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Algebra and geometry in Pietro Mengoli (1625–1686) ☆

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Abstract

An important step in 17th-century research on quadratures involved the use of algebraic procedures. Pietro Mengoli (1625–1686), probably the most original student of Bonaventura Cavalieri (1598–1647), was one of several scholars who developed such procedures. Algebra and geometry are closely related in his works, particularly in *Geometriae Speciosae Elementa* [Bologna, 1659]. Mengoli considered curves determined by equations that are now represented by $y = K \cdot x^m \cdot (t - x)^n$. This paper analyzes the interrelation between algebra and geometry in this work, showing the complementary nature of the two disciplines and how their combination allowed Mengoli to calculate quadratures in a new way.

Résumé

L'un des plus grands pas en avant, au XVIIe siècle, dans la recherche de nouvelles méthodes de quadrature fut l'introduction des procédures algébriques. Pietro Mengoli (1625–1686), probablement le plus intéressant des élèves de Bonaventura Cavalieri (1598–1647), fut l'un de ceux qui développa ce type de procédures dans ses travaux mathématiques. Algèbre et géométrie sont étroitement liées dans les ouvrages de Mengoli, en particulier dans les *Geometriae Speciosae Elementa* [Bologna, 1659]. Mengoli emploie des procédures algébriques pour résoudre des problèmes de quadrature de curves déterminées par des ordonnées que nous noterions par $y = K \cdot x^m \cdot (t - x)^n$. Le but de cet article est d'analyser les rapports entre algèbre et géométrie dans l'ouvrage ci-dessus, de montrer leur complémentarité et d'indiquer comment celle-ci a permis à Mengoli de mettre en oeuvre une nouvelle méthode dans le calcul des quadratures.

☆ An earlier version of this article was presented at the University Autònoma of Barcelona on June 26, 1998 for my doctoral thesis in the history of science.

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Introduction

An important innovation in 17th-century mathematics was the introduction of algebraic procedures to solve geometric problems. Two fundamental advances in mathematics during that century were the invention of what is now called analytic geometry and the development of infinitesimal calculus. Both achieved their exceptional power by establishing connections between algebraic expressions and curves, and between algebraic operations and geometrical constructions.[1]

The publication in 1591 of *In Artem Analyticen Isagoge* by François Viète (1540–1603) drew attention to these connections. Viète used symbols not only to represent unknown quantities but also to represent known ones. In this way he was able to investigate equations in a completely general form. Viète solved equations geometrically using the Euclidean theory of proportions; he equated algebraic equations with proportions by means of the product of the medians and extremes of a proportion, thus introducing a new way of solving equations.[2] As Viète's work became known during the early years of the 17th century, mathematicians began to consider the utility of algebraic procedures in solving geometric problems. Among these scholars was Pierre de Fermat (1601–1665),[3] although the most influential figure in the research on the relationship between algebra and geometry was René Descartes (1596–1650), who published *La Géométrie* in 1637.[4]

In the 100 years following 1637 mathematics to a very considerable degree became algebraized.[5] This process involved a change from a mainly geometrical way of thinking to a more algebraic or analytical approach and was implemented in a slow and irregular manner.[6] Not all mathematicians in this period

[1] In the early 17th century a tradition had already developed in Italy of using algebra as an "art" to solve equations. The connection between algebra and geometry is present in most Italian algebrists—Leonardo da Pisa (1180–1250), Luca Pacioli (1445–1514), Niccolò Tartaglia (1500–1557), Girolamo Cardano (1501–1576), and Rafael Bombelli (1526–1573)—but these algebrists of the "cinquecento" only produced geometric demonstrations to justify the solutions of algebraic equations.

[2] On Viète see Viète [1970, 12], Freguglia [1999], and Giusti [1992].

[3] Fermat did not publish during his lifetime and his works circulated in the form of letters and manuscripts. On Fermat see Fermat [1891–1922, 65–71 and 286–292], Mahoney [1973, 229–232]. However, parts of his work are explained in other publications. For instance, Hérigone's course contains an exposition of Fermat's work on tangents; see Hérigone [1644, 59–69] and Cifoletti [1990, 129].

[4] The interpretation of Descartes' program gives rise to conflicting opinions even today. On the one hand Bos, Boyer, and Lenoir state that, for Descartes, algebra is merely a labor-saving instrument. "For Descartes the equation of a curve was primarily a tool and not a means of definition or representation" [Bos, 1981, 323]. Besides, the equation is a tool that permits classification of the curves. For these historians, Descartes' purpose in writing *La Géométrie* was to find a method for solving geometric problems, as was usual at that time, and the equation is not the last step on the way toward the solution. Giusti, on the other hand, says that for Descartes the curve is the equation. Giusti emphasizes the algebraic component of *La Géométrie* as the key to Descartes' program. Among the many studies of this program the following are particularly useful: [Mancosu, 1996, 62–84; Bos, 2001, 225–412; Giusti, 1987, 409–432].

[5] On this process of algebraization see Bos [1998, 291–317], Mancosu [1996, 84–86], Pycior [1997, 135–166], Panza [2004, 1–30].

[6] A detailed analysis of this change in thought can be found in Mahoney [1980, 141–155].

adopted algebraic procedures. Some regarded these new techniques as an "art" and tried to justify them according to a more "classical" form of mathematics; others disregarded algebra because their research evolved along other paths. Finally, a few accepted these new techniques as a complement to their mathematical procedures.[7]

Pietro Mengoli (1625–1686),[8] a mathematician from Bologna and a student of Bonaventura Cavalieri (1598–1647), can be included in the last of these groups. In his work *Geometriae Speciosae Elementa* [1659], algebra and geometry are used in complementary ways in the investigation of quadrature problems. At the beginning of this work he claimed that his geometry was a combination of those of Cavalieri and Archimedes obtained using the tools that Viete's "specious algebra" offered him:

> Both geometries, the old form of Archimedes and the new form of indivisibles of my tutor, Bonaventura Cavalieri, as well as Viète's algebra, are regarded as pleasurable by the learned. Not through their confusion nor through their mixture, but through their perfect conjunction, a somewhat new form [of geometry will arise]—our own—which cannot displease anyone.[9]

The quadratures Mengoli wished to investigate were known from the method of indivisibles, but he wanted to derive them using an algebraic approach. His principal aim was to square the circle, a goal he achieved by means of his new method in a later work, *Circolo* [1672]. This method was based on the underlying ideas of the method of indivisibles and Archimedes' method of exhaustion, combined using algebraic tools suggested by a study of Viète.

In Section 1 we examine the "specious" language in Mengoli's works and describe his notation and algebraic tools. In Section 2 we explore the relationship between algebra and geometry expressed in his system of coordinates, the geometric figures or "forms," the triangular tables of geometric figures, and the calculation of their quadratures.

1. Mengoli's "specious" language

In 1655, Mengoli wrote a book in verse dedicated to Queen Christina of Sweden,[10] *Via Regia ad Mathematicas per Arithmeticam, Algebram Speciosam, & Planimetriam, ornata Maiestatae Serenissimae D. Christinae Reginae Suecorum*, in which he showed her a "royal road" to understanding mathematics. The book is divided into three parts: arithmetic, in which he explains operations with numbers; "specious" algebra, in which he shows how to use letters to solve equations; and planimetry, in which he deals with plane figures and their properties. It is clear that he assumed algebra to be a part of mathematics alongside

7 On this subject see Høyrup [1996, 3–4], Massa [2001, 708–710].

8 The name of Pietro Mengoli appears in the register of the University of Bologna in the period 1648–1686. He studied with Bonaventura Cavalieri and ultimately succeeded him in the chair of mechanics. He graduated in philosophy in 1650 and three years later in canon and civil law. He took holy orders in 1660 and was prior of the church of Santa Maria Madalena in Bologna until his death. For more biographical information on Mengoli, see Natucci [1970–1991, 303], Massa [1998, 9–26], Baroncini and Cavazza [1986, 1].

9 Ipsae satis amabiles litterarum cultoribus visae sunt utraque Geometria, Archimedis antiqua, & Indivisibilium nova Bonaventura Cavallerij Praeceptoris mei, necnon & Viettae Algebra: quarum non ex confusione, aut mixione, sed coniuntis perfectionibus, nova quaedam, & propria laboris nostri species, nemini poterit displicere [Mengoli, 1659, 2–3].

10 This work for the queen was commissioned on the occasion of her visit to Bologna.

arithmetic and geometry. In this work he did not define the terms arithmetic and planimetry, but he did explain "specious algebra" and stressed its usefulness.[11] Indeed his attitude to algebra differed sharply from that of his master Cavalieri, Torricelli, and others in whose works algebraic calculus was deliberately excluded.[12] At the beginning of the second part of *Via Regia*, in the part devoted to "specious" algebra, Mengoli describes it as an art in the following way:

> About the utility of Specious [*Speciosa*] Algebra
>
> One alone among mathematics is called "speciosa algebra," by which art nothing is hidden from the questioner. If you ask "is it yes or no," it gives the true answer; if you ask "how great is it," this art does this satisfactorily, as one would expect since by general numbers it constructs methods fit for making, for things made, and for things said. Of course it is important that there should be both [these] general numbers: the one that you seek, and the one that you can give.[13]

At this stage in the development of his thinking, Mengoli considered algebra primarily as an art for demonstrating results that were already known rather than as a method for obtaining new results. In his later writings, as we shall see, he would come to view algebra more broadly, and would use it both to devise new proofs and to obtain new results.

In the *Via Regia* Mengoli adopted Viète's algebraic symbols. He explained that numbers would be represented by letters and algebra would be presented as a language. Metaphorically he compared linguistic and algebraic expressions: consonants represented data; vowels, unknowns; syllables, algebraic expressions of one letter; punctuation signs, rules of addition, subtraction; words, algebraic expressions of several letters; text, equalities; and verses, equations. He did not give examples with letters or with numbers to illustrate these comparisons.[14] His originality lay in this explicit description rather than in any new contribution to the formation of symbolic language.

Mengoli's aim in introducing these metaphorical comparisons was evidently didactic, to lay out a "royal road" to mathematics for the Queen. His views on symbolic language would be better explained in his later writings where he developed Viète's algebra to obtain new results.

[11] Mengoli thought that the queen already knew the significance of arithmetic and planimetry, but felt that "specious algebra" was a new part of mathematics that required some supplementary explanation.

[12] Also, in England, Thomas Hobbes (1588–1679), in his *Examinatio et emendatio mathematicae hodiernae* (1660), emphatically condemned the new algebra. In his opinion geometry and its subordinate arithmetic were sciences, whereas algebra, which he essentially regarded as symbolic reasoning, was an art able to record the inventions of geometry efficiently and quickly, but not a science. Isaac Barrow (1630–1677) who also opposed algebra, considered arithmetic as one part of geometry, geometry being the only true science and algebra being only a tool of logic. On this subject see Pycior [1997, 135–166].

[13] De Utilitate Algebrae Speciosae. Una, Mathematicas inter, Speciosa vocatur Algebra: quaerenti qua nihil arte latet. Sive rogas, utrum sic, vel non, dicere verum est; sive rogas, quantum est: ars facit ista satis. Utpote quae numeris generalibus instruit aptos, ad facere, ad facta, & dicta probare, modos. Scilicet intererit generalis uterque fuisse; Quem-quaeris numerus, quem-dare cunque potes [Mengoli, 1655, 19].

