Homework 10 Solution

Math 461: Probability Theory, Spring 2022
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Due date: Apr 22, 2022

1. If X and Y are independent and identically distributed with mean μ and variance σ^2, find $\mathbb{E}[(X-Y)^2]$.

Solution: We have $\mathbb{E}[X-Y]=0$ and thus $\mathbb{E}[(X-Y)^2]=\mathrm{Var}(X-Y)=\mathrm{Var}(X)+\mathrm{Var}(-Y)=2\sigma^2$.

2. If $\mathbb{E}[X]=1$ and $\mathrm{Var}(X)=4$, find (a) $\mathbb{E}[(2+X)^2]$ and (b) $\mathrm{Var}(4+2X)$.

Solution: (a) $\mathbb{E}[(2+X)^2]=\mathbb{E}[4+4X+X^2]=4+4\cdot\mathbb{E}[X]+(\mathbb{E}[X]^2+\mathrm{Var}(X))=13$ and (b) $\mathrm{Var}(4+2X)=4\,\mathrm{Var}(X)=16$.

3. The random variables X and Y have a joint density function given by

$$f(x,y)=\begin{cases}2e^{-2x}/x & \text{if } 0<x<\infty, 0<y<x\\ 0 & \text{otherwise.}\end{cases}$$

Compute $\mathrm{Cov}(X,Y)$.

Solution: We have

$$\mathbb{E}[X]=\int_0^\infty\int_0^x x\cdot 2e^{-2x}/x\,dy\,dx=\int_0^\infty x\cdot 2e^{-2x}dx=1/2$$
$$\mathbb{E}[Y]=\int_0^\infty\int_0^x y\cdot 2e^{-2x}/x\,dy\,dx=\int_0^\infty x\cdot e^{-2x}dx=1/4$$
$$\mathbb{E}[XY]=\int_0^\infty\int_0^x xy\cdot 2e^{-2x}/x\,dy\,dx=\int_0^\infty x^2\cdot e^{-2x}dx=1/4.$$

Thus $\mathrm{Cov}(X,Y)=\mathbb{E}[XY]-\mathbb{E}[X]\,\mathbb{E}[Y]=1/4-1/2\cdot 1/4=1/8$.

4. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1,2,\ldots,i$. Find
(a) the expected number of urns that are empty;
(b) the probability that none of the urns is empty.

Solution: Let X_j equal 1 if urn j is empty and 0 otherwise. Then

$$\mathbb{E}[X_j]=\mathbb{P}(\text{ ball } i \text{ is not in urn } j, i\geqslant j)=\prod_{i=j}^{n}(1-1/i)=\frac{j-1}{n}.$$

Hence,

$$(a) \qquad \mathbb{E}[\text{Number of empty bins}] = \sum_{j=1}^{n} \frac{j-1}{n} = \frac{n-1}{2}.$$

$$(b) \qquad \mathbb{P}(\text{None are empty}) = \mathbb{P}(\text{ball } j \text{ is in urn } j, \text{ for all } j) = \prod_{j=1}^{n} \frac{1}{j} = \frac{1}{n!}.$$

5. Consider n independent flips of a coin having probability p of landing on heads. Say that a changeover occurs whenever an outcome differs from the one preceding it. For instance, if $n = 5$ and the outcome is $HHTHT$, then there are 3 changeovers. Find the expected number of changeovers.

Solution: Let X_i equal 1 if a changeover occurs on the i-th flip and 0 otherwise. Then

$$\mathbb{E}[X_i] = \mathbb{P}((i-1) \text{ is H}, i \text{ is T}) + \mathbb{P}((i-1) \text{ is T}, i \text{ is H}) = 2p(1-p), i \geqslant 2.$$

Thus, expected number of changeovers is $(n-1) \cdot 2p(1-p)$.

6. A group of 20 people consisting of 10 married couples is randomly arranged into 10 pairs of 2 each. Compute the mean and variance of the number of married couples that are paired together.

