

ENDOMORPHISMS OF HOMOGENEOUS SPACES OF LIE GROUPS

ERKKI LAITINEN

(Received May 6, 1993)

If H is a closed subgroup of a topological group G it is well-known that there is a bijection

$$\text{Map}_G(G/H, G/H) \simeq (G/H)^H$$

which is actually a homeomorphism when the mapping space is equipped with compact-open topology. Homeomorphisms correspond to the subspaces

$$\text{Homeo}_G(G/H) \simeq NH/H.$$

Our main purpose is to prove

Theorem. *If G is a Lie group and H is a closed subgroup then NH/H is open in $(G/H)^H$.*

In [tD, Ch. IV.1] Tammo tom Dieck defines a universal additive invariant $U(G)$ of pointed finite G -CW-complexes for arbitrary topological groups G and computes it for compact Lie groups. As a corollary we obtain that his result is valid for arbitrary Lie groups, too.

Corollary. *$U(G)$ is a free abelian group on elements $u(G/H^+)$ where H runs through a complete set of conjugacy classes of closed subgroups H in G for any Lie group G .*

The condition

$$(O) \quad NH/H \text{ is open in } (G/H)^H$$

was introduced in a study with Wolfgang Lück [LL] in order to define the equivariant Lefschetz class of a G -endomorphism $f: X \rightarrow X$ of a finite G -CW-complex.

The inverse images of the subspaces $NH/H \subset (G/H)^H \subset G/H$ are

$$NH = \{g \in G \mid g^{-1}Hg = H\} \text{ and } SH = \{g \in G \mid g^{-1}Hg \subset H\}$$

and we claim that NH is open in SH , when G is a Lie group. It is well-known that $NH = SH$ when H is compact. As H is closed in G , both NH and SH are always closed in G , so that for Lie groups G the fixed point space splits as a topological sum

$$(G/H)^H = NH/H + (G/H)^{>H}.$$

The Lie theory we need can be found e.g. in the books Helgason [H, Ch.II] or Kawakubo [K, Ch.3].

Reduction to a discrete subgroup

Let G be a Lie group and H be a closed subgroup. We first claim that it suffices to prove the Theorem for all Lie groups G in the case where H is discrete. Indeed, let H_0 denote the unit component of H . Then H_0 is a closed and open subgroup of H and $H/H_0 = \pi_0(H)$. If $g^{-1}Hg \subset H$ then $g^{-1}H_0g \subset H$ is a connected set which contains e , whence $g^{-1}H_0g \subset H_0$. Then it holds for the Lie algebras that $L(g^{-1}H_0g) \subset L(H_0)$, but as they have the same dimension they must coincide. By connectedness $g^{-1}H_0g = H_0$ and $NH \subset SH \subset N(H_0)$. We can therefore assume that $G = N(H_0)$, i.e. that H_0 is normal in G . Then the normalizer of the discrete subgroup $\pi_0(H) = H/H_0$ of G/H_0 is $N\pi_0(H) = NH/H_0$, $S\pi_0(H) = SH/H_0$ and it clearly suffices to prove the claim for the subgroup $\pi_0(H)$ of G/H_0 .

Lie algebra of the centralizer

Let G be a Lie group and let H be a discrete closed subgroup of G . The centralizer

$$ZH = \{g \in G \mid ghg^{-1} = h \text{ for } h \in H\}$$

is a closed subgroup of G and is normal in NH for any closed H . When H is moreover discrete, then it holds $(NH)_0 = (ZH)_0$: each $g \in (NH)_0$ can be connected to e by a path g_t in NH . The corresponding conjugations $c_{g_t}: H \rightarrow H$ give a homotopy from c_g to $c_e = id_H$. As H is discrete, the homotopy is constant and therefore $c_g = id_H$, i.e. $g \in ZH$. We conclude that $LNH = LZH$.