[14] For instance, Mengoli defined a word as an algebraic expression this way: "One word is composed of a certain number of letters, the same number of exponents, only one sign and one multiple. So the character that is produced by the product of letters I have pleasure in calling word" [Mengoli, 1655, 22]. Finally, Mengoli made a classification of equations up to the third degree in accordance with the degree and with the signs. Although Viète's classification was more complete there are some similarities in the words used: *antithesi*, which meant transposition of terms of one equation, *subgraduales*, which referred to the terms with a lesser degree than the equation, etc.

Mengoli published *Geometriae Speciosae Elementa*, in 1659, a 472-page book on pure mathematics with six *Elementa*, in which algebra became an essential part. The title already suggests this development of "specious" algebra, which Mengoli named "Specious Geometry."[15] Using Viète's symbolic language, he created new algebraic tools to determine the quadratures of geometric curves. Mengoli wanted to create a new field, a "specious geometry" modeled on Viète's "specious algebra." In fact, he had unintentionally created a new part of the new mathematical field that was beginning to emerge at that time, inspired by the works of Descartes and Fermat.

Mengoli's main algebraic sources were texts by Viète, Pierre Hérigone (1580–1643),[16] and Jean Beaugrand (1595–1640),[17] as is implied by comments at the beginning of the book:

> On the other hand as François Viète and other Analysts...; To those symbols that Viète, Hérigone, Beaugrand....[18]

In the second book of his six-volume textbook *Cursus Mathematicus* (1644), Hérigone had included a 296-page treatise entitled *Algebra* composed of 20 chapters. He dealt with equations and their solutions using algebra that was clearly inspired by Viète but that employed a very different notation and presentation.[19]

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15 *Geometriae Speciosae Elementa* (1659) has an introduction entitled *Lectori elementario*, which provides an overview of the six *Elementa*, or individually titled chapters, that follow. In the first *Elementum*, *De potestatibus, à radice binomia, et residua* (pp. 1–19), Mengoli gives the first 10 powers of a binomial given with letters for both addition and subtraction, and says that it is possible to extend his result to higher powers. The second, *De innumerabilibus numerosis progressionibus* (pp. 20–94), contains calculations of numerous summations of powers and products of powers in Mengoli's own notation, as well as demonstrations of certain identities. In the third, *De quasi proportionibus* (pp. 95–147), he defines the ratios "quasi zero," "quasi infinity," "quasi equality," and "quasi a number." With these definitions, he constructs a theory of quasi proportions on the basis of the theory of proportions found in the fifth book of Euclid's *Elements*. The fourth *Elementum*, *De rationibus logarithmicis* (pp. 148–200), provides a complete theory of logarithmical proportions. He constructed a theory of proportions between the ratios in the same manner as Euclid did with magnitudes in the fifth book of *Elements*. From this new theory in the fifth *Elementum*, *De propriis rationum logarithmis* (pp. 201–347) he found a method for calculation of the logarithm of a ratio and deduced many useful properties of the ratios and their powers. Finally, the sixth *Elementum*, *De innumerabilibus quadraturis* (pp. 348–392) calculates the quadratures of curves determined by algebraic expressions now represented by $y = K \cdot x^m \cdot (t - x)^n$. A detailed analysis of this work can be found in Massa [1998, 1–300].

16 On Hérigone's algebra see Hérigone [1644, second and sixth book] and Cifoletti [1990, p. 129].

17 Beaugrand was also a mathematician; in 1635 he spent an entire year in Italy and visited Cavalieri in Bologna. He published a version of *In Artem analyticem Isagoge*, which was in fact the work of Viète extended with some "scolies" and a mathematical compendium. More references appear in Cifoletti [1990, pp. 114–128].

18 Porrò cum Francisco Viettae, alijsque placuerit Analystis, ...; Quibus characteribus à Vietta, Herigonio, Beaugrand... [Mengoli, 1659, 11–12].

19 Notice that Hérigone distinguished between vulgar algebra, which dealt with numbers, and specious algebra, which dealt with species. He defined Algebra in this way: "La doctrine analytique ou l'Algebra est l'art de trouver la grandeur incognue, en la prenant comme si elle estoit cognue, & trouvant l'egalité entre icelle & les grandeurs données." He also defined specious algebra: "Mais l'Algebre Specieuse n'est pas limitée par aucune genre de probleme, & n'est pas moins utile à inventer toutes sortes de theoremes, qu'à trouver les solutions & demonstrations des problemes" [Hérigone, 1644, 1]. Also, in the sixth book of his *Cursus*, Hérigone wrote two parts about algebra, "supplement of algebra" (73 page) and "isagoge of algebra" (74–98). In this supplement Hérigone published Fermat's method of maximum and minimum [Hérigone, 1644, 59–69].

Mengoli did not cite Descartes as a source, nor does the treatment of algebraic symbols throughout his book suggest that he had read him.[20] Fermat's manuscripts and letters had circulated among Parisian mathematicians and reached Italy through Beaugrand and Mersenne.[21] It is possible that Mengoli knew Fermat's results: Ricci, Torricelli, and Cavalieri certainly did. He may also have known Fermat's method of maximum and minimum, which was published in the *Sixth Book* of the *Cursus Mathematicus* [Hérigone, 1644, 59–69]. Although Mengoli did not cite Fermat as a source in his *Geometriae*, this work could have been inspired by a reading of Fermat's method in Hérigone or in Fermat's manuscripts.

1.1. Mengoli's notation

One of the main difficulties in understanding Mengoli's book concerns the notation; it is original and becomes more complicated as the text progresses.[22] On a separate page, under the title *Explicationes quarundam notarum*, before the first Theorem in the *Elementum primum*, Mengoli outlined the basic notation that he would use throughout the book: addition, subtraction, the equals sign, and ratio. He also named all the letters and algebraic expressions contained in his analysis.

There are certain differences between these signs and those of Viète, Descartes, and Hérigone. For instance, equality was represented with two points, whereas Viète used an abbreviation of the word *aequalis*, Descartes wrote the symbol $\propto$, and Hérigone wrote 2/2. To multiply, Viète used the word *in*, whereas Mengoli, Descartes, and Hérigone wrote one letter next to the other. Mengoli used a semicolon to express the ratio between two quantities; Viète used the expression *ad*, Descartes *à*, and Hérigone the symbol π.

To represent quantities by symbols Mengoli did not distinguish between vowels and consonants, which could represent data, unknowns or variables. He used both capitals and lower case letters; in general, lower case represented data and capital letters variables. He invented names for the letters and expressions he used. In some cases these names were the same as Viète's, such as the word *radix* (the first power); others, such as *triprimam* (a^3r), *unisextam* (ar^6), and so forth, are original creations. To represent powers, Viète retained the words *A quadratus*, *A cubus*, and so on. Descartes wrote the exponents as they are written today, with one exception: he wrote xx to represent the square. Mengoli wrote the exponents on the right side of the letter, $x2$, as had Hérigone.[23] For instance, to represent one proportion Mengoli [1659, 8] wrote

$$\text{“}a\,;\ r : a2; ar\text{”} \quad \text{for } a : r = a^2 : ar.$$

It should be noted that in the 17th century there were no standard criteria either for symbols or for mathematical terms.[24]

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[20] According to Luigi Pepe, Descartes' *Géométrie* did not reach a wide readership in Italy. Pepe claims to have found two references, one in Giannantonio Rocca (1607–1659), a pupil of the Jesuit College of Parma, who possessed the translation of Descartes' *Géométrie* [Pepe, 1982, 263]. Mengoli wanted to square geometric figures as an answer to a question proposed by Rocca [Mengoli, 1659, 348]. This is the only association with Descartes' *Géométrie* that we have found.

[21] On the diffusion of Fermat's works in Italy see Mahoney [1973, 56].

[22] In a letter to Collins, Isaac Barrow said that Mengoli's style was harder than Arabic [Gregory, 1939, 49].

[23] On the same page Mengoli also explained how he represented a proportion, a composition of ratios and a power of a ratio. He defined the composition of ratios as a ratio obtained by multiplying the antecedents and the consequents.

[24] On the origins of algebraic language see Malet and Paradis [1984, 169–179].

1.2. Algebraic tools

As far as the definitions of the *Elementum primum* are concerned, Mengoli defined the powers of a quantity in continuous proportion to unity, u, as did Descartes [1979, 138]. When Mengoli used these definitions in demonstrations, he wrote

$$u : a = a : a^2 = a^2 : a^3 = \cdots.$$

In the fourth definition, he introduced the "rationalis," or unit u,

> 4. Quantity, from which the progression of the continuously proportional is ordered to infinity, will be called "Rationalis" and it will be represented by the symbol u.[25]

Then in the fifth definition, Mengoli introduced the radix a, and in the sixth definition powers of a.[26]

> 5. And the first quantity after "Rationalis" will be called Radix or first Power and it will be represented by a letter of the alphabet.
> 6. And the following remainders will be called the second, third and so on powers, in accordance with their order. And any [power] will be represented by the letter of its radix with the number of the order on the right side. For example from radix "a," second power "a^2," third "a^3," and so on.[27]

Mengoli put these quantities in a triangular table, the table "of proportionals" [*proportionalium*], to make their identification easier.[28] The table presents numbers expressed by letters so that in every row the first two elements always have the same ratio $a : r$, a and r both being integers. They also have the same ratio in the diagonals $1 : a$ and $1 : r$, respectively, because the letter u placed in the vertex represents unity or one (see Fig. 1).

Throughout the book triangular tables served as useful algebraic tools for calculations. In the *Elementum primum*, the terms of the triangular tables are numbers and they are used to obtain the development of any binomial power. In the *Elementum secundum*, the terms are summations used to obtain the sum of the pth powers of the first $t - 1$ integers. Finally, in the *Elementum sextum*, the terms are geomet-

[25] 4. Quantitas, unde progressio continuè proportionalium, ordinatur in infinitum, dicetur, Rationalis.& significabitur charactere u [Mengoli, 1659, 4].

[26] Curiously, though Mengoli never mentioned zero, either as a power or as a number, he defined the order of u as one unit less than the first power [Mengoli, 1659, 4].

[27] 5. Et prima consequens à rationali, dicetur, Radix, vel Potestas prima.& significabitur, charactere cuiusq; litterae alphabeti. 6. Et reliquae consequentes, dicentur Potestates radicis, Secunda, Tertia, & deinceps, iuxta suum cuiusque ordinem. Et significabitur unaquaeque, eidem litterae suae radicis, adscriptoque ordinis numero. Ut radicis a, secunda potestas $a2$, tertia $a3$, & sic deinceps [Mengoli, 1659, 4].

[28] Mengoli noted its similarity to a table said to be found in Euclid VII.2. We have not found this table in Euclid's *Elements*, but there is a reference to a similar table in a 13th-century Latin edition of the *Elements* published by Johan Ludvig Heiberg and H. Menge in Bosmans [1924, 22].

