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The First Lectures in Italy on Galois Theory: Bologna, 1886–1887

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During the academic year 1886–1887, Cesare Arzelà (1847–1912) gave a course on Galois theory at the University of Bologna, the first in Italy on this subject. That year, the audience included the future mathematician and historian, Ettore Bortolotti (1866–1947), who took notes on the lectures. Here, the lectures are analyzed first in the context of the development of Galois theory in Europe and then in light of institutional developments at Bologna, especially following the unification in 1861. Arzelà emerges as a creative and effective teacher and mathematician in the discussion of the actual content of the lectures. © 1999 Academic Press

Nell'anno accademico 1886–1887, Cesare Arzelà (1847–1912) tenne un corso sulla teoria di Galois all'Università di Bologna, il primo in Italia su tale argomento. Quell'anno l'uditorio includeva il futuro matematico e storico, Ettore Bortolotti (1866–1947), che trascrisse gli appunti delle lezioni. Nel presente articolo le lezioni sono analizzate prima nel contesto dello sviluppo della teoria di Galois in Europa e poi alla luce degli sviluppi istituzionali a Bologna, soprattutto quelli successivi all'unificazione del 1861. Dalla discussione del contenuto delle lezioni, Arzelà emerge come un insegnante e matematico creativo ed efficace. © 1999 Academic Press

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THE LECTURES

During the academic year 1886–1887, Cesare Arzelà (1847–1912), most noted for his work in analysis, gave a course on Galois theory at the University of Bologna. The audience that year included a student who later became famous as a mathematician and historian of mathematics, Ettore Bortolotti (1866–1947). Bortolotti compiled the text of the lectures; the set of notes forms a substantial volume, entitled *Teoria delle sostituzioni*, now held in the Bortolotti Library of the University of Bologna's Department of Mathematics [9].

Arzelà joined the Bologna faculty in 1880. In 1884, he became a Full Professor, assuming the Chair of Higher Analysis. The course he gave in 1886–1887 was thus ostensibly on higher analysis but actually on the theory of substitutions and Galois theory. Contrary to what is commonly believed (see, for example, [17, 184]) Enrico Betti (1823–1892) was not the first in Italy to offer a public cycle of lectures on Galois theory. Although he was the first Italian mathematician to devote himself to the study of Galois theory, he never taught it as part of the curriculum at the University of Pisa. As Betti wrote in a letter to Placido Tardy (1816–1914) in 1859,[1] he did give lectures on the most important parts of algebra,

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[1] The letter has never been published but is held in the "Fondo Betti" at the *Scuola Normale Superiore* in Pisa.

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but only to a few talented students in his home and not in the public forum of the university.[2] Others had assumed that Luigi Bianchi's (1856–1928) course on Galois theory at the *Scuola Normale Superiore* of Pisa in the academic year 1896–1897 was the first such presentation because it was the first actually to be published [16]. Arzelà gave his course, however, 10 years before Bianchi.

Bortolotti's notes on Arzelà's lectures are written in standard late-19th-century Italian and appear quite modern.[3] He did not divide his notes into chapters, but the course can be subdivided, according to the subjects treated, into four parts. The first part includes the study of symmetric functions, of the discriminant and its properties, and of two-valued functions. The second part consists of the exposition of the theory of substitutions and includes an introduction to multiple-valued function theory, a discussion of substitution representations, a detailed treatment of group theory, an introduction to the study of families of functions, and a presentation of congruence theory. The third part treats the resolution of equations and covers a method for solving equations of the first four degrees by radicals, the presentation of algebraic functions, a treatment of the impossibility of solving equations of degree higher than 4 algebraically (in which Arzelà states and proves the so-called Ruffini–Abel theorem), and a detailed discussion of abelian, binomial, and reciprocal equations. The last part focuses on Galois theory and conditions for solvability by radicals (see the section "Mathematical Overview of the Lectures" below for details).

GALOIS THEORY IN THE EUROPEAN CURRICULUM

The resolution of the general equation of degree *n* in one unknown represented a main problem in mathematics in the period following 1770, the year in which the French mathematician, Joseph Louis Lagrange (1736–1813), published his *Réflexions sur la résolution algébriques des équations* [44]. In this treatise, Lagrange pointed out a method for solving algebraic equations of degree 3 and 4 and tried unsuccessfully to extend that method to equations of degree higher than 4 [27 : 1, 72–73]. In fact, Lagrange's approach fails for equations of degree greater than 4, but his innovative reflections on the relation between the given equation and an appropriate auxiliary equation were later reconsidered by Évariste Galois (1811–1832), who ultimately resolved the problem of the solvability of algebraic equations by radicals.[4]

[2] As Betti explained to Tardy, he gave the twice-weekly lectures to four students, "per esporre loro le parti più elevate dell'algebra che non posso esporre nel corso che fo all'Università. Per ora ho esposto la teorica delle equazioni abeliane all'applicazione [sic] alla teorica della divisione del circolo. Passerò presto ad esporre la teorica della risoluzione algebraica in tutta la sua generalità [in order to convey to them the highest parts of algebra which I am not able to include in my university course. Up to now, I have spoken on the theory of abelian equations as applied to [?] the theory of the division of the circle. I will soon move to the theory of algebraic resolution in all of its generality]."

[3] The essential parts of the lectures (the theory of substitutions, groups, and algebraic equations, as well as Galois theory) are available from the Department of Mathematics, University of Siena, Via del Capitano 15, 53100 Siena, Italy.

[4] The problem already had a long history in 1770. In the first years of the 16th century, Scipione dal Ferro (1465–1526) had found the general solution of the algebraic equation of degree 3 with zero quadratic term. Afterwards, Gerolamo Cardano (1501–1576) and Niccolò Tartaglia (?–1557) extended the formula to the general equation of degree 3. Cardano's pupil, Ludovico Ferrari (1522–1565), succeeded in finding a method to solve every algebraic equation of degree 4. Cardano gathered together all these new, definitive results in his masterpiece, *Ars magna*, published in 1545. For a complete account on the history of the theory of algebraic equations, see [31].

The year 1799 represented an important turning point in the history of algebraic equations. The German mathematician, Carl Friedrich Gauss (1777–1855), gave the proof of the fundamental theorem of algebra,[5] and the Italian mathematician, Paolo Ruffini (1765–1822), proved that the general equation of degree $n > 4$ is not solvable by radicals [56]. Twenty-five years later, the Norwegian prodigy, Niels Henrick Abel (1802–1829), showed that the general quintic equation is not algebraically solvable [2], and in 1826, he gave a new proof of the impossibility of the solution by radicals of the general equation of degree $n > 4$, independent of Ruffini's work [1]. However, a basic problem remained unsolved; it is actually possible to point out infinitely many particular equations of degree $n > 4$ that *are* algebraically solvable.[6] Therefore, the question reduced to finding the properties common to all algebraic equations which are solvable by radicals. Galois did just that. His ingenious idea was to associate to any equation a group of substitutions and to study the nature of the group. He obtained the following result: an algebraic equation of degree n is solvable by radicals if and only if its associated group, in modern terminology, is solvable. It should be noted that Galois's purpose was not to obtain methods for solving equations, but rather to know whether or not an algebraic equation of degree n was algebraically solvable. The aim of Galois theory is, in fact, to find necessary and sufficient conditions for an algebraic equation to be solvable by radicals.

As is well known, Galois presented his research to the *Académie des Sciences de Paris* for the first time on May 25 and June 1, 1829. His work consisted of two memoirs, entitled "Recherches algébriques" and "Recherches sur les équations algébriques de degré premier," respectively. The report on these contributions was supposed to have been presented by Augustin Louis Cauchy (1789–1857) during the meeting of the *Académie* on January 18, 1830. However, Cauchy did not participate in the meeting and, a week later, he presented to the *Académie* one of his own works. In the meetings that followed, Cauchy never mentioned Galois's papers, and it seems that the manuscripts in his care were lost [60, 33–34].

