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A problem on the approximation of n-roots based on the Viète's work

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In this paper, after giving a brief approach about the historical question of how different authors and cultures handled approximations of the n-th root of $N := A^n + R$, we consider the particular case of the approximation of $\sqrt[n]{N}$ by $A + \frac{R}{d}$ (here n, A, d are positive integers, with $n \geq 2$, $d = (A+1)^n - A^n - 1$, and $0 < R \leq d$), explicitly considered by Viète in his *De Numerosa Potestatum*. Viète established in one of the precepts of his work that the n-th root is placed between $A + \frac{R}{d+1}$ and $A + \frac{R}{d}$. Indeed, it is easily seen that, of course, $A + \frac{R}{d+1} \leq \sqrt[n]{N}$, but unfortunately $\sqrt[n]{N}$ can be greater than $A + \frac{R}{d}$ when $n \geq 3$. Then the question arises whether we are able or not of determining what values of R (fixing the integer values n and A, so also d) give a true approximation by excess when $\sqrt[n]{N}$ is estimated as $A + \frac{R}{d}$, that is, to find those values of R producing an excess in relation with the exact value of the root. We give a complete answer to the cases $n \leq 7$, and we present without proof the main result for a general n.

1 Introduction

There is not doubt that the History of Mathematics allows us to improve both our research and docent tasks. In the first case, the reading of ancient

François Viète, image from [24], https://rcin.org.pl/dlibra/publication/16160/edition/11661 texts or the compilation of works of some authors, as well as the rediscovering of some ideas hidden in those texts, provide us a valuable source for starting new investigations and for fitting them into a more general framework, thus avoiding the centrifugal tendency of specialization in the different fields of mathematics, or at least to make us see a certain internal unity in mathematics. Respect to our docent role, it is clear that the introduction of historical aspects in our different lectures can influence in the positive reception by our students and, therefore, it contributes to trap their attention and to bring closer a subject considered a priori rather arid. Another possible benefit is the eventual elaboration of mathematical materials for breaking the monotony of classical classroom presentations. A lot of literature is developed in this direction, let us to mention, for instance, [18], [21]. Bearing all this in mind, in this paper we are going to give a brief historical note about the approximation of n-th roots by French mathematician François Viète (1540–1603) in his *De Numerosa Potestatum*. This historical note will be the starting point for analyzing a simple mathematical question arising from the reading of the above mentioned Viète's work, and dealing with the possibility of finding approximations by excess of the n-roots.

The paper is organized as follows. After presenting some well-known historic episodes on the numerical computation of square and cubic roots, we present in Section 3, in a broad manner, the procedure employed by Viète in his *De Numerosa Potestatum* to compute general n-roots as well as a precept for locating irrational roots between two approximations by default and excess. This precept applies for square roots, but for greater roots the proposed approximation by excess can be in fact less than the searched root. This gives us occasion for proposing in Section 4 a mathematical question about the validity of the precept depending on the value $N = A^n + R$ whose n-root we are looking for. The question is completely solved in Section 5 for $3 \leq n \leq 7$ by using elementary mathematics and the Descartes' rule of signs, whereas in Section 6 we advance without proof the result for a general value n. Our paper finishes with some comments about the Viète's work, and the importance of a suitable implementation of the History of Mathematics in our mathematical task.

2 Historical approach of n-th roots

Let us present some milestone about the topic of extraction of square, cubic, and general n-roots. For the reader interested in the topic of the numerical treatment of roots of equations along the history, we strongly recommend [7], a very complete treatise of arithmetical methods from the Antiquity to the end of 19-th century, as well as [23].

The computation of square and cubic roots just appears as a practical subject of undoubted value in the Babylonian and Egyptian cultures. For instance, in the Babylonian culture we can find, by a numerical procedure, the computation of the measure of the diagonal of a door in the clay tablet BM 96957 +VAT 6598, see [25]. To do it, they used the approximation $a + \frac{1}{2}\frac{B}{a}$ for the square root of $a^2 + B$ (for a geometrical explanation of this approach, the reader is referred to [14], see also Figure 1).