[29] He composed this table "of proportionals" with the table of binomial coefficients to obtain a new triangular table. Its elements are the development of the powers of the binomial $a + r$ or $a - r$, adding the corresponding signs depending on whether the binomial contains an addition or subtraction. He demonstrated these developments in Theorems 8 and 10 of the first *Elementum* [Mengoli, 1659, 15].

ric figures or forms and they are used to obtain the quadratures of these figures. Mengoli's originality stemmed not from the presentation of these tables but rather from his treatment of them. On the one hand, he used them and Viète's algebra to create other tables with algebraic expressions, stating clearly their laws of formation; on the other hand, he employed the relations between these expressions and the binomial coefficients of the arithmetic triangle to prove results. It is significant that he used the symmetry of triangular tables and the regularity of their rows in order to generalize the proofs. Mengoli took it for granted that if a result was true for one row of the table, this result was also true for all rows and there was no need to prove it in the remaining rows. For instance, he proved the development of the powers of the binomial $a+r$, for the second row,

$$u : a = a : a^2 = r : ar = a + r : a^2 + ar,$$
$$u : r = r : r^2 = a : ar = a + r : r^2 + ar,$$
$$u : a + r = a + r : a^2 + 2ar + r^2,$$
$$a^2 + 2ar + r^2 \text{ is the second power of } a + r.$$

Note that here Mengoli is using propositions in the theory of proportions from the fifth book of Euclid's *Elements* [Mengoli, 1659, 16]. It is evident that the derivation can be easily adapted to obtain the third, fourth, etc. powers of $a+r$.

The arithmetic manipulation of algebraic expressions helped Mengoli to obtain new results and new procedures. In *Elementum secundum* he invented a manner of writing and calculating finite summations of powers and products of powers. He did not give them values or write them using the sign $+$ and suspension points $(\ldots)$, but rather represented the numbers by letters. In this way he created an innovative and useful symbolic construction that would allow him to calculate these summations, which he assumed as new algebraic expressions. He considered an arbitrary number or *tota*, represented by the letter t, and divided it into two parts, a (abscissa) and $r = t - a$ (residua).[30] In his words,

> The parts of *tota* will be called the separated part [*abscissa*] and the remaining part [*residua*] and the separated part will be represented by the letter a and the remainder by r.[31]

$$\begin{array}{ccccccc} & & & u & & & \\ & & a & & r & & \\ & a^2 & & ar & & r^2 & \\ a^3 & & a^2r & & ar^2 & & r^3 \end{array}$$

Fig. 1. Tabula Proportionalium.[29]

[30] Mengoli referred to an "arbitrary number" [quantitas utcunque] although here he only gave examples with integers. As we will show later, in the quadratures he divided the unit into t parts of side $1/t$; that is to say, $a = 1/t$, and $r = 1 - 1/t$.

[31] Et partes totae, dicentur, Abscissa, & Residua: & significabitur abscissa, charactere a; & residua, r [Mengoli, 1659, 21].

$$\begin{array}{c} O.u \\ O.a \quad O.r \\ O.a^2 \quad O.ar \quad O.r^2 \\ O.a^3 \quad O.a^2r \quad O.ar^2 \quad O.r^3 \end{array}$$

Fig. 2. Tabula Speciosa.

He then took *tota* equal to 2, 3, ..., and gave examples up to 10. That is to say, if t is 2, a is 1, and r is 1. If t is 3, a may be 1 or 2 and r is then 2 or 1, respectively. He also calculated the squares and cubes of a, the products of a and r, of the squares of a and r, and so on. He then proceeded to add all the numbers a that he separated from the same number t. For instance, if t is 3, the summation will be 3, because it is the sum of 1 and 2; if t is 4, the summation will be 6, because it is the sum of 1, 2, and 3, and so on. He wrote *O.a*[32] to express this sum from $a = 1$ to $a = t - 1$,

$$O.a = \sum_{a=1}^{a=t-1} a.$$

Mengoli put all these summations of powers and products of powers in a triangular table which he called the "table of symbols" [*Speciosa*] (see Fig. 2).

The terms or "species" of this table are summations of the type

$$\begin{aligned} &O.u = (t-1), \\ &O.a = 1 + 2 + 3 + \cdots + (t-1), \\ &O.r = (t-1) + (t-2) + (t-3) + \cdots + 1, \\ &O.a^2 = 1^2 + 2^2 + 3^2 + \cdots + (t-1)^2, \\ &O.ar = 1.(t-1) + 2.(t-2) + 3.(t-3) + \cdots + (t-1).1. \end{aligned}$$

Mengoli combined his table of symbols with the table of binomial coefficients to obtain a new table. He then used new relations between the terms of these tables to calculate the summations of positive integers and summations of products of powers indefinitely.[33] Specifically, in Theorem 22 of *Elementum*

[32] Obviously "O." meant *Omnes* and originated with Cavalieri and his *Omnes lineae*.

[33] The summation formula for powers was, in fact, not new. The first recognition of it as a general rule was apparently made in 1636 by Fermat, who announced that he had solved "what is perhaps the most beautiful problem of all arithmetic" [Fermat, 1891–1922, 69], namely, given an arithmetic progression, to find the sum of any power of the elements of the progression. Fermat stated the rules but gave neither the formula nor the demonstration. Later, Bernoulli (1654–1705), in the *Ars Conjectandi* (1713), deduced and wrote the general formula on the basis of rules for polygonal numbers. See the third volume in [Bernoulli, 1975, 164–168].

Secundum, he proved that

$$(m+n+1).\binom{m+n}{n}.\sum_{a=1}^{a=t-1} a^m.(t-a)^n = t^{m+n+1} - P\big(t^s\big).$$[34]

Here $P(t^s)$ is a polynomial in t of degree less than or equal to $m+n$, with coefficients of the same type as the "Bernoulli numbers," depending on the binomial coefficients.

Mengoli, like Pascal [1954, 166–171] and Fermat [1891–1922, 65–71], found a rule in which the value of the sum of the pth powers is given in terms of the sum of the $(p-1)$th powers, $(p-2)$th powers, etc.:

$$t^p = \sum_{a=1}^{a=t-1}\binom{p}{1}a^{p-1} + \cdots + \sum_{a=1}^{a=t-1}\binom{p}{p}a^0 + 1^p.$$

Mengoli based the demonstration of this rule on earlier theorems. In Theorem 1, he established the symmetry of the table of summations, for example, $O.a^3 = O.r^3$, $O.ar^2 = O.a^2r$. In Theorem 2, he found two differences which he called *incrementa*. He proved that

$$\left[\sum_{a=1}^{a=t} a^p\right] - \left[\sum_{a=1}^{a=t-1} a^p\right] = t^p,$$

and also showed that

$$\left[\sum_{a=1}^{a=t}\big((t+1)-a\big)^p\right] - \left[\sum_{a=1}^{a=t-1}(t-a)^p\right] = \left[\sum_{a=1}^{a=t-1}\binom{p}{1}a^{p-1}\right] + \cdots + \left[\sum_{a=1}^{a=t-1}\binom{p}{p}a^0\right] + 1^p.$$

Since by Theorem 1 the two *incrementa* are equal, the rule is demonstrated.

To get a sense of this result, consider the examples

$$t^2 = \sum_{a=1}^{a=t-1} 2a + \sum_{a=1}^{a=t-1} a^0 + 1; \qquad t^3 = \sum_{a=1}^{a=t-1} 3a^2 + \sum_{a=1}^{a=t-1} 3a + \sum_{a=1}^{a=t-1} a^0 + 1.$$

In addition to stating and demonstrating the rule Mengoli, in Theorem 22, performed 36 calculations. Using the preceding expressions, he obtained

$$\sum_{a=1}^{a=t-1} 2a = t^2 - t; \qquad \sum_{a=1}^{a=t-1} 6a^2 = 2t^3 - 3t^2 + t.$$

He ended with the statement

[34] Actually Mengoli did not write a general formula with m and n. Instead he performed 36 summations. On this subject see [Massa, 1997, 266–268].

> And in infinity, it can be demonstrated, with the method shown above, that every summation is equal to some *tota*.[35]

He took advantage of the properties of the binomial coefficients to find and verify the value of the sum of the pth powers of the first $t-1$ integers. Mengoli reached this result using Viète's algebra to express the summations, a method that allowed him to achieve a certain level of generalization.

Another of Mengoli's original contributions was the justification and use of the notion of variable in the *Elementum tertium*. His idea was that letters could represent not only given numbers or unknown quantities, but variables as well: that is, determinable [but] indeterminate quantities. For example, summations were indeterminate quantities but they were determinate when the value of t was known. To clarify this idea, Mengoli stated that

> When I write $O.a, \ldots$ you have the summation [*massa*] of all the abscissae: but what value this summation is you do not yet know if I do not write what number the summation is. But if I assign $O.a$ to the summation of the number t, you do not know either how much it is if at the same time I do not assign the value of the letter t. But when I allow you to fix a value for the letter t, and you, using this licence, say that t is equal to 5, immediately you will accurately assign $O.a$ equal to 10, t^2 equal to 25, t^3 equal to 125, and $O.r$ equal to 10, and if the letters t are determinate, the quantities $O.a$, $O.r$, t^2, t^3, [are] determinable [but] indeterminate quantities.[36]

Mengoli applied his idea of variable to calculate the "quasi ratios" of these summations. The ratio between summations is also indeterminate but is determinable by increasing the value of t. The ratio does not really reach this limiting value, which can be interpreted as its actual value; instead, it tends toward it as t increases. It is in this sense that Mengoli understood the expression "determinable indeterminate ratio."

Mengoli proceeded to give examples and to clarify his notion of "ratio quasi a number." From this idea he constructed the theory of quasi proportions, which would prove important in his study of quadratures.

2. Algebraic treatment of geometric figures

Mengoli developed his algebraic analysis of geometric figures in the *Elementum sextum* of *Geometriae*.[37] This chapter, entitled *De innumerabilibus quadraturis* involves calculating quadratures of plane curves in the interval $(0, t)$ determined by equations now represented as $y = Kx^m \cdot (t-x)^n$.

[35] Et in infinitum, eadem methodo supra tradita, potest demonstrari, qualiter acceptis totis, quaeque massa est aequalis [Mengoli, 1659, 44].

[36] Cum scripsero $O.a\ldots$ habes massam ex omnibus abscissi: sed quota sic haec massa, nondum habes, nisi scripsero cuius numeri sit massa. Quod si assignavero $O.a$, numeri t massam esse; neque sic habes, quota sit, nisi simul assignavero, quotus est numerus, valor litterae $t\ldots$ Cum verò licentiam dedero, ut quotum quemque litterae t valorem taxes; tuque huiusmodi usus licentia dixeris, t valere quinario: statim profecto assignabis & $O.a$, valere 10; & $t2$, valere 25; & $t3$, valere 125; & $O.r$, valere 10; & determinatae litterae t, determinatas esse quantitates $O.a$, $O.r$, $t2$, $t3$. Quare data licentia antequam usus fueris, habebas profecto $O.a$, $O.r$, $t2$, $t3$, quantitates indeterminatas determinabiles [Mengoli, 1659, 61].

[37] This sixth *Elementum*, with the title *De innumerabilibus quadraturis* (pp. 348–392), contains (besides a letter to Cassini) three triangular tables, 36 definitions, 11 propositions (4 of them he named problems), and last, two pages on barycenters.