Solution: Let $X_i = \mathbf{1}_{\{\text{pair } i \text{ consists of a married couple}\}}$. Thus $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = 1/19, \mathrm{Var}(X_i) = \frac{1}{19}\left(1 - \frac{1}{19}\right)$, $\mathrm{Cov}(X_i, X_j) = \mathbb{P}(X_i = 1, X_j = 1) - \mathbb{P}(X_i = 1)\,\mathbb{P}(X_j = 1) = \frac{1}{19 \cdot 17} - \left(\frac{1}{19}\right)^2$ for $i \neq j$. Hence

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_{10}) = 10 \cdot \frac{1}{19}\left(1 - \frac{1}{19}\right) + 10 \cdot 9 \cdot \left[\frac{1}{19 \cdot 17} - \left(\frac{1}{19}\right)^2\right] = \frac{180 \cdot 18}{19^2 \cdot 17}.$$

7. Let $X_1, X_2, \ldots$ be independent random variables with common mean μ and common variance σ^2. Set $Y_n = X_n + X_{n+1} + X_{n+2}$, $n \geqslant 1$. For $j \geqslant 0$, find $\mathrm{Cov}(Y_n, Y_{n+j})$.

Solution:

$$\begin{aligned}
\mathrm{Cov}(Y_n, Y_n) &= \mathrm{Var}(Y_n) = 3\sigma^2 \\
\mathrm{Cov}(Y_n, Y_{n+1}) &= \mathrm{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\
&= \mathrm{Cov}(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) = 2\sigma^2 \\
\mathrm{Cov}(Y_n, Y_{n+2}) &= \mathrm{Cov}(X_{n+2}, X_{n+2}) = \sigma^2 \\
\mathrm{Cov}(Y_n, Y_{n+j}) &= 0 \text{ when } j \geqslant 3.
\end{aligned}$$

8. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-x/y - y}/y & \text{if } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\mathbb{E}(X^2 | Y = y)$.

Solution: We have

$$f_Y(y) = \int_0^\infty \frac{e^{-x/y - y}}{y} dx = e^{-y}$$

for $y > 0$, so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}e^{-\frac{x}{y}} & x > 0 \\ 0 & x \leqslant 0. \end{cases}$$

Now, we have

$$\mathbb{E}(X^2|Y) = \int_0^\infty \frac{x^2}{y} e^{-\frac{x}{y}} dx = 2y^2.$$

9. The joint density of X and Y is given by

$$f(x, y) = \begin{cases} e^{-y}/y & \text{if } 0 < x < y, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Compute $\mathbb{E}(X^3|Y = y)$.

Solution: We have

$$f_Y(y) = \int_0^y \frac{e^{-y}}{y} dx = e^{-y},$$

so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0, y) \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that

$$\mathbb{E}(X^3|Y = y) = \int_0^y \frac{x^3}{y} dx = \frac{y^3}{4}.$$

10. The number of people who enter an elevator on the ground floor is a Geometric random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all of its passengers.

Solution: Let Y_i be one if the elevator stops at the i-th floor, for $i = 1, \ldots, N$. Let $Y = Y_1 + \cdots + Y_{10}$. Let X be the number of passengers, i.e., X is Geometric with parameter $p = 1/10$. We have $\mathbb{E}(Y_i = 1|X = k) = 1 - \left(\frac{N-1}{N}\right)^k$, so that

$$\mathbb{E}(Y|X = k) = N\left(1 - \left(\frac{N-1}{N}\right)^k\right).$$

We have

$$\begin{aligned} \mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) &= \mathbb{E}(N\left(1 - \left(\frac{N-1}{N}\right)^X\right)) \\ &= N - N\sum_{k=1}^{\infty}\left(\frac{N-1}{N}\right)^k \frac{1}{10}\left(\frac{9}{10}\right)^{k-1} \\ &= N - \frac{N}{10} \cdot \frac{(N-1)/N}{1 - 9(N-1)/(10N)} = N - \frac{N(N-1)}{N+9} = \frac{10N}{N+9}. \end{aligned}$$