Crux Mathematicorum

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EDITORIAL

In problem solving, I am often reminded of a Russian proverb: "Do not shoot a sparrow with a cannon." The meaning is subtle. It is not just about overkill and exerting more power than needed. The fact is, you might not actually be able to accomplish the task at hand since the chosen weapon, while powerful, is simply ill-suited: a heavy awkward cannon to aim with versus a quick little bird that is fast to get away.

I thought of this saying after my recent calculus exam. Here is a part of one problem from it:

> Rainbow trout in Deer Lake can no longer reproduce due to habitat destruction, so city officials consider stocking the lake with fish and allowing locals to fish them out. As such, the fish population satisfies the differential equation $\frac{dF}{dt} = s - rF$, where $F(t)$ is the number of fish at time t (in months), s is the stocking rate (in number of fish per month) and r is the fishing rate (proportion of fish population that gets fished out every month).
>
> a) Fishing is prohibited between October 1st and March 1st, but the stocking continues at the rate of 100 fish per month (occurring always in the 2nd of a month). If there are an estimated 1500 trout in the lake on October 1st, how many fish will there be on March 1st?
>
> b) ...

My class was stumped! They all realized that they can plug in $r = 0$ but that left them with a form of a differential equation we haven't yet studied (in this course, differential equations come before antiderivatives). They pulled out just about every weapon from their differential equations ammunition: I saw phase diagrams, analysis of steady states, slope fields, ... All of that for a poor little linear growth, which in the end successfully escaped many of their attacks.

Lesson for my students and the rest of us: read and think before reaching for a bazooka. In math and otherwise.

Kseniya Garaschuk

THE CONTEST CORNER

No. 62

John McLoughlin

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'un concours mathématique de niveau secondaire ou de premier cycle universitaire, ou en ont été inspirés. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er juillet 2018**.

La rédaction souhaite remercier André Ladouceur, Ottawa, ON, d'avoir traduit les problèmes.

CC306. On considère un cube $5\times5\times5$ dont la surface extérieure est peinte en bleu. Biz coupe le cube en 5^3 cubes unités, puis il en prend un au hasard. Sachant que le cube a au moins une face bleue, quelle est la probabilité que ce cube ait exactement deux faces bleues ?

CC307. Déterminer toutes les solutions entières (x, y) de l'équation

$$x^2 - xy + 2017y = 0.$$

CC308. On définit la *matrice de Pascal* $n \times n$ comme suit : $a_{1j} = a_{i1} = 1$; $a_{ij} = a_{i-1,j} + a_{i,j-i}$ lorsque $i, j > 1$. Par exemple, la matrice de Pascal 3×3 est

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Démontrer que toute matrice de Pascal est inversible.

CC309. Soit $P(x)$ et $Q(x)$ des polynômes avec coefficients réels. Déterminer des conditions nécessaires et suffisantes sur N de manière que si le polynôme $P(Q(x))$ est de degré N, il existe une valeur réelle de x telle que $P(x) = Q(x)$.

CC310. On donne

$$\tan x + \cot x + \sec x + \csc x = 6.$$

Déterminer la valeur de

$$\sin x + \cos x.$$

CC306. Consider a $5 \times 5 \times 5$ cube with the outside surface painted blue. Buzz cuts the cube into 5^3 unit cubes, then picks a cube at random. Given that the cube Buzz picked has at least one painted blue face, what is the probability that the cube has exactly two blue faces?

CC307. Find (with proof) all integer solutions (x, y) to

$$x^2 - xy + 2017y = 0.$$

CC308. Define the $n \times n$ *Pascal matrix* as follows : $a_{1j} = a_{i1} = 1$, while $a_{ij} = a_{i-1,j} + a_{i,j-i}$ for $i, j > 1$. So, for instance, the 3×3 Pascal matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Show that every Pascal matrix is invertible.

CC309. Suppose $P(x)$ and $Q(x)$ are polynomials with real coefficients. Find necessary and sufficient conditions on N to guarantee that if the polynomial $P(Q(x))$ has degree N, there exists real x with $P(x) = Q(x)$.

CC310. Suppose

$$\tan x + \cot x + \sec x + \csc x = 6.$$

Find the value of

$$\sin x + \cos x.$$

CONTEST CORNER SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2017 : 43(2), p. 44–45.

CC256. All vertices of a polygon P lie at points with integer coordinates in the plane (that is to say, both their co-ordinates are integers), and all sides of P have integer lengths. Prove that the perimeter of P must be even.

Originally question 5 from The University of Melbourne Department of Mathematics and Statistics School Mathematics Competition, 2012 (Senior Division).

We received five correct solutions. We present the one by Steven Chow.

Let n be the number of vertices of P, and denote by (x_j, y_j) the vertices of P in clockwise order (where $1 \le j \le n$), with the additional convention that $x_{n+1} = x_1$ and $y_{n+1} = y_1$.

If a is any integer then $a \equiv a^2 \pmod 2$. Using this observation, as well as the fact that the side lengths of P are integers, the perimeter of P is

$$\begin{aligned}\sum_{j=1}^{n} \sqrt{(x_j - x_{j+1})^2 + (y_j - y_{j+1})^2} &\equiv \sum_{j=1}^{n} \left((x_j - x_{j+1})^2 + (y_j - y_{j+1})^2\right) \\ &\equiv \sum_{j=1}^{n} ((x_j - x_{j+1}) + (y_j - y_{j+1})) \pmod 2 .\end{aligned}$$

In the last line, all coordinates appear once with a positive sign and once with a negative sign, so the sum is equal to zero. Therefore, the perimeter of P is even.

CC257. It is asserted that one can find a subset S of the nonnegative integers such that every nonnegative integer can be written uniquely in the form $x + 2y$ for $x, y \in S$. Prove or disprove the assertion.

Originally question 6 from The University of Melbourne Department of Mathematics and Statistics School Mathematics Competition, 2012 (Senior Division).

We received three correct solutions. We present the solution of the Missouri State University Problem Solving Group.

We prove that there is such a set, and our construction shows there is only one such set S. The only way to write 0 as $x + 2y$ for nonnegative integers x, y is $x = y = 0$. Therefore $0 \in S$. The only way to write 1 in this form is $x = 1$, $y = 0$, and therefore $1 \in S$. We may write $2 = 2 + 2(0)$ or $2 = 0 + 2(1)$, but if $2 \in S$, then we do not have uniqueness. Therefore, $2 \notin S$. Now $3 = 3 + 2(0)$ and $3 = 1 + 2(1)$, so to ensure uniqueness we must have $3 \notin S$. Continuing in this way we see that S

must contain $0, 1, 4, 5, 16, 17, 20, 21, 64, 65$, and we notice that, other than 0, each of these numbers is a sum of powers of 4. For example, $17 = 4^2 + 4^0$, $20 = 4^2 + 4^1$, and $21 = 4^2 + 4^1 + 4^0$. Let S be the set consisting of 0 together with all possible sums of powers of 4 :

$$S = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \cdots\}$$

To see that every nonnegative integer N can be written in the form $N = x + 2y$ for x, $y \in S$, consider the binary representation of N :

$$N = \sum_{i=0}^{t} b_i 2^i \text{ with each } b_i \in \{0, 1\}.$$

Now,

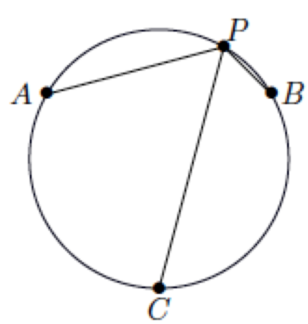
$$N = \sum_{i=0}^{\lfloor t/2 \rfloor} b_i 2^{2i} + \sum_{i=0}^{\lfloor (t-1)/2 \rfloor} b_i 2^{2i+1} = \sum_{i=0}^{\lfloor t/2 \rfloor} b_i 2^{2i} + 2 \sum_{i=0}^{\lfloor (t-1)/2 \rfloor} b_i 2^{2i}.$$

Taking

$$x = \sum_{i=0}^{\lfloor t/2 \rfloor} b_i 2^{2i} \quad \text{and} \quad y = \sum_{i=0}^{\lfloor (t-1)/2 \rfloor} b_{2i+1} 2^{2i},$$

we see that $N = x + 2y$, and $x, y \in S$. The uniqueness follows from the uniqueness of the base 2 representation.

CC258. The three points A, B and C in the diagram are vertices of an equilateral triangle. Given any point P on the circle containing A, B and C, consider the three distances AP, BP and CP. Prove that the sum of the two shorter distances gives the longer distance.



Originally question 7 from The University of Melbourne Department of Mathematics and Statistics School Mathematics Competition, 2015 (Intermediate Division).

We received 25 correct solutions, representing 12 solvers. We present the solution by Andrea Fanchini.

By Ptolemy's theorem, $AB \cdot CP = BC \cdot AP = CA \cdot BP$, but $AB = BC = CA$, so $CP = AP + BP$.

CC259. If you are told that a rectangle has area A and perimeter P, is that sufficient information to determine its side lengths?

Originally question 2 from The University of Melbourne Department of Mathematics and Statistics School Mathematics Competition, 2013 (Senior Division).

We received ten correct solutions. We present an amalgamation of many.

Let x and y be the side lengths of the rectangle. Then $A = xy$ and $P = 2(x + y)$ so that $y = \frac{P}{2} - x$. Plugging into the area

$$A = x\left(\frac{P}{2} - x\right) \iff x^2 - \frac{P}{2}x + A = 0.$$

Using the quadratic formula, we obtain

$$x = \frac{P}{4} \pm \frac{\sqrt{P^2 - 16A}}{4} \quad \text{and} \quad y = \frac{P}{4} \mp \frac{\sqrt{P^2 - 16A}}{4}.$$

Since

$$P^2 - 16A = 4(x + y)^2 - 16xy = 4(x - y)^2 \geq 0,$$

there is always a real solution to our quadratic equation, and furthermore, there is only one solution for each A and P, once we account for rotation. Hence, the side lengths are uniquely determined as a pair from the specified area and perimeter of a rectangle.

CC260. Assume you have a 9-faced die, appropriately constructed so that when the die is thrown, each of the faces (which are numbered 1 to 9) occurs with equal probability. Determine the probability that after n throws of the die, the product of all the numbers thrown will be divisible by 14.

Originally question 6 from The University of Melbourne Department of Mathematics and Statistics School Mathematics Competition, 2013 (Senior Division).

We received six correct solutions and one incorrect solution. We present the solution of the Missouri State University Problem Solving Group.

Let A be the set of n tosses that do not contain an even number and B the set of n tosses that do not contain 7. The product of numbers shown in n tosses is a multiple of 14 if and only if both an even number and 7 appear among the n tosses. Thus we need to to count $|A^c \cap B^c|$. Now, $|A| = 5^n$ (since each toss can be any of the five outcomes, $1, 3, 5, 7, 9$), $|B| = 8^n, |A \cap B| = 4^n$. Thus,

$$|A \cup B| = |A| + |B| - |A \cap B| = 5^n + 8^n - 4^n,$$

and

$$|A^c \cap B^c| = |(A \cup B)^c| = 9^n - |A \cup B| = 9^n - 5^n - 8^n + 4^n.$$

Thus the required probability is

$$\frac{9^n - 5^n - 8^n + 4^n}{9^n}.$$

THE OLYMPIAD CORNER

No. 360

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale. Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er juillet 2018**.

