Finitism and the Cantorian Theory of Numbers

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This thesis examines finitism and the Cantorian theory of numbers. It gives an overview of relevant mathematical philosophies, in particular the finitistic controversy, and then presents Cantor's transfinite theory and the three principles behind his transfinite theory, namely the domain principle, the enumeral principle, and the abstraction principle. In presenting the principles various objections specific to the principles are raised. The major arguments against Cantor's theory--arguments relating to the endorsement of free mathematics, the use of non-constructive proof, the need to justify his weak reductionism, the existence of non-Cantorian sets, intension in an extensional theory, and tension of increasable infinity with absolute infinity, are made after this.

本論文論述有限主義和康托數論。第一部份簡介相關數學哲學學派和分類，包括有限主義論爭。第二部份介紹康托的超限數論及其背後的三個原則：定義域原則、序次原則和抽象原則；介紹這三個原則的同時亦指出各原則的缺失。最後一部份提出反對康托數論的六個理由：自由數學的不穩、非構造性證明的可質疑性、弱還原主義的問題、非康托集的存在、外延理論中內涵的使用、可增無限與絕對無限的理論衝突。

Contents

Chapter 1

Introduction and Preliminary Discussions

1.1 Introduction

In 1925, Hilbert says before a congress of the Westphalian Mathematical Society that

> [Disputes about the foundations of analysis] have not terminated because the meaning of the *infinite*, as that concept is used in mathematics, has never been completely clarifiied. [38, p.134]

Despite this the situation has not been much improved since. The concept of the infinite is what we will be concerned with in this paper.

The problems we are interested in are simple to state, though perhaps not as simple to make clear: what are infinite numbers, and should or should not they be allowed? And, on a higher level, what do comparisons of infinities mean, and should or should these be allowed? Why, and why not? The philosophical school prohibiting both is known as finitism. Thus in this paper finitism will be one of our foci.

Historically speaking, finitism had not really been a philosophical school until Cantor published his epoch-making papers, for before that there was no finitism because there was nothing as clear and as definite as Cantor's theory to oppose to. For a historical account of related developments, the reader is referred to Section 3.0.1 of this thesis and to Tiles [67].

1.1.1 Overview of the Thesis

Position

My position is that Cantor's justification for transfinite numbers via his theory of numbers does not stand because of the following problems detailed in the thesis, namely the endorsement of free mathematics, the use of non-constructive proof, the need to justify his weak reductionism, the existence of non-Cantorian sets, intension in an extensional theory, and, finally, tension of increasable infinity with absolute infinity.

Contribution

My thesis addresses the issue of infinity in a way that is seldom done. It combines philosophical reflection and technical survey of the relevant concepts. It organises and re-explores the now relatively dormant side of finitists in the debate of finitism vs. Cantor's transfinite theory.

It presents an original system of analysis for analysing mathematical philosophies and casts a clear light on the similarities and differences of the major schools of the philosophy of mathematics which are seldom precisely articulated in any way in discussions in the field. (Chapter 2)

It brings to clear view the philosophically suspect assumptions of his theory in a precise and concise fashion drastically improving on the existing formulations (the three principles in Chapter 3).

It critically examines these assumptions and presents reformulated arguments against them (the six problems in Chapter 4). Of these six objections, free mathematics has not been explicitly raised against Cantors theory of transfinite numbers; the section about non-constructive proof gives a rigorous philosophical discussion of the technical problem, re-organising the myriad of controversy involved; weak reductionism is a problem that texts in the literature do not pay due attention to; the ontological problem caused by non-Cantorian sets to Cantor's transfinite theory has not been looked into at all; Wittgenstein's objection about the use of intension in an extensional theory is relatively well-known, but the subsidiary section about the infinite as a rule in relation to cardinal and ordinal theories is original; tension with absolute infinity has not been presented as an issue against Cantor's transfinite theory.

Reformulated arguments are at once critical and original, for others have surely used similar ones before, but not in this context. The thesis gives a well-articulated account of the whole issue which has not been brought together before and which enables the reader to decide his stance in the matter.

1.1.2 Background

In Section 1.2, we will first try to discuss and clarify philosophically some of the key concepts involved in mathematics and the philosophy of mathematics.

And then in Chapter 2, we will present a brief overview of various schools of mathematical philosophies before diving into the corresponding attitudes towards infinities and transfinite numbers. Firstly, in Section 2.1, we will briefly introduce the main schools of mathematical philosophy, namely nominalism, conceptualism, intuitionism, realism, empiricism, logicism, neo-logicism, formalism, and practicism. After that we will give an examination and explication of mathematical philosophies in the context of metaphysics (Section 2.3), semantics (Section 2.4), epistemology (Section 2.5), foundations of mathematics (Section 2.6), and finally finitistic considerations (Sections 2.7 and 2.8).

1.1.3 About Chapter 3: Details of the Theory

In Chapter 3, we will give some historical and theoretical notes on Cantor's theory (Sections 3.0.1 and 3.0.2). After that we will articulate Cantor's three principles behind his transfinite theory, the domain principle, the enumeral principle, and the abstraction principle.

The Domain Principle

Section 3.1 deals with the domain principle. The domain principle says that for any variable to be meaningful in a mathematical context, there has to be a domain for it to range over. Its consequence is that any potential infinity presupposes a corresponding actual infinity. In mathematics, if an equation with a variable x does not have a domain, the x in the equation would be meaningless. Frequently the variable x is said by classical finitists to denote a potentially infinite quantity. In the context of Cantor's theory, for a variable quantity that is "potentially infinite", actual infinity is its domain. The consequence of the domain principle that any potential infinity presupposes a corresponding actual infinity put advocates of potential infinity in a dilemma, for because of this they cannot coherently endorse potential infinity, while at the same time shunning actual infinity.

Section 3.1.2 deals with the problems associated with the domain principle. The problems with the domain principle is, firstly, the paradoxical nature of an infinite totality, for it is reasonable to doubt if any given "whole" could be genuinely infinite.

The second problem is the primacy of actual infinity. Cantor's argument via the domain principle presupposes that it is not possible for potential infinity to be meaningful without presupposing actual infinity as its domain. Endorsing the primacy or independence of potential infinity at least have the merit of being a weaker claim and thus easier to justify.

The Enumeral Principle

Section 3.2 deals with the enumeral principle. The enumeral principle contends that being a natural number is being the enumeral of a well-ordered set. An enumeral e of a well-ordered set $(E, <)$ stands in such a relation to it if and only if the set of predecessors of e is isomorphic to $(E, <)$. Loosely speaking, the enumeral principle says that numbers are the counting numbers of ordered objects. We know the magnitude of sets because we have ordered their members and we can tell from the structure how big it is. We know how big a set is in much the same way as the way we know how big a hotel is through checking the room number of the last room if its room numbers are given consecutively. Cantor thinks that to be a natural number is to be an ordinal number, and in turn an ordinal number is the enumeral of a well-ordered set. Therefore, he argues, a finite number is not inherently different from an infinite number because each one is respectively the enumeral of a well-ordered set which has the same sort of structure and capable of undergoing the same set-theoretical operations regardless of whether it is a finite well-ordered set or an infinite well-ordered set.

Section 3.2.5 deals with the problems associated with the enumeral principle. The first problem with the enumeral principle is that an ordinal number is in its original sense a *counting* number. But it is not at all possible that transfinite numbers could be counted. Thus the enumeral principle accounting for the natural numbers via the ordinal numbers does not stand very well conceptually.

The second problem is that the well-ordering principle which is presupposed in the enumeralist account is independent in a well-received axiomatic system. If even such an "obvious" principle cannot be proved axiomatically in a canonical system, what ground does it have other than its obviousness? If the well-ordering principle were provable in the system, then the enumeral principle could be a well-grounded, well-fitted account of the natural numbers. But if it were not, then the enumeral principle cannot very well claim precedence over other accounts. If the enumeral principle cannot claim precedence over other accounts, then the equal status that it gives to the finite numbers and the infinite numbers cannot be established convincingly by means of it.

The Abstraction Principle

Section 3.3 deals with the abstraction principle. The abstraction principle says that a number is a cardinal number. A cardinal number is a "pure", definite set composed of abstract units to which all sets with the same cardinality (number of elements) will be equivalent (one-one correspondent). Cantor's argument via the abstraction principle goes like this: if numbers are construed as cardinal numbers, then since the comparison and manipulations of cardinal nmubers is done by means one-one correspondence which is as meaningful and determinate between infinite cardinal numbers as between finite cardinal numbers, infinite numbers have the same status as finite numbers.

Section 3.3.5 deals with the problems associated with the abstraction principle. The first problem with the abstractionist account is that these abstract "ones", these abstract units, are rather problematic. How can we distinguish among the abstract ones and use them to count if they are really abstract and presumable indistinguishable, without presupposing some numeric concept? Notice that no technique bypassing some sort of numeric concept has been employed in Cantor's account, and by the extensionality principle, anything that is indistinguishable with something is identical with that thing.

The second problem with the abstractionist account is that in Cantor's formulation the cardinal number $\overline{\overline{M}}$ of a set M is actually a set in one-one correspondence with M and that it is ontologically more cumbersome than the Frege-Russell logicist formulation.

1.1.4 About Chapter 4: Defects of the Theory

After a brief discussion of problems associated with each principle, we will give more detailed arguments against Cantor's theory in Chapter 4. These include the endorsement of free mathematics, the use of non-constructive proof, the need to justify his weak reductionism, the existence of non-Cantorian sets, intension in an extensional theory, and, finally, tension of increasable infinity with absolute infinity.

Section 4.1 discusses Cantor's endorsement of free mathematics and the use of non-constructive proof. This part's focus is on Cantor's generosity with existence and proof.

Structure and Procedure: Free Mathematics

Section 4.1.1 deals with Cantor's endorsement of free mathematics. Free mathematics is the doctrine that endorses a maximum ontology, allowing existence whenever no inconsistencies result. Cantor's theory seems to be based on a preference for free mathematics. Advocates of free mathematics argue that mathematical objects are free creations of the mind, which is the only constraint, apart from the law of contradiction, to what can be said to exist. Whether this generosity with ontology is appropriate is a difficult question. The advantage of this position is that there is more creative space for the mathematician to work with, while the disadvantage of it is that mathematics under this doctrine has less security, and it gives rise to more difficult foundational questions. The lack of existence proofs and the violation of the simplicity principle also pose problems for Cantor.

Structure and Procedure: Non-constructive Proof

Section 4.1.2 deals with the use of non-constructive proof in Cantor's theory. A constructive proof is a proof in which the existence of a mathematical object or function etc. is not simply proved by establishing that its non-existence is contradictory, but instead proved by showing that algorithmic construction of that object from some accepted primitives is possible in principle. An algorithm is a specification of a stepwise computation which a human being or a machine can, in principle, perform in a finite period of time. Cantor's proofs are non-constructive. Of course, to be fair, he is not in the minority. Most mathematicians prove non-constructively.

The problem of non-constructive proof has its source in the conflict between the realist tendencies of the classical mathematician on the one side and the requirement of an executable algorithm on the part of the intuitionist on the other. This in turn comes from the difference in their aims and ontological views. Classical mathematicians go by the law of the excluded middle, and intuitionist constructivists say that the acceptance of the law of the excluded middle is too metaphysical. The classical mathematician points out that using the Brouwerian counter-example as a criteria for non-constructability entails that whether something is constructive changes with human knowledge, because the Brouwerian counter-example depends on the present stage of mathematical knowledge, and the intuitionist constructivist replies that a mathematical assertion is generally about the construction or the constructedness of a certain mathematical object so that this change of state is not counter-intuitive at all. The classical mathematician laments the confusions and sloppiness caused by the renouncement of logical laws in order to account for human epistemic states, and the intuitionist constructivist replies that even formalists use contentual reasoning instead of exact and mechanical derivations, when they are doing metamathematics. The classical mathematician questions the intuitionist constructivist's sole reliance on intuition which sometimes seems only intuitive to themselves, and the intuitionist constructivist replies that formal logic itself needs ground and it is ultimately our intuition that decides the day. The argument goes on into more minor alleys but the gist is outlined above.

Section 4.2 discusses the need to justify Cantor's weak reductionism, the existence of non-Cantorian sets, and the use of intension in an extensional theory. It questions the conceptual role, the structure, and the specification of sets in Cantor's transfinite theory.

Number and Numerosity: Weak Reductionism

Section 4.2.1 deals with the fact that Cantor holds a kind of weak reductionism. It is *weak* in that he does not simply reduce numbers to sets, but it is *reductionistic* in that numbers and their existence are explained and justified in terms of sets. As we have seen, the ordinal account relies on the well-ordered *set*, and the cardinal account cannot do without the doubly abstracted *set* of units. The problems with this weak reductionism are, firstly, the problem of whether the reduction is philosophically appropriate in terms of ontology, and whether it is pragmatically useful in terms of its relationship with the commonly accepted terrain of mathematics, and, secondly, the problem of the existence of sets. Regarding the reduction there is the problem of definition and construal of numbers, while regarding the existence of sets we have to be concerned with the questions as to whether it is justifiable to postulate sets and why, or why not.

Number and Numerosity: Non-Cantorian Sets

Section 4.2.2 deals with the problem posed by the existence of non-Cantorian sets in some systems. A "non-Cantorian set" as we use it here refers to a set that is not equivalent to the set of its *unit subsets*. Now it is a fact that systems such as Quine's NF [58] admit non-Cantorian sets. This constitutes a problem for Cantor's transfinite theory because it depends on the abstraction principle by which it is defined that a cardinal number $\overline{\overline{M}}$ is a "pure", definite set composed of abstract units to which all sets with the same cardinality (number of elements) will be equivalent (one-one correspondent). Any theory that admits non-Cantorian sets endorses a fundamentally different ontology from Cantorian set theory and renders powerless the abstraction principle which accounts for numbers by means of cardinal numbers and which accounts for the comparison and manipulations of cardinal numbers by means of one-one correspondence, as the non-equivalence of a set and the set of its unit subsets constitutes an insurmountable theoretical difficulty for the abstraction principle.

Number and Numerosity: Intension in an Extensional theory

Section 4.2.3 deals with the use of intension in Cantor's theory which is inevitably extensional. By definition, the extension of an infinite concept cannot be completely listed and, more specifically, the objects in an infinite class cannot be completely listed. Therefore one has to have recourse to intensional definitions, that is, specifying the property which allows and ensures the membership of an element. However, this brings in the problem of the equivalence of intensional definitions and extensional definitions, for one needs this equivalence in that set theory is basically a theory of extensionality, as in general axiomatic set theories explicitly contain an axiom of extensionality. Wittgenstein explained his objections clearly.

Conceivability and Comparability: Tension with Absolute Infinity

Section 4.3 presents the last aspect of arguments against Cantor. Its focus is on conceivability and comparability of infinities.

Section 4.3.1 deals with the tension of increasable infinity with absolute infinity. This tension has to do with the dubious role of absolute infinity and its clash with the domain principle. Cantor argues that natural and real number operations make existence of transfinite numbers inevitable because of the domain principle, and he states on the other hand that the transfinite numbers themselves form a universe (but not a domain) of mathematical forms which constitute absolute infinity. But then why do the transfinite numbers not form a domain likewise, via the domain principle? Cantor does not have a way of satisfactorily resolving this tension between the numerability of transfinite numbers and the unnumberability of absolute infinity, other than invoking God as the only one who can understand absolute infinity, and mentioning the undesirable consequence that this uniqueness would be destroyed if absolute infinity were a domain and could be mathematically determined in the same way as transfinite numbers.

After this synopsis of our arguments it seems also necessary to give before the main parts of this thesis some preliminary discussions and clarifications of the key concepts involved.

1.2 Preliminary Discussions

We state some of the important concepts that we are going to touch on in this thesis. We will give some preliminary discussions and clarifications of those concepts, which are: number, mathematical existence and abstract reality, finite vs. infinite, actually infinite vs. potentially infinite, and denumerability.

1.2.1 number

What's in a number? This is the problem that underlies any view about the foundations of mathematics and mathematical philosophy. Is it some entity in an abstract and eternal realm? Or is it merely a linguistic convenience and adequately reducible to other, arguably more fundamental, entities? We will look at the issues that are related to this problem and, in particular, Cantor's view in Chapter 3.

1.2.2 mathematical existence and abstract reality

What is mathematical existence? Numbers are said to exist as abstract entities, but what does that mean? Is there really a difference between such an existence and no existence at all? This would be one of the recurring themes of this paper, as the ancient opposition between realists and nominalists translates itself into that between abstractist and non-abstractist camps (see Section 2.3), and as constructivists contend that mathematical existence should coincide with constructibility (see Section 4.1.2).

1.2.3 finite/infinite

An infinite set has been characterised as one which can have one-one correspondence with a proper subset of it (*Dedekind* *infinite*), or a non-empty set for which there does not exist a natural number n such that between the elements of S and the elements of the set $N_n = \{x \mid (x \in N) \wedge (1 \leq x \leq n)\}$ there exists a one-one correspondence.

1.2.4 actually/potentially infinite

Actual infinity refers to a completely given, existent (abstractly or not) infinite collection. *Potential infinity* refers to something like an unending operation.

With the development of set-theoretical conceptions, it is sometimes speculated that those who assert that the actually infinite exists mean to say that one can really keep on counting physically forever, while those who assert that only the potentially infinite exists mean to say that only one-one correspondence among infinite sets can be talked of, because it is not a physical procedure as infinity cannot be realized physically. If that is the case, then despite what is popularly believed, it turns out that what physically is possible does have a bearing on what mathematical operations are allowed, perhaps?