Galois continued to study algebraic equations, and in February 1830, he presented his work to the *Académie* again. The submission of Galois's memoirs was duly recorded in the *Académie*'s protocol books, and this time the manuscripts were entrusted to Joseph Fourier (1768–1830) as examiner. However, Fourier died, and the manuscripts were never found among his effects. Galois submitted his memoir to the *Académie* for the third time on January 17, 1831. He had corrected and enlarged his work, now entitled "Sur les conditions de résolubilité des équations par radicaux" [33]. This time the paper was examined by Silvestre François Lacroix (1765–1843) and Siméon Denis Poisson (1781–1840), who in their report to the *Académie* pronounced Galois's work incomprehensible. The manuscript of the two academicians was given to Galois who, the night before his death, made some corrections and brief additions to his work. That night Galois wrote a letter to his friend, Auguste Chevalier, in which he sketched the main results he had obtained in the theory of algebraic equations. Upon his death in 1832, the name "Galois" was forgotten [60, 34–35]. (For a fascinating and accurate biography of Galois, see [59].)

[5] Gauss proved the fundamental theorem of algebra in his first published paper, his Helmstadt thesis of 1799. See *Werke*, ed. Gesellschaft der Wissenschaften zu Göttingen, 12 vols. (Göttingen: Akademie der Wissenschaften 1, 1863–1929; reprint ed., Hildesheim/New York: Georg Olms Verlag, 1981), 3:1–30.

[6] In 1801, Gauss analyzed the equation $x^n - 1 = 0$ and indicated a method of solution by radicals for every natural number n, making specific calculations for $n = 17$ and $n = 19$ [34].

Ten years later, some of Galois's friends[7] persuaded the French mathematician, Joseph Liouville (1809–1882), founder (in 1836) of the *Journal des mathématiques pures et appliquées*, to study Galois's works. Liouville was convinced of their importance and decided to publish them in the *Journal*'s December 1843 number. At the last minute, however, he replaced them with papers by Serret and others, and Galois's memoir only appeared in 1846 together with a fragment of another unpublished paper, his last letter to Chevalier, and all of his previously published works. All subsequent developments in Galois theory were based on this publication [45, 130–131].

The first Italian mathematician to study questions related to the solvability of algebraic equations by radicals and Galois theory was Enrico Betti. His first note on the subject, "Sopra la risolubilità per radicali delle equazioni irriduttibili di grado primo," appeared in 1851 [14]. A year later, he published "Sulla risoluzione delle equazioni algebriche" [15]. The first part of this commentary consists of the exposition of the theory of substitutions; the second is entirely devoted to the development of Galois theory. Betti's writings, neither always clear nor completely correct, nevertheless deserve attention since they represent the first attempt to interpret and develop both Galois theory and the recently conceived theory of groups [60, 54, 58–59]. Given that Betti assumed the Chair of Algebra at the University of Pisa in 1857, Italy could have become an early European leader in Galois theory and group theory, but Betti taught only the traditional algebraic topics, never including Galois theory in his university courses [61, 243].[8] Moreover, he never wrote a textbook on the subjects to which he had dedicated himself during the first years of his research activity, and he never had research students in algebra.

In 1859, Betti moved into the Chair of Higher Analysis at Pisa, succeeded in the Chair of Algebra by Giovanni Novi. Novi planned to write a three-volume treatise on higher algebra but only one volume of the *Trattato di algebra superiore* appeared in 1863 [47]. In the preface, he explained that he had followed Betti's lecture notes in compiling the treatise; thus, once again, Galois theory failed to reach a broader Italian audience [60, 66]. In Germany and elsewhere, the situation was quite different.

The first university course on Galois theory was given by the German mathematician, Richard Dedekind (1831–1910), at the University of Göttingen in the winter semester of the academic year 1856–1857 [26]. So interested was Dedekind in the topic that he gave a second course on it the following winter semester. His written text of the lectures provided not only the first organic exposition of a large part of Galois theory (at that time the conditions for solvability by radicals were not completely clear) but also a basic contribution to group theory of which he, together with Galois, is considered a founder. The text of Dedekind's lectures, however, was only published in 1981 [57].

Five years after Dedekind gave his second course on Galois theory in Germany, Ludvig Sylow (1832–1918) lectured on the subject at the University of Oslo (at that time called

[7] Galois's friends included his brother, Alfred, and Auguste Chevalier. For more details, see [45, 560–564].

[8] Betti treated the following subjects in his algebra course: numerical series, algebraic series, theory of derivatives (Taylor series), theory of homogeneous functions, invariant theory, general principles of equations of any degree, symmetric functions of the roots of an equation, equations with more than one unknown, limits of roots, Descartes's theorem, separation of roots, irreducible equations, Newton's method as improved by Fourier, numerical resolution of equations by continuous fractions, manipulation of equations, binomial equations, algebraic resolution of third degree equations, and algebraic resolution of fourth degree equations [20, 245].

Christiania) in Norway. In his presentation, Sylow gave the criterion of solvability for irreducible equations of prime degree, but his exposition of the condition for the general equation of degree higher than 4 was not clear. Among the students who heard his explanations, however, was the 20-year-old Sophus Lie (1842–1899) [17].

In France, the first university text to include a chapter on Galois theory was the third edition of Joseph Alfred Serret's (1819–1885) *Cours d'algèbre supérieure* published in 1866. Serret's treatise was widely used as a textbook. As early as 1867, it had been adopted in the United States, and a German translation appeared a year later in 1868 [60, 87]. Serret's *Cours*, in a seventh and final edition in 1928, had a great impact on students of algebra well into the 20th century. As we shall see below, Cesare Arzelà based part of his university lectures on Galois theory in Italy on the fifth edition (published in 1885) of this influential book [58].

Serret's text was soon followed by Camille Jordan's (1838–1922) ground-breaking *Traité des substitutions et des équations algébriques* of 1870 [39]. There, Jordan gave the first cogent explanation of the conditions for solvability by radicals. Moreover, he recognized that the concept of a group could be fruitfully applied outside the theory of algebraic equations. In the third chapter of book III of the *Traité,* he detailed the role Galois theory would play in geometry.

Like Serret's work, Jordan's *Traité* proved extremely influential to a generation of mathematicians in France, Europe, and abroad. Beginning in the mid-1870s, for example, Julius Petersen (1839–1910) gave courses on the theory of algebraic equations at the Polytechnic School of Copenhagen in Denmark. Based on his lectures, Petersen wrote a two-volume book, which was published in Copenhagen in 1878 and which treated the theory of algebraic equations, the theory of substitutions, and Galois theory [49].

In Germany, studies on Galois theory and group theory also proliferated. In 1881, Paul Bachmann (1837–1920), who had been a student in that first course by Dedekind in 1856–1857, published the article "Ueber Galois' Theorie der algebraischen Gleichungen" in the *Mathematische Annalen* [12]. There, he based his analysis of Galois theory not on the concept of a group, but on the new concept of a division ring. Almost immediately, Eugen Netto (1846–1919), a former student of Leopold Kronecker (1823–1891) in Berlin, published his textbook, *Substitutionentheorie und ihre Anwendungen auf die Algebra,* in 1882 [46]. In 1881, Kronecker had written a very long memoir entitled "Grundzüge einer arithmetischen Theorie der algebraischen Grössen," which appeared in Crelle's *Journal* the following year [41]. In this important work, after introducing the concept of field of rationality,[9] Kronecker defined the notion of a family as an enlargement of the field of rationality. Netto's text advocated Kronecker's notion of a family over Bachmann's concept of a division ring [60, 117] and emphasized that in the 1880s Galois theory was developing in a number of different ways. Moreover, the brief discussion of Galois theory in the European curriculum thus far shows that by 1880s the subject was well entrenched in France and Germany and even in Norway and Denmark.

