delving into such questions might then be experienced as a continuation, rather than a "reversal" (to paraphrase Otero et al., as cited above), of the teachers' mathematical development.

References

Barquero, B., Bosch, M., & Romo, A. (2018). Mathematical modelling in teacher education: dealing with institutional constraints. *ZDM The International Journal on Mathematics Education, 50*(1-2), 31-43. https://doi.org/10.1007/s11858-017-0907-z

Bosch, M. & Winsløw, C. (2016). Linking problem solving and mathematical contents: the challenge of sustainable study and research processes. *Recherches en Didactique des Mathématiques* 35 (3), 357-401.

Chevallard, Y. & Matheron, Y. (2002). Travaux Personnels Encadrés : un cadre d'analyse didactique pour un changement majeur dans l'enseignement au Lycée. *Actes des Journées de la commission inter-IREM Didactique*, pp. 141-150. Dijon: IREM. Located January 25, 2022 at: http://yves.chevallard.free.fr/spip/spip/IMG/pdf/TPE_-_un_cadre_d_analyse_didactique.pdf

Chevallard, Y. (2015). Teaching Mathematics in tomorrow's society: A case for an oncoming counter paradigm. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education,* pp. 173–187). Switzerland: Springer. https://doi.org/10.1007/978-3-319-12688-3_13

Eisenberg, T. (1977). Begle Revisited: Teacher Knowledge and Student Achievement in Algebra. *Journal for Research in Mathematics Education* 8 (3), pp. 216-222. https://doi.org/10.5951/jresematheduc.8.3.0216

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement. *American Educational Research Journal*, 42(2), 371-406. https://doi.org/10.1037/e494732006-007

Huo, R. (in preparation). Drawing on a computer algorithm to advance future teachers' knowledge on real numbers.

Klein, F. (2016). *Elementary mathematics from a higher standpoint* (G. Schubring, Trans.). Berlin: Springer. (Original work published 1908) https://doi.org/10.1007/978-3-662-49445-5

Monk, D. H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review*, 13(2), 125–145. https://doi.org/10.1016/0272-7757(94)90003-5

OECD (2014) *Education at a Glance 2014: OECD Indicators.* OECD Publishing, Paris. https://doi.org/10.1787/eag-2014-en

Otero, M., Arlego, M, & Llanos, V. (2019). Development of research and study paths in the pre-service teacher education. *European Journal of Education Studies* 3(8) 214-240.

Mesa V., Leckrone L. (2020) Assessment of Mathematics Teacher Knowledge. In: Lerman S. (eds) *Encyclopedia of Mathematics Education*. Cham: Springer. https://doi.org/10.1007/978-3-030-15789-0_13

Schmidt, W., Burroughs, N. & Cogan, L. (2013). *World Class Standards for Preparing Teachers of Mathematics.* Working paper, Michigan State University. Located (January, 25, 2022) at https://education.msu.edu/epc/library/documents/WP37WorldClassStandardsforPreparingTeachersofMathematics.pdf

Winsløw, C. & Grønbæk, N. (2014). Klein's double discontinuity revisited: contemporary challenges for universities preparing teachers to teach calculus. *Recherches en Didactique des Mathématiques* 34 (1), 59-86.

Winsløw, C., Matheron, Y. Mercier, A. (2013) Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics* 83 (2), 267-284. https://doi.org/10.1007/s10649-012-9453-3

Winsløw, C. (2020). Professional and academic bases of university mathematics teaching for the 21st century: the anthropological approach to practice based research. In T. Hausberger, M. Bosch & F. Chellougui (Eds.), *Proceedings of the Third Conference of the International Network for Didactic Research in University Mathematics*, pp. 8-27. University of Carthage and INDRUM.

Statistics teacher education at secondary school level in the paradigm of questioning the world

Janielly Verbisck[1]

[1]Facultat d'Educació, Universitat de Barcelona, Spain and Universidade Federal de Mato Grosso do Sul, Brazil

Marilena Bittar[2], Marianna Bosch[3], Berta Barquero[3] and Ricardo Benito[4]

[2]Universidade Federal do Mato Grosso do Sul, Brasil

[3]Facultat d'Educació, Universitat de Barcelona, Spain

[4]Universidade Federal de Sergipe, Brasil

Abstract. This paper presents the design and implementation of a "study and research paths for teacher education" (SRP-TE) based on the Anthropological Theory of the Didactic. The SRP-TE approaches a school exercise about the distribution of water resources in Brazil and turns it into an open activity. The activities promoted by this inquiry foster different dimensions of statistical work and can be used to analyse curriculum and textbook proposals. When asked about the multidisciplinary knowledge works that intervene in the inquiry, preservice teachers disregard many dimensions of the statistical work, especially those related to data handling, which they do not seem to consider as part of the involved mathematical–or statistical– activity.

Resumen. Este trabajo presenta el diseño y la implementación de un "recorrido de estudio e investigación para la formación del profesorado" (REI-FP) basado en la Teoría Antropológica de lo Didáctico. El REI-FP aborda un ejercicio escolar sobre la distribución de los recursos hídricos en Brasil y lo convierte en una actividad abierta. Las actividades promovidas por esta indagación fomentan diferentes dimensiones del trabajo estadístico y pueden ser utilizadas para analizar propuestas curriculares y de libros de texto. Cuando se les pregunta por los trabajos de conocimiento multidisciplinar que intervienen en la indagación, los profesores en formación desestiman muchas dimensiones del trabajo estadístico, especialmente las relacionadas con el manejo de datos, que no parecen considerar como parte de la actividad matemática – o estadística – implicada.

Liste des editeurs (Eds)

El paradigma del cuestionamiento del mundo en la investigación y en la enseñanza (pp. xx-yy)

VII congrès international de la TAD (Autrans, 22-26 janvier 2018)

Axe 1. *Analyse et évaluation des usages de la TAD dans la recherche et la Formation en didactique*

Axe 2. *Le paradigme du questionnement du monde et la question curriculaire*

Axe 3. *La TAD et la professionnalisation du métier d'enseignant*

Editorial, año

1. Introduction

We present a PhD project in progress at the Universidade Federal de Mato Grosso do Sul (Brazil) and the Universitat de Barcelona (Spain). The research topic arose from the first author's perception of a deficit in the preservice teacher education process, specifically in the area of statistics, with a strong double discontinuity (Winsløw & Grønbæk, 2014) between the statistics that are taught at the university and the needs she experienced as a novel teacher. Our starting point is the teacher education problem:

> $P_{0\text{-TE}}$: What praxeological needs do lower secondary school teachers have about statistics?

that gives rise to our research question:

> RQ: What educational proposal is it possible to implement with a group of preservice mathematics teachers in Brazil and how this proposal contributes to providing future teachers with tools to design, analyse and implement new didactic processes for the teaching of statistics in lower secondary school?

The strategy we follow starts with the consideration of a mathematics textbook exercise about the distribution of hydric resources in Brazil that focuses on a highly topical and paradoxical issue: Brazil has a lot of water, yet a lot of water scarcity. However, the exercise only points at trivial aspects of the graph summarising the situation. We decided to use this case as the origin of a *study and research path for teacher education* (SRP-TE, Barquero et al., 2018) to address a double issue. On the one side, the SRP-TE assumes the need for teachers to live an SRP where data plays an important role as a basis to introduce them into the didactic analysis. On the other side, the exercise represents an illustrative case of the consequences of the pedagogical paradigm of visiting works, where critical questions are sacrificed for the sake of the expected answers. We present here the design of the SRP-TE and a pilot study implemented with a group of secondary school preservice teachers of Brazil at the end of 2021.

2. Hypotheses and SRP-TE design

Our first hypothesis corresponds to the assumption that SRPs-TE have the capacity to lead teachers to question the knowledge to be taught and even, sometimes, the related scholarly knowledge (Barquero et al., 2022). A second hypothesis, sustained by our previous work (Verbisck, 2019; Verbisck et al., 2022) and also other authors (Sorto, 2006; Batanero et al., 2011; Gould et al., 2018; Zapata-Cardona & Escobar, 2019), affects the nature of school statistics and its dominant model in secondary education institutions. We assume that school statistics is mainly composed of numerical calculations of statistical measures (frequencies, means, medians, modes, deviations, ranges, quartiles, etc.) and standardised graphical representation of distributions (pie charts, bar charts, histograms). Moreover, these measures and graphs are mainly introduced to summarise or represent already available and clean data, without systematically linking them to the need of solving open problems.

From the ATD, we propose an epistemological reference model that, within the statistical activity, incorporates aspects such as:

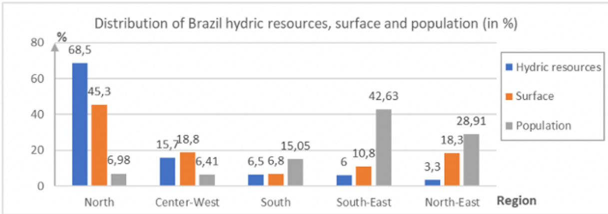
(1) The collection, cleaning, organisation, summary, representation, and exploration of data as a strategy for studying problematic questions.

(2) The study of data variability and the use of probabilities as theoretical distributions of observed frequencies, that is, as models of certain characteristics of the observed frequencies (Wozniak, 2005).

(3) The role of conjecture production and validation tool including the idea of “confidence level”.

When looking through Brazilian lower secondary school textbooks, it is clear that the prescribed statistical tasks primarily consist of completing calculations (frequencies, means, medians, etc.) and reading graphs and tables. Figure 1 depicts one of these activities, which is very widespread in Brazilian textbooks and clearly fits within the paradigm of visiting works. It does not ask any open questions, nor does it require students to examine the data, organise and summarise it, or investigate other issues linked to the topic, although it presents a relevant topic for discussion. Students just have to glance at the graph, pick the largest or smallest bar, and perform some percentage calculations to answer the questions. Our goal was to turn the scholarly activity into an investigative problem, moving it away from the paradigm of visiting works and toward the paradigm of questioning the world.

Brazil has about 13.7% of the total fresh water in the world, being considered a territory rich in water terms. However, the country is experiencing serious problems related to both the degradation of water quality, especially in the urban areas' proximity, and the lack of control of excess and lack of water, which affect several Brazilian locations. Not just the floods affect Brazilian cities: water scarcity also imposes serious restrictions and high costs on economic and social development of large cities. Looking at the chart below, answer in your notebook:



Information obtained in: Ministério do Meio Ambiente <http://wwww.mma.gov.br/estruturas/sedr_proecotur/_publicaco/140_publicacao09062009025910.pdf>. Accessed: 1st July 2018.

a) What kind of chart is this?
b) Indicate the Brazilian region:
- with the largest surface;
- with more water resources;
- with the second lowest population concentration.

c) Which region has the lowest percentage rate of water resources in our country?
d) In which region is there the greatest concentration of population?
e) Can it be said that the larger the surface of the region, the greater the number of inhabitants? Justify your answer.
f) How many percent of the world's fresh water is in the Southeast region of Brazil? Explain how you elaborate your answer.
g) Can it be said that the region with the most water resources is the one with the largest population?

Figure 1. Brazilian exercise (Giovanni Júnior & Castrucci, 2018, p. 26, our translation)

Therefore, for the SRP to be developed with the group of preservice teachers, we propose to start with a generating question Q_0:

> Q_0: How can we explain the contradiction between the abundance of water resources and the water problems (scarcity, quality degradation, lack of control, etc.) of Brazilian locations, as illustrated in the graph?

In our a priori analysis of Q_0, we included four initial derived questions with their related subquestions:

> Q_{1_Water}: What do we know about water distribution in Brazil?
>
> Q_{2_Graph}: What information can we draw from the graph?
>
> $Q_{3\_Data\ source}$: What data is used to make the graph?
>
> $Q_{4\_Data\ work}$: How to obtain data to start working with it?

We can see how the inquiry process incorporates dimensions of statistical work that are typically absent from secondary education, such as data collection, cleaning, debugging, organising, summarising, and representing data, as well as the questioning of the reliability of data taken from real media, i.e., a critical reading of the given quantitative information, by placing ourselves in the paradigm of questioning the world. A new statistical perspective emerges, less influenced by the school institution and more aligned with the supposed reference epistemological paradigm.

However, this was an a priori analysis of a possible generating question concerning the chosen school exercise. We did not know its potential with preservice teachers or secondary school students in terms of other possible derived questions. Nor did we know what possible constraints might arise during the study (access to data, interesting conclusions, motivating themes for the trainee group, etc.). We decided to carry out a pilot study with preservice teachers from the Pedagogical Residency Program of the Federal University of Sergipe.

3. Experimentation

3.1. Context and organisation of the sessions

The Pedagogical Residency Program is one of the actions that integrate the National Policy for Teacher Education in Brazil and this programme aims to encourage preservice teachers to engage as in-service teachers in compulsory education schools. This encourage must contemplate, among other activities, classroom regency and pedagogical interventions, accompanied by an experienced schoolteacher and tutorised by a lecturer from the university implementing the program: the last author of this contribution. In this context, the SRP-TE activity was proposed to a group of 16 preservice teachers during the 90-minute sessions devoted to the university teacher tutorials, which were organised virtually (through Google Meet) due to the Covid19 pandemic. We used the last four sessions of the academic year, in November and December 2021. The first author and the regular lecturer led the sessions, with the other two authors acting as observers. At the time, the students were already familiar with some elements of the ATD, especially the SRP device, as the lecturer had already carried out some implementations in different contexts with this group. Students were also used to autonomous teamwork out of the classroom, from one session to the other. The development of the four sessions can be seen in the Table 1:

1st session – 11/11/21 (3 moments)
The educators presented the school exercise of Figure 1, containing only the introductory statement and the graph. In groups of 4, students are required to discuss the following proposal: *Based on the situation presented in the school exercise, what questions can you raise about the considered theme, and also about the data used?*
In teams, the students write down the issues raised and organise them to be presented; they elect a communicator for the common sharing moment.
Each team of students present the questions raised and discuss them with the educators and other students.
2nd session – 25/11/21 (2 moments)
The researcher presents the complete school exercise (with the textbook questions and expected answers), comments on the strong limitations of the statistical work proposed (especially when compared to the questions raised by the students) and presents a report of the questions raised in the previous sessions made of a list of questions grouped by themes and a question-answer map that summarises them (Figure A from Annex 1).
The researcher distributes a main derived question to each team and asks them to start studying them: - Group 1. $Q_{Resources}$: How much water is there in the five regions of Brazil? How is it measured? - Group 2. $Q_{Climate}$: Where does the water come from in the five Brazilian regions? Rain? Other factors? - Group 3. $Q_{Consumption}$: What is water consumption like in the five regions of Brazil? What factors cause scarcity? - Group 4. $Q_{Sharing}$: What strategies exist to take water from one region to another? Students are required to note down all the sources consulted, both those that were used and those that were discarded; appoint a secretary who notes down everything that is being done and a communicator who prepares a summary with the main results to be presented.
There was no time to share results at the end of the session.
3rd session – 08/12/21 (2 moments)
Each team presents the question addressed and the proposed answer elements, along with one or two questions arising from the study (to complete the map). The other groups are also asked to ask questions to the presenting group: both clarifications and questions they consider most important.
Teamwork to propose a project activity adapted to lower secondary school students, which may be a project or an activity of a longer duration. The design of the project/activity must answer the following questions: - What is the initial question to approach? - How is the question posed, what is proposed to be done, what tools are available to the students? - How is the study planned to be managed in class: approximate timetable? - How is the activity expected to be completed? - Which content themes, domains, or areas (not only in mathematics) are addressed by the proposed activity?
Students started the work during the session and continued it at home, ending up with a presentation to share their results in the last session.
4th session – 16/12/21 (1 moment)

Presentations of the teaching proposals of each group. Final discussion and didactic analysis of the students' proposals and the initial activity proposed by the researcher.

Table 1. Organisation of the SRP-TE sessions

3.2. Sessions 1 and 2: questions raised by the student teams

The first two sessions aimed at raising questions related to the theme presented in the exercise in the book. To prepare the next session, we grouped the questions in six topics: water resources (distribution of these resources), sharing water resources (from one region to another), quality of water resources, climate conditions (rainfall), consumption of water resources (scarcity, deforestation, etc.), and politics (citizen actions, responsibility of governments). However, questions related to the data presented in the exercise situation such as, "What variables does the exercise present and how were they measured? Where were these data taken from? Is the source reliable? How to validate?" did not appear.

At the beginning of the second session, we presented the school exercise (Figure 1), including the questions proposed in it. It was interesting to see that the preservice teachers were surprised by the poverty of the textbook questions as opposed to the richness of the questions they had prepared. Afterwards, we presented their questions grouped into topics that were most repeated and questions that they had not proposed, as a way to incorporate and enrich the work done so far (see Annex 1)[1].

Due to the time constraint, it was not possible to work on finding answers to all the questions raised, although very interesting and relevant. Thus, we chose one question for each group to research (Table 1 – second moment of session 2), collect some elements os answers and reraise other questions.

3.3. Sessions 3 and 4. Elements of answers and didactic design

In this session, one representative from each group presented their elements of answers, and possible derived questions, to the question

[1] Annex 1 can be found here: (https://docs.google.com/document/d/1rJtYp5eUzb5km91ERSr4I4FOF7b6K8aNUfQDWKsDSjE/edit?usp=sharing).

assigned to them in the last session. We organised these questions and answers into a new Q-A map, presented in Annex 2[2].

