# HIGHER ORDER MODES

The spatial variation of the intensity (and electric field!) in the plane transverse to propagation is denoted by the

MODE ORDER

$TEM_{mn}$

# of nodes in "X" direction

# of nodes in "Y" direction

$\therefore$ $TEM_{00}$ is the lowest order or "Gaussian" mode.

It is also known as the **FUNDAMENTAL MODE**.

Higher order mode beams may be produced for a variety of reasons:

1. A large volume gain medium with high gain (excimer, $CO_2$ lasers)

2. Portion of the optical axis is obscured.

3. Flawed mirror(s)

- 
- 
- 

TEM$_{mn}$:

$$E(x,y,z) = E_{mn} \cdot H_m\left\{\frac{\sqrt{2}\,x}{w(z)}\right\}$$

$$\cdot H_n\left\{\frac{\sqrt{2}\,y}{w(z)}\right\} \cdot \frac{w_0}{w(z)} \cdot e^{-\frac{(x^2+y^2)}{w(z)^2}}$$

$$\cdot e^{-jkz} \cdot \exp\left\{\frac{-jk(x^2+y^2)}{2R(z)}\right\}$$

$$\cdot \exp\left\{-(1+m+n)\tan^{-1}\left(\frac{z}{z_0}\right)\right\}$$

where the function $H_n(x)$ is known as a

## <u>HERMITE POLYNOMIAL</u> <u>OF ORDER</u> n:

$$H_n(x) \equiv (-1)^n e^{+x^2} \frac{d^n}{dx^n} \left\{ e^{-x^2} \right\}$$

So $\quad H_0(x) = (-1)^0 \, e^{x^2} \, e^{-x^2} = 1$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

and $\quad H_3(x) = 8x^3 - 12x$

As an example, let's consider the $TEM_{00}$ and $TEM_{01}$ modes. Writing only the spatially-varying portion of E(r, z):

$$U_{00} = H_0\left(\frac{\sqrt{2}\,x}{w}\right)H_0\left(\frac{\sqrt{2}\,y}{w}\right)$$

$$\cdot\, e^{-(x^2+y^2)/w^2}$$

$$= e^{-(x^2+y^2)/w^2}$$

But

$$U_{01}(x,y) = H_0 \cdot H_1 \cdot e^{-r^2/w^2}$$

$$= 1 \cdot 2\left(\frac{\sqrt{2}\,y}{w}\right) e^{-r^2/w^2}$$

Notice the clear extension of the mode volume outwards!

Similarly,

$$U_{11}(x,y) = \frac{8xy}{w^2} e^{-r^2/w^2}$$

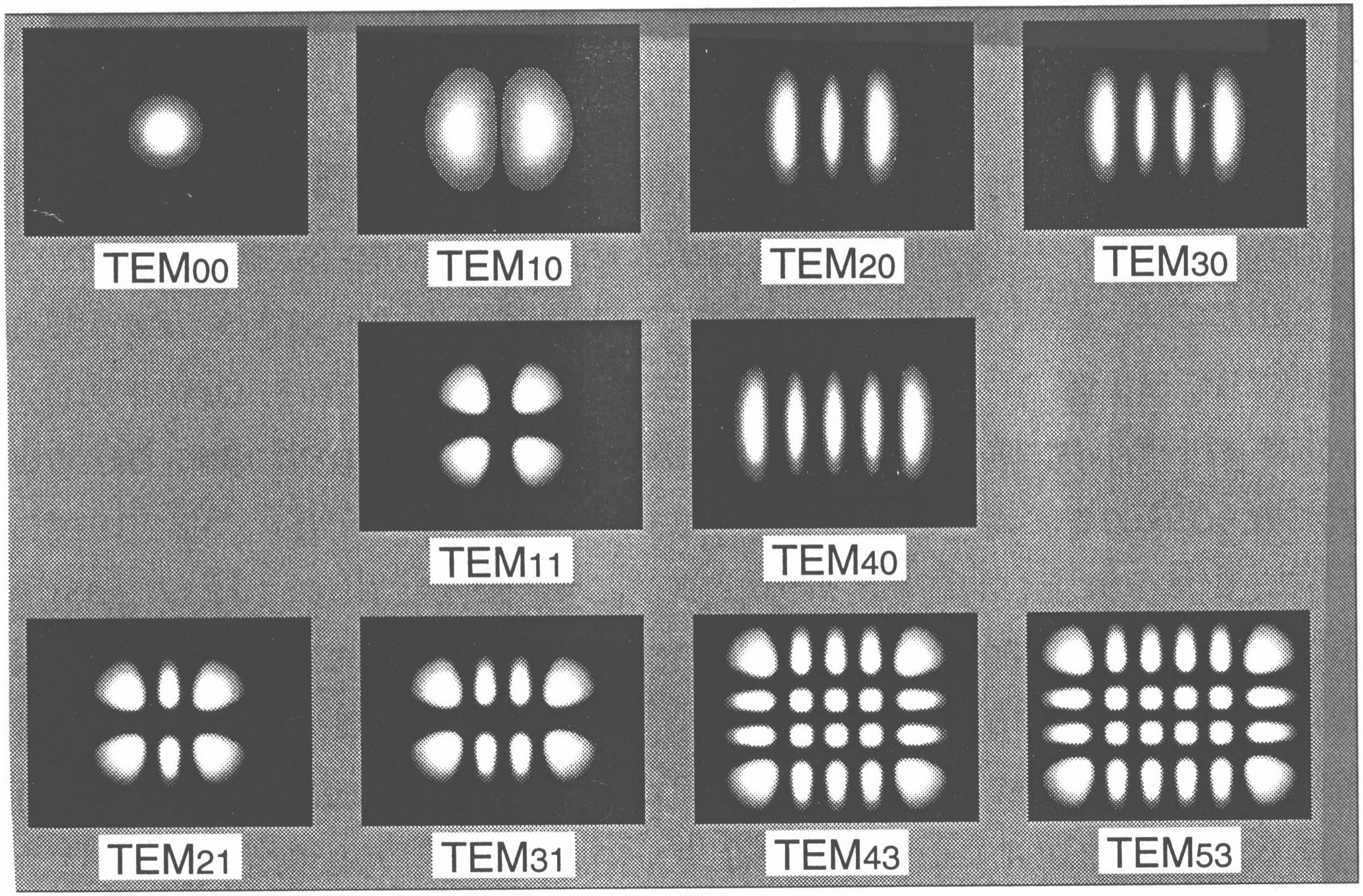
The implications of this are profound:

1. Higher order modes diverge more rapidly than $TEM_{00}$.

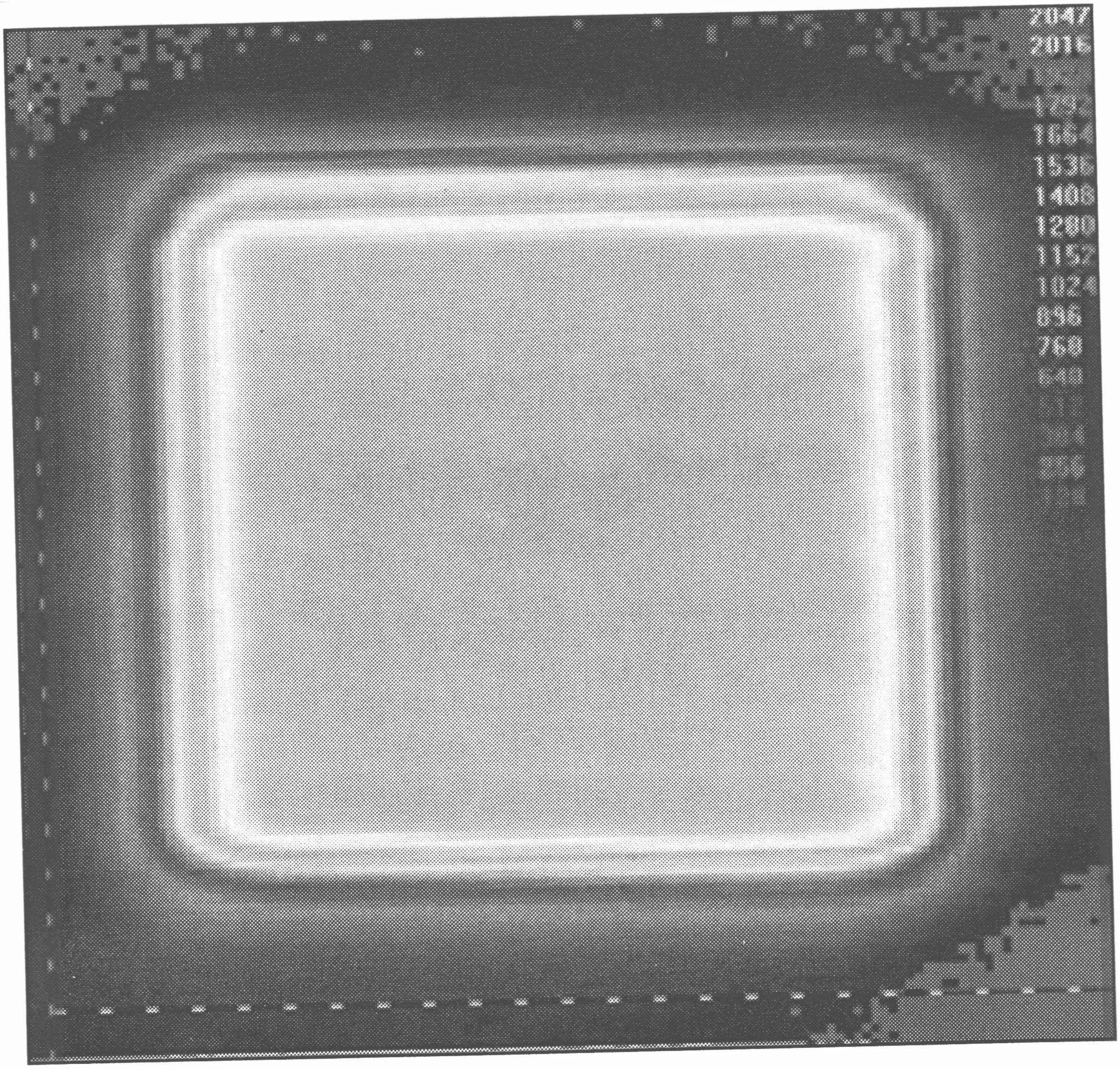
2. $\therefore$ $TEM_{00}$ can be focused more tightly than $TEM_{mn}$ $(m, n \neq 0)$.

