Probability in Computing

CS 237

LECTURE 15

Last time

- Pairwise and Mutual Independence

Today

- Finish independence of random variables
- Expected value of a random variable (can be interpreted as the average value of a random variable)

Reminders

- HW 7 is due Thursday

Reading

- LLM 19.4-19.5
  P 3.2.2

Independent Random Variables

Definition: Independent Random Variables

Random variables X and Y are independent if **for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$,** the events $[X \leq x]$ and $[Y \leq y]$ are independent, i.e.,

$$\Pr([X \leq x] \wedge [Y \leq y]) = \Pr(X \leq x) \cdot \Pr(Y \leq y).$$

- This definition applies to both discrete and continuous RVs.
- For discrete random variables, it is equivalent to:

Definition: Discrete Independent Random Variables

Discrete random variables X and Y are independent if **for all $\boldsymbol{x} \in \text{range}(\boldsymbol{X})$ and $\boldsymbol{y} \in \text{range}(\boldsymbol{Y})$,** the events $[X = x]$ and $[Y = y]$ are independent.

Examples: 1) X and Y are the results of two rolls of a die.

2) X and Y the are distances of two darts from the center of the target.

Review Exercise

- Let X and Y be **independent** random variables, each taking on the values -1 and 1 with probability 1/2.
- Let $Z = X \cdot Y$.

Find the PMF of Z

Top Hat question (Join Code: 033357)

- Let X and Y be **independent** random variables, each taking on the values -1 and 1 with probability 1/2.
- Let $Z = X \cdot Y$.

Are $X, Y,$ and Z pairwise independent?

A. YES

B. NO

Mutually Independent Random Variables

- Definition of mutual independence carries over from events to RVs

Definition: Mutually Independent RVs

Random variables $X_1, X_2, \ldots, X_n$ are mutually independent if **for all values** $\boldsymbol{x_1, \ldots, x_n} \in \mathbb{R}$, the events $[X_1 \leq x_1], [X_2 \leq x_2], \ldots, [X_n \leq x_n]$ are mutually independent

- For discrete random variables, we can replace the events with

$$[X_1 = x_1], [X_2 = x_2], \ldots, [X_n = x_n]$$

Examples: 1) $X_1, \ldots, X_n$ are the results of n rolls of a die.

2) $X_1, \ldots, X_n$ are the distances of n darts from the center of the target.

Top Hat question (Join Code: 033357)

- Let X and Y be **independent** random variables, each taking on the values -1 and 1 with probability 1/2.
- Let $Z = X \cdot Y$.

Are $X, Y,$ and Z **mutually** independent?

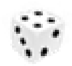
A. YES

B. NO

New Topic

Expectation
of Random Variables

Top Hat question (Join Code: 033357)

You roll one die. 

Let X be the random variable that represents the result.

What value does X take on average?

A. 1/6

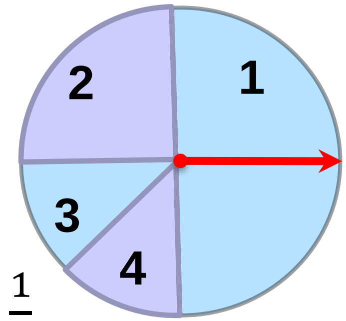
B. 3

C. 3.5

D. 6

E. None of the above.

Example: Spinner



- Spin the dial of the spinner.
  Let Y be the number of the region where it stopped.
  - ➢ $\text{Range}(Y) = \{1,2,3,4\}$
  - ➢ $\Pr(Y=1) = \frac{1}{2},\ \Pr(Y=2) = \frac{1}{4}, \Pr(Y=3) = \Pr(Y=4) = \frac{1}{8}$
- What value does Y take on average?
  - It is NOT $\frac{1+2+3+4}{4} = \frac{10}{4} = 2.5$
- Suppose we spin the dial N times, where N is huge. Then we expect to see about $\frac{N}{2}$ ones, $\frac{N}{4}$ twos, $\frac{N}{8}$ threes, and $\frac{N}{8}$ fours.
- If we add them up and divide by N, we get

$$\frac{\frac{N}{2}\cdot 1 + \frac{N}{4}\cdot 2 + \frac{N}{8}\cdot 3 + \frac{N}{8}\cdot 4}{N} = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{2} = \frac{15}{8} = 1.875$$

$\neq 2.5$

We want a *weighted average*:
each value is counted the number of times proportional to its probability.

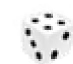
Random variables: expectation

Definition: Expectation

The expectation (also called the expected value or mean) of a discrete random variable X over a sample space Ω is

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega).$$

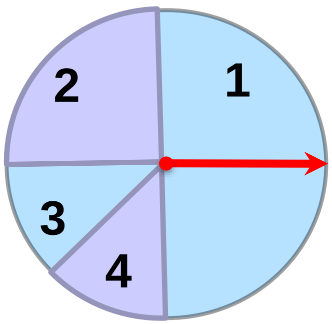
- Example 1: $X =$ number obtained when rolling a die



$$\mathbb{E}(X) = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{21}{6} = 3.5$$

