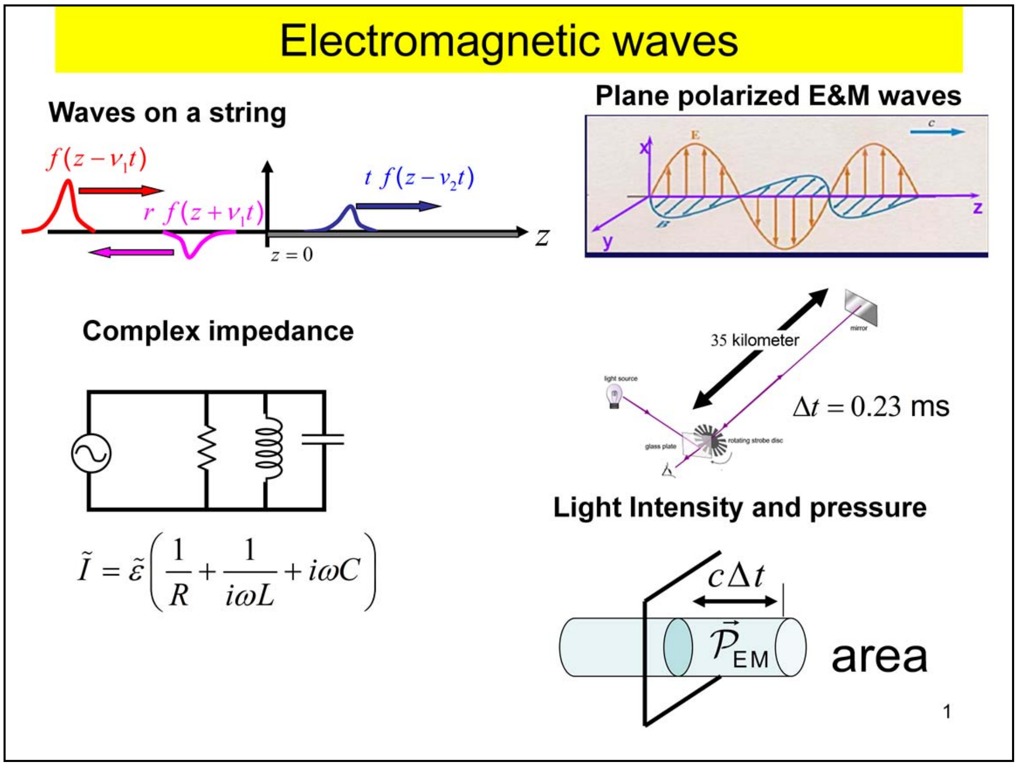
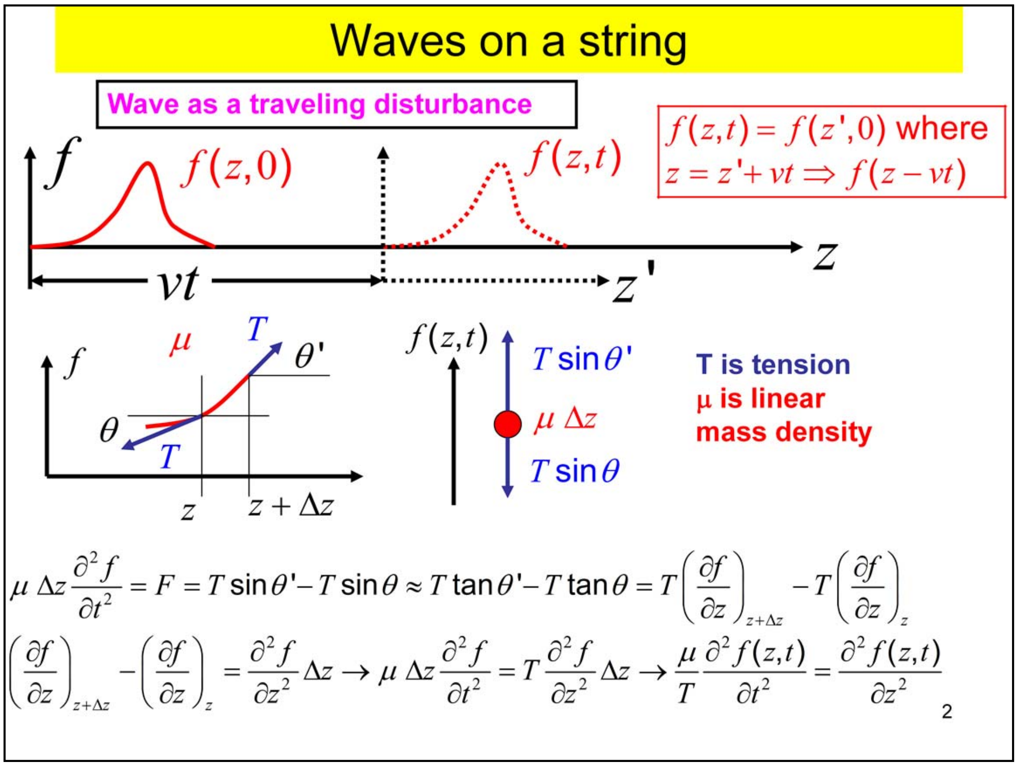


This chapter is our first on electromagnetic waves. We begin with a discussion of mechanical, transverse waves on a string to get you use to the concept of a wave equation and wave parameters such as wave length, wave number and frequency. We also discuss wave reflection and transmission from the boundary of between a light and heavy string. A tool we use in the wave reflection is the use of the complex representation for sinusoidal functions. As an example we introduce complex impedances as an alternative to the phaser method for AC circuits that you used for RLC circuits in Physics 212. The complex impedance technique allows one to analyze any AC circuit (such as this **parallel** RLC circuit) as a resister network but with complex numbers as the resistances. Recall in Physics 212 we only analyzed the **series** RLC circuit. We next obtain the wave equation for electromagnetic waves traveling in a vacuum. This is remarkably simple using the differential form of the four Maxwell's Eq. and we get the same wave equation that describes waves on a string as well as a correct prediction for the speed of light in terms of the "lab" bench constants: epsilon_0 and mu0. This is perhaps the greatest triumph of 19th century physics. The wave equation is not sufficient to describe EM waves since there are solutions to the wave equation with fields that that violate some Maxwell's equations. Hence only transverse waves where the electric and magnetic field are perpendicular to the direction of propagation and perpendicular to each other are possible and the amplitude of the magnetic field wave is tied to the amplitude of the electric field wave. We conclude by discussing the energy density of an electromagnetic wave, its Poynting vector, and momentum density and pressure.



To get us use to the concept of waves, we consider classical transverse waves on a string. The mathematics of mechanical waves and electromagnetic waves are very similar– they essentially obey the same wave equation. The wave is described by a wave function that gives some property of the wave as a function of time and space. For waves on a string, stretched (and traveling) along the z-direction, this would be the transverse displacement or the displacement in the y or x direction. For electromagnetic waves this would be the electric or magnetic field (or possibly the vector potential). For ideal waves (which satisfy the wave equation) the wave keeps the same shape as it travels. For mechanical waves traveling along the z-axis this means that the transverse displacement is a function of z and time and in fact is a function of z - vt for waves traveling along the positive z axis where v is the "velocity of propagation". For x-displacements we write the wave as x = f(x-vt). As shown in the figure, the f(x-vt) form means one can essentially take a snap shot of the wave at t=0 and superimpose it on the wave displacement at a later time. They have exactly the same shape. We now discuss the dynamics (e.g. the wave equation) that causes idealized wave motion for the case of a wave on a string. The idea is to look at the forces and acceleration of segment of the string from z to z + delta z. The mass of this segment is mu delta z where mu is the linear mass density or kg/m. This segment will accelerate due to a net force. The net force is due to a force imbalance – the force is the due to the string tension (T), and the imbalance is due to the curvature or double derivative of the displacement. This means the "displacement" component of the force on the left of the segment is unequal to force in the displacement direction on the the right. These force components are proportional to the tension times the sin of the angle but for "small" displacements, the sine is equal to the tangent which is the slope or partial of f wrt z. Setting the force to the mass times the acceleration (or double time derivative of the displacement) , and canceling the segment length (delta Z) we get a wave equation where the double space derivative of f is proportional to the double time derivative of f.

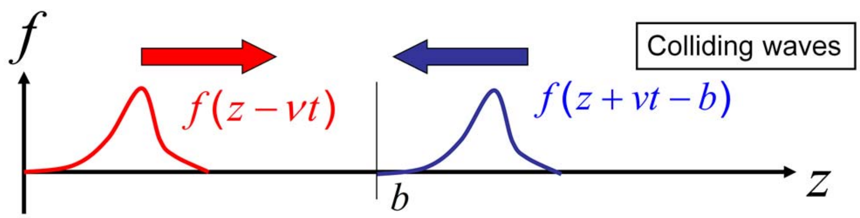
Wave equation solutions

$$\boxed{\frac{\mu}{T}\frac{\partial^2 f(z-vt)}{\partial t^2}=\frac{\partial^2 f(z-vt)}{\partial z^2}}\quad \text{Let } u=z-vt\rightarrow\frac{\partial f(z-vt)}{\partial t}=\frac{df(u)}{du}\frac{\partial u}{\partial t}=\frac{df(u)}{du}\frac{\partial z-vt}{\partial t}=-v\frac{df(u)}{du}$$

$$\frac{\partial^2 f(z-vt)}{\partial t^2}=-v\frac{\partial}{\partial t}\left\{\frac{df(u)}{du}\right\}=-v\frac{\partial u}{\partial t}\frac{d}{du}\left(\frac{df(u)}{du}\right)=-v\frac{\partial z-vt}{\partial t}\frac{d^2 f(u)}{du^2}=v^2\frac{d^2 f(u)}{du^2}$$

$$\frac{\partial f(z-vt)}{\partial z}=\frac{df(u)}{du}\frac{\partial z-vt}{\partial z}=\frac{df(u)}{du};\quad\frac{\partial^2 f(z-vt)}{\partial z^2}=\frac{\partial z-vt}{\partial z}\frac{d}{du}\left\{\frac{df(u)}{du}\right\}=\frac{d^2 f(u)}{du^2}$$

$$\frac{\mu}{T}\frac{\partial^2 f(z-vt)}{\partial t^2}=\frac{\partial^2 f(z-vt)}{\partial z^2}\rightarrow\frac{\mu}{T}v^2\frac{d^2 f(u)}{du^2}=\frac{d^2 f(u)}{du^2}\rightarrow v=\pm\sqrt{\frac{T}{\mu}}\rightarrow\text{Both } f(z\pm vt)\text{ are solutions}$$