Recall that the adjoint representation of G in LG is defined by attaching to an element $g \in G$ the differential $Ad(g): LG \rightarrow LG$ of the conjugation $c_g: G \rightarrow G$.

Lemma. $LZH = \{X \in LG \mid X = Ad(h)X \text{ for } h \in H\}$.

Proof. A closed subgroup H of a Lie group G is itself a Lie group with Lie

algebra

$$LH = \{X \in LG \mid \exp(tX) \in H \text{ for } t \in \mathbf{R}\},$$

see [H, Theorem II 2.3] or [K, Theorem 3.36]. Hence

$$\begin{aligned} LZH &= \{X \in LG \mid \exp(tX) \in ZH \text{ for } t \in \mathbf{R}\} \\ &= \{X \in LG \mid c_h(\exp(tX)) = \exp(tX) \text{ for } h \in H, t \in \mathbf{R}\} \end{aligned}$$

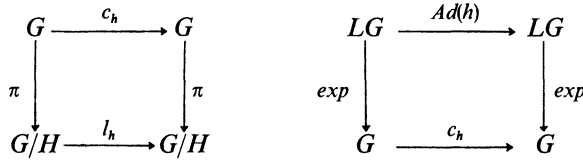
as $Ad(h)$ is the differential of c_h , the last set equals to

$$\begin{aligned} &= \{X \in LG \mid \exp(tX) = \exp(tAd(h)X) \text{ for } h \in H, t \in \mathbf{R}\} \\ &= \{X \in LG \mid X = Ad(h)X \text{ for } h \in H\} \end{aligned}$$

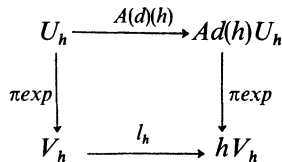
as \exp is a diffeomorphism near the origin. This proves the Lemma.

Proof of the Theorem

Let G be a Lie group and let H be a closed discrete subgroup. The quotient space G/H is then a smooth manifold and the projection $\pi: G \rightarrow G/H$ is a smooth covering projection. Then the diagrams



commute for each $h \in H$, where $l_g: G/H \rightarrow G/H$ is left translation by g and $c_g: G \rightarrow G$ is conjugation by g . As the exponential map \exp is a local diffeomorphism at the origin 0 , $\exp(0) = e$ and similarly π is a local diffeomorphism at the unit element e and $\pi(e) = eH$, the composite $\pi \exp$ is a diffeomorphism of a small enough open disk $U_h \subset LG$ onto its image $V_h \subset G/H$. V_h is an open neighborhood of eH and the diagram



commutes. In particular $U_h^{Ad(h)}$ is diffeomorphic to V_h^h .

By the Lemma $LZH = \{X \in LG \mid X = Ad(h)X \text{ for } h \in H\}$. As the spaces in question are finite-dimensional vector spaces, we can choose a finite set $h_1, h_2, \dots, h_n \in H$

such that $LZH = \{X \in LG \mid X = Ad(h_i)X \text{ for } i = 1, \dots, n\}$. Let $U = \bigcap_{i=1}^n U_{h_i}$ and $V = \bigcap_{i=1}^n V_{h_i}$. The map πexp restricts to a diffeomorphism $U \rightarrow V$, which induces a diffeomorphism

$$U \cap LZH = \bigcap_{i=1}^n U_{h_i}^{Ad(h_i)} \simeq \bigcap_{i=1}^n V_{h_i}^{h_i} = V^{(h_1, \dots, h_n)}.$$

Choose U and consequently V is so small that $\pi exp(U \cap LZH) = V \cap (ZH/H)$ holds. Then the neighborhood V of the point $eH = G/H$ satisfies

$$V \cap (ZH/H) = V \cap (G/H)^{(h_1, \dots, h_n)}.$$

But clearly we have $ZH/H \subset NH/H \subset (G/H)^H \subset (G/H)^{(h_1, \dots, h_n)}$ so in fact equality

$$V \cap (NH/H) = V \cap (G/H)^H$$

holds. Hence $NH/H \subset (G/H)^H$ is open at the point $eH \in NH/H$. Using the left action of NH/H we see that NH/H is open in $(G/H)^H$. This proves the Theorem.