In Italy, however, Betti's Galois-theoretic work of the 1850s represented only an isolated case. Group theory did not enter the Italian research arena until the mid-1870s, when Alfredo Capelli (1855–1910) published over a dozen memoirs on groups of substitutions and on the

[9] For Kronecker, a field of rationality or *Rationalitäts-Bereich* of magnitudes $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \ldots$ was the collection of all rational functions of $\mathfrak{R}', \mathfrak{R}'', \mathfrak{R}''', \ldots$

theory of algebraic equations [23]. By 1885, Giovanni Frattini (1852–1925) had also added his algebraic works, notably "Intorno alla generazione di gruppi di operazioni," in which he characterized that subgroup of a group that has been named after him [32]. Also in 1885, the Italian translation by Giuseppe Battaglini (1826–1894) of Netto's *Substitutionentheorie* appeared. It was followed in 1891 by an Italian version of Petersen's treatise as well. In this atmosphere of renewed interest in group theory, and almost 50 years after Betti's studies, Italy finally saw the publication of its first lectures on Galois theory. During the 1896–1897 academic year, Luigi Bianchi gave a course on this topic at the *Scuola Normale Superiore* in Pisa; the text of his lectures appeared in print in 1899 [16].

While Bianchi's course in Pisa may have been the first to be published on Italy, it was not, as noted above, the first to be given. That distinction belongs to Cesare Arzelà a decade earlier at the University of Bologna. Reflective of the active interest in Galois theory outside of Italy, Arzelà consciously drew from the available European texts in introducing his students to this subject. It is to a discussion of Arzelà's Bologna and of his lectures on Galois theory that we now turn.

ALGEBRA AT BOLOGNA IN THE 1880s

In 1797, the city of Bologna, which had been part of the Papal State, was ceded to Napoleon. After the Congress of Vienna (1814–1815), it returned to Papal control and remained part of the Papal State until 1859. The last years of Papal rule had left the mathematical school of Bologna in a miserable state. Indeed, the Papal reign had marked a period of progressive decline for the entire university. Its best professors teaching after the turn of the 19th century were those who had received their training during the Napoleonic period (1797–1815). When they began to die or retire, they were replaced not by mathematicians of equal caliber but by candidates chosen solely on the basis of political considerations. Those who showed too much originality or who wished to promote scientific relations with foreigners (that is, scholars outside the Papal State) were immediately suspicious to the Holy Congregation that nominated candidates for professorships. The faculty thus consisted of professors who were devoted to their work but whose performance, in many cases, was mediocre from a scientific point of view [19, 202–203]. Moreover, the Restoration (1815–1831) witnessed ultimately negative official changes at the universities belonging to the Papal State. On August 28, 1824, Pope Leo XII issued the *Quod divina sapientia*, a reform that reconstituted the narrowly defined Mathematical–Philosophical Faculty into the broader Philosophical Faculty. Therefore, on the eve of national unity (1861), the Philosophical Faculty trained neither real engineers nor professors of mathematics [48, 19].

After the unification and its associated modifications of the political regime, the regulations governing the universities and their teaching staffs changed. Italy, at last unified as a nation, strove to revive the intellectual power of the state by improving its university studies. In 1860, chairs of higher mathematics were founded in the country's principal universities: Enrico Betti (1823–1892) and Francesco Brioschi (1824–1897) obtained the Chairs of Higher Analysis in Pisa and Pavia, respectively, while Giuseppe Battaglini and Luigi Cremona (1830–1903) inaugurated their courses in higher geometry in Naples and Bologna [19, 211; 48, 20]. Following the unification, in fact, the Faculty of Mathematics at Bologna gained three new professors. In addition to Cremona, Quirico Filopanti (1812–1894) joined the faculty as Professor of Applied Mathematics and Eugenio Beltrami (1835–1900) as

Extraordinary Professor of Complementary Algebra. Together with Domenico Chelini (1802–1878), who had served as Professor of Mechanics and Hydraulics since 1851, these men constituted the new Faculty of Mathematics at the University of Bologna [48, 21–22]. Under their guidance, the first 10 years of unification saw intense scientific activity at Bologna.

First, the University had three of Italy's best mathematicians on its faculty: Cremona, Beltrami, and Chelini. Such a concentration of talent had been unknown for more than a century. Unfortunately, from an institutional standpoint, things did not markedly improve. Despite Cremona's attempts to bring Bologna's mathematics teaching up to international standards, the University was still unable to offer a full baccalaureat course of study. This only began to change in the early 1880s when Cesare Arzelà (1847–1912) was named Professor of Higher Analysis in 1880–1881 and Salvatore Pincherle (1853–1936) followed as Professor of Algebra and Analytical Geometry a year later. In 1880, Luigi Donati (1846–1932), who had been teaching at the Engineering School of Bologna for three years, was named Professor of Mathematical Physics in the Faculty of Science [48, 22–23]. A new age in Bolognese mathematics finally began with their arrival.

Cesare Arzelà was born in S. Stefano di Magra (La Spezia, Italy) on March 6, 1847 and died there on March 15, 1912. He attended the Gymnasium of Sarzana and the Lyceum of Pisa.[10] In November 1861, having won a competition, he was admitted to the *R. Scuola Normale Superiore* of Pisa as a student of physical and mathematical sciences. During the four years that followed, he attended university courses and, at the same time, the complementary courses given at the *Scuola Normale*. He graduated in physical and mathematical sciences, defending a dissertation on potential theory that had been directed by Enrico Betti.

During the academic year 1869–1870, Arzelà continued to attend courses in higher analysis, mathematical physics, and higher mechanics and, in July 1870, obtained his teaching certificate. Two months later, he became a schoolteacher at the Lyceum of Macerata, where he remained for two years. After obtaining a leave of absence, he returned to Pisa to attend courses in elasticity theory in which Betti was the principal lecturer. He also attended the lectures Ulisse Dini (1845–1918) gave on the theory of functions of a real variable, a subject that later became his main research field. During this year in Pisa, Arzelà wrote an elegant paper on the deformation of an elastic ellipsoid and solved an important problem on the study of the elastic deformation of the earth. He returned, however, to his teaching post in 1873 and devoted himself with enthusiasm for the next five years to his work there. During this time, he had the good fortune to encounter two exceptional students: Vito Volterra (1860–1940) and Rodolfo Bettazzi (1867–1941).

On the basis of the paper he had written in Pisa in 1873, Arzelà was awarded the professorship of algebra at the University of Palermo in 1878. Two years later, he moved to the University of Bologna, where he was named Professor of Infinitesimal Calculus. In four years, he rose to the rank of Full Professor and obtained the Chair of Higher Analysis. Arzelà did his most important scientific work during his Bolognese period. He elaborated the concept of stepwise uniform convergence which gives a necessary and sufficient condition for a series of continuous functions to converge to a continuous function (1883), and he proved

[10] In 19th-century Italy, students of the classical curriculum attended a Gymnasium for two years and then moved on to the Lyceum for three more years.

the termwise integration theorem for a series of functions using the Riemann integral (1885) [8:1]. Despite these strong and demonstrated analytic interests, Arzelà offered a course on Galois theory during the 1886–1887 academic year. One of his most notable students,[11] Ettore Bortolotti (1866–1947), attended this course and compiled the set of notes under discussion here.

During the 1880s, the Bolognese school of mathematics focused primarily on research that would now be classified as analysis. Arzelà concentrated on the theory of functions of a real variable, as mentioned, and Salvatore Pincherle dealt with the theory of analytic functions following Weierstrass. In fact, it would seem that the best mathematicians of the Bolognese school did not concern themselves with purely algebraic topics. (For a complete list of the courses given by the Faculty of Science of the University of Bologna in the years 1860–1940, see [29, 433–474].) Arzelà's lectures on Galois theory, with their detailed exposition of number theory and group theory, thus represent a certain anomaly, especially since they were given under the rubric of higher analysis. That these more algebraic topics were somewhat foreign to Arzelà is suggested by the strong influence that Eugen Netto's treatise, *Substitutionentheorie und ihre Anwendung auf die Algebra*, clearly had on the contents of Arzelà's lectures (see the section "The Lecture Notes and Their Sources" below). As noted, Netto's treatise had appeared in Italian translation in 1885. Arzelà most likely read the book, was fascinated by the interesting and innovative subjects treated therein, and decided to give a course on algebra instead of his usual course on analysis.[12] Besides, in 1886, Arzelà in a letter written to Volterra stated his intention to use Netto's treatise as a textbook for his course on higher analysis [35, 268].