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OC366. Démontrer qu'il existe un nombre infini de triplets (a, b, c) d'entiers strictement positifs tels que a, b et c soient premiers entre eux deux à deux et $ab + c, bc + a$ et $ca + b$ soient premiers entre eux deux à deux.

OC367. Un concours de mathématiques est composé de 3 problèmes, chacun pouvant recevoir une note entière de 0 à 7. On sait qu'étant donné n'importe quels deux concurrents, il existe au plus un problème pour lequel les concurrents ont reçu la même note (par exemple, il n'y a pas deux concurrents qui ont reçu, dans l'ordre, les notes $7, 1, 2$ et $7, 1, 5$, mais il peut y avoir deux concurrents qui ont reçu, dans l'ordre, les notes $7, 1, 2$ et $7, 2, 1$). Déterminer le nombre maximal de concurrents.

OC368. Soit n un entier strictement positif. Déterminer, en fonction de n, le nombre de solutions de l'équation

$$x^2 + 2016y^2 = 2017^n.$$

OC369. Soit I le centre du cercle inscrit dans le triangle ABC. Soit D le point d'intersection de AI avec le côté BC et S le point d'intersection de AI avec le cercle circonscrit au triangle ABC $(S \neq A)$. Soit K et L les centres des cercles inscrits dans les triangles respectifs DSB et DCS. Soit P l'image de I par une réflexion par rapport à l'axe KL. Démontrer que $BP \perp CP$.

OC370. Soit deux entiers n et k tels que $n \geq k \geq 2$. Vous jouez au jeu suivant contre un génie maléfique. Le génie tient $2n$ cartes, numérotées d'un côté de 1 à n, deux cartes pour chaque valeur de i, $i = 1, \ldots, n$. Au départ, le génie aligne les cartes à l'envers dans un ordre quelconque. Vous montrez du doigt n'importe quelles k cartes. Le génie remet alors ces cartes à l'endroit. Si deux des cartes ont le même numéro, le jeu est terminé et vous avez gagné. Autrement, vous devez fermer les yeux pendant que le génie permute les k cartes choisies et les remet à l'envers. C'est ensuite votre tour à nouveau.

On dit que ce jeu est gagnable s'il existe un entier strictement positif m et une stratégie qui garantit une victoire en m tours ou moins, peu importe comment le génie joue. Pour quelles valeurs de n et de k le jeu est-il gagnable ?

. . . .

OC366. Prove that there exist infinitely many positive integer triples (a, b, c) such that a, b, c are pairwise relatively prime, and $ab + c, bc + a, ca + b$ are pairwise relatively prime.

OC367. A mathematical contest had 3 problems, each of which was given a score between 0 and 7, inclusive. It is known that, for any two contestants, there exists at most one problem in which they have obtained the same score (for example, there are no two contestants whose ordered scores are $7, 1, 2$ and $7, 1, 5$, but there might be two contestants whose ordered scores are $7, 1, 2$ and $7, 2, 1$). Find the maximum number of contestants.

OC368. Let n be a positive integer. Find the number of solutions of

$$x^2 + 2016y^2 = 2017^n$$

as a function of n.

OC369. Let I be the incenter of $\triangle ABC$. Let D be the point of intersection of AI with BC and let S be the point of intersection of AI with the circumcircle of ABC ($S \neq A$). Let K and L be incenters of $\triangle DSB$ and $\triangle DCS$. Let P be a reflection of I with respect to KL. Prove that $BP \perp CP$.

OC370. Integers n and k are given, with $n \geq k \geq 2$. You play the following game against an evil wizard. The wizard has $2n$ cards ; for each $i = 1, \ldots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order. You may repeatedly make moves of the following form : you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face-down. Then, it is your turn again.

We say this game is winnable if there exist some positive integer m and some strategy that is guaranteed to win in at most m moves, no matter how the wizard responds. For which values of n and k is the game winnable ?

OLYMPIAD SOLUTIONS

Les énoncés des problèmes dans cette section paraissent initialement dans 2016 : 42(10), p. 425–426.

OC306. Find all positive integers n such that

$$\frac{10^n}{n^3+n^2+n+1}$$

is an integer.

Originally Problem 1 of the 2015 Japan Mathematical Olympiad.

We received four correct submissions ; we present the solution by Steven Chow.

Note first that $n^3+n^2+n+1|10^n \implies (n+1)(n^2+1)|2^n5^n$, so both $n+1$ and n^2+1 are products of powers of 5 and 2. If $n+1 \equiv 0 \pmod 5$, then $n \equiv -1$ implies that $n^2+1 \equiv 2 \pmod 5$, so n^2+1 does not contain a power of 5 and hence is a power of 2. Let $n^2+1 = 2^t$, where $t \in \mathbb{N}$. Since $n^2 \equiv 0$ or $1 (\bmod 4)$, we have

$$n^2+1 \neq 0 \pmod 4 \implies t=1 \implies n^2+1=2 \implies n=1,$$

which does not satisfy the given condition,. We conclude that $n+1=2^a$ for some $a \in \mathbb{N}$, $a \geq 2$. Then

$$n^2+1=(2^a-1)^2+1=2^{2a}-2^{a+1}+2=2(2^{2a-1}-2^a+1),$$

so $2^{2a-1}-2^a+1=5^b$ for some $b \in \mathbb{N}$. Hence, $2^a(2^{a-1}-1)=5^b-1$.

Let $b=2^c d$ where $c,d \in \mathbb{Z}$ where $c \geq 0$, and $d>0$ is odd. We claim that $5^b-1=(5-1)P \cdot S$, where

$$P=\prod_{j=0}^{c-1}\left(5^{2^j}+1\right) \quad \text{and} \quad S=\sum_{k=1}^{d} 5^{b-2^c \cdot k}.$$

If $c=0$, then we define $P=0$.

Note that

$$\begin{aligned}(5-1)P &= (5-1)(5+1)(5^2+1)\cdots(5^{2^{c-1}}+1)\\ &= (5^2-1)(5^2+1)\cdots(5^{2^{c-1}}+1)\\ &= \cdots = 5^{2^{c-1}}-1,\end{aligned}$$

and since $b=2^c d$,

$$\begin{aligned}S &= 5^{b-2^c}+5^{b-2^{c+1}}+\cdots+5^{b-2^c d}\\ &= \left(5^{2c}\right)^{d-1}+\left(5^{2c}\right)^{d-2}+\cdots+\left(5^{2c}\right)^0\\ &= \left(5^{2c}\right)^d-1.\end{aligned}$$

Hence,

$$2^a\left(2^{a-1}-1\right)=5^b-1=(5-1)\left(\prod_{j=0}^{c-1}\left(5^{2^j}+1\right)\right)\sum_{k=1}^{d}5^{b-2^c\cdot k}. \tag{1}$$

Since $5^{2^j}+1\equiv 2 \pmod 4$ for all j and $\sum_{k=1}^{d}5^{b-2^c\cdot k}$ is odd (therefore d is odd), by comparing the coefficients of the powers of 2 from both sides of (1), we see that $a=c+2$. Then from (1) we obtain

$$2^{c+1}-1=\left(\prod_{j=0}^{c-1}\frac{5^{2^j}+1}{2}\right)\sum_{k=1}^{d}5^{b-2^c\cdot k}\geq\frac{5^{2^{c-1}}+1}{2}>\frac{\left(2^2\right)^{2^{c-1}}}{2}=2^{2^c-1}. \tag{2}$$

But it is easy to see that for $c\geq 2$

$$2^c-1\geq c+1,\quad \text{so}\quad 2^{2^c-1}\geq 2^{c+1}>2^{c+1}-1.$$

Hence, (2) can only hold if $c=0$ or 1.

If $c=0$, then $a=c+2=2$, so $n=2^2-1=3$, and if $c=1$, then $a=c+2=3$, so $n=2^3-1=7$. It is readily verified that $n=3$ and 7 satisfy the given condition.

OC307. Several small villages are situated on the banks of a straight river. On one side, there are 20 villages in a row, and on the other there are 15 villages in a row. We would like to build bridges, each of which connects a village on the one side with a village on the other side. The bridges must not cross, and it should be possible to get from any village to any other village using only those bridges (and not any roads that might exist between villages on the same side of the river). How many different ways are there to build the bridges ?

Originally Problem 5 of the 2015 South Africa National Olympiad.

Two correct solutions, presented below, were received.

Solution 1, by Mohammed Aassila.

We prove by induction that, more generally, if there are a villages on one side of the river and b on the other, then the answer is

$$\binom{a+b-2}{a-1}=\frac{(a+b-2)!}{(a-1)!(b-1)!}.$$

With $(a,b)=(20,15)$, this equals $\binom{33}{14}=818809200$.

If either $a=1$ or $b=1$, there is exactly one configuration of bridges in which a bridge runs from the single village on one side of the river to each of the others. Now suppose that $a>1$, $b>1$, the villages on one side of the river are labelled $A_1,A_2,\ldots,A_a$ and on the other side $B_1,B_2,\ldots,B_b$ in parallel order, and the induction hypothesis holds for every smaller value of $a+b$.

There cannot be at the same time a bridge joining A_1 to B_r for $r > 1$ and a bridge joining B_1 to A_s for $s > 1$, since bridges must not cross. Therefore A_1 and B_1 are joined, and either (1) B_1 is joined only to A_1 or (2) A_1 is joined only to B_1.

In case (1), we can ignore B_1 and consider all the configuration of bridges between $B_2, \ldots, B_b$ and $A_1, A_2, \ldots, A_a$. There are

$$\binom{a+b-3}{a-1}$$

of these, each corresponding to a configuration for all the villages. In case (2), we can similarly ignore A_1 and find

$$\binom{a+b-3}{a-2}$$

configurations. Thus, in all, there are

$$\binom{a+b-3}{a-1} + \binom{a+b-3}{a-2} = \binom{a+b-2}{a-1}$$

ways to build the bridges.

Solution 2, by Steven Chow.

With the notation of the first solution, for $1 \leq i \leq a$, let c_i be the number of bridges connecting A_i to a village on the other side of the river. If there are nonconsecutive integers m and n for which A_i is connected to B_m and B_n, then A_i is connected to every village B_k between B_m and B_n, for otherwise there would be no route of bridges from B_k to any other village.

A_1 must be connected to B_1, A_a to B_b, and for each $i \geq 2$, A_i to B_d, the village with the largest index connected to A_{i-1}.

The total number t of bridges can be counted in two ways. First, $t = c_1+c_2+\cdots+c_a$. Secondly, there is at least one bridge to each village B_j. Also, each village A_i $(i \geq 2)$ is connected by a bridge to an extreme B_d connected to A_{i-1}, accounting for all the additional bridges to B_d. Thus

$$t = b + (a-1) = a+b-1.$$

Furthermore, for each choice of positive integers c_i $(1 \leq i \leq a)$ for which $c_1 + c_2 + \cdots + c_a = a+b-1$, there is a suitable configuration of bridges. Such a choice can be made in

$$\binom{(a+b-1)-1}{a-1} = \binom{a+b-2}{a-1}$$

ways.

OC308. Let n be a positive integer and let $d_1, d_2, \ldots, d_k$ be its positive divisors. Consider the number

$$f(n) = (-1)^{d_1} d_1 + (-1)^{d_2} d_2 + \cdots + (-1)^{d_k} d_k.$$

Assume $f(n)$ is a power of 2. Show that if m is an integer greater than 1, then m^2 does not divide n.