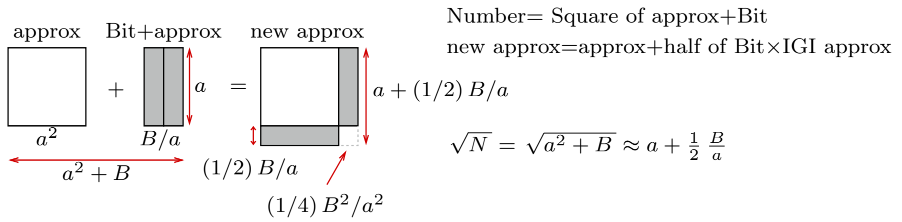


Figure 1: Geometrical explanation for the approximation by excess of $\sqrt{N} = \sqrt{a^2 + B}$ by $a + \frac{1}{2}\frac{B}{a}$, image composed from [14, p. 371]

Later, we find the topic in the Greek civilization. A simple way to proceed to the approximation of an irrational square root $\sqrt{N}$ consists in approaching N by a suitably nearly ratio $\frac{m^2}{n^2}$, with m, n natural numbers, therefore $\sqrt{N} \approx \frac{m}{n}$. This idea was applied by Aristarchus of Samos (c. 310 B.C.–c. 230 B.C.) in his work *Aristachi de magnitudinibus et distantiis Solis et Lunæ Liber* [3], when dealing with the approximation for the ratio between the distance of the Sun to the Earth and the distance between the Moon and the Earth ([3, Prop. VII]); to this effect, as a preliminary step in his study, he gives $\frac{7}{5}$ as an estimate of the square root of 2 (notice that $2 \approx 49/25$). For further information, consult [3], a Spanish translation of Aristarchus' work, with appropriate comments, by the third author of this paper.

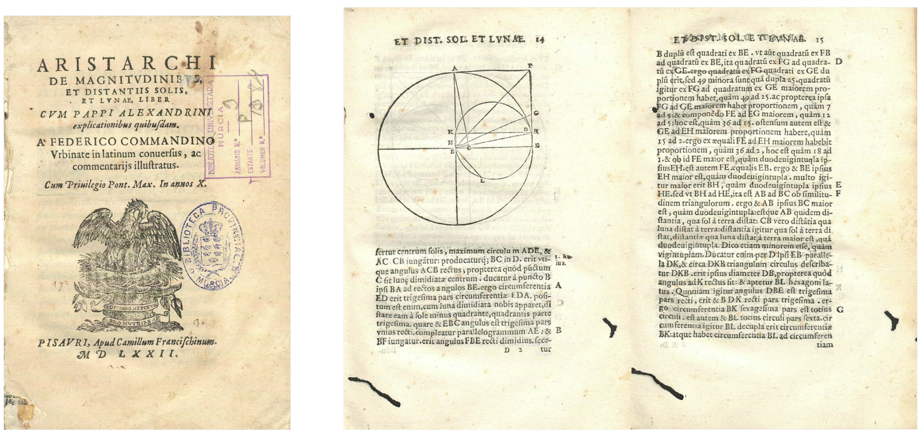


Figure 2: Front of Aristarchus' work, and part of Proposition VII, image from https://bibliotecafloridablanca.um.es/bibliotecafloridablanca/handle/11169/975

We can mention Archimedes (287 B.C.–212 B.C.) who in his "Measurement of a circle"[1] shows us his ability for manipulating numerical computations in order to give an approximation of the ratio between the circumference and the diameter of a circle; he needs to estimate the values of several square roots in order to achieve his task and, for instance, he obtains that $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$[2].

Also, in the Greek culture, let us cite to Hero of Alexandria (about first century A.D.); he provides us in his *Metric*[3] the method of averages for approximating square roots: in [1, Livre I, VIII, p. 165] Hero illustrates his well known formula for the area of a triangle (namely, if $p = \frac{a+b+c}{2}$ denotes the semiperimeter of a triangle of sides a, b, c, the corresponding area S is given by Hero's formula $S = \sqrt{p(p-a)(p-b)(p-c)}$) taking sides of lengths $7, 8, 9$ and, therefore, he needs to compute the value of $\sqrt{720}$. This is done by taking 27 as the closer integer to $\sqrt{720}$, and then giving the first approximation $\frac{1}{2}\left(27 + \frac{720}{27}\right) = 26\ \frac{1}{2}\ \frac{1}{3}$ as a more accurately approach; even, Hero says us that if we do the same operations with the new approach we will obtain a better approximation. Notice that Hero presents an iterative method; in current notation, if x_n is the approximation of $\sqrt{N}$, the next term is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$, the average between x_n and its inverse; realize that this recurrence is nothing else but the Newton-Raphson method applied to the equation $x^2 - N = 0$ [4].