*

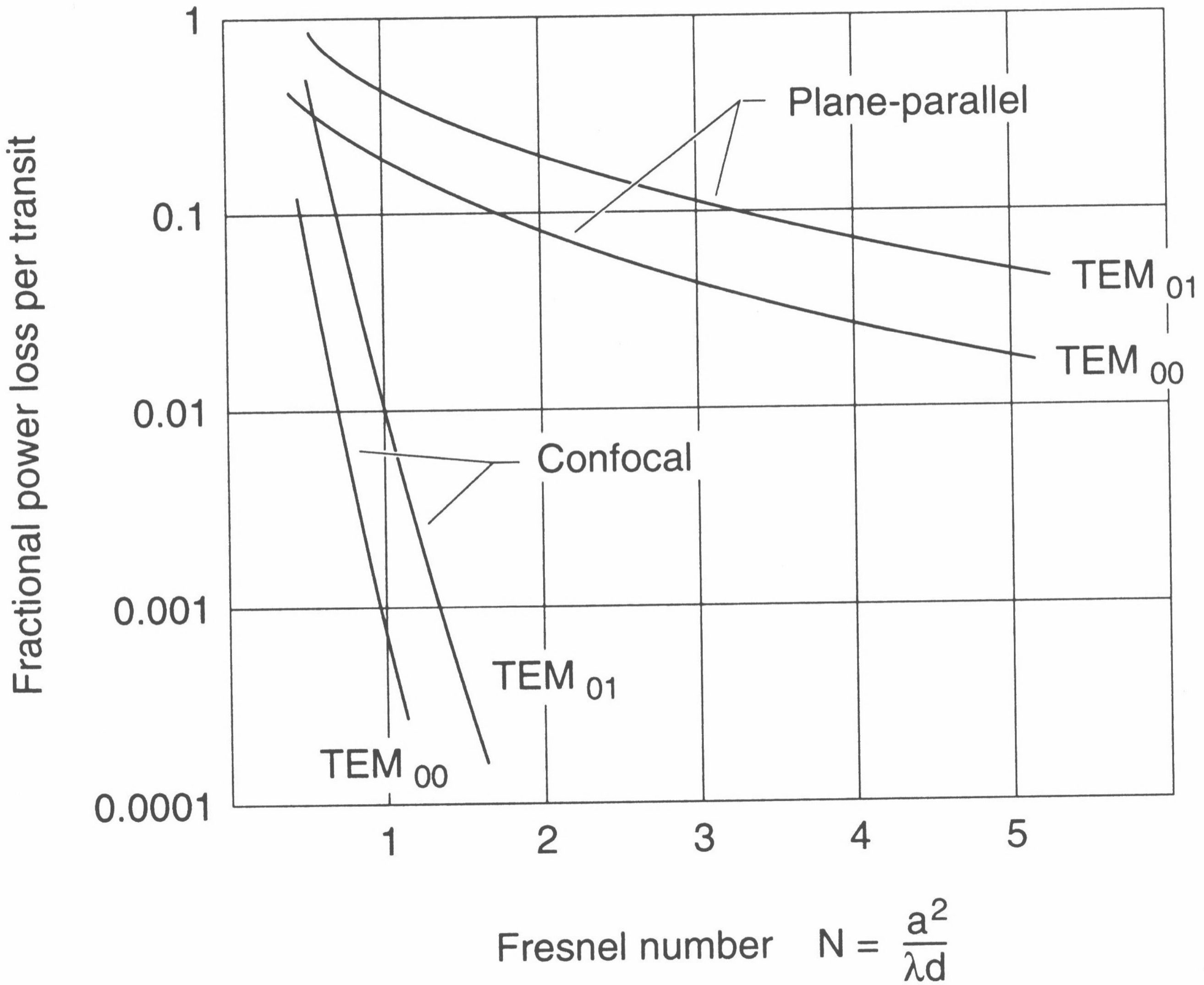
> If focal intensity is important, get rid of high order modes which couple with $TEM_{00}$.



$TEM_{00}$
$TEM_{10}$
$TEM_{20}$
$TEM_{30}$
$TEM_{11}$
$TEM_{40}$
$TEM_{21}$
$TEM_{31}$
$TEM_{43}$
$TEM_{53}$

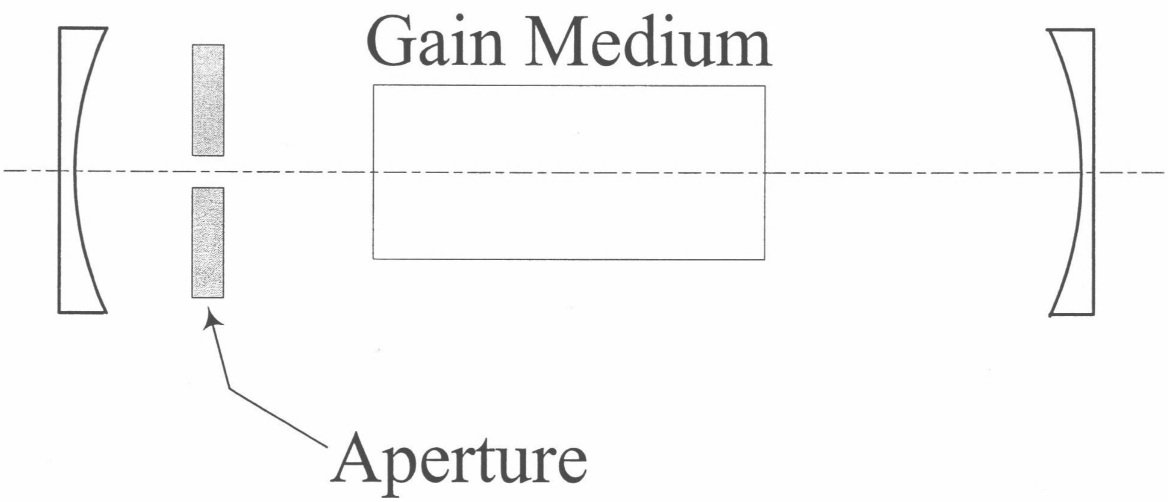



*Homogenized 193 nm excimer beam profile used for dose calibrations, measured with a pyroelectric array camera. The beam uniformity of the laser beam is 1.3 % (determined using ISO 13694 procedures).*




a = MIRROR RADIUS
d = MIRROR SEPARATION
SILFVAST

# FORCING OPERATION ON TEM$_{00}$:

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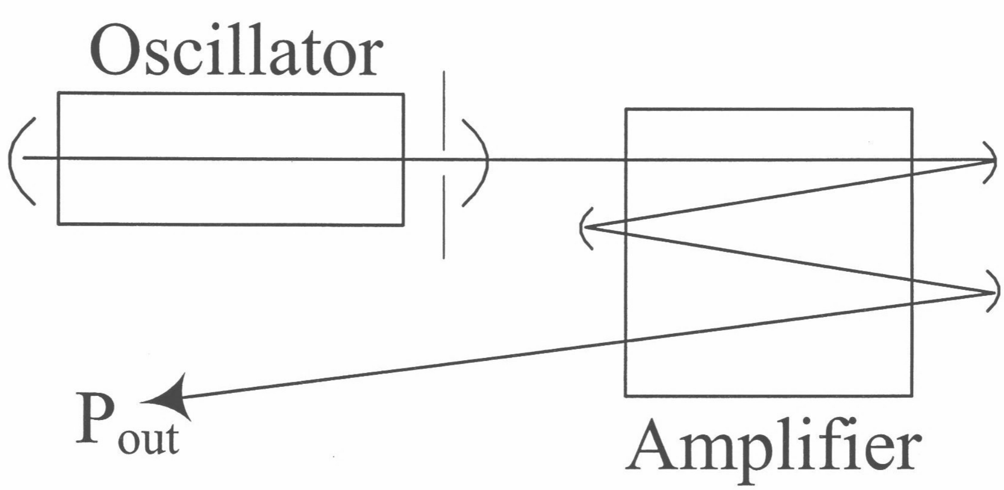
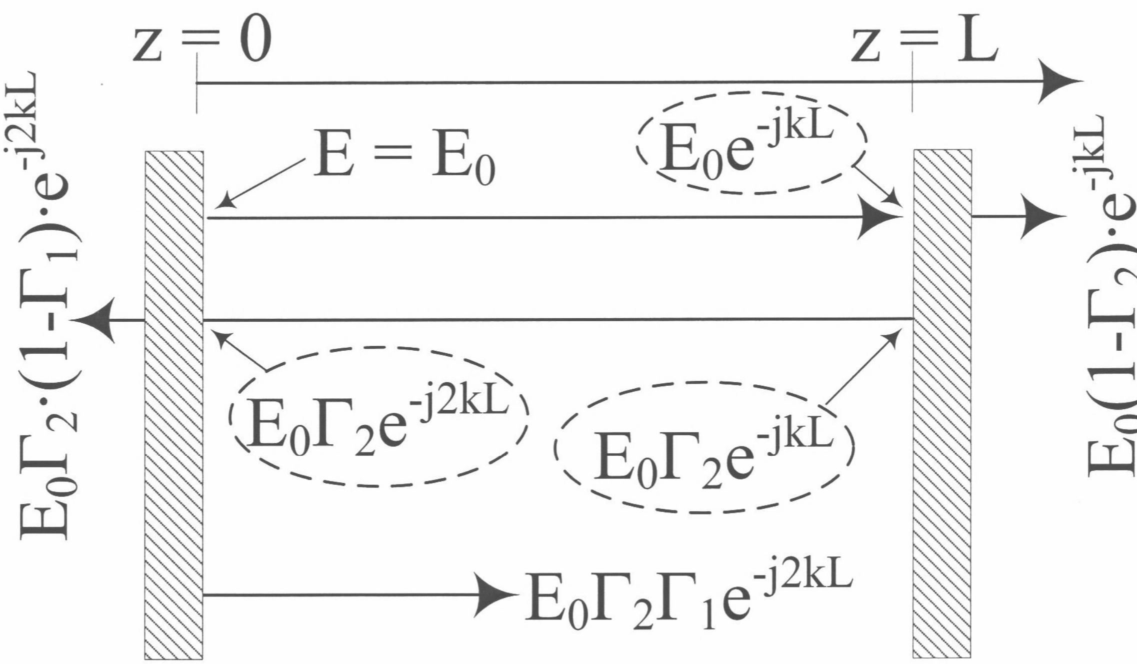
## Master Oscillator-Power Amplifier (MOPA)



# Resonant Optical Cavities

Let us now consider the electric field within a plane parallel resonator.




