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COOPERATIVE BIRTH PROCESSES WITH LINEAR OR SUBLINEAR INTENSITY

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Let $\{Z_t\}$ be an increasing Markov process on $\mathbb{N}^n$ and $\{\sigma(k)\}$ the corresponding sequence of jump times. Let the increments of Z_t be i.i.d. with finite expectation and covariances, and let

$$E(\sigma(k+1)-\sigma(k)|Z_0, Z_{\sigma(1)}, \ldots, Z_{\sigma(k)}) = \left(h\left(\frac{Z_{\sigma(k)}}{|Z_{\sigma(k)}|}\right)f(|Z_{\sigma(k)}|)\right)^{-1},$$

where h and f are sufficiently smooth positive functions and $|Z_t| = \sum_{j=1}^{n} Z_t(j)$, $Z_t = (Z_t(1), \ldots, Z_t(n))$. While a linear f results in asymptotically exponential growth, a suitable class of sublinear f leads to a growth asymptotically at most that of a power. Covering both cases, we obtain analoga of the strong LLN, the CLT and the LIL.

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1. Introduction

We prove limit theorems for a Markovian multi-type birth process with identically distributed offsprings in which the occurrence of births is in a general way state-dependent, admitting cooperation of particles in the production of new particles as well as nonexponential asymptotic growth. Both these features appear to be more realistic than the independent branching and asymptotically exponential growth in ordinary branching models. The problem of cooperation in Markovian multi-type processes, in case of discrete time, has been treated by Kesten [6, 7, 8]. He has given stability results for the type distribution and strong laws for asymptotic behaviour (cf. his references for further deterministic or stochastic models considering sex interaction). Limit theorems for Markovian birth and death processes with state-dependent transition probabilities, admitting nonexponential growth, have been studied by Barbour [5] whose proofs are based essentially on the property of 'skip free upwards'. Our starting point has been the two-type model with exponential growth studied by Asmussen [3]. The methods of proofs correspond to those of ordinary one-dimensional Markov branching processes (Athreya and Karlin [4]). We obtain the statements about asymptotic growth of the Markov process by studying the limiting behaviour of the embedded Markov chain and the split time process.

2. Definition of the model

Let $\{Z_t, t \in \mathbb{R}^+\}$ be an increasing Markov process on $\mathbb{N}^n$ with jump times $\sigma(k) < \sigma(k+1)$, $k \in \mathbb{N}$, such that the increments

$$\Delta_k = Z_{\sigma(k)} - Z_{\sigma(k-1)}$$

are i.i.d. with expectation $\mu \neq 0$ and finite covariance matrix C. Then the conditional distribution of

$$\delta_k = \sigma(k+1) - \sigma(k)$$

given $Z_0, Z_{\sigma(1)}, \ldots, Z_{\sigma(k)}$ is exponential with parameter λ_k,

$$\lambda_k^{-1} = E(\delta_k | Z_0, Z_{\sigma(1)}, \ldots, Z_{\sigma(k)}),$$

and the $\{\lambda_k \delta_k\}$ are independent. In the case of independent branching with intensities $\lambda(j)$, $j = 1, \ldots, n$, we simply have

$$\lambda_k = \sum_{j=1}^{n} \lambda(j) Z_{\sigma(k)}(j),$$

$Z_t = (Z_t(1), \ldots, Z_t(n))$. Here we admit cooperation of particles in the production of new particles by assuming

$$\lambda_k = h(\tilde{Z}_{\sigma(k)}) f(|Z_{\sigma(k)}|),$$

where

$$\tilde{Z}_t = Z_t / |Z_t|, \qquad |x| = \sum_{i=1}^{n} x(i), \quad x = (x(1), \ldots, x(n)) \in \mathbb{R}^n,$$

h and f positive functions. More precisely, we assume throughout that

$$h(x) > 0, \quad x \in (0, 1)^n,$$

and for some $c < \infty$ and $0 < p \leq 1$

$$\|h(x) - h(y)\| \leq c \|x - y\|^p, \quad x, y \in (0, 1)^n$$

where $\|\cdot\|$ denotes the usual norm in l_2. Note that $|x|$, $x \in \mathbb{R}^n$, as defined above is not a norm, $|x| \in \mathbb{R}$.

Concerning f we distinguish two cases:

Model 1: $f(x) = x$.