In a preliminary calculation, in the dedicatory letter to Giandomenico Cassini,[38] Mengoli derived values for the quadratures of these curves using Cavalieri's method of indivisibles.[39] He outlined that he had determined these values 12 years before (1647) and he enunciated 25 results.[40] For example, he derived (in modern notation)

$$6.\int_0^t x.(t-x)\,\mathrm{d}x=\int_0^t t^2\,\mathrm{d}x;\qquad 12.\int_0^t x.(t-x)^2\,\mathrm{d}x=\int_0^t t^3\,\mathrm{d}x;\qquad 20\int_0^t x.(t-x)^3\,\mathrm{d}x=\int_0^t t^4\,\mathrm{d}x.$$

Afterward, Mengoli wondered if by adding these results he could obtain a new quadrature:

> Having demonstrated these [quadratures by indivisibles], I thought whether I could calculate some other quadrature which would be obtained from those known, so that I could solve some significant quadratures in the same manner that Archimedes solved the parabolas with triangles.[41]

For instance, he indicated the quadrature obtained by adding the preceding quadratures,

$$\int_0^1 x\,\mathrm{d}x+\int_0^1 x.(1-x)\,\mathrm{d}x+\int_0^1 x.(1-x)^2\,\mathrm{d}x+\int_0^1 x.(1-x)^3\,\mathrm{d}x+\cdots$$
$$=1/2+1/6+1/12+1/20+\cdots=1.$$

He stated that he derived the value of this summation from the results obtained by indivisibles and from Proposition 17 [Mengoli, 1650, 21] of his work *Novae Quadraturae Arithmeticae seu de Additione Fractorum*.[42] In Proposition 17 he had proved that

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)}\frac{}{(n+2)}=1/2+1/6+1/12+\cdots=1.$$

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[38] Giandomenico Cassini (1625–1712) was a professor of astronomy at the University of Bologna from 1650 to 1669, before moving in the latter year to Paris. On the relation between Cassini and Mengoli see Baroncini and Cavazza [1986, 37].

[39] Cavalieri's method of indivisibles is largely set forth in two works: *Geometria indivisibilibus continuorum nova quadam ratione promota* (1635) and *Exercitationes geometricae sex* [1647]. The derivation of the quadratures of the parabolas $y=x^m$ for m any positive integer was published by Cavalieri in this last book. On Cavalieri's indivisibles, see Cavalieri [1966], Andersen [1984/1985], Giusti [1980], Malet [1996], and Massa [1994].

[40] Mengoli proved three of these results as examples. Interestingly, he did this using a lemma and three quasi-algebraic propositions of Jean Beaugrand, stating that he would use this algebraic technique with indivisibles because the procedure was shorter. These Beaugrand's propositions are found in Cavalieri's *Exercitatione quarta*. In the introduction to this part Cavalieri explained that when he was working on quadratures he told father Nicerone of his discoveries; during a subsequent visit to Paris, Nicerone then passed on this information to Beaugrand. Later Cavalieri learned of Beaugrand's death, from Mersenne; Mersenne also told him of the solutions that Beaugrand had found to the proposed quadratures. Cavalieri incorporated these solutions so that they would not be lost [Cavalieri, 1647, 243–245].

[41] His demonstratis, cogitabam si possent aliae quadraturae inveniri ex inventis compositae, in quas insignis aliqua resolvatur, quaemadmodum in triangula, parabolam Archimedes resolvit [Mengoli, 1659, 363].

[42] Mengoli had already published this work, in which he worked with infinite series, adding them together and giving them suitable properties. On this subject see Giusti [1991, 195–213].

He presented two more examples but he did not find any new quadrature, only relations between quadratures that were already known by means of indivisibles.[43] He therefore proceeded to develop a new and more fruitful method. He acknowledged that he did not publish this research on account of the attacks often leveled against quadrature methods:

> Meanwhile I left aside this addition that I had made to the Geometry of Indivisibles, because I was afraid of the authority of those who think false the hypothesis that the infinity of all the lines of a plane figure is the same as the plane figure. I did not publish it not because I agreed with them, but because I was doubtful of it, and I tried ... to establish new and secure foundations for the same method of indivisibles or for other methods, which were equivalent.[44]

Mengoli believed that the basis of Cavalieri's method of indivisibles was not sufficiently sound. He wanted to provide a solid foundation for the application of this method to square the given figures, new figures, and, especially, the circle. He sought to make his procedure for introducing algebra into geometry clear from the beginning. First, using his own system of coordinates, he expressed geometric figures by algebraic expressions. Second, to classify these algebraic expressions he placed them in a triangular table. Third, he used these algebraic expressions as part of a method for the geometrical construction of ordinates of these figures, and finally, he used triangular tables and quasi proportions to find new quadratures and to produce general demonstrations of quadrature results.

2.1. Mengoli's system of coordinates

In the first definitions of *Elementum Sextum* Mengoli described his own system of coordinates. He proposed a line segment, which he named "Rationalis," whose measure is any quantity. He then put this segment in a straight line and named it "Tota."

> 1. One of the line segments will be taken, of any quantity, which will be called Rationalis. 2. And [one] will be put in a straight line equal to Rationali, which will be called Tota.[45]

Next Mengoli defined a base as a straight-line segment the length of which is t or one. He used the word abscissa[46] for our x, but in a segment measuring the unit u or t. Mengoli always worked within a finite base in which the abscissa was represented by the letter "a" and the remainder was represented by the letter "$r = t - a$" or "$1 - a$," depending on whether the base was a given value t or the unit u.

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43 We can suppose that this "insignis" quadrature which he looked for was the quadrature of the circle. In fact, at the beginning of his later work *Circolo* [1672] he stated that he had found the quadrature of the circle in 1660.

44 Ipsam interim accessionem, quam Geometriae Indivisibilium feceram, praeterivi: veritus eorum authoritatem, qui falsum putant suppositum, omnes rectas figurae planae infinitas, ipsam esse figuram planam: non quasi hanc sequens partem; sed illam quasi non prorsus indubiam debitans: tentandi animo, si possem demum eandem indivisibilium methodum, aut aliam equivalentem novis, & indubijs prorsus constituere fundamentis [Mengoli, 1659, 364].

45 1. Assumatur inter lineas, una quaelibet quantitas; quae, Rationalis, dicetur. 2. Et exponatur quaedam recta linea, rationali aequalis; quae dicetur, Tota [Mengoli, 1659, 367].

46 The word abscissa appears in Fermat [1891–1922, 195], in Torricelli [1919, III, 366], in Cavalieri [1966, 858–859], and in Angeli [1659, 175–179]. Another word used with the same meaning was "diameter."

> 3. And a position is given, which will be called Base. 4. And one of the ends [of the base] will be called the end of the abscissae. 5. And the other one the end of the remainders. 6. And the quantity [that goes] from any point of the base to the end of the abscissae, as far as the same base is extended, will be called *abscissa*.[47]

He considered a base AR:

A---------B-------------------------R.

A is the end of the abscissae, R is the end of the remainders, AB is the abscissa, and BR is the remainder.

As for the word "ordinate,"[48] Mengoli first defined the ordinates of known figures, such as the square (or rectangle) and the triangle, from his construction on every point of the base. For instance, in the square (or rectangle) he stressed how to draw these lines:

> 10. Over a base is described a square, and I suppose that from any of the points of the base a straight line will be drawn to the opposite side, maintaining itself parallel at all times to the sides of the square; this will be called *ordinate in* [*the*] *square*.[49]

He defined the ordinates traced in a triangle consisting of half of a square:

> 15. The diagonal of the square, traced from the end of the abscissae, makes a half-square triangle. [In] which I suppose that from any of the points of the base a straight line will be drawn to the aforementioned diagonal, once again parallel to the sides [of the square]; this [line] will be called *ordinate in triangle*.[50]

Mengoli did not define the ordinates in the case of "mixed-line" or curved figures through his constructions, but he explained that they are equal to abscissae or powers of abscissae and named them "ordinate in form." The equality between ordinates and the powers of abscissae was expressed by means of proportions as follows:

$$1 : y = (1 : x)^n.$$

---

47 3. Sitque data positione; quae dicetur, Basis. 4. Eiusque alterum extremorum punctorum, dicetur, Finis abscissarum. 5. Alterum, Finis residuarum. 6. Et ab unoquoque puncto in basi sumpto, usque ad finem abscissarum, quatenus ipsa basis extenditur, quantitas dicetur Abscissa [Mengoli, 1659, 367].

48 Mengoli used the word "ordinata" instead of the word "applicata," which was commonly used at the time. Descartes defined the ordinates as "celles qui s'appliquent par ordre" [Descartes, 1954, 67]. In the 1954 edition there is the following editorial note: "The equivalent of 'ordination application' was used in the 15th century on translating Apollonius." The note also states that Hutton's *Mathematical Dictionary* of 1796 gave "applicata" as the word corresponding to the ordinate and explained that the expression "ordinata applicata" was also used. In fact Fermat and Cavalieri used "applicata." Mengoli in *Circolo* [1672] named them "ordinatamente applicate" [Mengoli, 1672, 5].

49 10. Super basi describatur quadratum: & ab uno quolibet puncto in basi sumpto, recta ducatur, usque ad oppositum latus, reliquis lateribus quadrati parallela: quae dicetur, Ordinata in quadrato [Mengoli, 1659, 368].

50 15. A fine abscissarum ducta diameter quadrati, facit semiquadratum triangulum: cuius ab unoquolibet puncto in basi sumpto recta ducatur, usque ad praedictam diametrum, alteri lateri parallela, quae dicetur, Ordinata in triangulo [Mengoli, 1659, 368].

2.2. Geometric figures as algebraic expressions

Mengoli described the figures that he wanted to square as “extended by their ordinates.” He called them “forms” and expressed them by an algebraic expression beginning with FO. He never mentioned the word “curve”—only the word figure or *forma*, which dates from the previous century and was identified by measuring the intensity of a given quality. The word appears in the work of Oresme (1323–1382) *Tractatus de latitudinibus formarum (1346)* among others.[51] A form was any quality that was variable in nature. The intensity or latitude was measured vertically over a base that measured the longitude, and the area of the described figure measured the quantity.

Mengoli began with known figures such as the square and the triangle and then progressed to mixed-line figures. He expressed the square and the triangle algebraically:

> 12. And the square, extended by its ordinates, is called “Form of all rationals,” and “Form of all totals,” and it will be represented by the characters $FO.u$ and $FO.t$.[52]
> 17. And in the same manner the triangle [made] by its ordinates extended will be called “Forma *omnes abscissae*” [Form of all abscissae] and it will be represented by the character $FO.a$.[53]

The first mixed-line figure that he defined was determined by one branch of the parabola, $y = x^2$, and the base.

> 20. If over the base a [geometric] figure is constructed, not extended more than by ordinates within the square but in which any ordinate is the “second” abscissa [a^2], it will be called “Form of all second abscissae,” and it will be represented by the character $FO.a^2$.[54]

When he used this definition in demonstrations he explained:

> The ratio of the base AR [u] to the ordinate by B [y] is “the double” of the ratio AR [u] to AB [x].[55]

(In modern notation $1 : y = (1 : x)^2$.) In the same way, he also defined the “Form of all products of the abscissa and the remainder” and the “Form of all second remainders,” representing them by the characters $FO.ar.$ and $FO.r^2$. The ordinates of the curves corresponding to these figures are given by the proportions $1 : y = (1 : x).(1 : (1 - x))$ and $1 : y = (1 : (1 - x))^2$, respectively. More generally, he defined the geometric figure extended by any ordinate.[56]

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[51] On Oresme see Clagett [1968, 91–92], Lindberg [1978, 231–241], and Crombie [1980, 82–95].

[52] 12. Et quadratum, per suas ordinatas extensum, dicetur, Forma omnes rationales, & Forma omnes totae. & significabitur characteribus $FO.u$, & $FO.t$ [Mengoli, 1659, 368].

[53] 17. Ipsumque triangulum per suas ordinatas extensum, dicetur, Forma omnes abscissae. & significabitur charactere, $FO.a$ [Mengoli, 1659, 368].