One of the members of that unique class was the 20-year-old Ettore Bortolotti. Following his experiences in Arzelà's course on Galois theory, Bortolotti went on to earn his degree in mathematics from the University of Bologna in 1889 with excellent marks. After serving as an assistant at Bologna and as a teacher at the Lyceum of Modica, Sicily, he completed his postgraduate studies in Paris (1892–1893) and then taught in Rome from 1893 to 1900. He moved to the professorship of infinitesimal calculus at the University of Modena in 1900, where he taught analysis and rational mechanics. His final position, from 1919 until his retirement in 1936, was the professorship of analytical geometry back at the University of Bologna.

Bortolotti's early research interests were in topology, but he later devoted himself to analysis, studying, among other topics, the calculus of finite differences, the convergence of infinite algorithms, the asymptotic behavior of series, and improper integrals. In his early work on topology, Bortolotti also showed a deep interest in the history of mathematics which increased during his stay in Rome. By the time he moved to Modena, he was dedicating himself almost exclusively to the history of mathematics, studying Paolo Ruffini's manuscripts [36:2, 320]. His first published historical work, *Influenza dell'opera matematica di Paolo Ruffini sullo svolgimento delle teorie algebriche*, appeared in 1902 [18]. It is in some sense fitting, then, that the lecture notes of the future historian would document the first university course on Galois theory given in Italy.

[11] Among Arzelà's other notable students were Giuseppe Vitali (1876–1932) and Leonida Tonelli (1885–1946).

[12] The Biblioteca Universitaria of the University of Bologna has two copies of the Italian translation of Netto's treatise (pressmark: BUT 481/TOR 64596; BUT 1181/TOR 120400), which further suggests the possibility that Arzelà had access to the work.

THE LECTURE NOTES AND THEIR SOURCES

The lecture notes Bortolotti took consist of a set of notebooks half-bound in leather into one sizeable volume of 650 pages, 230 mm long and 165 mm wide. It is held in the Bortolotti Library of the Mathematics Department of the University of Bologna and carries the pressmark B.B.X.5.

The cover page bears the title, *Teoria delle sostituzioni*, the name of the professor who gave the lectures, Arzelà, and the place and the year in which the course took place, Bologna 1886–1887. At the bottom of the page, Bortolotti signed his name as he did at the end of almost every set of lecture notes. While the notes are not divided into chapters, Bortolotti subdivided them into 200 numbered sections and drafted an unfinished table of contents page following the cover. He also numbered and dated a large part of the notes and, at the beginning of the discussion of every new subject, included a heading indicating the subject Arzelà was going to lecture about. The last 160 pages of the volume consist of a recapitulation; it seems that that was not an integral part of the course but rather that Bortolotti wrote this review of the lectures for his own personal use.

The notes are written in a formal style of penmanship and the handwriting is quite legible almost everywhere. Since the notes are those of a young student, they are also peppered with personal comments and funny and sarcastic remarks typical of the language of a 20-year-old university student. Bortolotti's annotations enliven the manuscript as they reveal a common denominator between students of the 19th and 20th centuries. At the same time, they reveal aspects of the personality and character of a young student, who later became a famous mathematician and historian of mathematics, that do not come through in his later published works.

The great impact the reading of Netto's treatise had on Arzelà is evident in a large part of the lectures. Arzelà utilized the work of the German mathematician as a principal reference for his course, and this emphasis may be reflected in the title Bortolotti chose for his notebook. Beginning with the opening lectures of the course, Arzelà accurately followed the main points of Netto's treatment, using the same arguments and, almost everywhere, the same notation. Thus, Arzelà clearly drew his discussion of symmetric, alternating, and two-valued functions from chapter I of Netto's treatise. The treatment of multiple-valued functions and the presentation of the theory of groups of substitutions correspond to Netto's chapters II and III. Arzelà went over all of the material in these chapters before proceeding to a discussion of one of the most interesting and original parts of Netto's treatise, namely, families of functions, to which Netto devoted all of chapter V. It should be noted that the concept of a family of functions was introduced by Kronecker in a memoir of 1879 [40], but it was Netto's *Substitutionentheorie* that made it widely known.

Arzelà then shifted gears somewhat to focus on number theory. For his presentation he switched to another source, P. G. Lejeune Dirichlet's (1805–1859) *Zahlentheorie* [28]. Dirichlet's lectures on number theory had gone into a third edition in 1879 and were translated into Italian in 1881. Arzelà's discussion of number theory, in general, and on Euler's φ function, on congruence theory, and on the theory of power residues, in particular, is drawn from pages 19–127 of Dirichlet's text.

Following this detour into number theory, Arzelà returned to Netto's treatise, this time to chapter VII, for his treatment of "certain special classes of groups" [46, 125]. He then moved to the analytical representation of substitutions as Netto presented it in chapter VIII.

This chapter closes part one of Netto's treatise; the second part is devoted to the application of the theory of substitutions to algebraic equations. Arzelà, taking his lead from the first three sections of Netto's chapter IX, proposed a method for solving equations of degree 2, 3, and 4 [46, 151–154].

He then shifted to a new subject and began to utilize a new source. In fact, for the presentation of algebraic functions, for the discussion of the impossibility of solving equations of degree higher than four by radicals, and for the treatment of abelian and binomial equations, he clearly drew from the 1885 edition of Serret's *Cours d'algèbre supérieure*, in particular from chapters II and III in section V of the second volume [58:2, 497–512]. Arzelà followed Serret's exposition up to the statement of the Ruffini–Abel theorem, and then switched back to Netto's text to present the proof (see the section "Arzelà, the Teacher: His Presentation of the Ruffini–Abel Theorem" below). At this point of the course, Arzelà had all the material he needed to start the presentation of Galois theory. His source for the very first part of the first lecture was once again Serret's *Cours*, but the principal guide for his lectures on Galois theory was, quite naturally, Jordan's *Traité des substitutions et des équations algébriques*. Arzelà's exposition of Galois theory came from Jordan's book III, "Des irrationnelles" [39, 257–270], while his presentation of the solvability conditions by radicals follows book IV, "De la résolution par radicaux" [39, 385–388].

Despite these clear influences, Arzelà included almost no explicit references to the mathematical literature in his lectures. He did mention P.G. Lejeune Dirichlet's *Zahlentheorie*, as evidenced by Bortolotti's title of the lecture on "The $\varphi(n)$ Function and the Congruences" [9, Sect. 98] as well as Serret's *Cours* in the context of binomial equations [9, Sect. 162]. Arzelà also named Serret, in reference to the terminology used, in his lecture on the definition of a group of substitutions [9, Sect. 19], while the Italian mathematician, Paolo Ruffini, surfaces at the end of the proof of the Ruffini–Abel theorem, but with no references to his work [9, Sect. 150]. Gauss appeared relative to binomial roots and the division of the circle (see, for example, [9, Sect. 151]), and Kronecker and Jordan came up in the context of abelian equations, but, again, no particular work was cited [9, Sects. 155, 160]. Finally, the Norwegian mathematician, Niels Henrik Abel, received due mention in the lecture on algebraic functions, where Arzelà proposed Abel's classification of algebraic functions according to order and degree [9, Sect. 144], as well as in the lectures on abelian equations, where Arzelà referred to him without providing further specifics [9, Sects. 151, 160]. In Bortolotti's notes, there are no explicit references to Galois's papers, and Netto's treatise is never mentioned.