Examples

(a) $\lambda_k = \lambda \min_{1 \leq i \leq n} Z_{\sigma(k)}(i), \quad \lambda > 0,$

(b) $\lambda_k = \sum_{i=1}^{n} \lambda(i) Z_{\sigma(k)}(i), \quad \lambda(i) \geq 0.$

(c) $\lambda_k = \lambda \prod_{i=1}^{n} Z_{\sigma(k)}(i)^{\beta(i)}, \quad \lambda > 0, \beta(i) \geq 0, \sum_{i=1}^{n} \beta(i) = 1.$

Model 1 is a direct extension of Asmussen [3] who treated the case $n = 2, |\Delta_k| = 1$; the case $n = 2$ being the one of most importance for modelling the growth of two-sex populations. On the other hand, it is useful to regard the case of $n > 2$ – e.g., one can imagine different types of females and males (fertilities) who influence the growth rate of the process. Certainly, this is not a model for populations whose characteristics are hereditary as we have assumed identical offspring distributions.

Model 2: $f(x)$ is positive for all $x > 0$ and, for $x \geq x_0$ sufficiently large, $f(x)$ is increasing in such a way that, for some $c \in \mathbb{R}^+$ and all $y > 0$,

$$1 \leq f(x+y)/f(x) \leq (x+cy)/x$$

and $g(x)f(x)/x$ is uniformly bounded, where

$$g(x) = \lambda^{-1} \int_1^x f(\tau)^{-1} \, d\tau, \quad \lambda = h(\mu/|\mu|)|\mu|.$$

Examples. Replace $Z_{\sigma(k)}(i)$ in (a) and (b) by $Z_{\sigma(k)}^{1-\delta}(i)$, $\delta \in (0, 1]$. Assume in (c) $\sum_{i=1}^{n} \beta(i) < 1$.

Next we shall prove a sufficient condition for the second assumption on f.

Proposition. *If $f(x) > 0$, and there is $c \in [0, 1)$ such that for $y > 0$, $x \geq x_0$*

$$1 \leq f(x+y)/f(x) \leq (x+cy)/x$$

then $g(x)f(x)/x$ is uniformly bounded.

Proof. For $y > 0$, $x \geq x_0$,

$$g(x+y)f(x+y)/(x+y) \leq g(x)f(x)x^{-1}(x+cy)(x+y)^{-1} \cdot \left(1+g(x)^{-1}\lambda^{-1}\int_x^{x+y} f(\tau)^{-1}\,\mathrm{d}\tau\right)$$
$$\leq g(x)f(x)x^{-1}(x+cy)(x+y)^{-1} + y/\lambda x.$$

Hence, $g(x)f(x)/x$ is a decreasing function once it is above $1/\lambda(1-c)$. As $c < 1$ it must be a bounded function.

Pure birth processes Z_t with increasing functions $f(j) = E(\delta_j)^{-1}$, $f(j) \to \infty$, have been treated by Waugh [10]. $\{\delta_j, j \in \mathbb{N}\}$ were allowed to be dependent. He distinguished three cases: see Table 1.

Table 1

	$\sum_{j=1}^{n} f(j)^{-1}$	$\sum_{j=1}^{n} \delta_j - f(j)^{-1}$
$\bar{H}$	convergent	convergent (a.s.)
H_c	divergent	convergent (a.s.)
H_d	divergent	divergent (a.s.)

Processes of class H_c, H_d have no explosive growth. Our models contain pure birth processes as special cases. They belong to H_c, H_d as $f(x) = \mathrm{O}(x)$. μ and $h(x)$ bounded establishes non-explosive growth also in the general case, i.e.. $\sigma(n) \to \infty$ a.s. While in case H_c Waugh has proved a general connection between the growth of $\sum_{j=1}^{n} f(j)^{-1}$ and Z_t he obtained some results for asymptotic behaviour of Z_t in H_d only if a law of iterated logarithm for $\sum_{j=1}^{n} \delta_j - f(j)^{-1}$ was valid. Waugh [11] and Lenz [9] have confined themselves to type H_c when they proved strong laws of large numbers for some special Markov processes with denumerable state space and nonlinear intensity function $f(j) = Aj^{\alpha}$, $\frac{1}{2} < \alpha \leq 1$.

For Model 1 and 2 we shall prove strong laws of large numbers. While the first will result in asymptotically exponential growth – as did ordinary independent branching – the assumption of the second model will lead to a growth asymptotically at most that of a power of t. If the function $f(x)$ is only slightly different from a power of x we obtain analoga of the central limit theorem and the law of iterated logarithm.

3. Almost sure limiting behaviour

Since the only difference between Model 1 and the multitype Markov branching processes (which have the same offspring distributions for all types) is the dependence of δ_k on a more general function $h(\tilde{Z}_{\sigma(k)})$ one would expect the type distribution to stabilize and the population size to grow exponentially. The situation in Model 2 should be similar except for slower growth.

