Generators of certain inner mapping groups

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3rd Mile High Conference on Nonassociative Mathematics, August 2013

Definitions

In a loop Q, the left and right translations by an element $x \in Q$ are the maps

$$L_x \, : \, y \mapsto xy \quad \text{and} \quad R_x \, : \, y \mapsto yx$$

respectively.

The *multiplication group* of Q, denoted by $\mathrm{Mlt}(Q)$, is the group of permutations generated by all of the left and right translations.

The *inner mapping group* of Q, denoted by $\mathrm{Inn}(Q)$, is the subgroup of $\mathrm{Mlt}(Q)$ consisting of all maps that leave the identity of Q fixed.

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Inner Mapping Group

Lemma [R. Bruck]

For a loop Q, $\mathrm{Inn}(Q)$ is the subgroup of $\mathrm{Mlt}(Q)$ generated by
the *left inner maps* $L(x,y) = L_x L_y L_{yx}^{-1}$,
the *right inner maps* $R(x,y) = R_x R_y R_{xy}^{-1}$ and
the *middle inner maps (conjugation maps)* $T_x = R_x L_x^{-1}$
where maps are composed from left to right.

Inner Mapping Group

Definition

A subloop of Q is said to be a *normal* subloop if it is stabilized by any element of $\mathrm{Inn}(Q)$.

Lemma [R. Bruck]

For any Moufang loop

$$
\begin{aligned}
&i) \quad L\left(x^{-1}, y^{-1}\right) = R(x, y) = R(y, x)^{-1} \\
&ii) \quad R(x, y) = R(x, xy) = R(yx, y) \\
&iii) \quad R\left(x^{-1}, y^{-1}\right) = R([y, x], y)\, R\left(x^{-1}, y\right)^{-1}
\end{aligned}
$$

where $[y, x] = y^{-1}x^{-1}yx$.

Let Q be a Moufang loop. If $x, y, u \in Q$ then

$$\left(xu^{3m}\right)\left(yu^{3n}\right) = (\ z\ )\, u^{3m+3n}$$

where $z \in Q$ is dependent on the inner map

$$f : Q \longrightarrow Q$$
$$g \longmapsto ugu^{-1}.$$

Namely, $z = f^{2m+n}\left(f^{2m+n}(x) f^{m-n}(y)\right)$.

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Inner maps of Moufang loops

Let Q be a Moufang loop with $x, y, u \in Q$ and

$$\begin{aligned} f : Q &\longrightarrow Q \\ g &\longmapsto ugu^{-1}. \end{aligned}$$

Observation

$$\begin{aligned} (y)R(x, u^3) &= ((yx)u^3)\,(u^{-3}x^{-1}) \\ &= ((yx)u^3)\,(f^{-3}(x^{-1})u^{-3}) \\ &= f\big(f^{-1}(yx)f^{-1}(x^{-1})\big) \end{aligned}$$

Observation

$$\begin{aligned} (u^3\ x\ )^{-1}\, y\, (u^3\ x\ ) &= (x^{-1}u^{-3})\, y\, (f^3(x)u^3) \\ &= f^{-2}\big(f^2(x^{-1})\, f^{-1}(y)\big)u^{-3} \cdot (f^3(x)u^3) \\ &= f^{-1}\big(f^{-1}\big(f^2(x^{-1})\, f^{-1}(y)\big)\, f(x)\big) \end{aligned}$$

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$$\begin{aligned} \left(u^3 \; x\right)^{-1} y \left(u^3 \; x\right) &= \left(x^{-1}u^{-3}\right) y \left(f^3(x)u^3\right) \\ &= f^{-2}\left(f^2\left(x^{-1}\right) f^{-1}(y)\right)u^{-3} \cdot \left(f^3(x)u^3\right) \\ &= f^{-1}\left(f^{-1}\left(f^2\left(x^{-1}\right) f^{-1}(y)\right) f(x)\right) \end{aligned}$$

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Observation

$$\begin{aligned} \left(u^3 f^{-2}(x^{-1})\right)^{-1} y \left(u^3 f^{-2}(x^{-1})\right) &= \left(f^{-2}(x)u^{-3}\right) y \left(f(x^{-1})u^3\right) \\ &= f^{-2}\left(x f^{-1}(y)\right)u^{-3} \cdot \left(f(x^{-1})u^3\right) \\ &= f^{-1}\left(f^{-1}\left(x f^{-1}(y)\right) f^{-1}(x^{-1})\right) \end{aligned}$$

Proposition

Let Q be a Moufang loop with $x, v \in Q$. If v can be written as the cube of another element of Q, say $v = u^3$, then $R(x, v)$ can be written as a product of conjugation maps, namely,

$$R\big(x, u^3\big) = T_x\ T_u^{-1}\, T_{u^3(x^{-1})T_u^2}\, T_u^{-2}.$$

Question

Let Q be a Moufang loop and let

$$S = \left\{ w \in Q \;\middle|\; R(x,w) \in \langle\, T_y \mid y \in Q \,\rangle \text{ for all } x \in Q \right\}.$$

By the previous proposition, S contains all elements that are cubes of other elements of Q.
Is S is closed under multiplication forming a subloop of Q?

Lemma

Suppose Q is a Moufang loop. Then

$$R(x, w_1 w_2) = R\big(x, w_2^{-1}\big)\, R\big(x w_2^{-1}, (w_1)T_{w_2}^{-1} w_2^3\big)\, R\big((x(w_1 w_2))w_2, w_2^{-1}\big)$$

for any $x, w_1, w_2 \in Q$.

Lemma

If Q is a Moufang loop then

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Theorem

Suppose Q is a Moufang loop (finite or infinite). If Q can be generated by elements that are cubes of other elements of Q then its inner mapping group can be generated by conjugation maps.

Corollary

Let Q be a Moufang loop that can be generated by elements which are cubes of other elements of Q. Then a subloop of Q is normal if and only if it is stabilized by conjugation maps.

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For any Moufang loop

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where $[y, x] = y^{-1}x^{-1}yx$.

Theorem

Let Q be a Moufang loop such that the subloop $N = \langle x^3 \mid x \in Q \rangle$ is of index three. Then the inner mapping group of Q is generated by conjugation maps. Moreover, any subloop of Q is normal if and only if it is stabilized by conjugation maps.

proof

$H = \left\{ w \in Q \;\middle|\; R(x,w) \in \langle\, T_y \mid y \in Q \,\rangle \text{ for all } x \in Q \right\}$ is a subloop of Q containing N. If $u \in Q \setminus N$ then for any $x \in N$ and any integer m

$$\begin{aligned} R(u^m x, u) &= R(x,u) \\ &= R(u,x)^{-1} \\ &\in \langle\, T_y \mid y \in Q \,\rangle . \end{aligned}$$

Hence, $u \in H$ and $H = Q$.

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