Mirror; field reflectivity of $\Gamma_1$

Mirror reflectivity of $\Gamma_2$

If we sum the contributions to the $\vec{E}$-field at z = 0 (moving to the right), we have:

$$E_+ = E_0\,(1 + \Gamma_1\Gamma_2 \exp(-jk2L)$$

$$+ (\Gamma_1\Gamma_2)^2 \exp(-jk4L) +$$

$$\ldots)$$

$$= E_0 \sum_{n=0}^{\infty} \{\Gamma_1\Gamma_2 \exp(-jk2L)\}^n$$

$$\equiv \boxed{\frac{E_0}{1-\Gamma_1\Gamma_2 \exp(-jk2L)} = E_+}$$

Similarly, for the electric field contributions moving to the left:

$$E_{-} = \frac{E_0 \Gamma_2 \exp(-jkL)}{1 - \Gamma_1 \Gamma_2 \exp(-jk2L)}$$

<u>Let's convert these field amplitudes to intensities</u>

$$I_{+} = \frac{E_{+} E_{+}^{*}}{2\eta}$$

So

$$I_+ = \frac{\left|E_0\right|^2 \cdot 1}{2\eta\left\{1 - \left|\Gamma_1\Gamma_2\right| \exp(-jk2L) - \left|\Gamma_1\Gamma_2\right| \exp(jk2L) + \left|\Gamma_1\Gamma_2\right|^2\right\}}$$

or

$$I_+ = \frac{E_0^2 / 2\eta}{1 - 2\left|\Gamma_1\Gamma_2\right| \cos(2kL) + \left|\Gamma_1\Gamma_2\right|^2}$$

So

$$I_+ = \frac{I_0}{1 - 2\left|\Gamma_1\Gamma_2\right|\left\{1 - 2\sin^2 kL\right\} + \left|\Gamma_1\Gamma_2\right|^2}$$

Let $\Gamma_1 = R_1^{\frac{1}{2}}$ where $R_1$ **is the intensity reflectivity**.

Note that **$R_1$ is not a radius of curvature**! Sorry about the "double usage" of R but this is the convention in the field.

So

$$\boxed{I_+ = \frac{I_0}{\left\{1-(R_1R_2)^{1/2}\right\}^2 + 4(R_1R_2)^{\frac{1}{2}}\sin^2 kL} \quad \text{where } I_0 = {E_0^2}/{2\eta}, \quad k = {2\pi n}/{\lambda}}$$

The question now is: **Under what conditions is $I_+$ maximum?**

In other words, for what values of $\lambda$ or L is the cavity "resonant"?

$I_+$ is max. when

$$\sin^2 kL = 0$$

$$\Rightarrow kL = m\pi$$

where m is an integer. So…

$$L \cdot \frac{2\pi n}{\lambda} = m\pi$$

or

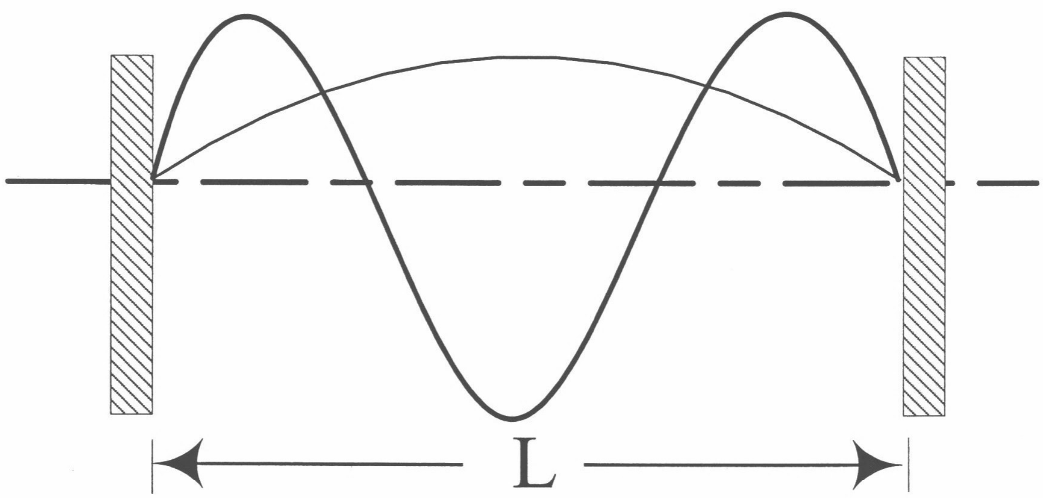
$$\boxed{\begin{aligned}\lambda_m &= \frac{2nL}{m}\\ \nu_m &= \frac{m \cdot c}{2nL}\end{aligned}}$$

The $\nu_m$ are frequencies at which the optical cavity is resonant.

Note that $\nu_m$ and $\nu_{m+1}$ (**known as longitudinal modes**) are separated by

$$\boxed{\Delta\nu \equiv \nu_{m+1} - \nu_m = \frac{c}{2nL}}$$

The $\nu_m$ represent those frequencies (for a fixed L) for which **an integral number of half-wavelengths fit between the two mirrors**.




$$L = m \cdot \left( \frac{n\lambda}{2} \right) \quad \text{or} \quad \nu = \frac{mc}{2nL}$$

Another way of looking at this is that the round trip phase shift must $= m \cdot 2\pi$

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## EXAMPLES

1. Consider a semiconductor laser

$L$ $\cancel{d} = 100\ \mu m$, $n = 3.6$ (GaAs)

$$\text{So } \Delta\nu = \frac{c}{2nL} \simeq 4.2 \cdot 10^{11}\ \text{Hz}$$

$$\boxed{\Delta\nu = 420\ \text{GHz}}$$

If $\lambda = 800$ nm, then

$$\frac{\Delta\nu}{\nu} = 10^{-3}$$

Also, $\Delta L$ (the separation between longitudinal modes in space) is

$$\Delta L = \frac{\lambda}{2n} = 0.11\ \mu\text{m}$$

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2. Now consider a larger laser: copper vapor.

$$\lambda = 510.6\ \text{nm}$$
$$\nu = 5.88 \cdot 10^{14}\ \text{Hz}$$

$n \simeq 1.0$

Let L = 100 cm;

$$\text{then } \Delta\nu = \frac{c}{2nL} = 1.5 \cdot 10^8\ \text{s}^{-1}$$

$$\text{or } \boxed{\Delta\nu = 0.15\ \text{GHz}}$$

$$\text{Also, } \frac{\Delta\nu}{\nu} = \frac{1.5 \cdot 10^8}{5.88 \cdot 10^{14}}$$

$$= 2.6 \cdot 10^{-7}$$

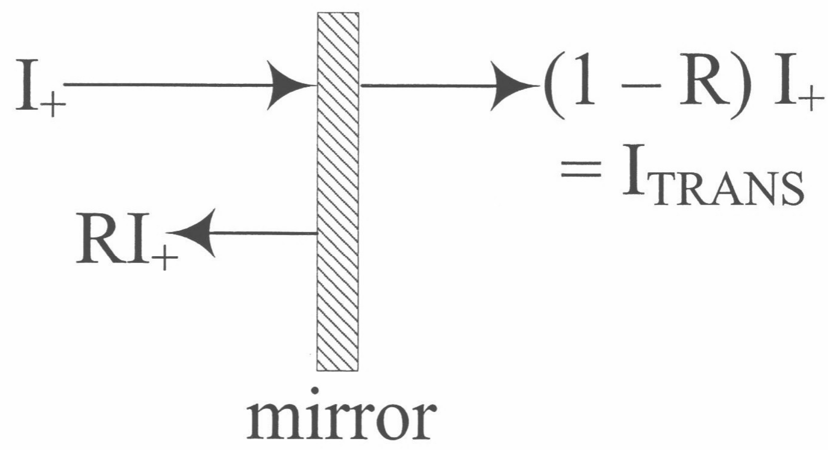
Similarly,

$$\Delta L = \frac{\lambda}{2n} = 0.26\ \mu\text{m!}$$

$$\Delta\lambda = \Delta\nu \cdot \frac{c}{\nu^2} = 1.3 \cdot 10^{-3}\ \text{Å}$$