Theorem 1. *Under the conditions of Model* 1, $Z_0 \neq 0$, *there exists a random variable* W *such that* $0 < W < \infty$ *and, as* $t \to \infty$,

$$\tilde{Z}_t = z + o(1) \quad a.s.,$$

$$|Z_t| = W\,e^{\lambda t} + o(e^{\lambda t}) \quad a.s.,$$

where $z = \mu/|\mu|$ *and* $\lambda = h(z)|\mu|$, *while under the conditions of Model* 2

$$\tilde{Z}_t = z + o(1) \quad a.s.,$$

$$|Z_t| = w(t)(1 + o(1)) \quad a.s.$$

where $w(t)$ *is the inverse function of*

$$g(t) = \lambda^{-1} \int_1^t f(\tau)^{-1}\,d\tau.$$

Proof. Let $M = |\mu|$. As $Z_{\sigma(k)} = Z_0 + \sum_{j=1}^{k} \Delta_j$ is a sum of i.i.d. random variables we can use the law of iterated logarithm and obtain

$$|Z_{\sigma(k)}| = Mk + o(k^{1/2+\varepsilon}), \qquad \tilde{Z}_{\sigma(k)} = z + o(k^{-1/2+\varepsilon}), \tag{3.1}$$

and thus $\tilde{Z}_t = z + o(1)$.

To get the rate of growth write

$$\sigma(k+1) = \sum_{j=0}^{k} (\delta_j - \lambda_j^{-1}) + \sum_{j=0}^{k} \lambda_j^{-1}. \tag{3.2}$$

In case of Model 1 we can argue similarly to [3] and obtain

$$\sigma(k+1) = \log|Z_{\sigma(k)}| - \log W + o(1) \tag{3.3}$$

which implies the first statement of the theorem.

Only the conditional means λ_k^{-1} are different in Model 2, and they affect only the split times $\sigma(k)$. By (3.1) and the properties of f

$$f(|Z_{\sigma(j)}|)/f(Mj) = 1 + o(j^{-1/2+\varepsilon}),$$

and

$$\lambda_j^{-1} = (h(z)f(Mj))^{-1} + o(j^{-p(1/2-\varepsilon)})/f(Mj). \tag{3.4}$$

Thus,

$$\left(\sum_{j=1}^{k} \lambda_j^{-1}\right)^{-1} \leq h(z)(f(Mk) + \max\{f(x)\,|\,x \leq x_0\})\,k^{-1}(1+o(1)),$$

and as

$$\sum_{j=1}^{\infty} (\delta_j - \lambda_j^{-1}) f(Mj) j^{-1} = \sum_{j=1}^{\infty} (\delta_j\lambda_j - 1) j^{-1}(h(z)^{-1} + o(1))$$

converges, Kronecker's lemma implies that

$$\left(\sum_{j=0}^{k} \lambda_j^{-1}\right)^{-1}\left(\sum_{j=0}^{k} \delta_j - \lambda_j^{-1}\right) \to 0 \quad \text{a.s.}$$

Combining this with (3.2) and (3.4),

$$\sigma(k+1) = \left(\sum_{j=0}^{k} h(z) f(Mj)^{-1}\right)(1+\mathrm{o}(1)),$$

where we can replace the summation by integration because g is a divergent function. Hence,

$$\sigma(k+1) = g(Mk)(1+\mathrm{o}(1)) = g(|Z_{\sigma(k)}|(1+\mathrm{o}(1)))(1+\mathrm{o}(1)),$$

and, by $\sigma(k+1) - \sigma(k) = \delta_k = \mathrm{o}(f(Mk)^{-1} \log k)$,

$$t = g(|Z_t|(1+\mathrm{o}(1)))(1+\mathrm{o}(1)).$$

Therefore, the theorem will be proved if we can show that for every zero convergent function $\gamma(t) = \mathrm{o}(1)$ the function $\tilde{\gamma}(t)$ with $g(t)(1+\gamma(t)) = g(t(1+\tilde{\gamma}(t)))$ converges to zero, too. $\gamma(t) \geq 0$ is equivalent to $\tilde{\gamma}(t) \geq 0$ and by monotonicity

$$\|g(t(1+\tilde{\gamma}(t))) - g(t)\| \geq \begin{cases} \|t\tilde{\gamma}(t)/f(t)\lambda\|, & \tilde{\gamma}(t) \leq 0, \\ t\tilde{\gamma}(t)/f(t)(1+c\tilde{\gamma}(t))\lambda, & \tilde{\gamma}(t) \geq 0. \end{cases}$$

As $g(t)f(t)/t$ is bounded the inequalities imply $\tilde{\gamma}(t) = \mathrm{o}(1)$. Hence, we can write

$$t = g(|Z_t|(1+\mathrm{o}(1))). \tag{3.5}$$

We get the last statement of the theorem by inversion.

