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Syzygies of modules with positive codimension

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Abstract

A higher syzygy of a module with positive codimension is a maximal Cohen–Macaulay module that plays an important role in Cohen–Macaulay approximation over Gorenstein rings. We show that every maximal Cohen–Macaulay module is a higher syzygy of some positive codimensional module if and only if the ring is an integral domain. Also we discuss the hierarchy of rings with respect to Cohen–Macaulay approximation by codimensions of modules.

Keywords: Cohen–Macaulay approximation; Cohen–Macaulay module

1. Introduction

Let $(R, \mathbf{m}, k)$ be a complete Gorenstein local ring of $\dim R = d$. Auslander and Buchweitz introduced the notion of Cohen–Macaulay approximation.

Theorem 1.1. *(See Auslander and Buchweitz [2].) Every finite R-module M has exact sequences*

$$0 \to Y_M \to X_M \xrightarrow{\xi_M} M \to 0 \quad (\textit{Cohen–Macaulay approximation}),$$
$$0 \to M \xrightarrow{\eta_M} Y^M \to X^M \to 0 \quad (\textit{finite projective hull}),$$

where Y_M and Y^M are of finite projective dimension and X_M, X^M are maximal Cohen–Macaulay modules.

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We may assume that no common summand exists between Y^M and X^M nor between Y_M and X_M via these maps. We easily observe the following:

- $\Omega_R^n(X_M) \cong \Omega_R^n(M)$ $(n > d)$ where $\Omega_R^n(M)$ denotes the nth syzygy module of M.
- $\mathrm{Ext}_R^i(M, R) \cong \mathrm{Ext}_R^i(Y^M, R)$ for $i > 0$.
- $X_M \cong \Omega_R^1(X^M)$ and $Y_M \cong \Omega_R^1(Y^M)$ up to projective summands.

Thus M shares syzygy property with X^M and homological property with Y^M. Different from Auslander and Buchweitz's original motivation, our concern is how to recover M from X^M and Y^M. A typical example is given by modules with positive codimension.

Theorem 1.2. *(See Kato [5].) Suppose M has a positive codimension. If a finite module N satisfies $X^N \cong X^M$ and $Y^N \cong Y^M$, then $N \cong M$.*

Motivated by this theorem, this paper gives a characterization of a maximal Cohen–Macaulay module X such that $X \cong X_M$ with some M of a positive codimension. This is not always the case for any maximal Cohen–Macaulay module.

Example 1.3. Let $R = k[\![x, y]\!]/(xy)$ and let $C = R/xR$ which is a maximal Cohen–Macaulay module. Then C is not isomorphic to X_M for any M with a positive codimension. Moreover, for every indecomposable module M with a positive codimension, $X_M^R \cong R/xR \oplus R/yR$.

Our result is the following:

Theorem 3.3. *Every maximal Cohen–Macaulay module is isomorphic to X_M with some M of positive codimension if and only if the ring is an integral domain.*

Before going to Theorem 3.3, in Section 2, among X_Ms with positive codimensional Ms, we give classification in terms of $\operatorname{codim} M$. If every maximal Cohen–Macaulay module C has some M with $\operatorname{codim} M = r$ such that $X_M \cong C$, we say that the ring satisfies SC_r-condition. Actually, we show the implication from SC_{r+1}-condition to SC_r-condition. Thus the SC_r-conditions give hierarchy of rings. From this point of view, Theorem 3.3 says that SC_1-condition is equivalent to being an integral domain. As a preceding result, Yoshino and Isogawa showed that SC_2-condition is equivalent to being UFD [8]. Their original statement is for two-dimensional normal rings, which we see is valid for general Gorenstein rings. Now finding an equivalent condition for SC_3 is a very tempting problem, which is posed by Yoshino. But no results are known for this.

In the last section, we give a method to recollect M from Y^M, which explains another reason for our interest in modules with positive codimensions.

Throughout the paper, R is a complete Gorenstein local ring. A finite R-module is simply called an R-module. The R-dual $\mathrm{Hom}_R(\ , R)$ is denoted by $(\ )^*$. The category of finite R-modules is denoted by $\operatorname{mod} R$, that of maximal Cohen–Macaulay modules by $\mathcal{CM}(R)$, and that of finite projective dimensional modules by $\mathcal{F}(R)$. Two R-modules M and N are said to be stably isomorphic and denoted as $M \overset{\mathrm{st}}{\cong} N$ if there are projective modules P and Q such that $M \oplus P \cong N \oplus Q$. (See [2].) A first syzygy $\Omega_R^1(M)$ of a module M is a kernel of the projective cover $P_M \to M$, and rth syzygy module is inductively defined as $\Omega_R^r(M) = \Omega_R^1(\Omega_R^{r-1}(M))$.