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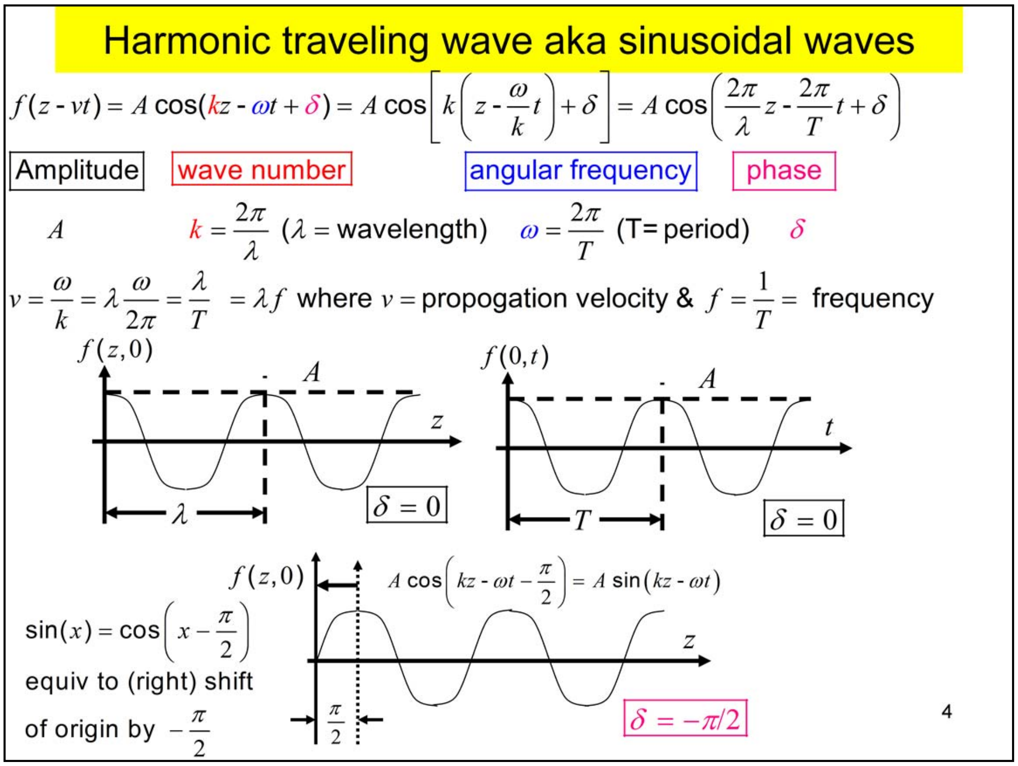
What is the solution of the wave equation? We show that for transverse waves traveling along the z-axis. That **any** function of time and space of the form f(z-vt) will solve the wave equation where v is the velocity of propagation. We show this by taking formal time and space derivatives of f(u) where u = z-vt using the chain rule. Any function of u will work as long as v^2 = T/mu or v = +/- sqrt(T/mu). The + solution means the waves will travel along the +z direction and the – solution means the waves travel along the –z direction. The critical point is **any** f(u) or wave shape will work and the shape is perfectly preserved as the wave travels. This is true of waves that obey an ideal wave equation. Any additional terms involving t or z will create “dispersion” and cause the shape to change (usually spread-out) as a function of time. One important class of examples is wave packet dispersion in quantum mechanics. An E&M example is dispersion of a pulse traveling along a lossy cable. Electromagnetic waves traveling in a vacuum obey a perfect wave equation and have no dispersion.



A very useful particular form for a wave is the “harmonic” wave where the function f(u) is a either a sine or cosine plus an offset called the phase. A harmonic wave is commonly described in terms of three constants. The amplitude A multiplies the trig function and describes the maximum “size” of the displacement. The wave number (or later wave vector) multiplies z or the coordinate along the direction of propagation. The (angular frequency) omega multiplies the time; and the phase delta is an offset to the trig function of the argument which is related when time is “zero-ed”. Our formulas are for a wave traveling along the positive z axis ; we could change the sign of the k or omega term to get waves traveling along the negative z axis. Alternatively we could parameterize the wave in terms of the wavelength lambda (rather than k) and the period T (rather than omega). I think of lambda as the “repeat” distance – if you shift your position from z to z + lambda you see the same wave shape. Similarly the period T is the “repeat” time.

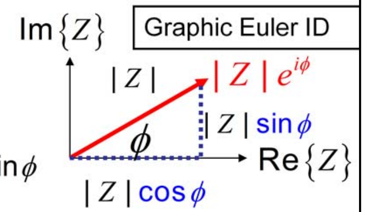
If we write the argument u = k z – omega t as u=k[z-(omega/k) t] we see we have a solution of the wave equation or a function of z – v t where v = omega/k – or v= lambda/T. We can write our “harmonic” traveling waves in terms of sines or cosines since one can get from sine to cosine by adding or subtracting pi/2 or 90 degrees.

Complex representation of harmonic functions

Euler's identity:

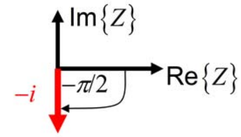
$$\exp(i\phi) = \sum_{n=0}^{\infty} \frac{1}{n!}(i\phi)^n = \sum_{n=even} \frac{1}{n!}(i\phi)^n + \sum_{n=odd} \frac{1}{n!}(i\phi)^n$$

$$= \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \cdots\right) + i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \cdots\right) = \cos\phi + i\sin\phi$$



We can write $f(z,t) = A\cos(kz - \omega t) = \text{Re}\{A\exp(ikz - i\omega t)\} = \text{Re}\{\tilde{f}\}$

where $\tilde{f} = A\exp(ikz - i\omega t)$



Principal advantage is "multiplicative" phases or phase factors

Example if $\tilde{f}$ solves W.E $\tilde{g} = -i\,\tilde{f} = -iA\exp(ikz - i\omega t)$ does as well. $-i = \exp(-i\pi/2)$

$$\tilde{g} = -iA\exp(ikz - i\omega t) = Ae^{-i\pi/2}\exp(ikz - i\omega t) = A\exp\left[i(kz - \omega t - \pi/2)\right]$$

$$g = \text{Re}(\tilde{g}) = A\cos(kz - \omega t - \pi/2) = A\,\text{Re}\left\{-i\left[\cos(kz - \omega t) + i\sin(kz - \omega t)\right]\right\} = A\sin(kz - \omega t)$$

