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# The Fundamental Groups of the Complements of Some Solid Horned Spheres

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# THE FUNDAMENTAL GROUPS OF THE COMPLEMENTS

# OF SOME SOLID HORNED SPHERES

by

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ABSTRACT

The Fundamental Groups of the Complements
of Some Solid Horned Spheres

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One of the methods used for the construction of the classical Alexander horned sphere leads naturally to generalization to horned spheres of higher order. Let $M_2$, denote the Alexander horned sphere. This is a 2-horned sphere of order 2. Denote by $M_3$ and $M_4$, two 2-horned spheres of orders 3 and 4, respectively, constructed by such a generalization.

The fundamental groups of the complements of $M_2$, $M_3$, and $M_4$ are derived, and representations of these groups onto the Alternating Group, $A_5$, are found. The form of the presentations of these fundamental groups leads to a more general class of groups, denoted by $G^k$, $k \geq 2$. A set of homomorphisms $\phi_k^\ell : G^k \to G^\ell$, $k \geq \ell \geq 2$ is found, which has a clear geometric meaning as applied to the groups $G^2$, $G^3$, and $G^4$.

Two theorems relating to direct systems of non-abelian groups are proved and applied to the groups $G^k$. The implication of these theorems is that the groups $G^k$, $k \geq 2$ are all free groups of countably infinite rank and that the embeddings of $M_2$, $M_3$, and $M_4$ in $E^3$ cannot be distinguished by means of fundamental groups.

(33 pages)

## INTRODUCTION

The classical "horned sphere," the best known and one of the first wild surfaces in $E^3$, was defined by Alexander. Several authors have studied Alexander's horned sphere and related surfaces. In 1966, L. O. Cannon gave a general definition for the solid horned sphere M in $E^3$.[1] Certain of the horned spheres are defined in such a way that the complementary domain $(E^3 - M)$ is not simply connected. The question arises as to which, if any, of the horned spheres are equivalently embedded in $E^3$.

One means available for testing this property is the use of algebraic techniques to examine the fundamental group, $\pi(E^3 - M)$, of the complementary domain. Let M, M' be two horned spheres; $\pi(E^3 - M)$, $\pi(E^3 - M')$, respectively, their fundamental groups. If it can be shown that the groups $\pi(E^3 - M)$ and $\pi(E^3 - M')$ are not isomorphic, then it follows that the spaces $(E^3 - M)$ and $(E^3 - M')$ are not homeomorphic and that M, M' are not equivalently embedded in $E^3$.[2] On the other hand, if the fundamental groups are isomorphic no conclusion can be drawn.

Let $M_2$ denote Alexander's horned sphere. $M_2$ is often called a 2-horned sphere, and it is obtained by a limiting process in $E^3$. This limiting process is sometimes thought of as the construction of a pair

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[1] L. O. Cannon, "Sums of Solid Horned Spheres," *Transactions of the American Mathematical Society*, CXXII, No. 1 (March, 1966), p. 203-228.

[2] Richard H. Crowell and Ralph H. Fox, *Introduction to Knot Theory* (New York: Blaisdell Publishing Company, 1965), p. 13-30.

of interlocking, unknotted, "loops" at each stage. The process suggests the possibility of using three, four, or more interlocking loops to construct other 2-horned spheres. See Figure 1 for a 2-horned sphere constructed with four loops.[3] This horned sphere can be called a 2-horned sphere of order 4, and we denote it by $M_4$. Observe that a horned sphere with three loops can be obtained from $M_4$ simply by omitting one of the loops from the sphere in Figure 1 at each stage of the limiting process. Such a horned sphere can be regarded as a 2-horned sphere of order 3, and we denote it by $M_3$. Furthermore, $M_2$ is obtained from $M_4$ by omitting one loop from each side at each stage of the limiting process.

In this paper, we restrict attention to the horned spheres $M_2$, $M_3$, and $M_4$, and the fundamental groups of their complementary domains, $G^2 = \pi(E^3 - M_2)$, $G^3 = \pi(E^3 - M_3)$, and $G^4 = \pi(E^3 - M_4)$, respectively. These fundamental groups will be derived, and representations of them onto the Alternating Group on five elements, $A_5$, will be found. The form in which these groups will be presented will suggest a more general set of groups related to $G^i$, $i = 2,3,4$, which will be denoted $G^k$, $k \geq 2$. A general set of homomorphisms of $G^k$ onto $G^\ell$, for all positive integers $k$ and $\ell$, $k \geq \ell$ will be derived. As applied to the groups $G^i$, $i = 2,3,4$, the homomorphisms will describe algebraically the process of omitting loops outlined above.

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[3]Let PQ represent that portion of a solid cylinder enclosed by the dotted lines in the figure. PQ is to be carried into each of the solid cylinders $P_iQ_i$ by a homeomorphism $h_i$ which maps each disc P onto $P_i$ and Q onto $Q_i$. The horned sphere $M_4$ is obtained by iteration of this process a countably infinite number of times in such a way that the diameter of $h_n(PQ)$ approaches zero for increasing n.

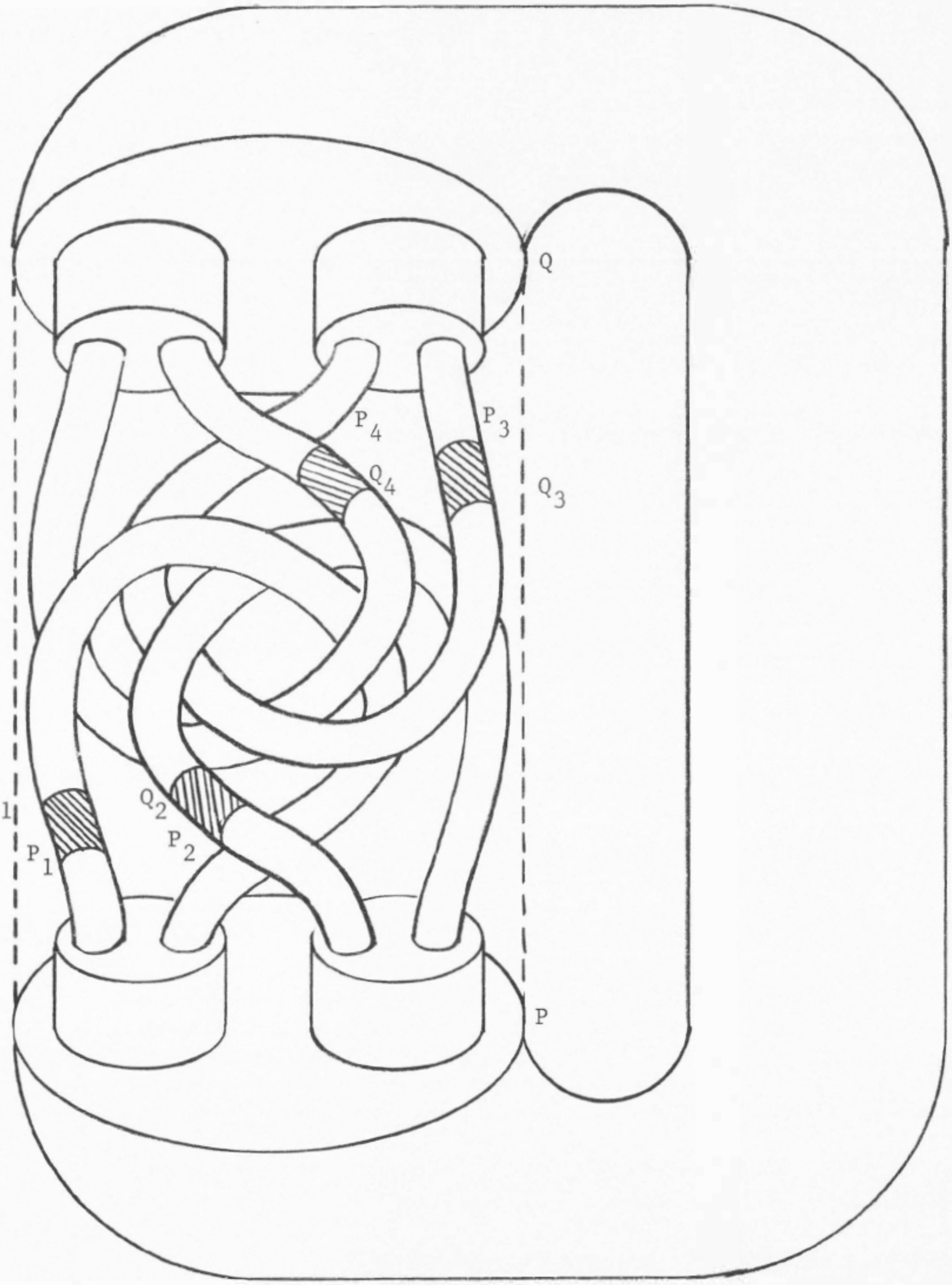


Figure 1. A 2-horned sphere of order 4 constructed by generalizing the Alexander horned sphere.