$$= 1.3 \cdot 10^{-4}\ \text{nm}$$

# TRANSMITTED POWER



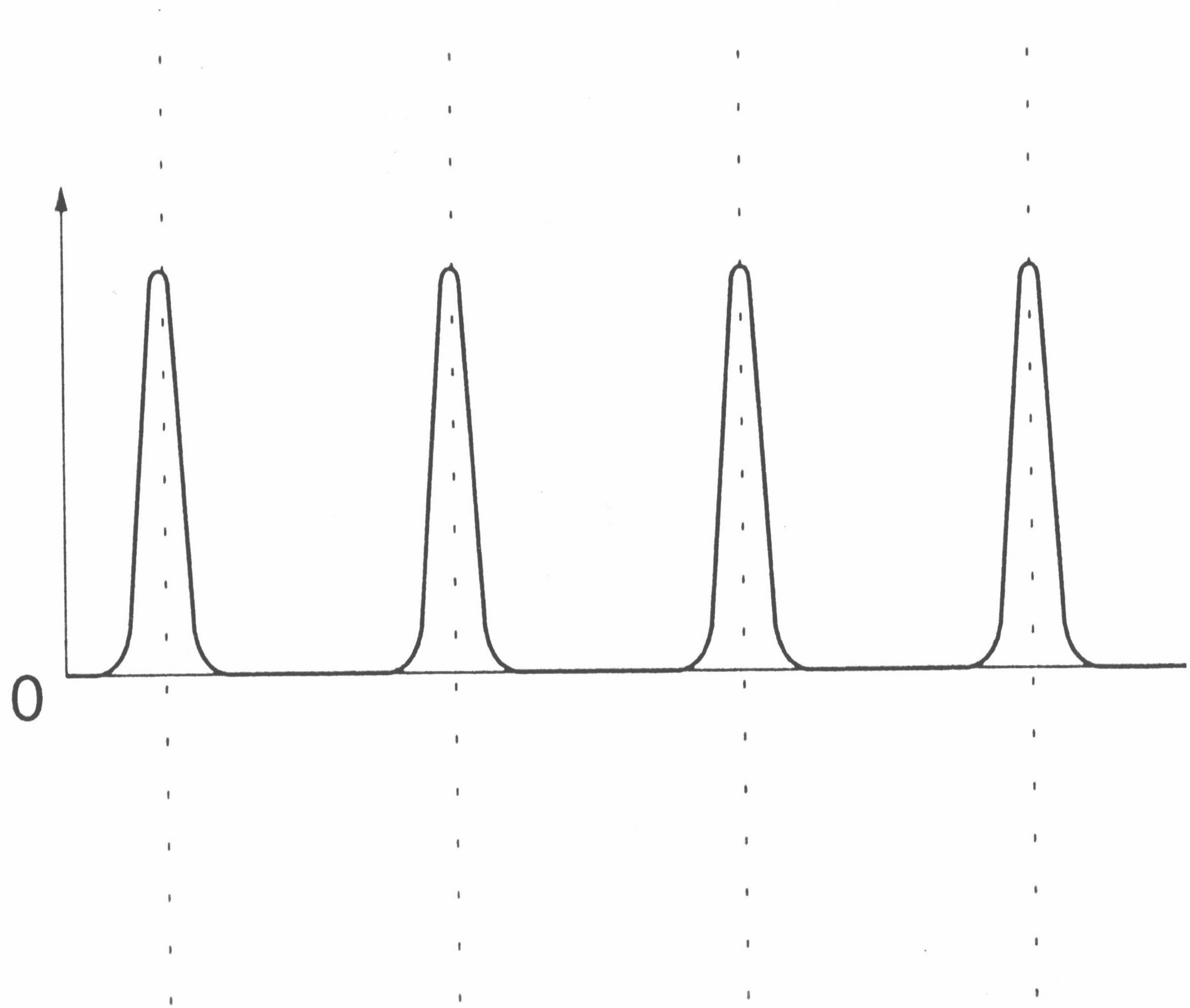

The intensity transmitted by the right-hand mirror is

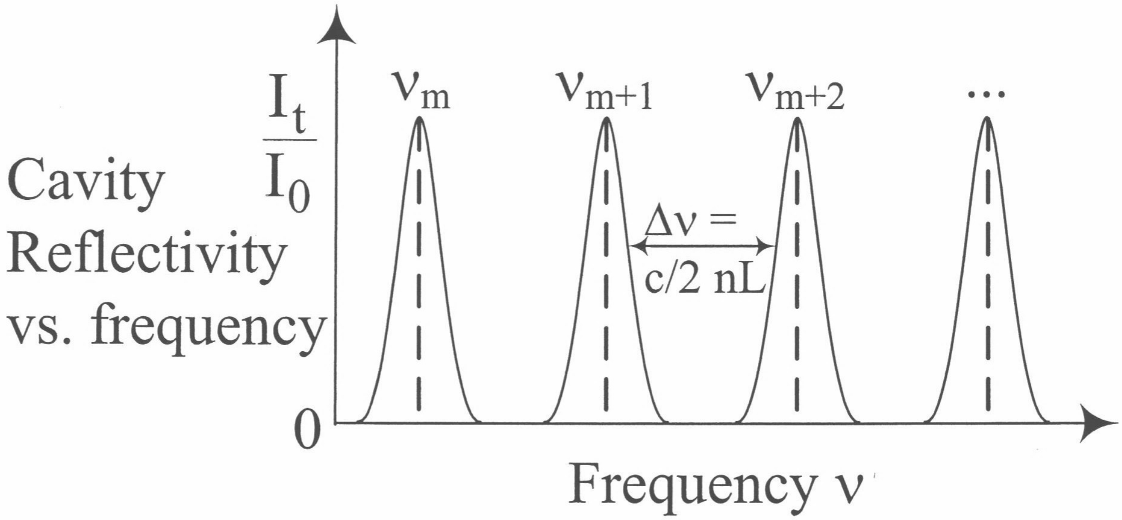
$$I_t = I_0 \cdot$$

$$\cdot \frac{1 - R_2}{\left\{1 - (R_1 R_2)^{\frac{1}{2}}\right\}^2 + 4(R_1 R_2)^{\frac{1}{2}} \sin^2 kL}$$

(b)

Cavity reflectivity vs. frequency






Hold L constant, vary $\nu$ (can also fix $\nu$ and "sweep" L)

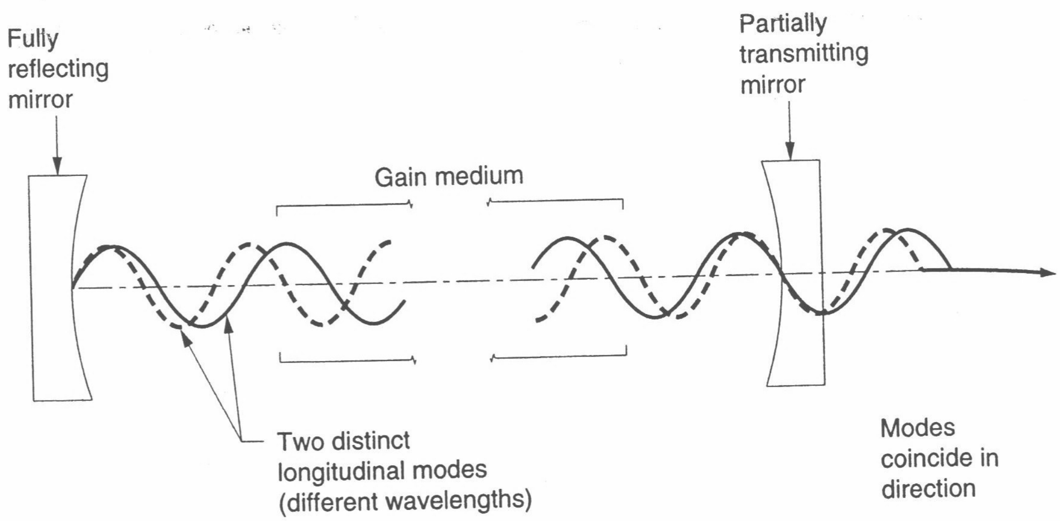



**Figure 10-5.** Diagram of two longitudinal laser modes

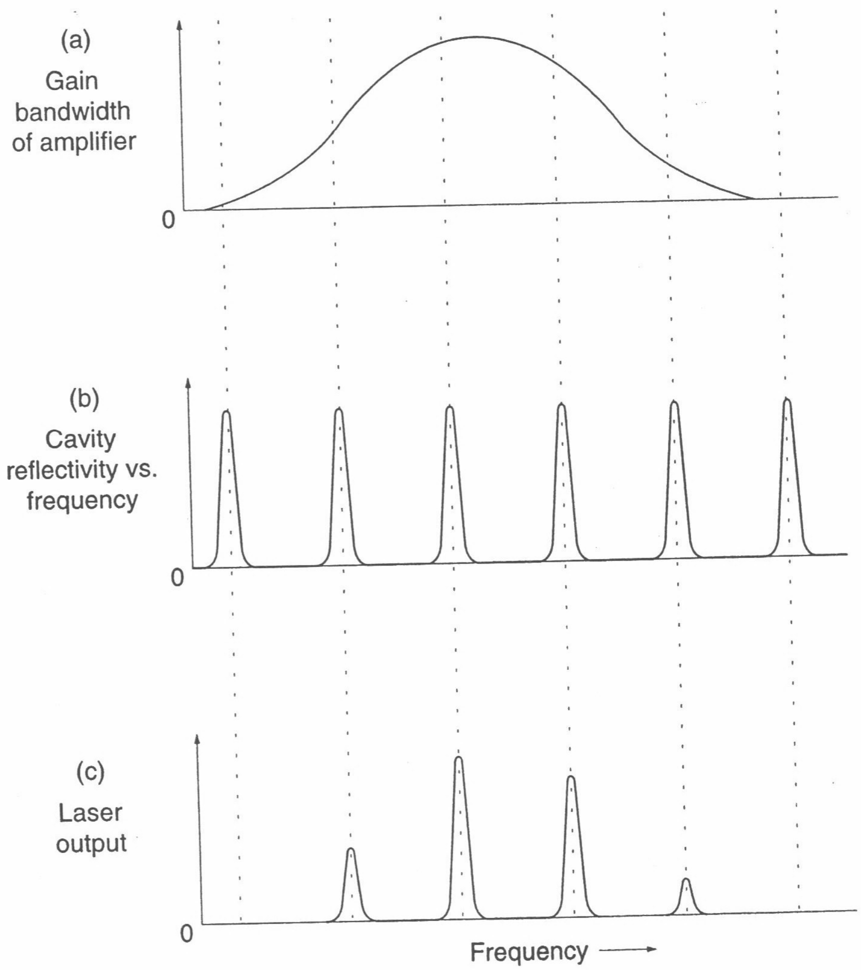
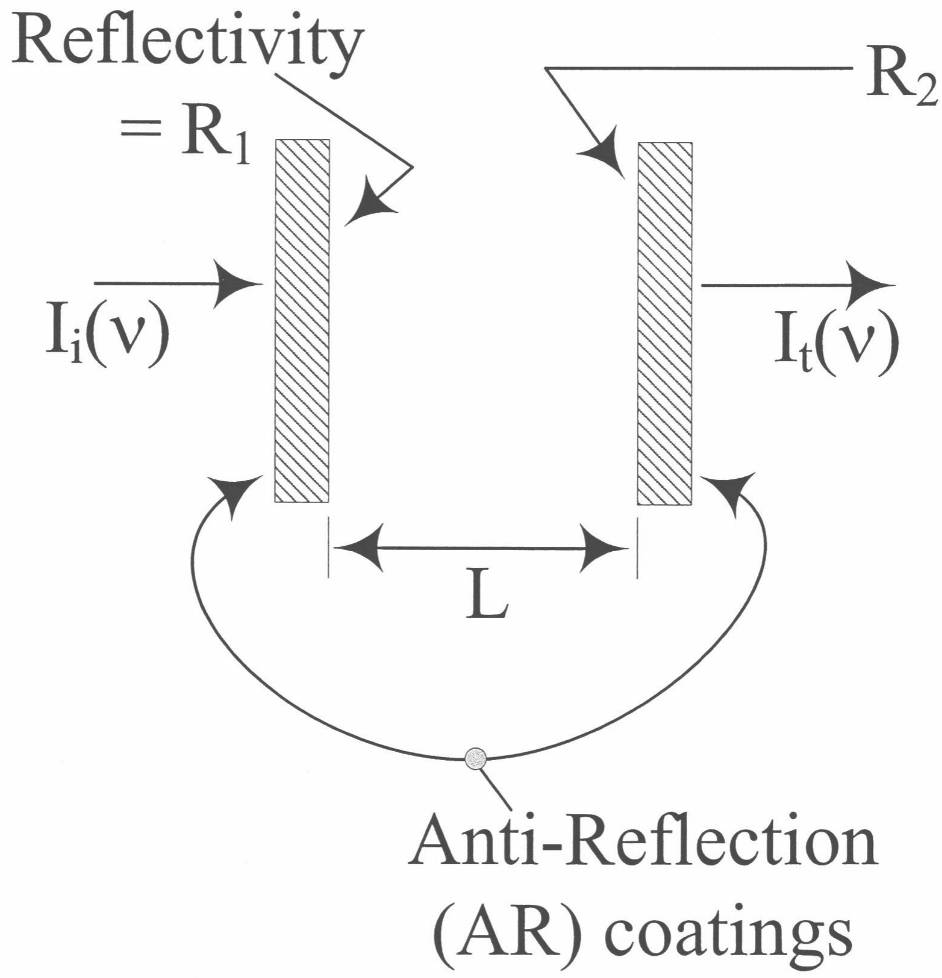
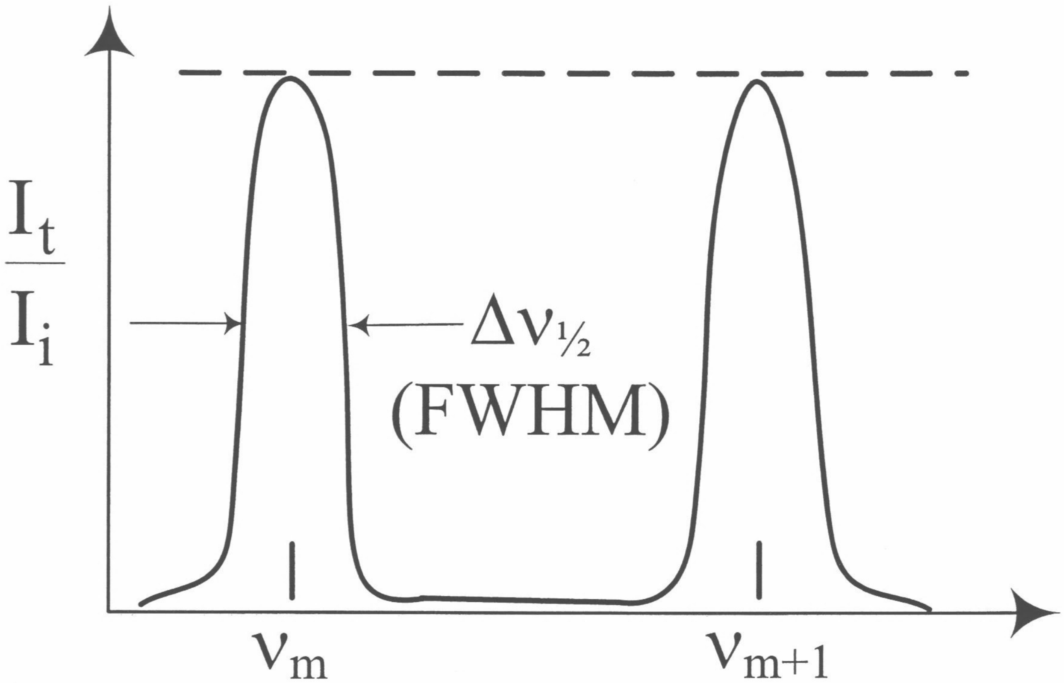