[54] 20. Si super basi concipiatur figura extensa non nisi per ordinatas in quadrato: sed in qua, unaquaelibet ordinata est abscissa secunda, dicetur, Forma omnes abscissae secundae. & significabitur charactere $FO.a2$ [Mengoli, 1659, 369].

[55] Basis AR, ad ordinatam per B, duplicata habet rationem eius, quàm habet ad AB [Mengoli, 1659, 372].

[56] In his later work, *Circolo* [1672], Mengoli defined the same ordinates as powers of abscissae by means of other proportions and named them “ordinatamente applicate.” “Et altresì sopra la Rationale s’intendano descritte tre figure, una nella quale le ordinatamente applicate alla base sono le terze proportionali della tota, e dell’abscissa, ch’io chiamo Abscisse seconde: l’altra

23. And generalizing, if over the base a figure is constructed, not extended more than by ordinates within the square, in which any ordinate is considered as some element of the proportional table [see Fig. 1]. [This figure] is called "Form of all possible proportionals" and an appropriate character will represent it. For instance, "Form of all third abscissae," $FO.a^3$, "Form of all products of the second abscissae and the remainders" *biprimae*, $FO.a^2r$, "Form of all products of the abscissa and second remainders," *unisecundae*, $FO.ar^2$, "Form of all third remainders," $FO.r^3$, and so on.[57]

2.3. Triangular tables of geometric figures

After defining the given geometric figures and assigning algebraic expressions to them, Mengoli proceeded to work with these new algebraic objects. His approach here was deeply original. He used these new symbols, such as $FO.a.$, which he had associated with geometric figures, in algebraic calculations. Mengoli explained that when these figures [forms] constructed over a base are put in a triangular table as he had done before, they become a new table which he called *Tabula Formosa*, or table of "forms" (see Fig. 3).

The figure at the vertex represented a square of side 1. The two figures of the first row represented two triangles. The first "$FO.a$" is determined by the diagonal of the first quadrant $y = x$, the axis of abscissae and the straight line $x = 1$, and the second triangle "$FO.r$" is determined by the straight $y = 1 - x$ traced from the point $(1, 0)$ to the point $(0, 1)$ and the axis of abscissae. The three figures of the third row are determined by the ordinates of a parabola, the axis of abscissae and the straight line $x = 1$. The first figure, "$FO.a^2$," is determined by the ordinates $y = x^2$, the second, "$FO.ar$," by the ordinates $y = x.(1 - x)$, the

$$
\begin{array}{c}
FO.u. \\[1ex]
FO.a. \qquad FO.r. \\[1ex]
FO.a^2. \quad FO.ar. \quad FO.r^2. \\[1ex]
FO.a^3. \quad FO.a^2r. \quad FO.ar^2. \quad FO.r^3.
\end{array}
$$

Fig. 3. Tabula Formosa.

nella quale le ordinatamente applicate alla base sono le quarte proportionali della tota dell'abscissa, e della residua, ch'io chiamo Uniprime: la terza nella quale le ordinatamente applicate alla base sono le terze proportionali della tota e della residua, ch'io chiamo Residue Seconde, ..." [Mengoli, 1672, 5]. Mengoli also defined the ordinates named "third abscissae" as the fourth proportional of the "tota," the abscissa, and the second abscissa. Afterwards, he defined the ordinates called "the products of the second abscissae and the remainders" again as the third proportional, and in this case he stressed that all the ordinates, "in infinity," could be defined in this way.

[57] 23. Et generaliter, si super basi concipiatur figura, extensa non nisi per ordinatas in quadrato; & in qua, unaquaelibet ordinata, est assumpta quaedam in tabula proportionalium: dicetur, Forma omnes tales proportionales aptoque significabitur charactere. Vt Forma omnes abscissae tertiae, $FO.a3$: Forma omnes biprimae, $FO.a2r$: Forma omnes unisecundae, $FO.ar2$: Forma omnes residuae tertiae, $FO.r3$. & sic deinceps [Mengoli, 1659, 369]. Note that Mengoli wrote the exponent without a superscript on the right side of the letter.

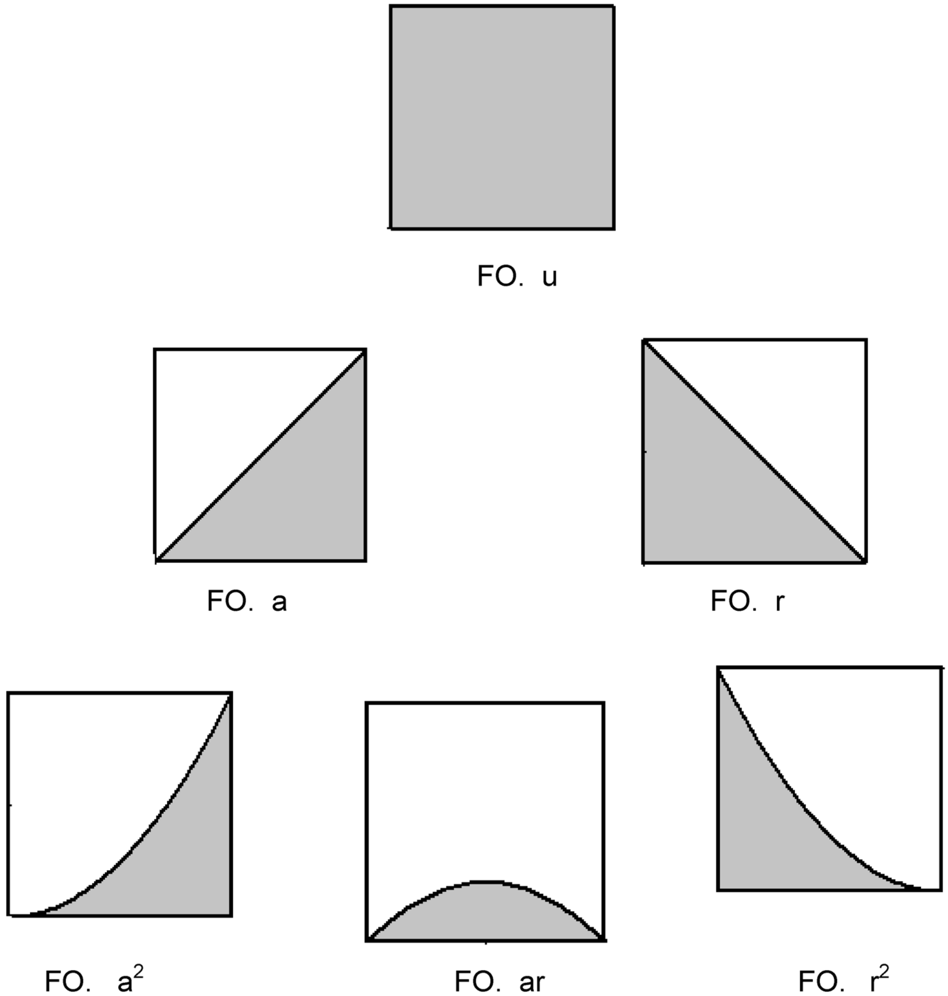

Fig. 4. My own sketches of geometric figures.

third, "$FO.r^2$," by the ordinates $y = (1 - x)^2$, and so on in the other rows.[58] Below are my sketches of these geometric figures arranged as a triangular table (see Fig. 4).

From this table of forms Mengoli derived a second table by multiplying the elements of each row, term by term, by the corresponding binomial coefficients. He called this the *Tabula subquadraturarum*, or "Table of subquadratures" (see Fig. 5).

Mengoli called the first row "of order one," the second "of order two," and so on. He then formed a third table by multiplying each of the rows of the second table by the order of the row plus one: he

[58] In his *Circolo* Mengoli defined these figures $FO.a2$ and $FO.r2$ in this way: "E sono le due $FO.a2$, $eFO.r2$, avanzi di due semiparabole dal quadrato della Rationale, del quale un lato è l'asse, e l'altro è la semibase della semiparabola [Mengoli, 1672, 5].

1	$FO.u.$
1 1	$FO.a.$ $FO.r.$
1 2 1	$FO.a^2.$ $FO.2ar.$ $FO.r^2.$
1 3 3 1	$FO.a^3.$ $FO.3a^2r.$ $FO.3ar^2.$ $FO.r^3.$

Fig. 5. Tabula subquadraturarum.

	$FO.u.$
First base	$FO.2a.$ $FO.2r.$
Second base	$FO.3a^2.$ $FO.6ar.$ $FO.3r^2.$
Third base	$FO.4a^3.$ $FO.12a^2r.$ $FO.12ar^2.$ $FO.4r^3.$

Fig. 6. Tabula quadraturarum.

multiplied the first row by two, the second one by three and so on. He called this new table the *Tabula quadraturarum*, or “Table of quadratures” (see Fig. 6).

Mengoli put the forms for the given curves in triangular tables in order to classify them and to work with them as a group. These expressions could be infinite in number; it is only necessary to increase the degree and to calculate the coefficients through the laws of formation of the table. The symmetry of the table and the regularity of its rows allowed Mengoli to generalize the proofs occurring in his theory.

2.4. Representation and geometrical construction of geometric figures

In the graphical representation of these geometric figures, Mengoli introduced a horizontal axis as a base, which he called rational. He did not use a vertical axis, and always drew the ordinates as lines perpendicular to the base. However, it should be emphasized that there are only three drawings of geometric figures in *Geometriae Speciosae Elementa*.[59]

In Mengoli’s work the graphical representation of a geometric figure was not so much a sketch as an accurate description of the curve corresponding to the figure that was informative enough to allow a sketch to be made. Mengoli did not draw these figures but made clear that their drawings could be deduced from their definitions and their positions in the triangular table. He considered three groups of geometric figures: the first, in the outside left diagonal of the *Tabula Formosa*, $FO.a^m$, the second, in the opposite diagonal of the table, $FO.r^n$, and the third, in the middle of the table, $FO.a^m r^n$. For each group he demonstrated its characteristics for only one specific entry, although he took this demonstration as true for all the entries on account of the table’s symmetry and the regularity of its rows.

[59] In his later work, *Circolo* [1672], in which he calculated the quadrature of the circle, he did not include any drawings.

In the First Theorem of *Elementum Sextum*, he demonstrated that in all the curves corresponding to the outside left diagonal of the table *Formosa*, $FO.a^m$ (determined by $y = x^m$; see Fig. 5), the ordinates increase with a and the maximum ordinate is found at the end of the base and is equal to it.

The demonstration is based on the definition of the ordinates: that is to say, for $n = 2$, he established the proportion $1 : y = (1 : x)^2$. In the proof, he started from the inequality of the abscissae and from there he obtained the inequality of the ordinates, through this same proportion. He also showed that all the curves corresponding to the entries in the opposite diagonal of the table, $FO.r^n$, were determined by ordinates that were always decreasing.

As for the entries in the middle of the table, in the Second Theorem he demonstrated that in the curves corresponding to $FO.a^m r^n$, determined by $y = x^m \cdot (1 - x)^n$, the ordinates first increase and then decrease, reaching their maximum value in an abscissa that divides the base AR in the ratio $m : n$. The demonstration is given for the curve corresponding to $FO.a^2 r^3$, where the abscissa B with $\mathrm{AB} : \mathrm{BR} = 2 : 3$ has the maximum ordinate, A is the end of the abscissae, R is the end of the remainders and D is any division of the base AR:

A---------------D-----B------------------R.