Moreover, in the problem XX of Book III of the *Metric* it is proposed the division of a pyramid by a plan parallel to the basis holding a certain proportion between the pyramids; in the development of the problem, it is necessary to obtain a numerical approximation, based on double false position, of the cubic root of 100, and Hero gives us the value $4 + \frac{9}{14}$, having, currently we will say, two exact decimals. In our current notation, if $a^3 < N < (a+1)^3$, and $d_1 = N - a^3, d_2 = (a+1)^3 - N$, then $\sqrt[3]{N}$ is approximated by $a + \frac{(a+1)d_1}{(a+1)d_1 + ad_2}$, so for $\sqrt[3]{100}$ we find —notice that $a = 4, d_1 = 36, d_2 = 25$— the value $4 + \frac{9}{14}$[5].

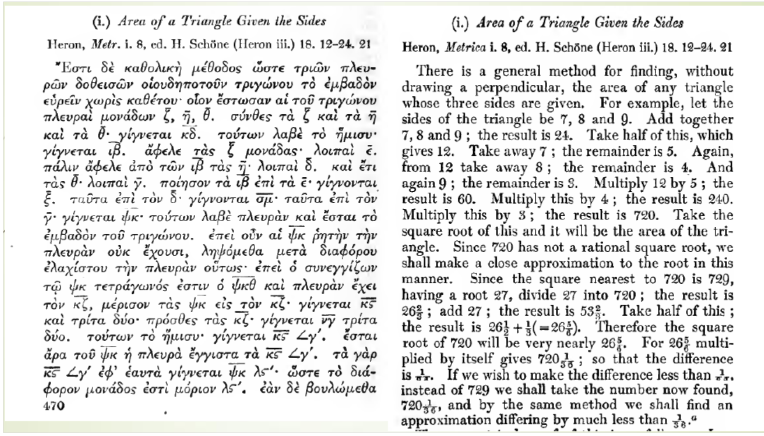
[1]The translation into English of the Archimedes' works can be consulted in [2], in particular the referred 'Measurement of a Circle'.

[2]For an explanation of this approximation, based on geometrical considerations, we recommend the reading of [6].

[3]The reader can consult [1], a French translation, jointly with a critical edition, of the Hero's *Metric*, one of his more important works.

[4]As a delightful curiosity, and based on the idea that the recurrence to obtain $\sqrt{N}$ can be viewed as the mean between a term x and $\frac{N}{x}$, in [15] the author establishes the following generalization for the approximation of $\sqrt[k]{N}$: by considering the average between $x, \overset{(k-1)\text{ times}}{\ldots}, x$ and Nx^{1-k}, it is easily seen that the recurrence $x_{n+1} = \frac{(k-1)x_n + Nx_n^{1-k}}{k}$ gives a sequence that converges quickly to $\sqrt[k]{N}$ if the initial condition is close to the desired k-root, and surprisingly if we apply the Newton-Raphson method to $f(x) = x^k - N$, we derive the same recurrence.

[5]Hero says us ([1, p.351]): "Mais comment faut-il prendre un côté cubique des 100 unités, nous le dirons maintenant. Prends le cube le plus proche de 100, aussi bien celui