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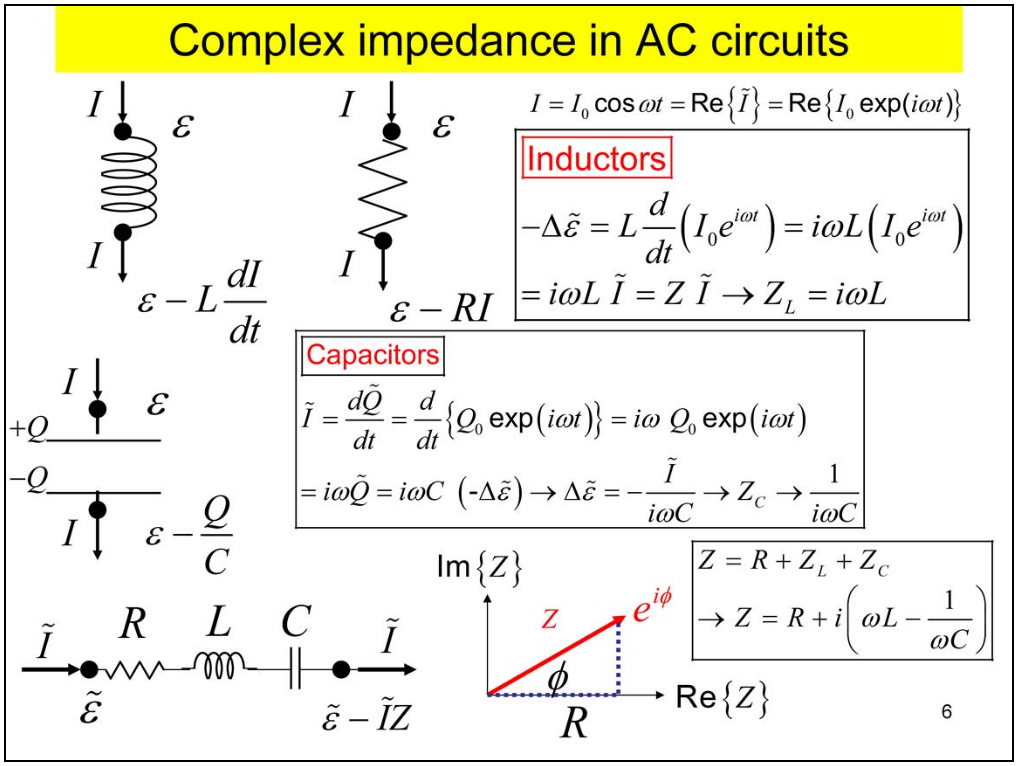
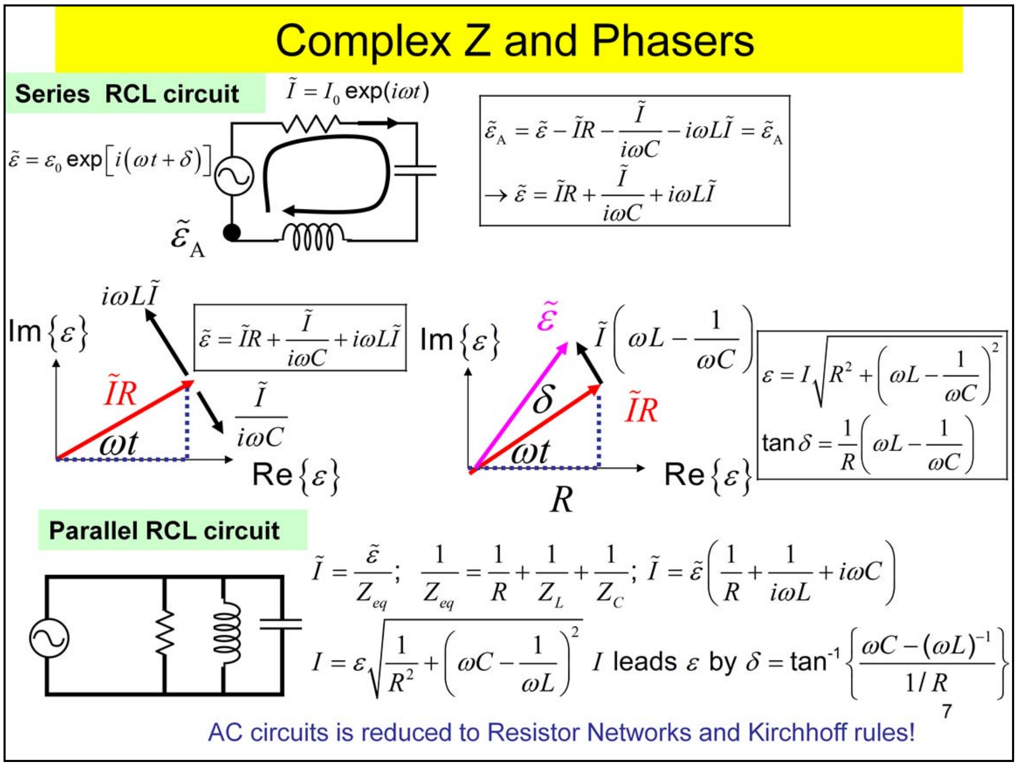
We will frequently write sinusoidal functions such as traveling waves in “exponential” form. The exponential form is based on Euler’s identity which says exp(i phi) = cos(phi) + i sin(phi) where phi is a real variable and i is sqrt(-1) or the imaginary number. It is easy to prove the identity using the (hopefully) well known Taylor expansion of an exponential which consists of the argument to the power n divided by n! (or n factorial), Odd powers of (i phi) will be imaginary and even powers of (i phi) will be real. Both the real and imaginary parts will be alternating series in phi^n/n! with the real part being a even function and the imaginary part being an odd function. The even and odd functions of phi^n/n! are the familiar cosine and sine functions of trigonometry. One nice use of Euler’s identity is writing a complex number Z in “polar form” As shown in the figure we can plot the complex number Z = |Z|exp(i phi) as a vector on an Argand diagram where the x coordinate is the real part of the complex number Z and the y-coordinate is the imaginary part of the complex number Z. If we expand | |Z|exp(i phi) using Euler’s identity we see the modulus |Z| is the length of the vector and phi is the angle of the vector with respect to the x-axis (sometimes called the phase). The principal advantage of the exponential form is based on the fact that the product of two exponentials is the exponential of the sum of their arguments. This is also the basis of multiplying by adding logarithms on a slide rule. We write a traveling wave such as f = A cos(k z – omega t) in exponential notation as ftilde=A exp(i k z – i omega t). Here A can be a complex number which incorporates phase information as well as the magnitude of the amplitude.To go from ftilde to a meaningful transverse displacement we take the real part or f = Re{ ftilde}. After all the displacement for a transverse wave, or the electric field for an E&M wave is an intrinsically real function. All classical theories are based on real numbers – only quantum theories are based on complex numbers. We could switch from a cosine traveling wave to a sine traveling wave by taking the imaginary part of ftilde. We can choose to take the real or imaginary part of the “complex” wave function but will stick to taking the real part as an (arbitrary) convention. Under the “real” convention, one switch from cosine waves to sine waves by changing A to -i A since Re {-i ftilde}= Re{ -i A exp(i k z – i omega t)}. We can write –i in polar or “Eulerean form” as exp(-i pi/2) since the complex number –i has unit modulus and shows up at -90 degrees with respect to the real axis in an Argand plot. We then use slide-rule trick to write make the multiplication by (- i) equivalent to adding a phase of – pi/2 to the argument. Essentially we have shown cos(k z –omega t - pi/2) = sin(k z – omega t) which can also be obtained using angular addition formulae but this illustrates some of them computational tricks of complex notation.

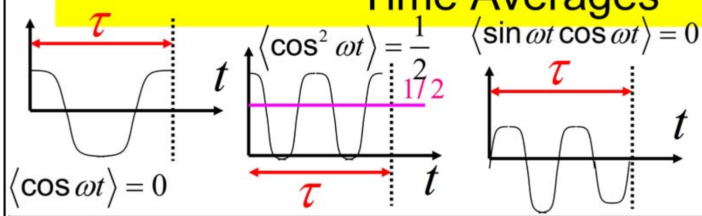


The use of complex impedances in analyzing passive AC circuits is another example of the advantages of complex notation of sinusoidal functions. The basic idea is that in an AC circuit the voltage across a passive circuit element such as a resistor, capacitor, or inductor has a magnitude proportional to the current flowing through the element but has a possibly different phase. In the complex notation, we can accommodate both the change in magnitude and phase by writing -delta E-tilde = Z i-tilde where Z and the current i-tilde are both complex numbers. Here –delta E-tilde is the voltage change in the direction of the current. We can get the Z expression for resistors, capacitors, or inductors by considering the basic physic relating the current to the voltage drop. The simplest case is a resistor where Ohm's law tells us voltage drop= -delta E = i R and hence Z = R and here the impedance is real. A positive voltage **drop**, means the potential of the current entering the resistor is larger than the potential exiting the resister – ie the current flows from a high potential to a low potential. The voltage change is the opposite of the voltage drop. Kirchoff's law says the sum of all voltage changes when walks along a circuit loop in the direction of assumed current flow sums to zero. Another example is the voltage drop across an inductor. Here the physics is Faraday's law which tells us the EMF across the inductor is given by the negative rate of change of the magnetic flux. The magnetic flux is the self inductance L times the current so the EMF is given by L times the rate of change of the current. Lenz's law provides the (–) sign so we know we the voltage change is also a voltage drop. If we insert the complex form for the sinusoidally varying current we find the voltage drop is i omega L times the complex current. Hence Z = i omega L. The voltage drop is thus 90 degrees advanced in phase with respect to the current implying a purely imaginary impedance. Finally for capacitors, the physics is the current is rate of change of on a capacitor plates and the voltage drop is the charge over the capacitance. We write the complex current through the capacitor as the rate of change of the complex charge Q. The charge is the capacitance times the voltage drop which implies a complex impedance of Z= 1/(i omega C). These are the same basic impedance rules you learned for "phasors" in series RLC circuits. The voltage drop across a resistor is in phase with the current; the voltage drop across an inductor is 90 degree phase ahead of the current' and the voltage across an inductor in 90 degrees behind the phase of the current. For a series RLC circuit, we can write the impedance as the resistance + i times [omega L – 1/(omega C)]. At the resonant frequency where omega = 1/sqrt(LC), the impedance is just R and the inductive reactance cancels the capacitive reactance. In general the voltage will lead the current by a phase of Arctan{ [omega L – 1/(omega C)]/R}.



The application of AC circuit laws you are probably most familiar with is the use of phasors in series RCL circuit. If we apply the voltage changes using the complex impedance we have a single complex current i-tilde that flows through each element. The sum of the voltage changes around the complete circuit is zero ; we take a clockwise path from point A back to point A. The first change is E-twiddle due to the AC generator, the remaining three changes are – i-twiddle Z where we use the complex impedances of the resistor, capacitor, and inductor. Hence the E-twiddle supplied by the generator equals the sum of the complex voltage drops. I also show the complex voltage changes can be viewed on an Argand plot. The construction is essentially identical to the construction for the EMF used in a Physics 212 phasor diagram. You can easily find the amplitude of the EMF provided by the generator as well phase by which the generator EMF leads the current by solving the reactance triangle with a base of i-tilde R and a height of i-tilde [omega L – 1/(omega L)]. The critical realization is we essentially solve the AC circuit just like we would solve an equivalent resistor network– the only difference is we use the complex impedance Z rather than the resistance R for each element. We turn next to the parallel RLC circuit which is generally not done using phasor methods in Physics 212 but can easily be done using Kirchoff laws for “resistors” except we use complex impedance Z for the inductors and capacitors. Say we know the complex generator EMF (E-tilde) and we want to know the complex current i-tilde which flows through the generator. We can think of the generator as driving a load with three “resistors” in parallel. We can find the equivalent impedance of this load by borrowing the familiar parallel resistor law 1/R_eq = 1/R_1 + 1/R_2 + 1/R_3 but modified to complex impedances to 1/Zeq = 1/Z_R + 1/Z_L+1/Z_C. We thus have a fairly simple expression for the complex current as the product of the generator EMF [1/R + 1/(i L omega) + (i omega C)] We can compute the current amplitude by taking the modulus of the complex current. We can compute this as |i-tilde| = sqrt{(i-tilde)* (i-tilde)} which is essentially the sqrt of the real part squared + the imaginary part squared. We can also find the phase by which the current leads the voltage by taking the arc tangent of the imaginary part divided by the real part.