MATHEMATICAL OVERVIEW OF THE LECTURES

Arzelà's year-long course on the theory of substitutions opened with a discussion of symmetric functions. He began, naturally, by stating the definition of a symmetric function of n elements and giving a few simple examples of symmetric functions. He then introduced the concept of an elementary symmetric function of n elements, regarding the n elements as roots of an equation of the nth degree. In particular, he stressed the special importance of the concept by stating and proving that every symmetric function of the roots of an equation can always be expressed in one and only one way as an integral function of elementary symmetric functions.

Symmetric functions also arose in the presentation of the theory of integral functions *per se*, in the form of the concept of the discriminant. Arzelà defined the discriminant of n quantities and proved that it was a particular kind of symmetric function. This led him to the analysis of the square root of the discriminant and to a discussion of two-valued functions and alternating functions. In particular, he proved the theorem that expresses the general form of an alternating function and of a two-valued function of n quantities. Arzelà also began his presentation of the theory of substitutions, introducing the concepts of permutation, substitution, and transposition.[13]

With this groundwork laid, Arzelà turned to multiple-valued functions and the theory of groups of substitutions. He defined a group,[14] noting in passing that in his *Cours* Serret called it a "system of conjugate substitutions" and stating his own preference for the term "group." Unlike some earlier mathematicians, Arzelà also emphasized the closure of the group with respect to the product of two substitutions, before defining the degree, namely, the number of elements on which the substitutions belonging to the group operate.

These preliminary definitions out of the way, Arzelà next defined a group of substitutions belonging to a given function and developed this subject in great detail, introducing the symmetric and alternating groups and dealing with group generation. He moved to a discussion of transitivity and permutability, first presenting primitive and non-primitive groups and simply transitive and k-fold transitive groups and then analyzing permutability between two substitutions, between a substitution and a group, and between two groups. This led to the definition of a normal subgroup which Arzelà termed a "sottogruppo singolare" [9, Sect. 74]. A detailed presentation of normal subgroups and their properties followed. He closed this part of the course with the definitions of simple and compound groups and of maximal normal subgroups, in order to define a composition series of the group G. He then proved the uniqueness of the orders of the factor groups of a composition series.

Arzelà opened the next part of the course with a theorem of great importance for the theory of equations, namely, that the composition series of the symmetric group of n elements consists, if $n > 4$, of the alternating group and the identical substitution. Moreover, the alternating group of more than four elements is simple. To make clear the importance of this theorem, he presented two examples, first considering the case of four and then that of three elements. He especially emphasized this part of the course, ending it with a treatment of the theory of algebraic equations and Galois theory. Given this objective, he defined the notion of isomorphism. Up to this point, Arzelà had been following Netto's presentation, but he departed from Netto to adopt Jordan's definition of isomorphism between two groups [39, 56]. In fact, Arzelà defined two different types of isomorphism that he called "isomorfismo meriedrico" and "isomorfismo oloedrico." They were the Italian translations of the French expressions "isomorphisme mériédrique" and "isomorphisme holoédrique" used by

[13] For mathematicians of the 19th century, the concepts of permutation and substitution were not always clearly delineated. Arzelà, however, did distinguish between the different concepts of permutation, substitution, and transposition: "Quando si passa da una disposizione particolare degli n elementi ad un'altra, si opera una *sostituzione*. Il risultato di questa operazione e' una *permutazione* ... Chiameremo *trasposizione* lo scambio di due elementi [When one moves from a particular arrangment of n elements to another, one performs a *substitution*. The result of this operation is a *permutation* .... We will call *transposition* the exchange of two elements]" [9, Sects. 9, 12].

[14] "Tutte le sostituzioni che hanno la proprietà di lasciare immutato un certo valore φ della funzione si dice costituiscono un *gruppo* [All the substitutions which have the property of leaving a certain value φ of the function fixed form a *group*]" [9, Sect. 19].

Jordan in his *Traité* and were used consistently throughout the rest of the course.[15] With isomorphism defined, Arzelà faced the problem of constructing groups which are isomorphic to a given group. In this part of his presentation, both Netto's treatise [46, 92–94 (English)] and Jordan's *Traité* [39, 56–60] served as mathematical guides.

The next two lectures, numbered XXIII and XXIV, treated families of functions. In this part of the course, the great impact on Arzelà of the German school of Kronecker and Netto is extremely evident. Following Netto, he tackled the problem of a group-theoretic classification of functions and introduced the concept of a family of algebraic functions as a collection of all functions belonging to the same group. A family is a Galois family if the associated group reduces to the identical operation.

The 20 sections that follow were devoted to the presentation of number theory. Arzelà introduced the theory of congruence as well as the function $\varphi(n)$—known as the Euler φ function—that gives the number of numbers K with $1 \leq K \leq n$ and K prime relative to n. Among other things, Arzelà explored congruences with unknowns, in particular congruences of the first degree, and proposed two methods of resolution and Euler's algorithm. He also analyzed the theory of power residues, especially in the case of composite moduli.

After this number-theoretic interlude, Arzelà returned to group theory with a discussion of what Netto termed "certain special classes of groups" [46, 125]. In particular, he discussed transitive groups whose degree and order are equal. Netto did not give a particular name to this special class, but Arzelà called them "gruppi tipo," following the Italian translation "tipo di un gruppo" of this part of Netto's text [46, 127]. Arzelà then proceeded to determine "gruppi tipo" with order prime, the product of two primes, and the square of a prime. Such special classes of groups are especially important in the study of Galois theory, since the "gruppi tipo" play a fundamental role in the process of extending the field of rationality of an equation and in the subsequent reduction of its associated group.

After devoting approximately 15 sections to the study of the analytical representation of substitutions, Arzelà finally began the presentation of the theory of algebraic equations, dealing in particular with the application of the theory of substitutions to algebraic functions. He opened with a method for solving equations of degree 2, 3, and 4. Given an equation of degree ≤ 4, the idea was to focus on the most general $n!$-valued function of the roots possible which can be expressed by the coefficients of the given equation. Assigning particular values to the coefficients of the $n!$-valued function, it was then possible to obtain the roots of the equation in terms of the coefficients. At the end of his exposition, Arzelà stated that the same method did not apply to the general equation of the fifth degree, since it was impossible to proceed beyond the construction of the two-valued functions. He also affirmed that the solution of general equations of degree higher than 4 failed not because of a defect in the method, but because of the nature of the equations considered.

Arzelà next analyzed algebraic functions. After a long and laborious proof, he found the general form of an algebraic function of order μ and degree m in order to discuss the question of the impossibility of solving equations of degree higher than 4. In the lecture of 1 May 1887, he proved that the algebraic functions of the coefficients involved in the general expression of an algebraic function are rational functions of the roots; in particular,

[15] Netto, in his discussion of isomorphism, used the expressions "manifold isomorphic" and "simply isomorphic" for Jordan's "mériédrique" and "holoédrique," respectively. It should be noted that, in a footnote, Netto mentioned the terminology Jordan used in his *Traité* [46, 92].

he showed that every algebraic function of the coefficients involved in the resolution of the general equation of degree *n* is a rational function of the roots. These theorems could be considered as lemmas for Arzelà's proof of the Ruffini–Abel theorem that followed (see the next section). He then turned to the study of abelian equations following Serret's discussion and terminology [58:2, 518ff].[16] In order to prove the algebraic solvability of abelian equations, he showed that the resolution of abelian equations depends on the resolution of those equations now termed "cyclic" that are solvable by radicals.

Arzelà singled out yet another class of equations solvable by radicals, namely, one isolated by Abel [3].[17] After stating and proving Abel's theorem, Arzelà explained to his class that, since the theorem is due to Abel, Jordan termed "abelian" equations so defined [39, 286].[18] Arzelà also pointed out two other classes of equations which are algebraically solvable, showing the solvability by radicals of binomial and reciprocal equations.

