The Hawking Energy on the Past Light Cone

Inhomogeneous Cosmologies IV

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Outline

- Definition and properties of the Hawking energy E_H
- The past light cone in cosmology & the cut locus
- Theorem & proof
- Comments on assumptions and applicability
- Summary and outlook

Definition of Hawking energy

- Given a spacetime (M, g), take a spacelike (topol.) 2-sphere S with area $A(S) = \int_S dS$.
- $\exists$ past-directed outgoing and ingoing null direction $\perp S$, represented by tangent vectors l and n.
- The expansion of each geodesic congruence is given by θ_l & θ_n.
- Idea: energy in 3-volume surrounded by S affects the light bending on S.

Def. Hawking Energy E_H:

Given a spacelike 2-sphere S, the Hawking energy E_H is defined as

$$E_H := \frac{A(S)}{(4\pi)^{3/2}} \left[2\pi - \int_S \rho\mu \, dS \right] \tag{1}$$

with $\rho := -\theta_l/2$ and $\mu := \theta_n/2$.

Properties of E_H

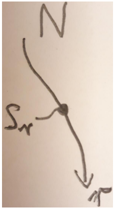
- In the limit of S degenerating to a point, $E_H(S) \to 0$.
- For small sphere of radius r around $p \in M$ in the limit $r \to 0$:
  - non-vacuum: $E_H \sim r^3 T_{ab} t^a t^b$
  - vacuum: $E_H \sim r^5 B_{abcd} t^a t^b t^c t^d \geq 0$

  T_{ab}: energy-momentum tensor, B_{abcd}: Bel-Robinson tensor, t^a: unit timelike vector
- For a metric 2-sphere in Minkowski: $E_H = 0$
- For Killing horizons: $E_H = M_{\mathrm{irr}}$
- For large spheres near $\mathcal{I}^+$: $E_H \to E_{\mathrm{Bondi-Sachs}}$
- For large spheres near i^0: $E_H \to E_{\mathrm{ADM}}$

Monotonicity of E_H on a null hypersurface

Let (S_r) be a 1-param. family of (topol.) 2-spheres foliating the outgoing null hypersurface N. For a special class of foliations (Eardley 1978):

$$\frac{dE_H(S_r)}{dr} \geq 0\,.$$



In more detail:

- Start with a spacelike 2-sphere S obeying $\rho < 0$ & $\mu < 0$ everywhere.
- Define a constant r on S: $r := \sqrt{\frac{A(S)}{4\pi}}$
- Rescale l^a such that $\rho = -1/r$ [most general form: $\rho = -1/P(r)$ with $P(r) > 0$]
- r extends to a distance along the outgoing past-directed null hypersurface $N \perp S$, defining a 1-param. family of level surfaces in N.

The Past Light Cone in Cosmology

Motivation:

- Given an observer at $p \in M$ with 4-velocity t^a at p.
- Past light cone $C^-(p)$ at p can be uniquely constructed
- All (light) signals that can be received at p travel on $C^-(p)$.

Cosmological effects on the topology of the past light cone:

- In Minkowski: $C^-(p) \simeq S^2 \times \mathbb{R}$
- If only weak gravitational lensing present: no multiple images of sources, but image distortions $\to$ still have $C^-(p) \simeq S^2 \times \mathbb{R}$
- Strong gravitational lensing: multiple images $\Rightarrow$ self-intersections of $C^-(p)$ $\Rightarrow$ topology changes!

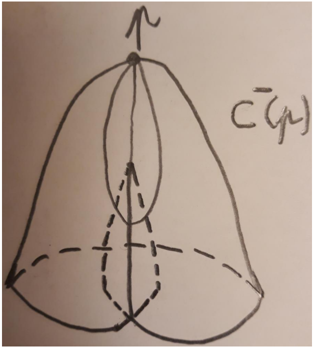
Can be made more precise by referring to the cut locus

Cut Locus

- $I^-(p)$:={points in the past of p that can be reached by a timelike curve}
  $\dot{I}^-(p)$:= boundary of $I^-(p)$
- Can show: $\dot{I}^-(p)$ immersed, achronal, 3-dim., C^{1-} (i.e. Lipschitz continuous) submanifold [everywhere except at p & cut locus]
- Consider a past-oriented null geodesic $\gamma(\lambda)$ issued at p, $\gamma(\lambda)$ is confined to $I^-(p) \cup \dot{I}^-(p)$, but may leave $\dot{I}^-(p)$
- **Cut point** of γ := last point of γ, still $\in \dot{I}^-(p)$.
  In other words: points on γ beyond cut point can also be reached by a timelike curve
- **Past cut locus** $L^-(p)$:= union of all cut points along past null geodesics from p
- In a globally hyperbolic spacetime, $L^-(p)$ is closed in M and has measure zero in $\dot{I}^-(p)$ [but might be dense in it].

Past light cone with spherical lens

- Thus, if $L^{-}(p) = \emptyset \Rightarrow$ no strong lensing
- For a globally hyperbolic spacetime: at a cut point, not being a conjugate point, two (globally) different null geodesics intersect
- The cut point comes always before or on a conjugate point



Positivity & monotonicity of E_H in cosmology

Theorem:

Let (M, g) be a globally hyperbolic spacetime satisfying the dominant energy condition and $p \in M$. Given a foliation (S_r) of $C^-(p) \cap \dot{I}^-(p)$ by 2-dim. level surfaces $r =$const. [i.e. $\bigcup_r S_r = C^-(p) \cap \dot{I}^-(p)$]. If

(i) $S_r \simeq S^2 \quad \forall r$,

(ii) $\rho < 0$ & $\mu \leq 0$ everywhere $\forall S_r$,

(iii) The foliation (S_r) is constructed as by Eardley,

then $E_H(S_r) \geq 0$ and $\frac{dE_H(S_r)}{dr} \geq 0$.