In addition to these derivations, two theorems relating to direct systems of non-abelian groups will be proved. The application of these theorems to the groups $G^k$ will imply that $G^k$ is a free group of countably infinite rank. Thus, it will follow that $G^k$ is isomorphic to $G^\ell$, for all $k, \ell \geq 2$, and that the embeddings of $M_2$, $M_3$, and $M_4$ cannot be distinguished by their fundamental groups.

DERIVATION OF THE FUNDAMENTAL GROUPS OF THE

COMPLEMENTS OF THE 2-HORNED SPHERES

OF ORDERS 2, 3, AND 4

The 2-horned sphere of order 2

Let $M_2 \subset E^3$ be the Alexander horned sphere. Blankenship and Fox have derived a presentation for the fundamental group $G^2 = \pi(E^3 - M_2)$, and have shown that the group is locally free and not finitely generated.[4] A relabeling of the graph, together with a slightly different projection, results in the presentation given here which is more suggestive of a general form.

The complementary domain of $M_2$ is homeomorphic to the complement of an infinite graph whose projection is shown in Figure 2. From the graph of Figure 2, a presentation of $G^2$ may be read by a standard method.[5] The generators of the group are x, a, and all elements $x_\alpha$, $a_\alpha$, $b_\alpha$, where $\alpha$ is an element of the set of all finite sequences of the integers 1 and 2. That is, $\alpha = p_1p_2\cdots p_\lambda$, where $\lambda = \lambda(\alpha)$ the length of the sequence, $p_i = 1$ or 2, for all $1 \leq i \leq \lambda$. Thus, if $\lambda(\alpha) = 1$, then $\alpha = 1$ or 2; if $\lambda(\alpha) = 2$, then $\alpha$ will denote one of the elements 11, 12, 21, or 22. We will further use the notation $\alpha i$, defined as follows: if $\alpha$ is a sequence of length $\lambda(\alpha) = n$, then $\alpha i$ will denote a sequence of length n + 1 such that the first n elements are the same as

[4] W. A. Blankenship and R. H. Fox, "Remarks on Certain Pathological Open Subsets of 3-Space and Their Fundamental Groups," *Proceedings of the American Mathematical Society*, I (October, 1950), p. 618-624.

[5] Crowell and Fox, p. 72-86.

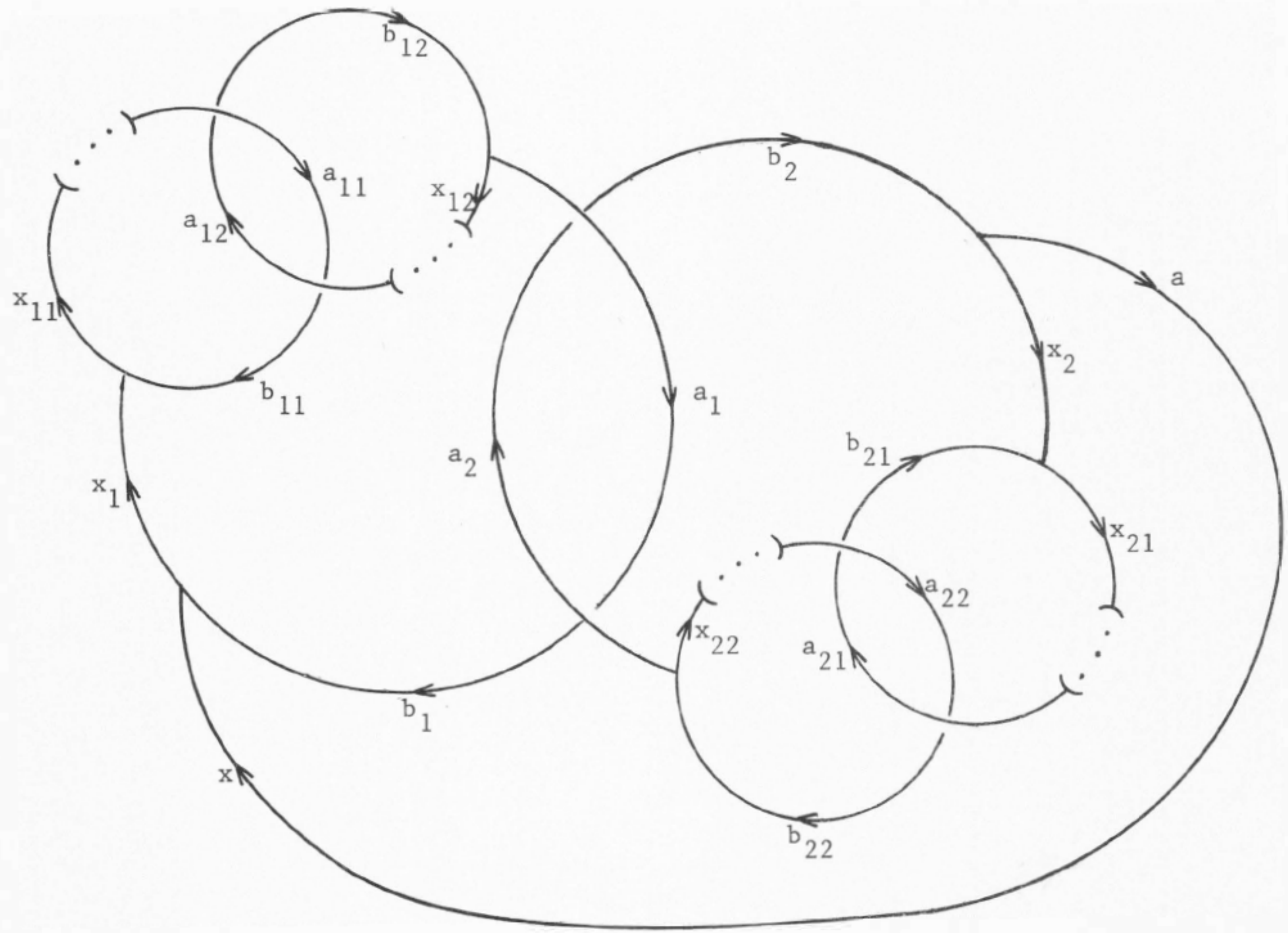


Figure 2. Projection of the infinite graph whose complement is homeomorphic to the space $E^3 - M_2$.

the sequence $\alpha$; the last element will be i, where i = 1 or 2. Thus, if $\alpha$ = 12212, say, then $\alpha 1$ will denote the sequence 122121, and $\alpha 2$ will denote 122122. For the sake of convenience, we also define a sequence of length zero, so that if $\lambda(\alpha) = 0$, then $x_\alpha = x$, and $a_\alpha = a$.

The defining relations are of three types.

At the crossings (see Figure 3):

$a_{\alpha 1}\ a_{\alpha 2}\ a_{\alpha 1}^{-1}\ b_{\alpha 2}^{-1} = 1$

$a_{\alpha 2}\ a_{\alpha 1}\ a_{\alpha 2}^{-1}\ b_{\alpha 1}^{-1} = 1$

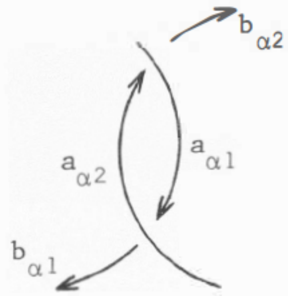


Figure 3. Graph for reading relations at the crossings in $E^3 - M_2$.