It was finally on 24 May, 1887 that Arzelà began his discussion of Galois theory. His presentation was always extremely clear and impeccable from a pedagogical point of view. After emphasizing the aim of Galois theory,[19] he considered an algebraic equation of degree *n* and its *n* roots (supposed distinct). He then constructed the Galois resolvent of the given equation and the associated group of the equation, calling it the "Galois group."[20] After proving the uniqueness of the Galois group of an equation, Arzelà drew the connection between the irreducibility of an equation and the transitivity of its associated group. He emphasized the close relation between an equation and its associated group, focusing in particular on those cases in which the equations are named after their groups and on the case of equations whose associated group is nonprimitive. He closed this part of the course with a very detailed discussion of how to reduce the Galois group of the equation by adding rational functions of the roots to the field of rationality of the equation. Given the fact that Arzelà followed Jordan in his treatment here, the lectures were extremely clear and well explained.

The final lecture of the course, dated 4 June 1887, dealt with the solvability conditions by radicals. After defining solvable groups as groups "che caratterizzano equazioni risolubili per radicali"[21] [9, Sect. 197], Arzelà gave three different and equivalent, necessary and sufficient conditions for an algebraic equation to be solvable by radicals. Once again, he drew from Jordan's work [39, 386–388], using Jordan's final condition to prove the impossibility

[16] It should be noted that the class of equations Serret called "abelian" does not coincide with that dealt with by Abel. It consists of the class of equations that Kronecker called "abelian" [42].

[17] The class of equations pointed out by Abel includes those equations whose roots can be expressed rationally by a function of one of them and for which, besides, the rational operators are permutable.

[18] It should be noted that after Jordan's contribution, "abelian" became synonymous with "commutative." Later, it would be proved that the commutativity of the rational operators was the same as the commutativity of the Galois group of the equation. This is the reason why we now term "abelian" those equations whose Galois group is commutative.

[19] "Si propone, con questa teorica, di ricercare le condizioni necessarie e sufficienti perche' una equazione sia risolubile algebricamente, od anche, come si suol dire, per radicali [The aim of this theory is to find necessary and sufficient conditions for an equation to be solvable algebraically, or, as we commonly say, by radicals]" [9, Sect. 171].

[20] The work in which the expression *Galois group* appeared, perhaps for the first time in the published mathematical literature, is [38].

[21] "which characterize equations solvable by radicals."

of solving general equations of degree higher than 4 by radicals, the culminating result of the course.[22]

ARZELÀ, THE TEACHER: HIS PRESENTATION OF THE RUFFINI–ABEL THEOREM

As mentioned above, Arzelà devoted the second part of his course to the question of the resolution of algebraic equations. Once he had concluded the long and detailed discussion of the theory of algebraic functions, he stated and proved the following theorem: "Le equazioni di grado superiore al quarto non si possono risolvere algebricamente" [9, Sect. 150],[23] which has been known, since the end of the 19th century, as the Ruffini–Abel theorem. Despite the fact that Arzelà intended to lecture about Galois theory and, therefore, to state and prove the solvability conditions by radicals, he gave the proof of the Ruffini–Abel theorem. In this way, he gave a somewhat more historical presentation of the material, moving from the result of Ruffini and Abel to the work of Galois. (It should be noted that at the end of the course he stated and proved the theorem again as a corollary of the conditions for solvability by radicals [9, Sect. 200].)

As is well known, Ruffini published the result of his first studies on the solvability of algebraic equations in 1799 in a two-volume work, entitled *Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto* [56].[24] In this work, he proved that the algebraic solution of the general equation of degree 5 is impossible, and later, in responding to the objections made by his contemporaries, he pointed out a more general proof of the impossibility of solving the general equation of degree higher than 4 algebraically [30, 753]. In fact, following the controversies provoked by the publication of his work, Ruffini put forth a new proof of the theorem that was published in 1803 under the title "Della insolubilità delle equazioni algebraiche generali di grado superiore al quarto" [51].[25] In order to respond to further objections, this time from the Italian mathematician, Gianfrancesco Malfatti (1731–1807), Ruffini published yet another proof of the impossibility of solving the quintic equation algebraically in 1804 [55]. He continued to work on his proof, eventually succeeding in proving the insolvability of algebraic equations of degree higher than 4 for certain classes of transcendental functions. He published the latter result in 1806 in the brief memoir, "Della insolubilità delle equazioni generali di grado superiore al 4°, qualunque metodo si adoperi, algebraico esso sia o transcendentale" [52].[26] Still refusing to drop the issue, Ruffini published his last proof in 1813 as part of the memoir, *Riflessioni intorno alla soluzione algebraica delle equazioni* [54].[27] This is the fifth, the simplest, and the clearest of Ruffini's proofs, and it essentially coincides

[22] "L'equazione generale di grado n, se é $n > 4$, non è risolubile per radicali, poichè i suoi fattori di composizione 2, $\frac{1 \cdot 2 \cdot 3 \cdots n}{2}$, 1 non sono tutti primi [The general equation of degree n, if $n > 4$, is not solvable by radicals, because the orders of its factor groups 2, $\frac{1 \cdot 2 \cdot 3 \cdots n}{2}$, 1 are not all prime]" [9, Sect. 200].

[23] "Equations of degree higher than the fourth cannot be solvable algebraically."

[24] *General Theory of Equations in Which It Is Shown That the Algebraic Solution of the General Equations of Degree Greater Than Four Is Impossible.*

[25] "On the Insolvability of the General Algebraic Equations of Degree Greater Than Four."

[26] "On the Insolvability of the General Equations of Degree Greater Than Four, Regardless of the Method Used, Algebraic or Transcendental."

[27] *Reflections on the Algebraic Solution of Equations.*

with what would later be called the modification of Abel's proof, published in 1845 by the French mathematician, Pierre Laurent Wantzel (1814–1848) [62].

Since the end of the 19th century, Ruffini's proof has been analyzed by several scholars. In an article that appeared in 1892, Heinrich Burckhardt reconstructed Ruffini's work on the theory of algebraic equations and also pointed out his numerous contributions to the theory of groups of substitutions [22]. Burckhardt's article rescued Ruffini's work from the oblivion into which it had fallen some 80 years before. In fact, in 1896, an article appeared in which the name of Ruffini is cited together with that of Abel in relation to the impossibility theorem. The American mathematician, James Pierpont, published a paper entitled "On the Ruffini–Abelian Theorem" [50], in order to present a proof that did not suffer from the defects present in the work of both Ruffini and Abel and "to give a demonstration of the Ruffini–Abelian Theorem which shall be as direct and *self-contained* as possible" [50, 201; his emphasis]. Later, in 1902–1903, the Italian mathematician and historian, Ettore Bortolotti, the very same man who took the lecture notes under discussion here, gave a very interesting reconstruction of Ruffini's life and scientific activity in his paper, "Influenza dell'opera matematica di Paolo Ruffini sullo svolgimento delle teorie algebriche" [18].[28] Probably inspired by Burckhardt's article, Bortolotti gathered Ruffini's papers together with his mathematical correspondence and published the first volume of his collected works in 1915 [53]. In the 1980s, Raymond G. Ayoub and R. A. Bryce wrote two interesting papers on Ruffini's contributions to the quintic equation [11; 21]. Particularly notable is Ayoub's article, which, using the modern tools of group theory and field theory, gives a reconstruction of Ruffini's first and last proofs. Also noteworthy is Jean Cassinet's discussion [25], which is closer to Ruffini's text.

Despite Ruffini's attempts to explain the validity of his work to his colleagues, his proofs were not completely accepted by the European mathematical community. Mathematicians who had not been convinced by Ruffini's work or who had not heard about it still believed in the possibility of solving algebraic equations of degree higher than 4 by radicals. For example, in 1811, the Polish mathematician, Josef Maria Hoëne Wronski (1767–1853), believed he had a demonstration of the solvability by radicals of general equations of any degree [63]. As Ludvig Sylow wrote in a comment on Abel's collected works, Abel still believed, in 1821, that he had found a solution by radicals of the quintic equation [4:2, 290–291]. The young Norwegian mathematician soon discovered his own error, and, in 1824, he proved that the quintic equation is not algebraically solvable. By 1826, he had a new proof of the impossibility of solving algebraic equations by radicals, independent of Ruffini's work.