Originally Problem 6 of day 2 of the 2015 Mexico National Olympiad.

There was one solution, submitted by Steven Chow, which is presented here.

If n is odd, then $f(n) = -\sigma(n) < 0$, where $\sigma(n)$ is the sum of the positive divisors of n. In this case, $f(n)$ is not a power of 2.

Let $n = 2^r s$ where $r \geq 1$ and $s = \prod p^a$, the product taken over all the odd prime divisors of n. Since the divisors of n have the form $2^u d$ where $0 \leq u \leq r$ and d divides s,

$$f(n) = (2^r + 2^{r-1} + \cdots + 2 - 1)\sigma(s) = (2^{r+1} - 3) \prod \sigma(p^a).$$

Suppose that $f(n)$ is a power of 2. Then so are $2^{r+1} - 3$ (which is odd) and $\sigma(p^a) = 1 + p + \cdots + p^a$ for each odd prime divisor p of n. Thus $r = 1$, a is odd and

$$\sigma(p^a) = (1 + p)(1 + p^2 + p^4 + \cdots + p^{a-1}).$$

Since $p \geq 3$, $p^2 \equiv 1 \pmod 4$, and $1 + p^2 + p^4 + \cdots + p^{a-1}$ is a power of 2, either $a = 1$ or the number of terms in the sum is a multiple of 4 and so divisible by $1+p^2$. But, $1+p^2$, being congruent to 2, modulo 4, cannot be a power of 2. Hence $a = 1$ and $n = 2\prod p$ is a product of distinct prime factors. Therefore, n cannot have any nontrivial square divisors.

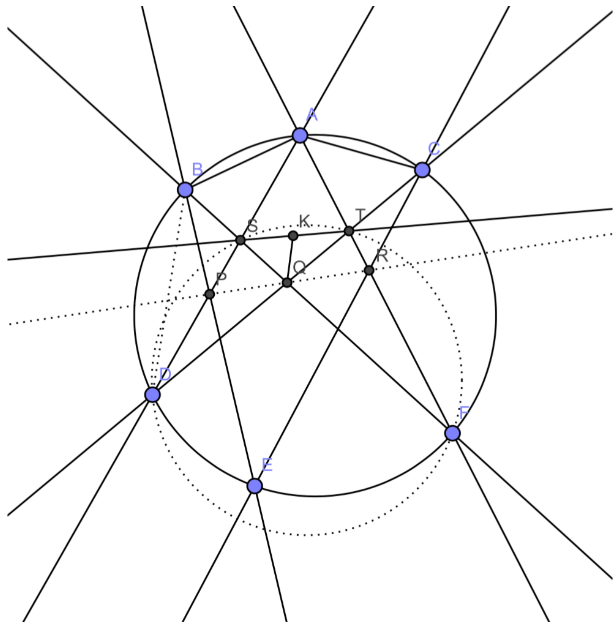
(Thus, $f(n)$ is a power of 2 if and only if n is the product of 2 and any number of distinct Mersenne primes.)

OC309. Let A, B, D, E, F, C be six points that lie on a circle (in order) and satisfy $AB = AC$. Let $P = AD \cap BE$, $R = AF \cap CE$, $Q = BF \cap CD$, $S = AD \cap BF$ and $T = AF \cap CD$. Let K be a point lying on ST satisfying $\angle QKS = \angle ECA$. Prove that

$$\frac{SK}{KT} = \frac{PQ}{QR}.$$

Originally Problem 2 of day 1 of the 2015 China National Olympiad.

The only solutions we received came from Mohammed Aassila and Steven Chow; we will present a composite of the two.



We shall be using directed angles. Since $AB = AC$, $\angle AFB = \angle CFA = \angle CDA$; thus, $\angle TFS = \angle TDS$, whence $DFTS$ is cyclic. Therefore,

$$\angle QSK = \angle FST = \angle FDT = \angle FDC = \angle FAC = \angle RAC.$$

Moreover, since we are given that $\angle SKQ = \angle ACE = \angle ACR$, $\triangle KQS \sim \triangle CRA$, which implies that $\frac{SK}{KQ} = \frac{AC}{CR}$.

Similarly, $\angle KTQ = \angle BAP$ and $\angle QKT = \angle QKS = \angle ACE = \angle ABE = \angle ABP$, so $\triangle KQT \sim \triangle BPA$, which implies that $\frac{KQ}{KT} = \frac{BP}{BA}$. Consequently,

$$\frac{SK}{KQ} \cdot \frac{KQ}{KT} = \frac{AC}{CR} \cdot \frac{BP}{AB} \left(= \frac{AC}{CR} \cdot \frac{BP}{AC} \right),$$

and we have

$$\frac{SK}{KT} = \frac{BP}{CR}. \tag{1}$$

The Sine Law applied to ΔQCR and ΔPBQ, respectively, yields

$$\frac{CR}{QR} = \frac{\sin \angle RQC}{\sin \angle QCR} \quad \text{and} \quad \frac{BP}{PQ} = \frac{\sin \angle BQP}{\sin \angle PBQ}. \tag{2}$$

Pascal's Theorem applied to the hexagon $ADCEBF$ implies that P, Q, and R are collinear, whence $\angle BQP = \angle FQR$. Furthermore,

$$\angle PBQ = \angle EBF = \angle ECF = \angle RCF,$$

so that equation (2) becomes

$$\frac{BP}{PQ} = \frac{\sin \angle FQR}{\sin \angle RCF}. \tag{3}$$

Finally, because FR bisects the angle at F in ΔCFQ, we have (by the sine version of Ceva's theorem)

$$\frac{\sin\angle RQC}{\sin\angle QCR} = \frac{\sin\angle FQR}{\sin\angle RCF}. \tag{4}$$

Putting together equations (2), (4), and (3), we have

$$\frac{CR}{QR} = \frac{BP}{PQ},$$

which together with equation (1) yields the desired conclusion, namely

$$\frac{SK}{KT} = \frac{BP}{CR} = \frac{PQ}{QR}.$$

OC310. For a positive integer k, let $n = \left(2^k\right)!$ and let $\sigma(n)$ denote the sum of all positive divisors of n. Prove that $\sigma(n)$ has at least one prime divisor larger than 2^k.

Originally Problem 8 of day 2 of the 2015 China Western Mathematical Olympiad.

We received 3 submissions, all correct. We present a composite solution based on similar proofs by Mohammed Aassila, Steven Chow, and the Missouri State University Problem Solving Group.

For any prime p and $m \in \mathbb{N}$, let $v_p(m!)$ denote the highest power of p which divides $m!$. It is well known that $v_p(m!) = \sum_{i=1}^{\infty} \left\lfloor \frac{m}{p^i} \right\rfloor$. Hence,

$$v_2(n) = v_2((2^k)!) = 2^{k-1} + 2^{k-2} + \cdots + 2 + 1 = 2^k - 1.$$

Since

$$\sigma(m) = \prod_{i=1}^{t} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}$$

when the prime power factorization of m is $m = \prod_{i=1}^{t} p_i^{\alpha_i}$, we see that $2^{2^k} - 1 | \sigma(n)$.

Since $2^{2^k} - 1 = (2^{2^{k-1}} - 1)(2^{2^{k-1}} + 1)$ we then have $2^{2^{k-1}} + 1 | \sigma(n)$.

Next, a well known theorem of Euler about Fermat numbers (see K.H. Rosen, *Elementary Number Theory and its Applications* (5th Ed.), Pearson-Addison Wesley ; Theorem 3.20, p. 128) states that all prime divisors p of $F_m = 2^{2^m} + 1$ satisfy $p \equiv 1 \pmod{2^{m+2}}$.

Hence, if p is a prime such that $p | 2^{2^{k-1}} + 1$, then $p = 2^{k+1}t + 1$ for some $t \in \mathbb{N}$. Since $p > 2^{k+1} > 2^k$ and $p | \sigma(n)$ we conclude that in fact, every prime divisor of F_{k-1} is greater than 2^k.

PROBLEM SOLVING 101

No. 3

Shawn Godin

This month, we will look at a problem from the last Canadian Open Mathematics Challenge, hosted by the CMS. You can check out the contest, and past contests on the CMS website at `cms.math.ca/Competitions/COMC`.

We will look at problem **C1** from the 2017 COMC :

> For a positive integer n, we define function $P(n)$ to be the sum of the digits of n plus the number of digits of n. For example, $P(45) = 4+5+2 = 11$. (Note that the first digit of n reading from left to right, cannot be 0).
> (a) Determine $P(2017)$.
> (b) Determine all numbers n such that $P(n) = 4$.
> (c) Determine with an explanation whether there exists a number n for which $P(n) - P(n+1) > 50$.

The problem is interesting not because it is overly difficult, but because of the precision of the argument needed. Many students were able to fully solve this problem, but many did not get full marks because of what they didn't say ...

Let's dive right in. Part (a) is a straight forward use of the definition of the function to get

$$P(2017) = 2 + 0 + 1 + 7 + 4 = 14.$$

In part (b) it gets interesting because it asks us to determine **all** numbers that have the stated property. There is an implication that not only should we find all such numbers, but we need to **prove** that there are not any more. We will proceed by defining two new functions $S(n)$ and $N(n)$ which give the sum of the digits and the number of digits of n, respectively. Then we have

$$P(n) = S(n) + N(n).$$

We are dealing with positive integers, so each n must have a non-zero digit and hence $S(n) \geq 1$. We also clearly have $N(n) \geq 1$. Since $P(n) = S(n) + N(n)$, we can write $P(n) \geq 1 + N(n)$, or $N(n) \leq P(n) - 1$. We want $P(n) = 4$ so we must have $N(n) \leq 4 - 1 = 3$. So any numbers that satisfy the given property must be at most 3 digits long. We can then look at the three cases to find the desired numbers : 3, 11, 20 and 100. Many students got these four solutions, but failed to show, or mention in any way, that there couldn't be any others.

On the other hand, when we look at the last part of the problem it is the opposite situation. There are many situations in mathematics where there are proofs that a certain thing exists, but with no indication of how to find the thing. An example would be an algorithm for producing primes. Mathematicians have long sought a easy way to predict or produce prime numbers.

Consider the following algorithm :

- start with a list of all primes that you know : $p_1, p_2, p_3, \ldots, p_n$,
- construct the number $N = p_1 \times p_2 \times p_3 \times \cdots p_n + 1$,
- clearly $p_i \nmid N$ for each $i = 1, \ldots, n$

hence either N is a prime, that was not previously on your list or N is composite. If N is composite, it factors uniquely into primes, but we saw that none of "our" primes are factors of N, so all the prime factors of N are "new".

This algorithm will work every time. Unfortunately, determining if N is prime can be difficult for large values of N. If N is composite, it is even harder to factor large numbers into their prime components. So the algorithm will generate new primes, but we may not be able to get at them easily.

On the other hand, we can show that there exists a number with a certain property by finding one. Thus, if we show

$$P(2\ 017\ 999\ 999) - P(2\ 018\ 000\ 000) = 74 - 21 = 53 > 50$$

then $n = 2\ 017\ 999\ 999$ is a number that satisfies the condition for part (c), and I am done.