- Example 2: $Y =$ region of the spinner selected

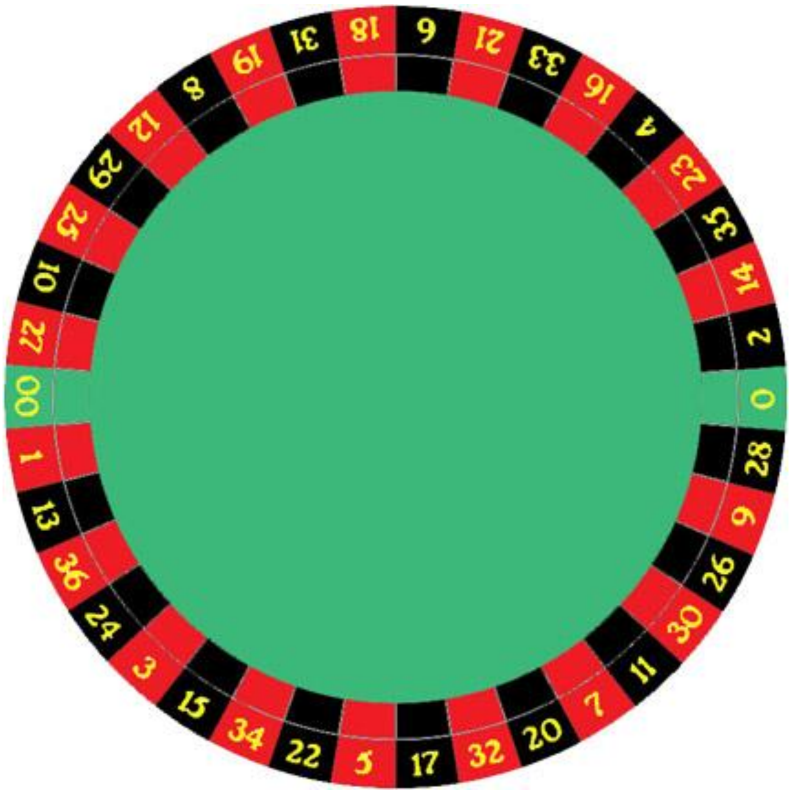
$$\mathbb{E}(Y) = \sum_{i=1}^{4} i \cdot \Pr(i) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{8} = \frac{15}{8}$$



Expectation of X does not have to be in Range(X)

Example: roulette

- 38 slots: 18 black, 18 red, 2 green.



- If we bet \$1 on red, we get \$2 back if red comes up. What's the expected value of our winnings?
- Let X be the value of winnings

It is tedious to consider each slot separately

We can combine outcomes on which X takes the same value

Random variables: expectation

Definition: Expectation

The expectation (also called the expected value or mean) of a discrete random variable X over a sample space Ω is

$$\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega).$$

- We can group together outcomes ω for which $X(\omega) = a$:

$$\mathbb{E}(X) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a).$$

This version of the definition is more useful for computations.

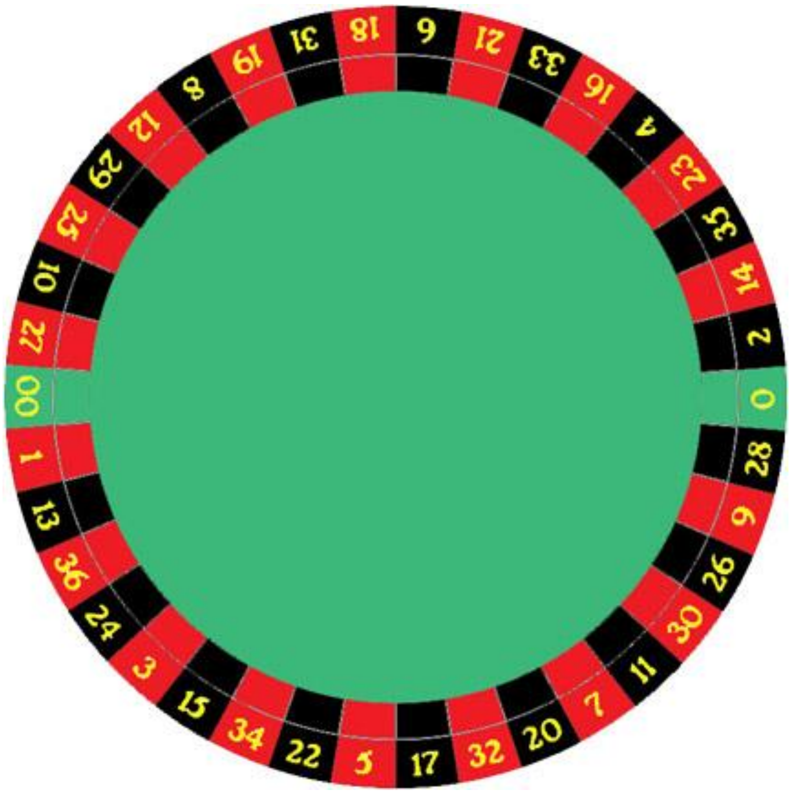
Proof: $\mathbb{E}(X) = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$

$$= \sum_{a \in \text{Range}(X)} \sum_{\omega \in \Omega : X(\omega) = a} X(\omega) \cdot \Pr(\omega) = \sum_{a \in \text{Range}(X)} \sum_{\omega \in \Omega : X(\omega) = a} a \cdot \Pr(\omega)$$

$$= \sum_{a \in \text{Range}(X)} a \cdot \sum_{\omega \in \Omega : X(\omega) = a} \Pr(\omega) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a)$$

Example: roulette

- 38 slots: 18 black, 18 red, 2 green.



- If we bet \$1 on red, we get \$2 back if red comes up. What's the expected value of our winnings?
- Let X be the value of winnings

$$\mathbb{E}(X) = \sum_{a \in \text{Range}(X)} a \cdot \Pr(X = a)$$

Top Hat question (Join Code: 033357)

You roll two dice. 

Let X_1 be the number on the 1st die, X_2 be the number on the 2nd die, and $X = |X_1 - X_2|$. Find $\mathbb{E}(X)$.

A. 0

B. 1

C. $\frac{70}{36}$

D. 2

E. 3.5

F. None of the above