As noted, until Burckhardt's article appeared, Ruffini's work seemed to be forgotten. Several mathematicians in the 19th century worked on the question of the solvability of algebraic equations, but they all referred mainly to Abel's research. In 1839, the Irish mathematician, William R. Hamilton (1805–1865), published a long paper, entitled "On the Argument of Abel, Respecting the Impossibility of Expressing a Root of Any General Equation Above the Fourth Degree, by Any Finite Combination of Radicals and Rational Functions" [37], which aimed to rectify the defects in Abel's proof. In France, Joseph Alfred Serret included the proof of the theorem of impossibility in the third edition of his *Cours d'algèbre supérieure*, which was published in 1866 [64, 131]; the proof he

[28] "Influence of the Mathematical Work of Paolo Ruffini on the Development of Algebraic Theories."

reported is essentially Wantzel's. In Denmark, Julius Petersen gave the impossibility proof in his *Theorie der algebraischen Gleichungen* of 1878 [50, 201; 49], while Joseph Antoine Carnoy (1841–1906) did the same in Belgium in his *Cours d'algèbre supérieure* published in 1892 [50, 201; 24]. Finally, Leopold Kronecker in the 1879 *Monatsberichte* of the Berlin Academy published the article "Einige Entwicklungen aus der Theorie der algebraischen Gleichungen," in which he closely followed Abel's proof [50, 201; 40].

Therefore, at the time in which Arzelà gave his lectures, he had at his disposal practically no studies on Ruffini's works and no papers in which Ruffini's proof was explicitly mentioned, except for Wantzel's article.[29] However, at the end of his proof, Arzelà stated categorically that "Questa dimostrazione fu data per la prima volta da Ruffini" [9, Sect. 150].[30] It is possible that Arzelà had read Ruffini's first proof which, after all, had been published privately in Bologna and which was well known to several of Ruffini's Italian contemporaries in mathematics. (Ruffini had sent copies of his work to selected individuals.)

As remarked above, for the presentation of the classification of algebraic functions according to order and degree, for the construction of the most general expression to represent an algebraic function of order μ and degree m, and for the study of the algebraic functions which satisfy a given equation, Arzelà followed Serret's exposition [58:2, 497–512]. Serret's next step into the theory of algebraic equations was the presentation of the "Démonstration de l'impossibilité de résoudre algébriquement les équations générales de degré supérieure au quatrième" [58:2, 512]. In the introduction that precedes the proof, Serret noted that "Ce théorème a été démontré pour la première fois, d'une manière rigoreuse par Abel; je présenterai ici la démonstration plus simple que l'on doit à Wantzel" [58:2, 512]. Thus, Serret presented the proof known as Wantzel's modification of Abel's proof, without mentioning Ruffini's works.

Since Arzelà was utilizing Serret's text, he most likely read the proof, but he decided not to present it to his students. As the good professor and teacher that he was, Arzelà sought the clearest and simplest proof of that theorem of fundamental significance in the theory of algebraic equations. He found such a demonstration in Netto's treatise. In fact, Theorem III in Chapter XII (on "The Algebraic Solution of Equations") of Netto's treatise states that "Le equazioni generali di grado superiore al quarto non sono risolubili algebricamente" [46, 245].[31] The structure of the proof is clearer and simpler than those of Serret and Ruffini, and probably this is the reason for Arzelà's choice. Moreover, having utilized Netto's text for his lectures on the theory of symmetric and multiple-valued functions, Arzelà had the right background and the tools to make the proof understandable to his students. As it is easy to note, however, Arzelà did not restrict himself to Netto's exposition; rather, he filled in the details that Netto took for granted for the benefit of his students. The result is an easily comprehensible proof, impeccable from a pedagogical point of view. In what follows, I present first Netto's proof and then Arzelà's presentation. Arzelà's elaborations will be immediately obvious.

[29] Wantzel wrote about Ruffini's work on the solvability of algebraic equations in these terms: "Plusieurs années auparavant, [with respect to Abel's work] Ruffini, géomètre italien, avait traité la même question d'une manière beaucoup plus vague encore, et avec des développements insuffisants, quoiqu'il soit revenu plusieurs fois sur le même sujet" [62, 57]. However, before he presented his proof, Wantzel stated his intention to face the problem from the same point of view "envisagé dans les mémoires d'Abel et de Ruffini" [62, 58].

[30] "This proof was given for the first time by Ruffini."

[31] "The general equations of degree higher than the fourth are not solvable algebraically."

Netto's exposition ran this way:

> Theorem III. The general equations of degree higher than the fourth are not algebraically solvable.
> For if the n quantities $x_1, x_2, \ldots, x_n$, which in the case of the general equation are independent of one another, could be algebraically expressed in terms of $\mathfrak{R}', \mathfrak{R}'', \ldots$, then the first introduced irrational function of the coefficients, V_ν, would be the p_νth root of a rational function of $\mathfrak{R}', \mathfrak{R}'', \ldots$. Since, from Theorem II,[32] V_ν is a rational function of the roots, it appears that V_ν, as a p_ν-valued function of $x_1, x_2, \ldots x_n$, the p_νth power of which is symmetric, is either the square root of the discriminant, or differs from the latter only by a symmetric factor. Consequently, we must have $p_\nu = 2$ (§ 57). If we adjoin the function $V_\nu = S_1\sqrt{\Delta}$ to the rational domain, the latter then includes all the one-valued and two-valued functions of the roots. If we are to proceed further with the solution, as is necessary if $n > 2$, there must be a rational function $V_{\nu-1}$ of the roots, which is $(2p_{\nu-1})$-valued, and of which the $(p_{\nu-1})$th power is two-valued. But such a function does not exist if $n > 4$ (§ 59). Consequently, the process, which should have led to the roots, cannot be continued further. The general equation of a degree above the fourth therefore cannot be algebraically solvable. [46, 245]

Arzelà presented his proof in this way:[33]

> [Theorem:] Equations of degree higher than the fourth cannot be algebraically solvable.
> Let
>
> $$x^m + a_1 x^{m-1} + \cdots = 0$$
>
> be an algebraic equation.
> To find the roots, that is, to find an algebraic expression of the coefficients which satisfies the given equation, we will start combining the coefficients rationally.
> But we know that it is possible to express only the roots of functions of the first degree by rational expressions of the coefficients.
> Thus, it is necessary to apply some radicals to the combination of algebraic operations
>
> $$\varphi(a_1 a_2 \ldots a_m)$$
>
> that we found.
> I can always suppose that the first radical I apply is of prime order m_1, I claim that it must be $m_1 = 2$.
> And, in fact, by the previous result,[34] it must be
>
> $$\sqrt[m_1]{\varphi(a_1 a_2 \ldots a_n)} = \psi(x_1 x_2 \ldots x_n)$$
>
> which means, a rational function of the roots.
> Namely, ψ has to be an m_1-valued function whose m_1th power
>
> $$\psi^{m_1}(x_1 x_2 \ldots x_n) = \varphi(a_1 a_2 \ldots a_n)$$
>
> is a single-valued function.

[32] "Theorem II. The explicit algebraic function x_0, which satisfies a solvable equation $f(x) = 0$, can be expressed as a rational integral function of quantities $V_1, V_2, V_3, \ldots, V_\nu$, with coefficients which are rational functions of the quantities $\mathfrak{R}', \mathfrak{R}''$. The quantities V_λ are on the one hand rational integral functions of the roots of the equation $f(x) = 0$ and of primitive roots of unity, and on the other hand they are determined by a series of equations $V_a^{pa} = F(V_{a-1}, V_{a-2}, \ldots, V_\nu; \mathfrak{R}', \mathfrak{R}'', \ldots)$. In these equations the $p_1, p_2, p_3, \ldots, p_\nu$ are prime numbers, and $F_1, F_2, F_3, \ldots, F_\nu$ are rational integral functions of their elements V and rational functions of the quantities $\mathfrak{R}', \mathfrak{R}'', \ldots$, which determine the rational domain" [46, 245].