It is a shame to leave such a nice problem hanging like that, so let's dig deeper and see what we can say about $P(n) - P(n+1)$. In most cases, n and $n+1$ differ by only their last digit. In that case, $P(n+1) = P(n) + 1$, so $P(n) - P(n+1) = -1$. For example $P(123) = 1 + 2 + 3 + 3 = 9$ and $P(124) = 1 + 2 + 4 + 3 = 10$, so $P(123) - P(124) = 9 - 10 = -1$.

On the other hand, if n ends in a 9, $n + 1$ will end in a 0, so there will be a large difference between $P(n)$ and $P(n+1)$. As the number of nines at the end increases, so does the difference between $P(n)$ and $P(n + 1)$:

$$\begin{aligned} P(20\ 179) - P(20\ 180) &= 8 \\ P(201\ 799) - P(201\ 800) &= 17 \\ P(2\ 017\ 999) - P(2\ 018\ 000) &= 26 \\ P(20\ 179\ 999) - P(20\ 180\ 000) &= 35 \end{aligned}$$

Upon closer inspection we see that if we look at the non-trailing nines and zeros, the sum of the remaining digits of $n + 1$ will be 1 more than the remaining digits of n. Unless n is composed solely of nines, n and $n + 1$ will have the same number of digits. So if we let n have k trailing nines, and let σ be the sum of the non-trailing nines of n and let d be its number of digits, then $P(n) = \sigma + 9k + d$ and $P(n+1) = \sigma+1+k\times 0+d = \sigma+d+1$, as long as n isn't made solely of nines. If n is totally made of nines $P(n) = 9d+d = 10d$ and $P(n+1) = 1+d\times 0+(1+d) = d+2$. Hence, if n is not made up of just nines we have

$$P(n) - P(n + 1) = \sigma + d + 9k - (\sigma + d + 1) = 9k - 1$$

and if n is made up of just nines we get

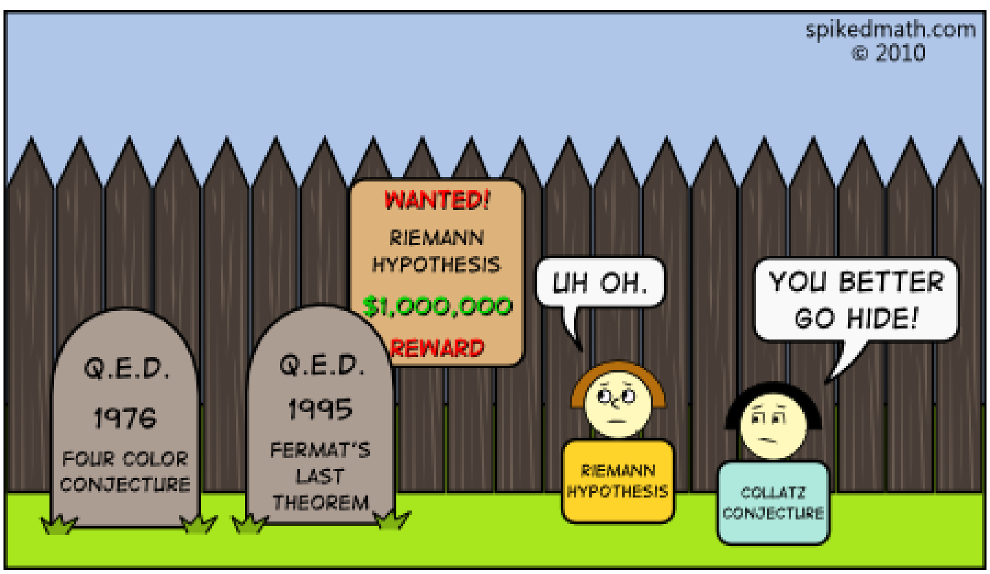
$$P(n) - P(n+1) = 10d - (d+2) = 9d - 2$$

so we can solve our problem completely by solving two inequalities

$$\begin{aligned} 9k - 1 &> 50 \\ 9k &> 51 \\ k &> \frac{51}{9} \end{aligned} \qquad\qquad \begin{aligned} 9d - 2 &> 50 \\ 9d &> 52 \\ d &> \frac{52}{9} \end{aligned}$$

which are equivalent to $k \geq 6$ and $d \geq 6$, since k and d are integers. This tells us that any number that ends in at least six nines has the desired property.

There are many things we can do with this function such as determining the largest or smallest n for which $P(n) = k$ for some k or even coming up with a way to enumerate the number of solutions to $P(n) = k$. Enjoy your explorations!



On the Centres of Root-Mean-Square Triangles

Michel Bataille

A triangle ABC with sides $BC = a, CA = b, AB = c$ is root-mean-square if the squares of its sides are in arithmetic progression, that is, if one of the equalities $2a^2 = b^2 + c^2,\ 2b^2 = c^2 + a^2,\ 2c^2 = a^2 + b^2$ holds. In such a triangle, one of the sides is the root mean square of the other two sides, a property explaining the name. Root-mean-square triangles regularly appear in geometry problems, as neatly shown by J. Chris Fisher's 2011 retrospective (**[5]**). See also **[3]** for a recent example.

We propose several characterizations of these triangles, some of them believed to be new, which involve the four most familiar centres of the triangle. Surprisingly, most of the results can be presented in pairs, according to a duality that evokes the isogonal conjugacy (see **[1]** p. 270 for details about this conjugacy).

In what follows, G, H, O, and K denote the centroid, the orthocenter, the circumcentre, and the symmedian point of the triangle ABC. Also let $\alpha = \angle BAC,\ \beta = \angle CBA,\ \gamma = \angle ACB$.

Without loss of generality, we restrict ourselves to root-mean-square triangles satisfying $2a^2 = b^2 + c^2$ that we call A-RMS triangles.

With the circles with diameters AH and AO

From now on, we denote by γ_H and γ_O the circles with diameters AH and AO, respectively. Our first characterization involves H and G :

> ABC is A-RMS if and only if G is on the circle γ_H.

Recalling that $\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\overrightarrow{OG}$, we can calculate the dot product $\overrightarrow{AG} \cdot \overrightarrow{GH}$ as follows :

$$\begin{aligned}\overrightarrow{AG} \cdot \overrightarrow{GH} &= 2\overrightarrow{AG} \cdot \overrightarrow{OG} = \frac{2}{9}(\overrightarrow{OB} + \overrightarrow{OC} - 2\overrightarrow{OA}) \cdot (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) \\ &= \frac{2}{9}(OB^2 + OC^2 - 2OA^2 + 2\overrightarrow{OB} \cdot \overrightarrow{OC} - \overrightarrow{OC} \cdot \overrightarrow{OA} - \overrightarrow{OA} \cdot \overrightarrow{OB}).\end{aligned}$$

Let $R = OA = OB = OC$ be the circumradius of ΔABC. Since $\angle BOC = 2\alpha$ if α is acute and $2(180^\circ - \alpha)$ otherwise, the dot product $\overrightarrow{OB} \cdot \overrightarrow{OC}$ is equal to $R^2 \cos 2\alpha$. Similar results hold for the dot products $\overrightarrow{OC} \cdot \overrightarrow{OA},\ \overrightarrow{OA} \cdot \overrightarrow{OB}$ and so

$$\overrightarrow{AG} \cdot \overrightarrow{GH} = \frac{2}{9}R^2(2\cos 2\alpha - \cos 2\beta - \cos 2\gamma) = \frac{1}{9}(b^2 + c^2 - 2a^2)$$

where the last equality is deduced from

$$2\cos 2\alpha - \cos 2\beta - \cos 2\gamma = 2(\sin^2\beta + \sin^2\gamma - 2\sin^2\alpha) = \frac{1}{2R^2} \cdot (b^2 + c^2 - 2a^2). \quad (1)$$

Thus, $\overrightarrow{AG}\cdot\overrightarrow{GH}=0$, that is, $\overrightarrow{AG}$ and $\overrightarrow{GH}$ are orthogonal, if and only if $b^2+c^2=2a^2$ and the result follows.

The dual theorem, formed by substituting K for G and O for H, also holds!

ABC is A-RMS if and only if K is on the circle γ_O.

Analogously, since $\sigma K = a^2A+b^2B+c^2C$ where $\sigma = a^2+b^2+c^2$, we see that

$$\begin{aligned}\sigma^2\overrightarrow{AK}\cdot\overrightarrow{OK} &= (b^2\overrightarrow{OB}+c^2\overrightarrow{OC}-(b^2+c^2)\overrightarrow{OA})\cdot(a^2\overrightarrow{OA}+b^2\overrightarrow{OB}+c^2\overrightarrow{OC})\\ &= R^2[b^2c^2(2\cos 2\alpha-\cos 2\beta-\cos 2\gamma)+m]\end{aligned}$$

where (with the help of the law of sines)

$$\begin{aligned} m &= b^2(b^2-a^2)(1-\cos 2\gamma)+c^2(c^2-a^2)(1-\cos 2\beta)\\ &= 2b^2(b^2-a^2)(\sin^2\gamma)+2c^2(c^2-a^2)(\sin^2\beta)\\ &= \frac{b^2c^2}{2R^2}(b^2+c^2-2a^2).\end{aligned}$$

Using (1) again we find that

$$\begin{aligned}\sigma^2\overrightarrow{AK}\cdot\overrightarrow{OK} &= R^2\left[b^2c^2\left(\frac{b^2+c^2-2a^2}{2R^2}\right)+\frac{b^2c^2}{2R^2}(b^2+c^2-2a^2)\right]\\ &= b^2c^2(b^2+c^2-2a^2),\end{aligned}$$

and conclude that $\overrightarrow{AK}\cdot\overrightarrow{OK}=0$ if and only if $2a^2=b^2+c^2$.

In the wake of the proofs above, let us remark that from $3\sigma G=\sigma A+\sigma B+\sigma C$ and $3\sigma K = 3a^2A+3b^2B+3c^2C$ we deduce

$$3\sigma\overrightarrow{KG}=3\sigma(G-K)=(b^2+c^2-2a^2)A+(c^2+a^2-2b^2)B+(a^2+b^2-2c^2)C.$$

As a result,

> A non-equilateral triangle ABC is A-RMS if and only if KG is parallel to BC,

a characterization that was the subject of a problem of Bankoff's in 1978 ([**2**]).

The proof of the first theorem above also shows that

> ABC is A-RMS if and only if G is on the circle γ_O.

Another way to obtain this result is to introduce the midpoints N and P of CA and AB and calculate $BP\cdots BA$ and $BG\cdots BN$. The details are left as an exercise for the reader, who is referred to Figure 3.

With the circles (BHC) and (BOC)

Before addressing another pair of characterizations, we consider two lemmas presenting interesting properties valid in any triangle.

> In an arbitrary triangle ABC, let H_1 be the orthogonal projection of H onto the median through A. Then B, H, H_1, C are concyclic.