4. Finer limit theorems

To get more precise results on the limiting behaviour of $\tilde{Z}_t - z$ and $|Z_t| - w(t)$ we need more regularity for h and f. We assume in addition that h is differentiable at z and can be expanded in the form

$$h(x) = h(z) + (x-z) \cdot \operatorname{grad} h(z) + \mathrm{O}(\|x-z\|^2).$$

Furthermore, let f be a positive function with second derivative such that, for some $\alpha \in (0, 1]$, $\delta > 0$, $A > 0$,

$$f(t) = At^{\alpha}(1+\mathrm{o}(t^{-\delta})), \qquad f'(t) = \mathrm{O}(t^{\alpha-1/2-\delta}), \qquad f''(t) = \mathrm{O}(t^{\alpha-1-\delta}).$$

Theorem 2. *Let the above assumption be satisfied and $Z_0 \neq 0$. Then*

$$\tilde{Z}_t = z + w(t - I(\alpha = 1)\Gamma)^{-1/2} A_t,$$

$$|Z_t| = \begin{cases} w(t - I(\alpha > \frac{1}{2})\Gamma) + w(t - I(\alpha = 1)\Gamma)^{1/2} B_t, & \alpha \neq \frac{1}{2}, \\ w(t) + (w(t) \log w(t))^{1/2} B_t, & \alpha = \frac{1}{2}, \end{cases}$$

where Γ is a random variable, $I(\cdot)$ the indicator function and $w(t)$ the inverse function of $g(t)$. The limiting distribution of (A_n, B_t) exists and is the $(n+1)$-dimensional normal distribution with mean zero and covariance matrix D. With the notation

$$B=(b_{ij})_{1\le i,j\le n}, \qquad b_{ij}=I(i=j)-z(i), \quad z=(z(1),\dots,z(n)),$$

$$h=\alpha h(z)^{-1}\operatorname{grad} h(z), \qquad e=(1,1,\dots,1)\in\mathbb{R}^n$$

matrix D is given by

$$(d_{ij})_{1\le i,j\le n}=|\mu|^{-1}BCB^{\mathrm{T}},$$

$$(d_{i,n+1})_{1\le i\le n}=|\mu|^{-1}(-I(\alpha\ne\tfrac12)hBCB^{\mathrm{T}}+I(\alpha<\tfrac12)(1-\alpha)^{-1}(e+hB)CB^{\mathrm{T}}),$$

$$\begin{aligned}d_{n+1,n+1}&=|\mu|(I(\alpha=\tfrac12)+I(\alpha\ne\tfrac12)|2\alpha-1|^{-1/2})\\&\quad+|\mu|^{-1}((I(\alpha=\tfrac12)+I(\alpha\ne\tfrac12)|2\alpha-1|^{-1/2})(e+hB)C(e+hB)^{\mathrm{T}}\\&\quad+I(\alpha\ne\tfrac12)hBCB^{\mathrm{T}}h^{\mathrm{T}}-I(\alpha<\tfrac12)2(1-\alpha)^{-1/2}hBC(e+hB)^{\mathrm{T}}),\end{aligned}$$

where C is the covariance matrix of Δ_1. Superscript T denotes the transposed of a matrix.

As we have assumed $f(t)=At^{\alpha}(1+o(t^{-\delta}))$ we can describe $w(t)$ in a more explicit form,

$$w(t)=\begin{cases}\exp(\lambda At(1+o(t^{-\delta}))), & \text{as } \alpha=1,\\ (\lambda A(1-\alpha)t)^{1/(1-\alpha)}(1+o(t^{-\delta})), & \text{as } 0<\alpha<1.\end{cases}$$

If $\delta>\frac12$ we can omit the terms $o(t^{-\delta})$ (see (4.4)). In case of $f(t)=t$ the time-lag Γ corresponds to the factor W in Theorem 1, $W=e^{-\lambda\Gamma}$.