[33] The original Italian of the passage that follows may be found in the Appendix.

[34] In the previous paragraph, Arzelà proved that all the algebraic functions of the coefficients involved in the resolution of an algebraic equation of degree m are rational functions of the roots of the equation [9, Sect. 149].

Such functions exist only if $m_1 = 2$.[35]

For the same reason, we could not continue to apply to φ a radical of index higher than the second; besides, applying quadratic radicals, we can solve equations only of the second degree because the corresponding functions of x are two-valued functions.

Thus, to solve equations of degree higher than the second, we must apply to the radical just found new radicals, for instance $\sqrt[p_1]{\ }$,

$$\sqrt[p_1]{\sqrt{\varphi(a_1 a_2 \ldots)}}.$$

But even this algebraic expression of the coefficients has to be a rational expression of the roots, that is:

$$\sqrt[p_1]{\sqrt{\varphi(a_1 a_2 \ldots)}} = \psi(x_1 x_2 x \ldots),$$

which implies

$$\sqrt{\varphi(a_1 a_2 \ldots)} = (\psi(x_1 x_2 \ldots))^{p_1}.$$

ψ has to be a $2p_1$-valued function, such that its p_1th power is a two-valued function.

Now, such functions do not exist if the number of the elements on which they operate is greater than four.[36]

The given function will be algebraically solvable only in the case in which we have either four roots or less than four roots.

This proof was given, for the first time, by Ruffini. [9, Sect. 150]

Arzelà's treatment of the proof of the Ruffini–Abel theorem is a clear example of his way of teaching mathematics. His attitude in teaching the course on Galois theory was always geared toward presenting the subjects as clearly as possible to his students. This clarity resulted from his years of teaching in the secondary school and in the university. It should be noted that, in 1880, Arzelà had written one of the most widely used secondary school texts. His *Trattato di algebra elementare* [10] went into many editions and was extensively utilized as a textbook for almost 30 years [35, 252]. He also wrote *Complementi di algebra elementare* [6] and *Aritmetica razionale* [5] for the secondary school audience in addition to the university-level text, *Lezioni di calcolo infinitesimale* [7], which encompasses the lectures on infinitesimal calculus given at the University of Bologna beginning in the academic year 1880–1881 [35, 252]. Arzelà thus commands an important position in the history of the teaching of mathematics in Italy both at the secondary and at the university level.

CONCLUSIONS

The unification of Italy marked a turning point not only in the political life of the country but also in the organization of secondary and university education. At the University of Bologna, for example, it sparked a deep discussion among the members of the mathematical community on the radical reform of mathematical studies.

[35] Arzelà previously had proved that the only functions which, when raised to a certain power can become symmetric, are alternating functions [9, Sect. 51].

[36] Previously, Arzelà had stated and proved the following theorem: it is not possible to find a function of more than 4 elements which, when raised to a prime power, can become a two-valued function [9, Sect. 53].

During the 1870s, students at Bologna could not obtain a degree in mathematics because of the lack of professors capable of teaching the high-level courses of the last two years of the curriculum. Beginning in 1881, students could finally complete these studies and earn their degrees in mathematics. This change resulted from the addition of excellent professors to the faculty with the express objective of bringing Bologna's mathematics teaching up to international standards.

In this atmosphere of renewal and intellectual ferment and growth, Cesare Arzelà, Professor of Infinitesimal Calculus and Higher Analysis, gave lectures on Galois theory that represented the first known course on the subject in Italy. Arzelà's decision to give a course on a subject outside of his main research interests attests to his mathematical range, while the exposition of these lectures emphasizes his uncommon ability as a teacher. In order to present a clear and, at the same time, substantial course on the theory of substitutions and Galois theory, Arzelà consulted the most significant published texts on the subject in Europe. In selecting the specific material for his presentation, he chose from the best works in print at the time—texts by Dirichlet, Serret, Netto, Jordan—but he did more than that. He assessed the various presentations; he chose what he viewed as the best of the best; he altered and elaborated on those presentations in full knowledge of the needs of his student auditors. In so doing, he succeeded brilliantly in organizing a cogent and pedagogically sound course of lectures that at the same time reflected his own understanding of the algebraic subject matter.

As evidence of his success, his student, the scribe of the lecture notes under discussion here, Ettore Bortolotti, went on to pursue some of his best historical work precisely on the mathematics and the mathematical influence of Paolo Ruffini. While Arzelà's course on Galois theory and algebraic equations may not have exerted great influence on the course of late 19th-century Italian mathematics, it was indicative of changes then underway in Italian higher education, changes that would result in the vibrant Italian mathematical research community of the early twentieth century.

APPENDIX

The following is the original Italian text of Arzelà's presentation of the Ruffini–Abel theorem [9, Sect. 150]:

[Teorema:] Le equazioni di grado superiore al quarto, non si possono risolvere algebricamente.

Si abbia una equaz algebrica

$$x^m + a_{\mathrm{I}}x^{m-1} + \cdots = 0$$

Per trovarne le radici, per trovare cioè un'espressione algebrica dei coeff che sostituita per x la renda identica si incomincierà col combinare raz fra loro i coefficienti

Ma noi sappiamo che mediante espressioni raz dei coeff non si possono esprimere che le radici di funzioni del primo grado

Quindi al complesso di operaz algebriche

$$\varphi(a_1 a_2 \ldots a_m)$$

trovato bisognerà applicare dei radicali.

Posso sempre supporre che il primo radicale che si impiega sia di ordine primo m_1, dico che deve essere $m_1 = 2$.

Ed infatti. Per quanto abbiamo trovato deve essere

$$\sqrt[m_{\mathrm{I}}]{\varphi(a_1 a_2 \ldots a_n)} = \psi(x_{\mathrm{I}} x_{\mathrm{II}} \ldots x_n)$$

cioè funz raz delle radici.

Ossia la ψ deve essere una funz ad m_1 valori la cui potenza m_1^{esima}

$$\psi^{m_{\mathrm{I}}}(x_1 x_2 \ldots x_n) = \varphi(a_1 a_2 \ldots a_n)$$

è ad un sol valore.

Tali funzioni non esistono se non nel caso di $m_1 = 2$.

Per la stessa ragione non si potrà continuare coll'applicare alla φ un radicale di indice sup. al secondo, e siccome d'altra parte con radicali quadrati non si possono risolvere che equaz di secondo ordine perchè le funz di x corrisp. sono a due soli valori.

Dunque per risolvere le radici di equ. di ordine sup. al secondo sarà giocoforza applicare nuovi radicali es $\sqrt[p_{\mathrm{I}}]{}$ al radicale trovato

$$\sqrt[p_{\mathrm{I}}]{\sqrt{\varphi(a_1 a_2 \ldots)}}$$

Ma anche questa espressione algeb. dei coeff dovrà essere raz nelle radici, sarà cioè:

$$\sqrt[p_{\mathrm{I}}]{\sqrt{\varphi(a_1 a_2 \ldots)}} = \psi(x_1 x_2 \ldots)$$

da cui:

$$\sqrt{\varphi(a_1 a_2 \ldots)} = (\psi(x_1 x_2 \ldots))^{p_{\mathrm{I}}}$$

La ψ dovrà essere una funzione a $2p_1$ valori e tale che la sua potenza p_1^{esima} sia una funzione a due soli valori.

Ora tali funzioni non esistono se il numero degli elementi su cui operano è maggiore di quattro.

La funz data sarà quindi risolubile algebricamente solamente nel caso che abbia o 4 sole radici, o meno di quattro radici.

Questa dimostraz. fu data la prima volta da Ruffini."

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