We use the convention that $\Omega_R^0(M) = M$. The Auslander transpose $\operatorname{Tr} M$ of an R-module M is defined as $\operatorname{Tr} M = \operatorname{Coker} \delta^*$ where $P_1 \xrightarrow{\delta} P_0 \to M$ is a projective presentation of M.

We close the section by a fundamental property of Cohen–Macaulay approximations.

Lemma 1.4. *Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. The following are commutative diagrams with exact rows and columns*:

(1)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_A & \longrightarrow & Y_B & \longrightarrow & Y_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_A & \longrightarrow & X_B & \longrightarrow & X_C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(2)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^A & \longrightarrow & Y^B & \longrightarrow & Y^C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X^A & \longrightarrow & X^B & \longrightarrow & X^C & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. (1) Let $P_{A\bullet}$, $P_{B\bullet}$ and $P_{C\bullet}$ be projective resolutions of A, B and C, respectively. Let $\theta : P_{C\bullet+1} \to P_{A\bullet}$ be a chain map that corresponds to the given sequence $0 \to A \to B \to C \to 0$. Then θ gives an R-linear map $\Omega_R^d(\theta) : \Omega_R^{d+1}(C) \to \Omega_R^d(A)$, which induces a chain map $P_{\Omega_R^d(X_C)_{\bullet+1}} \to P_{\Omega_R^d(X_A)_\bullet}$ because $\Omega_R^d(X_C) = \Omega_R^d(C)$ and $\Omega_R^d(X_A) = \Omega_R^d(A)$. From this chain map we can construct a chain map $\theta_X : P_{X_C\,\bullet+1} \to P_{X_A\,\bullet}$ since X_A and X_C are maximal Cohen–Macaulay modules. Hence we have a commutative diagram of chain maps

$$\begin{array}{ccc}
P_{C\,\bullet+1} & \xrightarrow{\ \theta\ } & P_{A\,\bullet} \\
\uparrow{\scriptstyle \xi_{C\,\bullet+1}} & & \uparrow{\scriptstyle \xi_{A\,\bullet}} \\
P_{X_C\,\bullet+1} & \xrightarrow{\ \theta_X\ } & P_{X_A\,\bullet}
\end{array}$$

where $\xi_{A\bullet}$ and $\xi_{C_\bullet}$ are chain maps with the property $\xi_{A_i} = \mathrm{id}$ and $\xi_{C_i} = \mathrm{id}$ for $i > d$. Notice that $\mathrm{H}_0(\xi_{A_\bullet}) = \xi_A$ and $\mathrm{H}_0(\xi_{C_\bullet}) = \xi_C$. Now we have a commutative diagram with termwise exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_{X_{A\bullet}} & \longrightarrow & C(\theta_X) & \longrightarrow & P_{X_{C\bullet}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi_{A\bullet}} & & \downarrow & & \downarrow{\scriptstyle \xi_{C\bullet}} & & \\ 0 & \longrightarrow & P_{A\bullet} & \longrightarrow & C(\theta) & \longrightarrow & P_{C\bullet} & \longrightarrow & 0 \end{array}$$

where $C(\theta)$ and $C(\theta_X)$ are mapping cones of chain maps θ and θ_X. It is easy to see that $C(\theta)$ is a projective resolution of B and $C(\theta_X)$ is that of X_B. Thus we have a desired diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_A & \longrightarrow & X_B & \longrightarrow & X_C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \xi_A} & & \downarrow{\scriptstyle \xi_B} & & \downarrow{\scriptstyle \xi_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

(2) Since X_A is a maximal Cohen–Macaulay module, there is an exact sequence

$$0 \to X_A \xrightarrow{\epsilon_{X_A}} F^1_{X_A} \to X' \to 0$$

with a projective module $F^1_{X_A}$ and a maximal Cohen–Macaulay module X'. The finite projective hull is obtained by a push-out of ξ_A and ϵ_{X_A} [2];

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y_A & \longrightarrow & X_A & \xrightarrow{\xi_A} & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \epsilon_{X_A}} & & \downarrow & & \\ 0 & \longrightarrow & Y_A & \longrightarrow & F^1_{X_A} & \longrightarrow & Y^A & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & X' & = & X^A & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Now that we have (1), to show (2) is a matter of diagram-chasing. □

2. The SC_r-conditions

Definition 2.1. Let r be a positive integer. A maximal Cohen–Macaulay module X is said to satisfy SC_r-condition if there exists $M \in \operatorname{mod} R$ such that $\operatorname{codim} M = r$ and $X_C \overset{\mathrm{st}}{\cong} X$. If every $X \in \mathcal{CM}(R)$ satisfies SC_r-condition, we say the ring R satisfies SC_r-condition.