Time Averages



$$\tau = \frac{2\pi}{\omega}\ ;\ \langle [\ ] \rangle = \frac{1}{\tau}\int_0^\tau [\ ]\,dt$$

$$\langle \cos\omega t \rangle = \langle \sin\omega t \rangle = 0$$

$$\langle \cos^2\omega t \rangle = \langle \sin^2\omega t \rangle = 1/2$$

$$\langle \sin\omega t \cos\omega t \rangle = 0$$

$$\langle \text{Power} \rangle = \langle \varepsilon I \rangle = \langle \varepsilon_0 \cos(\omega t + \delta_\varepsilon) I_0 \cos(\omega t + \delta_I) \rangle = \varepsilon_0 I_0 \langle \cos(\omega t + \delta_\varepsilon)\cos(\omega t + \delta_I) \rangle$$

$$\langle (\cos\omega t \cos\delta_\varepsilon - \sin\omega \sin\delta_\varepsilon)(\cos\omega t \cos\delta_I - \sin\omega \sin\delta_I) \rangle =$$

$$\cos\delta_\varepsilon \cos\delta_I \langle \cos^2\omega t \rangle - (\sin\delta_\varepsilon \cos\delta_I + \sin\delta_I \cos\delta)_\varepsilon \langle \sin\omega t \cos\omega t \rangle + \sin\delta_\varepsilon \sin\delta_I \langle \sin^2\omega t \rangle =$$

$$\frac{\cos\delta_\varepsilon \cos\delta_I + \sin\delta_\varepsilon \sin\delta_I}{2} = \frac{\cos(\delta_I - \delta_\varepsilon)}{2} \rightarrow \langle \text{Power} \rangle = \frac{\varepsilon_0 I_0}{2}\left[\cos(\delta_I - \delta_\varepsilon)\right]_{\text{Power factor}}$$

Complex form $\langle \text{Power} \rangle = \frac{1}{2}\text{Re}\{\varepsilon^* I\} = \frac{1}{2}\text{Re}\left\{\left(\varepsilon_0 e^{i\delta_\varepsilon} e^{i\omega t}\right)^* \left(I_0 e^{i\delta_I} e^{i\omega t}\right)\right\} = \frac{1}{2}\text{Re}\left\{\left(\varepsilon_0 e^{-i\delta_\varepsilon} e^{-i\omega t}\right)\left(I_0 e^{i\delta_I} e^{i\omega t}\right)\right\}$

$$= \frac{\varepsilon_0 I_0}{2}\text{Re}\left\{e^{i(\delta_I - \delta_\varepsilon)}\right\} = \frac{\varepsilon_0 I_0}{2}\cos(\delta_I - \delta_\varepsilon)$$

Poynting $\langle \vec{S} \rangle = \left\langle \frac{\vec{E}\times\vec{B}}{\mu_0} \right\rangle \xrightarrow[\vec{B}=\text{Re}\{\tilde{B}\}\ ;\ \tilde{B}=\tilde{B}e^{i\omega t}]{\vec{E}=\text{Re}\{\tilde{E}\}\ ;\ \tilde{E}=\tilde{E}e^{i\omega t}} \frac{1}{2\mu_0}\text{Re}\langle \tilde{E}^* \times \tilde{B} \rangle$ or $\langle \vec{S} \rangle = \frac{1}{2\mu_0}\text{Re}\langle \vec{E}^* \times \vec{B} \rangle$

$$\vec{\nabla}\bullet\vec{S} + \frac{\partial u_{EM}}{\partial t} + \vec{E}\bullet\vec{J} = 0 \rightarrow \langle \vec{\nabla}\bullet\vec{S} \rangle + \cancel{\frac{\partial \langle u_{EM} \rangle}{\partial t}} + \langle \vec{E}\bullet\vec{J} \rangle = 0 \xrightarrow{\text{Harmonic } \tilde{E}, \tilde{B} \& \tilde{J}} \frac{\vec{\nabla}\bullet\text{Re}\langle \vec{E}^* \times \vec{B} \rangle}{2\mu_0} + \frac{\text{Re}\{\vec{E}^* \bullet \vec{J}\}}{2} = 0$$

For AC circuits and for light or radio waves we are often interested in the time average of useful quantities. After all EM light waves oscillate with essentially immeasurable frequencies. Often these systems involve sinusoidal oscillations and the standard time average is over a complete period. I show three frequently used sinusoidal time averages which can either be obtained by doing the integral or thinking "graphically" about the time dependence. We use these sinusoidal averages to calculate the time average power in an AC circuit where the current has a phase of delta_I and the EMF has a phase of delta_E. We just expand the cosines using the angular additional formulae and use our sinusoidal averages. We find that the average power expression involves the product of the emf and current amplitude and the cosine of the phase difference between current and emf. In Physics 212 , this cosine was called the "power factor" . A much simpler way uses the complex notation. The time average is just ½ of the Real part of the complex conjugate of the emf times the current. We can use the same trick to time-average the Poynting vector for the case where both the electric and magnetic fields have sinusoidal time variation and can thus be written in complex form. The "length" of the time average Poynting vector is called the intensity. It is also worth noting that the "time averaged" Poynting theorem becomes simpler since a time-averaged quantity has no remaining time dependence – all of the time was integrated out. Hence the time averaged Poynting theorem has no rate of change of the electromagnetic energy density. If the E, B, and J terms undergo sinusoidal oscillates (i.e. harmonic variation), we can compute these time averages as ½ Re E*B and ½ Re E*J. We will frequently use these complex averaging techniques in Physics 436.

Adding Harmonic waves

$$f(z,t)=A_1\cos(kz-\omega t+\delta_1)+A_2\cos(kz-\omega t+\delta_2)=\text{Re}\left\{\left(A_1e^{i\delta_1}+A_2e^{i\delta_2}\right)\exp\left[i\left(kz-\omega t\right)\right]\right\}=$$

$$f(z,t)=\text{Re}\left\{A_3e^{i\delta_3}\exp\left[i\left(kz-\omega t\right)\right]\right\}=A_3\cos(kz-\omega t+\delta_3)$$

Eulerian method is often the best way to find A_3 amplitude and r.m.s. amplitude

$$A_3=\left|A_1e^{i(kz-\omega t)+i\delta_1}+A_2e^{i(kz-\omega t)+i\delta_2}\right|=\sqrt{\left(\tilde{A}_1+\tilde{A}_2\right)^*\left(\tilde{A}_1+\tilde{A}_2\right)}=\sqrt{\tilde{A}_1^*\tilde{A}_1+\tilde{A}_1^*\tilde{A}_2+\tilde{A}_2^*\tilde{A}_1+\tilde{A}_2^*\tilde{A}_2}$$

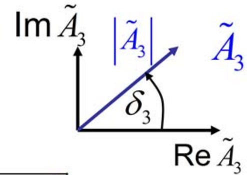
$$\tilde{A}_1^*\tilde{A}_1+\tilde{A}_1^*\tilde{A}_2+\tilde{A}_2^*\tilde{A}_1+\tilde{A}_2^*\tilde{A}_2=\left|\tilde{A}_1\right|^2+\left(\tilde{A}_1^*\tilde{A}_2\right)+\left(\tilde{A}_1^*\tilde{A}_2\right)^*+\left|\tilde{A}_2\right|^2=\left|\tilde{A}_1\right|^2+2\text{Re}\left(\tilde{A}_1^*\tilde{A}_2\right)+\left|\tilde{A}_2\right|^2$$

We used $z+z^*=\text{Re}\,z+i\,\text{Im}\,z+\left(\text{Re}\,z+i\,\text{Im}\,z\right)^*=\text{Re}\,z+i\,\text{Im}\,z+\text{Re}\,z-i\,\text{Im}\,z=2\,\text{Re}\,z$

$$\left|\tilde{A}_1\right|=\left|A_1e^{i(kz-\omega t)+i\delta_1}\right|=A_1\;;\;\left|\tilde{A}_2\right|=A_2\;;\;2\text{Re}\left(\tilde{A}_1^*\tilde{A}_2\right)=2A_1A_2\,\text{Re}\left\{e^{i(\delta_2-\delta_1)}\right\}=2A_1A_2\cos\left(\delta_2-\delta_1\right)$$

$$\Rightarrow\boxed{A_3=\left|A_1e^{i\delta_1}+A_2e^{i\delta_2}\right|=\sqrt{A_1^2+A_2^2+2A_1A_2\cos\left(\delta_2-\delta_1\right)}}\to\left|A_1-A_2\right|\le A_3\le\left|A_1+A_2\right|$$

$$A_3^{(\text{r.m.s})}=\sqrt{\left\langle f(z,t)^2\right\rangle_\tau}=\sqrt{\frac{\text{Re}\left\{\tilde{A}_3^*\tilde{A}_3\right\}}{2}}=\frac{\sqrt{A_1^2+A_2^2+2A_1A_2\cos\left(\delta_2-\delta_1\right)}}{\sqrt{2}}$$



Eulerian method is often the best way to find δ_3 phase

$$A_3e^{i\delta_3}=A_3\cos\delta_3+iA_3\sin\delta_3=\text{Re}\,\tilde{A}_3+i\,\text{Im}\,\tilde{A}_3$$

$$\Rightarrow\tan\delta_3=\frac{\text{Im}\,\tilde{A}_3}{\text{Re}\,\tilde{A}_3}=\frac{\text{Im}\left\{A_1e^{i\delta_1}+A_2e^{i\delta_2}\right\}}{\text{Re}\left\{A_1e^{i\delta_1}+A_2e^{i\delta_2}\right\}}\to\boxed{\delta_3=\tan^{-1}\left[\frac{A_1\sin\delta_1+A_2\sin\delta_2}{A_1\cos\delta_1+A_2\cos\delta_2}\right]}$$

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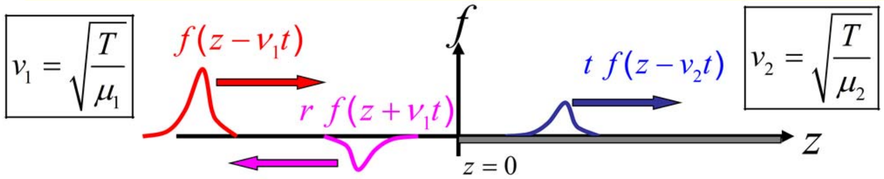
Here is another demonstration of the power of the complex representation for traveling waves. Here we add two harmonic traveling waves with different amplitudes and phases. We write the sum of the two waves as A_3-tilde exp(I delta_3) exp(i kz –i omega t). Since we have explicitly written the phases as exp(i delta) we are thinking of A1, A2, and A3 as real numbers. The amplitude of this wave is |A_3-tilde| which we compute by sqrt{(A1+A2)*(A1+A2)}. We distribute this product and note that A1*(A2) + A1(A2*) = 2 Re{(A1*)A2} since the two terms being summed are complex conjugates of each other. We thus are able to work through the amplitude of the modulus of A1 + A2 in terms of the moduli of 1 and 2 complex waves and and the cosine of the 1 and 2 phase difference. The same technique can be used to compute the r.m.s average displacement for the sum of the two harmonic waves. Since the actual time average of a sinusoidal varying displacement is zero – we use the r.m.s average which is the sqrt of the average of the square of the displacement. We can easily do the math by writing f(z,t) in complex form and then borrow the time average power method which involves the Real part of A*A. We essentially did this math to calculate the A3 amplitude and can just steal the result. We find that the r.m.s. amplitude is just the A3 amplitude/ sqrt(2). Although our r.m.s. average is just an average over time -- the result does not depend on the position z. One can mistakenly get the impression we are averaging over z as well as time. In homework you will show using trig explicitly that “ a miracle occurs” and the time average eliminates all z dependence as well for the traveling string waves and the E&M waves that we will discuss shortly. Complex methods based on Euler’s identity is significantly easier than computing the phase , modulus, and r.m.s. averages using trigonometry and calculus.