At the points of order 3 (see Figure 4):

$x_\alpha\ b_{\alpha 1}\ x_{\alpha 1}^{-1} = 1$

$x_{\alpha 2}^{-1}\ a_\alpha^{-1}\ b_{\alpha 2} = 1$


Figure 4. Graph for reading relations at points of order 3 in $E^3 - M_2$.

Around each singular point (see Figure 5):

$x_\alpha a_\alpha^{-1} = 1$

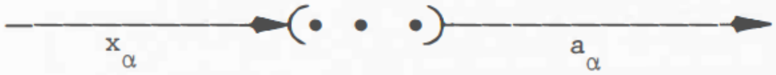


Figure 5. Graph for reading the relation around each singular point in $E^3 - M_2$.

By straightforward substitution, these relations reduce to:

$x_\alpha = x_{\alpha_1} x_{\alpha_2} x_{\alpha_1}^{-1} x_{\alpha_2}^{-1} = [x_{\alpha_1}, x_{\alpha_2}]$.[6]

Thus, the following presentation is obtained:[7]

$G^2 = \pi(E^3 - M_2)$

$= |x_\alpha, \alpha = p_1p_2\ldots p_\lambda, p_i = 1 \text{ or } 2, \lambda(\alpha) \geq 0: x_\alpha = [x_{\alpha_1}, x_{\alpha_2}]|$

See page 15 for a representation of this group into the 3-cycles of $S_5$ (and hence onto the Alternating Group, $A_5$).

The 2-horned sphere of order 3

Let $M_3 \subset E^3$ be the 2-horned sphere of order 3 as described in the introduction. By the same method used above for $G^2$, we can derive a presentation for the fundamental group, $G^3 = \pi(E^3 - M_3)$, of the complementary domain of $M_3$. Figure 6 shows a projection of a general singular point in the infinite graph whose complement is homeomorphic to the space $E^3 - M_3$.

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[6] The notation $[u, v] = uvu^{-1}v^{-1}$ will be used consistently.

[7] cf., Crowell and Fox, p. 40, for the presentation notation used here.

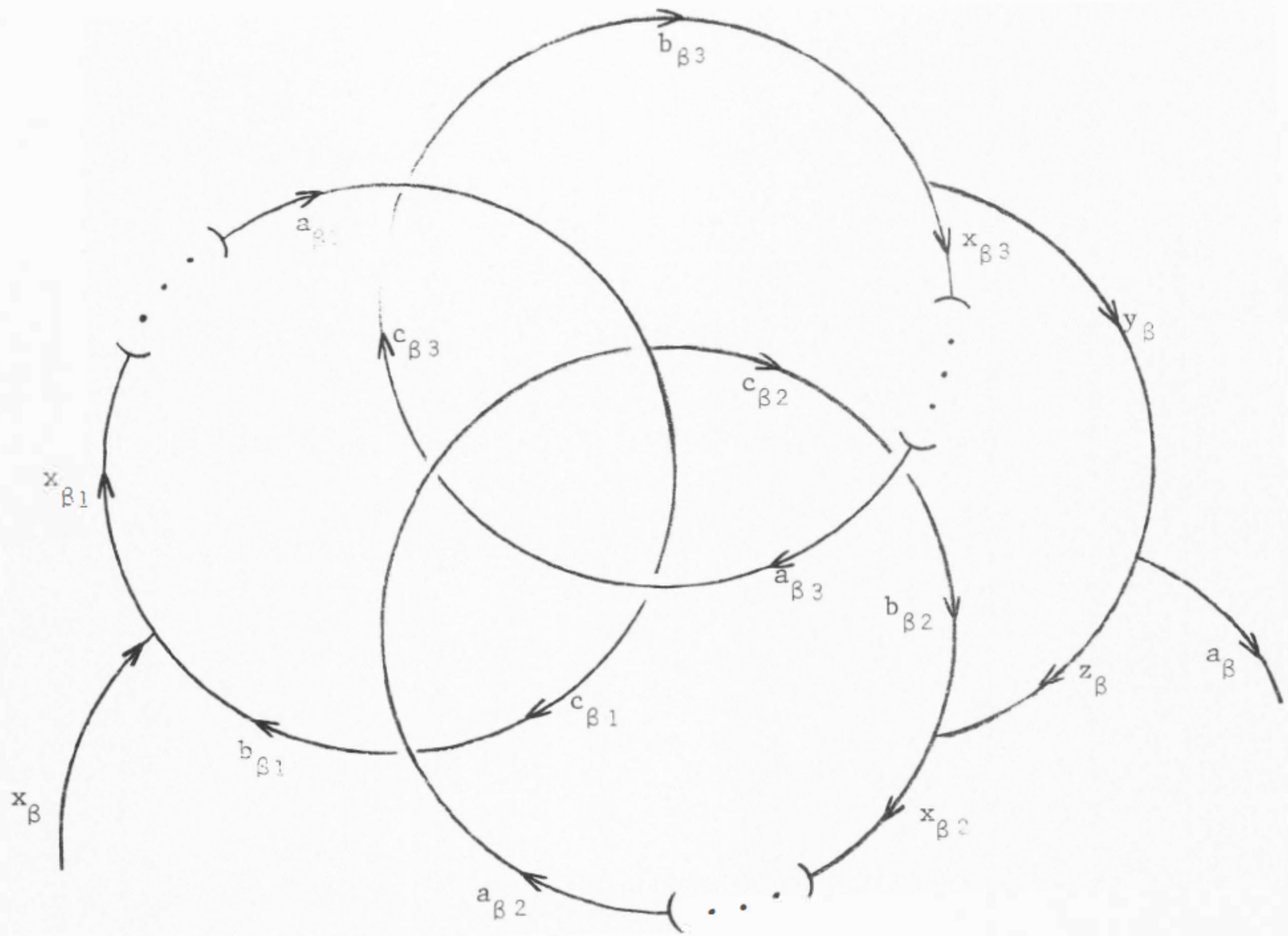


Figure 6. Projection of the graph of a general singular point for reading the relations in $G^3 = \pi(E^3 - M_3)$.

The generating elements of $G^3$ are: x, y, z, a, and all elements $x_\beta$, $y_\beta$, $z_\beta$, $a_\beta$, $b_\beta$, $c_\beta$, where $\beta$ is a member of the set of all finite sequences formed from the integers 1, 2, 3. That is $\beta = p_1 p_2 \cdots p_\lambda$, where $\lambda = \lambda(\beta)$ the length of the sequence, $p_i$ = 1, 2, or 3, for all $1 \leq i \leq \lambda$. As before, we define $\beta i$ as that sequence of length $\lambda(\beta) + 1$ formed by adjoining to $\beta$ on the right one of the integers 1, 2, or 3, and if $\lambda(\beta) = 0$, $x_\beta = x$, $y_\beta = y$, $a_\beta = a$.

The defining relations can be read from Figure 6 as follows:

At the crossings there are six relations:[8]

$a_{\beta i}\ c_{\beta j}\ a_{\beta i}^{-1}\ b_{\beta j} = 1$, $(i,j) = (2,1)$, $(1,3)$, or $(3,2)$

$a_{\beta i}\ a_{\beta j}\ a_{\beta i}^{-1}\ c_{\beta j} = 1$, $(i,j) = (2,3)$, $(1,2)$, or $(3,1)$

At the points of order 3:

$x_\beta\ b_{\beta 1}\ x_{\beta 1}^{-1} = 1$

$b_{\beta 3}\ x_{\beta 3}^{-1}\ y_\beta^{-1} = 1$

$b_{\beta 2}\ x_{\beta 2}^{-1}\ z_\beta = 1$

$y_\beta\ z_\beta^{-1}\ a_\beta^{-1} = 1$

Around each singular point:

$x_\beta\ a_\beta^{-1} = 1$

Again by straightforward substitution these relations reduce to:

$$x_\beta = x_{\beta 1}\ x_{\beta 2}\ x_{\beta 3}\ x_{\beta 1}^{-1}\ x_{\beta 3}^{-1}\ x_{\beta 2}^{-1}$$

$$= [x_{\beta 1},\ x_{\beta 2} x_{\beta 3}]$$

Thus, $G^3$ has the presentation:

---

[8]First reading the graph clockwise for the "outer" crossings; then, clockwise for the "inner" crossings.