However, infinite collections themselves are already problematic in the eyes of some finitists, especially strict finitists, see Section 2.7.1.

1.2.5 denumerability

If a set has $\aleph_0$ elements, that is, if it is one-one correspondent with the set of natural numbers, then we call it *denumerable*.

Alternatively, if a set is of order type ω, that is, if it is of the same order type as the set of natural numbers, then we say it is *denumerable*.

1.3 Concluding Remarks

We have given a preview of the main points of this thesis and preliminary discussions of some of the relevant concepts. We will now proceed to discuss mathematical philosophies and their stance in relation to the problem of infinity. It will serve as a background to our discussion of Cantor's theory.

Chapter 2

Mapping Mathematical Philosophies

The various existing schools of mathematical philosophies form a truly chaotic scene at first glance. Moreover, there has not been a very thorough and fundamental treatment of the subject in an elementary approach. In view of this, we will attempt a taxonomical study, or, in other words, a conceptual analysis of the confusing terrain of philosophies of mathematics in this chapter. After a preview of various schools of mathematical philosophies, we will give an examination and explication of mathematical philosophies in the context of metaphysics, semantics, epistemology, foundations of mathematics, and finitistic considerations, in order to make clear the background for our focus, Cantor's transfinite theory and its problems.

This is an original system of analysis. It casts a clear light on the similarities and differences of the major schools of the philosophy of mathematics which are seldom precisely articulated in any way in discussions in the field.

2.1 Preview

First of all, we will give a brief preview of the mathematical philosophies that will be analysed.

2.1.1 Nominalism

Nominalism in general is the doctrine that abstract concepts, general terms, universals and the like have no independent existence but in names. Nominalism has traditionally been the opposing camp of realism, which asserts that abstract concepts, general terms, universals and the like have corresponding Platonic Forms in an abstract realm. Mathematical nominalism holds that numbers are not independent entities, and what apparently talks about numbers really talks about other rather concrete things, such as mental images, numerals, or some sort of physical objects rather than some sort of abstract objects. Thus mathematical nominalism is an opposing camp of mathematical realism in the same way nominalism is that of realism.

2.1.2 Conceptualism

Conceptualism in general holds that universals exist, but only exist in the mind when they are instantiated in individual objects, and that it has no substance, nor external reality. Historically it is an intermediate view between the extremes of over-liberal realism and over-reductionist nominalism. Kantian conceptualism holds that universals have no external reality because they are exclusively produced by our a priori mental framework. Mathematical conceptualism asserts that numbers are abstract entities created by this a priori mental framework for its understanding of the world. Therefore mathematical conceptualism also, in a certain sense, stands in the middle ground between the reductionist tendencies of mathematical nominalism and the allowing spirit of mathematical realism.

2.1.3 Intuitionism

Kronecker [41] was probably the first person befitting the name enough to be called an intuitionist, for he was a precursor to Brouwer in asserting that natural numbers and their operations are *intuitively founded*, and that real numbers cannot have such a foundation. Brouwer's intuitionism [7, 8, 9] maintains that the truth of a mathematical statement is equivalent to the mathematician's being able to intuit the statement. It also maintains that numbers are creatures of the mind and truths about them are known through pure intuition. This school of mathematical philosophy could be regarded as a branch of conceptualism except that it has explicit methodological commitments that conceptualism is not known to endorse, e.g. that all definitions and proofs should be *constructive*, which means that a definition of a mathematical entity should give a rule which enables one to construct it from mathematical elements already known to exist.

Heyting summarises Brouwer's position thus

> The idea that for the description of some kinds of objects another logic may be more adequate than the customary one has sometimes been discussed. But it was Brouwer who first discovered an object which actually requires a different form of logic, namely the mental mathematical construction [6]. The reason is that in mathematics from the very beginning we deal with the infinite, whereas ordinary logic is made for reasoning about finite collections. [37, p.1]

It has been suggested that Brouwer's position is merely a methodological maxim in Ambrose [1, p.610]. Anyway, methodological maxim or ontological creed, a distinguishing characteristic of intuitionism lies in its so-called rejection of the (universal applicability of the) law of the excluded middle, for example Heyting says that

> [Intuitionists] consider an integer to be well defined only if a method for calculating it is given. Now this line of thought leads to the rejection of the principle of excluded middle [...] [37, p.2]

(Well-trained logicians emphasize their never committing to the stance that any sentence can be substituted into the formula $p \vee \neg p$ which is tautological in classical propositional logic. For more about intuitionism and the law of the excluded middle, see Section 4.1.2.)

2.1.4 Realism

Realism in general could be described as the view that statements describe a mind-independent reality. Metaphysical or Platonistic realism grants universals and such like Platonic existence. Its allowance of an immense abstract realm is directly opposed to the nominalistic tendency towards accounting for the use of universals and names by means of concrete individuals. Mathematical realism refers to the school of thought that takes numbers to be mind-independent entities (the broad sense of "mathematical realism"), and frequently, furthermore, the school of thought that not just takes numbers to be mind-independent entities, but also takes them to be Platonic entities the truths about which are known through a priori rational insight (the narrow sense of "mathematical realism").

Mathematical realism in the narrow sense (in order to avoid ambiguity, we will call this *Platonistic realism*) and mathematical conceptualism, although both are, so to speak, rationalist in epistemological commitments, are nevertheless different in that their rationalist knowledge comes from different realms—

the former from the mind-independent, and the latter from the mind-dependent.

2.1.5 Empiricism

Empiricism in general is the view that the role of experience, especially sensory perception, is indispensable in the formation of ideas and the verification of truth. It is a branch of mathematical realism in the broad sense of the term because it grants mathematical entities such as numbers (and sets?) the same status of being as theoretical entities such as quarks and black holes.

Mathematical empiricism holds that, contrary to popular belief, mathematical "truths" are theoretical hypotheses about the natural world, that they are part of the holistic web of knowledge. Contemporary proponents of mathematical empiricism, notably Quine and Putnam [56, 57] (Putnam favours the use of the term "pure realism"), hold that mathematics owes its justification in its indispensability in scientific enquiry. (See Colyvan's account [19] for an extended discussion of this school.)

Although mathematical realism in the narrow sense (Platonistic realism) is rationalist in epistemological principles, mathematical empiricism is undoubtedly empiricist in epistemological principles, since the two schools really have nothing to do with each other, as mathematical empiricism is only a branch of mathematical realism *in the broad sense.* Much confusion is apt to arise if the distinction between mathematical realism in the broad sense and in the narrow sense is not made clear.

2.1.6 Logicism

Logicism, exemplified in the monumental work of Whitehead and Russell [72], is the view that mathematics can be reduced to logic, and that mathematical truths are analytic logical truths. Logicism claims that mathematics can be reduced to logic, by which is signified the propositional calculus, plus the quantificational calculus, plus set-theoretic operations and some axioms which assert the existence of, e.g. representative sets (axiom of choice). Whether that should be called logic will be a point of contention we do not intend to go into yet.

Historically, the logicist enterprise first started to be seen as a viable approach to the foundations of mathematics in the work of Frege [27], in which he tried to create a less ambitious version of Leibniz's *lingua characterica* (cf. the historical accounts of this in van Heijenoort [70, p.2] and Mayberry [47, p.214]), a precise symbolic language of logic for expressing content (a "concept script", or an "ideography" in Russell's wording in [60]), not just a *calculus ratiocinator*, a formal system only for computations. The *Begriffsschrift* was deliberately made distinct from the language of arithmetic by using different symbols. It is in such a type of symbolic system that Frege intended to provide a foundation for arithmetic. Frege went on to formulate the basic laws of arithmetic in [28, 29]. But, as we all know, his system, which employs the unrestricted comprehension principle (commonly referred to as *Frege's fifth axiom*) resulting in a so-called naïve set theory, was inconsistent because it is susceptible to Russell's paradox.

Whitehead and Russell [72] tried to avoid this by means of the theory of types, necessitating the addition of a further axiom, that of reducibility, adding yet another burden on the problematic nature of the axioms of the system. This is no doubt a great drawback of their system, for if the axioms of the system itself are problematic, how can it serve to be the foundation for mathematics? More on this in Section 2.6.2.

2.1.7 Neo-logicism

Neo-logicism is commonly characterised as the view which asserts that mathematics is applied set theory. It purports to replace logicism, which has severe difficulty in maintaining that what it assumes is logic. (Some argue that it should be third-order non-modal object theory instead of applied set theory, because it most closely answers to the goals of the original logicist programme. See Linsky and Zalta [43] for a presentation of this argument.) Neo-logicists try to draw a larger terrain for logic or loosen the criteria for what it is to be "reducible to logic." Neo-logicists such as Wright [75] and Hale [33] try to replace Frege's fifth axiom with other principles, for example *Hume's principle* which asserts that the number of Fs is equal to the number of Gs if there is a one-one correspondence between the Fs and the Gs.

2.1.8 Formalism

Probably in order to accommodate competing axioms systems such as those in geometry, and to evade attacks from critics expressing uneasiness regarding the suspect character of mathematics and its lack of universally accepted foundations, Hilbert's formalism maintains that mathematics can be regarded as but a meaningless game with marks, and that formal consistency is all that needed for its playability. He tried to prove the consistency of arithmetic within a finitary formal system (see Section 2.7), but we would use "formalism" loosely to refer to the use and endorsement of formalisation for providing a foundation for mathematics and securing it against paradoxes and inconsistencies.

In fact, formalists can "hedge their bets", for "the formal development of ZFC"[1]—and other systems really—"makes sense

[1] Zermelo-Fraenkel set theory with axiom of choice; see Tiles [67, pp.121–134] for a brief account of the system.

from a strictly finitistic point of view: the axioms of ZFC do not say anything, but are merely certain finite sequences of symbols. The assertion ZFC $\vdash \Phi$ means that there is a certain kind of finite sequence of finite sequences of symbols—namely, a formal proof of Φ. Even though ZFC contains infinitely many axioms, notions like ZFC $\vdash \Phi$ will make sense, since one can recognize when a particular sentence is an axiom of ZFC". [42, p.7] So that formalists can do mathematics as uninhibitedly as a mathematical realist, "but if challenged about the validity of handling infinite objects, he can reply that all he is really doing is juggling finite sequences of symbols". [42, p.7] We will call this premeditated reply to projected challenge *finitary formalism* and treat it in Section 2.7, while using *formalism* in the sense of maintaining that mathematics *can be regarded as* merely a meaningless game with marks, with the ontological implications of mathematics *being* merely a meaningless game with marks *which can only be finite in number* not asserted.

In fact, the professed ontological commitment of formalists is null while they do the same mathematics as mathematical realists, but that is because formalists treat mathematics as void of meaning. But this void of meaning is set down in marks, if not physical then realisable if required, so that the metatheory that talks about these marks is in effect finitistic.

Formalism was, together with intuitionism and logicism, one of the three main schools of philosophy of mathematics in the twentieth century. However, since Gödel shown in his second incompleteness theorem [32] that the consistency of a system of arithmetic cannot be proved within itself, formalism has lost much of it charm, for if one has to rely on another system to prove its consistency, then one might as well give up insisting on its purely formal character, as surely one cannot accept the meta-system merely on formal ground. This is because, as the meta-system is of a higher level, surely again its consistency cannot be proved within itself, so that there is the danger of infinite regress unless one accepts the system on some other ground or one does not bother about consistency at all. But consistency is very important in a system, for in inconsistent systems any well-formed formula could be proved, rendering proof creditless and the system creditless. Therefore one has to accept the meta-system on some other ground.

2.1.9 Practicism

Practicism, the word we coined for an easy reference of the school of thought that regards mathematics as a group of truths about counting procedures, is a view that emerges after the so-called demise of formalism. One *caveat*, however: one seems to be able to regard mathematics as a group of truths about counting procedures only until transcendental numbers, as transcendental numbers can never be said to be actually used in counting or measurement because even if one can keep counting and never stop, allowing for a potential infinity of natural numbers, it seems that one cannot measure the infinitesimal as there are inevitably marginal errors.

Now we are going to start our analysis.

2.2 Central Problem of Philosophy of Mathematics

The central problem of philosophy of mathematics is the nature of a number.[2] But the nature of a number can be probed from various depths and from various angles. The nature of a number *per se* is the metaphysical problem of number in the philosophy of mathematics; the meaning and reference of the numerals and

[2]Of course there are philosophical problems surrounding geometry and other branches of mathematics, but this is at least traditionally the central one. Anyway this is the one that concerns us here.

the meaning of mathematical statements is the semantical problem of number in the philosophy of mathematics; the possibility of knowledge about numbers and of the truth of mathematical statements is the epistemological problem of number in the philosophy of mathematics; the need or futility of a systematic theory of numbers is the foundational problem of mathematics; a consideration of all the above-mentioned sides of the problem of number results in a view concerning the finitistic problem of mathematics.

Therefore the following is an analysis of views regarding the nature of numbers and mathematical statements in terms of metaphysics, semantics, epistemology, foundations of mathematics, and finitistic considerations in that order, and they are philosophically speaking the most interesting and relevant issues for our purpose.

2.3 Metaphysics

The ontological problem in the philosophy of mathematics is whether numbers as such exist (in various senses of the word), and, if they do, in what form.

2.3.1 Abstractism

The abstractist view of numbers takes numbers to be genuine abstract entities, not to be identified with any spatio-temporal objects or to be taken as shorthand for counting procedures or similar operations. (We made up the term *abstractism* because neither the term "Platonism" nor the term "realism" seems to be adequate for our purpose here, for both imply some definite views about the sort of existence the mathematical objects lead.)

2.3.2 Abstractist Schools

Kantian conceptualism asserts that numbers are abstract entites created by the mind, and owes their existence to postulation. They are mind-dependent, but still objective. Therefore its view of numbers should be abstractist.

Similarly, for Brouwer's intuitionism, numbers are creatures of the mind—to exist is to be constructed by the mathematician, according to Brouwer's student Heyting [37, p.2]. Mathematical objects are constructed, but do exist nevertheless.

Those philosophers who hold Platonistic realism unconditionally take numbers to be mind-independent abstract objects, Forms. It might be said to be the most abstractist of all.

Logicism, a kind of softened Platonistic realism, takes numbers to be logical constructs, which are also mind-independent abstract objects.

Neo-logicism adheres to the same ontological commitments as logicism.

Mathematical empiricism regards numbers as some sort of being which is akin to the theoretical entities postulated in science, such as quarks, so that numbers are regarded as real provided that the mathematical empiricist also embraces scientific realism, which is frequently the case for otherwise it is pointless to account for mathematics in this way as its status would not be made less questionable.

2.3.3 Non-abstractism

The non-abstractist view of numbers does not take numbers to be abstract entities, and rather identifies numbers with mental images, ideas, or even physical objects.

2.3.4 Non-abstractist Schools

Nominalism denies that numbers are abstract entities, and, instead, identifies them with mental images, psychological ideas, physical objects, the corresponding numerals, etc., and contends that numbers exist but in names, i.e. it is merely a way of talking about other objects.

Formalism has no ontological commitment regarding numbers, as mathematical systems are only formalized systems without attestation of content in the formalist's view.

Practicism, the view that mathematics is a body of truths regarding counting methods, does not endorse the existence of numbers as abstract entities. This touches on the problem of what to be a rule is and what to be a number is, more discussion of which will appear in Section 4.2.3.

2.4 Semantics

The semantical problem in the philosophy of mathematics is whether mathematical statements are literally construed, and how are they to be interpreted. The problem of construal concerns our understanding of mathematical objects, as well as the problem of interpretation our understanding of them in relation to the world.

2.4.1 Literalism

The literalistic view of mathematics sees mathematical statements as literally construed, i.e. mathematical statements actually talk about some sort of objects, whatever they might be.

2.4.2 Literalistic schools

Nominalism, Kantian conceptualism, intuitionism, realism, logicism, neo-logicism, and empiricism all hold the literalistic view.

Nominalism takes numbers as various concrete objects, so that even though numbers are not construed as abstract entities, they are construed as entities nevertheless.

Conceptualism and intuitionism take them as mentally constructed abstract entities; therefore they also belong to the literalistic camp.

Realism regards them as Platonic objects. Thus it is quintessentially literalistic.

Logicism regards them as logical constructs, with an abstract existence. Therefore, numbers are literally construed in this case. However, as regards interpretation, it brings out the problem that logical truths are supposed to be true in all models (at least the ones with at least one object) in model-theoretic semantics, whereas classical number theory (we are not even talking about mathematical analysis) needs a model with $\aleph_0$ objects.

Neo-logicism also regards numbers as logical constructs which exist as abstract entities.

Empiricists think numbers are much like other theoretical entities in science. Hence they are construed literally and interpreted as scientific terms are.

2.4.3 Non-literalism

The non-literalistic view of mathematics does not regard mathematical statements as literally construed.

2.4.4 Non-literalistic schools

Formalism, in its most defensive moments, holds that mathematics has no meaning and is but a game with marks, and that the axiom systems such as ZF[3], NF[4], and NBG[5] are simply different games with marks. Indeed this variety troubles the formalist least, and is one of the prominent advantages of this view.

Practicism, likewise, does not regard mathematics as literally construed because it merely talks about counting procedures in a roundabout way. Again the niceties regarding this will be treated in Section 4.2.3.

2.5 Epistemology

The epistemological problem in the philosophy of mathematics is whether we have mathematical knowledge, and if we do, by what means.