Reflection from string junction



For harmonic waves:
$f(z<0,t) = A_I e^{ik_1 z - i\omega t} + A_R e^{-ik_1 z - i\omega t}\ ;\ f(z>0,t) = A_T e^{ik_2 z - i\omega t}$

$$f(0^-,t) = f(0^+,t) \rightarrow A_I e^{-i\omega t} + A_R e^{-i\omega t} = A_T e^{-i\omega t} \rightarrow A_I + A_R = A_T$$

$$T\left(\frac{\partial f}{\partial z}\right)_{0+} - T\left(\frac{\partial f}{\partial z}\right)_{0-} = \mu \Delta z \left(\frac{\partial^2 f}{\partial t^2}\right)_0 \rightarrow 0 \Rightarrow A_T\left(\frac{\partial e^{ik_2 z}}{\partial z}\right)_0 = A_I\left(\frac{\partial e^{ik_1 z}}{\partial z}\right)_0 + A_R\left(\frac{\partial e^{ik_1 z}}{\partial z}\right)_0$$

$$ik_1 A_I - ik_1 A_R = ik_2 A_T \Rightarrow A_I + A_R = A_T \text{ \& } A_I - A_R = \frac{k_2}{k_1} A_T$$

$$A_I - A_R = \frac{k_2}{k_1}(A_I + A_R) \text{ Let } r = \frac{A_R}{A_I}\ ;\ \frac{1-r}{1+r} = \frac{k_2}{k_1} \Rightarrow r = \frac{1 - k_2/k_1}{1 + k_2/k_1} = \frac{1 - k_2/k_1}{1 + k_2/k_1} = \frac{k_1 - k_2}{k_1 + k_2}$$

$$\frac{A_I + A_R}{A_I} = \frac{A_T}{A_I} \Rightarrow 1 + r = t \Rightarrow t = \frac{2k_1}{k_1 + k_2} \text{ Using the } v = \frac{\omega}{k} \text{ form:}$$

$$r = \frac{v_2 - v_1}{v_1 + v_2} \text{ and } t = \frac{2v_2}{v_1 + v_2}$$

Both r and t are real and independent of ω

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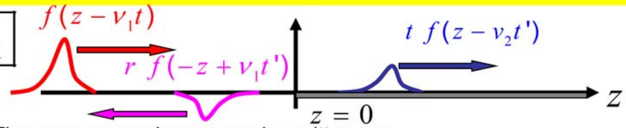
Here is a dry run of some of the techniques we will use to understand reflections of electromagnetic waves from a dielectric boundary. We consider the reflections of a traveling wave from the junction between a light and heavy string which join at z=0. The string in the region z<0 will satisfy a wave equation with a string velocity v1 = sqrt{T/mu_1} and the region z>0 will have a string velocity v2=sqrt{T/mu_2}. The two halves have the same string tension T but different mass densities per unit length. We think of the incident wave as having a transverse displacement of A_i exp(i k z – i omega t) If A_i is real, this means the incident wave is a cosine wave of the form A_i cos(kz-omega t). When this wave strikes the heavy-light junction at z = 0 there will be a reflected traveling wave moving along the negative z axis of the form A_r exp(-i k z –i omega t) as well as a single transmitted wave along the positive z axis of the form A_t exp (i k_2 z – i omega t). Note all three waves are given the same time dependence of exp (-i omega t). The left going transmitted wave has a negative k while the incident and transmitted wave have a positive k. The waves for z < 0 (the incident and reflected wave) are assigned the same magnitude |k_1| but the transmitted wave has different magnitude of |k_2|. They must have a different magnitude since v_1=omega/k_1 and v_2 = omega/k_2 and v1 ne v2. Assuming we know the incident amplitude and phase (or A_i), the two unknowns are the complex amplitudes A_R and A_T. Two solve for these two unknowns we need two complex equations which we we will call boundary conditions since the follow from comparing the displacements on either side of the z=0 boundary. Our first BC is that the string is continuous across the boundary and thus we require that f(0-,t) = f(0+,t) where z=0- is a point just to the left of z=0, and 0+ is a point just to the right. This “continuity” BC must be true for all times which implies A_I + A_R = A_T. The second equation that we need follows from “derivative continuity” across the boundary. Derivative discontinuity follows from F=ma applied to the point at z=0. This point has zero mass which implies the displacement-direction component of force just left of z=0 must equal the displacement component to the right to avoid infinite acceleration. As we discussed in our derivation of the wave equation, the force component is essentially T (partial f/partial z) and since T is the same for z<0 as z>0, our F=ma argument concludes that the slope of the string (or derivative of the displacement with respect to z) is continuous as well as the displacement. We an manipulate the two continuity conditions to get an equation for A_R in terms of A_I. It is customary to solve for the reflection amplitude ratio r = A_R/A_I and rather than A_R. We get a simple expression for r in terms of K_1 and K_2 . Defining the transmission amplitude ratio as t = A_T/A_I, and using the continuity condition A_R + A_I = A_T, we can get a simple expression for t as well. We can also cast this expression in terms of velocity ratios rather than wave vector ratios. Sorry for double booking the t symbol as time and transmission coefficient!

Adding a knot

$v_2 < v_1$

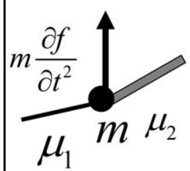


Fourier transform Thm says general wave can be written as:

$$f_I(z-vt)=\int_0^\infty a(\omega)e^{ikz-i\omega t}d\omega=\int_0^\infty a(\omega)\exp\left[i\omega\left(\frac{z-vt}{v}\right)\right]d\omega$$

$$f_R(z-vt)=\int_0^\infty r\,a(\omega)e^{-ikz-i\omega t'}d\omega=\int_0^\infty r\;a(\omega)\exp\left[i\omega\left(\frac{-z-vt'}{v}\right)\right]d\omega=r\;f_I(-z-vt')\;;\;t'\text{-}t$$ is time for f_I to reach boundary

Hence we will get same shapes --apart from $f_R(-z-vt')$ inversion and r scaling



$$f(0^-,t)=f(0^+,t)\Rightarrow A_I+A_R=A_T\;;\;T\left(\frac{\partial f}{\partial z}\right)_{0+}-T\left(\frac{\partial f}{\partial z}\right)_{0-}=m\left(\frac{\partial^2 f}{\partial t^2}\right)_0$$

$$\Rightarrow A_T\left(\frac{\partial e^{ik_2z}}{\partial z}\right)_0-\left\{A_I\left(\frac{\partial e^{ik_1z}}{\partial z}\right)_0+A_R\left(\frac{\partial e^{ik_1z}}{\partial z}\right)_0\right\}=\frac{m}{T}\left(\frac{\partial^2 A_Te^{-i\omega t}}{\partial t^2}\right)_0$$

$$ik_2A_T-\left[ik_1A_I-ik_1A_R\right]=-\frac{m\omega^2}{T}A_T\Rightarrow A_I+A_R=A_T\;\&\;A_I-A_R=\left[\frac{k_2}{k_1}-\frac{im\omega^2}{Tk_1}\right]A_T$$

$$A_I-A_R=\left[\frac{k_2}{k_1}-\frac{im\omega^2}{Tk_1}\right]\left(A_I+A_R\right)\;\Rightarrow r=\frac{k_1-k_2+\frac{im\omega^2}{T}}{k_1+k_2-\frac{im\omega^2}{T}}=\frac{v_2-v_1+i\frac{m\omega v_1v_2}{T}}{v_2+v_1-i\frac{m\omega v_1v_2}{T}}$$

Both r and t are complex and are functions of ω

$$f_R(z-vt)=\int_0^\infty r(\omega)\;a(\omega)\exp\left[i\omega\left(\frac{-z-vt}{v}\right)\right]d\omega\;\neq\text{const}\,f_I(-z-vt')$$ thus shape change

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We note that r and t are just real functions of the wave velocities and are independent of frequencies. If v_2 < v_1 (such that the light string is on the left and the heavy string is on the right) , t will be positive and r will be negative. We can write any shape incident wave f_I(z, t=0) as a combination of harmonic waves using Fourier methods. Since r and t are independent of frequency, the same harmonic components with the same relative amplitudes will be present in the transmitted and reflected waves. Hence all three waves will have the same shape as the incident wave, apart from the velocity differences, but the reflected and transmitted waves will be smaller and the reflected wave will be inverted in displacement and in z as shown in the figure. You will soon see the same mathematics used for the reflection and transmission coefficients for E&M waves striking dielectric boundaries (such as glass) and much later for the scattering of electron waves from potential barriers in quantum mechanics.

We next consider the case where the light and heavy strings are joined by a knot which we model as a point with mass m at z=0. We still have continuity (or A_I + A_R = A_T) but now we need to modify the derivative discontinuity condition which was predicated on the idea that the light and heavy strings were joined at a massless point. The modified expression follows from F=ma where F is the difference between partial f/partial z slopes on either side of the boundary times the tension, m is the mass of the point, and a is the double time derivative of f at z = 0. The continuity and discontinuity equations are easy to solve, except now the r (and t) coefficients are complex ratios which depend on the frequency omega as well as the heavy and light string wave numbers (k_1 and k_2) or string velocities (v1 and v2). The fact that the r coefficient has real and imaginary parts means the reflected wave will have a different phase for the case of a pure, harmonic incident wave. The frequency dependence of r means that the reflected wave will not have the same shape as the incident wave.