Proof. For the more precise results in Theorem 2 we need sharper estimates for the remainder terms in (3.3), respectively (3.5). The notation used will indicate that Γ_i are finite random variables adding up to Γ, Δ_k^i are normed sums of i.i.d. random variables. As

$$\begin{aligned}|Z_{\sigma(k)}|^{-1}&=(Mk)^{-1}\left(1+(Mk)^{-1}\sum_{j=1}^{k}|\Delta_j-\mu|\right)^{-1}-|Z_0|(Mk)^{-2}(1+o(1))\\&=(Mk)^{-1}\left(1-(Mk)^{-1}\sum_{j=1}^{k}|\Delta_j-\mu|+o(k^{-1+\varepsilon})\right)\end{aligned}$$

we can write

$$\tilde{Z}_{\sigma(k)}=z+(Mk)^{-1}\sum_{j=1}^{k}(\Delta_j-\mu)-z|\Delta_j-\mu|+o(k^{-1+\varepsilon}). \tag{4.1}$$

Hence,

$$h(\tilde{Z}_{\sigma(k)})=h(z)+(Mk)^{-1}\left(\sum_{j=1}^{k}(\Delta_j-\mu)-z|\Delta_j-\mu|\right)\cdot\operatorname{grad} h(z)+o(k^{-1+\varepsilon})$$

and by Taylor's theorem and conditions on the derivatives of f, (3.4) can be replaced by

$$\lambda_j^{-1}=(h(z)f(Mj))^{-1}+\sum_{k=1}^{j}|\Delta_k-\mu|\frac{\mathrm{d}}{\mathrm{d}(Mj)}(h(z)f(Mj))^{-1}$$
$$-h(z)^{-2}(Mjf(Mj))^{-1}\left(\sum_{k=1}^{j}(\Delta_k-\mu)-z|\Delta_k-\mu|\right)\cdot\operatorname{grad}h(z)$$
$$+\sum_{j=1}^{k}o(j^{-1/2-\alpha-\delta+\varepsilon}).$$

Summing up, we get

$$\sum_{j=1}^{k}\lambda_j^{-1}=\sum_{j=1}^{k}(h(z)f(Mj))^{-1}+\sum_{j=1}^{k}|\Delta_j-\mu|h(z)^{-1}\sum_{l=j}^{k}\frac{\mathrm{d}}{\mathrm{d}(Ml)}f(Ml)^{-1}$$
$$-h(z)^{-2}(Mjf(Mj))^{-1}\left(\sum_{k=1}^{j}(\Delta_k-\mu)-z|\Delta_k-\mu|\right)\cdot\operatorname{grad}h(z)$$
$$+\sum_{j=1}^{k}o(j^{-1/2-\alpha-\delta+\varepsilon}).\tag{4.2}$$

Using $f(x)=Ax^{\alpha}(1+(x^{-\delta}))$ we can replace the summation by integration,

$$\sum_{l=j}^{k}\frac{\mathrm{d}}{\mathrm{d}(Ml)}f(Ml)^{-1}=M^{-1}(f(Mk)^{-1}-f(Mj)^{-1})+O(j^{-1/2-\alpha-\delta}),$$

$$\sum_{l=j}^{k}(Mlf(Ml))^{-1}=-\alpha^{-1}M^{-1}(f(Mk)^{-1}-f(Mj)^{-1})+O(j^{-\alpha-\delta}),$$

and

$$\sum_{j=1}^{k}(h(z)f(Mj))^{-1}=g(Mk)+d+O(f(Mk)^{-1}),$$

where

$$d=\lim_{k\to\infty}\left(\sum_{j=1}^{k}(h(z)f(Mj))^{-1}-\lambda^{-1}\int_1^{Mk}f(\tau)^{-1}\,\mathrm{d}\tau\right)$$

exists. By the strong law of large numbers for independent random variables

$$\sum_{j=1}^{k}(\Delta_j-\mu)O(j^{-\alpha-\delta})=\begin{cases}o(k^{1/2-\alpha}), & \text{as } \alpha<\tfrac{1}{2},\\ O(1), & \text{as } \alpha=\tfrac{1}{2},\end{cases}$$

$$\sum_{j=k+1}^{\infty}(\Delta_j-\mu)O(j^{-\alpha-\delta})=o(k^{1/2-\alpha}),\quad \text{as } \alpha>\tfrac{1}{2}.$$

Use Lemma 2 in [2] to get the rate of convergence of tail sums. Combining these relations with (4.2) it follows after some easy calculations that

$$\sum_{j=1}^{k} \lambda_j^{-1} = \begin{cases} g(Mk)+(Mk)^{1/2}(\lambda f(Mk))^{-1}(\Delta_k^1+\Delta_k^2+\mathrm{o}(1)), & \alpha<\frac{1}{2}, \\ g(Mk)+(Mk\log Mk)^{1/2}(\lambda f(Mk))^{-1}(\Delta_k^2+\mathrm{o}(1)), & \alpha=\frac{1}{2}, \\ g(Mk)+\Gamma_1+(Mk)^{1/2}(\lambda f(Mk))^{-1}(\Delta_k^1+\Delta_k^2+\mathrm{o}(1)), & \alpha>\frac{1}{2}, \end{cases} \tag{4.3}$$