We can restrict our attention from all the modules with positive codimension to Cohen–Macaulay modules with positive codimension. Let us denote the category of Cohen–Macaulay modules with codimension r as $\mathcal{CM}_r(R)$, whose object $M \in \mathcal{CM}_r$ is characterized with the property $\mathrm{Ext}^i_R(M,R)=0$ $(i \neq r)$.

Proposition 2.2. *Let X be an object of $\mathcal{CM}(R)$, and let r be a positive integer. The following are equivalent.*

(1) *X satisfies SC_r-condition; there is $M \in \operatorname{mod} R$ with $\operatorname{codim} M = r$ such that $X_M \overset{\mathrm{st}}{\cong} X$.*
(2) *There is $C \in \mathcal{CM}_r(R)$ such that $X_C \overset{\mathrm{st}}{\cong} X$.*

The proof is given after the following two lemmata.

Lemma 2.3. *For given $X \in \mathcal{CM}(R)$, if there is $M \in \operatorname{mod} R$ with positive codimension such that $X_M \overset{\mathrm{st}}{\cong} X$, then there exists $C \in \mathcal{CM}_s(R)$ such that $X_C \overset{\mathrm{st}}{\cong} X$ where $s = \operatorname{codepth} M$.*

Proof. Let us put $s = \operatorname{codepth} M$, $r = \operatorname{codim} M$, and use an induction on $s-r$. If $s=r$, then $M \in \mathcal{CM}_s(R)$. Suppose $s>r$. Note that $\mathrm{Ext}^i_R(M,R)=0$ $(i<r)$. Set $E=\mathrm{Ext}^r_R(M,R)$ and $P_{E\bullet}$ as a projective resolution of E. Since $\mathrm{Ext}^i_R(E,R)=0$ $(i<r)$,

$$0 \to {P_{E0}}^* \to {P_{E1}}^* \to \cdots \to {P_{Er}}^* \to \operatorname{Tr}\Omega_R^{r-1}(E) \to 0$$

gives a projective resolution of $Y^r = \operatorname{Tr}\Omega_R^{r-1}(E)$. We have $\operatorname{pd}(Y^r) \leqslant r$, $\mathrm{Ext}^i_R(Y^r,R)=0$ $(0<i<r)$, and $\mathrm{Ext}^r_R(Y^r,R) \cong E = \mathrm{Ext}^r_R(M,R)$. Now we want to make a map $M \to Y^r$ that induces an isomorphism $\mathrm{Ext}^r_R(Y^r,R) \cong \mathrm{Ext}^r_R(M,R)$. As for a projective resolution of M

$$\cdots \to P_{M\,r+1} \xrightarrow{d_{r+1}} P_{M\,r} \xrightarrow{d_r} P_{M\,r-1} \to \cdots \to P_{M0} \to 0,$$

we have an exact sequence

$$0 \to E \to \operatorname{Coker} d_r^* \to \operatorname{Im} d_{r+1}^* \to 0.$$

The map $E \to \operatorname{Coker} d_r^*$ induces a chain map of projective resolutions

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & P_{Er} & \longrightarrow & \cdots & \longrightarrow & P_{E1} & \longrightarrow & P_{E0} & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {P_{M0}}^* & \longrightarrow & \cdots & \longrightarrow & {P_{M\,r-1}}^* & \longrightarrow & {P_{Mr}}^* & \longrightarrow & \operatorname{Coker} d_r^* & \longrightarrow & 0 \end{array}$$

whose R-dual gives a map $M \to Y^r$, inducing $\mathrm{Ext}^r_R(Y^r,R) \cong \mathrm{Ext}^r_R(M,R)$. Grade the mapping cone $C_\bullet$ of $P_{M\bullet} \to (P_E)^*$ as

$$0 \to {P_{E\,r-1-i}}^* \to C_i \to P_{Mi} \to 0$$

are exact. We have $\mathrm{H}_i(C_\bullet)=0$ $(i>0)$ and $\mathrm{H}_j(C_\bullet^*)=0$ $(j\leqslant r)$, hence $M'=\operatorname{Coker} d_{C1}$ has $\Omega_R^r(M')\overset{\mathrm{st}}{\cong}\Omega_R^r(M)$ and $\operatorname{Ext}_R^i(M',R)=0$ $(i<r+1)$. Therefore $X_{M'}\overset{\mathrm{st}}{\cong}X_M\overset{\mathrm{st}}{\cong}X$, $\operatorname{codepth} M'=\operatorname{codepth} M=s$ and $\operatorname{codim} M'>\operatorname{codim} M$. Now we get a conclusion from the inductive hypothesis. □