$G^3 = \pi(E^3 - M_3)$

$= |x_\beta, \ \beta = p_1p_2\cdots p_\lambda, \ p_i = 1,2, \text{ or } 3, \ \lambda(\beta) \geq 0:$

$$x_\beta = [x_{\beta 1}, \ x_{\beta 2} \ x_{\beta 3}]|$$

This group also has a representation onto the group $A_5$ as shown on pages 14 and 15.

The 2-horned sphere of order 4

Let $M_4 \subset E^3$ be the 2-horned sphere of order 4 as described in the Introduction. As before, the complementary domain $E^3 - M_4$ is homeomorphic to the complement of an infinite graph, the projection of a general singular point of which is shown in Figure 7.

The generating elements of $G^4 = \pi(E^3 - M_4)$ are x, u, v, w, z, a, and all elements $x_\gamma$, $u_\gamma$, $v_\gamma$, $w_\gamma$, $z_\gamma$, $a_\gamma$, $b_\gamma$, $c_\gamma$, $d_\gamma$, where $\gamma$ is an element of the set of all finite sequences of the integers 1, 2, 3, and 4. As before, $\gamma = p_1p_2\cdots p_\lambda$, $p_i = 1,2,3, \text{ or } 4$, $\lambda = \lambda(\gamma)$ the length of the sequence; $\gamma i$ is a sequence of length $\lambda(\gamma) + 1$ formed by adjoining to $\gamma$ on the right one of the integers 1, 2, 3, or 4; and if $\lambda(\gamma) = 0$, $x_\gamma = x$, $u_\gamma = u$, etc.

There are 12 defining relations arising from the crossings and 6 from the points of order 3 as follows:

At the crossings:[9]

$a_{\gamma i} \ d_{\gamma j} \ a_{\gamma i}^{-1} \ b_{\gamma j}^{-1} = 1$, where $(i,j) = (2,3), (3,2), (4,3), \text{ or } (1,4)$

$a_{\gamma i} \ c_{\gamma j} \ a_{\gamma i}^{-1} \ d_{\gamma j}^{-1} = 1$, where $(i,j) = (3,1), (4,2), (1,3), \text{ or } (2,4)$

$a_{\gamma i} \ a_{\gamma j} \ a_{\gamma i}^{-1} \ c_{\gamma j}^{-1} = 1$, where $(i,j) = (4,1), (1,2), (2,3), \text{ or } (3,4)$

---

[9]First, reading the graph clockwise for the "outer" crossings; second, counterclockwise for the "middle" crossings; third, counterclockwise for the "inner" crossings.

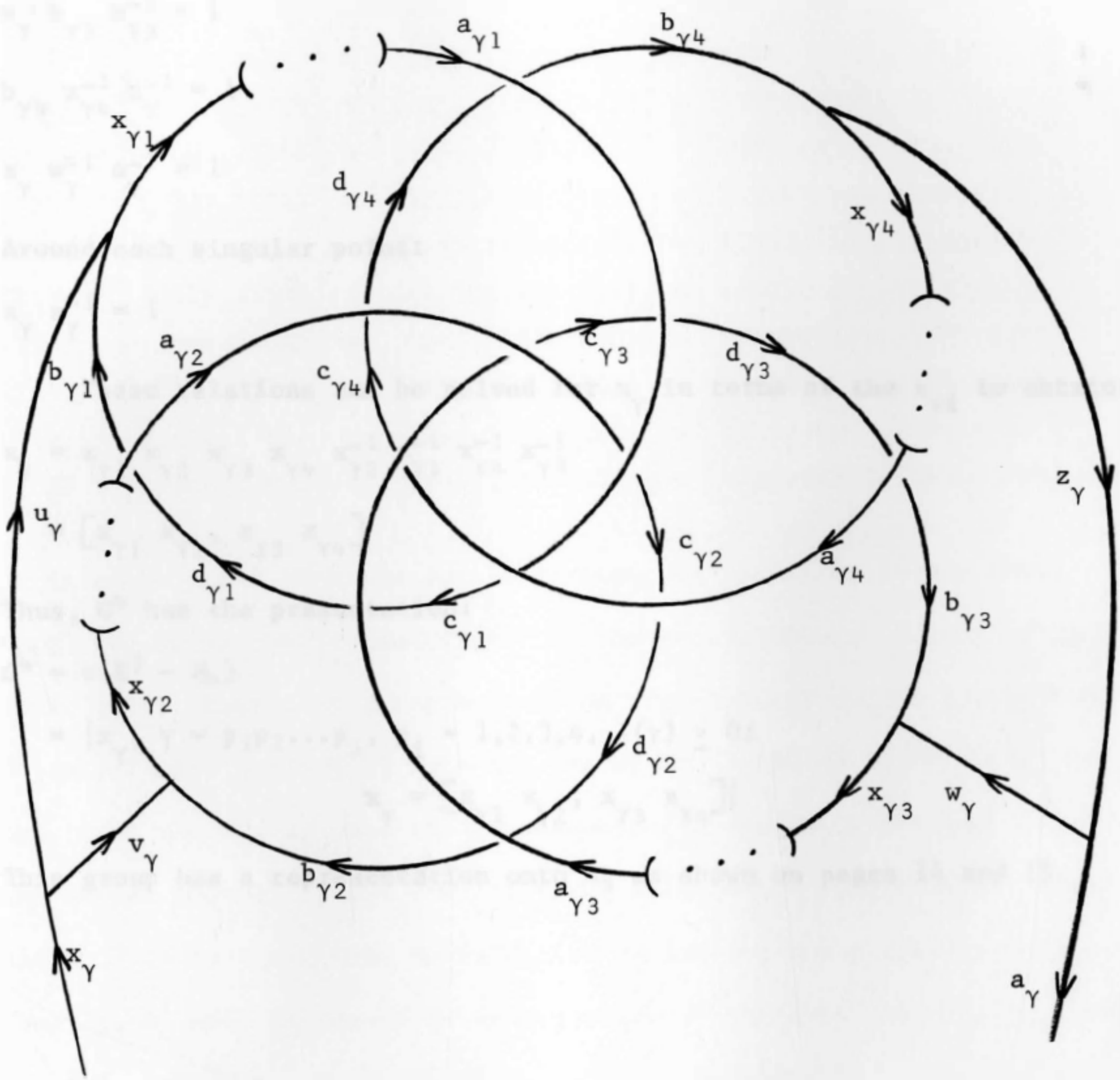


Figure 7. Projection of the graph of a general singular point for reading the relations in $G^4 = \pi(E^3 - M_4)$.

At the points of order 3:

$$x_\gamma v_\gamma u_\gamma^{-1} = 1$$

$$u_\gamma b_{\gamma 1} x_{\gamma 1}^{-1} = 1$$

$$b_{\gamma 2} x_{\gamma 2}^{-1} v_\gamma^{-1} = 1$$

$$w_\gamma b_{\gamma 3} x_{\gamma 3}^{-1} = 1$$

$$b_{\gamma 4} x_{\gamma 4}^{-1} z_\gamma^{-1} = 1$$

$$z_\gamma w_\gamma^{-1} a_\gamma^{-1} = 1$$

Around each singular point:

$$x_\gamma a_\gamma^{-1} = 1$$

These relations can be solved for $x_\gamma$ in terms of the $x_{\gamma i}$ to obtain:

$$x_\gamma = x_{\gamma 1} x_{\gamma 2} x_{\gamma 3} x_{\gamma 4} x_{\gamma 2}^{-1} x_{\gamma 1}^{-1} x_{\gamma 4}^{-1} x_{\gamma 3}^{-1}$$

$$= [x_{\gamma 1} x_{\gamma 2}, x_{\gamma 3} x_{\gamma 4}]$$

Thus, $G^4$ has the presentation:

$$G^4 = \pi(E^3 - M_4)$$

$$= |x_\gamma, \gamma = p_1 p_2 \cdots p_\lambda, p_i = 1,2,3,4, \lambda(\gamma) \geq 0:$$

$$x_\gamma = [x_{\gamma 1} x_{\gamma 2}, x_{\gamma 3} x_{\gamma 4}]|$$

This group has a representation onto $A_5$ as shown on pages 14 and 15.