where

$$\Delta_k^1 = (Mk)^{-1/2} \sum_{j=1}^{k} (\Delta_j-\mu)(e+hB)^{\mathrm{T}},$$

$$\Delta_k^2 = \begin{cases} -f(Mk)(Mk)^{-1/2} \sum_{j=1}^{k} f(Mj)^{-1}(\Delta_j-\mu)(e+hB)^{\mathrm{T}}, & \alpha<\frac{1}{2}, \\ -f(Mk)(Mk\log Mk)^{-1/2} \sum_{j=1}^{k} f(Mj)^{-1}(\Delta_j-\mu)(e+hB)^{\mathrm{T}}, & \alpha=\frac{1}{2}, \\ f(Mk)(Mk)^{-1/2} \sum_{j=k+1}^{\infty} f(Mj)^{-1}(\Delta_j-\mu)(e+hB)^{\mathrm{T}}, & \alpha>\frac{1}{2}, \end{cases}$$

$$\Gamma_1 = I(\alpha>\tfrac{1}{2})\left(d-\sum_{j=1}^{\infty} \lambda^{-1} f(Mj)^{-1}(\Delta_j-\mu)(e+hB)^{\mathrm{T}} - \sum_{j=1}^{\infty} \mathrm{o}(j^{-\alpha-\delta})\right),$$

with e, h, B as defined in Theorem 2. We have used the relation

$$(\Delta_j-\mu)(e+hB)^{\mathrm{T}} = |\Delta_j-\mu| + \alpha h(z)^{-1}(\Delta_j-\mu-z|\Delta_j-\mu|)\cdot \operatorname{grad} h(z).$$

To find an expression for $\sum_{j=1}^{k}(\delta_j-\lambda_j^{-1})$ define

$$\Delta_k^3 = \begin{cases} Mf(Mk)(Mk)^{-1/2} \sum_{j=1}^{k} f(Mj)^{-1}(\delta_j\lambda_j-1), & \alpha<\frac{1}{2}, \\ Mf(Mk)(Mk\log Mk)^{-1/2} \sum_{j=1}^{k} f(Mj)^{-1}(\delta_j\lambda_j-1), & \alpha=\frac{1}{2}, \\ -Mf(Mk)(Mk)^{-1/2} \sum_{j=k+1}^{\infty} f(Mj)^{-1}(\delta_j\lambda_j-1), & \alpha>\frac{1}{2}. \end{cases}$$

As $\lambda_j^{-1} = M(\lambda f(Mj))^{-1} + \mathrm{O}(j^{-\alpha-\delta+\varepsilon})$ we can write

$$\sum_{j=1}^{k} (\delta_j-\lambda_j^{-1}) = \begin{cases} (\lambda f(Mk))^{-1}(Mk)^{1/2}(\Delta_k^3+\mathrm{o}(1)), & \alpha<\frac{1}{2}, \\ (\lambda f(Mk))^{-1}(Mk\log Mk))^{1/2}(\Delta_k^3+\mathrm{o}(1)), & \alpha=\frac{1}{2}, \\ \Gamma_2+(\lambda f(Mk))^{-1}(Mk)^{1/2}(\Delta_k^3+\mathrm{o}(1)), & \alpha>\frac{1}{2}, \end{cases}$$

where

$$\Gamma_2 = I(\alpha>\tfrac{1}{2}) \sum_{j=1}^{\infty} (\lambda_j\delta_j-1) f(Z_{\sigma(j)})^{-1}.$$

Thus, putting $\Gamma = \Gamma_1 + \Gamma_2$,

$$\sigma(k+1) = \begin{cases} g(Mk) + (\lambda f(Mk))^{-1}(Mk)^{1/2}(\Delta_k^1 + \Delta_k^2 + \Delta_k^3 + o(1)), & \alpha < \frac{1}{2}, \\ g(Mk) + (\lambda f(Mk))^{-1}(Mk \log Mk)^{1/2}(\Delta_k^2 + \Delta_k^3 + o(1)), & \alpha = \frac{1}{2}, \\ g(Mk) + \Gamma + (\lambda f(Mk))^{-1}(Mk)^{1/2}(\Delta_k^1 + \Delta_k^2 + \Delta_k^3 + o(1)), & \alpha > \frac{1}{2}. \end{cases} \quad (4.4)$$