2.5.1 Scepticism

Mathematical scepticism does not see mathematical systems as providing knowledge. Obviously, if one accepts the tripartite definition of knowledge, there can be but two cases in which mathematics fails to be knowledge, the first in which it is meaningless; and the second in which it is meaningful, but unjustified, or false. The problem of meaning is in part taken care of in Section 2.4, and we will proceed on the basis of it.

2.5.2 Scepticist Schools

Nominalism, by identifying numbers with various concrete objects, forfeits mathematics of a true interpretation, as no concrete objects are numerous enough for infinite numbers, and thus no theorems assuming their existence can be true.

[3]Zermelo-Fraenkel set theory with or without axiom of choice depending on context, cf. note on ZFC in Section 2.1.8.

[4]Quine's "New Foundations", see his [58].

[5]Von Neumann-Bernays-Gödel set theory, see Gödel [31].

Formalism, denying that mathematics has meaning, does not hold that it expresses knowledge. It is, at most, to be viewed as a guide for manipulating empirical statements, and its formal consistency is its highest attainable merit. As mentioned before, this accounts best for the competing system of axiomatisations in use. A modern version of formalism is the advocation of mathematics as a study of axiomatic systems and their consequences, providing us with a type of meta-level "knowledge."

2.5.3 Non-scepticism

The non-scepticist view of numbers asserts that mathematics is indeed a body of knowledge, that it consists of meaningful, true, and justified beliefs.

2.5.4 Non-scepticist Schools

Conceptualism holds that mathematics contains a priori synthetic truths about abstract entities created by the mind.

Intuitionism holds that mathematical laws hold true of things as the mind intuits them, i.e. they are truths known by pure intuition, and that there are no unknowable (in other words, not constructively provable) truths in mathematics. In other words, intuitionism maintains that we know mathematical truths and that all mathematical truths are knowable, that mathematical truths and mathematical knowledge are co-extensional.

Realists think that what we know in mathematics is discovered through a priori rational insight which sees into the realm of abstract beings, and therefore axioms and theorems are true statements about abstract entities.

Logicists maintain that mathematical laws are disguised logical truths known through rational insight.

Neo-logicism holds that mathematics is reducible to an extended logic or set theory and that we know its truth through rational insight.

Empiricism argues that mathematics is a section of scientific knowledge.

Practicism asserts that mathematics is a body of truths about counting methods.

All of the above schools are non-scepticist.

2.6 Foundations of Mathematics

The foundational problem in mathematics[6] is whether there is a foundation for mathematics, and what, if there is any, it is. A useful distinction in related discussions is that between *intuition* and *ingenuity* made by Turing.

Turing articulated about this distinction between *intuition* and *ingenuity* in [68, Section 11]. The following is an expansion of this idea. Intuitive judgments are the results of frequently implicit trains of reasoning. Intuitive judgment of, e.g., whether positive integers are uniquely factorisable into primes usually needs some other means by which we verify the judgment. Ingenuity, on the other hand, aids the mathematician in finding the suitable arrangements of statements, etc. when he wants to verify his intuitive judgment. The use of ingenuity consists in making a well-arranged collection of statements in which the validity of the intuitive steps is beyond reasonable doubt.

In a formalized system, the role of intuition is confined to the stated formal rules abiding by which the inferences made will always be agreed to be intuitively valid. (However, as consistent systems complicated enough to express arithmetic are not complete, it might be necessary to introduce new axioms from

[6]The key terms "foundation", "foundations", "foundational" in its various combinations with the word "mathematics" differ slightly in meaning but the author strived to be clear about what is being meant in the context when such combinations are used. Nevertheless due to variations and confusions in the field it is impossible to give a fruitful and precise definition of those combinations.

time to time, and this is an exercise that necessarily employ the faculty of intuition, so that it cannot be simply dispensed with once the formal rules have been laid.) In a formalized system, the role of ingenuity, on the other hand, consists in deciding, among a variety of legitimate steps, which ones are more efficient for proving the would-be theorem.

A liberal allowance of a vast number of stated axioms and rules can be view as a characteristic of advocates of intuition, by means of which those axioms and rules are accepted and justified. This promotes efficiency of proof.

In contrast, a puritan urge to derive all the desired theorems in a system with the smallest number of axioms, however difficult and cumbersome proofs would be, can be seen as a trait of stereotypical supporters of ingenuity. And this promotes economy of the system.

In reality, however, this is but a dramatic presentation of this pair of concepts, and the reader must not be misled into thinking that we are trying to say that these could only be the opposing sides taken by participants in a controversy. In the contrary, these two frequently appear as rivalling tendencies in one and the same person.

When one reflects on the respective foundational views of intuitionists and formalists one will see that they agree in the role of ingenuity, but the former affirms the role of intuition while the latter thinks that its role is, or should be completely replaced by formal rules.

2.6.1 Foundationalism

Mathematical foundationalism is the view that there is something that serves as a foundation for mathematics. However, foundationalist schools differ widely in their views regarding the nature of the foundations of mathematics.

2.6.2 Foundationalist Schools

Nominalism contends that mental images, numerals and such like are what mathematics is talking about, and they are to be the things that numbers are to be interpreted as. These can be said to serve as the foundation of mathematics.

Conceptualism and intuitionism assert that the foundation of mathematics lies in intuition, by which means mathematical truths are known and justified.

Realism sets the foundation of mathematics in the Platonic realm in which mathematical entities dwell, and axioms are true because they describe the things in it.

Logicism reduces mathematics to logic, by which is signified the propositional calculus, the quantificational calculus, set-theoretic operations, and some axioms of existence. As hinted before (Section 2.1.6), despite its attractive appearance, proponents of logicism have severe difficulty convincing others that the system they use is unquestionable, that it can be properly called a system of logic, and that mathematical truths can be called logical truths.

Neo-logicism places the foundation of mathematics on set theory or third-order logic. The former seems well-attested by the practice of mathematicians, though they do not try to derive most of their work in set theory.

Empiricism lays the foundation of mathematics in its share in the tremendous power of science to which it is claimed to be indispensable. Hartry Field, however, constructed an axiomatisation of Newtonian mechanics without the use of numbers. (To examine this feat see his book, [24].)

Practicism holds that mathematics embodies what we know about counting procedures, and its foundation lies in its being a correct description of these procedures, providing a foundation for non-transcendental numbers.

2.6.3 Non-foundationalism

Mathematical non-foundationalism is the view that no external foundation needs to be laid for mathematics, that mathematics should be done "as is".

2.6.4 Non-foundationalist schools

Formalism seems to be the only school that demands no external foundation for mathematics. It affirms the merely formal character and the self-sufficiency of mathematics, and sought to prove its consistency within itself, which enterprise has later been shown unaccomplishable by Gödel's second incompleteness theorem. [32] But of course with its having proved that it is impossible to establish a proof of the consistency of a mathematical system within itself, formalism seems obliged to look for something external to a system.

We now proceed to the finitistic problem of mathematics, the position regarding which is often the result of a consideration of all the above-mentioned sides of the problem of number.

2.7 Finitistic Considerations

The finitistic problem of mathematics is whether potential infinity and actual infinity should be allowed. In Chapters 3 and 4 we will discuss Cantor's theory of transfinite numbers and the principles behind and specific objections to his theory, but here and in the next section we merely give a brief presentation of the positions of major philosophical schools on infinity and the diagonal proof respectively in order to provide some context to our later discussions.

As one can guess from the name, finitism refers to the doctrine that reference to infinite collections is to be eschewed because their meaning is uncertain and their formation problematic.

Finitary Formalism

In Hilbert's finitism (his finitary foundation for formalised arithmetic), for example, this eschewal of infinite collections comes with the epistemically motivated desire to replace all abstract concepts with concrete, visualisable notation.[7] When abstract concepts are abandoned and replaced with concrete, visualisable notation, no implicit inconsistency could arise, as it is simply a game with marks, and inconsistent rules in a game are relatively easy to spot. One might find Hilbert's finitism similar to concrete nominalism, the particular brand of nominalism equating talk of numbers with talk of numerals (cf. Section 2.1.1). But the similarity is only on the surface, for finitary formalism admits ontological interpretations. Hilbert's idea was that a metamathematics with its subject matter, the mathematics confined to the realm of the visualizable would be secure against paradoxes, while conventional theory about proofs would be at best precarious. Hilbert thinks that conventional theory about proofs would be at best precarious because it does not confine the proper objects of the theory to concrete symbols alone, so that it would be harder to spot any paradoxes in the theory. Skolem's primitive recursive arithmetic [61] and Yessenin-Volpin's ultraintuitionism [77] are two examples of twentieth-century finitistic schools besides Hilbert's finitism.

The most liberal formulation of these finitisms would be that only natural numbers or items encodable as natural numbers are said to be well-defined. (More strict formulations do not even allow that.) Advocating whichever variety of the formulations, these finitists do not conceive numbers as constituting an

[7]The discussion in this section highlights distinctions which are frequently confused in the literature, especially introductory texts.

infinite totality but as, for example, individually realisable physical signs, or tally marks (i.e., "1", "11", "111" etc.). Hilbert says that in finitary formalism natural numbers are construed as strings of tally marks. He wrote that

> The subject matter of mathematics is, in accordance with this theory, the concrete symbols themselves whose structure is immediately clear and recognizable. [38, p.142]

Hilbert envisages this as the foundations for a complete arithmetic. When this is supplemented with *ideal statements* (see below) he would be able to formulate all of arithmetic and avoid very complicated logical laws. If there were no ideal statements, complicated logical laws would have resulted from the requirement that existential statements have to be analysable into finitary disjunctions and universal statements into finitary conjunctions, as this requirement renders unbounded existential statements and unbounded universal statements inadmissible. Ideal statements are statements making assertions of a wide scope such that their content is not reducible to finitary conjunctions. With the help of ideal statements one can reintroduce unbounded statements, for example the denials of general statements such as

$$\neg\forall x(x+1=1+x)$$

(it is not the case that for all x, $x+1$ equals $1+x$), and unbounded existential statements such as (with "$Prime(x)$" meaning "x is a prime number")

$$\exists x(Prime(x) \wedge x > p)$$

(there exists some x such that x is prime and x is larger than p) which is derived from

$$\exists x(Prime(x) \wedge x > p \wedge x < p!)$$

(there exists some x such that x is prime and x is larger than p and x is smaller than $p!$).

"$\neg\forall x(x+1 = 1+x)$" is ideal because the search for a counter-example to the universal statement "$\forall x(x+1 = 1+x)$" that gets negated is not bounded. That is to say, a counter-example to $\forall x(x+1 = 1+x)$, a counter-example a such that

$$\neg(a+1 = 1+a),$$

if there is one, can occur anywhere in the vast ocean of numbers.

By a different "mechanism", "$\exists x(Prime(x) \wedge x > p)$" is also ideal because the search for a specimen confirming the claim, some b such that

$$Prime(b) \wedge b > p,$$

is again not bounded. Both types of ideal statement serve the function of preserving classical logic as the logic of arithmetical thinking. Ideal statements such as "$\neg\forall x(x+1 = 1+x)$" maintain the law of the excluded middle for unnegated universal statements because otherwise the negations of those unnegated universal statements, which are unbounded statements, could not be made in a finitary formalistic framework; ideal statements such as"$\exists x(Prime(x) \wedge x > p)$" preserve the classically valid inference scheme of simplification in a quantificational context—the scheme of deducing "$\exists x(\phi x)$" from "$\exists x(\phi x \wedge \psi x)$".

Finitism and the Philosophy of Mind

One of the most intriguing problems in philosophy has to do with the limits of the mind.[8] Related problems have been and continue to be probed by means of different approaches. In recent

[8]The following passages until Section 2.7.1 are of interest for it has implications that are to a certain extent significant in relation to the finitistic problem. The following passages put forward concepts that are crucial to an inspection to the limits of rational knowledge and this inspection is in a sense prior to a proper treatment of Section 2.5 and the problem of conceptualist stance in Section 2.7.2. Thus it is tangentially related to our main focus, finitism and Cantorian's theory of numbers. However those implications would have to be left for another essay.

years, in particular, there is the debate about the significance of Gödel's theorems in philosophy of mind between, on the one side, J. R. Lucas [44, 45] and R. Penrose [53, 54, 52] trying to establish definitive consequences of those theorems and, on the other, E. R. Nagel and J. R. Newman [50, 51] being sceptical of those consequences.

Finitistic thoughts in philosophy of mathematics are highly relevant to philosophical investigations into the limits of the mind. This is because rational thinking and rational choice involve computation. And unless it is mostly made up of totally non-formalisable and non-articulatable "insight", it falls under the governance of metamathematical theorems to the extent that it can be formulated as computation, distributed or otherwise. This is due to the fact that if a subject is precise enough, then it would be readily subsumable under discussions in mathematical logic and computational theories, and to the fact that if something is subsumable under discussions in mathematical logic and computational theories, then all the theorems about the type of systems that it belongs to would be applicable to it. For example, game theory is a branch of mathematics, and therefore many metamathematical results are applicable to it.

The modern development of metamathematics has much to do with tackling the epistemological problems of mathematics and thus surveying our power to know the truths of mathematics genuinely.

Finite Constructibility

We could start our discussion with finite constructibility. It is common to require that a proof in logic to consist of a finite number of steps, but it is not common to require that an object be constructible in a finite number of steps. By so requiring, finitism or constructivism (in one of the uses of these words) gives a stricter-than-normal criterion for mathematical objects and operations to be accepted as meaningful.

However, this restriction arguably represents a more realistic estimation of the power of the mind, because finitely constructible objects are something that the mind can surely grasp, while objects that are not finitely constructible seem not so secure in this respect. Representing a more realistic estimation of the power of the mind might not intrinsically be an advantage of this school of thought, as whether it is an advantage depends on ontology and a whole bunch of other philosophical issues, but it is an important point because we want mathematical knowledge to be absolutely secure.

From constructibility arise considerations of effective methods, because when one prescribes the construction of a set, it might involve giving an effective method for deciding, given any object, whether it is an element of the set.

Effective Method

Intuitively, an *effective method* (for solving a problem) is a mechanical method that requires no insight and is precisely specified. This method is logically bound to yield the right answer to the problem in a finite number of steps, if followed correctly. An example of an effective method is the Euclidean algorithm for determining the greatest common divisor of two integers.

The various formulations of effective method in mathematical logic give a paradigm for procedural determinability. One of these formulations is Church's thesis of identifying recursiveness[9] and the intuitive notion of effectiveness. One might have hoped that these precise formulations also capture all of rational thought. Sadly, it has already been proved that there

[9] A recursive function can be said to be a definition by means of mathematical induction. Roughly speaking it consists of a set of equations such that one of them gives the value of the function for the argument 0, and other equations give the value of the function for the argument $k+1$ in terms of its value for the argument k.

is no effective method for deciding, in general, given any well-formed formula in a first-order predicate logic (except in first-order monadic predicate logic), whether it is a theorem of it. (Moreover, it has also been proved that it is not possible to prove formally every truth in elementary number theory using one axiomatisation.)

The proof that there is no effective method for deciding, given any well-formed formula in a first-order theory, whether it is a theorem of it shows that the idea of effective method does not capture all of the workings of the mind, as humans can look for a proof of the formula and find out if it is a theorem. Therefore, it shows at the same time that the way the mind thinks provides rational knowledge beyond the reach of an effective method.

What about machine computation, can it capture and theoretise once and for all the rational knowledge that the mind is able to work out?

Turing Machines

The Turing test sets the goal for computational simulation. If a programme simulating human conversation is able to "fool" people into believing that its responses come from a real, conscious being, it means that not only does a Turing machine (or equivalent theoretical elucidations/reductions such as λ-calculus or recursive functions) achieve flawless computation, but it also thinks like a conscious being, for all that other minds can judge based on its "behaviour". However, a deterministic Turing machine would not be able to pass the Turing test because it cannot satisfactorily simulate human behaviour.

For effective computability, a computation have to halt in less than ω steps. On the other hand, if one does not limit oneself to effective computability, one could explore the possibility of allowing the sequence of steps to have the order type of e.g. $\omega+n$, which would constitute a first grade of hypercomputation.

However, it is important to recognize that removing an arbitrary finite ceiling on the number of steps of computations—treating computations as potentially infinite—in the informal idea of computation is quite different from allowing transfinite orderings of steps—allowing actual infinities of computational operations—in formal definitions of computation. Treating computations as potentially infinite in the informal idea of computation is tantamount to turning a blind eye to the problematic carrying out of a potentially infinite number of steps, for it averts, successfully or not, this problem by not specifying a particular curtailing point, without actually advocating anything about potential infinity. Its just a subdued theoretical point that says that the computation goes on indefinitely. In contrast, allowing actual infinities of computational operations in formal definitions of computation places this problem—a big problem for finitists—in the open.

Anyway, though non-deterministic or accelerating Turing machines[10] are not subject to the same constraints as deterministic Turing machines, complexity theorists themselves find the non-deterministic or accelerating machines too fantastic.

This means that, as recursive functions and axiomatisations of number theory have proved "disappointing", now, due to the "mathematical objection", deterministic Turing machines cannot simulate satisfactorily the power of the mind, so that if machine-state functionalism is construed as the view that "thought is computation of a deterministic Turing machine" it does not stand. Furthermore, as the mind supervenes on the brain and the brain is subject to physical constraints which accelerating Turing machines seem to defy, it is as yet unclear whether any type of physically constructible Turing machines would pass the Turing test. Non-deterministic ones might be a candidate, as one could let it do a fair dice roll, and this

[10]The reader is referred to [20] for a succinct exposition.

genuinely random seed (assuming that the universe is not deterministic) would enable it to appear to be able to answer the above-mentioned question, with occasional mistakes. To err is human, and this cleverly programmed non-deterministic Turing machine might be able to err like a human using a random seed along the way. But more research is surely needed on this point before these issues are cleared up.