Electromagnetic Waves

We start with Maxwell's Eqn. in absence of charge or currents

(i) $\vec{\nabla}\bullet\vec{E}=0$ (ii) $\vec{\nabla}\bullet\vec{B}=0$ (iii) $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ (iv) $\vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$

decouple by taking curl of both sides of (iii) or (iv)

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=-\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}=-\varepsilon_0\mu_0\frac{\partial}{\partial t}\left(\frac{\partial\vec{E}}{\partial t}\right)=-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=\vec{\nabla}\left(\vec{\nabla}\bullet\vec{E}\right)-\vec{\nabla}\bullet\left(\vec{\nabla}\vec{E}\right)=-\nabla^2\vec{E}=-\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}\rightarrow\boxed{\nabla^2\vec{E}=\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}}$$

Starting from $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right)=\varepsilon_0\mu_0\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E}\rightarrow\boxed{\nabla^2\vec{B}=\varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}}$

We have six wave equations (3 for $\vec{E}$ and 3 for $\vec{B}$)

$$\nabla^2\vec{E}=\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}\rightarrow$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(E_x\hat{x}+E_y\hat{y}+E_z\hat{z}\right)=\varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\left(E_x\hat{x}+E_y\hat{y}+E_z\hat{z}\right)$$

$$\nabla^2E_x=\varepsilon_0\mu_0\frac{\partial^2E_x}{\partial t^2}\ ;\ \nabla^2E_y=\varepsilon_0\mu_0\frac{\partial^2E_y}{\partial t^2}\ ;\ \nabla^2E_z=\varepsilon_0\mu_0\frac{\partial^2E_z}{\partial t^2}$$

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Having discussed the use of BC and complex representation tools, we are ready to obtain and solve the wave equation for E&M waves. These are in some sense more complicated than transverse waves on a string since we need to describe all three components of the electric and magnetic field or two vectors rather than a simple scalar displacement. In this chapter we will discuss propagation in a vacuum and leave propagation in material (such as a transparent dielectric like glass) for next chapter. Since we are in vacuum there are is no rho which means the divergence of the E-field is zero (and, of course, the divergence of the B-field is always zero). Since there is no current density (J), either the Ampere's law piece of the magnetic curl vanishes and we have a simplified form where the curl of E is related to the rate of change of B and vice versa. Our approach is to solve for E and B using the two curl equations (iii and iv), and refine our answers by throwing out any bogus solutions where either the E and B field has a non-zero divergence. The first step is to decouple the E-field and B-field equations. This is remarkably simple to do by taking the curl of (iii) and (iv). The curl of a curl is the gradient of a divergence minus the Laplacean and of course the divergence piece vanishes. We thus a wave equation for E and B which is identical to the wave equation for the transverse wave on a string. The only difference is we have a separate wave equation for each of the six E,B components and the partial^2/partial z => the Laplacean. This means the electromagnetic wave can propagate in any direction – not just the string direction. Again the physics gives the wave velocity. In the case of the electromagnetic wave the wave velocity is 1/sqrt(epsilon_0 mu_0) which are the static electric and magnetic Coulomb and Biot-Savart constants.

No longitudinal waves

Lets look at $E_x = a_x \cos(kz - \omega t)$ solution

$$\nabla^2 E_x = \varepsilon_0 \mu_0 \frac{\partial^2 E_x}{\partial t^2} \;;\; \nabla^2 E_x = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) E_x = \frac{\partial^2 E_x}{\partial z^2}$$

This is exactly the wave equation: $\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 E_x}{\partial t^2} \rightarrow v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$$v = \frac{1}{\sqrt{8.85\times 10^{-12}\left(4\pi \times 10^{-7}\right)}} \approx 3\times 10^8\, m/s \equiv c \;\; \text{(to 3 sig figs)}$$

Presumably we have $E_y = a_y \cos(kz - \omega t)$ & $E_z = a_z \cos(kz - \omega t)$

or do we?? $E_z = a_z \cos(kz - \omega t)$ is actually a bogus solution because

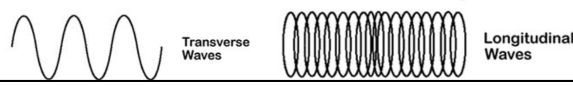
$\vec{\nabla}\bullet\vec{E} = \frac{\partial E_z}{\partial z} = -a_z k \sin(kz - \omega t) = \rho/\varepsilon_0$ but we started with $\rho = 0$!

The other solutions $\vec{\nabla}\bullet\left(a_x \cos(kz - \omega t)\, \hat{x} + a_y \cos(kz - \omega t)\, \hat{y}\right) = 0$ (so ok)

$\vec{E} = a_x \cos(kz - \omega t)\, \hat{x} + a_y \cos(kz - \omega t)\, \hat{y}$ is a transverse wave.

The field "motion" is $\perp$ to direction of propagation $\vec{v} = \hat{z}\, c$

Our bogus $\vec{E} = a_z \cos(kz - \omega t)\, \hat{z}$ would be a longitudial wave.



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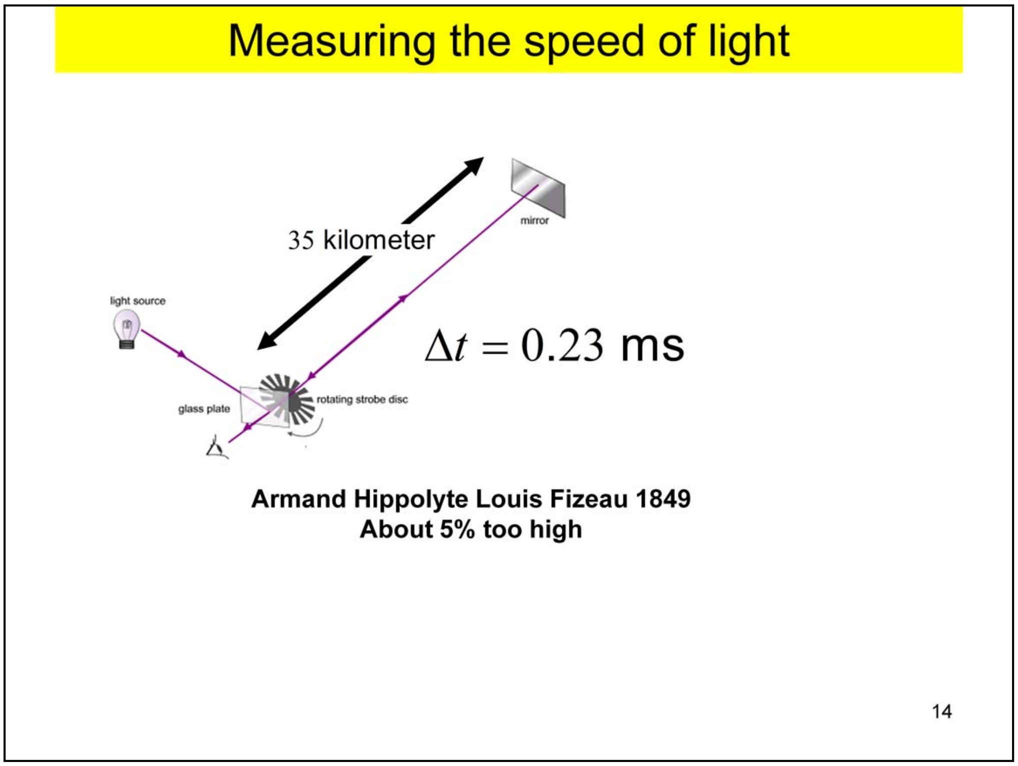
One can thus use the experimental values of Faraday’s constant and the Ampere constant that defines the magnetic field in terms of the current to predict the velocity of an E&M wave or the velocity of light! We get a prediction of very nearly 3 x 10^8 m/s – a very fast but measurable velocity which is usually given the symbol c because of its paramount importance in physics. Lets define the z – direction as direction of E&M wave propagation. Since we have three wave equations for each component we expect to find three, harmonic electric field solutions which could be E_x , E_y , E_z propto cos(k z-omega t). We think of E_x and E_y as transverse waves since fields in the x or y direction are transverse to the z-propagation direction just like for the transverse string waves we had x or y displacements which were transverse the string axis. E_z propto cos(k z –omega t) would be a longitudinal wave closer to a mechanical wave on a slinky where the displacement is along the slinky axis. But the E_z propto cos(k z –omega t) has a non-zero divergence since the field component is in the direction where E_z varies at fixed time. This would imply a non-zero charge density according to Gauss’s law but we are in a vacuum with zero charge density. Hence the longitudinal waves are bogus solutions. The electric (and magnetic fields) must be pointed transverse to the propagation direction.



Perhaps the most stunning prediction of the electromagnetic wave nature of light is that the speed of light is predicted to be 1/sqrt(epsilon_0 mu_0) where the Faraday and Ampere constant had well measured values. The speed of light had been measured with reasonable accuracy about 15 years before Maxwell's electromagnetic theory by Hippolyte and Fizeau in the best 19th century tradition. A beam of light was reflected from a mirror on a mountain that had traveled 70 km. Light travels so fast that the 70 km round trip took only 230 microseconds. In the 19th century this is a small interval to measure accurately. H and F devised a mechanical chopper to measure the time interval. Basically the light from the source had to pass between the teeth of a rapidly rotating gear in order to be sent to the mountain. The light reflected from the mountain mirror had to pass through the next gap between the teeth to be seen by an observer through a ½ silvered mirror. The observer increased the gear rotation until he could first clearly see the reflected light and noted the rpm of the gear. After a little simple algebra one obtains a reasonably accurate measurement of the speed of light that later was used to confirm Maxwell's prediction. Now one could easily measure the speed of light using a modern oscilloscope and a phototube on a lab bench!

Additional bogus solutions

$\vec{B} = b_x \cos(kz - \omega t)\ \hat{x} + b_y \cos(kz - \omega t)\ \hat{y}$ will work as well since longitudinal

How about $\vec{E} = a_x \cos(kz - \omega t)\ \hat{x}$ coupled with $\vec{B} = b_x \cos(kz - \omega t)\ \hat{x}$?