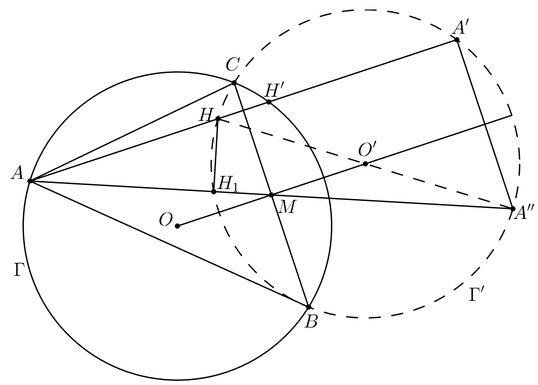


Figure 1

Let H', O' and A' be the reflections of H, O and A in the line BC, respectively. Since H' is on the circumcircle Γ of ΔABC (a well-known result), the points H and A' lie on the reflection Γ' of Γ and so does the reflection A'' of A' in the diameter OO' of Γ' (Figure 1). Now, A'' is the reflection of A in the midpoint M of BC and $\angle HA'A'' = \angle AA'A'' = 90°$, hence HA'' is a diameter of Γ'. Finally, either $H_1 = H$ or $\Delta HH_1A''$ is right-angled at H_1, hence H_1 is on Γ'.

The dual lemma holds as well :

> In an arbitrary triangle ABC, let O_1 be the orthogonal projection of O onto the symmedian through A. Then B, O, O_1, C are concyclic.

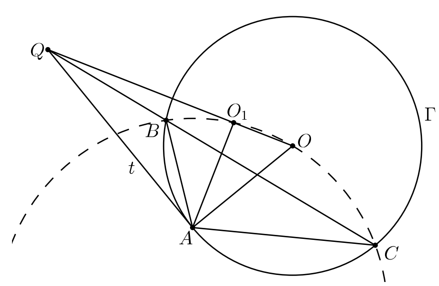


Figure 2

This results from a property of the symmedian proved in [**4**]. For convenience, we sum up the argument : Let the tangent t to Γ at A intersect BC at Q (Figure 2). The symmedian through A is the polar of Q with respect to Γ so that O_1 is the inverse of Q in Γ. Thus, O_1 is on the inverse of the line BC, that is, on the circle through B, C and O.

We can now state and prove a second pair of dual theorems.

> ABC is A-RMS if and only if G is on the circle through B, H, C.

Consider the first lemma above and figure 1. Note that the circle through B, H, C is the circle Γ' and must be so understood when $H = B$ or $H = C$. Also note that the reflection A'' of A in M is different from G and diametrically opposite to H on Γ'.

If ABC is A-RMS, then G is on γ_H, hence HG is orthogonal to the median AM. Thus $G = H_1$ and G is on the circle (BHC). Conversely, if G is on $(BHC) = \Gamma'$ (with diameter HA''), then HG is orthogonal to the median AA''. Thus, G is on γ_H and ABC is A-RMS.

Dually, we also have

> ABC is A-RMS if and only if K is on the circle through B, O, C.

The proof is similar (using the second lemma and Figure 2) and the details are left to the reader.

One more characterization

From our theorems so far, we deduce that if ABC is a scalene A-RMS triangle, then the circle γ_H and the circle (BHC) intersect at H and G while the circle γ_O and the circle (BOC) intersect at O and K (Figure 3).

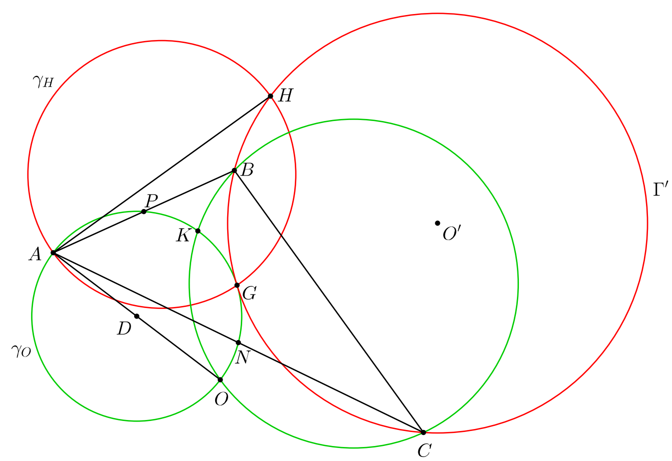


Figure 3

A close examination of Figure 3 suggests the tangency of two of the drawn circles. Indeed, we have the following theorem :

> ABC is A-RMS if and only if the circles γ_O and (BHC) are externally tangent. If this is the case, the point of tangency is G.

As before, $(BHC) = \Gamma'$ is the reflection of Γ in BC and O' denotes its centre. We first suppose that ABC is A-RMS, so that G is on Γ' and γ_O (Figure 3). Let D and P be the midpoints of AO and AB, respectively. Note that P is on γ_O whose centre is D. Consider the homothety $\mathfrak{h}$ with centre G and scale factor -2. Since

$\mathfrak{h}(O) = H$, $\mathfrak{h}(P) = C$ and $\mathfrak{h}(G) = G$, we have $\mathfrak{h}(\gamma_O) = \Gamma'$ and so $\mathfrak{h}(D) = O'$. Thus, D, G, O' are collinear with G between D and O' so that γ_O and Γ' are externally tangent at G.

Conversely, suppose that γ_O and Γ' are externally tangent at, say, T (Figure 4).

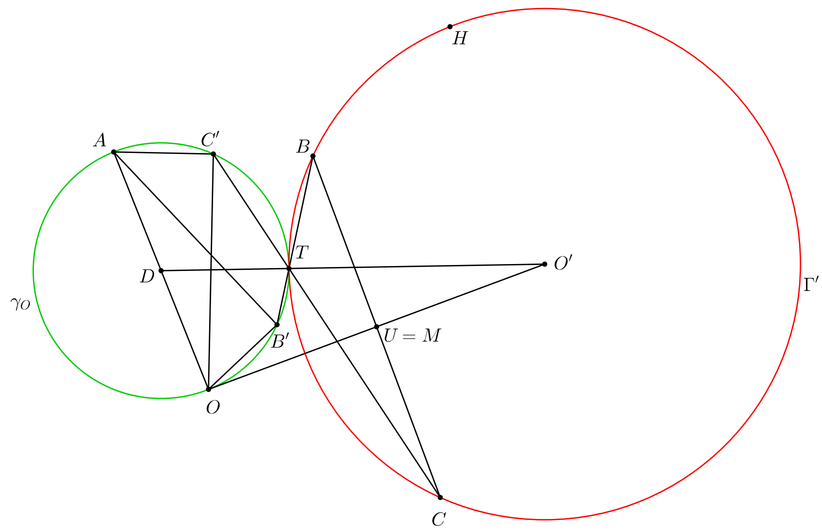


Figure 4

Let $\mathfrak{h}_T$ be the homothety with centre T and scale factor $-\frac{1}{2}$, which transforms the circle Γ' into the circle γ_O. Let $\mathfrak{h}_A$ be the homothety with centre A and scale factor 2. Clearly, $\mathfrak{h}_T(O') = D$ and $\mathfrak{h}_A(D) = O$. Let $B' = \mathfrak{h}_T(B)$, $C' = \mathfrak{h}_T(C)$ and $B'' = \mathfrak{h}_A(B'), C'' = \mathfrak{h}_A(C')$. The transformation $\mathfrak{h}_A \circ \mathfrak{h}_T$ is the symmetry S_U about the midpoint U of OO' (since $-\frac{1}{2} \times 2 = -1$ and $\mathfrak{h}_A \circ \mathfrak{h}_T(O') = O$). Observing that U is also the midpoint M of BC, we see that $B'' = \mathrm{S}_U(B) = C$ and $C'' = \mathrm{S}_U(C) = B$. Now, $C = \mathfrak{h}_A(B'), B = \mathfrak{h}_A(C')$, hence B' is the midpoint of AC and C' is the midpoint of AB. Thus, the point of intersection T of BB' and CC' coincides with the centroid G. Since G is on γ_O, a prior theorem ensures that ABC is A-RMS.

Exercises

To conclude we propose a few exercises, providing the reader with an opportunity to take another look at the results above.

1. Using the fact that $\angle CAG = \angle KAB$ (Figure 3), prove again that if ABC is A-RMS, then GK and BC are parallel.

2. Let ABC be A-RMS. The symmedians BK and CK intersect the circle γ_O at B_1 and C_1 (other than K), respectively. Prove that $OB_1 = OC_1 = \frac{Ra^2}{2bc}$.

3. Let ABC be a triangle and let the circle with diameter AG intersect AB and AC

respectively at E and F (distinct from A). The line EF intersects the perpendicular to BC through G at U. Show that ABC is A-RMS if and only if the reflection of A about U is on BC.

4. Use results of the article to prove that ABC is A-RMS if and only if $\angle GAB = \angle GBC$. (Compare with **[3]**.)

5. Let ABC be A-RMS. Prove that the circumcircles of ΔABG and ΔACG are tangent to BC and that the product of their radii equals the square of the circumradius of ΔABC.

References

[1] N. Altshiller-Court, *College Geometry*, Dover, 1980.

[2] L. Bankoff/O. Bottema, H. Charles, Problem 313, *Crux Mathematicorum*, **4** :7 (Aug.-Sept. 1978), p. 207.

[3] M. Bataille/C.R.Pranesachar, Problem 4114, *Crux Mathematicorum*, **43** :2, February 2017, p. 77.

[4] M. Bataille, Characterizing a Symmedian, *Crux Mathematicorum*, **43** :4 (April. 2017), p. 149.

[5] J.C. Fisher, Recurring Crux Configurations, *Crux Mathematicorum*, **37** :5 (Sept. 2011) 304–307.

PROBLEMS

Nous invitons les lecteurs à présenter leurs solutions, commentaires et généralisations pour n'importe quel problème présenté dans cette section. De plus, nous les encourageons à soumettre des propositions de problèmes. S'il vous plaît vous référer aux règles de soumission à l'endos de la couverture ou en ligne.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **1er juillet 2018**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4311. *Proposé par Mihaela Berindeanu.*

Soient A et B deux matrices dans $\mathfrak{M}_3(\mathbb{Z})$ telles que $AB = BA$ et $\det A = \det B = 1$. Déterminer les valeurs possibles de $\det\left(A^2 + B^2\right)$ sachant que

$$\det\left(A^2 + 2AB + 4B^2\right) - \det\left(A^2 - 2AB + 4B^2\right) = -4.$$

4312. *Proposé par William Bell.*

Démontrer que

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tanh\left(\frac{x}{2^r}\right) = \coth x - \frac{1}{x}.$$

4313. *Proposé par Marian Cucoanes and Leonard Giugiuc.*

Soit I le centre du cercle inscrit du triangle ABC, et soient H_a, H_b et H_c les orthocentres des triangles IBC, ICA et IAB, respectivement. Démontrer que les triangles ABC et $H_aH_bH_c$ ont la même surface.

4314. *Proposé par Michel Bataille.*

Soit n un entier positif. Évaluer l'expression qui suit, en forme close

$$\sum_{k=1}^{n} k2^k \cdot \frac{\binom{n}{k}}{\binom{2n-1}{k}}.$$

4315. *Proposé par Moshe Stupel, modifié par les éditeurs.*

Soit H l'orthocentre du triangle ABC et soient R, r et r' le rayon du cercle circonscrit, le rayon du cercle inscrit et le rayon du cercle exinscrit opposé au sommet A, respectivement. Démontrer que $HA + r' = 2R + r$.