The preservice teachers reported difficulties when searching for information on the theme related to the question they were trying to answer, as well as difficulties in finding information on water resources divided into Brazilian regions and a lack of more recent studies on the investigated themes. Much of the information is repetitive and very old (from decades past) and looks as if it needs to be updated.

After the presentations, we proposed one last activity for the groups of students. As we had discussed earlier, our goal for this SRP-TE activity was to develop with these preservice teachers an attitude (change of paradigms) converting a textbook exercise into an investigative activity, into an inquiry problem. So, having in mind the themes discussed in the previous sessions, the last action of each group was to design an instructional proposal adapted for students of the lower secondary school level (it could be a project or an activity of a longer duration). For the design, the students should be guided by the following question:

> $Q_{0\text{-}FP}$: What is the initial question to approach? How is the question posed, what is proposed to be done, what tools are available to the students? How is the study planned to be managed in class: approximate timetable? How is the activity expected to be completed? Which content themes, domains, or areas (not only in mathematics) are addressed by the proposed activity?

Groups started the work of drawing proposals in this session but, due to the short time remaining, they had to continue this work at home. The sharing of the results was destined for the last session.

The four groups presented different teaching proposals and with the following generative questions:

> $Q_{0\_Group\_1}$: Why is water consumption limited in certain regions of Brazil?
>
> $Q_{0\_Group\_2}$: Is it possible to collect rainwater to use at home?
>
> $Q_{0\_Group\_3}$: Given the problems related to water quality degradation, what would be the percentage rate of degraded water resources?

[2] Annex 2 can be found here: (https://docs.google.com/document/d/1rJtYp5eUzb5km91ERSr4I4FOF7b6K8aNUfQDWKsDSjE/edit?usp=sharing).

$Q_{0\_Group\_4}$: If Brazil is a country with great availability of water resources, why is there water scarcity and/or rationing in all regions of the country?

Due to limited space, we focused on presenting one of the teaching proposals that interested us the most, and it follows below:

Q_0: Is it possible to collect rainwater to use at home?

Context:

We can use the context seen in our experience with this SRP of water distribution, we start from how water is distributed in our city, making students realise that in our region we have, relatively, little water. This way we raise the idea of trying to save water since water distribution is a process little used in Brazil. To do this we will talk about the climate and how rainfall can help us alleviate this problem. Then we can introduce our Q_0 to them. Along the way, students can use whatever tools they have available, whether they are books, mobile phones, etc.

NOTE: If most of the students are from rural areas, they should generally use rainwater, as there is not always piped water in the homes.

Questions that may arise when answering Q_0

- How to store/treat rainwater and distribute it for use in most household activities?
- Can I use rainwater for drinking purposes? Is it possible to make it drinkable?
- How do I collect as much rainwater as possible? Will I use a spout?
- How much water can I capture?
- Would this saving change the amount I pay on my water bill?
- How can I clean the rainwater to make the most of it?
- How many gallons of chlorine would I have to use if I stored 300 gallons of water?
- Is the amount of rainfall in my region enough for one person to live on rainwater alone?
- How much water does it take for a person to use in their daily life?
- Etc.

Subjects and contents

Considering the previous questions, we expect to inquire on the topics pertaining to the area of chemistry, biology, mathematics and geography. We initially thought about the following contents:

- Mathematics: ratio, magnitudes and measures, functions, areas of figures, volume.
- Geography: Climate of the region.
- Chemistry: Stoichiometry.
- Biology: Micro-organisms and bacteria, water cycle.

[…].

Estimated time

We believe it will take about a month of class time to complete this inquiry, since we anticipate two weeks of discussions, one to learn the issues that arise in the process and another to prepare and present the final answer.

Conclusion

We hope that the study will conclude with an answer to the initial question, in which the students will use the knowledge acquired throughout the lessons to say whether it is possible to carry out what is asked or not. If the answer is yes, it would be necessary to show a project that in practice fulfils what is asked in Q_0 and estimates how much water can be collected. If it is not possible or feasible, students should argue why and under what conditions it could be.

Table 2: SRP proposal by Group 2

4. Discussion and conclusion

We are conscious that the pilot study was carried out in limited conditions, with few sessions and using the very last classes of the academic year, on top of the online interactions due to the pandemic situation. However, they provided us with interesting learnings about the teacher education proposal and its possible future exploitation. The change of paradigms proposed by Chevallard (2015) seems particularly appropriate in the case of statistical inquiries for the specific descriptive tools it provides.

The virtual sessions highlighted the difficulty in observing the work within the teams, and also because the most important work was done outside the classroom. However, we were also able to identify some favourable conditions for the development of the activity. First of all, the group of teachers in training already knew about the ATD and had even participated in other SRPs. Students were already used to generating questions, as well as not being restricted to the mathematical content. In this respect, the students did better than in our a priori analysis. They broadened the range of questioning from geographical (resources and climate) to social, political and environmental aspects (sharing and quality), allowing, as Chevallard said:

> A *co-disciplinary* symphony in which mathematics contributes with other disciplines to elucidating the conditions and constraints of all kinds that determine the production of answers *A* to questions *Q*. (Chevallard, 2004, p.12).

However, and somewhat surprisingly for us, among the contents activated by the SRP, aspects related to statistics and data processing appeared only tangentially. Students did not focus on the search and use of data, as we did in the a priori analysis. Maybe the fact of having little time to do the study explains the students' choices to focus on topics for which they found studies more easily accessible. More generally, we can also point at the fact that the statistical work that underpins many studies is not very visible and difficult to access. This fact raises the problem of defining more clearly "what it is to study a question Q_0", which experimental field is the most appropriate and why we should not be content with "copying" directly accessible answers. To set up an appropriate media-milieu dialectic seems to be an important challenge to be addressed.

Finally, the invisibility of data processing reappears when students are asked to associate curricular knowledge with the development of an SRP. This is a return to the invisibility of many statistical activities they incorporated in the proposal but did not identify as part of the knowledge to be taught (while they did identify many others). We relate it to the *transparency of didactic facts* discussed by Ruiz-Higueras & Fernández (1999). This points to yet another challenge for the implementation of future SRP-TE for teachers who teach statistics.

Acknowledgments

Funded by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brazil (CAPES) – Finance Code 001 and by the Spanish project RTI2018-101153-B-C21 (MCIU/AEI/FEDER, UE).

References

Barquero, B., Bosch, M., & Romo, A. (2018). Mathematical modelling in teacher education: Dealing with institutional constraints. *ZDM Mathematics Education*, *50* (1/2), 31-43.

Barquero, B., Bosch, M., & Florensa, I. (2022). Contribuciones de los recorridos de estudio e investigación en la universidad: el caso de la formación del profesorado. *Avances de Investigación en Educación Matemática* (under revision).

Batanero, C., Burrill, G., & Reading, C. (2011). *Teaching Statistics in School-Mathematics-Challenges for Teaching and Teacher Education: A Joint ICMI/IASE Study*. Springer.

Chevallard, Y. (2004). 'La place des mathématiques vivantes dans l'éducation secondaire: transposition didactique des mathématiques et nouvelle épistémologie scolaire.' *3e Université d'été Animath*, Saint-Flour (Cantal), 22-27, APMEP (pp. 239-263).

Chevallard, Y. (2015). Teaching mathematics in tomorrow's society: a case for an oncoming counter paradigm. In S. J. Cho (Ed.), *The Proceedings of the 12th International Congress on Mathematical Education: Intellectual and attitudinal challenges* (pp. 173–187). Springer.

Giovanni Júnior, J. R., & Castrucci, B. (2018). *A conquista da matemática. 8o ano ensino fundamental: anos finais* (4e), FTD.

Gould, R., Peng, R., Kreuter, F., Pruim, R., Witmer, J., & Cobb, G. (2018). In D. Ben-Zvi, K. Makar & J. Garfield (Eds.), *International Handkook of Research in Statistics Education* (pp. 415-432). Springer.

Higueras, R., & Rodríguez Fernández, J. (1999). *La transparencia de los hechos didácticos en la enseñanza de las matemáticas*. SUMA, 32, 69-78.

Sorto, M. A. (2006). Identifying content knowledge for teaching statistics. In A. Rossman, & B. Chance (Eds.), *Proceedings of the International Conference on Teaching Statistics*, International Association for Statistical Education.

Verbisck, J.T.S. (2019). *Uma análise praxeológica da proposta de ensino de probabilidade em livros didáticos da educação básica* (Master tesis). Universidade Federal de Mato Grosso do Sul, Brazil.

Verbisck, J. T., Bittar, M., & Bosch, M. (2022). Learning to teach statistics through study and research paths. In *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education*. Italy: Free University of Bozen-Bolzano and ERME (to appear).

Winsløw, C., & Grønbæk, N. (2014). Klein's double discontinuity revisited: contemporary challenges for universities preparing teachers to teach calculus. *Recherches en didactique des mathématiques, 34*(1), 59-86.

Wozniak, F. (2005). Conditions et contraintes de l'enseignement de la statistique en classe de seconde générale. Un repérage didactique (Doctoral dissertation, Université Claude Bernard-Lyon I, France).

Zapata-Cardona, L., & Escobar, M. (2019). Critical Citizenship in Colombian statistics Textbooks. In G. Burrill & D. Ben-Zvi (Eds.), *Topics and trends in current statistics education research: International perspectives* (pp. 373-389). Springer.

Study and research paths in teacher education: Naturalization of knowledge as an obstacle to questioning the knowledge to be taught

Heidi Strømskag

Norwegian University of Science and Technology, Trondheim; heidi.stromskag@ntnu.no

Abstract. This paper presents an analysis of inquiries in the form of unfinalized study and research paths conducted by 14 student teachers enrolled in a teacher education programme for Grade 8–13. The inquiries concerned the contents of differential calculus taught in Grades 11 and 12 in upper secondary school in Norway. The research question addresses issues related to operating in the paradigm of questioning the world. It is shown how the phenomenon of naturalization of knowledge is an obstacle to student teachers' questioning the knowledge to be taught. Further, a graphic device to display the path of an inquiry is introduced.

Keywords: Didactic milieu, differential calculus, milieu dynamics chart, naturalization of knowledge, unfinalized study and research path.

Résumé. Cet article présente une analyse d'enquêtes sous forme de parcours d'étude et de recherche non finalisés menées par 14 enseignants en formation inscrits à un programme de formation des enseignants de la 8e à la 13e année du secondaire. Les enquêtes concernaient le contenu du calcul différentiel enseigné en 11e et 12e année dans une école secondaire supérieure en Norvège. La question de recherche aborde les problèmes liés au fonctionnement dans le paradigme du questionnement du monde. On montre comment le phénomène de naturalisation des connaissances est un obstacle à la remise en question par les enseignants stagiaires des connaissances à enseigner. En outre, un dispositif graphique permettant de montrer le cheminement d'une enquête est introduit.

Mots-clés : Milieu didactique, calcul différentiel, graphique de la dynamique du milieu, naturalisation des savoirs, parcours d'étude et de recherche non finalisé.

Resumen. Este artículo presenta un análisis de las investigaciones en forma de recorridos de estudio e investigación no finalizados realizados por 14 estudiantes de magisterio matriculados en un programa de formación de profesores para los grados 8 a 13. Las indagaciones se referían a los contenidos del cálculo diferencial que se imparten en los grados 11 y 12 en la escuela secundaria superior de Noruega. La pregunta de investigación aborda cuestiones relacionadas con el funcionamiento en el paradigma del cuestionamiento del mundo. Se muestra cómo el fenómeno de la naturalización del conocimiento es un obstáculo para que los estudiantes de magisterio cuestionen el conocimiento por enseñar. Además, se introduce un dispositivo gráfico para mostrar el camino de una indagación.

Palabras clave: Medio didáctico, cálculo diferencial, gráfico de la dinámica del medio, naturalización del saber, recorrido de estudio e investigación no finalizado.

1. Introduction

This study involves inquiries in teacher education on differential calculus taught in upper secondary school. The research is conducted in the framework of the anthropological theory of the didactic (ATD, Chevallard, 2019), and the inquiries are framed within the paradigm of "questioning the world" (Chevallard, 2015), using the methodology of Study and Research Paths (SRPs; Bosch, 2018). The notion of praxeological reference model (PRM) is used with a descriptive purpose for students to analyse what kind of differential calculus is taught in school, what elements are left out, and what elements could have been included (under which conditions). This is informed by the notion of reference epistemological model as conceptualized by Bosch (2015) and Florensa et al. (2021); it is explained in Section 3.1 why "epistemological" is replaced by "praxeological".

The research question considered is: *What are the obstacles encountered and outcome achieved during* unfinalized *SRPs in teacher education?* On the notion of "unfinalized" SRP, see Chevallard (2022, p. 217). Data sources are the following: assignment and guidelines about the SRPs, 7 preliminary reports, 5 draft reports, 7 final reports, questions sent to the teacher educator (i.e., me), and students' evaluation of the mathematics course in which the SRPs took place (anonymous survey). Significant work has already been devoted to the theory and practice of SPRs in teacher education (see e.g., Barquero et al., 2019), in secondary education (see e.g., Jessen et al., 2022), and in engineering education (see e.g., Bartolomé et al., 2018). However, unlike the present study, these contributions focus on *finalized* SPRs only, which makes a difference, as we will see later.

2. The didactic system and some institutional conditions

The study reported here took place within the programme "Natural science with teacher education" (MLREAL, n.d.) at the Norwegian University of Science and Technology. This is a 5-year master's programme geared towards Grade 8–13 with five different fields of study: biology and chemistry; mathematics and biology; mathematics and chemistry; mathematics and informatics; and mathematics and physics. One of the two subjects in these fields of study is called Subject 1—this is the one in which the master's thesis is written; the other is called Subject 2. After 3 weeks on the programme, the student teacher (hereafter, 'student') chooses what will be Subject 1 and Subject 2. Subject 1 requires a minimum of 90 ECTS credits in the discipline and a minimum of 30 ECTS credits in the didactics of the discipline, before writing a master's thesis of 30 ECTS credits (in the discipline or in the didactics of the discipline). Subject 2 requires 60 ECTS credits in the discipline and 15 ECTS credits in the didactics of the discipline. All courses except two are 7.5 ECTS credit points.

The observed didactic system, denoted by $S = S(X, Y, Q)$, consisted of 14 students, $X = \{x_1, \ldots, x_{14}\}$, and a lecturer, $Y = \{y\}$, who is the author of this paper. In X, 5 students were in Year 4 and 9 students were in Year 5 of the programme. S was operating within the teaching unit *Design and Analysis of Mathematics Teaching* (MA3061, n.d.), in the autumn of 2021. This course was one fourth of the semester study load, so the students took three other courses simultaneously. For 11 weeks, X was split into 7 teams, based on students' own suggestions for companion, working in the format of SRPs. The teams were composed as shown in Table 1.

Table 1: **The student teams inquiring into Q**

Team	Student	Gender	Academic year	Subject 1	Subject 2
1	x_1	Male	5	Mathematics	Physics
	x_2	Male	5	Mathematics	Physics
2	x_3	Female	5	Mathematics	Physics
	x_4	Female	5	Mathematics	Biology
3	x_5	Female	5	Informatics	Mathematics
	x_6	Female	4	Mathematics	Chemistry
4	x_7	Female	4	Mathematics	Physics
	x_8	Male	5	Mathematics	Informatics
5	x_9	Male	5	Mathematics	Physics
	x_{10}	Female	5	Mathematics	Biology
6	x_{11}	Male	5	Mathematics	Physics
	x_{12}	Male	5	Mathematics	Physics
7	x_{13}	Female	4	Mathematics	Chemistry
	x_{14}	Male	4	Mathematics	Physics

We see that $X \setminus \{x_5\}$ have mathematics as Subject 1 and that x_5 has mathematics as Subject 2. Since all students of X had completed the first three years on the programme, they have studied the following topics of mathematics, which constitute the minimum requirements to be qualified as a mathematics teacher for Grade 8–13 in Norway: 'Basic Calculus 1 and 2', 'Vector Calculus', 'Linear Algebra', 'Geometry', 'Number Theory' (replaced by 'Discrete Mathematics' for x_5 and x_8), 'Statistics' or 'Probability and Statistics', and 'Complex Analysis, Differential Equations and Fourier Analysis'. At the time of the observation, $X \setminus \{x_5\}$ had, in addition to the topics mentioned, also studied these topics: 'Linear Algebra with Applications', 'Numerical Methods' or 'Numerical Mathematics', and 'Algebra' or 'Linear Methods'. Year 5 students of X had also studied more advanced mathematics.

3. The generating question Q and its study conditions

The generating question of the SRPs was the following:[1]

> Q: *What are the most important elements in an introduction to differential calculus in Mathematics 1T and Mathematics R1.*[2] *Why?*

The Herbartian schema had been introduced to X with this symbolism (adapted from Chevallard, 2019): $[S(X, Y, Q) \Rightarrow M] \Rightarrow A^{\heartsuit}$, where the *didactic milieu* $M = \{A_1^{\lozenge}, A_2^{\lozenge}, \ldots, A_m^{\lozenge}, W_{m+1}, W_{m+2}, \ldots, W_n, Q_{n+1}, Q_{n+2}, \ldots, Q_p\}$. Note that the question Q studied by these student teachers, who had learnt calculus for a long time at university, is clearly not the question Q^*, studied in depth by several authors (e.g., Lucas, 2015; Lucas et al; 2020), of the learning of calculus by high school students or

[1] The Norwegian texts used in the study reported here have been translated into English by the author.

