Homotopy and Homology Functors Commute with Direct Limits

Lemma 1 Let $\{X_\alpha,\phi_{\alpha\beta}\}$ be a directed system(on a poset $(I,\leq)$) of Hausdorff topological spaces. Suppose that $C\subset \lim_{\rightarrow} X_\alpha$ is compact. Then $C\subset i_\alpha(X_\alpha)$ for some $\alpha\in I$, where $i_\alpha : X_\alpha \rightarrow \lim_{\rightarrow} X_\alpha$ denotes the canonical map.

Proof Suppose not. Then we have a sequence of elements $\{x_n\}$ on $C\subset X=\lim_{\rightarrow} X_\alpha$ such that $x_n\in i_{\alpha_n}(X_{\alpha_n})\backslash i_{\alpha_{n-1}}(X_{\alpha_{n-1}})$ and $\alpha_1<\alpha_2<\alpha_3<\cdots$. This sequence is contained in a compact set C, and does not have a limit, thus compact and discrete, hence finite. Contradiction.

Lemma 2 Let R be a commutative ring and $Cplx_R$ be the category of chain complexes of R-modules. Let $\{C_\alpha,\phi^n_{\alpha\beta}\}$ be a directed system in $Cplx_R$. Then we have $\lim_{\rightarrow} H_*(C_\alpha)=H_*(\lim_{\rightarrow} C_\alpha)$.

Proof We have a natural map $i_\alpha : C_\alpha \rightarrow \lim_{\rightarrow} C_\alpha$. This induces a natural map $(i_\alpha)_* : H_*(C_\alpha)\rightarrow H_*(\lim_{\rightarrow} C_\alpha)$. The system of such maps satisfies the direct system condition, so we have a natural map $i_* : \lim_{\rightarrow} H_*(C_\alpha)\rightarrow H_*(\lim_{\rightarrow} C_\alpha)$.
This map is clearly bijective.

Theorem 3 Let $\{X_\alpha,\phi_{\alpha\beta}\}$ be a directed system of Hausdorff topological spaces. Let H_* denote the singular homology functor on a commutative ring R. Then $H_*(\lim_{\rightarrow} X_\alpha)=\lim_{\rightarrow} H_*(X_\alpha)$.

Proof By Lemma 1, every compact subset of $X=\lim_{\rightarrow} X_\alpha$ is contained in some X_α, so $C(X)=\lim_{\rightarrow} C(X_\alpha)$. By Lemma 2, this directed system induces an isomorphism on homology modules. Therefore $H_*\lim_{\rightarrow}=\lim_{\rightarrow} H_*$.

Theorem 4 Let $\{X_\alpha,\phi_{\alpha\beta}\}$ be a directed system of Hausdorff topological spaces. Let π_n denote the nth homotopy group functor. Then $\pi_n \lim_{\rightarrow}(X_\alpha)=\lim_{\rightarrow}\pi_n(X_\alpha)$.

Proof By definition, an element in $\Omega_n(X)$(nth loop space) is a map $S^n\rightarrow X$. Since S^n is compact, the image is contained in some X_α by Lemma 1. A homotopy is by definition a map $S^n\times I\rightarrow X^n$. Since $S^n\times I$ is also compact, its image is contained in some X_α by Lemma 1. Hence $\Omega_n(X)=\lim_{\rightarrow}\Omega_n(X_\alpha)$ and the homotopy relation on $\Omega_n(X)$ is the direct limit of the homotopy relations on $\Omega_n(X_\alpha)$. Therefore $\pi_n\lim_{\rightarrow}=\lim_{\rightarrow}\pi_n$.