# REPRESENTATIONS OF $G^2$, $G^3$, AND $G^4$ ONTO $A_5$

Consider the following list of 3-cycles and commutators in the Symmetric Group on five elements, $S_5$:[10]

$$
\begin{aligned}
(123) &= [(124),(135)] = [(124),(125)(132)] = [(154)(125),(125)(132)] \\
(124) &= [(123),(145)] = [(123),(125)(142)] = [(153)(125),(125)(142)] \\
(132) &= [(135),(124)] = [(135),(154)(125)] = [(125)(132),(154)(125)] \\
(135) &= [(132),(154)] = [(132),(124)(152)] = [(152)(135),(124)(152)] \\
(142) &= [(145),(123)] = [(145),(153)(125)] = [(125)(142),(153)(125)] \\
(145) &= [(142),(153)] = [(142),(123)(152)] = [(152)(145),(123)(152)] \\
(153) &= [(154),(132)] = [(154),(152)(135)] = [(124)(152),(152)(135)] \\
(154) &= [(153),(142)] = [(153),(152)(145)] = [(123)(152),(152)(145)] \\
(125) &= [(123),(154)] = [(123),(124)(152)] = [(153)(125),(124)(152)] \\
(152) &= [(154),(123)] = [(154),(153)(125)] = [(124)(152),(153)(125)]
\end{aligned}
$$

It is to be noted that the set of 3-cycles in the first column above is closed under the "commutations" in the second through fourth columns. Thus, it is clear that a representation onto these 3-cycles exists for $G^2$ as follows: map $x \in G^2$ onto any one of the 3-cycles in the first column, then map $x_1$, $x_2$ onto the corresponding elements in the commutator in the second column. Since the first two columns are closed under "commutation," it is clear that an inductive definition for a mapping exists. That is, $x_\alpha$ will be mapped to some element of the first column; $x_{\alpha 1}$ and $x_{\alpha 2}$ will be mapped to the elements in the corresponding commutator in the second column. Therefore, there is a representation of $G^2$ into $S_5$.[11]

---

[10] The multiplication of elements here is from right to left.

[11] Note that only the first eight elements of column one are needed for the representation.

In a similar way, the remaining columns can be used to obtain representations of $G^3$ and $G^4$ into $S_5$.

The following products are sufficient to show that the elements (123), (124), and (135) generate all the 3-cycles of $S_5$:

(123)(135) = (235)

(152)(124)(125) = (145)

(123)(142) = (143)

(153)(145)(135) = (354)

(123)(135)(132) = (152)

(124)(145) = (152)

(123)(124)(132) = (234)

Since the 3-cycles of $S_n$ generate the Alternating Group $A_n$, it follows that the 3-cycles in the list on page 14 generate all of $A_5$.[12] Thus, a representation is shown to exist for each of the groups $G^2$, $G^3$, and $G^4$ onto $A_5$.

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[12] Joseph J. Rotman, *The Theory of Groups: An Introduction* (Boston, Massachusetts: Allyn and Bacon, Inc., 1965), p. 38.

## DEFINITION OF $G^k$ AND CONSTRUCTION OF HOMOMORPHISMS OF $G^k$ ONTO $G^\ell$ FOR $k \geq \ell \geq 2$

Definition of $G^k$

For each integer $n \geq 0$ and each integer $k \geq 2$, we define the following sets:

$$A_n^k = \{\alpha : \alpha = p_1 p_2 \ldots p_n, \text{ a sequence of integers, } \lambda(\alpha) = n, \quad p_i \in \{1, 2, 3, \ldots, k\}\}$$

$$G_n^k = \{x_\alpha : \alpha \in A_n^k\}$$

Thus, $A_n^k$ consists of all sequences of the integers 1, 2, 3, ... , $k$ of length n, and $G_n^k$ is a set of elements in one to one correspondence with the elements of $A_n$, i.e., $\alpha \leftrightarrow x_\alpha$. $G_n^k$ can then be regarded as a set of generators for a free group of rank $k^n$.[13] Let $*G_n^k$ denote the free group generated by the elements of $G_n^k$. Now define the following set:

$$R_n^k = \{x_\alpha^{-1}[x_{\alpha 1} x_{\alpha 2} \cdot \ldots \cdot x_{\alpha i}, x_{\alpha(i+1)} \cdot \ldots \cdot x_{\alpha k}] : x_\alpha \text{ is a generator of } *G_n^k,\ x_{\alpha j} \text{ is a generator of } *G_{n+1}^k,\ \alpha \in A_n^k,\ \alpha j \in A_{n+1}^k,\ i = [k/2]^{14}\}$$

Let $F^k = \prod_{n=0}^{\infty}{}^* *G_n^k$ denote the free product of the groups $*G_n^k$, for fixed $k$,[15] and let $G_*^k$ be a group with the following presentation:

---

[13] William S. Massey, *Algebraic Topology: An Introduction* (New York: Harcourt, Brace, and World, Inc., 1967), p. 102-105. (cf., Rotman, p. 235-241; Crowell and Fox, p. 31-35.)

[14] The notation $[k/2]$ is used to denote the "bracket" function: the largest integer less than or equal to $k/2$.

[15] Massey, p. 97-100; cf., Rotman, p. 247-249.

$$G_*^k = |\bigcup_{n=0}^{\infty} G_n^k : \bigcup_{n=0} R_n^k|$$

We note also, that each $*G_n^k$ is a free group, and hence $F^k$ is also a free group. For $k = 2$, $\bigcup_{n=0}^{\infty} G_n^k$ is simply the set of all $x_\alpha$, where $\alpha$ ranges over the set of all finite sequences of the integers 1,2. $\bigcup_{n=0}^{\infty} R_n^k$ is then a set of elements $x_\alpha^{-1}[x_{\alpha 1}, x_{\alpha 2}]$. It is clear then, that for $k = 2$, $G_*^2 \simeq G^2$.[16] Similarly, $G_*^3 \simeq G^3$ and $G_*^4 \simeq G^4$. Since there will be no confusion, hereafter we will write $G^k$ for $G_*^k$.

Homomorphisms of $G^k$ onto $G^\ell$, for $k \geq \ell \geq 2$

Our objective is to obtain an homomorphism $\phi_k^\ell : G^k \to G^\ell$, onto, for each $k \geq \ell \geq 2$. That such an homomorphism exists is immediate from Theorem 3, page 25. However, the homomorphism constructed here has a particularly nice geometric interpretation.

Theorem 1. Let $k \geq \ell \geq 2$, there exists a homomorphism $\phi_k^\ell$ of $G^k$ onto $G^\ell$.

Proof: If $k = \ell$, let $\phi_k^k$ be the identity homomorphism. So let $k > \ell$. First, we define a map $f_k^\ell : F^k \to F^\ell$ of $F^k$ onto $F^\ell$ as follows: let $i = [k/2]$, $j = [\ell/2]$. Since $k > \ell$, $i \geq j$. Let any $x_\alpha \in F^k$ be given, where $\alpha = p_1 p_2 \cdots p_n$, $1 \leq p_1 \leq k$. If $p_m$, $1 \leq m \leq n$, is one of the integers that makes up the sequence $\alpha$, it is clear that $1 \leq p_m \leq j$, $j + 1 \leq p_m \leq i$, $i + 1 \leq p_m \leq i + \ell - j$, or $i + \ell - j + 1 \leq p_m \leq k$. (a) Suppose that for some $p_m$ we have $j + 1 \leq p_m \leq i$ or $i + \ell - j + 1 \leq p_m \leq k$. Then define $f_k^\ell(x_\alpha) = 1 \in F^\ell$. (b) Suppose that for each $p_m$ either $1 \leq p_m \leq j$

---

[16] In the presentation of a group, it is understood that the relations may be written in a number of different ways. In our case here, it is understood that $x_\alpha = [x_{\alpha 1}, x_{\alpha 2}]$ and $x_\alpha^{-1}[x_{\alpha 1}, x_{\alpha 2}]$ are equivalent. c.f., Crowell and Fox, p. 37-38; Wilhelm Magnus, Abraham Karrass, and David Solitar, *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations* (New York: Interscience Publishers, 1966), p. 7.

or $i + 1 \leq p_m \leq i + \ell - j$. Define $f_k^\ell(x_\alpha) = y_\beta \in F^\ell$, where $\lambda(\beta) = n$, and $\beta = q_1q_2\ldots q_n$ as follows: if $1 \leq p_m \leq j$, then $q_m = p_m$; if $i + 1 \leq p_m \leq i + \ell - j$, then $q_m = j - i + p_m$.[17]

We assert that this definition of $f_k^\ell$ is a one to one mapping of a subset $H_n^k \subset G_n^k$ onto $G_n^\ell$, where

$$H_n^k = \{x_\alpha \in G_n^k : \alpha = p_1p_2\ldots p_n,\ 1 \leq p_m \leq j \text{ or } i + 1 \leq p_m \leq i + \ell - j\}$$

which extends to a homomorphism from $*G_n^k$ onto $*G_n^\ell$ (or to an isomorphism when restricted to $*H_n^k$) for each n.