Lemma 2.4. *Let X be an object of $\mathcal{CM}(R)$, and let r be a positive integer. If there is $C\in\mathcal{CM}_{r+1}(R)$ such that $X_C\overset{\mathrm{st}}{\cong}X$, then there exists $C'\in\mathcal{CM}_r(R)$ such that $X_{C'}\overset{\mathrm{st}}{\cong}X$.*

Proof. There exist an R-regular sequence $a_1,\dots,a_r,a_{r+1}\in\operatorname{ann}_R C$. Set $\overline{R}=R/(a_1,\dots,a_r)R$ and take the Cohen–Macaulay approximation of C over $\overline{R}$;

$$0\to Y_C^{\overline{R}}\to X_C^{\overline{R}}\to C\to 0$$

where $Y_C^{\overline{R}}\in\mathcal{F}(\overline{R})$ and $X_C^{\overline{R}}\in\mathcal{CM}(\overline{R})$. Since $Y_C^{\overline{R}}$ has a finite projective dimension also as an R-module, we have $X_{X_C^{\overline{R}}}\overset{\mathrm{st}}{\cong}X_C\overset{\mathrm{st}}{\cong}X$. Obviously, $X_C^{\overline{R}}$ is an object of $\mathcal{CM}_r(R)$. □

Proof for Proposition 2.2. We have only to show (1) ⇒ (2). Suppose there exists a module M with $\operatorname{codim} M=r$ such that $X_M\overset{\mathrm{st}}{\cong}X$. From Lemma 2.3, there is a module $D\in\mathcal{CM}_s(R)$ with $s=\operatorname{codepth} M$ such that $X_D\overset{\mathrm{st}}{\cong}X$. Since $s\geqslant r$, applying Lemma 2.4, we get a module $C\in\mathcal{CM}_r(R)$ with $X_C\overset{\mathrm{st}}{\cong}X$. □

Proposition 2.5. *Let X be an object of $\mathcal{CM}(R)$, and let r be a positive integer. If X satisfies SC_{r+1}-condition, then X satisfies SC_r-condition.*

Proof. Straightforward from Proposition 2.2 and Lemma 2.4. □

Lemma 2.6. *Let X be an object of $\mathcal{CM}(R)$, and let r be a positive integer. If X satisfies SC_r-condition, then $X_{\mathbf{p}}$ is free $R_{\mathbf{p}}$-module for every prime ideal $\mathbf{p}$ with $\operatorname{ht}\mathbf{p}<r$.*

Proof. Let us take $C\in\mathcal{CM}_r(R)$ with $X_C\overset{\mathrm{st}}{\cong}X$. Let $\mathbf{p}$ be a prime ideal of height less than r. Take the Cohen–Macaulay approximation

$$0\to Y_C\to X_C\to C\to 0$$

of C, and localize this with $\mathbf{p}$. Then we have $(Y_C)_{\mathbf{p}}\cong(X_C)_{\mathbf{p}}$. In particular $(X_C)_{\mathbf{p}}$ is a maximal Cohen–Macaulay $R_{\mathbf{p}}$-module with a finite projective dimension hence is an $R_{\mathbf{p}}$-free module. □

The following is an immediate corollary of Lemma 2.6.

Proposition 2.7. *If R satisfies SC_r-condition, then $R_{\mathbf{p}}$ is regular for each prime ideal $\mathbf{p}$ with $\operatorname{ht}\mathbf{p}<r$.*

However SC_r-condition is stronger than the last condition in Proposition 2.7.

Theorem 2.8. *(See Yoshino and Isogawa [8, Theorem 2.2].) A normal Gorenstein complete local ring of dimension two satisfies SC_2-condition if and only if R is a unique factorization domain.*

Actually we do not need the assumption of dimension two.

Theorem 2.9 *(Generalized version of Yoshino–Isogawa's theorem). A complete Gorenstein local ring satisfies SC_2-condition if and only if R is a unique factorization domain.*

Proof. We can omit the assumption "normal" by Proposition 2.7. And we need only slight modifications to their original proof in [8, Theorem 2.2].