Intuitively clear, since matter can only leave $I^-(p)$ to the future but not enter!

Proof:

Given the above set-up and assume that (i)-(iii) are true. Then:

$$\frac{dE_H(S_r)}{dr} = \frac{1}{4\pi}\int_{S_r}\left[\Phi_{11} + \frac{1}{8}R + |\alpha + \bar{\beta}|^2 - r\mu(|\sigma|^2 + \Phi_{00})\right] dS_r$$
$$\geq 0$$

since because of the DEC $\Phi_{11} + \frac{1}{8}R \geq 0$ and $\Phi_{00} \geq 0$
$\Rightarrow \frac{dE_H(S_r)}{dr} \geq 0$.

In the limit $r \to 0$:

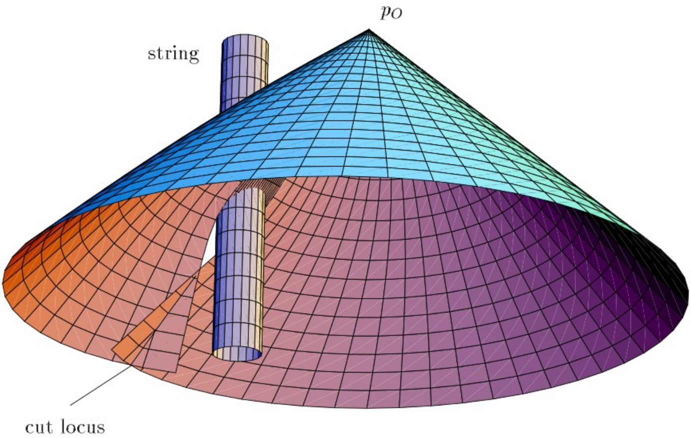
$$E_H(S_r) = \frac{4\pi}{3} r^3 T_{ab} t^a t^b \geq 0 \quad \text{(DEC)} \qquad \square$$

Foliation, Why $C^-(p) \cap \dot{I}^-(p)$, Expansion Scalars

- **Foliation:** only the ones by Eardley allowed, 'gauge freedom' encoded in function $P(r) > 0$, otherwise $E_H(S_r)$ not monotonous
- **Why $C^-(p) \cap \dot{I}^-(p)$ and not just $C^-(p)$?**
  $\rightarrow$ In general, $C^-(p)$ has many self-intersections and a slice fails even to be a submanifold
- $\rho < 0$ & $\mu \leq 0 \Leftrightarrow$ outgoing null congruence expanding, ingoing congruence contracting

Global Hyperbolicity

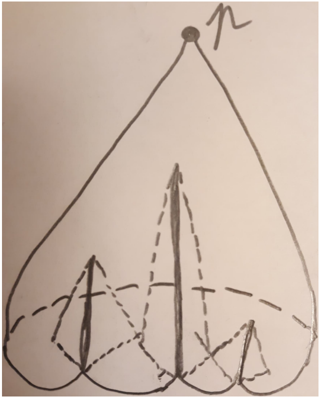
needed in order to exclude non-transparent lenses by cutting out its worldline/tube



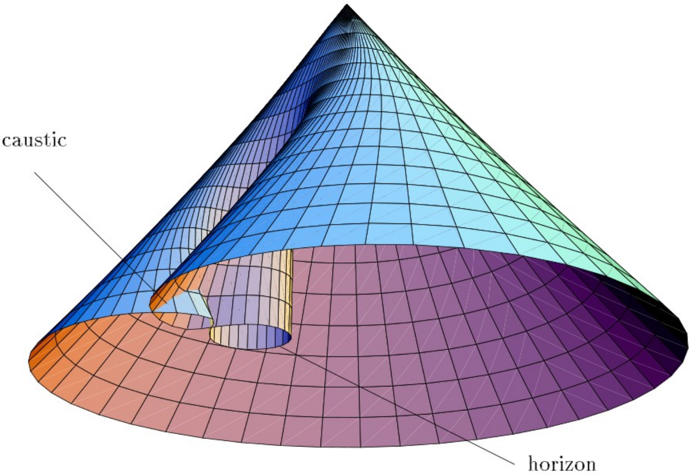
Picture credit: Volker Perlick

Topology of Slices $S_t \simeq S^2$ & Expansion Scalars

- 2-sphere needed in order to be able to define $E_H(S_t)$
- Taking $C^-(p) \cap \dot{I}^-(p)$ + global hyperbolicity + $L^-(p) \cap S_t$ measure zero in $S_t \overset{?}{\Rightarrow} S_t \simeq S^2$



Schwarzschild: S_r can conist of two S^2's



Picture credit: Volker Perlick

Summary

- Hawking Energy E_H has nice properties, however, positivity & monotonicity only given in special cases
- E_H is shown to be positive and monotonously increasing for certain foliations of the past light cone in a suitable spacetime
  $\rightarrow$ how generic are the assumptions?

Thank You!