[2] By "introduction" is meant the whole of differential calculus contained in Mathematics 1T and Mathematics R1. (This footnote was included in Q.)

first-year university students for example: Q merely points to a set of conditions and constraints that might inform a possible answer to Q^*.

3.1. Existing answers and a request for a reference model

One element of the initial didactic milieu for the inquiries consisted of textbooks for Mathematics 1T and Mathematics R1;[3] they were explained to be existing answers $A^\Diamond$ to Q. So, for each team, two sets of textbooks for each of Mathematics 1T and Mathematics 1R were provided. This was to ensure that all teams had access to textbooks written for the current national curriculum, in force since 2020 (Directorate for Education and Training, 2019, 2020). Distribution of textbooks was as follows: Teams 1–6: *Sinus 1T* (Oldervoll et al., 2020) + *Sinus R1* (Oldervoll et al., 2021) and *Matematikk 1T* (Borge et al., 2020) + *Matematikk R1* (Borgan et al., 2021). Team 7: *Matematikk 1T* + *Matematikk R1* and *Mønster 1T* (Kalvø et al., 2020) + *Mønster R1* (Kalvø et al., 2021).

In order to *study* these works, rather than just regarding them as giving unquestionable answers to Q and be able to answer the *why*-part of Q, each team was asked to develop their praxeological reference model (PRM for short) of modern differential calculus (starting with Newton and Leibniz)—aiming at "detachment" from the mathematical organisations presented in the textbooks. This involved considering 4 questions (Assignment, 15 October 2021):

1) What is differential calculus made of? (its *nature*)
2) What were the questions that motivated its development? (its *origin*)
3) In which areas is this knowledge used today? (what is it *for*)
4) Why should it be taught at school? (its *legitimacy*)

Since the first question can be addressed by sketching a praxeological organisation of differential calculus, the reference model aimed at is considered *praxeological*. It was informed that only *partial answers* to the 4 questions were expected. Further, it was stated that:

> The model you develop will be a reference model on which you will *base the SRP*; the model shall not be included in the SRP report in its entirety, but parts of it will naturally appear in some chapters of the report. (Assignment, 15 October 2021)

For the construction of reference models, two electronic sources were provided in the initial didactic milieu: the book *The Historical Development of the Calculus*, written by Charles Henry Edwards (1979), and the chapter "The Creation of the Calculus" in Morris Kline's book *Mathematical Thought from Ancient to Modern Times*, published in 1972. It was commented that these works could surely be supplemented by other resources.

3.2. The setup of the SRPs

The inquiries comprised the following activities, which amount to a total of approximately 40 working hours for the students: study and research in teams on answering Q; 7 lectures given by y (on mathematics, the SRP process, and the report); 2 seminars with presentation of preliminary

[3] These courses are given in Grades 11 and 12 respectively, preparing for university studies in science, technology, engineering, and mathematics.

reports and feedback from other teams and *y*; one "question session" in which *y* answered questions submitted by the teams beforehand; and voluntary submission of a draft of the final report for feedback from *y* (5 of the 7 teams did submit drafts).

4. Analysis and results

After 3 weeks of working on the SRPs, each team sent their preliminary report to *y* and to another team for discussion in class, following a pregiven permutation. This report was supposed to contain: 1) a draft of the first two elements of the "Results" section in the final report as detailed in the guidelines (see Section 3.2), and 2) a brief account of challenges they had faced and how they planned to tackle them.

4.1. Not questioning the origins of the mathematics to be taught

The preliminary reports contained lengthy extracts from the contents of differential calculus in the textbooks considered. Partial answers to the *why*-part of *Q* were made by 4 teams (Teams 3, 5, 6, 7), using the national curriculum in mathematics as justification. One of the teams, who had used the national curriculum in mathematics as providing an existing answer, indicated that they would use the Education Act to answer the *why*-part of *Q*:

> We have used the national curriculum for 1T and R1 to find an existing answer to what are the most important elements in an introduction to differential calculus in Mathematics 1T and R1. As concerns *why* these elements are the most important ones, we think it may be useful to take as a starting point the preamble of the Education Act and the overriding principles of the Core Curriculum. (Team 2, in preliminary report)

These observations are symptoms that the students' analyses do not extend to scholarly mathematics; they remain at the level of the noosphere, built on textbooks, the mathematics curriculum and the core curriculum. In the preliminary reports, traces of PRMs were barely visible; only 3 teams had made (scarce) references to Kline (1972) and/or Edwards (1979).

Accounting for the origins and history of the mathematics to be taught, helped by a PRM, was such an unusual task that most teams had, halfway into the inquiry, disregarded it when studying *Q*. The following question by Team 1 indicates that for them, mathematical concepts on the one hand, and their history and applications on the other hand, are rather unrelated as concerns the mathematics to be taught:

> Are the answers [to *Q*] mainly about mathematical concepts or can one cover broader topics such as the history behind differentiation, what applications it has, or how the theme could be presented? (Team 1, in preliminary report)

Even if the aspects mentioned by Team 1 were contained in the second and third questions in the assignment about a PRM (cf. Section 3.1), they wanted confirmation that it would be relevant to study these aspects. Further evidence that the utility of a praxeological reference model was not well conceived, is this statement taken from the final evaluation of the mathematics course:

> I remember I did not understand much of what we were going to do with the reference model, and our reference model was therefore only partially constructed. (Student, in anonymous survey)

4.2. Diverse roles of elements in the didactic milieu

There were several teams asking questions about the difference between existing answers and works to study these answers. Team 1 wrote in their preliminary report: "Where is the demarcation line between works and existing answers?" And here is a question from Team 5:

> You mentioned [in the question session] that we should treat the textbooks only as existing answers rather than treating them as works, as we had done. That's okay. … Now, when I write about the existing answers, I therefore wonder how much is expected to write about the existing answers before [presenting] the derived questions? (x_9, in an email to y on 29 November 2021)

These questions about the diverse roles of the elements in the milieu is a symptom that many teams didn't see the need to question the contents in the textbook, and they more or less ignored the second part of the generating question Q, that is, *why* the elements in the textbooks were important in an introduction to differential calculus. The request for constructing a PRM was assumed by y to help answering the *why*-part of Q. However, there were no sessions in the first three weeks allocated for discussing the PRMs and their relation to Q. This was discussed in y's lecture after the presentations, but by then most teams had already worked for three weeks with the view that the textbooks and the curriculum were adequate answers to Q, as commented in the previous section.

4.3. A tool to display the milieu of an ongoing inquiry

Several teams expressed that it was hard to know how they should write up the final report on the SRPs. For example, before the presentations of preliminary reports, x_9 wrote:

> What we struggle with is whether we should write chronologically how we proceed in the SRP and present preliminary answers, works and derived questions in the order we work, or whether we should try to divide the results section according to the Herbartian schema. (x_9, in an email to y on 27 October 2021)

Team 3 wrote in their preliminary report that "it may become a messy setup" and then proposed a solution, using a directed graph as shown in Figure 1. They explained the graph this way:

> The graph shows the path of our exploration of new derived questions. Dotted line means that we have come to this stage in the inquiry and continue working there. Underline means that we have chosen to stop the series of derived questions there… (Team 3, in preliminary report)

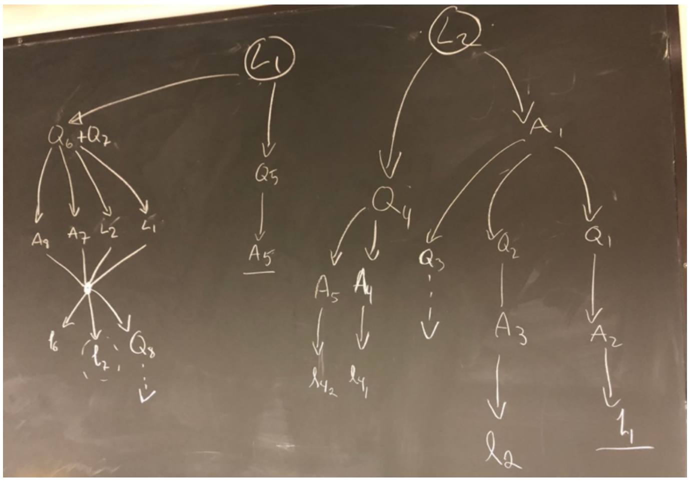


Figure 1: Directed graph representing an ongoing inquiry (taken from Team 3's preliminary report)

In Figure 1 (and in Figure 2 below), L_i means *existing answer* ("eksisterende løsning" in Norwegian); l_j means *provisional answer* ("foreløpig løsning" in Norwegian); A_j means *work* ("arbeid" in Norwegian); Q for *question* (as in English). This graphic device displays three features of an ongoing inquiry: the *elements of the didactic milieu* at a certain point in time of the inquiry, the *path* that has led to this stage, and *directions* for further steps in the inquiry.

Team 3 did not show this invention during their presentation, but in the follow-up lecture, *y* showed it to the class and encouraged all teams to adapt it for use in their final reports, along with a table listing the elements of the didactic milieu that had evolved during the inquiry. Figure 2 and Table 2 show how this was solved by Team 4 in their final report. They had used the software *Inkscape* to generate the graph.

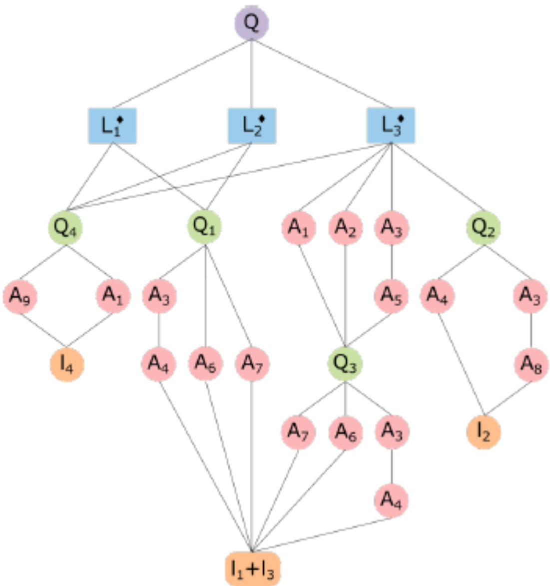


Figure 2: Directed graph displaying an inquiry into *Q* (taken from Team 4's final report)

Table 2: Elements in the didactic milieu created and drawn upon by Team 4

Existing answers, $L_i^\diamond$
$L_1^\diamond$: Sinus 1T and Sinus R1 (publisher Cappelen Damm)
$L_2^\diamond$: Mathematics 1T and Mathematics R1 (publisher Aschehoug)
$L_3^\diamond$: Articles on derivation at National Digital Learning Arena's website for Mathematics 1T
Works, A_i
A_1: Calculus 1 by Adams & Essex (2013)
A_2: Video on limits from NTNU Undervisning (YouTube Channel)
A_3: "Pupils' understanding of the derivative: A case study of R2 pupils' understanding of graphical representations of the derivative." Master's thesis by Fandrem (2016)
A_4: "Students' understanding of differentiation" by Orton (1983)
A_5: "Concept image and concept definition in mathematics with particular reference to limits and continuity" by Tall & Vinner (1981)
A_6: "The historical development of the calculus" by Edwards (1979)
A_7: "The changing concept of change: The derivative from Fermat to Weierstrass" by Grabiner (1983)
A_8: "Introduction to diagnostic teaching in mathematics" by Brekke (1995)
A_9: "Teaching mathematics in tomorrow's society: A case for an oncoming counter paradigm" by Chevallard (2015)
Derived questions, Q_i
Q_1. What strength does a graphical approach have and what strength does an algebraic approach have, and how do they supplement each other in an introduction to derivation?
Q_2. Do simplifications appear about derivation in the textbooks $L_1^\diamond$, $L_2^\diamond$, and at $L_3^\diamond$, which can create confusion, and how could they have been avoided?
Q_3. Should pupils first get a proper understanding of limits before they are introduced to derivation?
Q_4. Is "growth speed" an appropriate concept for instantaneous rate of change in a point?

4.4. Preliminary reports being instrumental for further steps taken by S

Based on the content and quality of the 7 preliminary reports, y prepared a lecture which, in addition to insisting on using the tool described in the previous section, focused on three elements: 1) Limits: Why is the limit concept important in differential calculus? 2) Differentials versus derivatives: What is a differential and how is it related to the derivative? 3) The Norwegian term "vekstfart" ("growth speed" in English) used in the mathematics curriculum and in the textbooks studied instead of "rate of change". It was commented by y that "endringsrate" in Norwegian—where "endring" means *change* and "rate" means the same as in English—would be more appropriate to describe the derivative, because change can be either positive or negative, and a rate is the quotient of two differences where the independent variable may be different from *time*.[4]

These three elements were taken up by y because they were either ignored or just mentioned by the teams, without further elaboration. All teams had claimed that limits, which were devoted to a separate chapter in the textbooks studied, were an important element of differential calculus since the derivative of a function is defined by the limit of a quotient of differences, $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$. However, none of them gave an account of the origins of this definition or why it is useful. Team 2

[4] Here is an example: What is the maximum volume of an open-top box made with a sheet of paper of width a and length b, where squares of side length x have been cut out of each corner? Here, *length* is the independent variable, so it doesn't make sense to talk about speed.

was the only one to mention the notion of *infinitesimal*. They stated that they had written about it in their PRM, but it was not further discussed in their preliminary report. In her lecture, *y* therefore tried to "model" how questioning the knowledge to be taught could be done, using limits as an example. She started with the question Q_{limits}: "What was the problem that gave rise to the development of the limit concept as we know it today?" With reference to Edwards (1979), *y* explained that the shortcomings in the theory of infinitesimal calculus paved the way for continuity and limits, developed independently by Bolzano and Cauchy. These developments made it possible for Cauchy to be the first mathematician, in 1823, to define the derivative as the limit of the quotient of differences as we know it today (Cauchy, 1823, p. 9). The theory of limits, where delta–epsilon proofs[5] (i.e., "the algebra of inequalities") are at the core, were instrumental in treating the problem of rigour in mathematical analysis.

The final reports showed that six teams had studied, in addition to other derived questions, what in essence was Q_{limits}, and limits were mentioned by quite a few in the final evaluation of the course (see two excerpts in Section 5.2). Other derived questions studied by the teams also had the characteristic of questioning the knowledge to be taught (see e.g., Q_1 and Q_2 in Table 2).

5. Discussion and final comments

In the following paragraphs, I answer the research question considered in the study reported here.

5.1. Obstacles to operating in the paradigm of questioning the world

Differential and integral calculus has been taught in Norwegian upper secondary school since the 1930s (Lyche, 1930). The task of questioning parts of this knowledge seemed unimaginable for the students involved in this study; they seemed to take it for granted. This is an incident of the phenomenon referred to by Chevallard (2022) as *naturalization of knowledge*: "Anything taught over a fairly long period of time… tends to be perceived by those who teach it as taken for granted, as 'natural'" (p. 28). The students' difficulties in distinguishing existing answers from works to study these answers can also be understood in light of this phenomenon. Further, the fact that *y* provided mathematics textbooks for the teams may have reinforced their conception of the knowledge to be taught as being something stable and unquestionable. Moreover, they got two different sets of textbooks, which may have been decisive for some teams interpreting Q as being more about a comparison of textbooks than about the nature, origins, and utility of the elements of differential calculus.

Another obstacle to the conducted inquiries is that the reference models, which were meant to help questioning the knowledge at stake, did not work as intended. The wording of the assignment that the SRPs should be *based on* the reference model was no guarantee that the teams constructed such as model, let alone that they were able to use it to answer Q in case they had actually constructed one. The fact that there was no session allocated for discussing the reference models in relation to the generating question Q was a weakness in the design of the inquiry. This gives rise to an open question

[5] According to Grabiner (1978), Cauchy's proof of one of his theorems on derivatives, Cauchy (1823, p. 27), was the first delta-epsilon proof published.

for future research in teacher education: How can a praxeological reference model help questioning the knowledge to be taught? This question, I believe, applies irrespective of the position of the inquirer (i.e., student teacher, teacher, teacher educator/researcher).

5.2. Outcome of the conducted inquiries

In order to represent the dynamics of the milieu *M*, a directed graph was used along with a table listing its nodes, as described in Section 4.3. This *milieu dynamics chart* shows the evolution of *M* generating the final (though provisional) answer to *Q*. Such a chart is an outcome of the conducted SRPs, that is, a creation of the didactic system. It displays not only the questions studied and answers arrived at in the course of an inquiry but also the works drawn upon. Such a representation of the dynamics of the milieu may be seen an extension of question–answer maps, as introduced by Winsløw et al. (2013) and frequently used by ATD researchers to put questions at the centre of study and research processes. Milieu dynamics charts (MDCs) as used here seem best appropriate to conducting *unfinalized* inquiries, where the evolution of the set of works available in the milieu *M* must be specified, just as the evolution of questions and answers is specified. More research is obviously needed to investigate the appropriateness of this tool.

Even if serious obstacles were encountered during the SRPs, as discussed in the previous section, the final evaluation of the course suggested that the students had gained important insight. Here are two (anonymous) statements:

> I have gained a better understanding of what differential calculus is about and what it can be used for. I have learned how it is related to limits. Derivation was very "instrumental" to me before we did the SRP, so I have actually learned incredibly much. (Student 1)
>
> I have learned to be a little more critical of mathematics textbooks: not all written in these books is "gold". … As concerns differential calculus, understanding what a limit is, is far more important than I was aware of. (Student 2)

As described in Section 4.4, there are indications that the SRPs, by questioning and studying the knowledge to be taught, have changed the students' *relations* to these items of knowledge and thus contributed to the creation of a new *topos* (Chevallard, 2019), that of student teacher, in which "well-known" notions are no longer taken for granted but are decidedly questioned.