4316. *Proposé par Daniel Sitaru.*

Soit $f : [0, 11] \to \mathbb{R}$ une fonction intégrable et convexe. Démontrer que

$$\int_3^5 f(x)dx + \int_6^8 f(x)dx \le \int_0^2 f(x)dx + \int_9^{11} f(x)dx.$$

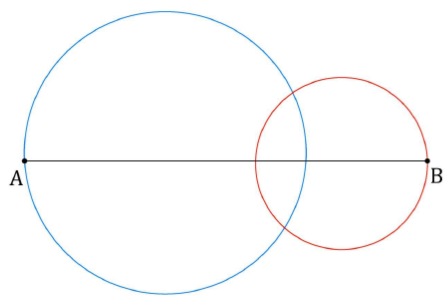
4317. *Proposé par Leonard Giugiuc.*

Soient a, b, c et d des nombres réels. Résoudre ce système d'équations

$$\begin{cases} a + b + c + d = 4, \\ abc + abd + acd + bcd = 2, \\ abcd = -\frac{1}{4}. \end{cases}$$

4318. *Proposé par Thanos Kalogerakis.*

Soient deux cercles qui intersectent, avec centres distincts, et soient A et B situés sur leur diamètre en commun, avec un point sur le premier cercle et un en dehors, et de même pour le deuxième cercle. Démontrer comment construire le mi point de AB, à l'aide d'une règle rectifiée seulement ; démontrer que votre construction est correcte.



4319. *Proposé par Marius Drăgan.*

Soient $x_1, x_2, \ldots, x_n \in (0, +\infty)$, $n \ge 2$, $\alpha \ge \dfrac{3}{2}$ tels que $x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha = 1$. Démontrer l'inégalité suivante

$$\prod_{i=1}^{n} (1 + x_i + x_i^{\alpha+1}) \le 3^\alpha.$$

4320. *Proposé par Abhay Chandra.*

Soient a, b, c et d des nombres réels positifs. Démontrer que

$$(a+b)(a+c)(a+d)(b+c)(b+d)(c+d) \ge 16\,(a+b+c+d)\,\sqrt[4]{a^5b^5c^5d^5}.$$

. . . .

4311. *Proposed by Mihaela Berindeanu.*

Let A and B be two matrices in $\mathfrak{M}_3(\mathbb{Z})$ with $AB = BA$ and $\det A = \det B = 1$. Find the possible values for $\det\left(A^2 + B^2\right)$ knowing that

$$\det\left(A^2 + 2AB + 4B^2\right) - \det\left(A^2 - 2AB + 4B^2\right) = -4.$$

4312. *Proposed by William Bell.*

Prove that

$$\sum_{r=1}^{\infty} \frac{1}{2^r} \tanh\left(\frac{x}{2^r}\right) = \coth x - \frac{1}{x}.$$

4313. *Proposed by Marian Cucoanes and Leonard Giugiuc.*

Let I be the incenter of triangle ABC, and denote by H_a, H_b and H_c the orthocenters of triangles IBC, ICA and IAB, respectively. Prove that triangles ABC and $H_aH_bH_c$ have the same area.

4314. *Proposed by Michel Bataille.*

Let n be a positive integer. Evaluate in closed form

$$\sum_{k=1}^{n} k2^k \cdot \frac{\binom{n}{k}}{\binom{2n-1}{k}}.$$

4315. *Proposed by Moshe Stupel, modified by the editors.*

Let H be the orthocenter of triangle ABC, and denote by R, r, and r' respectively the circumradius, inradius, and radius of the excircle that is opposite vertex A. Prove that $HA + r' = 2R + r$.

4316. *Proposed by Daniel Sitaru.*

Let $f : [0, 11] \to \mathbb{R}$ be an integrable and convex function. Prove that

$$\int_3^5 f(x)dx + \int_6^8 f(x)dx \leq \int_0^2 f(x)dx + \int_9^{11} f(x)dx.$$

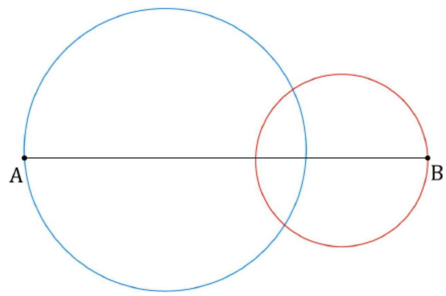
4317. *Proposed by Leonard Giugiuc.*

Solve the following system of equations over reals :

$$\begin{cases} a + b + c + d = 4, \\ abc + abd + acd + bcd = 2, \\ abcd = -\frac{1}{4}. \end{cases}$$

4318. *Proposed by Thanos Kalogerakis.*

Given a pair of intersecting circles (just their circumferences, not their centres), let AB be the common diameter with one end on each circle and neither end inside either circle. Show how to construct the midpoint of AB using only a straightedge and prove that your construction is correct.



4319. *Proposed by Marius Drăgan.*

Let $x_1, x_2, \ldots, x_n \in (0, +\infty)$, $n \geq 2$, $\alpha \geq \dfrac{3}{2}$ such that $x_1^\alpha + x_2^\alpha + \cdots + x_n^\alpha = 1$. Prove the following inequality :

$$\prod_{i=1}^{n}(1 + x_i + x_i^{\alpha+1}) \leq 3^\alpha.$$

4320. *Proposed by Abhay Chandra.*

For positive real numbers a, b, c, d, prove that

$$(a+b)(a+c)(a+d)(b+c)(b+d)(c+d) \geq 16\,(a+b+c+d)\,\sqrt[4]{a^5b^5c^5d^5}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2017 : 43(2), p. 67–71.

4211. *Proposed by Michel Bataille.*

Let A and M be two $n \times n$ matrices with complex entries such that A is invertible and M has rank 1.

a) Evaluate $\mathrm{tr}(A^{-1}M)$ if $\det(A + M) = 0$.

b) Find $(A + M)^{-1}$ if $\det(A + M) \neq 0$.

We received 6 correct solutions and will feature just one of them here, by Leonard Giugiuc.

Denote $A^{-1}M = B$. Since $\mathrm{rank}(A) = n$ and $\mathrm{rank}(M) = 1$, then $\mathrm{rank}(B) = 1$.

a) Define the polynomial $f : \mathbb{C} \longrightarrow \mathbb{C}$ as $f(x) = \det(xI_n + B),\ \forall x \in \mathbb{C}$.

Since $\mathrm{rank}(B) = 1$, it follows that $f(x) = x^{n-1}(x + \mathrm{tr}(B))$. On the other hand, we have that $\det(I_n + B) = 0$, so $f(1) = 0$, which implies $\mathrm{tr}(B) = -1$.

Note that $\det(A + M) = 0$ if and only if $\mathrm{tr}(A^{-1}M) = -1$.

b) Define the matrix C as

$$C = I_n - \frac{B}{1 + \mathrm{tr}(B)}.$$

Observe that since $\mathrm{rank}(B) = 1$, we have $B^2 = \mathrm{tr}(B) \cdot B$. Thus

$$(I_n + B)C = (I_n + B)\left(I_n - \frac{B}{1 + \mathrm{tr}(B)}\right) = I_n + B - \frac{B(1 + \mathrm{tr}(B))}{1 + \mathrm{tr}(B)} = I_n.$$

From this we get :

$$\begin{aligned}
(A + M)^{-1} &= [A(I_n + B)]^{-1} \\
&= (I_n + B)^{-1}A^{-1} \\
&= \left(I_n - \frac{B}{1 + \mathrm{tr}(B)}\right)A^{-1} \\
&= A^{-1} - \frac{A^{-1}MA^{-1}}{1 + \mathrm{tr}(A^{-1}M)}.
\end{aligned}$$

4212. *Proposed by Florin Stanescu.*

Let a, b and c be the sides of a triangle, r the inradius and R the circumradius. Show that

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}+\frac{r}{R}\leq 2.$$

We received 18 submissions, all of which are correct. Unfortunately, as pointed out by solvers Michel Bataille, and Mircea Lascu and Titu Zvonaru (jointly), the same problem by the same proposer has appeared in Math. Reflections, 2016, No. 4 as Problem S382 (with a solution published on page 10 of the solution part), as well as problem 27298 in Romanian "Gazeta Mathematica", 11/2016. However, we have decided to publish the following solution, modified slightly, by Mircea Lascu and Titu Zvonaru (who actually submitted two different proofs) since it is different from the one in Math Reflections and uses less-known results.

Let Δ and s denote the area and semiperimeter of the triangle, respectively. By Ravi's substitution ($a=y+z$, $b=z+x$, $c=x+y$, with $x,y,z>0$), we have $x+y+z=\frac{1}{2}(a+b+c)$ so $x=\frac{1}{2}(b+c-a)=s-a$, $y=\frac{1}{2}(c+a-b)=s-b$, and $z=\frac{1}{2}(a+b-c)=s-c$. Then

$$\frac{r}{R}=\frac{4(rs)^2}{4Rrs^2}=\frac{4\Delta^2}{abcs}=\frac{4s(s-a)(s-b)(s-c)}{abcs}=\frac{4xyz}{abc},$$

so the given inequality is equivalent to

$$\frac{y+z}{2x+y+z}+\frac{z+x}{x+2y+z}+\frac{x+y}{x+y+2z}+\frac{4xyz}{(x+y)(y+z)(z+x)}\leq 2. \qquad (1)$$

Let L denote the expression on the left hand side of (1). By the Engel's form (or Titu's lemma) of the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\frac{y+z}{2x+y+z}&=\frac{1}{y+z}\frac{(y+z)^2}{(x+y)+(x+z)}\\&\leq\frac{1}{y+z}\left(\frac{y^2}{x+y}+\frac{z^2}{x+z}\right)\\&=\frac{x(y^2+z^2)+yz(y+z)}{(x+y)(y+z)(z+x)}.\end{aligned}$$

It follows that

$$L\leq\frac{2(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2+2xyz)}{(x+y)(y+z)(z+x)}=2,$$

so (1) holds and the proof is complete.

4213. *Proposed by Oai Thanh Dao and Leonard Giugiuc.*

Let ABC be a triangle with no angle more than $120°$ and let I be its incentre. Consider points $D \in AI$, $E \in BI$ and $F \in CI$ such that

$$AD = \left(s - a - \frac{r}{\sqrt{3}}\right)\cos\frac{A}{2},$$
$$BE = \left(s - b - \frac{r}{\sqrt{3}}\right)\cos\frac{B}{2},$$
$$CF = \left(s - c - \frac{r}{\sqrt{3}}\right)\cos\frac{C}{2},$$

where a, b and c are sides opposite of angles A, B and C, respectively, s is the semiperimeter and r is the inradius of ABC. Prove that triangle DEF is equilateral.

The definitions of AD, BE, CF that appeared in the published statement of the problem incorrectly had angles $\frac{A}{3}$, $\frac{B}{3}$, $\frac{C}{3}$ instead of $\frac{A}{2}$, $\frac{B}{2}$, and $\frac{C}{2}$. Two of the submissions made the correction and, like the proposers, supplied a complete solution. A fourth correspondent noted the error and provided a counterexample. We present the solution by C.R. Pranesachar.

