MATHEMATICS

POLAR GEOMETRY. V

BY

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In this paper we give some additions and extensions to our previous paper [4] [2]). In section 1 some remarks are made on trace-valued forms, in 2 a simpler formulation of one of our axioms is given, in 3 a special kind of polar geometries is considered, viz. those in which two maximal elements span the whole space, in 4 and 5, finally, polar geometries in spaces of infinite rank are discussed.

1. In [4] we confined ourselves to null systems and polarities that can be represented by a trace-valued hermitian form. (See Ch. I, section 7, of [4].) That this restriction is by no means essential when strictly isotropic subspaces are in question can be readily seen in the following way:

Let σ be a polarity. Let V be the subspace that is spanned by the N-points with respect to σ. Take a subspace M of V such that M is also spanned by N-points and

$$V = V \cap V^{\sigma} \oplus M.$$

We then have:

$$0 = M \cap V \cap V^{\sigma} = M \cap V^{\sigma} = M \cap M^{\sigma} \cap (V + V^{\sigma}) = M \cap M^{\sigma}.$$

If X is a maximal strictly isotropic subspace with respect to σ, then of course $X \leqslant V$. From the maximality of X it follows that

$$X = V \cap V^{\sigma} \oplus X \cap M.$$

As M is non-isotropic, the restriction of σ to M is, again, a polarity; as the maximal strictly isotropic subspaces are determined by their intersection with M, we can confine ourselves to the consideration of the restriction of σ to M. That this restriction has the properties we required in I, 7 of [4] is a consequence of the following proposition:

PROPOSITION. *If σ is a polarity such that there exists a basis of N-points with respect to σ, then σ is a null system or it can be represented by a trace-valued hermitian form.*

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[2]) References at the end of the paper.

Proof. We may suppose σ to be represented by a semi-bilinear form f that is hermitian or skew-symmetric (Cf. [4], I, 5 or [1], ch. I, § 6).

If f is skew-symmetric, σ is a null system. For there exists a basis of N-points and if $f(x, x)=f(y, y)=0$, then

$$\begin{aligned} f(\lambda x+\mu y, \lambda x+\mu y) &= \lambda f(x, y)\mu+\mu f(y, x)\lambda \\ &= \lambda f(x, y)\mu-\lambda f(x, y)\mu=0. \end{aligned}$$

If f is a hermitian α-form, the line of reasoning is as follows: From $f(x, x)=\varrho+\varrho^{\alpha}$ and $f(y, y)=\sigma+\sigma^{\alpha}$ it follows that

$$\begin{aligned} f(\lambda x+\mu y, \lambda x+\mu y) &= \lambda(\varrho+\varrho^{\alpha})\lambda^{\alpha}+\mu(\sigma+\sigma^{\alpha})\mu^{\alpha}+\lambda f(x, y)\mu^{\alpha}+\mu f(y, x)\lambda^{\alpha} \\ &= \tau+\tau^{\alpha} \text{ where} \\ \tau &= \lambda\varrho\lambda^{\alpha}+\mu\sigma\mu^{\alpha}+\lambda f(x, y)\mu^{\alpha}. \end{aligned}$$

As there is a basis of N-points, f must be trace-valued.

2. Axiom VII of III, 2 [1]) can be formulated in a simpler way, viz.:

VII. *Let a and b be maximal and $a \wedge b=0$. To every point $x<a$ there exists an $x'<b$ with rank $r(x')=i(\mathbf{S})-1$ such that $x \vee x'$ exists.*

The property mentioned in III, 2 as axiom VII, is a consequence of the above axiom.

3. Instead of axiom VIII of III, 11 one may consider the following axiom VIII*, which implies VIII.

VIII*. *There exist two maximal elements a and b such that every point in $\mathbf{S}$ lies on an imaginary line pq where $p<a$ and $q<b$.*

From axiom VI follows that $a \wedge b=0$.

If t is an element with $r(t)=i(\mathbf{S})-1$ and r and s are points that are joined to t, then every point of the imaginary line rs is joined to t, as follows from the definition of imaginary lines. Now in a system $\mathbf{S}$ satisfying axioms I–VII and VIII* this assertion has a converse:

(3.1) Proposition. *Let $\mathbf{S}$ satisfy axioms* I–VII *and* VIII*. *If t is an element of rank $i(\mathbf{S})-1$ and r and s are points such that $r \vee t$ and $s \vee t$ exist but rs does not contain a point $<t$, and if $x>t$ is a maximal element, then there exists a point $y<x$, $y \in rs$.*

Proof. Take a and b such as in axiom VIII*. We distinguish several cases.