He proved that the ordinates of the curve increased to this maximum value and then decreased to the ordinate of the end of the base. We present only an outline of the demonstration. We know that $u = 1$, $a = x =$ abscissa, $r = 1 - x =$ residua, and we denote by $\mathrm{Ord}\,\mathrm{B} = y$ the ordinate of the abscissa B, $\mathrm{AR} = 1 =$ base. The following proportions are thus established:

$$\mathrm{AR} : \mathrm{AB} = 1 : x; \qquad \mathrm{AR} : \mathrm{BR} = 1 : 1 - x;$$
$$\mathrm{AR} : \mathrm{Ord}\,\mathrm{B} = 1 : \mathrm{Ord}\,\mathrm{B} = (1 : y) = (1 : x)^2.\big(1 : (1 - x)\big)^3.$$

Moreover, taking the abscissa $x_1 = \mathrm{AD}$ as any division of the base smaller than x, and using the letter y_1 as the ordinate of this abscissa, we find that

$$\mathrm{Ord}\,\mathrm{D} : \mathrm{AR} = \mathrm{Ord}\,\mathrm{D} : 1 = (y_1 : 1) = (x_1 : 1)^2 \cdot \big((1 - x_1) : 1\big)^3.$$

By operating on and composing the two proportions, it follows that

$$(\mathrm{Ord}\,\mathrm{D}) : (\mathrm{Ord}\,\mathrm{B}) = (y_1) : (y) = \big((x_1)^2 \cdot (1 - x_1)^3\big) : \big((x)^2 \cdot (1 - x)^3\big).$$

Mengoli proved that the antecedent—Ord D—is smaller than the consequent—Ord B—for any abscissa D, and he was thus able to affirm that the ordinate of the abscissa B is a maximum.[60]

[60] For this demonstration he needed to use some results from the *Elementum quintum* in which he constructed the logarithm of a ratio. Through the property of the logarithm that the product of the power of a ratio and its logarithm is equal to the logarithm of the ratio raised to this exponent, he obtained a relation between the ratios and their powers under certain specific conditions. The proposition that Mengoli used is "Given four quantities, disposed arithmetically, if it is verified that the first to the last is as one number to one number, then the first to the second raised to the number homologous to the first will be bigger than the third to the fourth raised to the number homologous to the fourth. If it is verified that the second to the third is as one number to one number, then the first to the second raised to the number homologous to the second will be smaller than the third to the fourth raised to the number homologous to the third" [Mengoli, 1659, 338]. Mengoli applied this theorem to four

Although some of the underlying ideas of this demonstration involve results about continuous magnitudes that we would today regard as part of differential calculus, Mengoli himself understood that he was using only algebraic procedures, Euclidean proportion theory, and properties of logarithms. We should also stress that his descriptions of curves belonging to geometric figures depended only on the type of the corresponding algebraic expressions in accordance with their positions in the triangular table.[61]

According to Bos [2001, 3–6], in the 17th century a curve was "known" or "given" when one could construct it starting from given elements.[62] Mengoli had to ensure that each of the expressions in the triangular table, which were new algebraic objects, could be associated with a definite geometric curve. He enunciated this Proposition Three as a Problem and demonstrated how to construct the ordinate to the curve corresponding to a geometric figure at a given point[63]:

Probl. I. Prop. 3.

Find the ordinate of a proposed [geometric] figure, at a given point and from a given base.[64]

Hypothesis

That is, given $FO.10a^2r^3$, over a given base AR, in which is given a point B. It is necessary to find the ordinate of B.[65]

Construction.[66]

Given AR, and given AB, BR, the recta BC will be found, to which AR is a ratio composed of given ratios AR to AB squared, AR to BR cubed, and of the ratio one tenth: and BC will be put perpendicular to AR. I say BC is the ordinate of B, in $FO.10a^2r^3$.[67]

Demonstration

The ratio AR to BC will be composed of ratios AR to AB squared, AR to BR cubed, and of one tenth; but AR is u; AB, is a; BR, is r. So the ratio AR to BC will be composed of ratios "u to a," squared, "u to

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quantities [segments], which have the same differences and of which two are in a specific ratio to each other. Mengoli named them arithmetical ordinates. He considered AD, AB, BR, and RD and proved that they were arithmetically ordinate quantities since $\mathrm{AB} - \mathrm{AD} = x - x_1 = \mathrm{RD} - \mathrm{BR} = (1 - x_1) - (1 - x) = \mathrm{BD}$, and besides, $\mathrm{AB} : \mathrm{BR} = 2 : 3$. He could then apply the theorem and set up the inequality $(\mathrm{AD} : \mathrm{AB})^2 = (x_1 : x)^2 < [(1 - x) : (1 - x_1)]^3 = (\mathrm{BR} : \mathrm{RD})^3$. Multiplying the antecedent of the first ratio by the consequent of the second ratio and vice versa, Mengoli demonstrated that the ordinate by D is smaller than the ordinate by B.

61 Mengoli defined the curves like Roberval, Fermat, and others by means of a proportion between ordinates and abscissae, but he could use the same demonstration for any curve of the same type. Information on Roberval may be found in [Auger, 1962, 18–21; Walker, 1986, 41–44].

62 Today the geometrical construction of algebraic expressions of curves presents no difficulty, but in Mengoli's time the geometrical construction was a very important subject. On this point see Bos [1981; 2001].

63 Mengoli here drew one horizontal axis AR and a perpendicular line (not in the middle) with the letter B over the base and the letter C at the top of the perpendicular line.

64 Formae propositae, in data basi, per datum punctum, ordinatam invenire [Mengoli, 1659, 377].

65 Esto proposita $FO.10a2r3$, super data basi AR, in qua datum punctum B. Oportet per B ordinatam invenire [Mengoli, 1659, 377].

66 Throughout the book Mengoli presented Theorems and Problems. In this case he wrote the word Construction, as Euclid did, before the demonstration and explained the construction used in it.

67 Data AR, datisque AB, BR, inveniatur recta BC, ad quàm AR, rationem habet compositam ex datis rationibus, AR ad AB duplicata, AR ad BR triplicata, & ex ratione subdecupla: & collocetur BC perpendiculariter ad AR. Dico BC, esse ordinatam per B, in $FO.10a2r3$ [Mengoli, 1659, 377].

> r," cubed, and of one tenth. But u to $10a^2r^3$ will be composed of these: then AR to BC is like u to $10a^2r^3$. But AR is u, so BC is $10a^2r^3$: then BC is the ordinate of B, in $FO.10a^2r^3$.[68]

Note here that Mengoli not only worked with proportions of segments but also equated segments with the letters of the triangular table. He equated the product of segments with the composition of ratios because he was familiar with the Euclidean theory of proportions. However, unlike Descartes, he did not define an algebra of segments. Rather, he demonstrated, for a given measure, how to construct the ordinate from the algebraic form corresponding to a curve using the composition of ratios. In this way, he established an isomorphic relation between algebraic objects and geometric figures that allowed him to study these geometric figures by their algebraic expressions.

2.5. Calculation of quadratures

Mengoli was able to compute quadratures using Cavalieri's method of indivisibles, but he was keen to find another way to verify the values so obtained. Using Viète's symbolic language he created new algebraic expressions and constructed triangular tables and a theory of "quasi proportions." Notice that the Euclidean theory of proportions is very important in the *Elementa*. Mengoli considered Euclid's *Elements* as the book of mathematics *par excellence* and developed his own theories, the theory of "quasi proportions" and the theory of logarithmic ratios, using as a model the Euclidean theory of proportions.[69]

In order to understand how Mengoli proved the given quadrature results, we consider the basic ideas of the theory of "quasi proportions." He set up this theory on the notion of "ratio quasi a number," which he clarified thoroughly. He considered values up to 10 in the ratio $O.a$ to t^2; for instance, if $t = 3$, then the ratio $O.a$ to t^2 is 3 to 9; if $t = 4$, then the ratio is 6 to 16; if $t = 5$, then the ratio is 10 to 25; ... if $t = 10$, then the ratio is 45 to 100. He argued that the ratio takes different values as the value of t increases.[70] Moreover, these values are eventually nearer to $1/2$ than is any other given ratio. Mengoli called it "ratio quasi $1/2$." The difference between $1/2$ and the ratio, which is determined when the value of t increases indefinitely, is smaller than the difference between $1/2$ and any other given ratio. The "limit" of this succession of ratios, as far as it is thus determinable, is $1/2$, and Mengoli uses the term "ratio quasi $1/2$" to denote this limit. The idea of "ratio quasi a number" suggests, though in a somewhat imprecise way, the modern concept of limit.[71]

This notion, together with the idea of determinable indeterminate ratio previously explained, was used in the definitions of ratio "quasi infinite," "quasi null," "quasi equality," and "quasi a number" in the *Elementum tertium*:

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[68] Ratio AR ad BC, componitur ex rationibus AR ad AB duplicata, AR ad BR triplicata, & ex subdecupla: sed AR, est u; AB est a; BR est r: Ergo AR ad BC ratio, componitur ex rationibus u ad a duplicata, u ad r triplicata, & ex subdecupla: sed ex ijsdem componitur u ad $10a2r3$: ergo AR ad BC est ut u ad $10a2r3$: sed AR est u: ergo BC est $10a2r3$: ergo BC est ordinata per B, in $FO.10a2r3$. Quod&c [Mengoli, 1659, 378].

[69] A knowledge of algebraic language enabled Mengoli to extend the Euclidean theory of proportions and create new theories. On the importance of Mengoli's work on the Euclidean theory of proportions see [Massa, 2003, 472–474].

[70] On these explanations see [Massa, 1997, 269–270].

[71] In his *Circolo* of [1672], Mengoli again uses quasi ratios and explains: "Dissi quasi, e volsi dire, che vadino accostandosi ad essere precisamente tali" [Mengoli, 1672, 49].

1. A determinable indeterminate ratio, which, when determined, can be greater than any given ratio, as far as is thus determinable, will be called quasi infinite.[72]
2. And one that can be smaller than any given ratio, as far as it is thus determinable, will be called quasi null.
3. And one that can be smaller than any given ratio greater than equality, and greater than any given ratio smaller than equality, as far as it is thus determinable, will be called quasi equality. Or otherwise, that which can be nearer to equality than any given ratio not equal to equality, as far as it is thus determinable, will be called quasi equality.
4. And one that can be smaller than any ratio larger than a given ratio, and larger than any ratio smaller than the same given ratio, as far as it is thus determinable, will be called quasi equal to this given ratio. Or otherwise one that can be nearer to any given ratio than any other ratio not equal to it, as far as it is thus determinable, will be called quasi equal to the same (given) ratio.
5. And the terms of ratios quasi equal between them will be called quasi proportional.
6. And (the terms) of quasi equality ratios will be called quasi equal.[73]

The sixth definition in light of the third definition can be read as follows: "And the terms of ratios that are nearer to equality than any other given ratio other than equality, as far as these ratios are determinable, will be called quasi equal." In calculating quadratures Mengoli used this interpretation of the definition of quasi equality ratio. In fact, he considered a "maior inaequalitas" ratio[74] and proved that he could find a number that allowed him to set up a ratio smaller than the given "maior inaequalitas" ratio.