For the "if" part, replacing "$\dim N = \operatorname{depth} N = 1$" with "$\operatorname{codim} N = \operatorname{codepth} N = 1$" on the fourth line, we can follow the original proof until the last two lines of (a) $\Rightarrow$ (b). If R is a Gorenstein complete local UFD of any dimension, along their proof, we can conclude that each $M \in \mathcal{CM}(R)$ has a module L with $\operatorname{codim} L \geqslant 2$ such that $\Omega_R^2(L) \overset{\text{st}}{\cong} M$. This equivalently says that each $X \in \mathcal{CM}(R)$ has a module L with $\operatorname{codim} L \geqslant 2$ such that $X_L \overset{\text{st}}{\cong} X$. From Proposition 2.2, there is $L' \in \mathcal{CM}_r(R)$ such that $X \overset{\text{st}}{\cong} X_{L'}$ with $\operatorname{codim} L = r \geqslant 2$, which implies the existence of $L'' \in \mathcal{CM}_2(R)$ such that $X \overset{\text{st}}{\cong} X_{L''}$ by Lemma 2.4.

For the "only if" part, it is enough to show that $c(\mathbf{p})$ is trivial in the divisor class group of R for any prime ideal $\mathbf{p}$ of height 1. Let d be the dimension of R. A maximal Cohen–Macaulay module $X_{\mathbf{p}}$ has some $L \in \mathcal{CM}_2(R)$ such that $X_L \overset{\text{st}}{\cong} X_{\mathbf{p}}$. This means $\Omega_R^d(L) \overset{\text{st}}{\cong} \Omega_R^d(\mathbf{p})$. Since a localization $L_{\mathbf{q}}$ vanishes for any prime ideal $\mathbf{q}$ of height less than two, $c(L) = 0$. Hence we have $c(\Omega_R^d(\mathbf{p})) = c(\Omega_R^d(L)) = 0$ and $c(\mathbf{p}) = 0$ from Proposition 16 of [4]. $\square$

3. SC_1-condition

In this section, we shall see the equivalent condition to SC_1 for the ring.

Lemma 3.1. *Let R be a noetherian local ring. If* $\operatorname{Tr} M$ *is of finite projective dimension, then M has a rank.*

Proof. Our assertion is that $M_{\mathbf{p}}$ is a free $R_{\mathbf{p}}$-module of constant rank for every minimal prime ideal $\mathbf{p}$. Clearly $M_{\mathbf{p}}$ is an $R_{\mathbf{p}}$-free module, so it remains to show that $\mu_{R_{\mathbf{p}}}(M_{\mathbf{p}})$ is independent of the choice of $\mathbf{p}$. Let $P_1 \xrightarrow{d} P_0 \to M \to 0$ be a projective presentation of M. Let t be the maximal size of non-vanishing minors of the matrix corresponding to d, and $I_t(d)$ an ideal generated by the t-minors of the matrix. We state that $\operatorname{ht}(I_t(d)) \geqslant 1$. Since $\operatorname{Tr} M = \operatorname{Coker} d^*$ is of finite projective dimension, we have $\operatorname{ht}(I_t(d^*)) \geqslant 1$ from Buchsbaum–Eisenbud's theorem [3]. Obviously $I_t(d) = I_t(d^*)$. Therefore $I_t(d) \not\subset \mathbf{p}$ for any minimal prime ideal $\mathbf{p}$. In other words, some t-minors of d is a unit in $R_{\mathbf{p}}$, so that $\mu_{R_{\mathbf{p}}}(M_{\mathbf{p}}) = \mu_R(M) - t$. $\square$

Lemma 3.2. *A maximal Cohen–Macaulay module X has a rank if and only if there exists $M \in \mathcal{CM}_1(R)$ such that $X_M \overset{\text{st}}{\cong} X$.*

Proof. Suppose X has a rank. Then there is an exact sequence

$$0 \to F \to X \to M \to 0$$

with a free module F and a torsion module M. This sequence is a Cohen–Macaulay approximation of M hence $X \overset{\text{st}}{\cong} X_M$. Dualize this sequence with R, we have $\operatorname{Ext}^i_R(M, R) = 0$ $(i > 1)$. Since $M^* = 0$, $M \in \mathcal{CM}_1(R)$.

To show the "if" part, take the Cohen–Macaulay approximation of M:

$$0 \to M \to Y_M \to X_M \to 0.$$

Since $M^* = 0$, $\operatorname{Tr} M$ has a projective dimension not larger than one. Hence from Lemma 3.1, both of Y_M and M have ranks, thus so does $X \overset{\text{st}}{\cong} X_M$. □

Theorem 3.3. *The following conditions are equivalent for a complete Gorenstein local ring R.*

(1) *R is an integral domain.*
(2) *Every finitely generated module has a rank.*
(3) *Every maximal Cohen–Macaulay module has a rank.*
(4) *Every maximal Cohen–Macaulay module is a higher syzygy of some module with a positive codimension.*
(5) *R satisfies SC_1-condition; for every maximal Cohen–Macaulay module C, there exists an R-modules M such that $X_M \overset{\text{st}}{\cong} C$ and* $\operatorname{codim} M = 1$.