In conclusion, this study demonstrates a need to counteract, in teacher education, the phenomenon of naturalization of knowledge. I believe one way to contribute in this respect is to help student teachers construct and use their own PRMs and inquire into the knowledge to be taught. How to proceed to achieve this is an object of ongoing research.

References

Barquero, B., Bosch, M., & Romo, A. (2018). Mathematical modelling in teacher education: Dealing with institutional constraints. *ZDM Mathematics Education, 50*, 31–43. https://doi.org/10.1007/s11858-017-0907-z

Bartolomé, E., Florensa, I., Bosch, M., & Gascón, J. (2018). A 'study and research path' enriching the learning of mechanical engineering. *European Journal of Engineering Education, 44*(3), 330–346. https://doi.org/10.1080/03043797.2018.1490699

Borgan, Ø., Borge, I. C., Engeseth, J., Heir, O., Moe, H., Norderhaug, T. T., & Vie, S. M. (2021). *Matematikk R1* (4th ed.). Oslo, Norway: Aschehoug.

Borge, I. C., Engeseth, J., Haug, H., Heir, O., Moe, H., Norderhaug, T. T., & Vie, S. M. (2020). *Matematikk 1T* (4th ed.). Oslo, Norway: Aschehoug.

Bosch, M. (2015). Doing research within the anthropological theory of the didactic: The case of school algebra. In S. J. Cho (Ed.), *Selected regular lectures from the 12th International Congress on Mathematical Education* (pp. 51–69). Springer. https://doi.org/10.1007/978-3-319-17187-6_4

Bosch, M. (2018). Study and research paths: A model for inquiry. In B. Sirakov, P. N. de Souza, & M. Viana (Eds.), *International Congress of Mathematicians* (Vol. 3, pp. 4001–4022). Singapore: World Scientific Publishing.

Cauchy, A.-L. (1823). *Résumé des leçons données à l'École Royale Polytechnique, sur le calcul infinitésimal*. Paris, France: Debure. Available at https://gallica.bnf.fr/ark:/12148/bpt6k62404287.texteImage

Chevallard, Y. (2015). Teaching mathematics in tomorrow's society: A case for an oncoming counter paradigm. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 173–187). Cham, Switzerland: Springer.

Chevallard, Y. (2019). Introducing the Anthropological Theory of the Didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education*, *12*, 71–114.

Chevallard, Y. (2022). Toward a scientific understanding of a possibly upcoming civilizational revolution. In Y. Chevallard, B. Barquero, M. Bosch, I. Florensa, J. Gascón, P. Nicolás, P., & N. Ruiz-Munzón, (Eds.), *Advances in the Anthropological Theory of the Didactic* (pp. 179–228). Cham, Switzerland: Birkhäuser. https://doi.org/10.1007/978-3-030-76791-4_16

Chevallard, Y. (2022, 18 January). *The Humble Seminar 2021–2022: A Notebook (4)*. Distributed by email to a network of ATD researchers.

Directorate for Education and Training. (2019). *Kompetansemål etter matematikk 1T* [Competence goals after Mathematics 1T]. https://www.udir.no/lk20/mat09-01/kompetansemaal-og-vurdering/kv42

Directorate for Education and Training. (2020). *Kompetansemål etter matematikk R1* [Competence goals after Mathematics R1]. https://www.udir.no/lk20/mat03-02/kompetansemaal-og-vurdering/kv293

Edwards, C. H. (1979). *The historical development of the calculus*. New York, NY: Springer-Verlag.

Florensa, I., Bosch, M., & Gascón, J. (2021). Question–answer maps as an epistemological tool in teacher education. *Journal of Mathematics Teacher Education*, *24*(2), 203–225. https://doi.org/10.1007/s10857-020-09454-4

Grabiner, J. (1978). The origins of Cauchy's theory of the derivative. *Historia Mathematica, 5*, 379–409.

Jessen, B. (2022). Study and research paths, ecology and in-service teachers. In Y. Chevallard, B. Barquero, M. Bosch, I. Florensa, J. Gascón, P. Nicolás, P., & N. Ruiz-Munzón, (Eds.), *Advances* *in the Anthropological Theory of the Didactic* (pp. 239–247). Cham, Switzerland: Birkhäuser. https://doi.org/10.1007/978-3-030-76791-4_18

Kalvø, T., Opdahl, J. C. L., Skrindo, K., & Weider, Ø. J. (2020). *Mønster: Matematikk 1T*. Oslo, Norway: Gyldendal.

Kalvø, T., Opdahl, J. C. L., Skrindo, K., & Weider, Ø. J. (2021). *Mønster: Matematikk R1*. Oslo, Norway: Gyldendal.

Kline, M. (1972). *Mathematical thought from ancient to modern times*. New York, NY: Oxford University Press.

Lucas, C. (2015). *Una posible razón de ser del cálculo diferencial elemental en el ámbito de la modelización funcional* [Doctoral dissertation]. Universidad de Vigo, Spain.

Lucas, C., Fonseca, C., Gascón, J., & Schneider, M. (2020). The phenomenotechnical potential of reference epistemological models: The case of elementary differential calculus. In M Bosch, Y. Chevallard, F. J. García (Eds.), *Working with the anthropological theory of the didactic in mathematics education: A comprehensive casebook* (pp. 77–97). Milton Park, UK: Routledge.

Lyche, R. T. (1930). *Begynnelsesgrunnene for differensial- og integralregningen* [Basic concepts of differential and integral calculus]. Oslo, Norway: Aschehoug.

MA3061. (n.d.). *Design and analysis of mathematics teaching*. https://www.ntnu.edu/studies/courses/MA3061#tab=omEmnet

MLREAL. (n.d.). *Natural science with teacher education*. https://www.ntnu.edu/studies/mlreal

Oldervoll, T., Svorstøl, O., Gustafsson, E., & Jacobsen, R. B. (2021). *Sinus R1: Matematikk* (4th ed.). Oslo, Norway: Cappelen Damm.

Oldervoll, T., Svorstøl, O., Vestergaard, B., Gustafsson, E., Osnes, E. R., Jacobsen, R. B., & Pedersen, T. (2020). *Sinus 1T: Matematikk*. Oslo, Norway: Cappelen Damm.

Winsløw, C., Matheron, Y., & Mercier, A. (2013). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics*, *83*(2), 267–284. https://doi.org/10.1007/s10649-012-9453-3

Study and research paths for teacher education within the pedagogical residence program in Brazil

Ricardo Nicasso Benito
Federal University of Sergipe, Brazil

Abstract
We aimed at investigating what proposals for activities, based on the paradigm of questioning the world, may arise when working within the Brazilian initial teacher training program: pedagogical residence. In order to do this, we designed and applied a study and research path for teacher education (SRP-TE) with a group of 24 pre-service teachers of the mathematics course at the Federal University of Sergipe (FUS) and 3 more preceptor teachers in public schools of Itabaiana / Sergipe / Brazil. At the end of the 4th SRP-TE module, graduating teachers had designed and implemented eight SRPs, distributed in four themes, for classes of elementary and high school.

Resumo
Nosso objetivo nesse trabalho foi investigar quais propostas de atividades, baseadas no paradigma de questionamento do mundo, podem surgir ao trabalharmos dentro do programa brasileiro de formação inicial de professores: residência pedagógica. Para isso, desenhamos e aplicamos um percurso de estudo e pesquisa para formação de professores (PEP-FP) com um grupo de 24 docentes em formação inicial do curso de matemática da Universidade Federal de Sergipe (UFS) e mais 3 professoras preceptoras em atividade em escolas públicas de Itabaiana/Sergipe/Brasil. Ao final do 4º módulo do PEP-FP, os professores em formação haviam desenhado oito PEP, distribuídos em quatro temas, para turmas do ensino fundamental II e ensino médio.

1. Introduction

The Pedagogical Residence Program is a Brazilian federal government action within the National Teacher Training Policy since 2012 and promotes the immersion of the licensing teachers in basic education schools under the supervision of a teacher from a higher education institution and a preceptor teacher from the school. In our case, despite the fact that the experiment with SRP-TE lasted about seven months, the

Liste des editeurs (Eds)
El paradigma del cuestionamiento del mundo en la investigación y en la enseñanza (pp. xx-yy)
VII congrès international de la TAD (Autrans, 22-26 janvier 2018)
Axe 1. *Analyse et évaluation des usages de la TAD dans la recherche et la Formation en didactique*
Axe 2. *Le paradigme du questionnement du monde et la question curriculaire*
Axe 3. *La TAD et la professionnalisation du métier d'enseignant*
Editorial, año

program continued for a total of 18 months (until March 2022) and the initial graduating teachers experienced other teaching strategies different from the study and research path (SRP).

According to Silva and Cruz (2018), the first debate on a residence program in teacher training courses in Brazil appeared in 2007. According to the authors, at that time different proposals were elaborated. However, none of them were approved until 2012 when the pedagogical residence project constituted by the coordination of improvement of higher-level personnel (CAPES) was finally implemented.

Since 2018, the pedagogical residence program has had a new proposal. Participants are pre-service teachers and can act as teachers in elementary school (6 to 14 years old) and/or high school (15 to 17 years old), always under supervision of a higher education teacher and a preceptor teacher that works at the school. In this new format, universities that are interested in participating in the program should write a project discriminating each teaching area of interest, such as pedagogy and the various bachelors (mathematics, languages, biology, geography, arts, theater, physical education, history, chemistry, physics, etc.).

Nowadays, the pedagogical residence program is in public and private universities of all Brazilian states, each scholarship student receives about 63 euros monthly (current exchange) and can remain in the program for up to 3 semiannual modules. Each module is composed of 138 hours of work/study divided by study of teaching theories and methodologies (86 hours), lesson planning (12 hours) and classroom practice/regency (40 hours).

The main objective of this work was to investigate which teaching proposals, based on the paradigm of questioning the world, can be developed by Brazilian students in mathematics course of the Federal University of Sergipe (FUS) who are also participants in the Pedagogical Residence Program, a national program for teacher training present in public and private universities from all states and in the Federal District. In view of this, our research question was Q_{RQ}: Which teaching proposals, based on the paradigm of questioning the world, can be elaborated by Brazilian teachers from the Pedagogical Residence Program?

In order to respond to Q_{RQ} we designed and experienced an SRP-TE with 24 students and three preceptor teachers. The students were attending the FUS Mathematics course at the campus Itabaiana-Brazil and had already attended at least 50% of the course by the time. All 27 participants received scholarships from the pedagogical residence program and took part in the experiment from October 2020 to May 2021, attending the five modules (M0, M1, M2, M3 and M4) of SRP-TE, according to Ruiz-Olarría (2015), Lícera (2017) and Benito (2019). In this work we are not going to deal with the last module, in which the experiments of the SRPs occurred in both elementary and high schools.

Inspired by Barquero et al. (2019), our initial question for SRP-TE was Q_{0TE}: How to teach proportionality? which allowed us to elaborate a study and research path (SRP) with the initial question Q_0: How are the sizes of T-shirts determined? which will not be shown in details in this article for space reasons.

After graduation, the future teachers have produced eight SRPs targeted at students both from elementary school, the 6th year (11 years old) and 9th year (14 years old), as well as high school, 1st year (15 years old) and 2nd year (16 years old).

This article introduces a little of the Brazilian teacher training program (pedagogical residence), a summary of what has occurred in each of the first four modules of SRP-TE, all the SRPs designed by students and our conclusions at the end of every experiment.

2. The experiment and results found

Due to the Covid-19 pandemic, all meetings were performed online through the Google Meet platform. For the first SRP-TE module, M0, after presenting the initial question "Q_{0TE}: How to teach proportionality?", students set up only three derivative questions (Table 1):

Q_{1TE}: What is in the schoolbook adopted by the school?
Q_{2TE}: Which applications could be used as problems to motivate students?
Q_{3TE}: Can we use the history of mathematics about proportionality to start the class?

Table 1. First questions after Q_{0TE}

Upon these first questions, the class and the investigating teacher discussed what themes a teacher should worry about after receiving the task of teaching a particular object of mathematics and everyone agreed to add five more questions (Table 2):

Q_{4TE}: Why is this content in the curriculum?
Q_{5TE}: Why teaching proportionality?
Q_{6TE}: How and where does this content appear in the curriculum?
Q_{7TE}: Which other content is it related to?
Q_{8TE}:How is this content introduced in didactic books? Why is it done this way?

Table 2. Second group of questions after Q_{0TE}

In order to answer these questions, participants analysed how the mathematical object in play appears in the Brazilian curricular official documents and textbooks used in public schools.

The analyses showed that curricular documents (Brasil, 2018) suggest that proportionality should be explored by solving problems involving direct and indirect proportional variation between two quantities; in troubleshooting rectangles triangles; and problems involving parallel straight lines cut by secants. Such suggestions were found in the problems presented by the textbooks that were analysed, but the problems are easily solved by the "rule of three" technique and the study of the mathematical object is reduced to the simple resolution of an equation. In addition, proportionality content does not appear connected to other objects, which allowed us to state that the reason for studying or teaching this object is quite weak, reduced to solve simple equations of incognito.

Confronted by this problem, we started the M1 module with an SRP, which future teachers have experienced as students, as a proposal for teaching proportionality in order to try to overcome the epistemological and didactic problems found in the previous module.

Our initial question was *"Q_0: How are the sizes of T-shirts determined?"*, which has already been experienced in the work of Barquero et al. (2019).

The 27 scholarship students were divided into three groups, a group for each of the project participating schools that, for data organisation purposes, we called School A, School B and School C. Thus, each school had 9 participants, out of which was one preceptor teacher who worked at school and 2 subgroups of 4 students.

The SRP about the T-shirts lasted for eight meetings and at the end future teachers concluded that there is no proportionality between the sizes of the T-shirts and that they are determined by the fashion industry.

We then moved to the M2 module and each of the six subgroups presented teaching proposals for various content, according to the class and grades chosen to work in each school, all designed according to the *paradigm of visiting works*. Through a comparison between its teaching proposals and the experienced SRP, pre-service teachers could perceive the main aspects that differentiate the *paradigm of visiting works* and the *paradigm of questioning the world* (Chevallard, 2013).

Each subgroup was divided into pairs and trios in the M3 module, and assuming the position of teachers, each group designed a SRP that was experienced, virtually, with school students. Initially graduating teachers presented their proposals for initial questions for all colleagues, which through the construction of the questions-answers's map, were able to discuss the need to improve and validate Q_0 questions based on the contents that should be worked on each case, according to the demand of each aware teacher. In Table 3 we can see the questions worked and their respective grades and schools, a total of eight experienced SRP.

School A	
Q_0: How are the sizes of T-shirts determined?	6th grade and 9th grade of elementary school (11 and 14 years old, respectively)
School B	
Q_0: How is the monthly charged value calculated in the electricity bill?	2nd grade of high school (16 years old)
Q_0: After vaccination against Covid-19, the school coordination wants to perform the internal school games. During the competition four sports modalities will be developed. What are the favorite sports of the students in the morning shift?	2nd grade of high school (16 years old)
Q_0: How are the sizes of T-shirts determined?	2nd grade of high school (16 years old)
School C	
Q_0: How are the sizes of T-shirts determined?	1st and 2nd grades of high school (15 and 16 year olds, respectively)
Q_0: What is the best investment so that in 12 months we have gotten the value to buy, each of us, a notebook?	2nd grade of high school (16 years old)

Table 3. SRP proposals by pre-service teachers

3. Final considerations

The SRP-TE designed by initial graduating teachers allowed us to conclude that, in spite of the distinct demands imposed by the different classes and schools, characteristic of the Pedagogical Residence Program, SRP-TE proved to be quite effective at evolving the *paradigm of visiting works* towards the *paradigm of questioning the world*.

We have concluded that initial graduating teachers were able to develop teaching strategies based on the *paradigm of questioning the world*, with several research themes, but many of these students presented insecurity when they thought of ministering a class based on an investigation, especially since they did not believe that basic education students would ask questions about the Q_0 question chosen, a fact that we consider to be the result of the current teaching model based on the *paradigm of visiting works*.

Finally, we look forward to further opportunities to work with the Pedagogical Residence Program so that we can implement new SRP-TE and where possible in partnership with new schools, so that the *paradigm of questioning the world* can be part of the daily routine of new pre-service teachers and new supervising teachers.

Acknowledgments

Funded by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brazil (CAPES).

References

Barquero, B., Bosch, M., & Romo, A. (2019). El uso del esquema herbartiano para analizar un REI online para la formación del profesorado de secundaria. *Educação Matemática Pesquisa*, *21*(4), 493-509. https://doi.org/10.23925/1983-3156.2019v21i4p493-509

Brasil. (2018). Base Nacional Comum Curricular. Ministério da Educação, Brasília.

Benito, R. N. (2019). *Construção de um Percurso de Estudo e Pesquisa para Formação de Professores: o ensino de cônicas*. 220p. [Doctoral dissertation, Pontifícia Universidade Católica de São Paulo], Repositorio PUCSP. https://tede2.pucsp.br/bitstream/handle/22544/2/Ricardo%20Nicasso%20Benito.pdf

Chevallard, Y. (2013). Enseñar matemáticas en la sociedad de mañana: Alegato a favor de un contraparadigma emergente. *REDIMAT-Journal of Research in Mathematics Education*, *2*(2), 161-182. https://doi.org/10.4471/redimat.2013.26

Licera, R. M. (2017). *Economía y ecología de los números reales en la Enseñanza Secundaria y la Formación del profesorado*. [Unpublished doctoral dissertation], Pontificia Universidad Católica de Valparaíso.