Let $y_\beta \in G_n^\ell$. Then $\beta = q_1q_2\ldots q_n$, and for each $q_m$, $1 \leq m \leq n$, we have $1 \leq q_m \leq \ell$. The inverse image of $y_\beta$ is then $x_\alpha$, where $\alpha = p_1p_2\ldots p_n$. Either $p_m = q_m$, for $1 \leq q_m \leq j$, or $p_m = q_m + i - j$, for $j + 1 \leq q_m \leq \ell$. Clearly, then, $x_\alpha$ is a unique element of $G_n^k$. Also, it is clear that $f_k^\ell$ is onto, and so it is a one to one correspondence between $H_n^k$ and $G_n^\ell$.

By the definition of a free group, the mapping $f_k^\ell$ extends to a homomorphism $f_{k*}^\ell: *G_n^k \to *G_n^\ell$, for each n, and its restriction to $*H_n^k$ is an isomorphism. Since $F^k$ is a free group, and $f_{k*}^\ell$ is defined for each positive integer n, it is clear that $f_{k*}^\ell: F^k \to F^\ell$ is a homomorphism onto. Furthermore, since $*H_n^k \simeq *G_n^\ell$ under $f_{k*}^\ell$ restricted to $*H_n^k$, it is clear that $\prod_{n=0}^{\infty}{}^* *H_n^k \simeq F^\ell$.

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[17] For example, suppose $k = 5$, $\ell = 2$, then,

$x_{13311} \to y_{12211}$

$x_{13312} \to 1$

$x_{13313} \to y_{12212}$

$x_{13314} \to 1$

$x_{13315} \to 1$

$x_{12331} \to 1$

$x_{13413} \to 1$

$x_{25413} \to 1$

Let $f^{\ell}_{k*}$ be defined above, and let

$$r_\alpha = x_\alpha^{-1}[x_{\alpha 1}x_{\alpha 2}\cdot \;\cdots\; \cdot x_{\alpha i},\, x_{\alpha(i+1)}\cdot \;\cdots\; \cdot x_{\alpha k}]$$

be any element of $R^{k}_{n}$, for any integer $n \geq 0$.

If $f^{\ell}_{k*}(x_\alpha) = 1$, then $\alpha = p_1p_2\cdots p_n$, $1 \leq p_m \leq k$, and by the definition of $f^{\ell}_{k}$, either $j + 1 \leq p_m \leq i$, or $i + 1 + \ell - j \leq p_m \leq k$, for some $1 \leq m \leq n$. But then $f^{\ell}_{k}(x_{\alpha s}) = 1$, $1 \leq s \leq k$, for each element of the commutator above. Thus $f^{\ell}_{k*}(r_\alpha) = 1$.

If $f^{\ell}_{k*}(x_\alpha) = y_\beta \neq 1$, then,

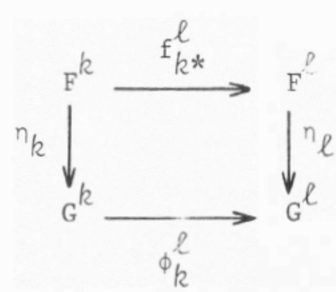
$$\begin{aligned} f^{\ell}_{k*}(r_\alpha) &= f^{\ell}_{k}(x_\alpha^{-1})[f^{\ell}_{k}(x_{\alpha 1})\cdot \;\cdots\; \cdot f^{\ell}_{k}(x_{\alpha i}),\, f^{\ell}_{k}(x_{\alpha(i+1)})\cdot \;\cdots\; \cdot f^{\ell}_{k}(x_{\alpha k})] \\ &= y_\beta^{-1}[y_{\beta 1}y_{\beta 2}\cdot \;\cdots\; \cdot y_{\beta j}\; f^{\ell}_{k}(x_{\alpha(j+1)})\cdot \;\cdots\; \cdot f^{\ell}_{k}(x_{\alpha i}), \\ &\qquad y_{\beta(j+1)}\cdot \;\cdots\; \cdot y_{\beta\ell} f^{\ell}_{k}(x_{\alpha(i+1+-j)})\cdot \;\cdots\; \cdot f^{\ell}_{k}(x_{\alpha k})] \\ &= y_\beta^{-1}[y_{\beta 1}y_{\beta 2}\cdot \;\cdots\; \cdot y_{\beta j},\, y_{\beta(j+1)}\cdot \;\cdots\; \cdot y_{\beta\ell}] \end{aligned}$$

and $f^{\ell}_{k*}(r_\alpha) \in R^{\ell}_{n}$.

Therefore, $f^{\ell}_{k*}(R^{k}_{n}) \subset R^{\ell}_{n}$, for each n, and hence

$$f^{\ell}_{k*}\left(\bigcup_{n=0}^{\infty} R^{k}_{n}\right) \subset \bigcup_{n=0}^{\infty} R^{\ell}_{n}$$

Thus, $f^{\ell}_{k*}$ is a presentation map which extends to a homomorphism $\phi^{\ell}_{k}$: $G^{k} \to G^{\ell}$, which makes the following diagram commutative




where $\eta_k$ and $\eta_\ell$ denote the natural homomorphisms onto the quotient group.[18]

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[18]Crowell and Fox, p. 41.

Since $f^{\ell}_{k*}$ is onto, it follows that $\phi^{\ell}_{k}$ is also onto. This completes the proof.

## The geometric meaning of the homomorphisms $\phi^{\ell}_{k}$

The geometric meaning of these homorphisms for $G^2$, $G^3$, and $G^4$ can be seen from Figures 1, 6, and 7. The homomorphism $\phi^2_3: G^3 \to G^4$ has the effect of ignoring all loops (horns) $x_3$, $a_3$, $b_3$, and all loops $x_\beta$, $a_\beta$, $b_\beta$, where $\beta$ is any sequence containing the integer 3. Furthermore, the homomorphism tells us that $G^2$ is contained as a subgroup of $G^3$. $\phi^2_4: G^4 \to G^2$ has the effect of ignoring all loops (horns) $x_\gamma$, $a_\gamma$, $b_\gamma$, $c_\gamma$, and $d_\gamma$, where the sequence $\gamma$ contains any integer 2 or 4. That is, it has the effect of ignoring one loop on each side. Similarly, $\phi^3_4$: $G^4 \to G^3$ has the effect of ignoring all loops $x_\gamma$, $a_\gamma$, $b_\gamma$, $c_\gamma$, and $d_\gamma$, where $\gamma$ contains the integer 2; thus, ignoring one loop on the "left."

Because these homomorphisms exist, it is clear that it is unnecessary to produce a non-trivial representation into a known group to show that $G^k$, $k > 2$ is non-trivial. We simply produce a representation for $G^2$ onto some group K, and the non-triviality of $G^k$ follows from the homomorphism $\phi^2_k: G^k \to G^2$.