1°. $t<a$, $r<a$, $s<b$.

Take a point y' such that $x=t \vee y'$. Then we can find points $p<a$ and $q<b$ such that $y' \in pq$. $y' \vee t$ and $p \vee t$ exist, hence so does $q \vee t$. Therefore $q=s$. By applying Proposition III, 7 we find a point $y \leqslant y' \vee (t \wedge (p \vee r))$, $y \in rs$; then $x=t \vee y$.

1) In the sequel a reference as III, 2 will mean: Ch. III, section 2 of [4].

2°. $t<a$, r and s arbitrary.

As it is impossible that both $r<a$ and $s<a$, one may suppose $r \nless a$. We then can find points $p<a$ and $q<b$ such that $r \in pq$. As $p \vee t$ and $r \vee t$ exist, so does $q \vee t$. From 1° it follows that $x=t \vee y'$ with $y' \in pq$, and that $t \vee s = t \vee s'$ with $s' \in pq$. Applying proposition III, 7 again we find a point $y \in rs$, $y \leqslant y' \vee (t \wedge (s \vee s'))$; but then $x = t \vee y$.

3°. The general case can be deduced step by step from the previous part of the proof with the aid of the following statement:

Let $r(t_1)=r(t_2)=i(\mathbf{S})-1$, $r(t_1 \wedge t_2)=i(\mathbf{S})-2$ and let $t_1 \vee t_2$ not exist. If the proposition we have to prove is true for t_1, so it holds for t_2.

To prove this we first choose two points r_1 and s_1 such that $r_1 \vee t_1$, $r_1 \vee t_2$, $s_1 \vee t_1$ and $s_2 \vee t_2$ exist and that $r_1 s_1$ does not contain a point $<t_1$ or $<t_2$. Now if $x>t_2$, we take $x'>t_1$ such that $r(x \wedge x')=i(\mathbf{S})-1$. Then there exists a point $y<x'$, $y \in r_1 s_1$, because of the assumption about t_1. From $y \in r_1 s_1$ it follows that $y \vee t_2$ exists. As $t_1 \vee t_2$ does not exist, $x' \ngtr t_2$. But then $y < x \wedge x'$. Thus we have found a point $y<x$, $y \in r_1 s_1$.

Now let r and s be arbitrary points such that $r \vee t_2$ and $s \vee t_2$ exist and that rs does not contain a point $<t_2$. There must exist a point $<r \vee t_2$ on $r_1 s_1$, say r_1, and a point $<s \vee t_2$ on $r_1 s_1$, say s_1. On $s \vee s_1$ we can find a point s_2 such that neither $r \vee s_2$ nor $r_1 \vee s_2$ exists, because of the non-existence of $r \vee s \vee s_1$ and $r_1 \vee s \vee s_1$.

If then $x>t_2$, there is a point $\in r_1 s_1$, that is $<x$. Applying proposition III, 7, again, we find a point $\in r_1 s_2$ that is $<x$, in the same way a similar point $\in rs_2$ and finally one $\in rs$, which completes the proof.

With the proposition we have just proved we can show:

(3.2) Proposition. *Let u and v be two maximal elements of $\mathbf{S}$ (satisfying I–VII and VIII*) with $u \wedge v = 0$. Then every point $x \in \mathbf{S}$ lies on an imaginary line pq where $p<u$ and $q<v$.*

Proof. We may assume $x \nless u$, $\nless v$, in the other case the proof being trivial. We take $u_1<u$ and $v_1<v$, both of rank $i(\mathbf{S})-1$, such that $x \vee u_1$ and $x \vee v_1$ exist. We then take points $p<u$ and $q<v$ such that $p \vee v_1$ and $q \vee u_1$ exist. Then we can prove that $x \in pq$:

If $p \vee q$ exists, $p<u_1$ and $q<v_1$. Then $x \vee u_1 \vee q$ exists, hence $x<u_1 \vee q$, for $u_1 \vee q$ is maximal; similarly $x<v_1 \vee p$. Hence $x<(u_1 \vee q) \wedge (v_1 \vee p)=$ $=p \vee q$, i.e. $x \in pq$.