Ruiz-Olarría, A. (2015). *La formación matemático-didáctica del profesorado de secundaria. De las matemáticas por enseñar a las matemáticas para la enseñanza*. [Doctoral dissertation, Universidad Autónoma de Madrid], Repositorio Institucional UAM. https://repositorio.uam.es/bitstream/handle/10486/665889/ruiz_olarria_alicia.pdf?sequence=1

Silva, K. A. C. P., & Cruz, S. P. (2018). A Residência Pedagógica na formação de professores: história, hegemonia e resistências. *Momento: diálogos em educação. 27*(2), 227-247, https://doi.org/10.14295/momento.v27i2.8062

Teaching from questions: the influence of the institutional position

María Rita Otero[1,2], María Paz Gazzola[1,2] and Viviana Carolina Llanos[1,2]

rotero@niecyt.unicen.edu.ar , mpgazzola@niecyt.uicen.edu.ar , vcllanos@niecyt.unicen.edu.ar

[1] NYECYT, Universidad Nacional del Centro de la Prov de Bs. As. Tandil, Argentina.

[2] CONICET, Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

This paper analyses the interaction and use of a question that involves mathematical knowledge typical of high school, carried out by 62 mathematics teachers in service during a mathematics didactics course at the university. The Anthropological Theory of Didactics is adopted to analyse the didactic-mathematical activity of these teachers in two different institutional positions: study and teaching. The results show the relevance of the study position for the questioning paradigm and that questioning almost disappears in the teaching position.

Keywords: Teacher training, mathematics didactics, ATD.

The problem and it background

This paper integrates a series of researches which objective is to inquire about the conditions so that teachers both in training and in service can carry out teaching by questioning the world and certain characteristics didactic gestures of this kind of teaching (Otero, 2021). Such gestures are typical of an emerging teaching paradigm, described and proposed by the Anthropological Theory of Didactics (ATD) Chevallard (1999, 2013). In addition to the notion of Paradigm of Research and Questioning the World the ATD has also created and developed a model for the study of a question, called the Herbartian model, which is materialized in the Study and Research Paths (SRP) (ibid). For the moment, SRPs are the most appropriate didactic devices to teach according to the questioning paradigm. In our first investigations (Llanos, Otero, Gazzola, 2019; Otero et al., 2016) we analysed how trainee teachers developed a co-disciplinary SRP involving mathematics and physics, they were students of the last year of university Mathematics teacher education, studying the generating question Q_0: Why did the Movediza Stone of Tandil fall? The greatest difficulties encountered are related to mathematical modelling, because in the university, modelling is conceived more as an application than as a generation of new knowledge. In the researches that we have carried out with in-service teachers (Otero, Llanos, 2019, 2021; Otero, Llanos and Arlego, 2019; Otero, Llanos, and Parra, 2019) we proposed them to study and teach from an SRP about the parabolic satellite dishes. In these works, we considered the differences between the institutional position of analysing and studying an SRP and that of teaching with it. When teachers move to the second position, that is, to the organization of teaching, it is observed that the questioning attitude disappears. We also identify that teachers propose teaching only the "topics" that they directly link to the taught program in high school. In our first works, we attributed the difficulties of the in-service teachers to the habit of completely controlling the teaching environment, to avoid the uncertainty of losing the didactic medium control or handing it over to others, as occurs in an SRP (Otero and Llanos, 2019). In addition, we warn that the issues involving a codisciplinary question are considered by teachers as too far from their usual practice, which is why the research we are presenting is proposing to study a question that refers to praxeologies closer to the real high school curriculum, such as the proposal in the pastry box problem (Chappaz and Michon, 2003), related to the mathematics taught as polynomial functions and to geometric sequences, but the latter are not usually taught.

Theoretical framework

The Anthropological Theory of Didactics (ATD) (Chevallard, 1999, 2012) advocates the emergence of a new paradigm named research and questioning paradigm that has to replace the dominant and entrenched paradigm, called monumental Chevallard (2013). Monumentalism is a metaphor that the ATD constructs to describe a didactic phenomenon characterized by treating mathematical knowledge as a monument, which must be admired, preserved, conserved, visited without question, as if knowledge were immutable.

The relationship of a person *p* with an object O occurs in a social institution I, where the connection between the person and the object is characterized by the type of practices that people who occupy a certain position into, I carry out with O. The ATD is interested in the institutional relations R_I (p, O), thus the relations with an object of knowledge are different in the position of student, that of a teacher or that of a father who helps his son study, or that of the didact.

The Study and Research paths (SRPs) are devices proposed by the ATD that allows to face the phenomenon of monumentalization. The SRPs are developed from a Q_0 question, called generating because it does not admit an immediate answer, but must be reconstructed.

Methodology

The research was carried out in two cohorts of a university course in mathematics didactic, with 62 in service mathematics teachers. The course is part of the undergraduate degree in Mathematics Education. Teachers work in various regions and provinces of the country and have different training paths in non-university tertiary institutions. Although the majority work in secondary education, their professional experience is dissimilar, ranging from 2 to 36 years. In the last month of the course, the teachers were proposed to study a generating question, and then organize a hypothetical teaching based on it, involving some didactic gestures of the questioning paradigm. The goal was never for teachers to develop an SRP. The question is related to the problem "La boîte du pâtissier" by Chappaz & Michon (2003). Taking in account the teachers difficulties observed while we interacted with each of the conformed groups, we decided to "correct" each written delivery making guiding comments and requesting reformulations of the answers. Table 1 shows the sequence of tasks proposed to the teachers. The teachers responded to all the tasks mentioned there. Tasks place teachers in two relatively different institutional positions: study and teaching. In the first one, it is about studying, analysing and solving the problem individually (RI) and in a group (RG). In the second, it is requested to propose a possible organization of teaching. In the second cohort, unlike the first, no written reformulations were requested, since the changes between the reformulations were minor.

	Study				Teaching		
CC1	T1 RI	T1bis Reformulation RI	T2 RG	T2 bis Reformulation RG	T3 PCG	T3 bis Reformulation PCG	T4 PCI
CC2	T1 RI		T2 RG		T3 Proposal CG		T4 PCI

Table 1: tasks in each cohort

The problem

You have to build boxes, following the instructions in the video:

https://www.youtube.com/watch?v=gxjpF4bUdDY

1. What are the height, width and length of the boxes that are obtained if any sheet is considered? For example: how would the Volume, the base area Sb, the total perimeter, etc. be calculated?

2. How can we make nested boxes with sheets A0, A1, A2, etc.?

The teachers could decide which questions to study in greater or less depth. We expected choices guided by the program they actually teach. This would have implied that they mostly adopted the functional framework to organize teaching. Although arithmetic and geometric sequences and series are part of the argentine secondary school curriculum, in general they are not usually taught.

Praxeological model of reference

The Praxeological Reference Model (PRM) developed by the researchers contains the knowledge involved in the study of the generating question according to the context of secondary school and also the expected didactic organizations (ODs), considering how in that institutional position, teachers could solve the problem regarding teaching.

The system that is required to be studied is a rectangular box built as the video indicates. The box arises from a sheet of dimensions L and H, where L is the dimension where the folds are made. Using the folded sheet and making some geometric considerations, the relationships represented in Figure 1 could be obtained.

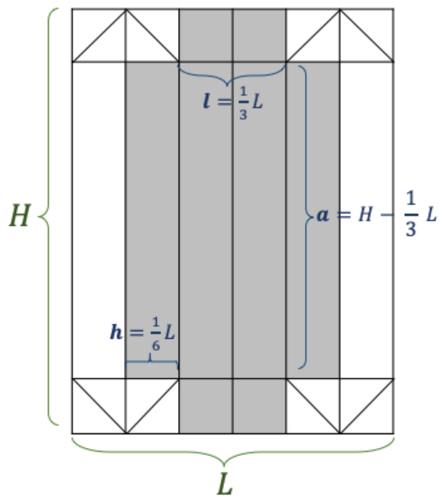


Figure 1: folded sheet

What is the length, height, and width of the box given L and H?

What surface of the base, side surface, perimeter or volume does the box have?

The height, length and width of the box could be written as follows:

$$h = \frac{1}{6}L, l = \frac{1}{3}L, a = H - \frac{1}{3}L$$

If $a > 0 \implies H > \frac{1}{3}L, L > 0$

The surface of the base:

$$S_b(L,H) = l.a = \frac{1}{3}L\left(H - \frac{1}{3}L\right) = \left(\frac{1}{3}L.H - \frac{1}{9}L^2\right)$$

The side surface:

$$S_{lat}(L,H) = \frac{1}{3}LH$$

La total surface:

$$S_{TOT}(L,H) = \frac{2}{3}LH - \frac{1}{9}L^2$$

The volume:

$$V(L,H) = \frac{1}{6}L.\frac{1}{3}L.\left(H - \frac{1}{3}L\right) = \frac{1}{18}L^2H - \frac{1}{54}L^3$$

The perimeter without cap:

$$P_{\text{without cap}}(L,H) = \frac{2}{3}L + 2H$$

The functional framework allows to emphasize the dependence of these magnitudes on the dimensions of the sheet. In all cases, the equations represent polynomial functions in two variables or polynomial equations in R^3. Considering that most of the course teachers work in secondary school, the geometric representation of surfaces in R^3, or the polynomial equations in three variables, are strange to that institution. Consequently, we suppose that it would be more feasible to study there, polynomial or rational equations in two variables, that is, in R^2.

In order to reduce the variables, it is possible to parameterize one or both sides of the sheet, or the surface, or the volume or the perimeter. In the first case, it is important to note that, if L were a parameter, all the functions to be studied will be linear, and this is too restricted and counterintuitive. For this reason, the parameter should be H.

If both sides of the sheet were parameterized, in order to build nested boxes according to the task instructions, a peculiar type of function could be studied, such as geometric sequences. If the volume or surface were parameterized, rational equations in two variables will be obtained, or hyperbolic functions of one variable, as it is displayed below.

If the surface were a parameter, a family of hyperbolic functions will be obtained, where each curve represents the boxes having the same surface, as it is drawn in Figure 2.

s= $\frac{1}{3}L.H - \frac{1}{9}L^2$ then $H(L) = \frac{L}{3} + \frac{3s}{L}, L > 0$

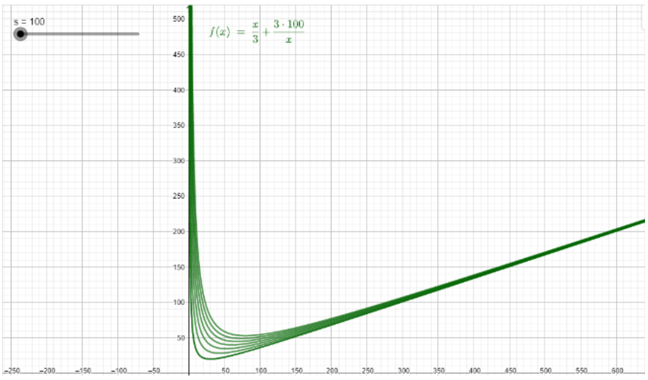


Figure 2: curves representing the boxes with the same surface

If the volume were parameterized, a family of hyperbolic functions will be obtained, where each curve represents the boxes having the same volume, as it is drawn in Figure 3.

$$H(L) = \frac{L}{3} + \frac{18\,v}{L^2},\ L > 0$$

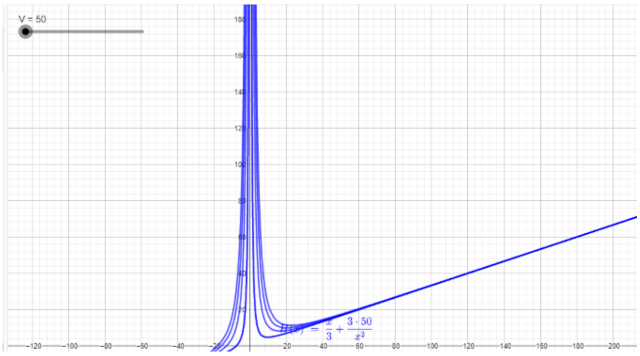


Figure 3: curves representing the boxes with the same volume

The recognition and distinction between constants, variables and parameters, as well as the relative and interchangeable nature of the latter two, becomes a considerable difficulty for teachers in training and in service. Specially, realising mathematical modelling activities and when they have to interpret the solutions of a differential equation (Otero, Llanos, 2018; Gazzola and Otero, 2021). This difficulty is the product of an extended and consolidated transposition process, which is observed even in university mathematics courses, when it is emphasized that the parameters are fixed. Furthermore, this idea prevents going beyond the first two levels of algebraic-functional modelling described by Ruiz, Bosch & Gascón (2007, 2015). The box problem seems to be appropriate showing teachers that praxeological and didactic analysis is essential to study and teach any question, because it is a fundamental tool to consider its mathematical and the didactic potential.

The nested boxes

Nested boxes can be generated with the DIN A series of sheets, defined by the ISO 216 standards (Figure 4): A given sheet has an area that is half the area of its predecessor, and successive sheets are similar rectangles, that is, their homologous sides are proportional. To build them the paper is folded by the mediatrix of the major side, obtaining thus two equal sheets of the following format. Consequently, it is easy to prove that the sides of any of these sheets retain the ratio $\frac{H}{L} = \sqrt{2}$ being H the major side.

The sides and areas of the successive sheets, the sides of the boxes obtained with them, the perimeters, the surfaces of their bases and the volumes, are respectively, geometric sequences.

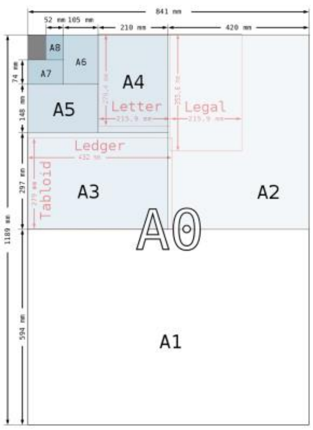


Figure 4: Sequence of the DIN A sheets

Naming α at the shorter side of A0 the sequence of the short sides of the sheets (see Figure 1) is $L_n(n) = \alpha\left(\frac{\sqrt{2}}{2}\right)^{n-1}$.

The dimensions of successive boxes are given by successions whose reason is $\frac{\sqrt{2}}{2}$:

$$h_n(n) = \frac{1}{6}\alpha\left(\frac{\sqrt{2}}{2}\right)^{n-1}, \quad l_n(n) = \frac{1}{3}\alpha\left(\frac{\sqrt{2}}{2}\right)^{n-1}, \quad a_n(n) = \left(\sqrt{2}-\frac{1}{3}\right)\alpha\left(\frac{\sqrt{2}}{2}\right)^{n-1}$$

The sequence of the base surfaces of the boxes, whose reason is $\frac{1}{2}$ could be written as:

$$Sb_n(n) = \frac{\alpha^2}{3}\left(\sqrt{2}-\frac{1}{3}\right)\left(\frac{1}{2}\right)^{n-1}$$

The sequence of the boxes volumes, whose reason is $\frac{\sqrt{2}}{4}$, could be written as:

$$V_n(n) = \frac{\alpha^3}{18}\left(\sqrt{2}-\frac{1}{3}\right)\left(\frac{\sqrt{2}}{4}\right)^{n-1}$$

The teachers' solutions

In this section we propose the categories constructed inductively from the solutions proposed by the teachers for all the tasks (Table 1). That is, firstly, tasks in which the teacher was in the position of study and had to solve the problem (T1 and T2), without being asked how or what could be taught with it and then, tasks where a teaching proposal was required (T3 and T4). The categories are:

Numerical solution: in a considerable number of responses given by teachers, they first assign numerical values to the sheets and then solve numerically. The subcategories analyse whether they do this only for the box, built with any sheet, or when they used the sheets of the DIN A series, or in both cases.

Algebraic Solution: It is considered whether teachers search for and use formulas to represent the relationships between the variables and if the solutions make explicit some functional relationship between the variables.

Geometric Sequence: the official curriculum contains the mathematical organization sequences and series, which involves arithmetic and geometric successions, but these are not usually taught. That is why, initially, very few teachers solved the task of nested boxes, relating it to geometric sequences.

Representation of the box: the importance that teachers give to the analysis of the folded sheet, both in the study and teaching position is considered. We want to know if teachers take into account the "disassembled" box to formulate the relations between the variables and if they analyse the geometric relations that justify them, when they formulate the dimensions of the box. Tables 2 and 3 show the relative frequencies of each category in the two cohorts respectively.