**Figure 10-6.** Resulting laser cavity modes when a gain bandwidth of a laser amplifier is combined with resonances of a two-mirror laser cavity

# Fabry-Perot Cavities

Two parallel, flat mirrors:






Q = quality factor of the cavity

$$\boxed{Q \equiv \frac{\nu_m}{\Delta\nu_{1/2}}}$$

At RF frequencies (3-30 MHz),
$Q \sim 10^2 - 10^4$

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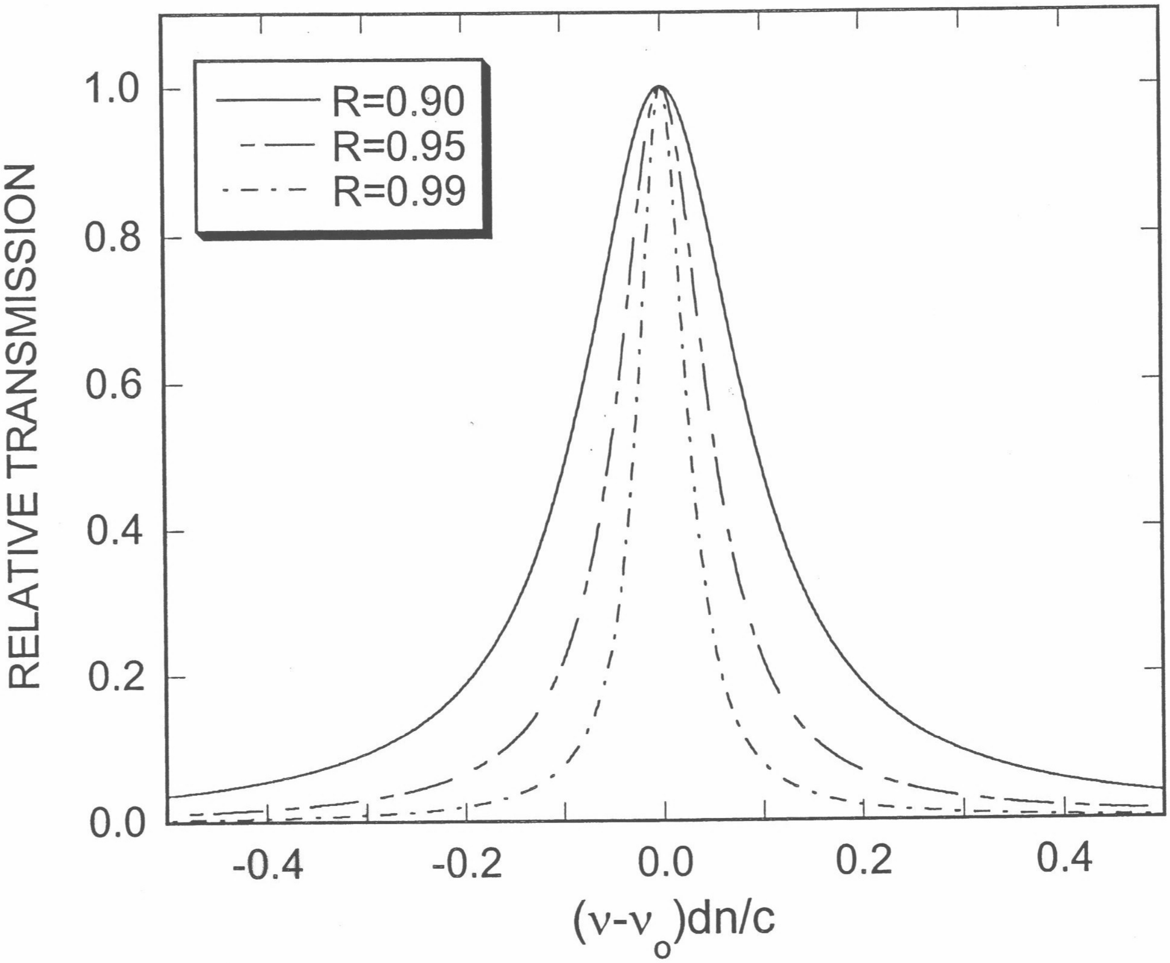
At optical frequencies
($\sim 10^{14} - 10^{15}$ Hz), $Q \sim 10^5 \rightarrow > 10^{11}$

Remember that

$$I_t(\lambda) = \frac{I_i(\lambda)\cdot(1-R_2)}{\left\{1-(R_1R_2)^{\frac{1}{2}}\right\}^2 + 4(R_1R_2)^{\frac{1}{2}}\sin^2\left(\frac{2\pi nL}{\lambda}\right)}$$

The wavelengths of the longitudinal modes (max. transmission of the cavity) occur when

$$\frac{2\pi nL}{\lambda} = m\pi$$




Transmission through a Fabry-Perot etalon as a function of intra-mode spacing for different mirror reflectivities.

**Find $\Delta\nu_{1/2}$**: If $\nu_{1/2}$ is the frequency corresponding to the 50% point for the $\nu_m$ mode and

$$\Delta\nu_{1/2} = (\nu_{1/2} - \nu_m) \cdot 2,$$

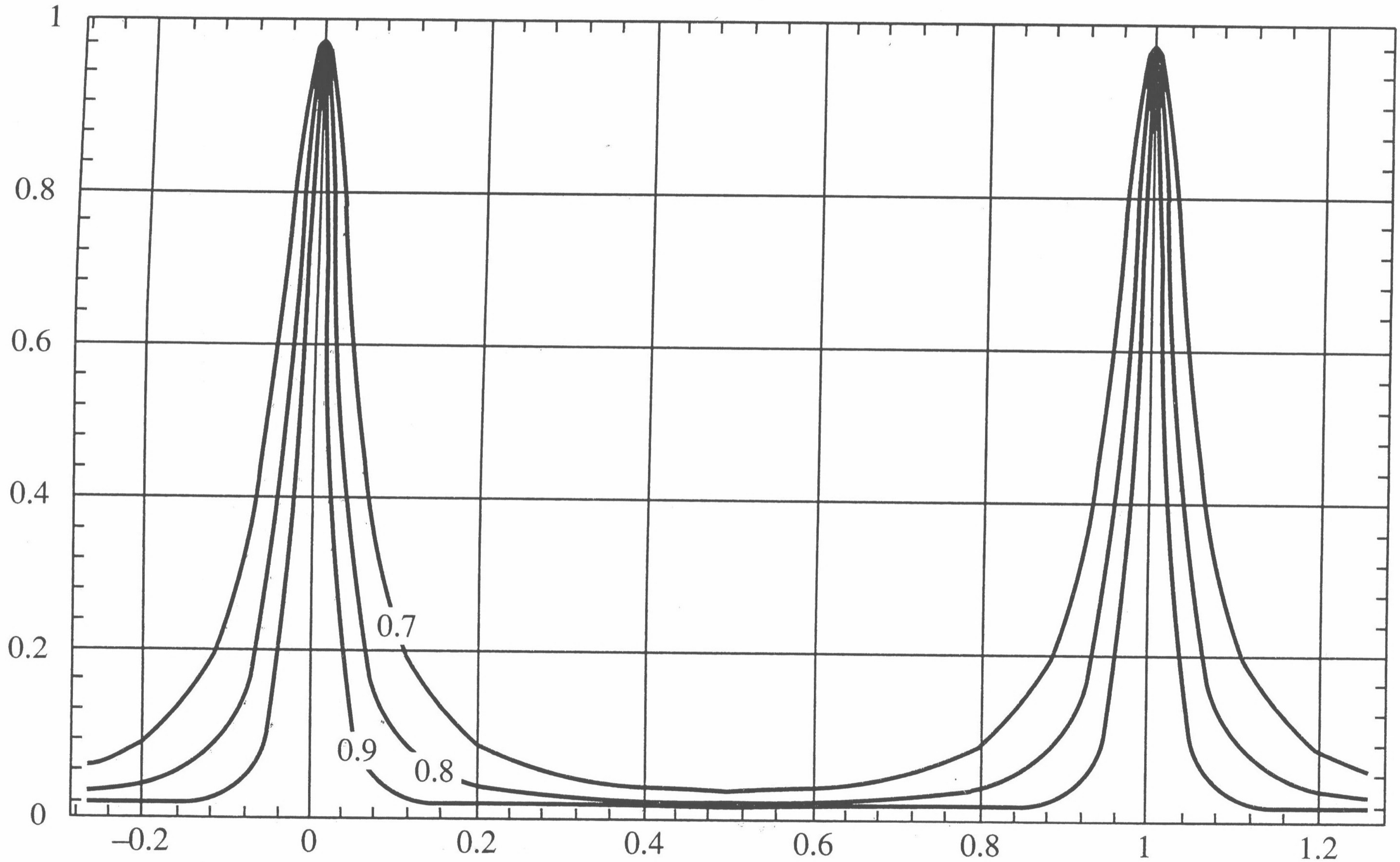
Then $\dfrac{I_t(\nu_{1/2})}{I_t(\nu_m)} = \dfrac{1}{2}$

$$\boxed{\Delta\nu_{1/2} = \frac{c\left(1-\{R_1R_2\}^{\frac{1}{2}}\right)}{2nL\,\pi(R_1R_2)^{\frac{1}{4}}}}$$

and

$$\boxed{Q = \frac{2nL\,\pi(R_1R_2)^{\frac{1}{4}}}{\lambda\left\{1-(R_1R_2)^{\frac{1}{2}}\right\}}}$$

Notice that $Q\uparrow$ as $R_1R_2 \rightarrow 1$.