Following the presentation of these six definitions Mengoli obtained ratios between all sorts of summations and the number t. (Recall that these are all constructed using t and that these summations have $t-1$ addends with different exponents.) He calculated what these ratios tend toward when t is very large, obtaining in this way all possible quasi ratios. Specifically, in Theorem 42, Mengoli demonstrated that

$$(m+n+1).\binom{m+n}{n}.\sum_{a=1}^{a=t-1} a^m.(t-a)^n$$

[72] To clarify the notion of "ratio quasi infinite" Mengoli considered values up to 10 in the ratio $O.a$ to t; for instance, if $t=4$, then the ratio is 6 to 4; if $t=7$ then the ratio is 21 to 7; if $t=10$ then the ratio is 45 to 10. He argued that the ratio takes greater and greater values as the value of t increases, so the ratio is quasi infinite. For the ratio quasi null he considered values up to 10 in the ratio $O.a$ to t^3 [Mengoli, 1659, 64–65].

[73] 1. Ratio indeterminata determinabilis, quae in determinari, potest esse maior, quam data, quaelibet, quatenus ita determinabilis, dicetur, Quasi infinita. 2. Et quae potest esse minor, quàm data quaelibet, quatenus ita determinabilis, dicetur, Quasi nulla. 3. Et quae potest esse minor, quàm data quaelibet minor inaequalitas; & maior, quàm data quaelibet minor inaequalitas, quatenus ita determinabilis, dicetur, Quasi aequalitas. Vel aliter, quae potest esse propior aequalitati, quàm data quaelibet non aequalitas, quatenus talis, dicetur, Quasi aequalitas. 4. Et quae potest esse minor, quàm data quaelibet non maior, proposita quadam ratione; & maior, quàm data quaelibet minor, propositâ eâdem ratione, quatenus ita determinabilis, dicetur, Quasi eadem ratio. Vel aliter, quae potest esse propior cuidam propositae rationi, quàm data quaelibet alia non eadem, quatenus talis, dicetur, Quasi eadem. 5. Et rationum quasi earundem inter se, termini dicentur, Quasi proportionales. 6. Et quasi aequalitatum, dicentur, Quasi aequales [Mengoli, 1659, 97].

[74] The *inaequalitas* of a ratio denotes a number other than unity, and so ratios *minor inaequalitas* and *maior inaequalitas* correspond to numbers smaller and larger than unity, respectively.

FO.u.

FO.2a . *FO.2r.*

FO.3a^2. *FO.6ar.* *FO.3r*^{2}.

FO.4a^3. *FO.12a*^{2}*r.* *FO.12ar*^{2}. *FO.4r*^{3}.

Fig. 7. Tabula quadraturarum.

tends to t^{m+n+1} when t tends to infinity, in the sense that their ratio can be made arbitrarily close to equality by making t sufficiently large.[75] He based this demonstration on Theorem 22 and on another theorem that he had previously demonstrated, which established that smaller powers could be ignored as t increases. In Theorem 22 of *Elementum Secundum* he had proved that

$$(m+n+1).\binom{m+n}{n}.\sum_{a=1}^{a=t-1} a^m.(t-a)^n = t^{m+n+1} - P\big(t^s\big).$$

Then, in Theorem 41 of *Elementum Tertium* he demonstrated the quasi equality ratio

$$t^{m+n+1} \quad \text{is quasi equal to } t^{m+n+1} - P\big(t^s\big).$$

It follows that the left side of the equation given in Theorem 22 is quasi equal to the first term of Theorem 41:

$$[m+n+1]\binom{m+n}{n}\sum_{a=1}^{a=t-1} a^m.(t-a)^n \quad \text{is quasi equal to } t^{m+n+1}.$$

This result is used in the calculation of the quadratures, as we explain further below.

We return now to Mengoli's treatment of the quadratures of the curves defined by the equations $y = x^m \cdot (1-x)^n$. These quadratures are given in terms of the entries in the *Tabula quadraturarum* (Fig. 7).

In this table the quadrature of $y = x^m \cdot (1-x)^n$ is multiplied by the product $(m+n+1).\binom{m+n}{n}$.[76] Mengoli knew by the method of indivisibles that the quadrature is equal to the inverse of this product:

$$\int_0^1 x^m.(1-x)^n\,\mathrm{d}x = \frac{1}{(m+n+1).\binom{m+n}{n}}.$$

[75] On this subject see Massa [1997, 271–275].

[76] Mengoli knew that one factor, the binomial coefficient, corresponded to the coefficient of the binomial development of $[x+(1-x)]^{m+n} = [1]^{m+n}$ and the other factor could be found through the relation between the summation of powers and the degree. For instance, if we wish to calculate the quadrature of curve $FO.x^{25} \cdot (1-x)^{30}$, it will be necessary to multiply by 56 and by the binomial coefficient $\binom{55}{30}$.

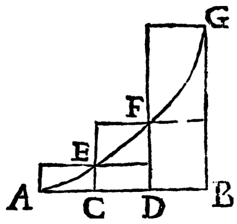

Fig. 8. Mengoli's figure in Proposition 4.

Hence each entry in the table of quadratures has the value one. Thus, all that remained was to prove that each entry is equal in area to a square of side 1 (in the limit as t tends to infinity). In modern notation,

$$(m+n+1).\binom{m+n}{n}.\int_0^1 x^m.(1-x)^n\,\mathrm{d}x = 1.$$

To demonstrate this result Mengoli used the theory of quasi proportions. He considered two ratios: the first one, between a new figure (the "ascribed" figure) and the figure or form which he wanted to square, and a second one, involving this "ascribed" figure and a square of side 1.[77] He showed that these two ratios are quasi equality ratios and then used a theorem that he had previously demonstrated, which showed that in quasi equality ratios with the same antecedents, the consequents of the ratios are also equal.

For the first quasi equality ratio he used Archimedes' definitions of inscribed and circumscribed figures. The inscribed figure is determined by all the greater rectangles included in the figure and the circumscribed figure is determined by all the smaller rectangles containing the figure.[78] The ascribed figure is determined by all the rectangles built over the ordinates of the divisions of the base. So, the ascribed figure is determined by $t-1$ rectangles when one divides the base in t parts.

> 33. The figures composed of just as many rectangles, as there are ordinates through the points of division and adjacent lines to these ordinates, will be called "ascribed" of the form.[79]

To get a sense of this, consider the geometric figures of the outside left diagonal of the table *Formosa*, $FO.a^m$ (see Fig. 8).

The inscribed figure is determined by the rectangles DE and BF; the circumscribed figure is determined by the rectangles AE, CF, and DG, and finally, the ascribed figure is determined by AE and CF or by DE and BF. In this case Mengoli demonstrated that the circumscribed figure is larger than the ascribed or inscribed figure by a rectangular quantity determined by the maximum ordinate and one of the equal parts of the base (Proposition 4).

[77] For these demonstrations Mengoli used the definitions from *Elementum tertium* of quasi equality.

[78] The circumscribed and inscribed figures were already known and used for instance by Luca Valerio, James Gregory [Malet, 1996, 83], Fermat, and later by Newton and others.

[79] 33. Figura vero ex tot parallelogrammis, quot sunt ordinatae per puncta divisionum, & ad ipsas ordinatas iacentibus composita, dicetur, Adscripta formae [Mengoli, 1659, 371].

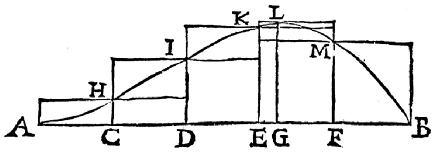

Fig. 9. Mengoli's figure in Proposition 5.

In the preceding example the inscribed and ascribed figures are identical. This will be true for any curve that is monotonically increasing. In general, the composite rectangles that make up the ascribed figure are sometimes smaller and sometimes larger than the associated curvilinear area elements of the figure. Hence in general the ascribed figure is larger than the inscribed figure. Such is the case for the entries in the middle of the table *Formosa*, $FO.a^m r^n$ (see Fig. 9).

The inscribed figure is determined by the rectangles HD, IE, and EM; the circumscribed figure is determined by the rectangles AH, CI, DK, ELF, and MB; the ascribed figure is determined by the rectangles AH, CI, DK, and EM or by the rectangles HD, IE, KF, and MB.

In this second example Mengoli demonstrated that the circumscribed figure is larger than the ascribed figure by a rectangular quantity (the area of the rectangle determined by the maximum ordinate and one of the equal parts of the base). He also proved that the ascribed figure is larger than the inscribed figure, but the difference in size is not greater than this rectangular quantity (Proposition 5). Immediately, using the theory of quasi proportions (Proposition 6), Mengoli proved for any figures in the table that the circumscribed and inscribed figures are "quasi equal." That is to say, he demonstrated that it is possible to find a number of divisions of the base so that the ratio between the circumscribed and the inscribed figures is nearer to equality than is any other given ratio (not equal to equality). With this result he was able to affirm that the ascribed figure, determined by rectangles, and the geometric figure or form, determined by ordinates, were quasi equal (Proposition 7).[80] Notice that Mengoli's ascribed, inscribed, and circumscribed figures are explicitly determined by a finite number of rectangles.

This demonstration follows Archimedes but uses the quasi-ratio method rather than *reductio ad absurdam*. Another difference is that in Archimedes the figure between the inscribed and circumscribed figures is used directly, whereas Mengoli introduced a new figure, the ascribed figure, determined by a finite number of rectangles. The number of rectangles making up the ascribed figure will increase indefinitely. The rectangles of the ascribed figure never actually become the ordinates of the curved figure, and the geometric figure exists independently of the existence of the successive ascribed figures. Mengoli needed the ascribed figure, determined by $t-1$ rectangles, to establish the proportion involving the ratio of the square of side 1 to the ascribed figure and the ratio of one power of t to a summation of $t-1$ powers.

In fact, like Newton in Lemma II of the *Principia* [Newton, 1972, 73–74], Mengoli might have stated that the ratios between the curvilinear, the inscribed and the circumscribed figures are ratios of equality. But it is evident that he needed the ascribed figure to be able to establish ratios with finite terms. For Mengoli the ascribed figure is a tool to clarify the nature of the curved figure, and furthermore to demonstrate in a general way results about the quasi ratio and the value of the quadrature.

80 He used Proposition 67 of *Elementum quintum*, which established ratios of quasi equality between two magnitudes that are situated between two quasi equals.

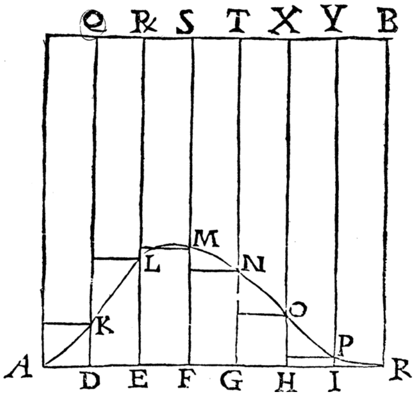

Fig. 10. Mengoli's figure in Proposition 8.

For the second quasi equality ratio involving the ascribed figure and the square of side 1, Mengoli used the ascribed figure that corresponds to the equation $y = \binom{m+n}{n}.(m+n+1).x^m.(1-x)^n$. He first established a proportion involving the ratio of the square of side 1 and the ascribed figure, and the ratio of a power of t to a summation of powers:

$$\frac{\text{Square (Side 1)}}{\text{Ascribed figure}} = \frac{t^{m+n+1}}{\binom{m+n}{n}} \frac{}{.(m+n+1)\sum_{a=1}^{a=t-1} a^m.(t-a)^n}.$$

He then applied the theory of quasi proportions to this proportion. He supposed implicitly that the proportion continues to hold when the number of rectangles on the left side is infinite and the number of addends on the right side is infinite. Since he knew from the theory of quasi proportions that the second ratio is a quasi equality ratio, it follows that the first ratio involving the square and the ascribed figure is also a quasi equality ratio.