Much of the workings of the mind and rational thought is "reducible" to models of computation, though of course it would be another problem whether one should take the reductions in the way scientific realists take them. (That one should do so is the stance of some of the functionalists.) Now for the processes that are "reducible" to models of computation,[11] there are many concepts and research in mathematical logic and computational theories that could be applied in discussing the issues involved; and for those that are not, the reason that they are not has much to do with the insight gained from such these subjects too.

To go back to the finitistic problem, these considerations illustrate the work and controversy about the limits of rational knowledge that we do not have the space to go into. But we would go on to consider the part of it that concerns Cantor's transfinite theory in Chapter 4.

2.7.1 Finitism

Finitism does not allow actual infinity. But there are relevant gradations of opinions that maintain this disallowance. A long-standing distinction of two varieties can be found in this camp.

[11]Whether and why, or why not, that are all that the mind has is again subject to controversy but is sadly out of the scope of this paper.

Classical Finitism

Classical finitists are against actual infinity. *Actual infinity* refers to a completely given, existent (abstractly or not) infinite collection. It has its first advocate in Aristotle. He only admits the existence of potential infinity. *Potential infinity* refers to something like an unending operation. For this position he argues that actual infinity would be an actualization of something which is never-ending in nature, and that an actualization of something which is never-ending in nature is a contradiction. Therefore, he concluded that talk of infinite sets is not coherent, and many philosophers and mathematicians have adopted this position since.

Strict Finitism

Strict finitism denies the use of any infinistic notions or methods as legitimate. Wright [76] gives an explication of such a position. It is arguably the most secure school in terms of ontology but it would be, at least in this stage (for in future mathematicians might be able to find very powerful tools in a strict finitist mathematics), very restrictive in the formulation and development of mathematical structures.

2.7.2 Finitist Schools

Nominalism is strict finitist, if it is indeed true that the universe has only a finite number of objects.[12] That it would then belong to this category is due to the nominalistic reduction of number to various sort of concrete objects.

[12]The sort of nominalism that identifying numbers with mental images (see Section 2.1.1) might make it seem that this sort of nominalism should be like conceptualism in its stance regarding the finitistic problem, but mental images can be arguably more limited in number than what a priori insight can know, but this is a fine point that we do not need to take a strong position about.

Conceptualism is classical finitist, since it allows potential infinity but not more, due to the limitation of the power of the mind. That the limitation is exactly at that point, not more, and not less, is because potential infinity is what traditionally has been allowed of the mind in being able to conceptualise. This is a contentious point, and something along the lines of the considerations at the beginning of this section (see the footnote on p.36) could resolve this problem but we do not have the space to go into it in this paper.

Intuitionism is in most cases strict finitist or classical finitist depending on which point on the spectrum of intuitionistic schools we are talking about, some allow the totality of natural numbers, and some do not. Those allowing the totality of natural numbers would be classical finitist, and those that do not would be strict finitist. What a brand of intuitionism allows would be determined by what it envisages as constructible by the mind, and "intuitionist" constructivists can be non-finitists. More about this in Section 4.1.2.

Theoretically speaking, empiricism is also strict finitist, if it is as science says that the universe has only a finite number of objects.[13] In spite of this, non-finitist mathematics might very well be used as a matter of convenience in calcultaions, if such mathematics is at this stage indispensable, given the pragmatic tendency of empiricists.

Practicism, conceiving of mathematics as a group of counting procedures, seems to be strict finitist. We should not be able to count an infinite number, for that is what "infinite" means, right?

[13]Detailed argument for the categorization of mathematical empiricism in this section and the next would take up too much space and thus is out of the scope of this paper. As this is not a major claim of this thesis we do not insist on it.

2.7.3 Non-finitism

Non-finitism refers to the stance that actual infinity should be allowed.

2.7.4 Non-finitist Schools

Realism is definitely non-finitist, as it postulates a whole Platonic world of an actually infinite number of objects. This is what makes realism appealing to classical mathematicians, for it does not set a restriction on what they are doing.

Logicism is non-finitist, as the logicist programme makes no pretence of a finitary proof theory. This is because the logicist programme only maintains a reduction of mathematics to basic logical and set-theoretical concepts. It is the same case with neo-logicism.

Formalism as stating that mathematics is and only is a formal system, would be non-finitist, as one has in most cases the axiom of infinity, except that finitary proof theory is finitist *de facto*, so that formalist metamathematics is finitist. However, the no-ontology position of formalists makes its mathematics utterly free and non-finitist in terms of what one may do in it while withholding attachment of any ontological significance.

2.8 Finitistic Reconsiderations

However, one may well ask, do we really know the difference between finitism and non-finitism as formulated above and the difference between each of its opponents? Indeed, the problem of finitism is part of a web of entangled problems. Tait [66], for example, distinguishes between the conceptual problem of finitism and the historical problem of finitism. The conceptual problem of finitism is the problem of "making sense of the idea of a 'finitist' function or 'finitist' proof of a finitist arithmetic proposition such as $\forall xy[x + y = y + x]$, which seems to refer to the infinite totality of numbers", that is, a finitist "redefinition" (as opposed to the customary unbounded sense) of such sentences. On the other hand, the historical problem of finitism is the problem of what Hilbert (and Bernays) meant by "finitism".

There is a way to be clearer about this web of entangled problems. And that is to think about the issue in terms of what those contenders make of precisely formulated operations used in mathematics and of the justifications offered. Therefore we now refine the concept of finitism to C-finitism.[14]

2.8.1 C-finitism

C-finitism refers to the camp against Cantor's method of proof. He shows that there is one-one correspondence between fractions and natural numbers by giving the former a well-ordering. And he shows by the diagonal argument that it is impossible to put the real numbers into one-one correspondence with the natural numbers. (For a brief illustration of his proof, see Chapter 3.)

2.8.2 C-finitist Schools

Nominalism does not admit Cantor's method of proof, as a result of nominalistic identification of numbers with objects such as mental images, numerals or other physical object which are generally believed to be finitely numerous. (See note on p.42.) It cannot even allow countably infinite sets, let alone uncountably infinite ones, so that the set of natural numbers, and, as a matter of course, the set of real numbers would not be legitimate.

Conceptualism allows only potential infinity, but not Cantor's liberal use of actual infinities, because they do not seem to be knowable through a priori insight nor constructible by our mental framework. Therefore, the set of natural numbers is

[14]Note that this is still different from what we will consider in Chapters 3 and 4.

problematic in conceptualist view, for conceptualism does not seem to tolerate a complete, given totality of natural numbers. The conceptualist position of C-finitism is, as mentioned in Section 2.7.2, in some sense an unstable position, but we are obliged to leave it as it is.

Intuitionism maintains that constructive methods have to be used in proofs, whereas the construction of any of Cantor's transcendental numbers by means of the diagonal procedure requires an infinite duration of time. So the diagonal procedure is not legitimate in an intuitionist's view, and thus intuitionism is C-finitist.

Empiricism would seem to belong to the C-finitist side, because of the finiteness of the universe as we know it at this stage of science.

Practicism understands mathematics in terms of counting procedures, and considering the fact that we cannot count beyond the countable (in the technical sense or otherwise), it seems to advocate C-finitism, because Cantor's proof is way above what it could allow.

2.8.3 Non-C-finitism

Non-C-finitism, the opposing camp of C-finitism, allows Cantor's diagonal method of proof.

2.8.4 Non-C-finitist Schools

Realism has been liberal from a historical point of view, and it seems to favour anything that most mathematicians allow, and as Cantor's sets do not contain inconsistencies (his formulation does not entail Russell's paradox as he did not explicitly allow unrestricted formation of just about any sets the way later theorists assume; he also knew of Burali-Forti paradox, cf. Section 4.3.1), realists should allow one to work on them. Thus realists endorse Cantor's diagonal procedure and therefore realism belongs to the non-C-finitist camp.

Logicism allows mathematical operations that can be phrased in terms of logical vocabulary and the ontological commitments thereof, so it should be comfortable with Cantor's work on transfinite numbers. But this is as the logicists intended it and not what they actually hold, for in the strict sense logic does not contain enough ontological commitments to allow mathematics. This is because it is not the case that axiomatic set theory is true in every possible world, if we adopt the stipulative use of the term "logic" in the fashion Tarski argued for.

On the other hand, neo-logicists can say that they are truly non-C-finitist. As they have made clear, they have all the requisite logical vocabulary and the ontological commitments to formulate mathematics, and this is something which logicists cannot not vouch for.

Formalism is also non-C-finitist because formalist mathematics only requires consistency. But again it should be noted that finitary proof theory is finitist *de facto*, so that formalist metamathematics is finitist, just that the no-ontology position of formalists makes its mathematics utterly free and non-finitist while withholding any ontological commitment to what it says literally.

2.9 Concluding Remarks

We have given a brief treatment of the finitistic problem in order to show it in the context of other branches of philosophy of mathematics. We will proceed to give an extended discussion in the coming chapters.

Chapter 3

Principles of Transfinite Theory

Before we can discuss finitism, we have to have an in-depth understanding of Cantor's theory and justification of transfinite numbers. Cantor's transfinite theory is, of course, non-finitist, and it is *the* instance of non-finitism which fueled the rise of modern finitism because it made precise discussions possible.

Cantor's theory of transfinite numbers is mainly based on three principles, namely the domain principle, the enumeral principle, and the abstraction principle. We will explain each of them in Sections 3.1, 3.2 and 3.3.

This summary and reformulation of Cantor's transfinite theory in terms of the three principles draw on Hallett [34] but improve vastly in terms of organisation.

The discussions following each principle develop from ideas many of which are merely hinted at in discussions in the field.

However, before introducing the three principles, first we will need a brief account of the historical and theoretical background.

3.0.1 Historical Notes on Infinity

From the time of the Greeks, infinity was known to be a tricky concept, and let us have a quick view of its history. And after that we will present Cantor's proof.

More than 2300 years ago, Aristotle argued that distance is not infinitely divisible, and he argued for that view because of Zeno's paradoxes, whose moral is that infinite divisibility and motion are not compatible. This view dominated the scene, through the antiquities and the middle ages, until Newton and Leibniz independently developed infinitesimal calculus towards the end of the seventeenth century. In 1734, Berkeley criticised those methods as "an infinite Difficulty to any Man whatsoever" [3, §5] because they operate on infinitely small quantities which are inconceivable. Gauss protested against "completed infinite magnitude" in a letter written in 1831 concerning non-Euclidean geometry:

> [...] [B]ut I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a *façon de parler*[...] [30, p.216][1]

It was when mathematicians were more or less in this frame of mind that Cantor published his works in the few decades before and after 1900.

3.0.2 Cantor's Proof

Cantor proved that the set of real numbers has a larger cardinality ("power", or can be informally thought of as "size") than the set of natural numbers or the set of rational numbers, the latter two having the same cardinality. His proof is outlined below.

First of all, one can show that there is one-one correspondence between fractions and natural numbers. One way to prove this is to give the fractions a well-ordering by arranging them in a two-dimensional array like this:

[1]The different versions seen in various books are careless, for example, see [40, p.146] and [22, p.71].

$$F = \begin{Bmatrix} \frac{1}{1}, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & \frac{1}{6}, & \cdots \\ \frac{2}{1}, & \frac{2}{2}, & \frac{2}{3}, & \frac{2}{4}, & \frac{2}{5}, & \frac{2}{6}, & \cdots \\ \frac{3}{1}, & \frac{3}{2}, & \frac{3}{3}, & \frac{3}{4}, & \frac{3}{5}, & \frac{3}{6}, & \cdots \\ \frac{4}{1}, & \frac{4}{2}, & \frac{4}{3}, & \frac{4}{4}, & \frac{4}{5}, & \frac{4}{6}, & \cdots \\ & & & \vdots & & & \end{Bmatrix} \tag{3.1}$$

One obtains a well-ordered sequence of fractions from this by by following an oblique arrow tracing through each element in the array, $\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{2}{4}, \frac{3}{3}, \dots\}$, and deleting the repeating elements such as $\frac{2}{2}, \frac{2}{4}$ and $\frac{3}{3}$ (because of $\frac{1}{1}, \frac{1}{2}$ and $\frac{1}{1}$ respectively), finally obtaining $\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \dots\}$.

And then after performing this operation one can establish a one-one correspondence between this complete and well-ordered sequence of fractions and the sequence of natural numbers in the ordinary order ($\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \dots\}$), with

$$\frac{1}{1} \leftrightarrow 1, \frac{1}{2} \leftrightarrow 2, \frac{2}{1} \leftrightarrow 3, \frac{3}{1} \leftrightarrow 4, \frac{1}{3} \leftrightarrow 5,$$

$$\frac{1}{4} \leftrightarrow 6, \frac{2}{3} \leftrightarrow 7, \frac{3}{2} \leftrightarrow 8, \frac{4}{1} \leftrightarrow 9, \frac{1}{5} \leftrightarrow 10, \dots$$

Thus the set of natural numbers and the set of rational numbers have the same cardinality, $\aleph_0$.

On the other hand, it is impossible to put the real numbers into one-one correspondence with the natural numbers, however you line them up. One of the ways to show this is by arranging the real numbers between 0 and 1 (in terms of binary decimal expansion) into an array, each real number occupying a row. A new number can always be defined by copying along the diagonal of the array and then interchanging zeros and ones, so that it is different from each one already listed.[2]

[2]The summary of Cantor's proof above in large parts follows the account in [67, pp.109–110].

In general, given a set M, the cardinal number $\overline{\overline{\mathfrak{P}(M)}}$ (two lines above a set denotes its cardinal number) of its power set $\mathfrak{P}(M)$ is $2^{\overline{\overline{M}}}$ (two to the power $\overline{\overline{M}}$ involves the notion of the covering aggregate, and it could be informally thought of as pairing up the elements of the set {0,1} and the elements of the set M so that each gets paired up with the other exactly once; the cardinality is the same whether it is written $\overline{\overline{2^M}}$ or $2^{\overline{\overline{M}}}$, so customarily one sticks to $2^{\overline{\overline{M}}}$). This is because for each subset of M included as a member in its power set $\mathfrak{P}(M)$ we can decide for each member of M whether that member of M is included in that subset of M, so that the number of possible combinations is 2 to the power the number of members of M, $2^{\overline{\overline{M}}}$, and thus the number of subsets of M or the number of members in the power set of M is $2^{\overline{\overline{M}}}$. In short, the power set of M, $\mathfrak{P}(M)$, has as its cardinal number $\overline{\overline{\mathfrak{P}(M)}} = 2^{\overline{\overline{M}}}$ because this exhausts the number of possible subsets of M, and a power set of M is the set of all the possible subsets of M. Cantor argued that this applies to infinite sets too, and therefore given that the cardinal number of the set of natural numbers $\overline{\overline{N}}$ is $\aleph_0$, the cardinal number of the power set of natural numbers $\overline{\overline{\mathfrak{P}(N)}}$ is $2^{\overline{\overline{\aleph_0}}}$.

Cantor [11, §4, pp.287–289] uses the more general concept of *Belegungsmenge* ("covering-aggregate") but it is enough for our purpose to consider the special case of the "coverings of aggregates" with elements of the aggregate $2 = \{0, 1\}$.

Now we begin to discuss the three principles.

3.1 The Domain Principle

Central to the development of the theory of transfinite numbers is the *domain principle*.[3] The domain principle behind the the-

[3]Coined by Micheal Hallett [34].

ory of transfinite numbers states that, for any variable to be meaningful in a mathematical context, there has to be a domain for it to range over. A domain in ordinary mathematical usage signifies the set of possible objects that can be put into the "independent variable" x in an equation. For example, in the equation

$$y = 2x,$$

if we say that the domain is $\mathbb{N}$ the natural numbers, it means that only natural numbers can be put into the place of x. The "independent variable" does not literally "varies", it merely means that as we put different values in place of x, the so-called "dependent variable" y evaluates to correspondingly different values, i.e. when we put 1 into x, y would be 2, when we put 2 into x, y would be 4, when we put 3 into x, y would be 6, etc.

The domain principle boils down to the claim that, for any mathematical term denoting some variable quantity to be meaningful in a mathematical context, there has to be a domain for it to range over. Its consequence is that any potential infinity presupposes a corresponding actual infinity (summarising Hallett's account in [34, p.7]). The domain principle justifies this presupposition in that it "forces" an equation with a variable x to have a domain of x. If the equation does not have a domain, the x in the equation would be meaningless, an empty place in the equation, and the equation itself would be meaningless, too. Applying this in the context of Cantor's theory, the variable quantity denoted by x is said by classical finitists to be potential infinite in an ordinary unbounded equation such as "$y = 2x$", for otherwise we would not know what "potential" means; and any variable in an equation has to have a domain, and for a variable quantity that is "potentially infinite", actual infinity is its domain. That can be thought of as meaning that an actually infinite set serves as the domain. (However, as we will see in Section 3.1.2, this argument has its flaws.)

The consequence of the domain principle that any potential infinity presupposes a corresponding actual infinity put advocates of potential infinity in a dilemma, for with this in view they cannot coherently endorse potential infinity, while at the same time shunning actual infinity. If this principle stands, classical finitism (Section 2.7.1) would not be a tenable position. All those in its camp would have to leave for the side of strict finitism (Section 2.7.1) or that of Cantor's.