C1 (31 teachers)		Study				Teaching		
		T1	T1 bis	T2	T2 bis	T3	T3 bis	T4
Box representation								
0	No	0,48	0,55	0	0	0,47	0,32	0,58
1	Three-dimensional or unfolded	0,27	0,21	0,47	0,32	0,35	0,50	0,25
2	Geometric Considerations	0,24	0,24	0,53	0,68	0,18	0,18	0,17
Numerical solution								
0	No	0,48	0,27	0,15	0,00	0,47	0,18	0,42
1	Boxes	0,12	0,06	0,00	0,00	0,00	0,15	0,46
2	Nested boxes	0,27	0,45	0,71	0,85	0,53	0,68	0,04
3	Both boxes	0,12	0,21	0,15	0,15	0,00	0,00	0,08
Algebraic solution								
0	No	0,06	0,06	0,00	0,00	0,15	0,29	0,38
1	Formula	0,85	0,79	0,82	0,68	0,85	0,71	0,33
2	Functional dependency	0,09	0,15	0,18	0,32	0,00	0,00	0,29
Geometric Sequence								
0	No	0,85	0,33	0,47	0,15	0,68	0,50	0,75
1	Yes	0,15	0,67	0,53	0,85	0,32	0,50	0,25

Table 2: Relative frequencies for the tasks of the first cohort

C2 (31 teachers)		Study		Teaching	
		T1	T2	T3	T4
Box representation					
0	No	0,26	0,00	0,23	0,03
1	Three-dimensional or unfolded	0,35	0,55	0,45	0,47
2	Geometric Considerations	0,39	0,45	0,32	0,50
Numerical solution					
0	No	0,65	0,29	0,29	0,47
1	Boxes	0,13	0,00	0,42	0,23

2	Nested boxes	0,10	0,61	0,00	0,23
3	Both boxes	0,13	0,10	0,29	0,07
Algebraic solution					
0	No	0,00	0,00	0,26	0,07
1	Formula	0,94	0,84	0,58	0,77
2	Functional dependency	0,06	0,16	0,16	0,17
Geometric Sequence					
0	No	0,97	0,39	0,61	0,47
1	Yes	0,03	0,61	0,39	0,53

Table 3: Relative frequencies for the tasks of the second cohort

Discussion

Regarding the values in Table 2, in the study position, only half of the teachers in the first cohort analysed the assembled and disassembled of the box from the beginning, which is key to mathematically model the relationships of the system. Later, in the teaching position, the teachers again set aside the box as a model and used mostly numbers. For the second cohort, as shown in Table 3, the assembly and disassembly of the box played a more important role in both positions. This could be due to the observations of the teaching staff, based on the experience of the previous cohort.

If the numerical solutions of the first cohort for the study position are analysed, in the first solution and in the reformulation requested, most of the teachers solved the problem numerically either for the box, the nested boxes or both. In task referred to nested boxes, numerical solutions increased and the relative frequency was 0.85. In the second cohort, the numerical solutions for task one decreased, compared to the previous cohort, while the numerical solutions for the nested boxes increased. In the teaching position, in both cohorts, it is observed that many teachers proposed that the students obtained and justified the formulas by generalizing the operations with numbers, being this mathematically complex and didactically inappropriate.

In the algebraic solutions category, in both cohorts, and in the study position, it was observed that almost all the solutions used formulas. But in the teaching proposal, as already mentioned, teachers went back to the numbers trying to obtain the formulas from them. It was also observed that, in both positions, few teachers considered the relationships between the variables as functions. They only did so later, at the request of the course teachers. This could due to the fact that the functions in two variables do not belong to the secondary education curriculum, therefore, they are not on the teachers' radar. The notion of function is ubiquitous in the knowledge taught in secondary school, but only polynomial functions of first and second degree in a variable are taught. Usually, an encounter with the "definition" is propitiated, more specifically with the polynomial algebraic expression and its parameters. Graphical representations are made from tables and parameters are rarely varied. The parameters are described verbally from some visual characteristics of the Cartesian graph. Although algebraically, the functions of a variable are equations in R^2, only the equations of a variable are treated, setting the value of the dependent variable at zero, as a natural fact, without any discussion. By leaving aside the equations in two variables, the potentiality of the algebraic calculation to be studied is considerably reduced and the equivalence relations that allow justifying the equation solving techniques are masked. These are presented as a set of unmotivated and unjustified rules (for example, expressions such as "rules of the passage of terms" are used, even in the university courses).

In the case of the box, the teachers did not ask themselves how to reduce the variables nor did they analyse what could be studied if they fixed any of them, and much less did they consider that by parameterizing the perimeter, or the volume or the surface of the box base, even the hyperbolic functions of third degree could have been studied. That is, in relation to the PMR mentioned above, the teachers only look for the formulas, they do not analyse how to carry out the reduction to a variable, they only do it. If unconsciously they have chosen well, they could arrive teaching polynomial functions of the first and second degree in a variable and to a lesser extent third degree.

It is important to note here that we do not attribute this procedure to a limitation of the teachers' mathematical knowledge, but rather to a retrocognitive school ideology, which shuns questioning due to certain well-established ways of acting. Consequently, when in the context of a teaching professional situation the possibility of using a certain device arises (SRA, SRP, question, etc.), this is directly linked to notions of the program actually taught and to the dominant praxeologies in relation to how to teach this or that school mathematical knowledge. That is, the problem doesn’t belong to the personal dimension, nor to some individual teacher, on the contrary, it is linked to the institutional teaching praxis. In the usual professional teaching practice, modelling tasks are not performed, nor knowledge to be taught according to the paradigm of questioning the world is questioned (Chevallard, 2013). Therefore, it is neither studied nor considered necessary to analyse the system to be modelled. In other words, the praxeological and didactic analysis of knowledge to teach or teach is not a habitual activity of teachers, since according to the dominant monumental paradigm of teaching (ibid), mathematical knowledge is transparent and indeed unquestionable. Regarding the geometric sequence category, in the study position, in the first encounter with the problem, this praxeology wasn't identified as part of the solution in any cohort. However, in task two, after the interaction with the course teachers, the geometric sequences were used to solve. In the teaching position, the results of the cohorts changed. In the first cohort, geometric sequences mostly disappeared, while in the second, half of the proposals tried to teach geometric sequences, placing themselves in the secondary-university transition.

Conclusion

In this work the interaction of 62 mathematics teachers with the problem called pastry box was analysed, considering study and teaching positions, during an in-service training course, in which it was intended to promote questioning in the sense of the ATD. The study position affects the scope of the teaching position and is essential in the questioning paradigm, because studying means questioning. The results show that in the teaching position, teachers reduce mathematical knowledge to teach, then the questioning goes back. This is because questioning knowledge is not part of the usual teaching activity and on the other hand teachers are strongly influenced by institutional practices and have great difficulty in going beyond the established program.

References

Chappaz, J. et Michon, F. (2003). Il était une fois…. La boite du pâtissier. Grand N, 72, 19-32.

Chevallard, Y. (1999). El análisis de las prácticas docentes en la teoría antropológica de lo didáctico. Recherches en Didactique des Mathématiques, 19 (2), 221-266.

Chevallard, Y. (2007). Passé et présent de la théorie anthropologique du didactique. http://yves.chevallard.free.fr/.

Chevallard, Y. (2013). Enseñar Matemáticas en la Sociedad de Mañana: Alegato a Favor de un Contraparadigma Emergente. Journal of Research in Mathematics Education, 2(2), 161-182. doi: 10.4471 /redimat.2013.26.

Gazzola, M. P. y Otero, M. R. (2021). Instrumentalización de problemas escolares de los profesores de matemática en servicio. PNA, en prensa.

Llanos, V. C., Otero, M. R., & Gazzola, M. P. (2019). Physic and Mathematics models in a co-disciplinary Study and Research Paths (SRPs) in the pre-service teacher education. International Journal of Physics & Chemistry Education, 11(3), 49-57.

Otero, M. R. (2021). La Formación de Profesores. Recursos para la enseñanza por indagación y el cuestionamiento. Universidad Nacional del Centro de la Provincia de Buenos Aires. Tandil, Argentina.

Otero, M. R., Gazzola, M. P., Llanos, V. C. y Arlego, M. (2016). Co-disciplinary Physics Mathematics research and study course (RSC) within three study groups: teachers-in-training, secondary school students and researchers. Review of science, mathematics and ICT education, 10 (2), 55-78.

Otero, M. R., y Llanos, V. C. (2019). Formación de profesores de matemática en servicio: La organización de una enseñanza basada en preguntas. REDIMAT – Journal of Research in Mathematics Education, 8(2), 193-225. doi: 10.4471/redimat.2019.3618.

Otero, M. R. y Llanos, V. C. (2021). The Gap Between Studying a Generating Question and Planning Lessons Based on It. In: Barquero B., Florensa I., Nicolás P., Ruiz-Munzón N. (eds) Extended Abstracts Spring 2019. Trends in Mathematics, vol 13. Birkhäuser, Cham. https://doi.org/10.1007/978-3-030-76413-5_8.

Otero, M. R.; Llanos, V. C.; Arlego M. (2019). Mathematics and Physics Study and Research Paths within two groups of pre-service teacher education. 6 Congrès International Sur La Theorie Anthropologique Du Didactique. Autrans-Grenoble. Educación Matemática Pesquisa. http://dx.doi.org/10.23925/1983-3156.2020v22i4p711-724

Otero, M. R.; Llanos, V. C.; Parra, V. Training in-service teachers: study of questions and the organization of teaching. 6e Congrès International Sur La Theorie Anthropologique Du

Didactique. Autrans-Grenoble. Educación Matemática Pesquisa. http://dx.doi.org/10.23925/1983-3156.2020v22i4p711-724

Ruiz Munzón, N.; Bosch, M.; Gascón, J. (2007). La modelización funcional con parámetros en un taller de matemáticas con Wiris. En Ruiz-Higueras L.; Estepa A.; García F. J. (eds.) Sociedad, Escuela y Matemáticas. Aportaciones de la Teoría Antropológica de lo Didáctico (TAD). Universidad de Jaen.

Ruiz Munzón, N., Bosch, M. y Gascón, J. (2015). El problema didáctico del algebra elemental: Un análisis macro-ecológico desde la teoría antropológica de lo didáctico. REDIMAT, Vol 4(2), 106-131. doi: 10.17583/redimat.2015.1386

Une AER comme implémentation du plan B de Klein pour l'intégrale

Gaëtan Planchon[1] and Thomas Hausberger[2]

[1]IMAG, Université de Montpellier & CNRS, France ; gaetan.planchon@umontpellier.fr

[2]IMAG, Université de Montpellier & CNRS, France ; thomas.hausberger@umontpellier.fr

Abstract

In the institutional context of secondary school teacher training in France, we have developed a study and research activity (A.E.R) which aims at creating links between the integral of the Lycée (upper secondary) and the theories of Riemann and Lebesgue taught at the University. In other words, it is an answer to the question of the implementation of Klein's plan B, in the case of the integral, as a strategy to overcome the second Klein discontinuity. The methodology is based on the exploitation of a praxeological model of reference for the different objects of knowledge.

Résumé

Dans le contexte institutionnel de la formation des enseignants du secondaire en France, nous avons élaboré une activité d'étude et de recherche (A.E.R) qui vise à créer des liens entre l'intégrale du Lycée (upper secondary) et les théories de Riemann et de Lebesgue enseignées à l'Université. En d'autres termes, il s'agit d'une réponse à la question de l'implémentation du plan B de Klein, dans le cas de l'intégrale, comme stratégie pour pallier à la seconde discontinuité de Klein. La méthodologie se fonde sur l'exploitation d'un modèle praxéologique de référence pour les différents objets de savoir.

Introduction

Dès 1908, Felix Klein a identifié un problème dans la formation des enseignants de mathématiques du premier et du second degré , en l'occurrence une double discontinuité lors de la transition du lycée à l'université, puis du retour à l'école de l'élève professeur pour enseigner. Afin d'y remédier, il entreprend dans une série d'ouvrages (Klein, 2016) de présenter les « mathématiques élémentaires d'un point de vue plus élevé » en s'appuyant sur trois grands principes : souligner les connexions entre les domaines mathématiques, montrer comment les mathématiques universitaires se rapportent aux mathématiques scolaires et relier les mathématiques aux applications, ou encore l'intuition au formalisme et à l'abstraction (Kilpatrick, 2019). Ces trois principes fondent son « plan B » pour l'enseignement des mathématiques.

Le champ de la didactique des mathématiques a émergé de ce projet dans les années soixante et des cours destinés à récapituler les connaissances dans une perspective intégratrice sont désormais la norme à la fin d'un cursus de formation. Pour autant, la seconde discontinuité semble subsister (Wasserman et al., 2018) et appelle à davantage de recherches sur le transfert des connaissances académiques en connaissances pertinentes pour les enseignants, lequel n'est pas automatique (Hoth et al.,2020).

Quelles connaissances mathématiques sont utiles à un futur enseignant ? Quels types de liens sont développés et à développer entre les connaissances universitaires et les mathématiques de l'école, dans les dispositifs de formation, pour favoriser le développement professionnel des enseignants ? Ce sont autant de débats qui font actuellement de la seconde discontinuité de Klein une question vive dans les recherches en éducation mathématique et un enjeu fort pour la profession d'enseignant et de formateur.

Dans cette direction, de nouveaux outils ont été récemment apportés (Winsløw et Grønbæk 2014 ; Winsløw 2020) en posant la problématique de la seconde discontinuité de Klein avec la théorie Anthropologique de la didactique (TAD; Chevallard, 2020). Winsløw utilise la notion de *rapport* d'un individu à un objet de savoir au sein d'une institution, en TAD. Il distingue les institutions lycée (L) et université (U), ainsi que 3 différentes *positions institutionnelles* : élève au lycée (s), étudiant à l'université (σ) et enfin enseignant au lycée (t). Un objet de savoir (dans le cas de cet article, nous prenons pour exemple l'intégrale), qui vit à travers les 2 institutions, sera noté o au lycée et ω lorsqu'il s'agit d'une théorie de l'intégration (Riemann ou Lebesgue, en lien avec la théorie générale de la mesure) enseignée à l'université. Winsløw (2014) propose alors la modélisation suivante des discontinuités :

$$R_L(s,o) \rightarrow R_U(\sigma,\omega) \rightarrow R_U{}^*(\sigma,\omega) \rightarrow R_U(\sigma,o) \rightarrow R_L(t,o)$$

où la réponse de Klein au problème du transfert consiste à établir un rapport $R_U{}^*(\sigma,\omega)$ tissant des liens entre o et ω dans l'optique du changement de position que traduit la dernière flèche. Dans une modélisation ultérieure, Winsløw (2010) note $R_U(\sigma,o\cup\omega)$ ce nouveau rapport intégrateur.

Dans un travail antérieur (Planchon et Hausberger, 2020), nous avons rédigé un problème du type de ceux proposés à l'écrit du CAPES[1] et portant sur l'intégrale du lycée dans ses liens avec la théorie de Riemann et la théorie de la mesure. Comme le stipule le Journal Officiel[2], "Les notions traitées dans ces programmes [du collège et lycée] doivent pouvoir être abordées avec un recul correspondant au niveau M1 du cycle master". Un rapport de type $R_U(\sigma,o\cup\omega)$ est donc attendu d'un futur enseignant, en France. Le travail d'ingénierie s'est inspiré de la méthodologie mise en oeuvre par Winsløw et Kondratieva (2018) : les liens sont décrits à travers des relations entre blocs de la *praxis* et du *logos* (voir Outils théoriques) relatifs aux connaissances mathématiques en jeu dans les institutions *L* et *U* (dans notre cas, les différents concepts d'intégrale). Par ailleurs, l'étude d'épistémologie historique a permis d'identifier dans les travaux de Lebesgue (1935) une axiomatique, plus élémentaire que la théorie générale de la mesure, propice à fonder la notion d'aire. Sa reprise par Perrin (2005), en renforçant toutefois le rôle des transformations géométriques dans l'esprit de l'algèbre moderne, confirme ces potentialités. Le problème de CAPES qui en est issu a été soumis à une promotion d'étudiants préparant le concours : les résultats de cette expérimentation ont montré que les étudiants,

---

[1] Certificat d'Aptitude au Professorat de l'Enseignement du Second degré. Il s'agit du principal concours de recrutement des enseignants, en France, lequel évalue la maîtrise de connaissances disciplinaires couvrant essentiellement les deux premières années d'université, ainsi que des capacités liées aux dimensions professionnelles

[2] du 8 décembre 2015, texte 8

dans leur majorité, ont réussi à se saisir de l'axiomatique dans les premières questions du problème mais qu'ils perdent de vue sa fonction lorsqu'il s'agit de s'engager dans une preuve du théorème fondamental de l'analyse en s'abstenant de lire les propriétés de l'aire sur la figure. Ceci vient confirmer la nécessité de dispositifs d'apprentissage ayant pour objectif le développement de liens entre connaissances mathématiques des institutions L et U.

Nous présentons dans cet article les premières étapes d'une seconde implémentation des idées fondant notre ingénierie sur un plan épistémologique, mais dans une toute autre modalité didactique : une *activité d'étude et de recherche* (AER, voir Outils théoriques). Nous faisons l'hypothèse que promouvoir le questionnement va favoriser la mise en regard des savoirs universitaires avec ceux du lycée. Les effets produits par cette nouvelle modalité ne pourront être discutés dans le format contraint de la présente communication. L'objectif de l'article est de fournir une analyse a priori qui met en lumière les potentialités du dispositif pour l'implémentation du plan de Klein pour l'intégrale. Cette analyse mobilise de nombreux outils de la TAD selon une méthodologie nouvelle par rapport aux travaux existants en TAD sur la seconde discontinuité de Klein.

Ces outils et la méthodologie de l'étude font l'objet des premières parties de l'article, puis nous présentons le dispositif et son analyse a priori. Nous concluons sur la portée de ces analyses et les perspectives offertes.