Proof. (1) ⇔ (2). It is well known that the equivalence holds for general noetherian rings.

(2) ⇒ (3). It is obvious.

(3) ⇒ (2). Let M be a finite R-module. Consider the minimal Cohen–Macaulay approximation of M

$$0 \to Y_M \to X_M \to M \to 0.$$

From the assumption, both Y_M and X_M have ranks hence so does M.

(3) ⇔ (5). It is straightforward from Lemma 3.2 together with Proposition 2.2.

(4) ⇒ (3). Let C be any object of $\mathcal{CM}(R)$. Let M be a module with positive codimension such that $\Omega^n_R(M) \overset{\text{st}}{\cong} C$ for some $n > 0$. We have an exact sequence

$$0 \to C \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to M \to 0$$

with projective modules P_is. From Lemma 3.1, M has a rank, hence so does C.

(5) ⇒ (4). Let C be an object of $\mathcal{CM}(R)$. There exists $C' \in \mathcal{CM}(R)$ such that $C \overset{\text{st}}{\cong} \Omega^d_R(C')$ where $d = \dim R$. From the hypothesis, there is an R-module M with a positive codimension such that $X_M \overset{\text{st}}{\cong} C'$. Now we have $C \overset{\text{st}}{\cong} \Omega^d_R(C') \overset{\text{st}}{\cong} \Omega^d_R(X_M) \overset{\text{st}}{\cong} \Omega^d_R(M)$. □

4. Category determined by Y^M

In this section, we explain another reason why we insist on positive codimension. For a given $Y \in \mathcal{F}(R)$, how do we find a non-trivial module M with $Y^M \overset{\text{st}}{\cong} Y$? We do this by a new method of getting a maximal Cohen–Macaulay module associated to Y.

First we shall fix the notations.

Definition 4.1. (See Auslander and Bridger [1].) The projective stabilization $\underline{\mathrm{mod}\, R}$ is defined as follows.

- Each object of $\underline{\mathrm{mod}\, R}$ is an object of $\mathrm{mod}\, R$.
- For objects A, B of $\underline{\mathrm{mod}\, R}$, a set of morphisms from A to B is $\underline{\mathrm{Hom}_R}(A, B) = \mathrm{Hom}_R(A, B)/\mathcal{P}(A, B)$ where $\mathcal{P}(A, B) := \{f \in \mathrm{Hom}_R(A, B) \mid f$ factors through some projective module$\}$. Each element of $\underline{\mathrm{Hom}_R}(A, B)$ is denoted as $\underline{f} = f \bmod \mathcal{P}(A, B)$.

For a finite module M, the Auslander transpose $\mathrm{Tr}\, M$ is defined as $\mathrm{Tr}\, M = \mathrm{Coker}\, \delta^*$ where $P_1 \xrightarrow{\delta} P_0 \to M$ is a projective presentation of M.

A morphism f in $\mathrm{mod}\, R$ is a stable isomorphism if and only if $\underline{f}$ is an isomorphism in $\underline{\mathrm{mod}\, R}$. (See [1] for example.)

Both $\mathrm{Tr}\, M$ and Ω_R^is are endo-functors on $\underline{\mathrm{mod}\, R}$. And there is a natural map $\varphi_{rM} : M \to \Omega^r \,\mathrm{Tr}\, \Omega^r \,\mathrm{Tr}\, M$ for each non-negative integer r. We call $T = \Omega^d \,\mathrm{Tr}\, \Omega^d \,\mathrm{Tr}$ and $\varphi_M = \varphi_{dM}$ for $d = \dim R$. Notice that TM is a maximal Cohen–Macaulay module, and $\varphi_M : M \to TM$ is the Auslander transpose of $\xi_{\mathrm{Tr}\, M} : X_{\mathrm{Tr}\, M} \to \mathrm{Tr}\, M$.