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_x \cos(kz - \omega t) & 0 & 0 \end{vmatrix} = -b_x k \sin(kz - \omega t)\hat{y} \neq +\frac{1}{c^2}\frac{\partial}{\partial t} a_x \cos(kz - \omega t)\ \hat{x}$$

But $\vec{B} = \hat{y}\ b_y \cos(kz - \omega t)$ will work

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & b_y \cos(kz - \omega t) & 0 \end{vmatrix} = +b_y k \sin(kz - \omega t)\hat{x} = \frac{\omega}{c^2} a_x \sin(kz - \omega t)\ \hat{x}$$

As long as $b_y = \frac{a_x}{c}$ $\left(\text{Note } k = \frac{\omega}{c} \text{ since } v = \frac{\omega}{k} = c\right)$

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Another class of bogus solutions are cases where the B field is not perpendicular to the E-field. We can show this violates the Maxwell displacement term in Ampere's law. For example lets say the wave has an electrical field which is polarized in the x direction. If we assume that there is a B_x propto f(k z –omega t), we find that the curl of B is not proportional to the time derivative of the E field and thus violates the Ampere-Maxwell law since the curl will be in y direction and the time derivative of E will lie in the x direction. We illustrate this with two cosine waves but we will get the violation with any phase choice. The only way we get consistency with the Ampere-Maxwell law is to put the B field in the y direction so that it is transverse to both the E-field and the direction of propagation along the z axis, and both fields must have the same phase (in this case they are cosine waves) If we put B perpendicular to E, the Ampere-Maxwell law relates the magnetic and electric amplitudes. The magnetic amplitude has to be the electric amplitude divided by the speed of light c.

Portrait of the transverse E&M wave

The moral is that solutions of the E and B wave equation are not enough. We must check each solution against all four Maxwell's Eqn.

$\tilde{\vec{E}} = \hat{n}\, E_0 \exp(i\vec{k}\bullet\vec{r} - i\omega t)$ where $\vec{k} = \frac{2\pi}{\lambda}\hat{k}$ and $\hat{k}$ points along $\vec{v}$ and $\vec{E} \parallel \hat{n}$

We note $\vec{\nabla}\exp(i\vec{k}\bullet\vec{r} - i\omega t) = i\vec{k}\exp(i\vec{k}\bullet\vec{r} - i\omega t)$. Similar to $\frac{\partial e^{ikx}}{\partial x} = ike^{ikx}$.

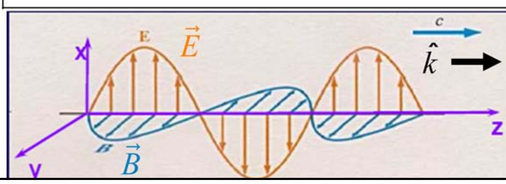
$\vec{\nabla}\bullet\tilde{\vec{E}} = \vec{\nabla}\bullet\hat{n}\, E_0 \exp(i\vec{k}\bullet\vec{r} - i\omega t) = E_0\hat{n}\bullet\vec{\nabla}\exp(i\vec{k}\bullet\vec{r} - i\omega t) = iE_0\hat{n}\bullet\vec{k}\exp(i\vec{k}\bullet\vec{r} - i\omega t) = 0$

$\rightarrow \hat{n}\bullet\vec{k} = 0$ or $\hat{n} \perp \vec{k}$ Hence $\vec{E} \perp \vec{k} \Rightarrow$ transverse polarization. Polarization $\parallel \vec{E}$

$\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}\vec{B}_0\exp(i\vec{k}\bullet\vec{r} - i\omega t) = i\omega\vec{B}_0\exp(i\vec{k}\bullet\vec{r} - i\omega t) = i\omega\vec{B}$

$\vec{\nabla}\times\tilde{\vec{E}} = \vec{\nabla}\times\hat{n}\, E_0\exp(i\vec{k}\bullet\vec{r} - i\omega t) = E_0\vec{\nabla}\exp(i\vec{k}\bullet\vec{r} - i\omega t)\times\hat{n} = i\vec{k}\times\hat{n}E_0\exp(i\vec{k}\bullet\vec{r} - i\omega t)$

$i\omega\vec{B} = i\vec{k}\times\hat{n}E_0\exp(i\vec{k}\bullet\vec{r} - i\omega t) \Rightarrow \vec{B} = \frac{\vec{k}}{\omega}\times\vec{E} \Rightarrow \boxed{\vec{B} = \frac{1}{c}\hat{k}\times\vec{E}}$ or $\vec{B} = \hat{k}\times\hat{n}\left(\frac{E_0}{c}\right)\exp(i\vec{k}\bullet\vec{r} - i\omega t)$



$\vec{E}$ and $\vec{B}$ are in phase
$\vec{E}\times\vec{B}$ points along $\hat{k}$
$|\vec{B}| = \frac{|\vec{E}|}{c}$

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The moral of the story is that the finding solutions to the electric or magnetic wave equations is not enough. Once must check that the solutions satisfy all four Maxwell's equations. To see what this implies we replace the concept of a wave number k by a wave vector vec-k. The wave vector is (2 pi/lambda) times a unit vector k-hat which points in the direction of propagation. This allows you to generalize the argument of a cosine or sine harmonic wave from k z – omega t to vec k dot vec r – omega t where r = (x,y,z) is the displacement vector to the point where the field is observed. We also generalize the polarization direction (which by convention is taken to be the direction of the electrical field). The n-hat direction is the polarization direction and n-hat multiplies complex amplitude E_0. The condition that del dot E =0 means n-vec dot k-vec = 0 which implies that the polarization (n-vec) is always perpendicular to the propagation vec k). As shown in the slide it is very easy to find B using Faraday's law and complex notation if we assume the same exponential time space dependence. The magnetic field at any point and time is the cross product of the propagation unit vector (k-hat) and the electric field divided by the speed of light. This means E and B are always in phase, are always perpendicular and the B field is down from the E-field by a factor of 1/c. The magnetic field is also perpendicular to k-vec since the vanishing of the B divergence implies k-vec dot vecB =0.

Energy and Poynting Vector

$$\tilde{\vec E} = \hat n\, E_0 \exp(i\vec k\bullet\vec r - i\omega t)\ ;\ \tilde{\vec B} = \hat k\times\hat n\left(\frac{E_0}{c}\right)\exp(i\vec k\bullet\vec r - i\omega t) \text{ or}$$

$$\vec E = \mathrm{Re}(\tilde{\vec E}) = \hat n\, E_0 \cos\left(i\vec k\bullet\vec r - i\omega t\right)\ ;\ \vec B = \mathrm{Re}(\tilde{\vec B}) = \hat k\times\hat n\left(\frac{E_0}{c}\right)\cos\left(i\vec k\bullet\vec r - i\omega t\right)$$

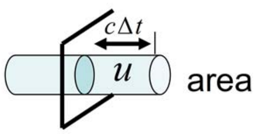
$$\langle u_e\rangle = \frac{\varepsilon_0\left|\vec E\right|^2}{2} = \frac{\varepsilon_0\left|\vec E_0\right|^2}{2}\left\langle\cos^2\left(i\vec k\bullet\vec r - i\omega t\right)\right\rangle = \frac{\varepsilon_0\,|E_0|^2}{4} = \frac{\varepsilon_0}{4}\mathrm{Re}\left\{\left(\tilde{\vec E}\right)^*\bullet\tilde{\vec E}\right\} = \frac{\varepsilon_0}{4}\mathrm{Re}\left\{\left(\hat n\, E_0 e^{i\vec k\bullet\vec r - i\omega t}\right)^*\bullet\left(\hat n\, E_0 e^{i\vec k\bullet\vec r - i\omega t}\right)\right\}$$

$$\langle u_m\rangle = \left\langle\frac{\left|\vec B\right|^2}{2\mu_0}\right\rangle = \frac{1}{4\mu_0}\mathrm{Re}\left\{\left(\tilde{\vec B}\right)^*\bullet\tilde{\vec B}\right\} = \frac{|E_0|^2}{4\mu_0 c^2} = \frac{|E_0|^2}{4\mu_0}(\mu_0\varepsilon_0) = \frac{\varepsilon_0}{4}|E_0|^2 = \langle u_e\rangle$$

Hence total energy density is $\boxed{\langle u\rangle = \langle u_e\rangle + \langle u_m\rangle = \frac{\varepsilon_0\,|E_0|^2}{2}}$ Now compute $\langle \vec S\rangle$.

$$\vec S = \frac{\vec E\times\vec B}{\mu_0} = \hat n\times\left(\hat k\times\hat n\right)\left(\frac{|E_0|^2}{\mu_0 c}\right)\cos^2\left(i\vec k\bullet\vec r - i\omega t\right)\ ;\ \hat n\times\left(\hat k\times\hat n\right) = \hat k\left(\hat n\bullet\hat n\right) - \hat n\left(\hat k\bullet\hat n\right) = \hat k \rightarrow \langle\vec S\rangle = \hat k\left(\frac{|E_0|^2}{\mu_0 c}\right)\left[\frac{1}{2}\right]_{\langle\cos^2(\ )\rangle}$$

$$\text{or } \langle\vec S\rangle = \mathrm{Re}\left\{\frac{\vec E^*\times\vec B}{2\mu_0}\right\} = \frac{\mathrm{Re}\left\{\left(\hat n E_0 e^{i\vec k\bullet\vec r}\right)^*\times\left[\hat k\times\hat n\left(\frac{E_0}{c}e^{i\vec k\bullet\vec r}\right)\right]\right\}}{2\mu_0} = \hat k\left(\frac{|E_0|^2}{2\mu_0 c}\right) = \frac{\hat k}{2}\left(\frac{\sqrt{\mu_0\varepsilon_0}}{\mu_0}\right)|E_0|^2 = \frac{\hat k}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,|E_0|^2$$



$$\text{energy} = \text{area}\,(c\,\Delta t)\,u$$

$$\Rightarrow \langle\vec S\rangle = \frac{\text{energy}}{\text{area}\times\Delta t} = c\,\langle u\rangle\,\hat k$$

$$\boxed{\langle\vec S\rangle = \frac{\hat k\,|E_0|^2}{2(377\Omega)} = \frac{\hat k}{2}\frac{\varepsilon_0\,|E_0|^2}{\sqrt{\varepsilon_0\mu_0}} = c\ \langle u\rangle\hat k}$$