3.1.1 Variables and Domain

The domain principle is justified by mathematics that involves variables ranging over, for example, natural numbers. Those variables are construed as potentially infinite by classical finitists. However, if there were not a *fixed* actually infinite domain for those potentially infinite *variables* to be "potential in", Cantor argues, how can the value of potentially infinite variables be defined?

Let us restate his argument in detail. As potential infinity is of fundamental importance in mathematics, especially in mathematical analysis, few people deny its use or presupposition. Potential infinity is so common in the subject, it is "used" in nearly all equations. However, it means also that actual infinity is presupposed in all those equations. Cantor argues for this by means of the reasoning that when we have a *variable* quantity in some mathematical study, it has to have a *fixed* domain. Therefore, as potentially infinite variables are employed in mathematics, those variable quantities which are potentially infinite require a fixed domain which is actually infinite. Therefore any potential infinity presupposes a corresponding actual infinity.[4]

Certainly to this point the argument is not yet conclusive.

[4] This is the gist of Cantor's argument as given in [14, pp.410–411].

There are several ways of attacking this principle and corresponding ways of defense.

3.1.2 Attack and Defense

Infinite totality

First of all, the idea of a completed infinity or an infinite totality is really quite mind-boggling if you think about it seriously. Could any given "whole" be genuinely infinite? If it is infinite, then it has no end, and if it has no end, how can it be a "whole"? Given the paradoxical nature of the term, do we really know what we are talking about when we say "an infinite whole"? In other words, could the mind really understand this term which refers to something at once infinite and whole?

On the other hand, is the mind powerful enough to know that the "actually infinite" is actually infinite? That infinity could be a completed whole all given at once, but not, for example, a rule for some sort of unending generation, is somehow hard to grasp, for the very word "infinity" suggests unboundedness, unendingness, uncountability (not necessarily uncountability *in the technical sense*, for even the act of counting the natural numbers which form a so-called "countable set" can never be fully accomplished) and the like.

Inspite of all these skeptical thoughts, the idea of an infinite domain has been shown to be very fruitful and coherent starting from the work of Cantor. He has shown that infinite sets are capable of mathematical determination and operations. Union and intersection of infinite sets, one-one correspondence between infinite sets etc. make perfect sense, and form as significant a part of set theory as finite sets.

However, capability of mathematical operations does not dissolve foundational questions. (This meta-level argument does not presuppose foundationalism in the sense of Section 2.6.1, though mathematicians in general would probably not feel at all troubled about foundational questions, even when these do not presuppose foundationalism, when there are operable mathematical functions to play with.) One can still wonder about its legitimacy; and if it is not legitimate, then those mathematical operations, however conceptually varied and fruitful, are problematic too, though they may be redeemable after appropriate modifications.[5]

Meaningful Potentiality

Secondly, why is it not possible for potential infinity to be meaningful without presupposing actual infinity? This really involves a whole lot of ontological and metaphysical speculations surrounding the problem of potentiality and infinity. Infinity seems by its nature something not actualisable, therefore why should actual infinity be conceptually more "fundamental" than potential infinity? Potential infinity seems to have the advantage of being ontologically simpler and thus easier to accept, at least for those that are not Platonistic realists, for it might be an irrelevant consideration for them (cf. Section 2.1.4).

A plausible reply to the above argument goes like this. That potential infinity presupposes actual infinity is inevitable because of the very nature of mathematical activity. What does that mean? It means that this presupposition is inevitable because in mathematics variable quantities simply *have* to have an expressly fixed domain that they are based on to be *meaningful* for people dealing with it, and the same reasoning that makes one deny the meaningfulness of a function if one does not know its assigned domain requires subscription to the position that potential infinity presupposes actual infinity. It would be sensible to suppose that Platonistic realists do not agree to this sort

[5]An original point in regard to the controversy.

of reasoning invoking knowledge of the assigned domain because a function is always meaningful if it is, and the fact that we do not know that it is now does not affect its meaningfulness, for it is mind-independent.

Is this counter-argument convincing? We think not. The reason is that the nature of mathematics is one of the issues under scrutiny, and therefore cannot be cited as something accepted.

More about this in Chapter 4. We will now explain and examine the other two principles first.

3.2 The Enumeral Principle

Cantor's enumeral principle[6] contends that being a natural number is being the enumeral of a well-ordered set. It would be like checking the ordinal numbers of the last item in an inventory in order to know how many items there are (provided that there is only one piece of each type of goods).[7] As the inventory only lists the goods once, it gives them a particular order, the order of being listed in the inventory, and any item in the inventory are ordered by the relation of being listed before another item. The numbers of the items do not necessarily presuppose numbers as used in mathematics in the customary way, for these are ordinal numbers, and conceptually they are definitely not the same as natural numbers. Moreover, they do not necessarily have to be symbolised by the arabic numerals, any symbols could do.

For example, given an inventory of clothes,

1. Blue shirt
2. Green shirt
3. Red shirt

[6]This principle comes from re-organizing the presentation in Hallett [34].

[7]An original illustration.

4. Mauve shirt
5. Violet shirt
6. Sepia skirt
7. Black skirt
8. Yellow skirt
9. Pink skirt

The following relations (but not only the following relations) subsist between the items:

$$\text{Blue shirt} < \text{Green shirt} \tag{3.2}$$
$$\text{Green shirt} < \text{Red shirt} \tag{3.3}$$
$$\text{Red shirt} < \text{Sepia skirt} \tag{3.4}$$
$$\text{Blue shirt} < \text{Pink skirt} \tag{3.5}$$
$$\text{Green shirt} < \text{Blue shirt} \tag{3.6}$$

The ordering relation is transitive, which means that, for example, if Blue shirt < Green shirt and Green shirt < Red shirt, then Blue shirt < Red shirt.

The relation is irreflexive, which means that ¬(Green shirt < Green shirt) and that the same can be said for any item in the inventory.

One of the "modern" definition of well-ordering says that a set is well-ordered if any subset of it has a least element, i.e. the element that bears the relation < to all other elements. If we inspect the subset $\{\text{Green shirt}, \text{Red shirt}, \text{Sepia skirt}\}$ of the inventory, we can see that it has a least element, and that is Green shirt, because Green shirt < Red shirt and Green shirt < Sepia skirt. We can satisfy ourselves that a least element can be found for any one of the 2^9 subsets of the inventory.

An alternative and perhaps better way to think of the concept well-ordering in terms of concrete lists is a dictionary. A dictionary has clear rules for order and it does not need to make use of numerals in order words. We could say that dictionary entries give a well-ordered set of words, provided that we do not count the words with the same spelling but are etymologically unrelated (like "bank" as in "river bank", and "bank" as in "investment bank") more than once.

3.2.1 Cantor's Ordinal Theory of Numbers

The enumeral principle states that being a natural number is being the enumeral of a well-ordered set. Now let us put in the fine details; Cantor thinks that to be a natural number is to be an ordinal number, and in turn an ordinal number is the enumeral of a well-ordered set. Therefore, he argues, a finite number is not inherently different from an infinite number because each one is respectively the enumeral of a well-ordered set which has the same sort of structure and capable of undergoing the same set-theoretical operations regardless of whether it is a finite well-ordered set or an infinite well-ordered set. (Zermelo proved the well-ordering theorem which says that there exists a well-ordering of S for any set S, assuming the axiom of choice.) In short, there are well-orderings for finite sets and infinite sets alike, and as a result there are enumerals of finite sets and infinite sets alike. Therefore, as being a natural number (finite or infinite) is "reduced to" being the enumeral of a well-ordered set, and if finite numbers exist, then transfinite numbers also exist because they are on an equal footing because the conceptual reductions of each are the same.

The key concepts involved in Cantor's ordinal theory of numbers are: an ordinal number, an enumeral of a well-ordered set, and a well-ordered set, in reverse order of the degree of being conceptually primitive. Below are explications of each.

3.2.2 A Well-ordered Set

A well-ordered set is a set whose every non-empty subset has a least element with regard to an irreflexive relation on that set. Intuitively, a well-ordered set is a set linearly ordered by a transitive relation, and subsets of it do not "go on forever" on the side of lesser values in the way the negative integers or an open interval of real numbers do. (An open interval is one in which the end-points are excluded.) The inventory as mentioned in Section 3.2 is an example of a well-ordered set, loosely speaking in a pedagogical way. Cantor's own definition of a well-ordered set differs slightly in wordings from the one in current usage, but is equivalent, as argued in Hallett [34, p.52].

3.2.3 An Enumeral

An enumeral (*Anzahl*) of a well-ordered set is a "picture" or "representational image" of a well-ordered set. It is a "canonical representative" of a well-ordered set, or of a class of isomorphic well-ordered sets.[8] A "canonical representative" of a well-ordered set or of a class of isomorphic well-ordered sets would be a representative inventory of the same length as the example in Section 3.2. And such an inventory would be the enumeral of the inventory in Section 3.2.

An enumeral e of a well-ordered set $(E, <)$ stands in such a relation to that set if and only if the set of predecessors of e is isomorphic to $(E, <)$. (This involves the iterative nature of the formation of well-ordered sets. We do not need to go into this in detail.)

[8]Paraphrasing Cantor's formulation in [15, pp.168–169].

3.2.4 An Ordinal Number

And, at last, an ordinal number is an enumeral of a well-ordered set, as explained above.

With this ends a brief account of the enumeral principle and we now discuss the philosophical issues involved.

3.2.5 Attack and Defense

Counting Number

The problem in the enumeral principle, however, is that an ordinal number is in its original sense a *counting* number.[9] But it is definitely not possible that transfinite numbers could be counted. Thus the enumeral principle accounting for the natural numbers via the ordinal numbers does not stand very well conceptually.

Of course the dialectic does not end there, it is only the beginning, for even though it is not possible to count infinite "numbers", Cantor initiated the alternative concept of one-one correspondence. One-one correspondence does what counting fails to do in an infinite context—one-one correspondence defines equivalence classes of sets of the same powers (or cardinalities), to put it in anachronistically modern parlance. But this goes from the enumeral principle to the abstraction principle, which will be explored in Section 3.3.

Well-ordering Principle

However, one can stick to the enumeral principle and defend the well-ordering principle instead of switching to the abstraction principle and endorsing the feasibility of one-one correspondence between sets, finite and infinite. So how does one stick to the

[9]This part articulates an objection to the principle that is original.

enumeral principle and defend the well-ordering principle?[10]

One can say in defense of the well-ordering principle that it is important for mathematical operations, that as it is equivalent to the axiom of choice (see below) which a significant portion of mathematics requires, the well-ordering principle should be upheld. Upholding the well-ordering principle means that the enumeral principle keeps its basis.

Let us explain the stakes involved.

There is a joke from Jerry L. Bona (a professor of mathematics at the University of Illinois) that says, "The axiom of choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?"

The axiom of choice states that there is a set (the "choice set") with exactly one element from each of an infinite number of sets sharing no common members. For example, there is a set of socks with one sock from each of an infinite number of pair of socks without assuming that there is a criteria by means of which one chooses a sock in the case of each pair. The common complaint against the axiom of choice is that it is arbitrary and counter-intuitive because no criteria whatsoever is needed for the formation of the choice set. It gives too much power to the mathematician.

The well-ordering principle states that there is a well-ordering function for any set. It means that there is always a way (though unspecified) to order a set into a well-ordered set. Again similar complaints are frequently voiced against this principle.

Zorn's lemma states that if every simply ordered subset of a partially ordered set has an upper bound, then that partially ordered set has at least one maximal element. A simply ordered set is a set with a complete ordering and the relation is irreflexive and transitive. A partially ordered set is ordered with respect to

[10]The connection between the enumeral principle and the well-ordering principle has not been explicitly noted in the controversy surrounding Cantor's transfinite theory.

a reflexive, anti-symmetric and transitive relation. (We do not need to go into this. One might refer to Mendelson [48, p.198] or Wilder [73, p.132].)

Despite what the joke says, the axiom of choice, the well-ordering principle, and Zorn's lemma are actually equivalent.[11] But mathematicians in general find the axiom of choice to be intuitive, the well-ordering principle to be counter-intuitive, and Zorn's lemma to be too complex for any intuition. Now the axiom of choice is consistent with but independent of the system ZF,[12] and that means that the well-ordering principle is likewise. But axiomatic treatment of arithmetic is incomplete anyway, so what is so devastating about the independence of the well-ordering principle for Cantor's programme?

The problem with the fact that the well-ordering principle is independent is that a Cantor-intuitive principle should be independent in a well-received axiomatic system. If even such an obvious principle cannot be proved axiomatically in a canonical system, what ground does he have other than its obviousness? If the well-ordering principle were provable in the system, then the enumeral principle could be a well-grounded, well-fitted account of the natural numbers. But if it were not, then the enumeral principle cannot very well claim precedence over other accounts. If the enumeral principle cannot claim precedence over other accounts, then the equal status that it gives to the finite numbers and the infinite numbers cannot be established convincingly by means of it.

Now it is established that the well-ordering principle cannot be proved in ZF, and that means that the enumeral principle cannot very well claim precedence over other accounts and that the equal status that it gives to the finite numbers and the infi-

[11] In fact one can refer to a book by Rubin and Rubin [59], that is entirely devoted to explicating the equivalents of the axiom of choice.

[12] ZF refers Zermelo-Fraenkel set theory with or without axiom of choice depending on context, cf. note on ZFC in Section 2.1.8.

nite numbers cannot very well be justified by this means. Then is there any other way to establish its precedence?

But that is not all, for the axiom of choice is disproved [64] in NF,[13] a respected system.

However, we will leave that part of the controversy for now, and will continue to inspect the enterprise in Chapter 4 after we have explicated the last of Cantor's three principles.

3.3 The Abstraction Principle

An alternative to the enumeral principle would be the abstraction principle.[14] The enumeral principle accounts for numbers by means of ordinal numbers, but the abstraction principle accounts for numbers by means of cardinal numbers. Now, if numbers are construed as cardinal numbers, then since the comparison and manipulations of cardinal numbers is done by means one-one correspondence which is as meaningful and determinate between infinite cardinal numbers as between finite cardinal numbers, infinite numbers have the same status as finite numbers.

Imagine you are trying to "count" a deck of playing cards.[15] The enumeralist way to count would be to arrange the cards by suit and then by number, and then check if there are any missing cards by referring to the representative deck (for the operation does not require that you know the suits and the numbers by heart). The abstractionist way, however, would be to just place the two decks in front of you, with back facing up, and pair off the deck being counted with the "good" deck, without looking at what is printed on the other side at all. If the two decks are successfully paired off, then the deck being counted has the right

[13]Quine's "New Foundations", see his [58].

[14]This principle comes from re-organizing the presentation in Hallett [34].

[15]This illustration is original.

number of cards.

3.3.1 Cantor's Cardinal Theory of Numbers

Cantor sees himself in formulating the cardinal number in an abstractionist way as continuing the tradition of Pythagoras and Euclid. Euclid writes, "a *number* is a multitude composed of units" and "a *unit* is that by virtue of which each of the things that exist is called one". [35, Vol.2, p.277] The only difference is that he replaces Euclid's "multitude" with "set".

> We denote the cardinal number or power of M, the result of this two-fold act of abstraction, by $\overline{\overline{M}}$. Since each individual element m if we disregard its nature becomes a "one", the cardinal number $\overline{\overline{M}}$ itself is a definite set composed of nothing but ones which exists in our mind as the intellectual image or projection of the given set M. [11, §1, pp.282–283][16]

The "two-fold act of abstraction" refers to the formation of the set M from members m and the formation of the cardinal number $\overline{\overline{M}}$ from the set M.

To go back to the cards metaphor, you use indiscriminate cards to count the cards in your hand. You pair off the cards in your hand with the indiscriminate cards, and you tell how many you have by looking at the indiscriminate cards (perhaps you arrange them into easily recognisable patterns, but that is not important). One asks, why not count the cards in your hands directly? Well, you use the indiscriminate cards as tokens. That is a device that has been used throughout history, so that even as a metaphor it bears a resemblance to how we do things in practical life.

The abstraction principle involves these concepts: a cardinal number, one-one correspondence, and "ones".

[16] Modified translation with reference to Cantor [16] and Hallett [34].

3.3.2 An Abstract One

In Cantor's conception the cardinal number of a set M is a definite set consisting of indiscriminate, abstract "ones" "which exists in our mind as the intellectual image or projection of the given set M".[17]

To use our metaphor of playing cards, it would be an indiscriminate card.

3.3.3 One-one Correspondence

Two sets M and N are in one-one correspondence if and only if there is a one-one function F with domain M and range (or co-domain) N.

In our cards metaphor, this would be the pairing off used in the counting process of the cards.

3.3.4 A Cardinal Number

A cardinal number $\overline{\overline{M}}$ is a "pure", definite set composed of abstract units to which all sets with the same cardinality (number of elements) will be equivalent (one-one correspondent). In other words, it is a set that, so to say, represents all sets with the same number of elements, regardless of what those elements are.

In our cards metaphor, this would be a quantity of indiscriminate cards used in one particular count.

3.3.5 Attack and Defense

An Abstract One

The first problem in the abstractionist account is that this "abstract one" is rather problematic. Cantor [11, §5, p.289] characterises what we call an "abstract one" thus:

[17] Paraphrasing Cantor [11, §1, p.283].