Now we are ready to introduce our method. Let Y be a module with a finite projective dimension. Then together with a projective cover $\rho_{TY} : P_{TY} \to TY$, we have an exact sequence

$$0 \to N_Y \to Y \oplus P_{TY} \xrightarrow{(\varphi_Y \rho_{TY})} TY \to 0$$

which is the finite projective hull of the module N_Y. Thus we get a module N_Y with the property $Y^{N_Y} \overset{\mathrm{st}}{\cong} Y$. Moreover, suppose $M \in \mathrm{mod}\, R$ satisfies $Y^M \overset{\mathrm{st}}{\cong} Y$. Then applying T to the map $f : Y^M \to X^M$, we get a diagram

$$\begin{array}{ccc} Y^M & \xrightarrow{\ f\ } & X^M \\ \downarrow{\scriptstyle \varphi_{Y^M}} & & \downarrow{\scriptstyle \varphi_{X^M}} \\ TY^M & \xrightarrow{\ Tf\ } & TX^M \end{array}$$

which commutes up to projective modules. Since φ_{X^M} is an isomorphism, we may say $\underline{f} = \underline{Tf \circ \varphi_Y}$ in $\underline{\mathrm{mod}\, R}$.

Theorem 4.2. *For a given $Y \in \mathcal{F}(R)$, there uniquely exists $N_Y \in \underline{\mathrm{mod}\, R}$ such that*

(1) $Y^{N_Y} \overset{\mathrm{st}}{\cong} Y$.

(2) *For any $M \in \mathrm{mod}\, R$ with $Y^M \overset{\mathrm{st}}{\cong} Y$, there exists an R-linear map $N_Y \to M$ which induces $\mathrm{Ext}_R^i(M, R) \cong \mathrm{Ext}_R^i(N_Y, R)$ for $i > 0$.*

Proof. We shall start with the existence. We have already seen (1). To see (2), let $M \in \mathrm{mod}\, R$ have $Y^M \overset{\mathrm{st}}{\cong} Y$ and let $f : Y^M \to X^M$ be a map with $\mathrm{Ker}\, f = M$. The equation $\underline{f} = \underline{Tf \circ \varphi_Y}$ in

$\underline{\operatorname{mod} R}$ implies that there exists $q : Y \to P_{X^M}$ such that $f = Tf \circ \varphi_Y + \rho_{X^M} \circ q$ in $\operatorname{mod} R$. We have a diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N_M & \cong & N_{Y^M} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M \oplus P_{TY^M} \oplus P_{X^M} & \longrightarrow & Y^M \oplus P_{TY^M} \oplus P_{X^M} & \xrightarrow{a} & X^M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle d} & & \| & & \\
0 & \longrightarrow & TM & \longrightarrow & TY^M \oplus P_{X^M} & \xrightarrow{c} & TX^M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $a = (f\ Tf \circ \rho_{TY^M}\ \rho_{X^M})$, $b = \left(\begin{smallmatrix} \varphi_{Y^M} & \rho_{TY^M} & 0 \\ q & 0 & 1 \end{smallmatrix}\right)$ and $c = (Tf\ \ \rho_{X^M})$.

The middle row is a finite projective hull of $M \oplus P_{TY^M} \oplus P_{X^M}$ and the middle column is that of N_{Y^M}. Looking at the leftmost column, we get (2).

To show the uniqueness, let N' be an R-module also satisfying (1) and (2). Then we have maps $h : N_Y \to N'$ and $g : N' \to N_Y$ such that $\operatorname{Ext}^i_R(h, R)$ and $\operatorname{Ext}^i_R(g, R)$ are isomorphisms for $i > 0$. All we need is to prove that h and g are stable isomorphisms, which comes from the following lemma. □

Lemma 4.3. *If an endomorphism f on an R-module M induces isomorphisms $\operatorname{Ext}^i_R(f, R)$ for $i > 0$, then f is a stable isomorphism.*

Proof. Let us take an exact sequence

$$0 \to K \to M \oplus P_M \xrightarrow{(f \rho_M)} M \to 0$$

with a projective cover $\rho_M : P_M \to M$. From the assumption, we have $\operatorname{Ext}^i_R(K, R) = 0$ $(i > 0)$ hence K is a maximal Cohen–Macaulay module. On the other hand, from Lemma 1.4, finite projective hulls make a commutative diagram with exact rows and columns.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M \oplus P_M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & Y^M \oplus P_M & \longrightarrow & Y^M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X^M & = & X^M & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Therefore K is of finite projective dimension and maximal Cohen–Macaulay at the same time, hence is projective.

The module K is what we call a pseudo-kernel $\underline{\mathrm{Ker}}\, \underline{f}$ of f (see [6]). By Theorem 4.12 of [6], since $K = \underline{\mathrm{Ker}}\, \underline{f}$ is a first syzygy, there exists an exact sequence

$$\theta : 0 \to M \xrightarrow{\binom{f}{\epsilon}} M \oplus Q \to K' \to 0$$

that has the following property:

(1) Q is a projective module.
(2) an R-dual

$$0 \to K'^* \to M^* \oplus Q^* \to M^* \to 0$$

is exact.
(3) $\Omega_R^1(K') \cong K$.