We now look in more detail at this demonstration, which Mengoli gave in Proposition 8 for the curve corresponding to the expression $FO.10a^2r^3$ from the fifth row of the table of subquadratures, or, alternatively the expression $FO.6 \cdot 10a^2r^3$ from the fifth row of the table of quadratures (see Fig. 10). (As we noted above, the proof can be generalized to any entry in these tables.) He divided the base of the square in t parts and on these constructed the ordinates of the curved figure and of the square. He also constructed the rectangles of the ascribed figure and of the square of side 1. First, he established a proportion for each rectangle of the ascribed figure and of the square. Notice that as each rectangle has the same base, for each division the ratio of rectangles is the same as the ratio of ordinates. That is,

$$\text{Rectangle of the square (AQ): rectangle of the ascribed figure (AK)} = \text{DQ} : \text{DK};$$
$$\text{DQ} = \text{ordinate of the square}; \ \text{DK} = \text{ordinate of the figure}.$$

But the ordinate of the square is equal to the base of the square. He could then apply the proportion between the base of the square, that is, one, and the ordinate of the geometric figure.

In the case of the first element of the division we have

$$\text{DQ} : \text{DK} = (1:10) \cdot \big(1 : (1/t)\big)^2 \cdot \big(1 : (1-1/t)\big)^3 = 1 : \big[10 \cdot 1^2 \cdot (t-1)^3\big]/t^5 = t^5 : 10 \cdot 1^2 \cdot (t-1)^3.$$

But rectangle (square) = AQ and rectangle (ascribed) = AK, so that

$$\mathrm{AQ} : \mathrm{AK} = \mathrm{DQ} : \mathrm{DK} = t^5 : 10 \cdot 1^2 \cdot (t-1)^3,$$
$$\mathrm{AQ} : \mathrm{AK} = t^5 : 10 \cdot 1^2 \cdot (t-1)^3.$$

In the case of the second element of the division we have rectangle (square) : rectangle (ascribed) = 1 : $[10 \cdot 2^2 \cdot (t-2)^3]/t^5 = t^5 : 10 \cdot 2^2(t-2)^3$, or

$$\mathrm{DR} : \mathrm{DL} = t^5 : 10 \cdot 2^2(t-2)^3,$$

and so on.

On the one side, Mengoli added all the rectangles in the antecedent t rectangles to obtain the square, and added all the $t-1$ rectangles in the consequent to obtain the ascribed figure. On the other side, in the antecedent, adding t^5 he obtained t^6, and in the consequent he obtained a finite sum. This yielded

$$\frac{FO.u}{\text{Ascribed } FO.10a^2r^3} = \frac{t^6}{10.\sum_{a=1}^{a=t-1} a^2.(t-a)^3}.$$

Mengoli then, in Proposition 10, stated that "All quadratures on the same base are equal to each other"[81] and used in the demonstration the preceding proportion with both consequents multiplied by 6; that is,

$$\frac{FO.u}{\text{Ascribed } FO.6 \cdot 10.a^2r^3} = \frac{t^6}{6 \cdot 10.\sum_{a=1}^{a=t-1} a^2.(t-a)^3}.$$

Because the second ratio is a quasi equality (Theorem 42), the first ratio, involving the square of side 1 and the ascribed figure, is also a quasi equality ratio. Notice that the justification of this proportion is based on the identification of the algebraic expression and the geometric figure by means of a proportion between segments and quantities.

Following Antoni Malet's interpretation [Malet, 1996, 68–71], the proportion derived by Mengoli may be regarded as an attempt to justify the result obtained by Cavalieri's method of indivisibles. This proportion can be interpreted as equating a ratio between finite sums of ordinates to a ratio between figures. Mengoli could then apply the quasi proportions, and thus did not have to establish proportions between infinity as Cavalieri did, because he established finite ratios which "tend" to other ratios, that is to say, quasi ratios.

One of the weak points of this demonstration is the step from a ratio of quasi equality between summation of powers and powers (numbers) to a ratio between figures. But Mengoli had based the theory of quasi proportions on the Euclidean theory of proportions, so for him the former theory was valid for any magnitude, figure, or number. It should be emphasized that this demonstration does not depend on the degree and can be used in all cases where the quasi ratio of the summation of powers is known.

As we have noted above, after 1650 through the influence of Viète and above all Descartes, algebraic methods became increasingly accepted in geometry. Other mathematicians of the period—such as

[81] Theor. 6. Prop. 10. Omnes quadraturae super eadem basi constitutae, sunt inter se aequales [Mengoli, 1659, 389].

Fermat, Gilles Personne de Roberval (1602–1675), John Wallis (1616–1703), and Blaise Pascal (1623–1662)—also used these methods, in various different ways. They aimed, among other things, to calculate the result that today would be written $\lim \frac{1^p+\cdots+t^p}{t^{p+1}} = \frac{1}{p+1}$ for t tending to infinity. This would have allowed them to square the parabolas $y = x^p$, for p any positive integer.[82] It is obvious that Mengoli, like Roberval and Wallis, knew the result to be $\frac{1}{p+1}$. But the latter authors carried out the summations of powers and verified the resulting values only in a few cases. From these results they inferred the general rule and then applied it directly to the quadrature problem by taking limits of ratios between sums of ordinates and areas under curves. Mengoli, on the other hand, constructed the theory of quasi proportions to handle these limits, and moreover to provide a demonstration for the results that were so obtained. He did not apply the theory directly to mixed-line figures but made an intermediate step and used the ascribed figure, which is determined by a finite number of rectangles. Another difference is that Mengoli's contemporaries determined the areas under the curves case by case whereas he obtained countless quadratures at once.

3. Concluding remarks

Mengoli, like Viète, considered his algebra as a technique in which symbols are used to represent abstract magnitudes. He dealt with species, forms, triangular tables, quasi ratios, and logarithmic ratios. But the most innovative aspect of his work was his use of letters to work directly with the algebraic expression of the geometric figure. On the one hand, he expressed a figure by an algebraic expression, in which the ordinate of the curve that determines the figure is related to the abscissa by means of a proportion, thus establishing the Euclidean theory of proportions as a link between algebra and geometry. On the other hand, he showed how the algebraic expression could be used to construct geometrically the ordinate at any given point. This allowed him to study geometric figures via their algebraic expressions and to derive the known values for the areas of a large class of curves at once.

The triangular table of quadratures that Mengoli constructed in *Elementum sextum* could be extended indefinitely. He knew the values of these quadratures and looked for a rule that allowed him to associate any geometric figure to an algebraic expression. Putting these expressions in the table with the appropriate coefficients, the quadratures of the new curves were given. He classified the figures by the curves that determine them in three types and studied the properties of each group, again using the theory of proportions. When he demonstrated a given quadrature result, the proof was independent of the graphical representation of the geometric figure and could be used in all cases where the quasi ratio of the summation of powers was known.

It seems unlikely that Mengoli was familiar with Descartes' *Géométrie*. In terms of both aims and methods the differences between the two were substantial. Mengoli introduced algebra into geometry to solve problems of quadratures; Descartes wanted to solve and classify geometrical problems and he used algebra as a tool. Mengoli did not produce an algebra of segments, as Descartes did; that is to say, he did not give a geometrical interpretation of each of the algebraic operations that he defined. Furthermore, when he demonstrated an algebraic identity such as $(a + b)^2 = a^2 + 2ab + b^2$, he developed the proof

[82] Information on these subject may be found in the following sources: on Fermat, see Mahoney [1973, 230]; on Roberval see Auger [1962, 18–21] and Walker [1986, 41–44]; on Wallis [1972, 365–392]; and on Pascal see Boyer [1943, 240] and Pascal [1954, 171].

using the properties of proportions. His introduction of algebra into geometry bore more similarities with Viète's procedures. Viète also used the theory of proportions as a link, but he produced diagrams without using coordinates systems and he verified the constructions of the solutions of second-degree equations without assuming any connection between the ordinates and the abscissae.

When the relation between the ordinates and abscissae in a geometric curve is mentioned, we immediately think of Fermat and his *Introduction to plane and solid loci* of early 1636. Although Mengoli may have drawn his inspiration from Fermat, he only established this relation for certain geometric curves such as $y = k \cdot x^m \cdot (t - x)^n$; he did not claim to have found a general principle, as Fermat did in his *Isagoge* [Fermat, 1891–1922, Book 1 91]. Mengoli did not deal with solid problems, nor with problems of geometric loci, as had Fermat; what is more, his algebraic method cannot be applied to solve these other geometric problems.

Although Mengoli's contributions were a step forward in the process of algebraization of mathematics, his principal aim was not to demonstrate the equivalence of algebraic expressions and geometric figures, but rather to develop a new and fruitful algebraic method for solving quadrature problems. One should not forget that Mengoli wished to square the circle by interpolating these tables of quadratures. This investigation appeared in his later publication *Circolo* [1672] in which he studied quadratures of curves determined by equations today represented as $y^p = k \cdot x^m \cdot (1 - x)^n$. Mengoli emphasized that these quadratures had never been found before. Indeed, any attempt to calculate quadratures geometrically would have to be done case by case.

Our study of Mengoli's work reveals that the basis of his new method of quadratures was not Cavalieri's method of indivisibles, but the triangular tables and the theory of quasi proportions, set out as a development of Viète's algebra. In this way he created a numerical theory of summations of powers and products of powers and *limits* of these summations which was unrelated to Cavalieri's *Omnes lineae*. It is not clear why Mengoli did not follow his master's path; perhaps it was because Cavalieri's method had received a great deal of criticism, a fact that Mengoli could not ignore. After showing that he was familiar with the method of indivisibles and could apply this method, Mengoli claimed that his purpose was to give solid foundations for a new method of calculating quadratures. To this end he constructed the triangular tables of geometric figures and applied the theory of quasi proportions. Unlike Cavalieri, he never compared two figures through the comparison of lines, nor did he superimpose figures; rather, he established quasi ratios between geometric figures. But what does it mean to say that a geometric figure is quasi equal to another? Mengoli defined the ascribed, inscribed and circumscribed figures determined by rectangles built on the divisions of the base. He worked at all times with a finite number of divisions. He demonstrated that for any given ratio it is always possible to find a number of divisions of the base so that the ratio between the circumscribed and inscribed figures is nearer to equality than is the given ratio. He also demonstrated that as the number of divisions increases the ascribed figure is quasi equal to the mixed-line figure determined by the ordinates; that is to say, a geometric figure determined by rectangles approximates to a mixed-line figure arbitrarily closely when the number of rectangles increases indefinitely. To an extent, this first quasi equality recalls Archimedes' method.

Mengoli also established a second quasi equality using algebraic procedures. He established a proportion in which the first ratio is between a summation of powers and a power and the second between a unit square and the ascribed figure. The step from the geometric figure to its algebraic expression is essential in his demonstration. The Euclidean theory of proportions is once again the link between figure and expression. It allowed him to operate with segments and to establish ratios and quasi ratios to determine the quadratures of these curves.

The use of the two quasi equalities (the ascribed figure and the square as well as the ascribed and the mixed-line figure) allows us to understand better Mengoli's words when he states that his geometry is a "perfect conjunction" of the geometry of indivisibles, the geometry of Archimedes (method of exhaustion) and the algebra of Viète. Algebraic and geometric methods complement each other, allowing one to obtain new and better results. Mengoli developed Viète's symbolic language using his triangular tables and quasi proportions, thereby arriving at an original theory to investigate geometric figures and to determine their quadratures.

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