## DIRECT LIMITS OF SYSTEMS OF GROUPS AND THEIR APPLICATION TO $\{G^k : k \geq 2\}$

### Preliminary comments

The original question of whether $E^3 - M$ and $E^3 - M'$ can be distinguished by their fundamental groups has not yet been answered in this paper, although the preceding discussion does help with examining their structure for $M_i$, i = 2, 3, 4.

It is clear that each of the groups $G^k$ is obtained by a limiting process using the free groups $\{*G_n^k : n \geq 0\}$ and the set of relations $\{R_n^k : n \geq 0\}$. Discussions of the properties of direct limits of groups are available such as that in Eilenberg and Steenrod.[19] However, the discussions are usually for abelian groups, and the proofs make use of the commutative property.[20] Therefore, it was decided to extend this theory for the non-abelian case and to apply the extended theory to the groups $G^k$.

The following exposition follows closely the development in Eilenberg and Steenrod.

### Direct limits of groups

Definition 1. A direct system of sets $\{X, \rho\}$ over a directed set M is a function which relates to each $\alpha \in M$ a set $X_\alpha$ and to each pair $\alpha, \beta \in M$ such that $\alpha < \beta$, a map $\rho_\alpha^\beta : X_\alpha \to X_\beta$, defined as follows:

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[19]Samuel Eilenberg and Norman Steenrod, *Foundations of Algebraic Topology* (Princeton, New Jersey: Princeton University Press, 1952), p. 212-232.

[20]*Ibid.*, p. 6 and 221; cf., the proof of Lemma 4.4.

$\rho_\alpha^\alpha$ is the identity map, and for $\alpha < \beta < \gamma$, $\rho_\beta^\gamma \rho_\alpha^\beta = \rho_\alpha^\gamma$.

Definition 2. Let $\{G,\pi\}$ be a direct system over the directed set M, where each $G_\alpha$ is a group and each $\pi_\alpha^\beta$ is an homomorphism. Let $\prod^*_{\alpha \in M} G_\alpha$ denote the free product of the groups in $\{G,\pi\}$.[21] Let $\alpha < \beta$, and let $g_\alpha \in G_\alpha$. The element $g_\alpha^{-1}\pi_\alpha^\beta(g_\alpha)$ will be called a relator, and

$R = \{g_\alpha^{-1}\pi_\alpha^\beta(g_\alpha): \alpha < \beta \text{ in } M,\ g_\alpha \in G_\alpha\}$

the set of all relators. Let Q be the normal subgroup of $\prod^*_{\alpha \in M} G_\alpha$ generated by R. Then the direct limit of $\{G,\pi\}$ is the factor group

$\overline{G} = \prod^*_{\alpha \in M} G_\alpha / Q$

Definition 3. Let $\eta: \prod^*_{\alpha \in M} G_\alpha \to \overline{G}$ be the natural homomorphism. Then $\eta | G_\alpha = \pi_\alpha : G_\alpha \to \overline{G}$ is a homomorphism which will be called the projection of $G_\alpha$.

Lemma 1. If $\alpha < \beta$, then $\pi_\beta \pi_\alpha^\beta = \pi_\alpha$.

Proof. Let $g_\alpha \in G_\alpha$, and $\alpha < \beta$. $\eta(g_\alpha^{-1}\pi_\alpha^\beta(g_\alpha)) = 1$. Thus, $\eta\pi_\alpha^\beta(g_\alpha) = \eta(g_\alpha)$, but by the definition of projection, we have $\pi_\beta\pi_\alpha^\beta(g_\alpha) = \pi_\alpha(g_\alpha)$.

Lemma 2. If $u \in \overline{G}$, there is an $\alpha \in M$ and a $g_\alpha \in G_\alpha$, such that $\pi_\alpha(g_\alpha) = u$.

Proof. u is an image of some $v \in \prod^* G_\alpha$ under $\eta$. That is, $\eta(v) = u$. Since $\prod^* G_\alpha$ is a free product, v is a finite product of elements of the $G_\alpha$: $v = g_{\alpha_1} g_{\alpha_2} \cdot \ldots \cdot g_{\alpha_k}$, $\alpha_i \in M$. Since M is directed, there is an $\alpha > \alpha_i$, for all $1 \leq i \leq k$. Define

$v_1 = \pi_{\alpha_1}^\alpha(g_{\alpha_1})\pi_{\alpha_2}^\alpha(g_{\alpha_2}) \cdot \ldots \cdot \pi_{\alpha_k}^\alpha(g_{\alpha_k})$

Then

$v^{-1}v_1 = g_{\alpha_k}^{-1} \cdot \ldots \cdot g_{\alpha_1}^{-1}\pi_{\alpha_1}^\alpha(g_{\alpha_1}) \cdot \ldots \cdot \pi_{\alpha_k}^\alpha(g_{\alpha_k})$

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[21] Each $g \in G_\alpha$ is identified in $\prod^*_{\alpha \in M} G_\alpha$ by its image under the injection map: $i_\alpha : G_\alpha \to \prod^*_{\alpha \in M} G_\alpha$, defined by

$$i_\alpha(g_\beta) = \begin{cases} g_\beta, & \alpha = \beta \\ 1, & \alpha \neq \beta \end{cases}$$

$$\eta(v^{-1}v_1) = \eta(g_{\alpha_k}^{-1}. \ \cdots \ .g_{\alpha_2}^{-1})\eta(g_{\alpha_1}^{-1}\pi_{\alpha_1}^{\alpha}(g_{\alpha_1}))\eta(\pi_{\alpha_2}^{\alpha}(g_{\alpha_2}). \ \cdots \ .\pi_{\alpha_k}^{\alpha}(g_{\alpha_k}))$$

But the image of a relator is the identity, and so the whole right member is the identity. Thus, $\eta(v^{-1}v_1) = 1$, and $\eta(v_1) = \eta(v) = u$. By definition, $v_1 \in G_\alpha$, thus $\eta(v_1) = \pi_\alpha(v_1) = u$.

Lemma 3. Let $g \in G_\gamma$ and $\pi_\gamma(g) = 1$. Then there is a $\delta \in M$ such that $\delta > \gamma$ and $\pi_\gamma^\delta(g) = 1$.

Proof. Since $\pi_\gamma = \eta|G_\gamma$, $\pi_\gamma(g) = \eta(g) = 1$. Thus $g \in \mathrm{Ker}(\eta) = Q$. If g is considered as an element of $\prod^* G_\alpha$, it has at least one expression as a product of conjugates of relators:

$$g = a_1 g_{\alpha_1}^{-1}\pi_{\alpha_1}^{\beta_1}(g_{\alpha_1})a_1^{-1}. \ \cdots \ .a_n g_{\alpha_n}^{-1}\pi_{\alpha_n}^{\beta_n}(g_{\alpha_n})a_n^{-1} \tag{1}$$

for some positive integer n; $\alpha_i, \beta_i \in M$, where $\alpha_i < \beta_i$; $a_i \in \prod^* G_\alpha$; $g_{\alpha_i}^{-1}\pi_{\alpha_i}^{\beta_i}(g_{\alpha_i}) \in R$; for all $1 \leq i \leq n$. Now $a_i$ is a reduced word in elements of the $G_\alpha$. Thus $a_i$ is a finite product of elements $b_{ij}$, where $b_{ij} \in G_{\zeta_j}$, for some $\zeta_j \in M$. Let $\varepsilon_i \in M$ be chosen such that $\varepsilon_i > \zeta_j$ for all elements $b_{ij}$ in $a_i$. Let $\delta \in M$ be chosen such that $\delta > \gamma$, $\alpha_i$, $\beta_i$, and $\varepsilon_i$, for all $1 \leq i \leq n$. Since $\delta > \gamma$, $\pi_\gamma^\delta(g)$ is defined; thus, g as expressed in (1) must also have an image in $G_\delta$. Let $\pi^\delta(a_i)$ denote the result of all the mappings $\pi_{\zeta_j}^\delta(b_{ij})$ for each $b_{ij}$ in $a_i$. Then

$$\pi_\gamma^\delta(g) = \pi^\delta(a_1)\pi_{\alpha_1}^\delta(g_{\alpha_1}^{-1})\pi_{\beta_1}^\delta\pi_{\alpha_1}^{\beta_1}(g_{\alpha_1})\pi^\delta(a_1^{-1}). \ \cdots$$

$$\pi^\delta(a_n)\pi_{\alpha_n}^\delta(g_{\alpha}^{-1})\pi_{\beta_n}^\delta\pi_{\alpha_n}^{\beta_n}(g_{\alpha_n})\pi^\delta(a_n^{-1}). \tag{2}$$

The whole right member of (2), clearly, is a product of elements in $G_\delta$. Consider a general relator in (2). By Definition 1 and the properties of each mapping as a homomorphism, we have

$$\pi_{\alpha_i}^\delta(g_{\alpha_i}^{-1})\pi_{\beta_i}^\delta\pi_{\alpha_i}^{\beta_i}(g_{\alpha_i}) = \pi_{\alpha_i}^\delta(g_{\alpha_i})^{-1}\pi_{\alpha_i}^\delta(g_{\alpha_i}) = 1$$

Thus each relator collapses to the identity, and so the whole right member of (2) collapses to the identity. Therefore, $\pi_\gamma^\delta(g) = 1$ as required.