1
0.8
0.6
0.4
0.2
0
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
0.7
0.8
0.9

Another measure of the spectral "resolution" of a Fabry-Perot cavity is its **FINESSE**:

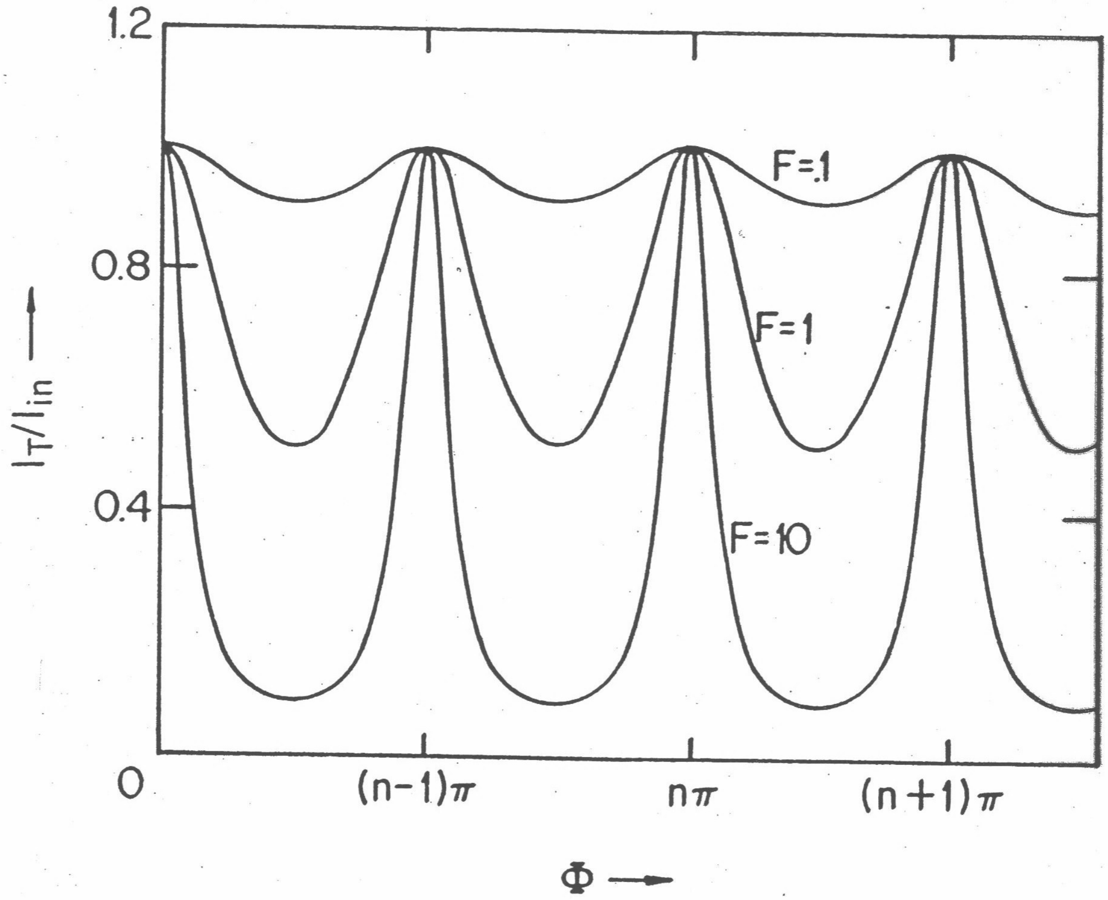
$$F \equiv \frac{\text{mode spacing}}{\Delta\nu_{1/2}}$$

$$= \frac{c/2nL}{\dfrac{c}{2nL}\dfrac{\left\{1-(R_1R_2)^{\frac{1}{2}}\right\}}{\pi(R_1R_2)^{\frac{1}{4}}}}$$

or

$$\boxed{F = \frac{\pi(R_1R_2)^{\frac{1}{4}}}{1-(R_1R_2)^{\frac{1}{2}}}}$$

$$\mathfrak{F} = \frac{\pi}{2}\sqrt{F} = \frac{\pi\sqrt{R}}{1 - R}$$



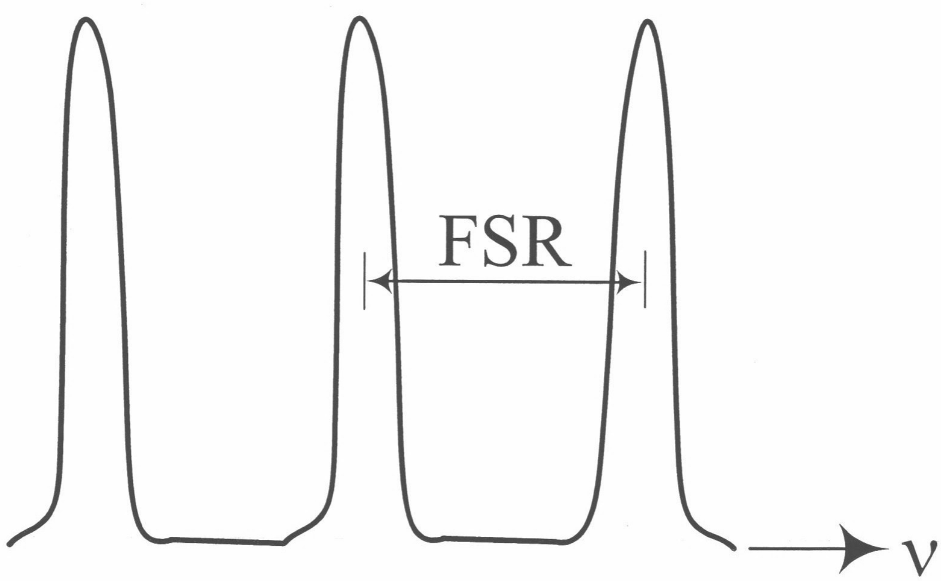

Transmission function of the Fabry–Perot etalon for three values of $\mathfrak{F}$.

# Another Definition

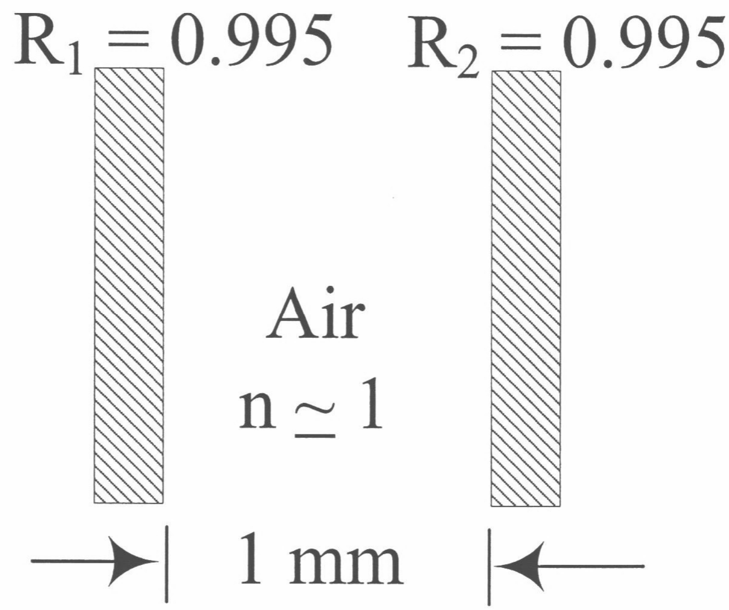
The frequency separation between two adjacent longitudinal modes of a resonator, previously termed the "mode spacing," is known as the

## Free Spectral Range

$$\boxed{FSR \equiv \frac{c}{2nL}}$$



# EXAMPLE




For $R_1 = R_2 = 99.5\%$ and $\lambda = 1\ \mu m$.

Then the resonator Q is

$$Q = \frac{2 \cdot 0.1\,\mathrm{cm}\,\pi\,(0.995)^{\frac{1}{2}}}{10^{-4}\,\mathrm{cm}\{1 - 0.995\}}$$

$$\boxed{Q = 1.25 \cdot 10^6}$$

$$\Delta\nu_{\frac{1}{2}} = \frac{c\left\{\ 1-(R_1R_2)^{\frac{1}{2}}\ \right\}}{2nL\pi(R_1R_2)^{\frac{1}{4}}}$$

$$\boxed{\Delta\nu_{\frac{1}{2}} = 239\ \text{MHz}}$$

$$\boxed{\text{FSR} = \frac{c}{2nL} = 150\ \text{GHz}}$$

$$\text{Finesse } F = \frac{\pi(R_1R_2)^{\frac{1}{4}}}{1-(R_1R_2)^{\frac{1}{2}}}$$

$$\boxed{F = 625}$$

Have you noticed that

$$\boxed{F \cdot \Delta\nu_{\frac{1}{2}} = \text{FSR}\,?}$$

If a Fabry-Perot resonator has "L" fixed, we generally refer to it as an

# <u>ETALON</u>

## <u>EXAMPLE</u>

Consider an $Al_{0.5}Ga_{0.5}As$ ($n \doteq 3.25$) ETALON, 500 μm in thickness.