Obviously, $\sigma(k+1)$ may be replaced by $\sigma(k)$. Now put $\Gamma = 0$ for $\alpha \leq \frac{1}{2}$, and by Taylor's formula, use $w'(x) = \lambda f(w(x))$, there is a random variable θ_k in $[0, 1]$ with

$$\begin{aligned} Mk = w(g(Mk)) &= w(\sigma(k) - \Gamma) \\ &\quad + (g(Mk) - \sigma(k) + \Gamma)\lambda f(w(g(Mk) + \theta_k(\sigma(k) - \Gamma - g(Mk)))) \\ &= w(\sigma(k) - \Gamma) + (g(Mk) - \sigma(k) + \Gamma)\lambda f(Mk)(1 + o(1)). \end{aligned} \quad (4.5)$$

Note that $f(x)$ and, for $\alpha < 1$, $w(x)$ are almost powers of x. Such functions satisfy

$$f(x(1 + o(1))) = f(x)(1 + o(1)).$$

For $\alpha = 1$, w is almost exponential function satisfying

$$w(x + o(1)) = w(x)(1 + o(1)).$$

Considering (4.4), (4.5) and

$$w(\sigma(k) - \Gamma) = w(\sigma(k)) - \Gamma\lambda f(w(\sigma(k))(1 + o(1)))$$

for $\alpha < 1$ one obtains

$$Mk = w(\sigma(k) - \Gamma) + o(k^{1/2+\varepsilon}) = w(\sigma(k) - I(\alpha = 1)\Gamma)(1 + o(k^{-\varepsilon})).$$

Now replace the l.h.s. of (4.5) by

$$|Z_{\sigma(k)}| = Mk + (Mk)^{1/2}\Delta_k^4,$$

$$\Delta_k^4 = (Mk)^{-1/2}\sum_{j=1}^{k}|\Delta_j - \mu| = (Mk)^{-1/2}\sum_{j=1}^{k}(\Delta_j - \mu)e^{\mathrm{T}}.$$

Using the above relations, we then obtain

$$|Z_{\sigma(k)}| = \begin{cases} w(\sigma(k)) + w(\sigma(k))^{1/2}(-\Delta_k^1 - \Delta_k^2 - \Delta_k^3 + \Delta_k^4 + o(1)), & \alpha < \frac{1}{2}, \\ w(\sigma(k)) + (w(\sigma(k)) \log w(\sigma(k)))^{1/2}(-\Delta_k^2 - \Delta_k^3 + o(1)), & \alpha = \frac{1}{2}, \\ w(\sigma(k) - \Gamma) + w(\sigma(k) - I(\alpha = 1)\Gamma)^{1/2}(-\Delta_k^1 - \Delta_k^2 - \Delta_k^3 + \Delta_k^4 + o(1)), & \\ & \alpha > \frac{1}{2}. \end{cases}$$

By (4.1),

$$\hat{Z}_{\sigma(k)} = z + (w(\sigma(k)) - I(\alpha = 1)\Gamma)^{-1/2}(\Delta_k^5 + o(1))$$

where

$$\Delta_k^5 = (Mk)^{-1/2}\sum_{j=1}^{k}(\Delta_j - \mu - z|\Delta_j - \mu|) = (Mk)^{-1/2}\sum_{j=1}^{k}(\Delta_j - \mu)B^{\mathrm{T}}.$$

The proposed expressions for $|Z_t|$ and $\hat{Z}_t$ follow by defining

$$A_t = \Delta_k^5 + \mathrm{o}(1)$$

$$B_t = \begin{cases} -\Delta_k^2 - \Delta_k^3 + \mathrm{o}(1), & \alpha = \frac{1}{2}, \\ -\Delta_k^1 - \Delta_k^2 - \Delta_k^3 + \Delta_k^4 + \mathrm{o}(1), & \alpha \neq \frac{1}{2}, \end{cases}$$

for $\sigma(k) \leq t < \sigma(k+1)$.

It remains to prove normality for the limiting d.f. of (A_t, B_t). For this it suffices to show that $C_t = a(A_t, B_t)$ is normally distributed with mean zero and variance aDa^{T} for all $a \in \mathbb{R}^{n+1}\backslash\{0\}$. The Δ_k^i have similar structures, therefore we can find i.i.d. random variables (Y_j^1, Y_j^2), $j \in \mathbb{N}$, such that

$$C_{\sigma(k)} = \begin{cases} k^{-1/2} \sum_{j=1}^{k} (Y_j^1 + f(Mk)f(Mj)^{-1} Y_j^2) + \mathrm{o}(1), & \alpha < \frac{1}{2}, \\ k^{-1/2} \sum_{j=1}^{k} (Y_j^1 + (\log Mk)^{-1/2} f(Mk)f(Mj)^{-1} Y_j^2) + \mathrm{o}(1), & \alpha = \frac{1}{2}, \\ k^{-1/2}\left(\sum_{j=1}^{k} Y_j^1 + f(Mk) \sum_{j=k+1}^{\infty} f(Mj)^{-1} Y_j^2 \right) + \mathrm{o}(1), & \alpha > \frac{1}{2}. \end{cases}$$