> A single thing e_0, if we subsume it under the concept of an aggregate $E_0 = (e_0)$, corresponds, as cardinal number, to what we call "one" and denote by 1; we have
> $$1 = \overline{\overline{E_0}}. \quad (3.7)$$
> Let us now unite with E_0 another thing e_1, and name the union-aggregate E_1, so that
> $$E_1 = (E_0, e_1) = (e_0, e_1). \quad (3.8)$$

How can we distinguish e_0 from e_1 if it is really abstract, without presupposing some numeric concept?[18] Notice that, unlike the Frege-Russell account which makes use of the non-identity of non-identical things,[19] no technique bypassing some sort of numeric concept has been employed in Cantor's account.

By the extensionality principle, anything that is indistinguishable with something is identical with that thing. If one accept the extensionality principle, as many do, then he could not consistently allow such "abstract ones" in a theory. Therefore the question whether one could use them to account for numbers satisfactorily inevitably arises, for it seems to be more justifiable to assume numbers as primitive, since they are at least distinguishable.

On a related note, it is actually possible to formulate ordinals in an abstractionist way. In [71, p.347], von Neumann tries to "avoid the vague notion 'type' " by defining ordinals in this

[18] An original objection to Cantor's abstractionist account.

[19] For example, the definition of "2" in Whitehead and Russell [72] is basically as a set β for which there exists some x and some y such that both belong to β and that x is not identical to y, and for which if any u and v and w all belong to β, then one of u and v and w is identical to one of the other two. In symbolic notation:

$$\{\beta : \exists x \exists y (x \in \beta \wedge y \in \beta \wedge \neg(x = y)) \wedge \\ \forall u \forall v \forall w ((u \in \beta \wedge v \in \beta \wedge w \in \beta) \rightarrow (u = v \vee u = w \vee v = w))\}.$$

form: "Every ordinal is the set of the ordinals that precede it" or symbolically

$$\lambda = [0, \lambda).$$

It is in a sense an ordinal version of the abstraction principle. But luckily it does not succumb to the same problem as the cardinal version. So maybe one should adhere to an abstractionist ordinal account of number. But the problem with this is that it also has to do with the well-ordering principle, so that the myriad of problems mentioned in Section 3.2.5 remain.

Interiority

The second problem with Cantor's cardinal theory is that it is "interior":

> Cantor's mathematical theory of cardinal number is as an *interior* theory with the number-classes as the interior representatives of power. [34, p.119]

The word "interior" signifies the fact that in Cantor's formulation the cardinal number $\overline{\overline{M}}$ of a set M is actually a set in one-one correspondence with M. Why would this "interiority" be a problem?

If $\overline{\overline{M}}$ is a set with cardinal number $\overline{\overline{M}}$, then we can say that the cardinal number of $\overline{\overline{M}}$, i.e. $\overline{\overline{\overline{\overline{M}}}}$, is $\overline{\overline{M}}$, which is same as the cardinal number of M. Aside from the confusion in notation, there is the more serious problem that cardinal numbers as formulated by Cantor are superfluous. That they are superfluous is because if the cardinal number $\overline{\overline{M}}$ of a set M is actually a set in one-one correspondence with M, then why should one bother with some sort of pure abstract set formulated particularly for the purpose of measuring cardinality? Just pick an existing set will do. One never needs to use sets of infinite cardinality that needs to be additionally formulated in measuring the cardinality of sets of physical objects anyway.

In contrast, in the Frege-Russell formulation of the cardinal number $\overline{\overline{M}}$ of a set M as the equivalence class of all sets in one-one correspondence with M, the cardinal number of a set is not in one-one correspondence with that set. Jourdain [16, p.203] argues in favour of the Frege-Russell logicist formulation because it is ontologically simpler as it avoids assuming the new and undefined entities called "numbers", and because it can be deduced that the class defined is not empty, so that the cardinal number of M exists in the sense signified in logic. In other words, Cantor's original account presupposes more existent objects.

One could settle with the Frege-Russell account instead, but though it solves the interiority problem, it is not immune to the other attacks in Chapter 4.

3.4 Concluding Remarks

What is philosophically significant with learning about these fine details in these slightly different versions of the ordinal theory and the cardinal theory is to enable one to find out if each theory is really an abstractionist account or an enumeralist account. This is because the two accounts presuppose different concepts.

But then comes the still more significant question whether at least one of the two accounts, the abstractionist account or the enumeralist account, stands. This is significant because it determines the success of Cantor's enterprise, provided that the domain principle does not fail. This provision is needed because if the domain principle fails, then potential infinity alone, which would probably sufficiently account for mathematical analysis, seems much easier to accept than Cantor's full-fledged transfinite theory.

We will look at how his accounts of numbers and thus his transfinite theory fare in the next chapter.

Chapter 4

Problems in Transfinite Theory

Now, against this backdrop of explicated mathematical philosophies and Cantor's theory, we can proceed to give our careful take on finitism. Finitists protest against Cantor's transfinite theory. What are the objections against Cantor's theory? We have divided the following problematic features of his theory in terms of the conceptual aspect involved, namely, structure and procedure (Section 4.1), number and numerosity (Section 4.2), and conceivability and comparability (Section 4.3).[1] We will start with the structural and procedural problems.

4.1 Structure and Procedure

Cantor's theory is problematic in view of the implied structure and procedure of mathematics as a whole.

There are two senses of the word "structure" which are relevant here, namely *structure qua entities* and *structure qua patterns*. "Structure qua entities" refers to complex entities, while "structure qua patterns" refers to the properties or patterns that similar entities exemplify. For example, model theorists generally mean the former when they say "structure". Of course sometimes these two senses are not strictly differentiated and

[1] There has not been any work in which the discussion of finitism and Cantor's theory of transfinite numbers brings together objections with such a wide spectrum and depth.

the same text might allow both sorts of interpretations, nevertheless we can comprehend the slight difference between these two sense.

Anyway, here we are more concerned with the *structure qua entities* among the two. Regarding the structure of mathematics in this sense, we are going to argue, in Section 4.1.1, that Cantor's proof sides with free mathematics, the ideological tendency of allowing any objects as long as they do not cause contradiction. Of course, not just the distinction between *structure qua entities* and *structure qua patterns* is delicate, even the distinction between *structure* and *procedure* is not so simple, for this view affects which procedures are found to be adequate and consequently adopted.

But that is not all. Loosely speaking, there are also two senses of the procedure which are relevant here. The first one is *procedure in a formal sense.* And the second one is *procedure in an informal sense.* "Procedure in a formal sense" refers to strictly formulated transformation of marks on paper, while "procedure in an informal sense" refers to any intuitively acceptable steps in mathematics or other related disciplines.

For our purposes in this chapter, we are using "procedure" primarily in the latter sense. Concerning the procedure of mathematics in this sense, there is the objection that Cantor's proof is non-constructive, on which we will elaborate in Section 4.1.2. However, just as free mathematics does not only affect the ontological structure of mathematics but also the procedure allowable, non-constructive proof does not only concern the procedure of mathematics but also the entities constructible and the structure formed from the inter-relationships between those entities. Attitude towards the constructive/non-constructive problem affects which mathematical objects can be fruitfully "investigated" because it limits the results obtainable.

Now let us look at the objection of free mathematics.

4.1.1 Free Mathematics

Free mathematics[2] is the doctrine that endorses a maximum ontology, allowing existence whenever no inconsistencies result.[3] Cantor's theory seems to be based on a preference for free mathematics. He says,

> Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. [12, §8]

Dedekind also wrote in his preface to [21] that

> In speaking of arithmetic [...] as merely a part of logic I mean to imply that I consider the number-concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought. [...] [N]umbers are free creations of the human mind [...] [21, p.31]

Advocates of free mathematics argue that mathematical objects are free creations of the mind, which is the only constraint, apart from the law of contradiction, to what can be said to exist. Whether this generosity with ontology is appropriate is a difficult question.

The advantage of this position is that there is more creative space for the mathematician to work with. He can investigate whatever he is able to come up with, given that it is not inconsistent.

[2]The term appears in Hallett [34].

[3]I have not seen anyone explicitly raise this objection against Cantor's theory of transfinite numbers.

The disadvantage of it is that mathematics under this doctrine has less security, and it gives rise to more difficult foundational questions. This is because the entities that he "creates" are frequently problematic for the more meticulously or philosophically minded. For example, he might make use of the axiom of choice when stipulating an entity. But many mathematicians find the axiom of choice suspect, and they might not accept this entity.

However this weighing of advantage and disadvantage goes, mathematicians less free in spirit are likely to question the practice of free mathematics, as it threatens the purity and integrity of the subject. More importantly, however, constructivists, having the advantageous claim of playing safe on their side, demands righteously the philosophically requisite proof of existence of mathematical objects (at least a relative proof on the basis of more commonly accepted entities) before applying the law of the excluded middle to the statements discussing them. The requirement of a proof of existence as opposed to the non-appearance of inconsistencies gives rise to a radically different form of mathematics, a restrained and "difficult" form of mathematics.

The lack of an existence proof is philosophically irritating. If one considers, for example, the view of Wittgenstein (constructive mathematicians also take a similar view to his, cf. Section 4.1.2), he does not hesitate to classify such statements as Cantor's theorems as nonsensical, and he would not busy himself straightaway, like others do, with the truth or falsehood of statements involving infinite numbers because of an overriding view of the law of the excluded middle.

> Let's imagine someone living an endless life and making successive choices of an arbitrary fraction from the fractions between 1 and 2, 2 and 3, etc. *ad. inf.* Does that yield us a selection from all those intervals?

> No, since he does not finish. But can't I say nonetheless that all those intervals must turn up, since I can't cite any which he wouldn't eventually arrive at? But from the fact that given any interval, he will eventually arrive at it, it doesn't follow that he will eventually have arrived at them all. [74, §146, p.167]

Probably anyone would agree that the law of the excluded middle applies to all and only the meaningful statements. But the problem remains as to which statements are meaningful. Constructivists are stricter with it, while in most cases non-constructivists are less strict with it. While a specification as to what is meaningful in this context that is at once appropriate and fits with our intuitive understanding of what is to be meaningful is yet to be found. Given this lack the prudence of those against free mathematics is more commendable.

An objection along a similar vein is that Cantor's proof has arguably violated a natural simplicity principle, and that is "do not invoke what is not necessary". If the non-finitists argue that real numbers as Cantor explicates them are necessary for mathematical analysis to retain all of its parts in classical mathematics, as opposed to the reconstruction of some and demolition of others in constructive mathematics—if necessity in this sense is meant, then non-finitists have to establish the insufficiency or inadequacy of constructive mathematics.

Of course, this is a problem with a wide scope and the point of contention goes back to the ontological and foundational commitments of the participants in the controversy. It depends on the weight put on the soundness of foundation vs. the value of applications. It would however be a safe claim to make that free mathematics seems not to be philosophically an advantageous position because of its potential for creating problems for the conceptual coherence of the subject. It makes it difficult to give a coherent account for its ontology. What this implies is that ontologically speaking, it is unlikely for intuitionists, realists and practicists, at least, to be advocates of free mathematics.

Closely related to the problem of the maximum ontology of free mathematics is Cantor's use of non-constructive proofs.

4.1.2 Non-constructive Proof

A constructive proof is a proof in which the existence of a mathematical object or function etc. is not simply proved by establishing that its non-existence is contradictory, but instead proved by showing that algorithmic construction of that object from some accepted primitives is possible in principle. An algorithm is a specification of a stepwise computation which a human being or a machine can, in principle, perform in a finite period of time.

The problem of constructibility is long-standing. Fraenkel et al. writes,

> The emphasis laid on the construction of mathematical entities and even the *identification between existence and constructibility in mathematics* is by no means a novelty. [26, p.221]

This emphasis gives rise to a variety of constructive thoughts with various degree of strictness in their specifications. In order of decreasing strength of construal of the word, constructivism [2, 4, 5, 23, 69] refers to the doctrine of *a*) accepting solely, *b*) promoting, or *c*) preferring, when there is a choice, constructive proofs in mathematics.

Brouwerian Counter-example

A heuristic way of finding out if a statement admits of a constructive proof is to see if it is impossible to construct a Brouwerian counter-example (C_B) to that statement. (See Mandelkern [46].) For example, suppose we have a binary sequence a, that is, a binary sequence in the constructive sense which means that there is *a finite routine by which each place in the sequence is assigned an element of* $\{0, 1\}$. Keeping this point in mind, consider the following group of statements as an example.

$$P(a) \ : \ a_n = 1 \text{ for some } n, \tag{4.1}$$
$$\neg P(a) \ : \ a_n = 0 \text{ for all } n, \tag{4.2}$$
$$P(a) \vee \neg P(a) \ : \ \text{Either } P(a) \text{ or } \neg P(a), \tag{4.3}$$
$$\forall a(P(a) \vee \neg P(a)) \ : \ \text{For all } a, \text{ either } P(a) \text{ or } \neg P(a), \tag{4.4}$$
$$C_B \ : \ a_n = \begin{cases} 1 & \text{if for all primes } p \geq n \\ & p+2 \text{ is not a prime.} \\ 0 & \text{if for some prime } p \geq n \\ & p+2 \text{ is a prime.} \end{cases} \tag{4.5}$$

"$P(a)$" means that 1 occurs somewhere in the sequence a, i.e. it might look like

$$0000000000100000001000000$$

so that

$$a_{11} = 1 \text{ and } a_{19} = 1.$$

"$\neg P(a)$" means that a_n is all 0, i.e. it may be

$$00000000000000$$

so that

$$a_1 = 0, a_2 = 0, a_3 = 0, \ldots, a_{14} = 0.$$

"$P(a) \vee \neg P(a)$" means that either "1" occurs somewhere in the sequence a, or it does not.

"$\forall a(P(a) \vee \neg P(a))$" means that for any binary sequence a, either "1" occurs somewhere in that sequence, or it does not.

"C_B" means that $a_n = 0$ if there is some prime p for which $p + 2$ is prime. Now $a_{229} = 0$ because there is some prime, 269, for which $269 + 2$, i.e. 271 is prime.

Now, if "$\forall a(P(a) \vee \neg P(a))$" could be constructively proved for all a and thus also a_n as in C_B, then an algorithm would have been given for deciding the twin prime conjecture.[4] This algorithm either provides a construction establishing the twin prime conjecture, so that for any n, there is some prime $p \geq n$ such that $p + 2$ is a prime, and so that "$\neg P(a)$" is true, or produces a construction that disproves the twin prime conjecture, to the effect that there is some n such that for all primes $p \geq n$, $p + 2$ is not a prime, and establishes "$P(a)$". (Remember that a is *ex hypothesi* a binary sequence for which there is a finite routine by which each place is assigned an element of $\{0, 1\}$, which means that a construction has to be given.)

Unless we do have an algorithm that decides the twin prime conjecture, C_B would constitute a Brouwerian counter-example to "$\forall a(P(a) \vee \neg P(a))$".[5]

While constructivism is an umbrella term that covers a variety of positive attitudes toward constructive proofs, intuitionism rejects a_n for any arbitrary n as a well-defined number specifically out of certain ontological considerations (cf. Section 2.1.3).

Cantor's proofs are non-constructive. Of course, to be fair, he is not in the minority. Most mathematicians prove non-constructively.

The Law of the Excluded Middle vs. Constructed Existence

To continue to use our example, classical mathematicians may argue that "the extent of our knowledge about the existence or non-existence of a last pair of twin primes is purely contingent and entirely irrelevant in questions of mathematical truth" [37, p.2], so that by the law of the excluded middle, either the twin

[4]The conjecture that there are infinitely many primes p such that $p + 2$ is also prime.

[5]If the conjecture were resolved in some future time in which case we might have an algorithm that decides the twin prime conjecture, we can simply refer to another open problem.

prime conjecture stands, then $\forall a \neg P(a)$, or it does not, then $\forall a P(a)$. And therefore, by first-order predicate logic, we have "$\forall a(P(a) \vee \neg P(a))$".

But the intuitionist constructivist promptly retorts that this argument is metaphysical in nature and presupposes that the relevant primes already exist outside of the human intellect, which is a point open to argument. So the classical mathematician stands at a more precarious position than the constructivist mathematician.

However, the classical mathematician returns fire and points out the undesirable consequence that this means that C_B is a "counter-example" while the twin prime conjecture is open but ceases to be one at exactly the moment when the conjecture is resolved. It ceases to be a Brouwerian counter-example because there would be then an algorithm for deciding the twin prime conjecture. This change of state is weird to say the least. Even more weird is the consequence that a_n for any arbitrary n is not a well-defined number while the twin prime conjecture is open but starts to be one at exactly the moment when the conjecture is resolved.[6]

The intuitionist constructivist does not find this counter-intuitive, for him a mathematical assertion is generally about the construction or the constructedness of a certain mathematical object. It "exists" in that it has been constructed. He clarifies that the resolution of the twin prime conjecture furnishes a method for constructing a_n for any arbitrary n. a_n for any arbitrary n does not necessarily already exist in some metaphysical realm before our construction. He emphasises that such metaphysical presupposition is unwarranted.

The classical mathematician laments the confusions and slop-

[6]This point is from Menger [49], and the presentation here is adapted from Heyting [37, p.2]. The arguments in the following debate is also constructed largely from the points found in Heyting's exposition [37], reformulated and illustrated in light of my explanations of constructive proofs above.

piness caused by the renouncement of logical laws in order to account for human epistemic states.