Because each of the angles A, B, C is less than $120°$,

$$s - a = \frac{r}{\tan\frac{A}{2}} > \frac{r}{\sqrt{3}}.$$

From $AI = r/\sin\frac{A}{2}$ we obtain

$$\begin{aligned} ID = AI - AD =& \frac{r}{\sin\frac{A}{2}} - \left(\frac{r}{\tan\frac{A}{2}} - \frac{r}{\sqrt{3}}\right)\cos\frac{A}{2} \\ =& \frac{r}{\sin\frac{A}{2}}\left(1 - \cos^2\frac{A}{2}\right) + \frac{r}{\sqrt{3}}\cos\frac{A}{2} \\ =& \frac{2r}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\sin\frac{A}{2} + \frac{1}{2}\cos\frac{A}{2}\right) \\ =& \frac{2r}{\sqrt{3}}\sin\left(\frac{A}{2} + \frac{\pi}{6}\right), \end{aligned}$$

with similar expressions for IE and IF. We may take $r = \frac{\sqrt{3}}{2}$ for the sake of convenience. Since $\angle EIF = \frac{\pi}{2} + \frac{A}{2}$, we get by the Cosine Rule that

$$EF^2 = \sin^2\left(\frac{B}{2} + \frac{\pi}{6}\right) + \sin^2\left(\frac{C}{2} + \frac{\pi}{6}\right) + 2\sin\left(\frac{B}{2} + \frac{\pi}{6}\right)\sin\left(\frac{C}{2} + \frac{\pi}{6}\right)\sin\frac{A}{2}.$$

Similarly

$$FD^2 = \sin^2\left(\frac{C}{2} + \frac{\pi}{6}\right) + \sin^2\left(\frac{A}{2} + \frac{\pi}{6}\right) + 2\sin\left(\frac{C}{2} + \frac{\pi}{6}\right)\sin\left(\frac{A}{2} + \frac{\pi}{6}\right)\sin\frac{B}{2}.$$

Hence,

$$
\begin{aligned}
&FD^2 - EF^2\\
&= \sin^2\left(\frac{A}{2}+\frac{\pi}{6}\right) - \sin^2\left(\frac{B}{2}+\frac{\pi}{6}\right)\\
&\quad + 2\sin\left(\frac{C}{2}+\frac{\pi}{6}\right)\cdot\left(\sin\left(\frac{A}{2}+\frac{\pi}{6}\right)\sin\frac{B}{2} - \sin\left(\frac{B}{2}+\frac{\pi}{6}\right)\sin\frac{A}{2}\right)\\
&= \frac{1-\cos\left(A+\frac{\pi}{3}\right)}{2} - \frac{1-\cos\left(B+\frac{\pi}{3}\right)}{2}\\
&\quad + 2\sin\left(\frac{C}{2}+\frac{\pi}{6}\right)\cdot\left[\left(\frac{\sqrt{3}}{2}\sin\frac{A}{2}+\frac{1}{2}\cos\frac{A}{2}\right)\sin\frac{B}{2} - \left(\frac{\sqrt{3}}{2}\sin\frac{B}{2}+\frac{1}{2}\cos\frac{B}{2}\right)\sin\frac{A}{2}\right]\\
&= \sin\left(\frac{A+B}{2}+\frac{\pi}{3}\right)\cdot\sin\left(\frac{A-B}{2}\right)\\
&\quad + 2\sin\left(\frac{C}{2}+\frac{\pi}{6}\right)\cdot\left(\frac{1}{2}\cos\frac{A}{2}\sin\frac{B}{2} - \frac{1}{2}\sin\frac{A}{2}\cos\frac{B}{2}\right)\\
&= \sin\left(\frac{A-B}{2}\right)\cdot\left(\sin\left(\frac{\pi}{2}-\frac{C}{2}+\frac{\pi}{3}\right) - \sin\left(\frac{C}{2}+\frac{\pi}{6}\right)\right)\\
&= \sin\left(\frac{A-B}{2}\right)\cdot\left(\sin\left(\frac{C}{2}+\frac{\pi}{6}\right) - \sin\left(\frac{C}{2}+\frac{\pi}{6}\right)\right)\\
&= 0.
\end{aligned}
$$

Thus $FD^2 = EF^2$ and, analogously, $EF^2 = DE^2$. Consequently, $FD = EF = DE$, and the triangle DEF is equilateral, as desired.

Alternatively, one can directly obtain the common value for the squares of the sides, namely,

$$DE^2 = EF^2 = FD^2 = \frac{r^2}{3}(3+\cos A+\cos B+\cos C+\sqrt{3}(\sin A+\sin B+\sin C)).$$

4214. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let ABC be a triangle with every angle bigger than $\frac{\pi}{6}$. Find $\min(\cos A\cos B\cos C)$.

We received 9 submissions all of which were correct. However, as pointed out by 5 of these solvers, the minimum value to be found actually does not exist unless the condition that "all angles are bigger than $\pi/6$" is replaced by "all angles are bigger than or equal to $\pi/6$." Otherwise, the value we find is $\inf(\cos A\cos B\cos C)$. *It is unclear whether the proposers were aware of this subtlety. We present the solution by Roy Barbara.*

Let $P = \cos A\cos B\cos C$. We prove that under the revised assumption that all angles are at least $\pi/6$, minimum$(P) = -3/8$. Without loss of generality, we may assume that $\pi/6 \le C \le B \le A$. If $A \le \pi/2$, then $P \ge 0 > -3/8$, so we can assume that $A > \pi/2$. Then clearly

$$\pi/6 \le C \le B < \pi/2 < A \le 2\pi/3,$$

so

$$-1/2 \leq \cos A < 0 < \cos B \leq \cos C \leq \sqrt{3}/2.$$

In particular, $|\cos A| \leq 1/2$. Hence

$$0 < |\cos A| \cos B \cos C \leq \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{8}$$

whence

$$P = \cos A \cos B \cos C = -|\cos A| \cos B \cos C \geq -3/8,$$

with equality if and only if $A = 2\pi/3$ and $B = C = \pi/6$.

4215. *Proposed by Gheorghe Alexe and George-Florin Serban.*

Find positive natural numbers a, b and c such that

$$\frac{a+1}{b}, \quad \frac{b+1}{c} \quad \text{and} \quad \frac{c+1}{a}$$

are all natural numbers.

We received 25 solutions, of which 24 were correct and complete. We present 2 solutions.

Solution 1, by Joseph DiMuro.

First, assume that two of a, b, and c are equal; without loss of generality, assume $a = b$. Then $\frac{a+1}{b} = 1 + \frac{1}{b}$ is an integer only if $a = b = 1$; if that is true, then $\frac{b+1}{c} = \frac{2}{c}$ is an integer only if $c = 1$ or 2. This gives us two solutions : $(a, b, c) = (1, 1, 1)$ or $(1, 1, 2)$.

Now, assume that a, b, and c are all distinct; without loss of generality, assume a is the smallest. For $\frac{a+1}{b}$ to be an integer, we must have $a + 1 \geq b$; since $a < b$, we must have $a + 1 = b$. The second fraction is then $\frac{a+2}{c}$; for that to be an integer, we must have $a + 2 \geq c$. Since $a < c$, we must have $c = a + 1$ or $c = a + 2$. If $c = a + 1$, then $\frac{a+2}{c} = 1 + \frac{1}{c}$; this will not be an integer, since $c > 1$. So we must have $c = a + 2$. The third fraction is then $\frac{c+1}{a} = 1 + \frac{3}{a}$, which is an integer if and only if $a = 1$ or 3. This gives us two solutions : $(a, b, c) = (1, 2, 3)$ or $(3, 4, 5)$.

Therefore, (a, b, c) must be a cyclic permutation of one of the following ordered triples : $(1, 1, 1), (1, 1, 2), (1, 2, 3),$ or $(3, 4, 5)$.

Solution 2, by Michel Bataille.

We show that the solutions for (a, b, c) are

$(1, 2, 3), (3, 1, 2), (2, 3, 1), (3, 4, 5), (5, 3, 4), (4, 5, 3), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1)$.

It is easily checked that each of these ten triples is a solution. Conversely, let (a, b, c) be any solution. Then,

$$a + 1 = bu, \; b + 1 = cv, \; c + 1 = aw \tag{1}$$

for some positive integers u, v, w. By addition, we obtain $a+b+c+3 = aw+bu+cv$, hence $3 = a(w-1) + b(u-1) + c(v-1)$. Since $a(w-1), b(u-1), c(v-1)$ are nonnegative integers, we are led to distinguish three cases.

(i) If one of $a(w-1), b(u-1), c(v-1)$ is equal to 3, the other two being equal to 0. For example, if $a(w-1) = 3, b(u-1) = c(v-1) = 0$, we must have $u = v = 1$ and ($w = 2, a = 3$ or $w = 4, a = 1$) so that $(a, b, c) = (3, 4, 5)$ or $(a, b, c) = (1, 2, 3)$ (using (1)). Taking $b(u-1) = 3$ instead gives $(a, b, c) = (5, 3, 4)$ or $(a, b, c) = (3, 1, 2)$ and taking $c(v-1) = 3$ leads to $(a, b, c) = (4, 5, 3)$ or $(a, b, c) = (2, 3, 1)$.

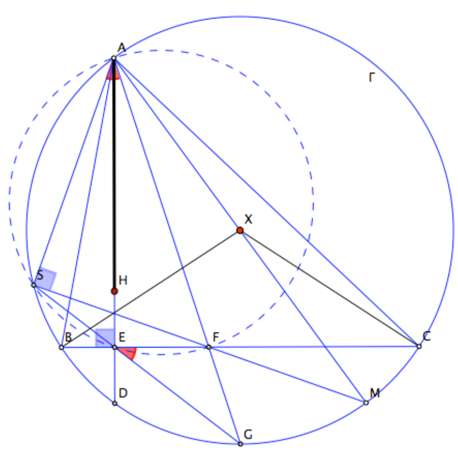
(ii) If $a(w-1), b(u-1), c(v-1)$ are $0, 1, 2$ in some order, suppose for example that $a(w-1) = 0$ so that $w = 1$ and $a = c+1$. In the case when $b(u-1) = 1$, we obtain $b = 1, u = 2$ and from (1), $a = 1$ which yields the contradiction $c = 0$. In the case when $c(v-1) = 1$, then $c = 1, v = 2$ and (1) gives $b = 1$ and $a = 2$ and $(a, b, c) = (2, 1, 1)$. We similarly obtain $(a, b, c) = (1, 2, 1)$ if $b(u-1) = 0$ and $(a, b, c) = (1, 1, 2)$ if $c(v-1) = 0$.

(iii) If $a(w-1) = b(u-1) = c(v-1) = 1$, then clearly $(a, b, c) = (1, 1, 1)$.

Editor's Comments. Many other solutions used the following approach. Since the three given numbers must be natural, then $b \leq a+1$, $c \leq b+1$ and $a \leq c+1$, which gives $c-1 \leq b \leq a+1 \leq c+2$, so $a \in \{c-2, c-1, c, c+1\}$. Now, using a case by case analysis you can find all the solutions (the proof is left to the reader).

4216. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let $AH \cap BC = \{E\}$, $AH \cap \Gamma = \{A, D\}$, the bisector of angle A cuts BC in F and Γ in G, $EG \cap \Gamma = \{G, S\}$, $SF \cap \Gamma = \{S, M\}$. If X is the midpoint of AM, prove that $\overrightarrow{AH} = \overrightarrow{XB} + \overrightarrow{XC}$.



We received eight submissions, all correct, and feature the solution by Steven Chow.

We shall see that the restriction to acute triangles can be dropped : The result holds for an arbitrary ΔABC. Directed angles are used (mod π).