As $f(Mk) = A(Mk)^{\alpha}(1 + \mathrm{o}(k^{-\delta}))$ and the variance of Y_1^2 exists we can replace $f(Mk)/f(Mj)$ by $(k/j)^{\alpha}$. We can substitute $(\log Mk)^{-1/2}$ by $(\log k)^{-1/2}$ as

$$\sum_{j=1}^{k} \operatorname{Var}(Y_j^2) j^{-1} (\log j)^{-3}$$

is uniformly bounded.

The central limit theorem for $C_{\sigma(k)}$ follows now easily by Lindeberg's criterion. Asymptotic normality still holds true for $C_t = C_{\sigma(n(t))}$, $n(t)$ are random indices. If $\alpha < 1$, $n(t)/w(t) \to 1$ a.s. As

$$P(\sup\{\|C_n - C_{n'}\| \mid (1-c)n \leq n' \leq (1+c)n\} < \varepsilon) > 1 - \eta$$

for all ε, $\eta > 0$, $n > n(\varepsilon, \eta)$, $c = c(\varepsilon, \eta) > 0$ (use Levy's inequality) we can apply Anscombe's theorem [1]. If $\alpha = 1$ $n(t)\,\mathrm{e}^{-AAt} \to W$ a.s. In this case we can prove the convergence in distribution of C_t similar to Asmussen [3].

We obtain the proposed expression of the covariance matrix D of the limiting distribution of (A_t, B_t) by straightforward calculations using the definition of the $\{\Delta_k^i\}$.

Besides the central limit theorem the random variables $\{C_t\}$ also satisfy some kind of law of iterated logarithm.

Theorem 3. *Let the conditions of Theorem* 2 *be satisfied. For every* $a = (a(1), \ldots, a(n+1)) \in \mathbb{R}^{n+1}\backslash\{0\}$ *define*

$$C_t = (A_t, B_t)a^{\mathrm{T}}, \qquad \sigma^2 = aDa^{\mathrm{T}},$$
$$\bar{\sigma}^2 = (a(1), \ldots, a(n), 0)D(a(1), \ldots, a(n), 0)^{\mathrm{T}};$$

$\log_i t = \log_{i-1} \log t$, $\log_0 t = t$ *denotes the iterates of logarithm. Then*

$$\limsup_{t\to\infty} C_t(2\sigma^2 \log_i t)^{-1/2} = 1 \quad a.s.,$$

$$\liminf_{t\to\infty} C_t(2\sigma^2 \log_i t)^{-1/2} = -1 \quad a.s.,$$

$i = 1$ *as* $\alpha = 1$, $i = 2$ *otherwise,*
if $\alpha \neq \frac{1}{2}$, $\sigma^2 \neq 0$;

$$\limsup_{t\to\infty} C_t(2\bar{\sigma}^2 \log_2 t)^{-1/2} = 1 \quad a.s.,$$

$$\liminf_{t\to\infty} C_t(2\bar{\sigma}^2 \log_2 t)^{-1/2} = -1 \quad a.s.,$$

if $\alpha = \frac{1}{2}$, $\bar{\sigma}^2 \neq 0$;

$$\limsup_{t\to\infty} C_t(2\sigma^2 \log_3 t)^{-1/2} = 1 \quad a.s.$$

$$\liminf_{t\to\infty} C_t(2\sigma^2 \log_3 t)^{-1/2} = -1 \quad a.s.,$$

if $\alpha = \frac{1}{2}$, $\bar{\sigma}^2 = 0$.

We omit the proof of Theorem 3 as it goes along the line of [3] with some natural modifications. For example, we have to change the $\{Y_j^i\}$ into truncated variables and have to use a special version of Berry Esseen's theorem for sums of centered variables with variance instead of the ordinary theorem under third moment assumption. If $\alpha = \frac{1}{2}$ the limiting behaviour of $\{C_{\sigma(n)}\}$ splits into two cases because the sums of the $\{Y_j^1\}$ obey (for $Y_j^1 \neq 0$) an ordinary law of iterated logarithm for i.i.d. random variables while the fluctuation of the sums with terms $\{Y_j^2\}$ is (for $Y_j^2 \neq 0$) of order $\log_3 k$.

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