If $p \vee q$ does not exist, there must exist a point $y \in pq$, $y<x \vee v_1$. As $p \vee u_1 = u$ and $q \vee u_1$ exist, so does $y \vee u_1$. If $x \neq y$, $(x \vee y) \wedge v_1 = q'$ would be a point $\neq q$ such that $u_1 \vee q'$ exists; but then $u_1 \vee q \vee q'$ would be of rank $i(\mathbf{S})+1$, which leads to a contradiction. Hence $x=y$, i.e. $x \in pq$.

4. *Polar geometry in spaces of infinite rank.*

4.1. So far we have only spoken about polar geometries corresponding to polarities in projective spaces of finite rank. The notion of polarity, however, can be generalized to that of *quasipolarity* in a space of arbitrary rank; see H. Lenz [3]. Such a quasipolarity is a mapping σ of a projective space $P(A)$ into its dual space, i.e. the space of hyperplanes in $P(A)$, with the properties:

(1) To every point X of $P(A)$ there is defined exactly one hyperplane X^σ.

(2) To every hyperplane Y in $P(A)$ there is at most one point X such that $Y = X^\sigma$.

(3) If X and Y are points such that $X < Y^\sigma$, then $Y < X^\sigma$.

If (2) is replaced by

(2′) To every hyperplane Y in $P(A)$ there is exactly one point X such that $Y = X^\sigma$,

then σ is a polarity and $P(A)$ has finite rank. If, conversely, σ is a quasipolarity in a space of finite rank, then σ is a polarity.

The definition of σ can, of course, be extended to subspaces of arbitrary rank by stating $V^\sigma = \bigcap X^\sigma$ where the intersection is taken over all points $X \leqslant V$, and $0^\sigma = A$.

Quasipolarities can be represented by semi-bilinear forms in exactly the same way as polarities.

N-points, N-subspaces and isotropic, strictly isotropic and non-isotropic subspaces with respect to a quasipolarity are defined in the usual way.

$(\sum\limits_{U \in \Theta} U)^\sigma = \bigcap\limits_{U \in \Theta} U^\sigma$ is also true for quasipolarities but if σ is not a polarity, $(U \cap V)^\sigma = U^\sigma + V^\sigma$ is not true for every U and V; we can only say $(\bigcap\limits_{U \in \Theta} U)^\sigma \geqslant \sum\limits_{U \in \Theta} U^\sigma$. But if P and Q are points, $(P^\sigma \cap Q^\sigma)^\sigma = P + Q$, as follows from H. Lenz [3], § 1, Hilfssatz 2.

$V^{\sigma\sigma} \geqslant V$ but not necessarily $V^{\sigma\sigma} = V$ if σ is a quasipolarity. If $V^{\sigma\sigma} = V$, V is called *closed* with respect to σ. If V is closed and F has finite rank, $V + F$ is closed; as 0 is closed, every subspace of finite rank is closed. If V has finite rank and $V \cap V^\sigma = 0$, then $A = V + V^\sigma$, where A is the whole space. For the proofs see e.g. I. Kaplansky [2].

If $V \cap V^\sigma = 0$ and we define δ for the subspaces of V by $X^\delta = X^\sigma \cap V$, then δ is clearly a quasipolarity, the *restriction of σ to V*.

4.2. An important property of quasipolarities is the following one (see [2]):

If V is a subspace of finite rank, then there exists a subspace $W \geqslant V$ of finite rank such that $W \cap W^\sigma = 0$.

(4.2.1) *We now suppose that σ is a null-system or that it is represented by a trace-valued hermitian form.* This restriction is not essential, for everything we said in section 1 of this paper on polarities is also true for quasipolarities.

Then the property that every subspace of finite rank is contained in a non-isotropic subspace of finite rank ensures that 1), 2) and 3) of I, 7 are also valid for quasipolarities, provided that the subspaces V and W

which we speak of in 1) and 3) have finite rank. Further, if the maximal rank $i(\sigma)$ of a strictly isotropic subspace is finite, then I, 8 remains true; so every maximal strictly-isotropic subspace has rank $i(\sigma)$.

I, 9–11 remains true with only some minor changes in the proofs.

4.3. The definition of *polar geometry* in II, 1 is extended to quasipolarities in the following way:

Let σ be a quasipolarity in a projective space $P(A)$ with property 4.2.1 such that the maximal rank $i(\sigma)$ of a strictly isotropic subspace is finite but not less than 3.

Let **S** be the system of strictly isotropic subspaces of A, partially ordered by inclusion ($\leqslant$). Then **S** is called the *polar geometry corresponding to* σ.

Instead of $i(\sigma)$ we speak again of $i(\mathbf{S})$, the index of **S**.