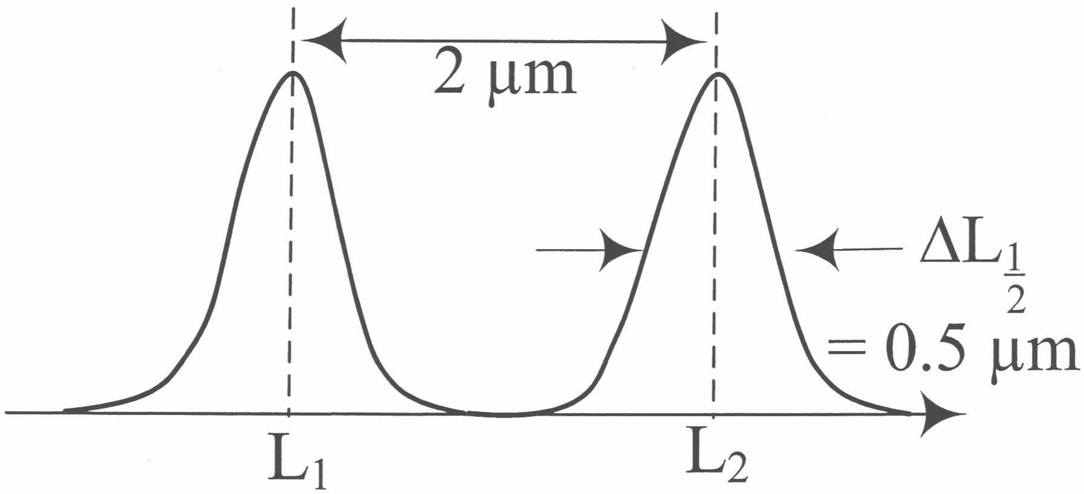
Then

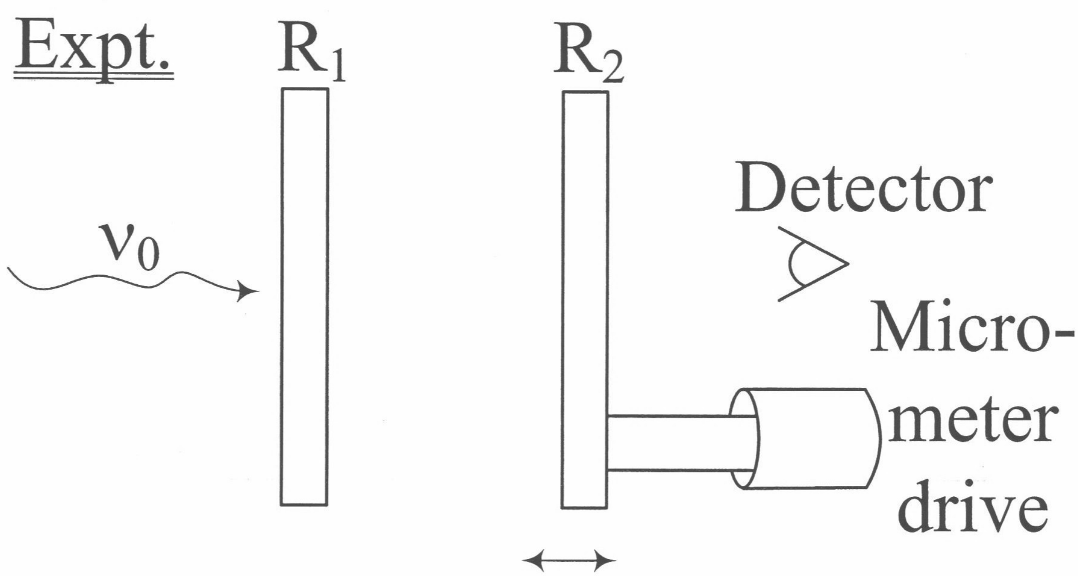
$$\boxed{\text{FSR} = \frac{3 \cdot 10^{10}\ \text{cm/s}}{2 \cdot 3.25 \cdot 500 \cdot 10^{-4}\ \text{cm}} = 92\ \text{GHz}}$$

# TWO MORE EXAMPLES

1. Consider an air-filled ETALON for which $R_1 = R_2$ and $L = 1$ mm.

We measure



**A.** What is $\lambda$? Remember that a longitudinal mode occurs when

$$\frac{2\pi nL}{\lambda} = m\pi$$

And $\boxed{\therefore\ L = \frac{\lambda}{2n} \cdot m}$

So, the separation between adjacent modes is

$$\boxed{\begin{array}{l} \Delta L = \dfrac{\lambda}{2n} = 2\ \mu m \\ \Rightarrow \lambda = 4\ \mu m \quad \text{(IR!)} \end{array}}$$

**B.** What is the mirror reflectivity?

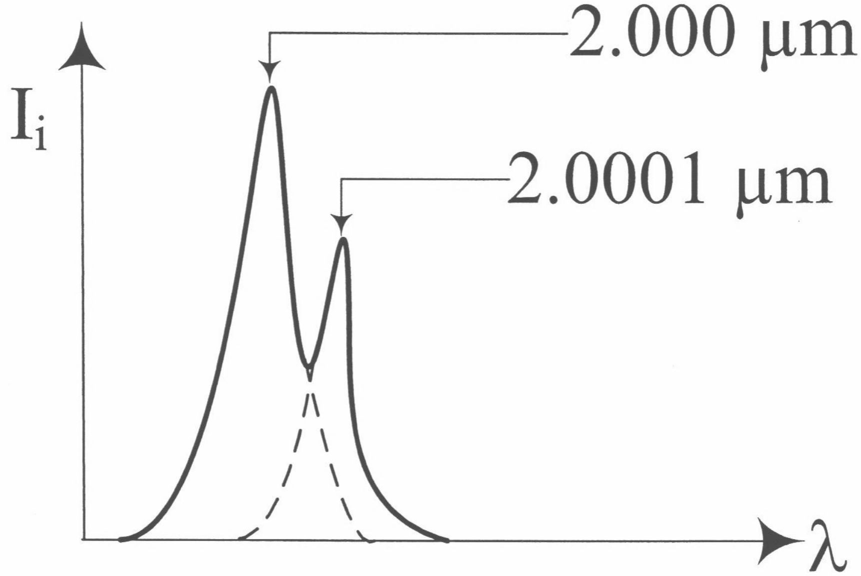
$$F = \frac{FSR}{\Delta\nu_{\frac{1}{2}}} = \frac{\Delta L}{\Delta L_{\frac{1}{2}}} = 4$$

$$= \frac{\pi (R_1 R_2)^{\frac{1}{4}}}{\left\{ 1 - (R_1 R_2)^{\frac{1}{2}} \right\}}$$

So

$$\boxed{R_1 = R_2 = 46\%}$$

C. Now suppose the radiation impinging on the FP ETALON is spectrally broad:




$$\therefore\ \Delta\nu_{\frac{1}{2}} << \frac{c}{\lambda_1} - \frac{c}{\lambda_2} = 7.5\ \text{GHz}$$

$$\text{Choose } \Delta\nu_{\frac{1}{2}} = \frac{7.5\ \text{GHz}}{20}$$

$$\therefore\ F = 4 = \frac{FSR}{\Delta\nu_{\frac{1}{2}}} = \frac{\overset{c/2L}{150\ \text{GHz}}}{\frac{7.5\ \text{GHz}}{20}}$$

where L = 0.1 cm.

$$\text{But } F = \frac{\pi(R_1R_2)^{\frac{1}{4}}}{1-(R_1R_2)^{\frac{1}{2}}}$$

$$\text{So } \boxed{R_1 = R_2 \doteq 85\%}$$

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## **SUMMARY**

1. **Resonance**:

$$\boxed{\frac{2\pi nL}{\lambda} = m\pi \quad \text{integer}}$$

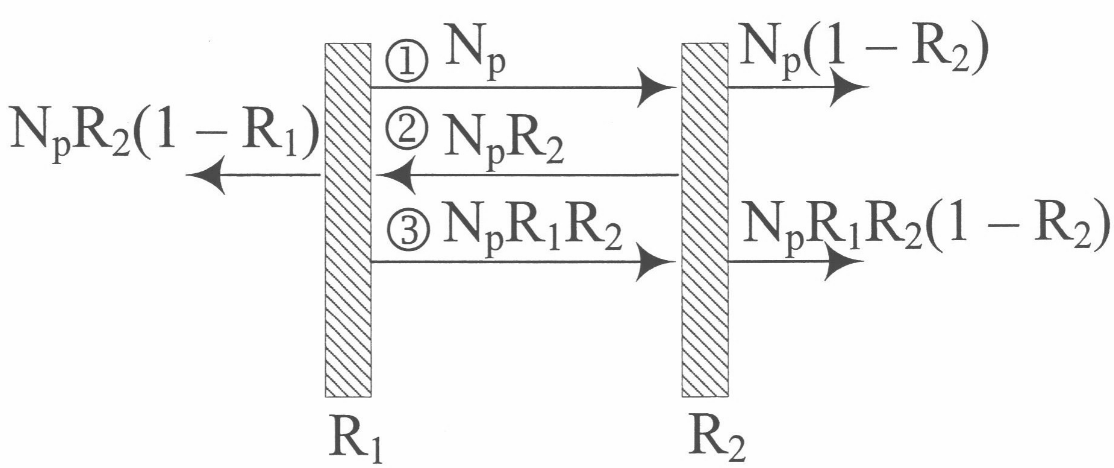
2. Can either fix L and vary $\lambda$ or fix $\lambda$ and vary L.

3. Find Q, FSR, F, and $\Delta\nu_{\frac{1}{2}}$.

# PHOTON (OR CAVITY) LIFETIME

Suppose we put 100 photons into the cavity below at t = 0. At time t, how many remain?

Let $N_p$ = number of photons in the cavity at time t




We are, of course, assuming that, aside from transmission losses ($R_{1,2} \neq 1$), the mirrors are perfect (i.e., no scattering or absorption).

$\therefore$ The time rate of change in the photon number within the resonator is

$$\frac{dN_p}{dt} = -\frac{N_p \cdot (1 - R_1 R_2)}{\Delta t_{\substack{\text{round}\\\text{trip}}}}$$

This is the fractional loss per round trip

where

$\Delta t_{\substack{\text{round}\\\text{trip}}}$ is the round trip transit time $= \dfrac{2nL}{c}$

$$\therefore \quad \frac{dN_p}{dt} \equiv -\frac{N_p}{\tau_{photon}}$$

where

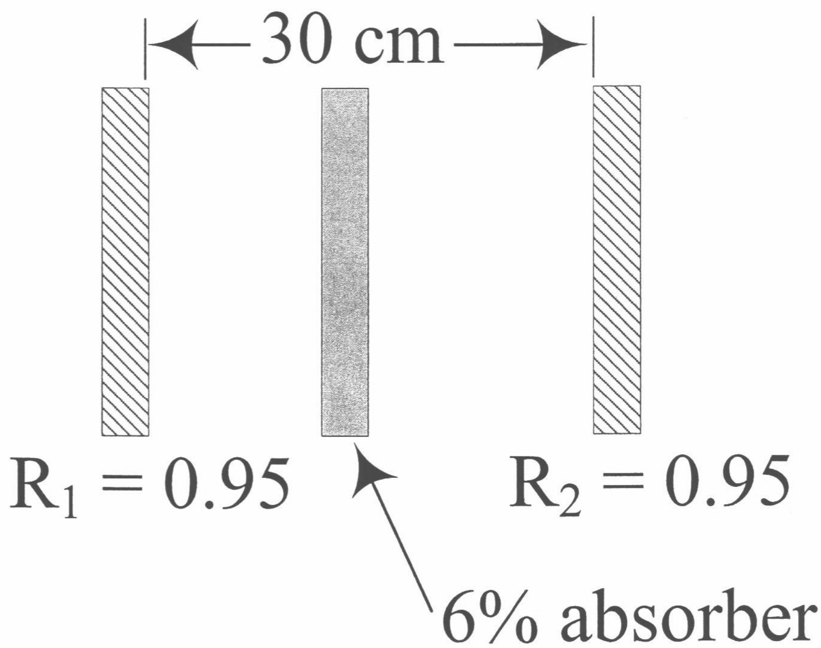
$$\boxed{\tau_{photon} = \frac{2nL}{c(1-R_1R_2)}}$$

<u>$\tau_p$</u> <u>is also known as the cavity</u>

<u>ringdown time</u>.

## EXAMPLES

1. What is $\tau_p$ for a cavity with $R_1 = R_2 = 95\%$, $L = 30$ cm, and a 6% absorber?




(for this calculation, it does not matter if the absorber is localized or distributed).