Lemma 4. $\pi_\alpha(g_\alpha) = \pi_\beta(g_\beta)$ if and only if there exists a $\gamma \in M$, $\gamma > \alpha, \beta$, such that $\pi_\alpha^\gamma(g_\alpha) = \pi_\beta^\gamma(g_\beta)$.

Proof. Suppose $\gamma$ exists such that $\pi_\alpha^\gamma(g_\alpha) = \pi_\beta^\gamma(g_\beta)$. The elements $g_\alpha^{-1}\pi_\alpha^\gamma(g_\alpha)$, $g_\beta^{-1}\pi_\beta^\gamma(g_\beta) \in Q$. But then

$g_\alpha^{-1}\pi_\alpha^\gamma(g_\alpha)\pi_\beta^\gamma(g_\beta^{-1})g_\beta = g_\alpha^{-1} \cdot g_\beta \in Q$

since by hypothesis, the product of the two middle factors is the identity. Thus, $\eta(g_\alpha^{-1})\eta(g_\beta) = \pi_\alpha(g_\alpha^{-1})\pi_\beta(g_\beta) = 1$, and $\pi_\alpha(g_\alpha) = \pi_\beta(g_\beta)$ as required.

Suppose $\pi_\alpha(g_\alpha) = \pi_\beta(g_\beta)$. Chose $\delta \in M$ such that $\delta > \alpha, \beta$. Let $g = \pi_\alpha^\delta(g_\alpha^{-1})\pi_\beta^\delta(g_\beta)$. Then

$\pi_\delta(g) = \pi_\delta\pi_\alpha^\delta(g_\alpha^{-1})\pi_\delta\pi_\beta^\delta(g_\beta) = \pi_\alpha(g_\alpha^{-1})\pi_\beta(g_\beta) = 1$

by Lemma 1. By Lemma 3, there is a $\gamma > \delta$, such that $\pi_\delta^\gamma(g) = 1$. Thus

$\pi_\gamma^\delta(g) = \pi_\delta^\gamma\pi_\alpha^\delta(g_\alpha^{-1})\pi_\delta^\gamma\pi_\beta^\delta(g_\beta) = \pi_\alpha^\gamma(g_\alpha^{-1})\pi_\beta^\gamma(g_\beta) = 1$

and $\gamma$ is the required element of M.

Theorem 2. If $\{G,\pi\}$ is a direct system of groups over the directed set M, and for each $\alpha < \beta \in M$, $\pi_\alpha^\beta : G_\alpha \to G_\beta$ has kernel one [or is a homomorphism onto], then for each $\alpha, \pi_\alpha : G_\alpha \to \overline{G}$ has kernel one [or is a homomorphism onto].

Proof. Let $g \in G_\alpha$ such that $g \in \mathrm{Ker}(\pi_\alpha)$. By Lemma 3, there is an element $\delta \in M$, $\delta > \alpha$, such that $\pi_\alpha^\delta(g) = 1$. Since $\pi_\alpha^\delta$ has kernel one by hypothesis, $g = 1$. Thus $\mathrm{Ker}(\pi_\alpha) = \{1\}$.

Suppose that $\pi_\alpha^\beta : G_\alpha \to G_\beta$ is a homomorphism onto for each $\alpha < \beta$. Let $u \in \overline{G}$ and $\delta \in M$. By Lemma 2, there is a $\gamma \in M$ and $g_\gamma \in G_\gamma$ such that $\pi_\gamma(g_\gamma) = u$. Let $\mathcal{E} > \gamma, \delta$. By hypothesis, $\pi_\delta^{\mathcal{E}}$ is onto, so there is a

$g_\delta \in G_\delta$ such that $\pi_\delta^\varepsilon(g_\delta) = \pi_\gamma^\varepsilon(g_\gamma)$. Thus, by Lemma 4, $\pi_\delta(g_\delta) = \pi_\gamma(g_\gamma) = u$, so that $\pi_\delta$ is also onto.

Theorem 3. Let $\{G,\pi\}$ be a direct system of groups over a directed set M, where each $G_\alpha$ is a free group, the $G_\alpha$ are pairwise disjoint, and each $\pi_\alpha^\beta : G_\alpha \to G_\beta$, $\alpha < \beta$, has kernel one. Then $\overline{G}$, the direct limit, is a free group.

Proof. By Theorem 2, $\pi_\alpha : G_\alpha \to \overline{G}$ has kernel one for each $\alpha \in M$. By Definition 3, $\pi_\alpha = \eta | G_\alpha$, where $\eta: \prod^*_{\alpha \in M} G_\alpha \to \overline{G} = \prod^*_{\alpha \in M} G_\alpha / Q$ is the natural homomorphism onto the quotient group. Thus $\eta$ also has kernel one. Since $\eta$ is onto by definition, it is an isomorphism onto, and $\prod^*_{\alpha \in M} G_\alpha \simeq \overline{G}$.

Each $G_\alpha$ is a free group by hypothesis, and the $G_\alpha$ are pairwise disjoint, so it follows that $\prod^*_{\alpha \in M} G_\alpha$ is a free group.[22] Thus, $\overline{G} \simeq \prod^*_{\alpha \in M} G_\alpha$, a free group, so that $\overline{G}$ is also free.

Application to the groups $G^k$

Theorem 3 can be applied directly to our groups $G^k$ (see pages 16 and 17 for the definition). Clearly, $G^k \simeq F^k/Q$, where Q is the normal subgroup of $F^k$ generated by $\bigcup_{n=0}^{\infty} R_n^k$. If $r_n \in R_n^k$, then

$$r_n = x_\alpha^{-1}[x_{\alpha 1} x_{\alpha 2} \cdot \cdots \cdot x_{\alpha i}, x_{\alpha(i+1)} \cdot \cdots \cdot x_{\alpha k}]$$

$\lambda(\alpha) = n$, and $r_n$ can be thought of as defining a function $\pi_n^{n+1}(x_\alpha) = [x_{\alpha 1} \cdot \cdots \cdot x_{\alpha i}, x_{\alpha(i+1)} \cdot \cdots \cdot x_{\alpha k}]$.

$\{n: n \geq 0\}$ is a directed set. Furthermore, it is clear that there is an inductive definition $\pi_m^n : {*G_m^k} \to {*G_n^k}$, for $0 \leq m < n$, and that because $\pi_m^{m+1}$ has kernel one, so does $\pi_m^n$. By definition, the groups $*G_n^k$ are

[22] Massey, p. 103.

pairwise disjoint. Thus all the hypotheses of Theorem 3 are satisfied, and it follows that $G^k$ is a free group.

It is clear that $G^k$ is generated by the elements $\pi_n(x_\alpha)$, $\lambda(\alpha) = n$, $n \geq 0$, and that the number of such elements is countably infinite for all $k \geq 2$.

Therefore, since $G^k$ and $G^\ell$ are both free groups of the same cardinality for all $k, \ell \geq 2$, it follows that $G^k \simeq G^\ell$.

In particular, the groups $G^k$, $k = 2, 3, 4$, are all isomorphic and thus the spaces $E^3 - M_i$, $i = 2, 3, 4$, cannot be distinguished by their fundamental groups.

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