The rest of II, 1 and II, 2 remains unchanged.

4.4. II, 3–12 remains true for quasipolarities with only some changes, the most important of which we shall enumerate here:

In II, 7 "$r(V)=r(U^\sigma)$" must be replaced by "$\mathrm{cor}(V)=r(U)$"; here $\mathrm{cor}(V)=$ co-rank of $V=r(A/V)$.

In II, 8 the consequence of the above change must be drawn. Further, at the end of this section we make use of the fact that if

$$P=(Q_1+Q_1')\cap(Q_2+Q_2'),$$

where P and the $Q-s$ are points, then $P^\sigma=(Q_1^\sigma\cap Q_1'^\sigma)+(Q_2^\sigma\cap Q_2'^\sigma)$. If this were not true, $Q_1^\sigma\cap Q_1'^\sigma=Q_2^\sigma\cap Q_2'^\sigma$, as lefthand and righthand side of the equation have both co-rank 2; but then we apply σ and find $Q_1+Q_1'=Q_2+Q_2'$, which is supposed not to be true.

In II, 9 and II, 10 we have to say that P^σ has a basis of N-points P and P_i, where i ranges over a certain set of indices.

In II, 11.3. and 5., we must take $V=\sum_i P_i$ as a possibly infinite sum.

4.5. It is readily seen that a polar geometry **S** such as defined in 4.3 satisfies the axioms I–VII, IX and X we stated in chapter III.

In chapter IV we have embedded a system **S** satisfying axioms I–X in a projective space, where it is a polar geometry with respect to a polarity (with exception of the case that the field of coordinates has characteristic 2). Exactly the same can be done with a system **S** that satisfies axioms I–VII, IX and X. The only place in chapter IV where we essentially made use of axiom VIII is IV, 18, in which section we showed that **S** can be embedded in a space of *finite* rank.

In the sections after IV, 18 some changes have to be made when we omit axiom VIII. So σ is no longer a polarity but a quasi-polarity.

In IV, 20–22, "$r(P^\sigma)\geqslant n-1$" and "$r(P^\sigma)=n-1$" must be replaced by "$\mathrm{cor}(P^\sigma)\leqslant 1$" and "$\mathrm{cor}(P^\sigma)=1$" respectively, and analogously in the proofs. Further, some more obvious changes.

The final result is then:

THEOREM. *The polar geometries* **S** *with respect to a quasipolarity* (*satisfying the condition* 4.2.1) *are characterized by the axioms* I–VII, IX *and* X *of chapter* III *of* [4]; *in case that the field of coordinates of the maximal elements of a system* **S** *that satisfies these axioms, has characteristic* 2 *it can only be shown that* **S** *is part of a polar geometry with respect to a quasipolarity.* **S** *is a polar geometry with respect to a polarity* (*in a space of finite rank*) *if, and only if, it satisfies, moreover, axiom* VIII.

5. We can now ask whether it is possible to characterize polar geometries with respect to a quasipolarity σ with *infinite* $i(\sigma)$. We can only say that our axioms of [4] completely fail to describe this situation. To show that all maximal strictly isotropic subspaces have the same rank and co-rank is not so very difficult in case of *countable* $i(\sigma)$. But axiom VI is certainly not true if $i(\sigma)=\infty$, i.e. two disjoint strictly isotropic subspaces are not necessarily contained in disjoint *maximal* ones. As axiom VI is often used in our proofs in an apparently essential way, it is clear that we have to look for quite different methods.

We shall, finally, show that not every two disjoint strictly isotropic subspaces are contained in two disjoint maximal ones. If VI were true, we could find two disjoint maximal strictly isotropic subspaces (for 0 and 0 are disjoint), say U and V. As $U+V$ is non-isotropic, we can restrict σ to $U+V$.

For $X \leqslant U$ we define $X^{\delta}=X^{\sigma} \cap V$. This is a mapping of U into the dual space of V. But as $r(U)=r(V)=\infty$, the dual space of V has a rank greater than $r(U)$, hence δ cannot be a mapping *onto*. So we can find a $Y \leqslant V, r(V/Y)=1$, such that there is no point $X \leqslant U$ with $X^{\delta}=Y$. Hence $Y^{\sigma}=V$. If we then take a point Z such that $V=Y+Z$, then Y and Z are disjoint strictly isotropic but the only maximal strictly isotropic subspace $\geqslant Y$ is V with $V \cap Z \neq 0$. This contradicts axiom VI.

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