From the Secant-Secant Angle Theorem,

$$\measuredangle SEF \;=\; \measuredangle SAB + \measuredangle GAC \;=\; \measuredangle SAB + \measuredangle BAG \;=\; \measuredangle SAF.$$

Consequently, $AFES$ is cyclic and

$$\measuredangle ASM = \measuredangle ASF = \measuredangle AEF = \frac{\pi}{2}.$$

From Thales Theorem, this implies that $\overline{AM}$ is a diameter of Γ, so X is the circumcentre of $\triangle ABC$. Because H is the orthocentre of $\triangle ABC$, it follows immediately that $\overrightarrow{XH} = \overrightarrow{XA} + \overrightarrow{XB} + \overrightarrow{XC}$. Therefore $\overrightarrow{AH} = \overrightarrow{XB} + \overrightarrow{XC}$.

4217. *Proposed by Dan Stefan Marinescu and Leonard Giugiuc.*

Let $n \geq 3$ and consider a cyclic convex polygon $A_0A_1 \dots A_{n-1}$ in which the circumcenter O coincides with the center of gravity. Let M and N be two distinct points such that O lies on the line segment MN and $ON = (n-1)OM$. Prove that

$$\sum_{k=0}^{n-1} MA_k \leq \sum_{k=0}^{n-1} NA_k.$$

We received one incorrect solution. We present the solution of the proposers.

[*Ed. : To clarify, the center of gravity in the question is the center of gravity of the vertices of the polygon, not the center of gravity of the polygon as a lamina.*]

Without loss of generality we may assume that the circumcenter O is the origin of the convex plane and that all the vertices $A_k = z_k \in \mathbb{C}$ lie on the unit circle. By the condition that O is also the center of gravity, we have $\sum_{k=0}^{n-1} z_k = 0$ and thus also $\sum_{k=0}^{n-1} \overline{z}_k = 0$ for the sum of the complex conjugates. We set $M = -z$ and $N = (n-1)z$ for some complex number z.

Since $|z_k| = 1$, we have $MA_k = |z_k + z| = |z_k + z||\overline{z}_k| = |1 + z\overline{z}_k|$ and similarly $NA_k = |z_k - (n-1)z| = |1 - (n-1)z\overline{z}_k|$.

Furthermore for any k,

$$|1+z\overline{z}_k| = \left|1 - z\sum_{\substack{m=0\\m\neq k}}^{n-1} \overline{z}_k\right| = \frac{1}{n-1}\left|\sum_{\substack{m=0\\m\neq k}}^{n-1}(1-(n-1)z\overline{z}_m)\right| \leq \frac{1}{n-1}\sum_{\substack{m=0\\m\neq k}}^{n-1}|1-(n-1)z\overline{z}_m|.$$

In conclusion,

$$\begin{aligned}\sum_{k=0}^{n-1} MA_k = \sum_{k=0}^{n-1}|1+z\overline{z}_k| &\leq \sum_{k=0}^{n-1}\frac{1}{n-1}\sum_{\substack{m=0\\m\neq k}}^{n-1}|1-(n-1)z\overline{z}_m| \\ &= \sum_{k=0}^{n-1}|1-(n-1)z\overline{z}_k| = \sum_{k=0}^{n-1} NA_k.\end{aligned}$$

4218. *Proposed by Daniel Sitaru.*

Prove that for all $a, b, c \in (0, \infty)$ and any natural number $n \geq 3$, we have

$$\frac{1}{n}\sqrt[n]{a+b+c} \geq \frac{3\sqrt[3]{abc}}{(a+b+c)^{n-1}+n-1}.$$

We received 14 solutions, out of which we present the one by Dan Daniel, slightly modified by the editor.

By the Arithmetic Mean - Geometric Mean (AM-GM) inequality, we have $a+b+c \geq 3\sqrt[3]{abc}$; hence, it is sufficient to prove that

$$\frac{\sqrt[n]{a+b+c}}{n} \geq \frac{a+b+c}{(a+b+c)^{n-1}+n-1}. \tag{1}$$

Applying the AM-GM inequality again to the n numbers $(a+b+c)^{n-1}, 1, \ldots, 1$ (where there are $n-1$ ones in the list),

$$\begin{aligned} &\frac{(a+b+c)^{n-1}+(n-1)}{n} \geq \sqrt[n]{(a+b+c)^{n-1}\cdot 1^{n-1}} = \frac{a+b+c}{\sqrt[n]{a+b+c}} \\ \Leftrightarrow\ &\frac{\sqrt[n]{a+b+c}}{n} \geq \frac{a+b+c}{(a+b+c)^{n-1}+n-1}; \end{aligned}$$

that is, (1) holds. From the two applications of AM-GM it follows easily that, in the given inequality, equality holds if and only if $a = b = c = \frac{1}{3}$.

4219. *Proposed by Nguyen Viet Hung.*

Let a, b, c and d be distinct positive integers such that

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+d}+\frac{d}{d+a}$$

is a integer. Prove that $a+b+c+d$ is not prime.

We received twelve correct solutions of varying complexity. We present the solution of the proposer.

Denote the displayed quantity by m. Then

$$m < \frac{a+c+d}{a+b+c+d}+\frac{b+d+a}{b+c+d+a}+\frac{c+a+b}{c+d+a+b}+\frac{d+b+c}{d+a+b+c} = 3$$

and

$$m > \frac{a}{a+b+c+d}+\frac{b}{b+c+d+a}+\frac{c}{c+d+a+b}+\frac{d}{d+a+b+c} = 1,$$

whence $m = 2$.

Since
$$m = \frac{a}{a+b} + \left(1 - \frac{c}{b+c}\right) + \frac{c}{c+d} + \left(1 - \frac{a}{d+a}\right),$$
it follows that
$$0 = a\left(\frac{1}{a+b} - \frac{1}{d+a}\right) + c\left(\frac{1}{c+d} - \frac{1}{b+c}\right) = \frac{(b-d)(a-c)(ac-bd)}{(a+b)(b+c)(c+d)(d+a)}.$$
Hence $ac = bd$, and so
$$a(a+b+c+d) = a^2 + ab + bd + ad = (a+b)(a+d).$$
If $a+b+c+d$ were prime, it would have to divide one of the smaller positive numbers $a+b$ and $a+d$. Since this cannot be, the sum is composite.

Editor's comment. Observe that, if $ac = bd$, then
$$\begin{aligned} m &= \frac{a}{a+b} + \frac{b}{b+c} + \frac{ac}{ac+ad} + \frac{bd}{bd+ab} \\ &= \frac{a}{a+b} + \frac{b}{b+c} + \frac{bd}{bd+ad} + \frac{ac}{ac+ab} \\ &= \frac{a}{a+b} + \frac{b}{b+c} + \frac{b}{a+b} + \frac{c}{b+c} = 2, \end{aligned}$$
so that the condition $ac = bd$ suffices for $m = 2$ and there are lots of examples. If we relax the condition that a, b, c, d are distinct, then $ac = bd$ may fail and $m = 2$ whenever any three are equal, $a = c$ or $b = d$, separately.

4220. *Proposed by Leonard Giugiuc, Daniel Sitaru and Hung Nguyen Viet.*

Let s and r be real numbers with $0 < r < s$ and let $a, b, c \in [s-r, s+r]$ be real numbers such that $a + b + c = 3s$. Prove that
$$ab + bc + ca \geq 3s^2 - r^2 \quad \text{and} \quad abc \geq s^3 - sr^2.$$

We received 11 correct solutions. We present the solution by Nghia Doan.

Let $x = a - s + r$, $y = b - s + r$, and $z = c - s + r$. Then $0 \leq x, y, z \leq 2r$ and $x + y + z = 3r$. We have
$$\begin{aligned} \sum ab &= \sum (x+s-r)(y+s-r) \\ &= \sum [xy + (x+y)(s-r) + (s-r)^2] \\ &= \sum xy + 2(s-r) \cdot \sum x + 3(s-r)^2 \\ &= \sum xy + 6r(s-r) + 3(s-r)^2 \\ &= \sum xy + 3s^2 - 3r^2. \end{aligned}$$

We can assume $x \geq y \geq z$. Then
$$r \leq x \leq 2r \Rightarrow \left(x - \frac{3r}{2}\right)^2 \leq \frac{r^2}{4}.$$
so that
$$xy + yz + zx = x(3r - x) + yz \geq x(3r - x) = \frac{9}{4}r^2 - \left(x - \frac{3r}{2}\right)^2 \geq 2r^2.$$
Thus
$$\sum ab \geq 3s^2 - r^2.$$
For the second inequality, we have
$$\begin{aligned}
abc &= (x + s - r)(y + s - r)(z + s - r)\\
&= xyz + (s - r)(xy + yz + zx) + (s - r)^2(x + y + z) + (s - r)^3\\
&= xyz + (s - r)(xy + yz + zx) + 3r(s - r)^2 + (s - r)^3\\
&\geq xyz + 2r^2(s - r) + 3r(s - r)^2 + (s - r)^3\\
&= xyz + 2r^2 s - 2r^3 + 3rs^2 - 6r^2 s + 3r^3 + s^3 - 3rs^2 + 3r^2 s - r^3\\
&= xyz + s^3 - sr^2\\
&\geq s^3 - sr^2.
\end{aligned}$$

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FROM THE ARCHIVES

*Did you know that **Crux** volumes before volume 39 are open to the public? You can find these archived materials at* `https://cms.math.ca/crux/`*. Let's take a look at some problems proposed 30 years ago.*

. . . .

1338. *Proposed by Jean Doyen, Université Libre de Bruxelles, Brussels, Belgium; J. Chris Fisher, University of Regina, Regina, Saskatchewan; and Gunter Kist, Technische Universitat, Munich, Federal Republic of Germany.*

In a theoretical version of the Canadian lottery "Lotto 6–49", a ticket consists of six distinct integers chosen from 1 to 49 (inclusive). A *t–prize* is awarded for any ticket having t or more numbers in common with a designated "winning" ticket. Denote by $f(t)$ the smallest number of tickets required to be certain of winning a t–prize. Clearly $f(1) = 8$ and $f(6) = \left[\begin{matrix}49\\6\end{matrix}\right]$. Show that $f(2) \leq 19$. Can you do better?

1345. *Proposed by P. Erdos, Hungarian Academy of Sciences, and Esther Szekeres, University of New South Wales, Kensington, Australia.*

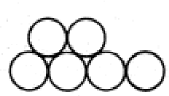
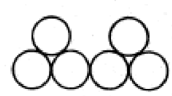
Given a convex n–gon $X_1X_2...X_n$ of perimeter p, denote by $f(X_i)$ the sum of the distances of X_i to the other n–1 vertices.

(a) Show that if $n \geq 6$, there is a vertex X_i such that $f(X_i) > p$.

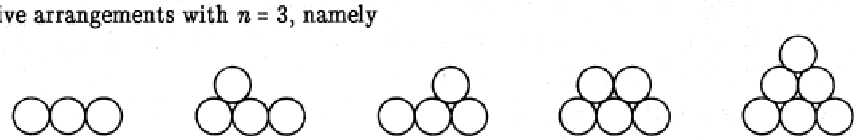
(b) Is it true that for n large enough, the *average value* of $f(X_i)$, $1 \leq i \leq n$, is greater than p?

1367. *Proposed by Richard K. Guy, University of Calgary.*

Consider arrangements of pennies in rows in which the pennies in any row are contiguous, and each penny not in the bottom row touches two pennies in the row below.

For example,  is allowed, but  isn't.

How many arrangements are there with n pennies in the bottom row? To illustrate, there are five arrangements with $n = 3$, namely

 .

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