Remember,

$$\tau_p = \frac{\Delta t_{\text{round trip}}}{\text{fractional loss per round trip}}$$

Fractional loss/RT

$$= 1 - (\underbrace{0.94}_{\text{absorber}} \cdot \underbrace{0.95}_{\text{mirrors}})^2$$

Double pass through medium

$\therefore$

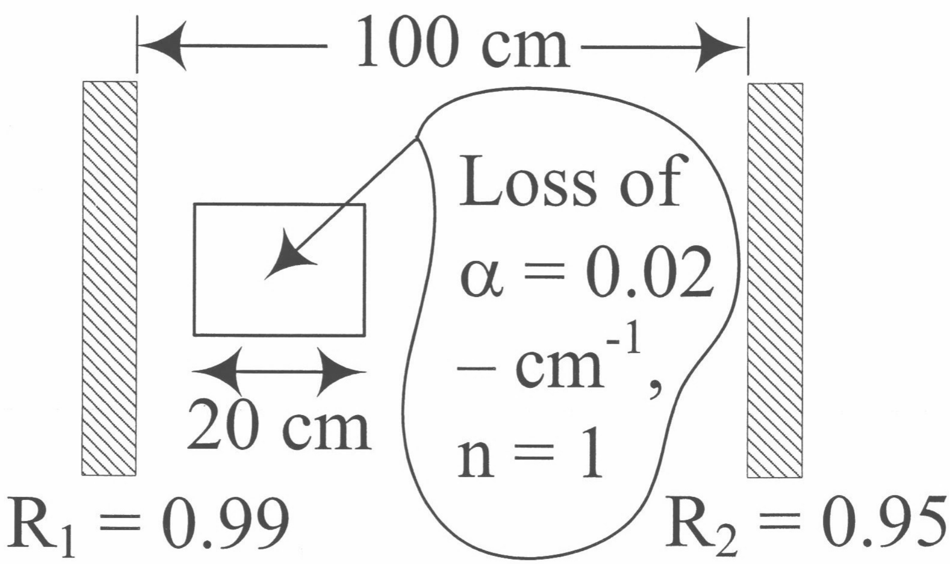
$$\tau_p = \frac{2 \cdot 30\ \text{cm}}{3 \cdot 10^{10}\ \text{cm s}^{-1}\left(1 - \{0.94 \cdot 0.95\}^2\right)}$$

$$\boxed{\tau_p \simeq 9.9\ \text{ns}}$$

Without the 6% absorber, $\tau_p$ is

$$\boxed{\tau_p = \frac{2 \cdot 30}{3 \cdot 10^{10}\left\{\ 1-(0.95)^2\ \right\}} = 20.5 \text{ ns!}}$$

2. What is $\tau_p$ for the resonator below?



Fractional loss through the absorber is

BEER–LAMBERT ABSORPTION → $e^{-\alpha\ell}$, $\ell = 20$ cm

$\therefore$ fractional loss/RT is

$$1-0.95\cdot 0.99\, e^{-2\cdot 2\cdot 10^{-2}\cdot 20}$$

and

$$\boxed{\tau_p = 3.85 \text{ ns}}$$

Notice that measuring $\tau_p$ is a convenient method of measuring small intracavity losses. **<u>This is the basis of CAVITY RINGDOWN SPECTROSCOPY.</u>**

# Resonant Frequencies of Hermite-Gaussian Modes

In a laser operating on a number of transverse modes, the situation is a bit more complicated.

*Remember*

$$E(x,y,z) = E_{mn} \cdot H_m(\cdots x) H_n(\cdots y)$$

$$\cdot \frac{w_0}{w(z)} \cdot e^{-(r/w)^2} \cdot e^{-jkr^2/2R(z)}$$

$$\underbrace{\cdot\, e^{-jkz} \cdot \exp\left\{-j(1+m+n)\tan^{-1}\left(\frac{z}{z_0}\right)\right\}}_{\text{axial phase shift}}$$

**POINT:**

The resonance condition is unchanged [*round trip phase shift* = $q \cdot 2\pi$] but the phase shift for each $TEM_{mn}$ mode is different.

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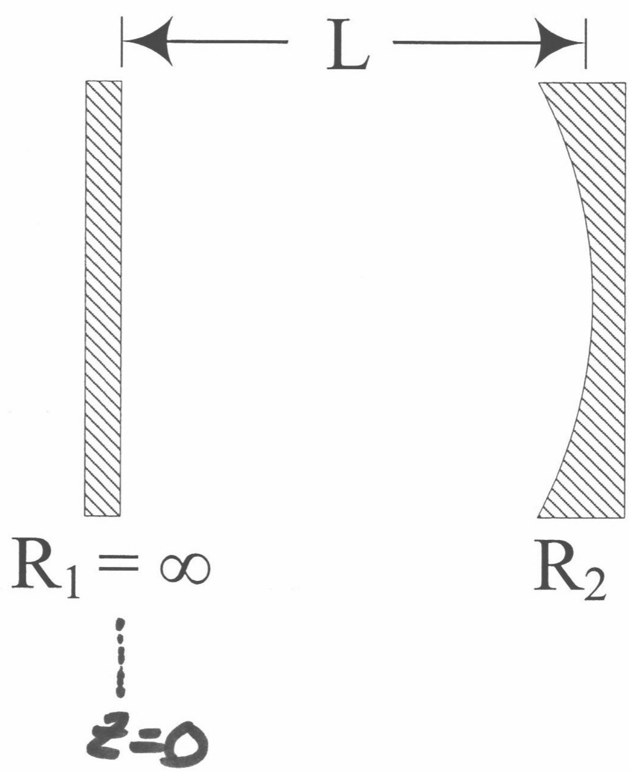

A beam propagating from mirror #1 to #2 suffers the phase shift:

$$\varphi(0) - \varphi(L) = kL$$
$$- \{1 + m + n\} \tan^{-1}\left(\frac{L}{z_0}\right)$$

$$\equiv q\pi$$

But $$z_0 = (LR_2)^{\frac{1}{2}} \left\{1 - \frac{L}{R_2}\right\}^{\frac{1}{2}}$$

And $$k \equiv \frac{2\pi n}{\lambda} = \frac{2\pi \nu n}{c}$$

Therefore, the $q^{th}$ longitudinal mode for the $TEM_{mn}$ transverse mode has a resonant frequency of

$$\boxed{\begin{aligned}\nu_{mnq} &= \frac{c}{2nL}\left\{q+\left(\frac{1+m+n}{\pi}\right)\cdot\tan^{-1}\left(\frac{L}{z_0}\right)\right\}\\ &= \frac{c}{2nL}\left\{q+\left(\frac{1+m+n}{\pi}\right)\cdot\right.\\ &\qquad\left.\cdot\cos^{-1}\left\{1-\frac{L}{R_2}\right\}^{\frac{1}{2}}\right\}\end{aligned}}$$

## <u>**TWO POINTS TO BE MADE:**</u>

1. The frequency separation $\Delta\nu$ between longitudinal modes <u>**with the same m & n is still**</u>

$$FSR = \frac{c}{2nL}$$

2. A number of $TEM_{mn}$ modes have the same $\nu_{mnq}$! <u>**THESE ARE KNOWN AS *DEGENERATE MODES*.**</u>

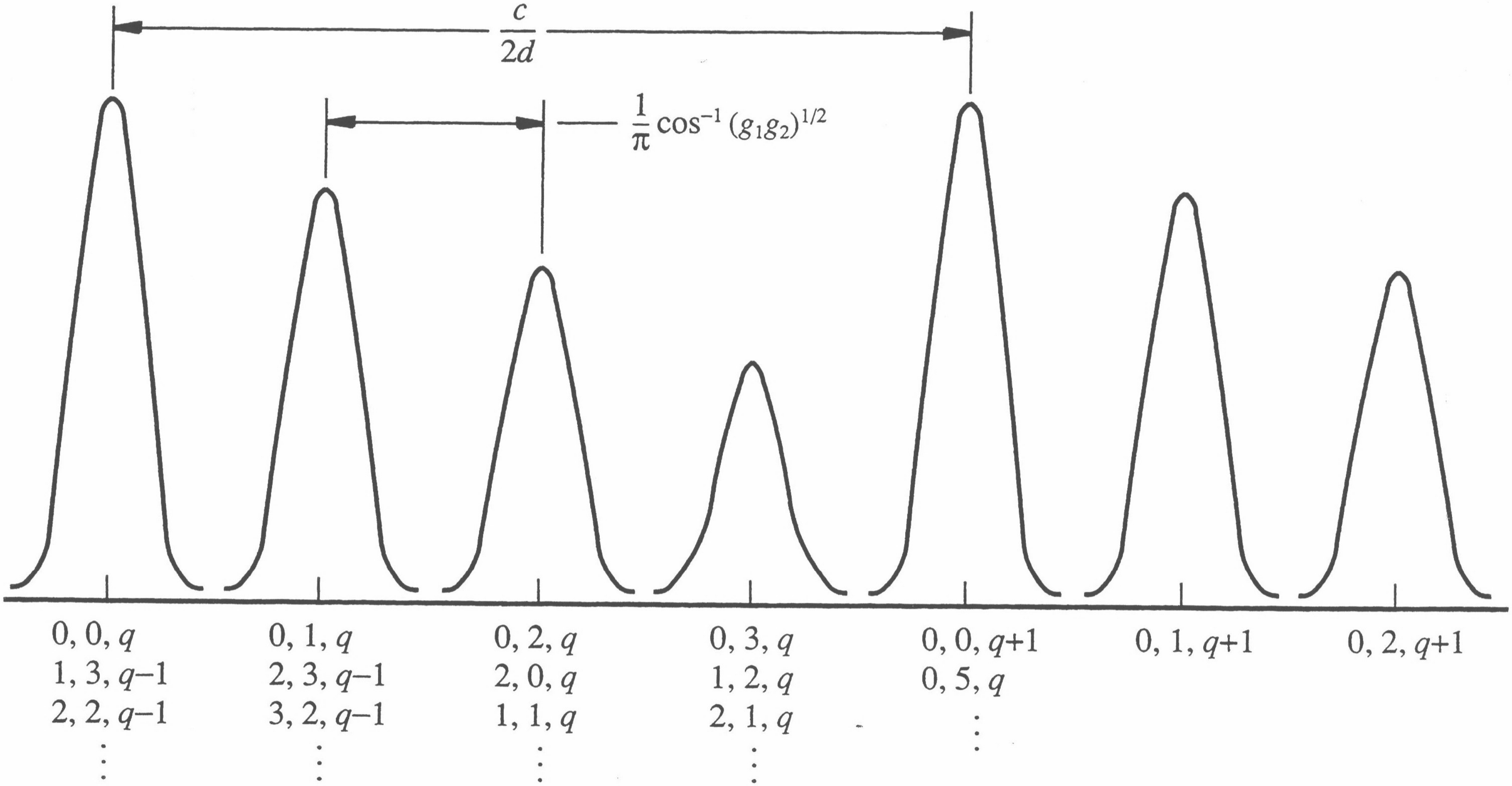


**FIGURE 6.6.** Frequency degeneracy in an optical cavity.