Outils théoriques

Le premier outil théorique de TAD utilisé est la notion de *praxéologie* (Chevallard, 2020), une unité que nous noterons P, composée d'une *praxis* Π et d'un *logos* Λ. La TAD postule que le rapport à un objet de savoir émerge des praxéologies où ce dernier intervient au niveau de la technique τ, technologie θ ou théorie Θ (les différents niveaux de la praxéologie P, qui vise la réalisation d'un type de tâches T). Ce rapport est relatif à une institution donnée et à une instance positionnelle dans l'institution.

Il s'agira donc de décrire les rapports institutionnels (voir Intoduction) $R_L(s,o)$ et $R_U(\sigma,\omega)$ à l'intégrale du Lycée o et de l'Université ω à travers des *modèles praxéologiques de référence* (MPR ; Florensa et al., 2015). Ces modèles sont des reconstructions du savoir enseigné, obtenus en considérant différents niveaux de la transposition didactique (via l'épistémologie historique, les programmes officiels, les études de manuels et documents de cours). Nous décrirons les MPR en mettant en évidence l'unification des praxéologies ponctuelles autour de technologies communes, pour constituer des *organisations mathématiques locales* (OML), elles-mêmes unifiées par le niveau théorique au sein d'organisations *régionales* (OMR). Les OMR correspondent en général aux *secteurs* et les OML aux *thèmes* qui structurent la présentation du cours. La mise en regard du cours et des exercices permet ainsi de construire des modèles qui rendent compte de l'écologie des objets de savoirs au sein des institutions.

Le second type d'outils de TAD utilisé dans cet article est relié aux paradigmes d'études en TAD. L'activité des étudiants (futurs enseignants) sera modélisée par le format de l'AER (*activité d'étude de recherche*), d'où a émergé la notion de PER (parcours d'étude et de recherche) en TAD et le paradigme du *questionnement du monde* (Chevallard, 2020). Alors que les liens entre connaissances du lycée et de l'université sont décrits à travers les blocs Π et Λ qui mobilisent ces connaissances, les processus d'apprentissage (qui visent de telles mises en relation) seront modélisés grâce aux outils des AER et PER : nous utiliserons notamment le *question-gramme* (les questions générées par le processus d'étude, décrites dans l'ordre chronologique d'apparition, et les éléments de réponses apportées) pour rendre compte de la *chronogénèse* et le *schéma Herbartien développé* (loc. cit.) pour rendre compte de la *mésogénèse* (comment les réponses apportées par l'institution, lesquelles conduisent à la visite d'éléments de praxéologies, sont déconstruites pour produire une réponse $R^{♥}$ de pair avec un enrichissement praxéologique).

Questions de recherche et méthodologie

Notre étude vise l'implémentation du plan B de Klein pour l'intégrale. Cet article rend compte d'une des premières phases du processus d'ingénierie : l'analyse a priori. La construction de l'activité se fonde sur la transformation du problème de CAPES évoqué dans l'introduction en un guide d'AER. Cette transformation a nécessité un approfondissement des MPR précédemment élaborés (notamment celui de l'Université, en théorie de la mesure, mais aussi celui du Lycée suite à un changement de programmes), de façon à préciser l'équipement praxéologique des étudiants. Ces nouveaux modèles sont présentés dans la section 4 de l'article.

L'utilisation d'outils permettant d'analyser l'évolution du milieu lors du processus d'étude (donc la chronogénèse et mésogénèse) nous a paru indispensable. En effet, les copies d'étudiants obtenues lors de la réalisation du problème de CAPES constituent des données insuffisantes pour comprendre les obstacles aux transferts des connaissances universitaires sur l'intégrale. Notre méthodologie se fonde ainsi sur une articulation fine entre la description du processus d'étude avec les outils des PER et les MPR fruits des analyses praxéologiques. Notamment, les œuvres visitées sont décrites en termes de blocs Π et Λ.

Cet article vise ainsi à répondre aux questions de recherche suivantes : *Comment décrire les liens entre connaissances mathématiques du lycée et de l'université, dans le cadre d'un processus d'étude ? Quelles sont les potentialités de l'activité que nous proposons (section Présentation du dispositif d'apprentissage) dans l'optique d'une implémentation du plan B de Klein pour l'intégrale* ? Ces deux questions sont liées ; la première est de nature méthodologique tandis que la seconde renvoie directement à l'analyse a priori de notre activité en vue de l'évaluation du dispositif d'apprentissage. La précision des réponses que nous pouvons apporter à la seconde question seront des indicateurs de la pertinence de notre réponse à la première.

Comme dernier élément méthodologique, nous introduisons de nouvelles notations pour affiner les descriptions praxéologiques des liens entre o et ω. Le modèle de Winsløw met en exergue la nécessité de créer un nouveau rapport $R_U*(\sigma,\omega)$ favorisant le transfert des connaissances avancées. Ceci nous amène à noter $P*$ une praxéologie qui trouve sa source dans le modèle praxéologique dominant de l'institution Université mais dont le travail d'ingénierie a modifié certaines composantes en vue de liens entre o et ω (lesquels apparaîtront ultérieurement dans le processus d'étude). Ces nouvelles praxéologies visent en engendrer le rapport $R_U*(\sigma,\omega)$. D'autre part, nous notons $P^{\sim}$ des praxéologies qui proviennent de l'institution Lycée mais qui se trouvent, au cours du processus de l'étude, enrichies par des éléments praxéologiques relatifs à ω. Cette dernière idée nous est venue de la lecture de Hochmuth (2022), lequel considère l'enrichissement de praxéologies élémentaires de calculus par intégration d'éléments praxéologiques tirés de mathématiques plus avancées (en analyse non linéaire).

Éléments saillants des MPR

Nous allons dans un premier temps présenter les principaux éléments du MPR correspondant à l'intégrale enseignée en classe de terminale en France, puis nous développerons un second MPR relatif à la théorie de la mesure enseignée à l'université de Montpellier, en se limitant aux principaux éléments qui serviront dans l'ingénierie.

MPR du lycée

L'intégrale au lycée figure depuis 2019 au programme de deux enseignements de niveau Terminale : en *mathématiques spécialité* et en *mathématiques complémentaires*. Au sein de ces deux programmes, la notion d'intégrale d'une fonction continue positive est définie comme aire sous la courbe, l'aire demeurant une notion intuitive renvoyant à une *praxis* introduite dès le cycle 3 du primaire : le dénombrement de carrés élémentaires constituant une surface données, après quadrillage (voir la figure 1 : extrait du programme de mathématiques spécialité).

- **Calcul intégral**

La définition de l'intégrale s'appuie sur la notion intuitive d'aire rencontrée au collège. Les élèves développent une vision graphique de l'intégrale et maîtrisent le calcul approché, en liaison avec la méthode des rectangles et le calcul exact par les primitives.

On met en regard les écritures $\int_a^b f(x)dx$ et $\sum_{i=1}^n f(x_i)\Delta x_i$.

Contenus

- Définition de l'intégrale d'une fonction continue positive définie sur un segment $[a,b]$, comme aire sous la courbe représentative de f. Notation $\int_a^b f(x)dx$.
- Théorème : si f est une fonction continue positive sur $[a,b]$, alors la fonction F_a définie sur $[a,b]$ par $F_a(x)=\int_a^x f(t)dt$ est la primitive de f qui s'annule en a.
- Sous les hypothèses du théorème, relation $\int_a^b f(x)dx = F(b) - F(a)$ où F est une primitive quelconque de f. Notation $[F(x)]_a^b$.
- Théorème : toute fonction continue sur un intervalle admet des primitives.
- Définition par les primitives de $\int_a^b f(x)dx$ lorsque f est une fonction continue de signe quelconque sur un intervalle contenant a et b.
- Linéarité, positivité et intégration des inégalités. Relation de Chasles.
- Valeur moyenne d'une fonction.
- Intégration par parties.

Figure 1 : Extrait du programme de mathématiques spécialité

Cette définition θ_{aire} génère trois praxéologies ponctuelles (calcul d'une intégrale à l'aide d'une formule d'aire, calcul de l'aire entre deux courbes, approximation d'intégrale par la méthode des rectangles). Le développement praxéologique s'accompagne d'un enrichissement du bloc théorique : la croissance de l'aire, l'invariance par symétrie sont de nouveaux éléments du *logos* qui apparaissent au fur et à mesure du développement de cette première OML dédiée à l'aire.

Le théorème fondamental de l'analyse (TFA, la dérivabilité de la fonction aire) fournit, en corollaire, le nouvel outil de calcul pour les intégrales des fonctions positives via les primitives. Sa preuve est au programme de la classe de terminale mathématiques spécialité, lequel contient une section « démonstrations » pour chaque thème. Nous notons t_L la tâche isolée « démontrer le TFA dans le cas d'une fonction continue et monotone ». De façon remarquable se produit alors un changement de perspective sur l'intégrale qui est définie, dans le cas d'une fonction continue de signe quelconque, via une primitive[3]. Le TFA devient ainsi un élément théorique relatif à trois nouvelles praxéologies (calcul d'intégrales via une primitive, intégration par partie, encadrement d'une intégrale) qui constituent une seconde OML unifiée par la nouvelle définition. Enfin, le *logos* s'enrichit encore de nouvelles propriétés de l'intégrale (linéarité et croissance).

MPR de théorie de la mesure

Pour notre étude, nous avons analysé les supports (feuilles de TD, corrigés des TD et notes de cours par un étudiant, programme de l'UE) de l'unité d'enseignement « mesure et intégration » de la L3 de mathématiques de l'université de Montpellier, en 2020. Le plan du cours suggère un découpage en 4 secteurs au sens de la TAD : la théorie de la mesure, la théorie générale de l'intégration, mesures images et mesures produits, espaces L^p.

Les notions de tribu et de mesure sont des concepts de la théorie de la mesure en tant que théorie structuraliste. Les dialectiques concret/abstrait et particulier/général, que Hausberger (2017) a subsumé sous le nom de dialectique des objets et des structures, s'appliquent ainsi dans ce contexte. La méthode structuraliste procède par des raisonnements en termes de classes d'objets, de stabilité des propriétés des structures par des opérations sur les structures, dont des opérations ensemblistes. Nous pouvons faire l'hypothèse que de tels principes structuralistes vont guider la présentation du cours donné par l'enseignant ainsi que les types de tâches proposés.

La notion de tribu d'un ensemble, c'est-à-dire la classe sur laquelle portera la notion de mesure, est définie de manière axiomatique dans le vocabulaire de la théorie des ensembles[4]. Cette notion unifie une première OML (montrer qu'un ensemble est une tribu, montrer qu'une tribu est engendrée par une partie). Le deuxième thème abordé est celui des ensembles et fonctions mesurables. La stabilité de la structure de tribu par image directe et réciproque est étudiée. De nouveaux éléments technologiques apparaissent en fonction de la structure de l'ensemble d'arrivée des fonctions étudiées (structure d'espace vectoriel, d'anneau, topologie), ce qui permet ainsi d'établir des propriétés de stabilité des fonctions mesurables vis à vis des opérations arithmétiques sur les fonctions et du passage à la limite.

Enfin, le troisième thème est centré sur la définition axiomatique[5] d'une mesure. Cette définition engendre une praxéologie ponctuelle (montrer qu'une application donnée est une mesure) qui est illustrée sur des cas particuliers : la mesure de Dirac, de comptage, de Lebesgue sur $\mathbf{R}^n$. Les feuilles de TD nous permettent d'identifier deux autres praxéologies ponctuelles au sein de l'OML mesure : tout d'abord, $P_{M,1}$ (mesurer un ensemble pour une mesure donnée) dont la technique consiste à

---

[3] Une fonction f continue sur un segment est minorée par un réel m. La fonction f-m est positive et admet donc une primitive G d'après le TFA. La fonction G(x)+mx est alors une primitive de f.

[4] Une tribu $\mathcal{A}$ sur un ensemble X est un ensemble de partie de X tel que, $\mathcal{A}$ est non vide, stable par passage au complémentaire, stable par union dénombrable

[5] Une mesure positive sur (X,$\mathcal{A}$), où $\mathcal{A}$ est une tribu sur X, est une application μ définie sur $\mathcal{A}$ à valeurs dans R+, telle que μ(Ø)=0 et, pour toute famille dénombrable (Ai) d'éléments de $\mathcal{A}$ telle que Ai∩ Aj=Ø , μ(∪Ai)=Σμ(Ai)

identifier comment est construit l'ensemble à mesurer vis à vis des opérations ensemblistes qui caractérisent les tribus, puis à utiliser les bonnes propriétés de la mesure considérées. Un exemple de réalisation de cette tâche est illustré par la figure 2 (question 2). Notons que, dans cette tâche singulière $t_{M,1}$, l'accent est mis sur une propriété qui sera caractéristique de la mesure de Lebesgue sur **R**, l'invariance par translation. La technique consiste ici, après avoir traité le cas de x entier et x rationnel, à utiliser la propriété de densité de **Q** dans **R** et d'écrire $[0,x[$ comme l'union croissante d'une suite d'intervalle $[0,q_n[$ puis à utiliser une propriété des mesures générales, liée à la σ-additivité : la limite de la mesure d'une suite croissante d'ensemble mesurable est la mesure de l'union dénombrable.

Ensuite, la praxéologie $P_{M,2}$ (montrer une propriété d'une mesure spécifiée) figure à travers de nombreuses instances : par exemple, montrer qu'une mesure sur **R** invariante par translation est diffuse (figure 2, question 1). La technique consiste à mobiliser les propriétés de la mesure considérée, par un raisonnement direct ou par l'absurde. La technologie θ_M contient les propriétés générales des mesures (définition axiomatique, croissance, sous-additivité, limites croissante et décroissante) et la théorie Θ_M les preuves de ces propriétés qui mobilisent la théorie des ensembles. Ce qui fonde le type de tâches tient de l'application de la méthode axiomatique : tirer profit, autant que possible, du point de vue généralisateur et simplificateur offert par les structures (ici la notion de mesure).

Exercice 11. Pour tout intervalle $I \subset \mathbb{R}$ et tout $a \in \mathbb{R}$, on note $I + a = \{x + a \mid x \in I\}$. Soit μ une mesure sur $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ telle que

- $\mu([0,1[) = 1$;
- Pour tout intervalle $I \subset \mathbb{R}$ et tout $a \in \mathbb{R}$ on ait $\mu(I + a) = \mu(I)$.

Le but est de montrer que μ est la mesure de Lebesgue.

1. Montrer $\mu(\{x\}) = 0$ pour tout $x \in \mathbb{R}$. *On dit que la mesure μ est diffuse.*
2. Montrer que $\mu([0,x[) = x$ pour tout $x \in \mathbb{R}_+^*$. *On pourra commencer par le montrer pour tout rationnel $x \in \mathbb{Q}_+^*$.*
3. En déduire que $\mu = \lambda$.

Figure 2. Extrait d'une feuille de TD portant sur la théorie de la mesure

En définitive, nous modélisons le secteur de la théorie de la mesure par une OMR constituée de trois OML autour, respectivement, de la notion de tribu, de la définition des fonctions mesurables, et de la définition d'une mesure. Suivra le développement praxéologique du secteur de la théorie générale de l'intégration, qui dépasse le cadre de cet article.

Présentation du dispositif d'apprentissage

La nouvelle modalité que nous proposons pour réaliser le plan de Klein pour l'intégrale est de type AER. Les étudiants travaillent en groupe de 4, à partir d'un document support (le guide d'AER) mettant en avant trois tâches et un extrait de manuel (figure 2). Ils ont pour consigne de noter sur un fichier partagé l'ensemble des questions qu'ils se posent et les éléments de réponse qu'ils peuvent apporter. Au bout de 30 minutes (par question) où les élèves travaillent en autonomie est prévue une phase de mise en commun orchestrée par l'enseignant.

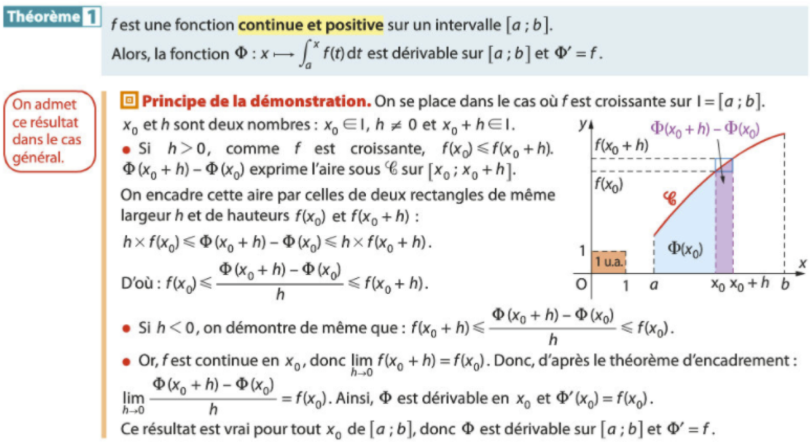
Le document pose une notion, appelée mesure des aires, définie axiomatiquement comme suit : on suppose qu'il existe un sous-ensemble Q de $\mathbf{R}^2$ qui contient les points, segments, polygones, stable par intersection et union fine. Une mesure des aires est une application μ définie sur Q à valeurs dans $\mathbf{R}^+$ simplement additive, invariante par isométrie, et telle que la mesure du carré [0,1[x[0,1[est 1.

Les trois tâches assignées sont :

t_1 : montrer que la mesure des aires est une mesure diffuse ;

t_2 : déterminer la mesure d'aire d'un rectangle en fonction de ses dimensions en justifiant ;

t_3: en appui sur la mesure des aires, réécrire la preuve du TFA extraite du manuel (Figure 3) avec la norme de rigueur de l'université.