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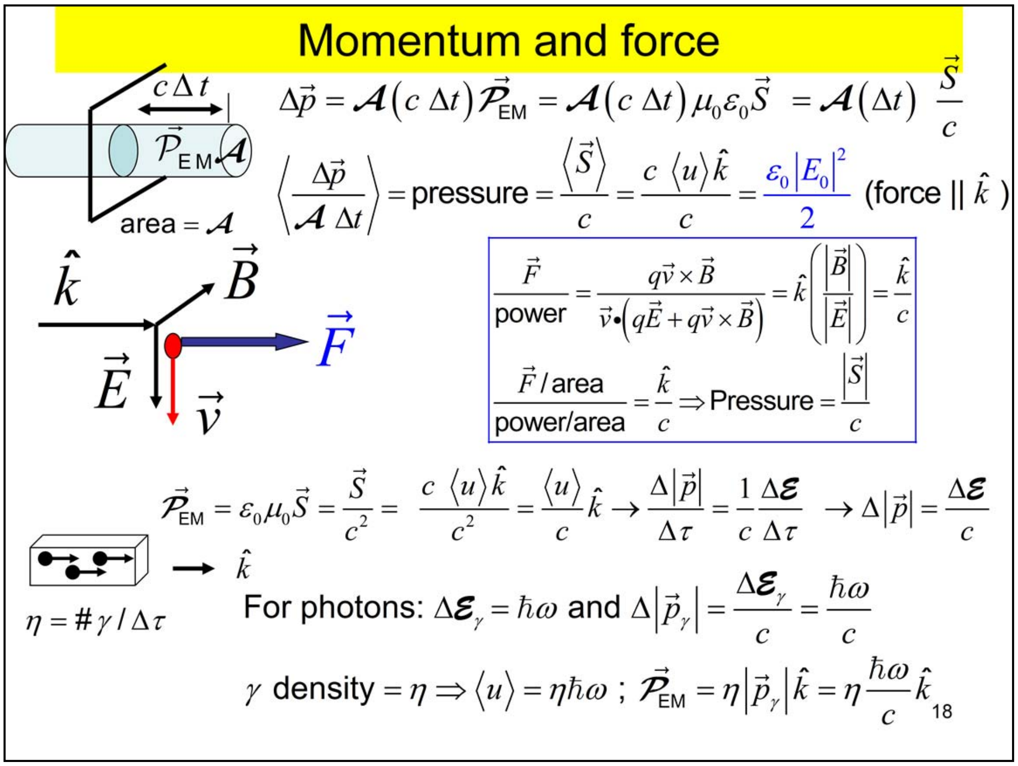
Now that we know how to get the B-field from the E-field, we can write general expressions for the electric and magnetic energy densities u_e and u_m and the Poynting vector for the general harmonic wave. We write the general electrical field in the form E-vec = n-hat E_0 exp(i vec k dot vec r – i omega t) where n-hat is the direction of polarization and vec k is parallel to the direction of propagation. Any phase shift relative to the cosine wave can be accommodated by putting an imaginary piece in the complex amplitude E_0. This is not really the most general harmonic wave but assumes that n-hat or polarization direction is constant so it is the most general **plane polarized** harmonic wave. We begin by calculating the electric and magnetic energy density using the formulae we developed in the Conservation chapter. We are calculating the time averaged u_e and u_m. We write these time averages with averaging brackets < u_e> and we mean the time average over a complete cycle. Our result gives <u_e> and <u_m> in terms of |E_o|^2 which is the squared modulus of the complex electric amplitude E_0. The averaging over the cos^2 (vec-k dot vec-r – omega t) is ½ for any observer coordinate vec-r which you show in HW. Alternatively we can get the harmonic average using ½ Re{E*E} or ½ Re{B*B}. We reach the interesting conclusion that the energy density stored in the electric fields of the E&M wave are always equal to the magnetic fields stored in the waves. We next compute the Poynting vector for the plane polarized E&M wave. We simplify the vector parts of E cross B which is n-hat cross (k-vec cross h-hat) using the BAC – CAB rule and get the simple answer that S points in the k-hat direction. We either use the average of cos^2 = ½ or better yet the complex algebra expression for the time average Poynting vector. We find that < vec-S> is proportional to |E_0|^2 and involves the constant sqrt{epsilon_0/ mu_0}. This is 1/377 Ohms which you show in HW. We further note that <S> is a constant and thus has zero divergence. This agrees with the our previous statement of the time average Poynting theorem which relates the divergence of <S> to <E dot J> but there is no J in the vacuum so <E dot J>=0 as is our divergence.

Many of the same factors in <S> show up in the total, time-averaged energy density formula and hence <vec-S> = c <u> k-hat. The lower illustration explains why you can get this relation between vec-S and u by imagining a cylinder of light piercing a plane. After an interval delta t, the light has moved a distance c delta t and the total energy that passed through the plane is the volume times the energy density where the volume is area*length = area*c delta t. The Poynting vector is the power per transverse area which is the energy / delta t divided by the area which gives us <S> = c < u> using a very simple argument. The magnitude of the time average Poynting vector is called the **intensity** of the light which has a very memorable form of the square of the rms E divided by 377 Ohms.



We next compute the pressure of light by using the momentum density that we first encountered in the conservation chapter. We again consider a cylinder of light that has penetrated a plane after delta t seconds. In the Conservation chapter we showed the momentum density was the epsilon_0 mu_0 times the Poynting vector and we now know epsilon_0 mu_0 = 1/c^2. This allows us to write the total momentum contained in the cylinder as the momentum density times the volume which is area c delta t. If this energy is totally absorbed from the material bounded by the plane, the material will also absorb all of the momentum and will experience a force of delta vec-p/ delta t. The pressure will be this force divided by the area of the cylinder we are considering. If we write the Poynting vector as c<u> we reach the very interesting conclusion that the pressure equals the energy density. We obtained the same conclusion that pressure = energy density for an electrostatic conductor using a very different argument in Physics 435. Of course if only 90% of the energy is absorbed in the material slab and the rest is transmitted the absorbed momentum , force, and pressure will also by down by a factor of 0.90. If 100% of the energy is reflected from the slab the pressure on the slab will double since change of momentum due to the slab will be double the momentum of the beam – if the beam travels in the z-hat direction P_slab = (delta P - - delta P) z-hat = 2 delta P z-hat.

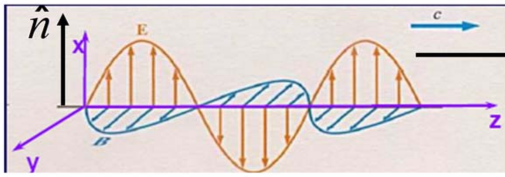
We now show a very simple way of getting the relationship between pressure and Poynting vector by considering the force over the power ratio absorbed by say electrons in the slab material. Assume the electrons have absorbed some energy and are traveling with a velocity vec-v. The absorbed power is vec-v dot force and the force is the both electric and magnetic in origin. But the magnetic force (eg Lorentz force) is perpendicular to vec-v and does no work and creates no power. In calculating vec-F , you can see I am using just the Lorentz force and neglecting the electric force. This is because the electric force which is proportional to the E-field changes direction every half cycle and thus averages to zero. Interestingly enough the Lorentz force is always in the direction of the charge times the k-hat vector since we are assuming v is parallel to E and thus changes direction every ½ period, but so does B. The figure is drawn assuming q > 0.We thus obtain the result that Force/power is B/E which is 1/c for an E&M wave. Dividing the numerator and denominator by the area of the beam we get that the pressure / intensity is 1/c assuming complete absorption of the beam and hence the pressure of the beam is the intensity /c as long as the beam is totally absorbed with no reflection. As a practical matter, since c is large the pressure is small and hence you don't get flung against a wall when you turn on a light in a dark room. Finally we give a quantum view of the light slab which consists of a density (eta) of photons each of which carries an energy of hbar omega. We can determine the momentum carried by the photon using our expressions for the momentum density in terms of the Poynting vector and the Poynting vector in terms of the energy density. This means if an electromagnetic volume carries an energy of E , it also carries a momentum of E/c. Hence the momentum of each photon is hbar omega/c. This allows us to think of the light beam as a density of photons which carries momentum and energy given by eta times the energy and momentum of each monochromatic photon. We will reach the same conclusion that p=E/c using relativity toward the end of the course where we think of the photon as a bullet-like projectile that travels at the speed of light.

Plane polarized EM wave summary

$$\vec{E} = \hat{n}\ \tilde{E}_0 \exp(i\vec{k}\bullet\vec{r} - i\omega t)$$

$$\vec{E} = \hat{n}\ E_0 \cos(i\vec{k}\bullet\vec{r} - i\omega t + \delta)$$

$$\boxed{\vec{B} = \frac{1}{c}\hat{k}\times\vec{E}} \qquad \vec{B} = \hat{k}\times\hat{n}\ \frac{E_0}{c}\cos(i\vec{k}\bullet\vec{r} - i\omega t + \delta)$$



$$\boxed{\hat{k}}$$

$$\left\langle \vec{S} \right\rangle = \frac{\hat{k}\ E_0^2}{2(377\Omega)} = c\ \langle u \rangle\ \hat{k}$$

$$\left\langle u_e \right\rangle = \left\langle u_m \right\rangle$$

$$\left\langle u \right\rangle = \left\langle u_e \right\rangle + \left\langle u_m \right\rangle = \frac{\varepsilon_0 \left|E_0\right|^2}{2}$$

$$\text{Pressure} = \frac{\left|\vec{S}\right|}{c}$$

$$\vec{\mathcal{P}} = \frac{\Delta\vec{p}}{\Delta\tau} = \frac{\hat{k}}{c}\frac{\Delta\mathcal{E}}{\Delta\tau} = \frac{u}{c}\hat{k}$$

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Here is a summary of what we learned about E&M waves. We give the forms of the electrical and magnetic fields. We concluded that the magnetic energy density equals the electric energy density. The total time averaged energy density is the same expression as the energy stored in an electric field. We conclude with expressions for the Poynting vector, light pressure, and the momentum density. Most of these expressions were covered in Physics 212 so this is meant as a review.