Formal Reasoning vs. Contentual Reasoning

The intuitionist constructivist replies that even formalists use contentual reasoning instead of exact and mechanical derivations, when they are doing metamathematics. (*Contentual reasoning* is the opposite of formal derivations in that it does not strip the expressions of their meanings as in the case of formal systems in which expressions are taken to be meaningless, merely marks to play games with.) They too "succumb" to those "confusions" and "sloppiness", it is just that they do so at a higher level. And, while the formalists want to separate the metamathematical reasoning from purely formal mathematics and "minimise" the former, intuitionist constructivists are not interested in this.

In fact, formalists would gladly investigate the formal characteristics or syntax of constructivist mathematics. But then there is the danger of treating it as merely part of mathematics, which the more radical constructivists would not be content with. It would be reasonable to suppose that the typical intuitionist constructivist views their enterprise an altogether different subject from classical mathematics.

This is because, for intuitionist constructivists, the sort of formal systems that formalists play with exemplifies a very ambiguous type of linguistic expression. It easily gives rise to misunderstanding and admits of more than one interpretation. This objection would seem rather bizarre for nearly anyone other than intuitionist constructivists, for formal systems are generally seen as the epitome of precision. But it is not as bizarre as it looks, as it has been proved that if a first-order theory of arithmetic has its intended model, then it has a non-standard model.[7] That means a first-order theory of arithmetic doing a good job would never be categorical—admitting of only one type of model. On the other hand, if formal systems are treated as simple mathematical structures, then fomalisations are, for intuitionist constructivists, a powerful mathematical tool, but they can never represent fully any domain of mathematics. But of course this fact has been made manifest to mathematical philosophers of any camp by Gödel's incompleteness proofs, which show that any consistent number-theoretic formal system can be extended consistently in more than one way.

However, even in the case of treating formal systems as simple mathematical structures, for intuitionist constructivists these formal systems are simply constructions made *after* building mathematics independently of the formalisation.

But this reliance on intuition seems not a little suspicious to the classical mathematician infused with classical logic, for classical mathematicians may not be formalist in tendency but it would be slightly more possible for the classical mathematician to accept the intuitionist-constructivist rejection of laws of classical logic if constructive mathematics were totally formalised.

The intuitionist constructivist replies that formal logic itself needs ground, and if mathematics were to be formalised on the basis of it then, as it involves principles more intricate and less direct than those of mathematics itself, mathematics would be put on problematic foundations, for the foundation of formal logic, if not problematic in itself, is at least doubtful ground for mathematics. This is because the intuitionist constructivist is of the opinion that a "mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever", and that one needs only "a clear scientific conscience" to know whether a reasoning is sound, without

[7] See the proofs in Henkin [36], Skolem [62] and Skolem [63].

using any logic.[8]

Put in another way, one may say that logic is part of mathematics, not its foundation, for so-called logical theorems (such theorems as: given $p \to q$ and $q \to r$, one then has $p \to r$) are really mathematical theorems of extreme generality. And the process by which one derives it does not differ in kind from mathematical proofs in general. By suitable juxtaposition one shows its obviousness. Therefore logical theorems do not claim precedence over other mathematical theorems.—The intuitionist constructivist goes on to argue.

Intuitionist Constructivists as Non-finitist but C-finitists

The intuitionist constructivist's emphasis on intuition and immediateness to the mind might make him seem a strict or a classical finitist, but it would be wrong to think that he is either. He is neither of the two. The intuitionist constructivist takes the natural numbers, as a given totality, for granted. His justification is *a*) that such a totality is intuitively clear enough, for even children understand what natural numbers are, *b*) that mathematicians know what it means when they use this notion, and *c*) that it is too demanding to demand more than this state of affairs. But the fact that the intuitionist constructivist is not inherently a strict or even a classical finitist does not deter him from finding faults with non-constructive proofs (cf. Section 2.7), and Cantor's proofs about transfinite numbers are non-constructive. This shows a point that is in a way obvious but easy to miss. The point is that *it does not take a strict or a classical finitist to be against Cantor's proofs*,[9] as we can see intuitionist constructivists are also against those proofs.

[8] This paragraph is quoting and paraphrasing the account in [37, p.6].
[9] An original point of mine.

Intuitionist Constructivists' Inconsistency?

The classical mathematician might attack the intuitionist constructivist's unjustified difference in attitude towards the logical law of the excluded middle and the unrestricted principle of mathematical induction, for to most people the former is at least as intuitive than the latter, if not more. And yet the intuitionist constructivist rejects the former while upholding the latter, without further justification. It seems dogmatic.[10]

The intuitionist constructivist replies in defense that mathematics is not about the external world, but about mental constructions and the process thereof, so that truth value should not be construed as mind-independent. Mathematical statements are reports of and about mental constructions, for example, "$3+3 = 8-2$" should be read as the mathematician reports that he has effected the mental constructions "$3+3$" and "$8-2$" and found that the result is the same. Others agree because they think in much the same way. This sort of agreement is found in other subjects and there is no fundamental difference between mathematics and other subjects in this particular point, as opposed to traditional accounts of the privileged epistemological status of mathematics.

The intuitionist constructivist goes on to argue that the value of mathematics is of the same kind as arts and letters, that it is a valuable *activity* of the mind. Its principal value does not lies in its being a conceptual calculus for science, as mathematical empiricists assert (cf. Section 2.1.5). Others attack this defense of intuitionist constructivism by pointing out that this school, despite its emphasis on the value of mathematics as an activity worth doing for its own sake, disowns the most precious mathematical work.[11]

The intuitionist constructivist now tries to justify directly his

[10] This paragragh is paraphrasing the account in [37, p.8].

[11] For an example of this position, see Hilbert [39].

"unjustified difference in attitude towards the logical law of the excluded middle and the unrestricted principle of mathematical induction". First of all, he emphasises that we have a clear notion of natural numbers, which could be explained as follows. We conceive of the notion of an entity through abstracting from the particular qualities of the object. Then we perceive the possibility of an indefinite repetition of entities.[12]

In the second place, he argues that the principle of mathematical induction stands because of the following proof: suppose that $E(x)$ is a predicate of natural numbers such that $E(1)$ is true, and that $E(n)$ implies $E(n')$ for any particular natural number n where n' is the successor of n. Now let p be any natural number. Investigating the numbers built up successively from 1 to p we see that the predicate E which holds for 1 will be preserved at every step in constructing p. Therefore $E(p)$ is true. One knows this by simple examination of the proof, in view of "evidence" rather than axioms and deductions.[13]

On the other hand, the logical law of the excluded middle that classical mathematicians hold unconditionally is unwarranted if we cannot not construct the number we are talking about.

Constructivistic Commonsense

Given this and similar explanations of his position, the intuitionist constructivist deplores the nonchalance of classical mathematicians in employing and relating unclear concepts. He thinks that his own conception is more natural and more disciplined. He advocates the "commonsense constructions" that we have before the "theoretical constructions", which he claims is analogous to the conviction that I see a tree versus the conviction that light waves reach my eyes and lead me to construct an image of

[12]This paragragh is paraphrasing the account in [37, p.13].
[13]This paragragh is paraphrasing the account in [37, p.14].

the tree.[14] He does not agree that his conception of mathematics as a subject is unruly and capricious. On the contrary, it is a conception that is untainted by theoretical speculations of a particular age, he argues.

Conclusion

This labyrinth of considerations might make one wonders what is to be the result of all of these. The provisional conclusion is that mathematics is a subject claimed by theorists of different ontologies and standards, and that the intuitionist constructivist points out rightly the unsatisfactoriness of certain classical mathematical methods. The use of non-constructive methods in the case of Cantor's proofs is particularly problematic because their effect is revolutionary, and they redefine what we are to think of infinity, and change the conceptual relationship between number theory and mathematical analysis—if not in itself, then as mathematicians conceive it. That these proofs is non-constructive allows room for the intuitionist constructivist argument that maybe what Cantor is doing is a confusion and abuse of infinite concepts.

But is it possible to reformulate his proofs constructively? For if it is possible to do so, then constructivists would not be able to have qualms about his theorems. Sadly, however, such proofs have not been found. The existence of transcendental numbers has been constructively proved (Liouville numbers), but that they are uncountable has not been proved. If they are countable it does not show that the set of real numbers is larger than the set of rational numbers, for the union of two countable sets is still countable.

Cantor places the basis and justification of his free mathematics and non-constructive proofs on the reduction of numbers

[14]This point about "commonsense constructions" and "theoretical constructions" is from [37, p.11].

to sets and the existence of sets. In the following section we will be looking at this reduction.

4.2 Number and Numerosity

The conceptual distinction between number and numerosity is highly relevant to our discussion. "Number" is the intuitive "object" of investigation in mathematics, while "numerosity" is a general word for quantity redefined as the quality shared among sets that can be put into one-one correspondence. Cantor's insight is to define number in terms of this newly-defined numerosity. This constitutes his cardinal account (Section 3.3). An alternative is the ordinal account (Section 3.2), but the gist of the two accounts is the same—set-theoretical reduction.

4.2.1 Weak Reductionism

As evident from the elucidation of his enumeralist and his abstractionist accounts of number in Sections 3.2 and 3.3, Cantor holds a kind of weak reductionism.[15] It is *weak* in that he does not simply reduce numbers to sets, but it is *reductionistic* in that numbers and their existence are explained and justified in terms of sets. As we have seen, the ordinal account relies on the well-ordered *set*, and the cardinal account cannot do without the doubly abstracted *set* of units.

This brings about, firstly, the problem of whether the reduction is philosophically appropriate and pragmatically useful, and, secondly, the problem of the existence of sets. Regarding the reduction there is the problem of definition and construal of numbers, while regarding the existence of sets we have to be concerned with the questions as to whether it is justifiable to postulate sets and why, or why not.

[15]Texts in the literature do not pay due attention to this.

Let us first take a look at Cantor's definition and construal of numbers. The enumeral and the abstraction principles as we explicated have tried to shift the problem of the meaning of numbers to the meaning of sets. Cantor claims that the cardinal numbers afford the most natural and rigorous foundation for the finite and transfinite numbers. [11, §5, p.289] It has its advantages, but as we shall see in Section 4.2.2, there are sets, i.e. non-Cantorian sets, that do not behave as the abstraction principle stipulates. This deals a blow to the reduction but for which it might have succeeded.

And now let us look at the existence of sets. The existence of sets is the fundamental and most crucial standpoint in Cantor's reductionist account of mathematics. It is the bottommost basis in his accounts of numbers, so that one has to either accept it or reject it, and can appeal no further. However, there seem to be plausible options between the primitive existence of numbers or that of sets, as one has to take some sort of entities as primitive, while which ones are is a contentious issue, involving ontological and practical considerations.

Atomic Theories of Numbers

Let us think about the primitive existence of numbers. Cohen mentions in [17, Chapter 2, §§1–2] a plausible type of theories that takes numbers as atoms or individuals, as *Urelemente*, i.e. a types of theories in which numbers are not viewed as sets. In that case, the axiom of extensionality has to be dropped or limited, as objects that do not contain things at all nevertheless have to stay different in this type of theory (otherwise there would be only one "number", as all of them are identical), while with the unlimited axiom of extensionality anything that contains nothing would all be identical—the one and only empty set $\varnothing$ in the theory. Let us explore the pros and cons of an atomic theory of numbers.[16]

The positive side of an atomic theory of numbers is that it is arguably more intuitive, in the sense that we learn to use and operate on numbers first, and sets later, so that numbers seem to be at least epistemologically more basic, if not conceptually.

But the negative side of an atomic theory of numbers seems to be that it is conceptually less simple or elementary because we have to have each and every number in the theory, while with set-theoretical reductions we add once and for all an axiom of infinity[17] that conjures into existence larger sets by itself. (By "an axiom of infinity" is not meant the axiom of infinity of infinite atoms of number as in Fraenkel [25].)

Now we are going to explain what non-Cantorian sets are. Non-Cantorian sets prove to be the Achilles's heel in Cantor's theory. That they exist seriously threatens Cantor's reductionism.

4.2.2 Non-Cantorian Sets

Non-Cantorian set theory is any set theory in which the axiom of choice or the continuum hypothesis is false. Thus "non-Cantorian sets" might mean "sets in any non-Cantorian set theory". But here we are not concerned with "non-Cantorian sets" in this sense. We are concerned with another, though related, sense of the phrase.

A "non-Cantorian set" as we use it here simply refers to a set

[16] Arguments not used in the context of Cantor's theory of numbers.

[17] It functions like the way mathematical induction does. In one representative formulation it specifies that there exists a set x containing the empty set and that if a set y belongs to set x then the union of y and $\{y\}$ also belongs to set x. Therefore this set is infinitely large. And so the existence of at least one infinitely large set is guaranteed. The axiom of infinity in symbolic notation is

$$\exists x(\varnothing \in x \land \forall y(y \in x \rightarrow y \cup \{y\} \in x).$$

that is not equivalent to the set of its *unit subsets*.[18]

On the other hand, a Cantorian set is a set that satisfies the requirement that it be equivalent to the set of its unit subsets. Sets of naïve set theory satisfy this requirement because there is a one-one correspondence between the set A and the special power set $\mathfrak{P}_1(A)$—the set of A's unit subsets—consisting of all and only the $\{a_i\}$'s. $\{a_i\}$'s are the unit subsets of A, and that one-one correspondence is one in which each member of A would correspond to its singleton. Using our example above, the one-one correspondence would be

$$\{\alpha\} \leftrightarrow \alpha, \{\beta\} \leftrightarrow \beta, \text{ and } \{\gamma\} \leftrightarrow \gamma.$$

But systems such as Quine's NF [58] admit non-Cantorian sets.[19] NF admits non-Cantorian sets because of stratification. Stratification refers to the hierarchisation of set-theoretical objects. In Whitehead and Russell [72] it was known as the theory of types. By whatever name the hierarchisation is known, it prevents the establishment of the equivalence which is possible in Cantorian sets as presented above because equivalence is not a relationship that can stand between sets of different "levels". It is simply prohibited in hierarchisised theories.

In view of Cantor's transfinite theory, it is clear that the very existence of non-Cantorian sets is an affront to the transfinite theory via the abstraction principle by which it is defined that a cardinal number $\overline{\overline{M}}$ is a "pure", definite set composed of abstract units to which all sets with the same cardinality (number of elements) will be equivalent (one-one correspondent). (See Section 3.3.4.)

[18]Intuitively, "unit subsets" of a set A are subsets of A that have exactly one member so that each member a_i of the set A under consideration "gives rise to" a set $\{a_i\}$ with a_i as its sole member, and the set of these unit subsets has as members all and only the $\{a_i\}$'s formed from each and every a_i of A. For example, given a set $\{\alpha, \beta, \gamma\}$ with members α, β and γ, its unit subsets would be $\{\alpha\}$, $\{\beta\}$ and $\{\gamma\}$, and the set of its unit subsets would be $\{\{\alpha\}, \{\beta\}, \{\gamma\}\}$.

[19]No one seems to be aware that this is a threat to Cantor's transfinite theory, judging from the literature.

This should be an affront because any theory that admits non-Cantorian sets endorses a fundamentally different ontology from Cantorian set theory and renders powerless the abstraction principle which accounts for numbers by means of cardinal numbers and which accounts for the comparison and manipulations of cardinal numbers by means of one-one correspondence, as the non-equivalence of a set and the set of its unit subsets constitutes an insurmountable theoretical difficulty for the abstraction principle. This non-equivalence is fatal to his theory of numbers.

This non-equivalence is fatal because sets in non-Cantorian set theories that have the same "number" (allow the provisional use of the term here) of elements can be of different level in the hierarchy, in which case those sets cannot be equivalent, so that cardinality cannot function as an adequate measurement of size. In such theories numbers cannot be reduced to sets, and the abstraction principle is not applicable. If the abstraction principle is not applicable in these theories, then there is one more weighty reason to be suspect of the adequacy of set-theoretical reduction of numbers.

Another important objection to Cantor's theory is against the inevitable use of intensions in a theory of extensionality. Unlike those in Sections 4.2.1 and 4.2.2, this objection has much to do with Cantor's domain principle (see Section 3.1).

4.2.3 Intension in an Extensional Theory

By definition, the extension of an infinite concept cannot be completely listed and, more specifically, the objects in an infinite class cannot be completely listed. Therefore one has to have recourse to intensional definitions, that is, specifying the property which allows and ensures the membership of an element. However, this brings in the problem of the equivalence of intensional definitions and extensional definitions, for one needs this equivalence in that set theory is basically a theory of extensionality, as in general axiomatic set theories explicitly contain an axiom of extensionality.

Wittgenstein is a prominent proponent of this stance of finding fault with "infinite extensions". As a staunch finitist, he repeatedly made his finitist arguments. One quote sums up his view neatly but, of course, with qualifications: "It's a question of the possibility of checking." [74, §174, p.212][20] It is this possibility of checking that underlines his arguably cryptical position, and his somehow unintuitive remarks. This crypticality and unintuitiveness and the reasons behind will be apparent anon.

He denies that "infinite extensions" are really extensions, for "[i]n truth, [...] it's impossible to talk of [the case in which all x happen to have a property] at all and the '$(x)\cdots$' in arithmetic cannot be taken extensionally" [74, §174, p.212] because it had to be specified by means of some property. Because of this denial of the extensionality of any universally quantified statement in arithmetic, which is at least very in keeping with his acute alertness towards minute differences in philosophical grammar and linguistic usage, he is antagonistic towards the consistency of the notion of an infinite set, as such a set presumably has as its members the extension given by such a universal statement. As a result of this antagonism, he deems it an abuse of language to compare the "sizes" of infinite sets, and, in particular, a flawed enterprise to compare the set of all transcendental numbers and the set of all algebraic numbers.[21] To him they seem to be different kinds of sets which cannot be compared with each other at all.