Théorème 1 f est une fonction **continue et positive** sur un intervalle $[a\,;b]$.
Alors, la fonction $\Phi : x \longmapsto \int_a^x f(t)\,dt$ est dérivable sur $[a\,;b]$ et $\Phi' = f$.

On admet ce résultat dans le cas général.

Principe de la démonstration. On se place dans le cas où f est croissante sur $I = [a\,;b]$.
x_0 et h sont deux nombres : $x_0 \in I$, $h \neq 0$ et $x_0 + h \in I$.
- Si $h > 0$, comme f est croissante, $f(x_0) \leqslant f(x_0+h)$.
$\Phi(x_0+h) - \Phi(x_0)$ exprime l'aire sous $\mathscr{C}$ sur $[x_0\,;x_0+h]$.
On encadre cette aire par celles de deux rectangles de même largeur h et de hauteurs $f(x_0)$ et $f(x_0+h)$:
$h \times f(x_0) \leqslant \Phi(x_0+h) - \Phi(x_0) \leqslant h \times f(x_0+h)$.
D'où : $f(x_0) \leqslant \frac{\Phi(x_0+h)-\Phi(x_0)}{h} \leqslant f(x_0+h)$.
- Si $h < 0$, on démontre de même que : $f(x_0+h) \leqslant \frac{\Phi(x_0+h)-\Phi(x_0)}{h} \leqslant f(x_0)$.
- Or, f est continue en x_0, donc $\lim_{h\to 0} f(x_0+h) = f(x_0)$. Donc, d'après le théorème d'encadrement :
$\lim_{h\to 0} \frac{\Phi(x_0+h)-\Phi(x_0)}{h} = f(x_0)$. Ainsi, Φ est dérivable en x_0 et $\Phi'(x_0) = f(x_0)$.
Ce résultat est vrai pour tout x_0 de $[a\,;b]$, donc Φ est dérivable sur $[a\,;b]$ et $\Phi' = f$.

Figure 3. Preuve du TFA extraite du manuel Transmath 2012

De surcroît, le document commence par un discours de nature méta problématisant les trois tâches, dont un extrait de l'épistémologue Blanché (2009) pour éclairer le rôle des axiomatiques dans l'activité mathématique contemporaine (dégager une théorie de son substrat intuitif pour faire apparaître le schéma logique abstrait). Ce discours vient poser la question génératrice de notre AER, qui est Q_0 : « comment fonder axiomatiquement la théorie intuitive des aires sur laquelle repose l'intégrale du lycée ? »

Analyse a priori de l'AER

Nous présentons relativement à chaque tâche le question-gramme a priori décrivant la chronogénèse ainsi que le schéma herbartien explicitant la mésogénèse. Les œuvres visitées sont liées aux MPR présentés dans la section 4.

Analyse de la première tâche

La tâche t_1 pose d'emblée la question de la technique en permettant la réalisation, soit Q_1 « comment démontrer que la mesure est diffuse? ». Il s'agit de démontrer une propriété de la mesure des aires. Les ostensifs « mesure » et « diffuse » suscitent une réactivation de la praxéologie $P_{M,2}$, en particulier son instanciation $t_{M,2}$ (voir partie 5) : se pose ainsi la question de l'application de la technique $\tau_{M,2}$ dans ce nouveau contexte. L'œuvre $\Pi_{M,2}$ (*praxis* de la praxéologie $P_{M,2}$) est visitée, d'où $Q_{1,1}$:

« Quelles sont les propriétés générales des mesures qui sont encore valables dans le cadre de la mesure des aires ? » Deux sous-questions $Q_{1,2}$ et $Q_{1,3}$ émergent pour identifier respectivement les propriétés générales des mesures et les ressorts des démonstrations de ces propriétés. Ainsi, l'œuvre Λ_M (définition et propriétés générales des mesures) est visitée : la réponse $R^{\Diamond}_{1,2}$ coïncide avec l'œuvre θ_M (voir 4.2) et la réponse $R^{\Diamond}_{1,3}$ est apportée par la visite de l'œuvre Θ_M. Le cours de L_3 joue ici le rôle de média.

L'étude des preuves permet ainsi de discriminer les propriétés des mesures générales qui s'appliquent dans le contexte particulier de la mesure des aires : l'invariance par translation et la croissance de la mesure sont conservées. Ceci constitue la réponse $R_{1,1}$ à la question $Q_{1,1}$ et permet la construction d'un nouveau bloc du *logos* Λ^*_M d'une nouvelle praxéologie $P^*_{M,2}$ fusionnant la *praxis* de $P_{M,2}$ avec le nouveau *logos*, qui peut être décrite comme suit :

$T_{M,2}$: montrer une propriété d'une mesure spécifique (la mesure des aires);

$\tau_{M,2}$: raisonner à partir des propriétés de la mesure spécifiée (ici celle des aires), par un raisonnement direct ou indirect

θ^*_M: propriétés de la mesure des aires : invariance par translation, croissance

Θ^*_M: axiomatique de la mesure des aires

Le travail de la technique est ensuite laissé aux étudiants avec la tâche : montrer que la mesure des aires ne charge pas les segments de $\mathbf{R}^2$, indiquée comme devoir à la maison.

Le schéma herbartien pour la première question est donc

$$[S(X;Y;Q_1) \rightarrow M_1] \rightarrow R_1^{\heartsuit} = P^*_{M,2}$$

où $M_1 = \{Q_1, Q_{1,1}, Q_{1,2}, Q_{1,3}, ,R^{\Diamond}_{1,2} ,R^{\Diamond}_{1,3} , \Pi_{M,2}, \Lambda_M, \Lambda^*_M \}$.

L'intégralité de la praxéologie $P^*_{M,2}$ est développée au cours de l'étude à travers la construction du milieu M_1.

Analyse de la deuxième tâche

La deuxième tâche rencontrée dans l'AER pose explicitement Q_2 : « que vaut la mesure d'aire d'un rectangle en fonction de ses dimensions ? » La tâche t_2 est une instanciation de $T_{M,1}$: « mesurer un ensemble pour une mesure donnée » dont $t_{M,1}$ est une autre instanciation rencontrée par les étudiants dans le contexte de la théorie de la mesure (voir 5.2). Ainsi, la *praxis* $\Pi_{M,1}$ est la première œuvre visitée. Conformément à la réalisation de $t_{M,1}$ (et à de nombreuses praxéologies d'analyse qui s'appuient sur l'articulation entre **N**, **Q** et **R**), les nouvelles sous-question produites par le milieu consistent à démontrer la formule de l'aire d'un rectangle lorsque les dimensions sont des entiers, des rationnels, puis des réels positifs quelconques x et y.

Après les procédures de découpage qui mobilisent l'additivité pour le cas entier et rationnel, l'œuvre $O_{\mathbf{R}}$ (densité de **Q** dans **R)** est visitée et conduit à la production d'une réponse à Q_2. La densité permet de justifier l'existence de quatre suites de rationnels (x_n), (X_n) et (y_n), (Y_n) qui convergent respectivement vers x et y, avec $x_n \leq x \leq X_n$ et $y_n \leq y \leq Y_n$; les rectangles $[0, x_n]$x$[0, y_n]$ et $[0, X_n]$x$[0, Y_n]$ conviennent.

L'œuvre θ^*_M est à nouveau visitée, notamment l'invariance par isométrie et la croissance de la mesure des aires, laquelle justifie de manière cruciale le passage à la limite. On obtient la réponse $R_2^\heartsuit$: un rectangle de côtés x et y réels a pour mesure d'aire xy.

Le schéma herbartien pour la deuxième question est :

$$[S(X;Y;Q_2) \Rightarrow M_2] \Rightarrow R_2^\heartsuit$$

avec $M_2 = \{Q_2, Q_{2,1}, Q_{2,2}, Q_{2,3}, R_{2,1}, R_{2,2}, R_{2,3}, \Pi_{M,1}, \theta^*_M, O_{\mathbf{R}}\}$. La réponse $R_2^\heartsuit$ vient enrichir la technologie θ^*_M avec la mesure de l'aire des rectangles.

Analyse de la troisième tâche

La dernière tâche porte sur une réécriture de la preuve du TFA (donc la tâche t_L qui est exigible d'un élève de lycée), telle qu'elle est extraite d'un manuel scolaire (Figure 3), mais dans la norme de rigueur de l'université. Il s'agit de mobiliser le *logos* Λ^*_M construit dans la réalisation des tâches t_1 et t_2. La question posée est Q_3 : « Comment rendre rigoureuse la preuve du TFA en s'appuyant sur l'axiomatique des aires? ».

Une première sous-question, induite par la mention dans l'énoncé de propriétés éventuellement lues sur la figure, est $Q_{3,1}$: « où intervient la notion intuitive d'aire dans la preuve du manuel ? » L'ostensif « aire » figure dans la démonstration du manuel. Par ailleurs, l'ostensif « intégrale » renvoie à l'aire sous la courbe à travers l'œuvre θ_{aire} (technologie qui unifie la première OML du lycée dédiée à l'aire, voir 4.1), d'où la réponse à la question $Q_{3,1}$.

La réponse à Q_3 est alors apportée : considérons l'ensemble $\Omega_t=\{(x,y) \in \mathbf{R}^2, 0 \leq x \leq t, 0 \leq y \leq f(x) \}$ pour écrire $\Phi(x_0+h)-\Phi(x_0)=\mu(\Omega_{x_0+h} \backslash \Omega_{x_0})$ par additivité de la mesure. La croissance de f sur l'intervalle $[x_0,x_{0+h}]$ justifie l'encadrement (au sens de l'inclusion) de $\Omega_{x_0+h} \backslash \Omega_{x_0}$ par deux rectangles de largeur h, et la croissance de la mesure des aires μ, ainsi que la mesure d'aire des rectangles, donne finalement l'encadrement attendu de $\Phi(x_0+h)-\Phi(x_0)$. La définition du nombre dérivé et un calcul de limite par encadrement permet alors de conclure. La visite de l'œuvre Λ^*_M élaborée lors des deux tâches précédentes, associée à un travail de formalisation consistant à désigner les objets géométriques en jeu et à les lier à des sous-ensembles de $\mathbf{R}^2$, permet de pointer les propriétés de l'aire qui sont mobilisées en complétant la preuve, donc de produire $R^\heartsuit_3$.

Le schéma herbartien pour cette dernière question est

$$[S(X;Y;Q_3) \Rightarrow M_3] \Rightarrow R_3^\heartsuit$$

avec $M_3 = \{Q_3, Q_{3,1}, R_{3,1}, \Lambda^*_M\}$.

Conclusions et perspectives

Dans cet article, nous avons montré comment les outils de la TAD peuvent permettre d'expliciter la construction de liens entre connaissances mathématiques du lycée et de l'université lors d'un processus d'étude. En effet, l'analyse a priori de notre AER sur l'intégrale, menée avec les question-grammes (chronogénèse) et le schéma herbartien (mésogénèse), met en évidence la rencontre avec des blocs du *logos* et de la *praxis* de praxéologies mobilisant les objet o et ω lors de l'évolution du milieu de l'étude.

Cette AER favorise le développement d'une organisation mathématique de type P^* relative à l'intégrale, avec des praxéologies universitaires de théorie de la mesure qui se trouvent modifiées pour s'adapter à l'objet o. Le temps de l'étude permet la construction d'un *logos* pertinent pour o, noté Λ^*_M.

La pertinence de P^* est saisie via le développement d'une organisation mathématique de type $P^{\sim}$, où cette fois une praxéologie issue du MPR relatif à o intègre le nouveau *logos* Λ^*_M, porteur de la norme de rigueur de l'Université. Ceci correspond à la dernière partie de l'activité, autour de la tâche t_3 que l'on note désormais $t^{\sim}_L$. Bien que la tâche $t^{\sim}_L$ soit isolée, le procédé de passage de t_L à $t^{\sim}_L$ que décrit $R^{\heartsuit}_3$ a une portée générale. Le type de tâche sous-jacent est $T^{\sim}$: justifier une *praxis* Π_L relative à l'intégrale du lycée vue comme une aire, selon la norme de rigueur de l'université. La technique associée à $T^{\sim}$ consiste à identifier où intervient la notion intuitive d'aire, à formaliser, et enfin à identifier les axiomes et propriétés de la mesure des aires qui sont en jeu. Le *logos* est Λ^*_M. On peut considérer que cette praxéologie $P^{\sim}$ se doit de figurer dans l'équipement praxéologique du professeur, selon la philosophie de Klein décrite dans l'introduction. D'un point de vue méthodologique et général, notre proposition d'implémentation du plan B de Klein consiste en le développement conjugué de praxéologies du type P^* et $P^{\sim}$, afin de générer le nouveau rapport que Winsløw note $R_U(\sigma, o \cup \omega)$ (voir figure 4).

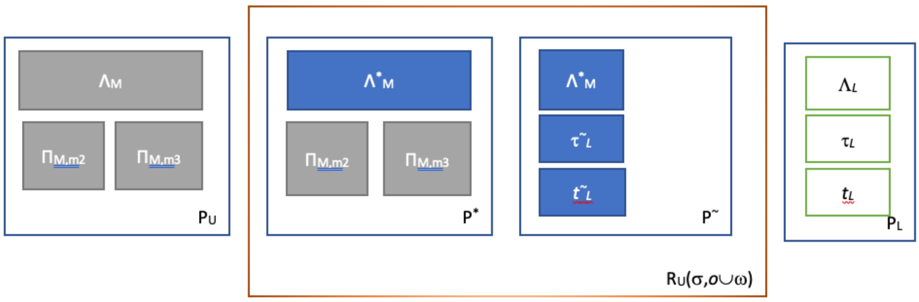


Figure 4 – Schéma de l'implémentation proposée du plan B de Klein

Pour revenir au cas de l'intégrale, la prochaine étape selon la méthodologie de l'ingénierie didactique consiste à analyser, avec les mêmes outils, les données empiriques recueillies lors de l'expérimentation avec les étudiants. La comparaison des question-grammes et des schémas herbartiens *a priori* et *a posteriori* permettra *in fine* de valider le dispositif, ou bien d'envisager de nouvelles évolutions à prévoir pour favoriser le développement des praxéologies P^* et $P^{\sim}$.

Références

Blanché, R. (2009). *L'axiomatique*. Presses Universitaires de France.

Chevallard, Y., Bosch, M. (2020). Anthropological Theory of the Didactic (ATD). In S. Lerman (Ed.), *Encyclopedia of Mathematics Education, Second Edition* (pp. 53-61). Springer.

Florensa, I., Bosch, M., & Gascón, J. (2015). The epistemological dimension in didactics: Two problematic issues. In Krainer, K., & Vondrová, N. (Eds.) *Proceedings of CERME9* (pp. 2635–2641).

Hausberger, T. (2017). La dialectique objets-structures comme cadre de référence pour une étude didactique du structuralisme algébrique. *Éducation & Didactique*, Presses Universitaires de Rennes, 2017, 11 (2), pp.131-151.

Hochmuth, R. (2022). Analysis Tasks Based on a Theorem in Nonlinear Approximation Theory. *International Journal of Mathematical Education in Science and Technology*. To appear in the INDRUM2020 Special Issue.

Hoth, J., Jeschke, C., Dreher, A. et al. (2020) Ist akademisches Fachwissen hinreichend für den Erwerb eines berufsspezifischen Fachwissens im Lehramtsstudium? Eine Untersuchung der Trickle-down-Annahme. *Journal für Mathematik Didaktik, 41*, 329–356.

Klein, F. (2016). Elementary mathematics from a higher standpoint: Arithmetic, algebra, analysis (Vol. 1, G. Schubring, Trans.). Springer. (Original work published 1933).

Kilpatrick, J. (2019). A Double Discontinuity and a Triple Approach: Felix Klein's Perspective on Mathematics Teacher Education. In H.-G. Weigand et al. (Eds.), *The Legacy of Felix Klein, ICME-13 Monographs*. Springer.

Lebesgue, H. (1935). *Sur la mesure des grandeurs. L'enseignement mathématique, Seconde édition*. Blanchard.

Planchon, G., Hausberger, T. (2020). Un problème de CAPES comme premier pas vers une implémentation du plan B de Klein pour l'intégrale. In T. Hausberger, M. Bosch & F. Chellougui (Eds.), *Proceedings of INDRUM 2020* (pp. 153-162).

Perrin, D. (2005). *Mathématiques d'école*. Cassini.

Wasserman, N., Weber, K., Villanueva, M., & Mejia-Ramos, J. P. (2018). Mathematics teachers' views about the limited utility of real analysis: A transport model hypothesis. *Journal of Mathematical Behavior*, *50*(1), 74-89.

Winsløw, C. & Grønbæk, C. (2014). Klein's double discontinuity revisited: contemporary challenges for universities preparing teachers to teach calculus. *Recherches en Didactique des Mathématiques, 34*(1), 59-86.

Winsløw, C. & Kondratieva, M. (2018). Klein's Plan B in the Early Teaching of Analysis: Two Theoretical Cases of Exploring Mathematical Links. *International Journal for Research in. Undergraduate Mathematics Education, 4*, 119-138.

Winsløw, C. (2020). Professional and academic bases of university mathematics teaching for the 21st century: the anthropological approach to practice based research. In T. Hausberger, M. Bosch & F. Chellougui (Eds.), *Proceedings of INDRUM 2020* (pp. 8-27).

Author Index