Proof of the Corollary

Recall tom Dieck's definition of the universal additive invariant of a topological group G . An additive invariant consists of an pair (B, b) , B an abelian group and b an assignement which associates to each pointed finite G -CW-complex X an element $b(X) \in B$ such that $b(X) = b(Y)$ if X and Y are pointed G -homotopy equivalent and that the condition

$$b(X) = b(A) + b(X/A)$$

holds when A is a pointed subcomplex of X . An additive invariant (U, u) is universal if every other additive invariant factors through it uniquely. A universal additive invariant is uniquely determined by the usual Grothendieck construction, and it is denoted by $(U(G), u)$.

It follows by an easy argument that $U(G)$ is always generated by the classes $u(G/H^+)$ [tD, Proposition IV.1.8]. Although not explicitly stated, the proof that there are no relations between the classes $u(G/H^+)$ uses implicitly the fact that G is a compact Lie group since it is based on the Euler characteristics $\chi(X^H/NH)$, which are guaranteed to exist if G is a compact Lie group since then X^H/NH is a compact ENR but not otherwise (cf. the example given below.)

Let G be a Lie group. Using our Theorem we can alternatively proceed as follows. As noted in [LL, p. 495], the condition (O) implies that a G -CW-complex structure on X induces a relative NH/H -CW-complex structure on the pair $(X^H, X^{>H})$. The quotient $(X^H/NH, X^{>H}/NH)$ is then an ordinary relative CW-complex (possibly non-Hausdorff) whose cells correspond to the G -cells of X of type G/H . If $n(X, H, i)$ is the number of such i -cells, it follows that the numbers

$n(X, H) = \sum_{i \geq 0} n(X, H, i)$ are G -homotopy invariants of X as

$$n(X, H) = \chi(X^H/NH, X^{>H}/NH).$$

Then $(\mathbf{Z}, n(X, H))$ is an additive invariant such that

$$n(G/H^+, H) = 1, n(G/K^+, H) = 0 \text{ for } K \text{ not conjugate to } H.$$

This proves the Corollary.

Example

We conclude with an example taken from Fuchsian groups. Let $G = PSL(2, \mathbf{R})$ considered as the group of Möbius transformations

$$g(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad a, b, c, d \in \mathbf{R}$$

of the complex plane. Let H be the discrete subgroup of translations

$$h(z) = z + n, \quad n \in \mathbf{Z}.$$

Then it is easy to check that the normalizer NH of H in G consists of translations

$$n(z) = z + b, \quad b \in \mathbf{R},$$

whereas SH equals the affine transformations

$$s(z) = mz + b, \quad m = 1, 2, \dots, b \in \mathbf{R}.$$

In particular NH/H is a circle S^1 and $SH/H = \mathbf{N} \times S^1$.

Taking $X = G/H$ gives an example of a finite G -CW-complex (a zero-cell) with X^H/NH a countable discrete set and therefore of infinite Euler characteristic.

References

- [tD] T. tom Dieck: Transformation Groups, de Gruyter Studies in Mathematics 8, Walter de Gruyter, Berlin-New York, 1987.
- [H] S. Helgason: Differential Geometry and Symmetric Spaces, Academic Press, New York-San Francisco-London, 1962.
- [K] K. Kawakubo: The Theory of Transformation Groups, Oxford University Press, Oxford-New York-Tokyo, 1991.
- [LL] E. Laitinen and W. Lück: *Equivariant Lefschetz classes*, Osaka J. Math. **26** (1989), 491–525.

Department of Mathematics,
P.O. Box 4 (Hallituskatu 15)
00014 University of Helsinki, Finland