We claim that K' is projective. First, from the assumption and (2), K' is a maximal Cohen–Macaulay module. And since $\Omega_R^1(K') \cong K$ is projective, K' is of finite projective dimension. □

Let Y be an object of $\mathcal{F}(R)$. The proof of Theorem 4.2 says that for any $M \in \operatorname{mod} R$ with $Y^M \overset{\mathrm{st}}{\cong} Y$, the map $Y^M \to X^M$ factors through the canonical map $Y \to TY$. This motivates us to characterize maximal Cohen–Macaulay modules of the form TY with $Y \in \mathcal{F}(R)$.

Proposition 4.4. *As subcategories of* $\underline{\operatorname{mod} R}$,

$$\{TY \mid Y \in \mathcal{F}(R)\} = \{X^W \mid W \in \mathcal{CM}_1(R)\}.$$

Proof. Set $S_1 = \{TY \mid Y \in \mathcal{F}(R)\}$ and $S_2 = \{X^W \mid W \in \mathcal{CM}_1(R)\}$. First we claim that S_1 equals to $\{TZ \mid \mathrm{pd}(Z) \leqslant 1\}$. Let $Y \in \mathcal{F}(R)$. Then from Lemma 3.1, $\mathrm{Tr}\, Y$ has a rank; there is an exact sequence

$$0 \to F \to \mathrm{Tr}\, Y \to V \to 0$$

where F is a free module and $V^* = 0$. This implies that $X_{\mathrm{Tr}\, Y} \overset{\mathrm{st}}{\cong} X_V$ hence $TY \overset{\mathrm{st}}{\cong} \mathrm{Tr}\, X_{\mathrm{Tr}\, Y} \overset{\mathrm{st}}{\cong} \mathrm{Tr}\, X_V \overset{\mathrm{st}}{\cong} T\, \mathrm{Tr}\, V$. And $\mathrm{Tr}\, V$ is of projective dimension at most one.

$S_2 \subset S_1$. For a one-codimensional Cohen–Macaulay module W, $W^\vee = \mathrm{Ext}_R^1(W, R)$ also belongs to $\mathcal{CM}_1(R)$ hence $\mathrm{pd}(\mathrm{Tr}\, W^\vee) \leqslant 1$. We shall show

$$X^W \overset{\mathrm{st}}{\cong} T\, \mathrm{Tr}\, W^\vee. \tag{4.1}$$

By Herzog–Martsinkovsky's formula [7], $X^W \overset{\mathrm{st}}{\cong} \Omega_R^1(\mathrm{Tr}\, \Omega_R^1(W^\vee))$. The right-hand side is $\mathrm{Tr}\,\mathrm{Tr}\, \Omega_R^1(\mathrm{Tr}\, \Omega_R^1(W^\vee)) = \mathrm{Tr}\, X_{W^\vee} = T\, \mathrm{Tr}\, W^\vee$.

$S_1 \subset S_2$. Let Z be a module with $\mathrm{pd}(Z) \leqslant 1$. Since $(\mathrm{Tr}\, Z)^* = 0$, there exists a non-zero-divisor $x \in \mathrm{ann}_R(\mathrm{Tr}\, Z)$. Set $\overline{R} = R/xR$, which is a Gorenstein ring with $\dim \overline{R} = \dim R - 1$. Since $\mathrm{Tr}\, Z \in \mathrm{mod}\, \overline{R}$, we have a Cohen–Macaulay approximation of $\mathrm{Tr}\, Z$ over $\overline{R}$; $0 \to U \to L \to \mathrm{Tr}\, Z \to 0$ where $U \in \mathcal{F}(\overline{R})$ and $L \in \mathcal{CM}(\overline{R})$. Since U is of finite projective dimension also as an R-module, $X_L \overset{\mathrm{st}}{\cong} X_{\mathrm{Tr}\, Z}$ hence $T\, \mathrm{Tr}\, L \overset{\mathrm{st}}{\cong} TZ$ in $\mathrm{mod}\, R$. From (4.1) we have $T\, \mathrm{Tr}\, L \overset{\mathrm{st}}{\cong} X^{L^\vee}$ hence $TZ \overset{\mathrm{st}}{\cong} X^{L^\vee}$. Obviously $L \in \mathcal{CM}_1(R)$. □

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