His colourfully delivered objection to set theory is worth quoting:

[20] And another similar one is "Every proposition is the signpost for a verification." [74, §148, p.174]

[21] Wittgenstein articulates this point in [74, §174, p.211].

> The theory of aggregates attempts to grasp the infinite at a more general level than a theory of rules. It says that you can't grasp the actual infinite by means of arithmetical symbolism at all and that therefore it can only be described and not represented. The description would encompass it in something like the way in which you carry a number of things that you can't hold in your hands by packing them in a box. They are then invisible but we still know we are carrying them (so to speak, indirectly). The theory of aggregates buys a pig in a poke. Let the infinite accommodate itself in this box as best it can. [74, §170, p.206][22]

The pig-in-a-poke metaphor signals his indignance of describing a structure amorphously. He finds it misleading to say the least. But we cannot have more than this when we deal with an infinite series.[23] Wittgenstein is of the opinion that this impossibility of representation by means of arithmetical symbolism makes the infinite merely a rule, and that there is no such thing as an infinite extension. He also gives a *reductio ad adsurdum* of the concept of an infinite *totality*:

> Let's imagine a man whose life goes back for an infinite time and who says to us: 'I'm just writing down the last digit of π, and it's a 2'. Every day of his life he has written down a digit, without ever having begun; he has just finished. [74, §145, p.166]

The idea of "counting" the members of an infinite set seems to be lurking behind the set-theoretic assertion about two infinite sets that they have one-one correspondence.[24] By establishing

[22] "Theory of aggregates" is an older term for set theory.

[23] This point is found in [74, §147, p.169].

[24] This paragraph consists of original points.

one-one correspondence between them, the set of natural numbers and the set of even numbers are found to contain the same "number" of elements even though the latter is a proper subset of the former. However, regarding the treatment of something of such delicate calibre, care must be taken and two points has to be noted,

1. if two infinite sets were said to contain the same "number" of elements whenever there is a one-one correspondence between them, and if the size of any set must be greater than that of its proper subset, then this "number" cannot be an adequate measure of its size;

2. this "number" is not something which could ever be reached by counting.

The second point set-theorists would not hesitate to admit. As to the first point, before Cantor put forth his theorem, it seemed doubtful whether the definition of equinumerosity between two infinite sets as a one-one correspondence between them would be fruitful at all, because it would not be if there were not sets of different infinite sizes to compare from,[25] for what good does it do to compare infinite sets if they were all of the same size? But then came Cantor's theorem, and comparison began to have (at least a semblance of) sense.

Despite the attraction and beauty of Cantor's paradise, however, Wittgenstein is staunchly opposed to the talk of cardinalities of infinite sets, as such "sets" seem to him to be rules rather than extensions. For him, what the notation $2^{\aleph_0}$ refers to is certainly not a number, and should not be subjected to operations as if it is a number. It is not something we can ever count and reach. And he sees the continuum hypothesis as plain nonsense. It would be easier to understand his view if we look at what he says regarding the nature of a number.

[25]Historical point mentioned in Potter [55], p.153.

Nature of a Number

Wittgenstein emphasizes that a number should not be seen as something independent of a number system. It always is a part of a particular number system, a structure.

> If, in the nature of the case, I cannot write down a number independently of a number system, that must also be reflected in the general treatment of number.
>
> A number system is not something inferior—like a Russian abacus—that is only of interest to elementary schools, while the higher, general discussion can afford to disregard it. [74, §171, p.207]

The sense of treating the cardinal numbers of the set of natural numbers and the set of real numbers as if they are ordinary finite numbers seems highly suspicious to Wittgenstein.

Wittgenstein holds the view that mathematics is a human construct and what human can construct are only finite numbers and infinite rules, but never infinite numbers. Because of his finite constructivism, he deems it nonsense to say that the maximum of a function is the largest value among all its values unless there are but finitely many, discrete points on the curve of the function.[26]

Wittgenstein attempts to show the nonsensical nature of taking "infinite numbers" as an extensional term by examples of ordinary usage and analysis of the concept of an extension, and it is pointed out that very different sorts of experience would be regarded as confirmation of the assertion "Suppose we travel out along a straight line into Euclidean space and that at 10m. intervals we encounter an iron sphere, *ad. Inf.*" and the assertion that we encounter 10,000 spheres in a row. [74, p.305] He then dismisses any comparison of the size of the sets of natural numbers and real numbers by means of cardinal numbers

[26]He expresses this in [74, §172, p.208].

as nonsense, because firstly, such sets may not be well-founded, and secondly, cardinal numbers are not genuine numbers, and thirdly, countably infinite and uncountably infinite sets are such different kinds of sets that it does not make sense to compare them. Such, in brief, was his argument against Cantor's cardinalities in [74].

Is the Infinite a Rule or a Number?

Cantor offers two accounts of numbers, namely the ordinal theory of numbers and the cardinal theory of numbers.[27] The ordinal theory of numbers is conceptually based on well-ordering while the cardinal theory of numbers is conceptually based on one-one correspondence.

In a sense, ordinal numbers represent the procedure of counting: sets are put into well-ordering and "quantitatively" represented in the mind via enumerals. For this reason, ordinal numbers can be said to be a philosophical compromise between a rule and a number, a kind of "static" counting.

On the other hand, cardinal numbers represent a roundabout way of counting—counting through comparison by way of finding a one-one function.

Whichever one of the ways Cantor explicates numbers, the infinite is not taken as a rule. It is an extension and subject to Wittgenstein's criticism. The infinite as a rule is intuitively attractive, while the infinite as a number gives unexpected and elegant results in Cantor's theory. While the fruitfulness of his theory makes it a convincing theory, its clash with ordinary understanding of quantity and comparison of size is a very serious problem.

The implication of this would be clearer if we framed it in terms of computability, but we may refer our readers to the

[27] This section offers a new perspective on the Cantor-Wittgenstein disagreement.

considerations in Section 4.1.2 to know the approach and consequences without explicitly reframing and repeating this part.

Let us now come to the last section, conceivability and comparability, which contains the last objection, tension with absolute infinity, which is also aimed against Cantor's domain principle, but via a different line of attack.

4.3 Conceivability and Comparability

Mathematics frequently touches on what is conceivable, what is comparable, and what is conceivably comparable; these have all much to do with the our conception of number and numerosity (which we have just discussed) because comparison in mathematics is nearly always done by means of number and numerosity, or some concepts closely related to them. Cantor's proofs push and mark the lines outlining what is conceivable, what is comparable, and what is conceivably comparable, all third. Infinite sets were not really convincingly conceivable until they were shown to be comparable (via set-theoretical operations) and the result of comparison shown to be fruitful (as infinite sets were of different sizes). Cantor's conceptual "inventions" established the conceivability of such comparisons. However, Cantor maintains that there is a type of collection which is not comparable with others. Cantor calls such a collection *absolute.*

Let us look at the problem it causes to Cantor's theory.

4.3.1 Tension with Absolute Infinity

Let us first review the domain principle, presented in Section 3.1. The domain principle states that, for any variable to be meaningful in a mathematical context, there has to be a domain for it to range over. Its consequence is that any potential infinity presupposes a corresponding actual infinity. The domain principle justifies this presupposition in that it "forces" an equation with a variable x to have a domain of x.

Cantor argues that natural and real number operations make existence of transfinite numbers inevitable because of the domain principle. And at the same time he maintains that the transfinite numbers themselves form a universe (but not a domain) of mathematical forms which constitute absolute infinity. But then why do the transfinite numbers not form a domain likewise, via the domain principle? And why do the transfinite numbers not require a domain for it to range over so as to be meaningful, as in the case of potential infinity?

Cantor does not have a way of satisfactorily resolving this tension between the "numerability" (or the comparability) of transfinite numbers and the "unnumberability" (or the incomparability, the absoluteness) of absolute infinity, other than invoking God as the only one who can understand absolute infinity, and mentioning the undesirable consequence that this uniqueness of God would be destroyed if absolute infinity were a domain and could be mathematically determined in the same way as transfinite numbers.[28]

Justification for Type Distinction

Cantor did not try to resolve this tension between the numerability of transfinite numbers and the unnumberability of absolute infinity, but maybe we could construe Cantor as implicitly assuming that there is a type distinction?

We could construe Cantor as implicitly assuming that absolute infinity cannot be conceived as a unity and thus nor can it be conceived as a set, simply because by definition there is an intrinsic *type distinction* between the increasable infinite (transfinite) and the absolute infinite in that the former can be a unity

[28] No one questions this horrible weakness of Cantor's theory in the literature, and there are only brief textual references to this theistic argument of Cantor's in Hallett [34].

while the latter cannot, but Cantor has not provided any justification for the difference between his treatments of transfinite numbers and absolute infinity. If there is a genuine type distinction, then non-theists do not need to stand aghast at the unconvincing invocation of God. Is there a justifiable type distinction between the increasable infinite (transfinite) and the absolute infinite?

Cantor ordains in a rather *ad hoc* way that

> The ordinal numbers do not form a set, but an absolute collection. [34, p.168]

To be sure, Cantor could appeal to the Burali-Forti paradox[29] to justify his absolute infinity, since this paradox was already known to him. He could appeal to the Burali-Forti paradox in order to justify a type distinction between the increasable infinite (transfinite) and the absolute infinite because by the Burali-Forti paradox, if the order type of all ordinal numbers is an ordinal number and can be compared as ordinal numbers are compared, then paradox arises. Therefore if "absolute infinity" were also comparable with transfinite numbers, there would be a paradox, as it will be larger than itself.

Indeed, in [13, p.114], Cantor tries to connect *absolutely infinite collections* and *inconsistent collections* (see below) together as referring to the same things. But if they refer to the same things, it might recommend an axiomatic set theory rather than the postulation of absolute infinity.

To gain insight into what Cantor calls *inconsistent collections*, let us look at his transfinite theory. Cantor [12] introduced his theory of transfinite numbers with something like this:

[29] The Burali-Forti paradox is the paradox that if the order type of all ordinal numbers is an ordinal number, then it is strictly less than itself, because it is an ordinal number and the order type of all ordinal numbers is larger than any ordinal number, including itself, as it is an ordinal number. See Burali-Forti [10] for his original publication on this.

> If Ω is any initial segment of numbers, then there is a least number $S(\Omega)$ which is greater than all the numbers in Ω.

Examples of this operation of taking supremum, or least upper bound, are (cf. Section 1.2.5)

$$S(\text{empty segment}) = 0 \tag{4.6}$$
$$S(0, \dots, n) = n + 1 \tag{4.7}$$
$$S(0, 1, 2, \dots) = \omega \tag{4.8}$$

Now if we were to take Ω to be the initial segment of numbers and operate on it, then we may form $S(\Omega)$, and whenever we have a initial segment of numbers Ω, we seem to be able to take its supremum $S(\Omega)$, and take it again, if we would like to.

Now we have to concern ourselves with the totality of all numbers. Could we take its supremum? But here we have a problem, and the problem is that if we were to admit S(all numbers) as a number, then we will have the absurd conclusion that it is less than itself:

$$S(\text{all numbers}) < S(\text{all numbers})$$

as in the Burali-Forti paradox, because S(all numbers) is larger than any number, and as S(all numbers) is also a number, it is larger than itself.

The arguments above show that the problem of distinguishing among those initial segments Ω of numbers to which upper bounds $S(\Omega)$ can be assigned and those to which upper bounds $S(\Omega)$ cannot be assigned is serious. If we cannot solve it we are going to end up with paradoxes.

Cantor knew the existence of this problem and later, in an attempt to solve it, called the segments which have no upper bound *inconsistent collections.*[30] These are now generally called

[30] He did this in [13, p.115].

proper classes. On the other hand, segments which do have upper bounds are called *sets* or sometimes *improper classes.* However, picking them out by name does not by itself solve the problem.

Condition on Taking Supremum

A way to solve it that is mentioned in Tait [65, p.90] would be to make the definition of the numbers precise by setting up some precise condition Φ on initial segments, and to admit its supremum $S(\Omega)$ only when it satisfies the condition Φ, so that it would no longer be possible to obtain *inconsistent collections.* One can define the new operation thus,

> If Ω is any initial segment of numbers satisfying the condition Φ, then there is a least number $S(\Omega)$ which is greater than all the numbers in Ω.

Let us call this a Φ-number. It does not lead to contradiction in admitting S(all Φ-numbers) because we can stipulate that the totality of all Φ-numbers, i.e. S(all Φ-numbers), does not satisfy the condition Φ. As it does not satisfy the condition Φ, S(all Φ-numbers) would not be a Φ-number and we could no longer derive the absurd conclusion that

$$S(\text{all } \Phi\text{-numbers}) < S(\text{all } \Phi\text{-numbers})$$

as before.

But we can add S(all Φ-numbers) as another number, provided that we do not keep using the condition Φ and that we switch a new condition, say Ψ, and have it as a Ψ-number. And then, when we want to take the supremum of all Ψ-numbers, we may add yet another number satisfying another condition, and continue this process indefinitely. Formulated in this manner, the theory of transfinite numbers would never be complete.

One either makes use of conditions like Φ or not, and if one makes use of them, the theory would be open-ended, while if one does not make use of them, one is left with absolute infinity as Cantor was.[31] Both seem to be very unsatisfactory, and this pair of alternative constitutes an important objection to Cantor's transfinite theory, for this is entailed by his comparable infinities.

The reason that this is entailed by his comparable infinities is that if infinity were from the outset not allowed of those operations Cantor formulated, there would not be the problem of open-endedness because of the conceptual characteristic of ordinals, and there would not be the problem of absolute infinity in a bid to put an end to the unending series of infinite ordinals. The problem of open-endedness causes unstability to the theory while its alternative, the problem of absolute infinity, causes inconsistency—not due to *inconsistent* collection *per se*, but due to the inconsisteny of standard. This inconsisteny of standard poses a very serious threat to the theory.

4.4 Conclusion

We have brought to clear view the philosophically suspect assumptions of his theory (the three principles in Chapter 3) and critically examined these assumptions and presented reformulated arguments against them (the six problems in Chapter 4). We have given a well-articulated account of the whole issue which has not been brought together before and which enables the reader to decide his stance in the matter.

In this chapter, after having given all the relevant backgrounds and explanations in the previous ones, we discussed various problems of Cantor's transfinite theory, and those are: the

[31] I have not seen this dilemma noted or used as an objection against Cantor in the literature.

endorsement of free mathematics, the use of non-constructive proof, the need to justify his weak reductionism, the existence of non-Cantorian sets, intension in an extensional theory, and, finally, tension of increasable infinity with absolute infinity.

First of all we discussed Cantor's endorsement of free mathematics which is the doctrine that endorses a maximum ontology, allowing existence whenever no inconsistencies result. We argued that free mathematics gives rise to more difficult foundational questions. The lack of existence proofs and violating the simplicity principle were also defects in his theory.

After that we discussed the use of non-constructive proof in Cantor's theory. A constructive proof is a proof in which the existence of a mathematical object or function etc. is not simply proved by establishing that its non-existence is contradictory, but instead proved by showing that algorithmic construction of that object from some accepted primitives is possible in principle. We went into the myriad of arguments between classical mathematicians and intuitionist constructivists, and presented the philosophical considerations against non-constructive proofs.

And then we discussed Cantor's weak reductionism. It is *weak* in that he does not simply reduce numbers to sets, but it is *reductionistic* in that numbers and their existence are explained and justified in terms of sets. We showed that his weak reductionism is unwarranted.

We went on to discuss the problem posed by non-Cantorian sets for Cantor's transfinite theory. Non-Cantorian sets were a problem for Cantor's transfinite theory because any theory that admits non-Cantorian sets endorses a fundamentally different ontology from Cantorian set theory and renders powerless the abstraction principle which accounts for numbers by means of cardinal numbers and which accounts for the comparison and manipulations of cardinal numbers by means of one-one correspondence.

The next problem that we discussed was use of intension in Cantor's theory which is inevitably extensional. By definition, the extension of an infinite concept cannot be completely listed and, more specifically, the objects in an infinite class cannot be completely listed. Therefore one has to use intensional definitions. The use of intensional definitions means that it is no longer truly an extensional theory, causing inconsistency.

Lastly we discussed the tension of increasable infinity with absolute infinity. This tension has to do with the dubious role of absolute infinity and its clash with the domain principle. Cantor does not have a way of satisfactorily resolving this tension between the numerability of transfinite numbers and the unnumberability of absolute infinity, and of explaining his difference in treatments of transfinite numbers and absolute infinity.

Considering the above arguments, beautiful as Cantor's paradise is, we may probably have to renounce it, if we are to act sensibly.

There are numerous problems with Cantor's theory, even though the controversy is not overwhelmingly against Cantor, as both sides of the argument have their points to offer. Cohen and Hersh [18] think that the development of set theory and that of geometry are analogous, and they hint that as non-Euclidean geometry found interpretation in the works of physics of Minkowsky and Einstein, non-Cantorian set theory might one day find its use outside mathematics and facilitate wider reception. (Of course Cantor is not as towering a figure as Euclid is, but his theory is nevertheless very commonly accepted.) Indeed quantum mechanics promises support for finitary mathematics. But the problem of application will have to be covered by another paper, for we had only the space to cover the problem on the theoretical level.

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