(i.) *Area of a Triangle Given the Sides*

Heron, *Metr.* i. 8, ed. H. Schöne (Heron iii.) 18. 12–24. 21

Ἔστι δὲ καθολικὴ μέθοδος ὥστε τριῶν πλευρῶν δοθεισῶν οἱουδηποτοῦν τριγώνου τὸ ἐμβαδὸν εὑρεῖν χωρὶς καθέτου· οἷον ἔστωσαν αἱ τοῦ τριγώνου πλευραὶ μονάδων ζ, η, θ. σύνθες τὰ ζ καὶ τὰ η καὶ τὰ θ· γίγνεται κδ. τούτων λαβὲ τὸ ἥμισυ· γίγνεται ιβ. ἄφελε τὰς ζ μονάδας· λοιπαὶ ε. πάλιν ἄφελε ἀπὸ τῶν ιβ τὰς η· λοιπαὶ δ. καὶ ἔτι τὰς θ· λοιπαὶ γ. ποίησον τὰ ιβ ἐπὶ τὰ ε· γίγνονται ξ. ταῦτα ἐπὶ τὸν δ· γίγνονται σμ· ταῦτα ἐπὶ τὸν γ· γίγνεται ψκ· τούτων λαβὲ πλευρὰν καὶ ἔσται τὸ ἐμβαδὸν τοῦ τριγώνου. ἐπεὶ οὖν αἱ ψκ ῥητὴν τὴν πλευρὰν οὐκ ἔχουσι, ληψόμεθα μετὰ διαφόρου ἐλαχίστου τὴν πλευρὰν οὕτως· ἐπεὶ ὁ συνεγγίζων τῷ ψκ τετράγωνός ἐστιν ὁ ψκθ καὶ πλευρὰν ἔχει τὸν κζ, μέρισον τὰς ψκ εἰς τὸν κζ· γίγνεται κϛ καὶ τρίτα δύο· πρόσθες τὰς κζ· γίγνεται νγ τρίτα δύο. τούτων τὸ ἥμισυ· γίγνεται κϛ ∠γ'. ἔσται ἄρα τοῦ ψκ ἡ πλευρὰ ἔγγιστα τὰ κϛ ∠γ'. τὰ γὰρ κϛ ∠γ' ἐφ' ἑαυτὰ γίγνεται ψκ λϛ'· ὥστε τὸ διάφορον μονάδος ἐστὶ μόριον λϛ'. ἐὰν δὲ βουλώμεθα

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(i.) *Area of a Triangle Given the Sides*

Heron, *Metrica* i. 8, ed. H. Schöne (Heron iii.) 18. 12–24. 21

There is a general method for finding, without drawing a perpendicular, the area of any triangle whose three sides are given. For example, let the sides of the triangle be 7, 8 and 9. Add together 7, 8 and 9; the result is 24. Take half of this, which gives 12. Take away 7; the remainder is 5. Again, from 12 take away 8; the remainder is 4. And again 9; the remainder is 3. Multiply 12 by 5; the result is 60. Multiply this by 4; the result is 240. Multiply this by 3; the result is 720. Take the square root of this and it will be the area of the triangle. Since 720 has not a rational square root, we shall make a close approximation to the root in this manner. Since the square nearest to 720 is 729, having a root 27, divide 27 into 720; the result is $26\frac{2}{3}$; add 27; the result is $53\frac{2}{3}$. Take half of this; the result is $26\frac{1}{2}+\frac{1}{3}(=26\frac{5}{6})$. Therefore the square root of 720 will be very nearly $26\frac{5}{6}$. For $26\frac{5}{6}$ multiplied by itself gives $720\frac{1}{36}$; so that the difference is $\frac{1}{36}$. If we wish to make the difference less than $\frac{1}{36}$, instead of 729 we shall take the number now found, $720\frac{1}{36}$, and by the same method we shall find an approximation differing by much less than $\frac{1}{36}$.[a]

Figure 3: Text of Hero's formula as collected in [30, v. II]

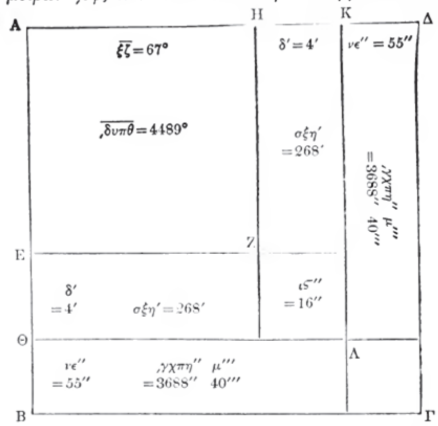
As another instance of the approximation of square roots, let us mention to Theon of Alexandria (c.335–c.405). In his *Commentary* on Ptolemy's *Syntaxis*[6], Theon explains how to make an approximation to the irrational side of a square, and he shows the approximation of $\sqrt{4500°}$ to be (in sexagesimal numeration) $67° \, 4' \, 55''$, that is, $\sqrt{4500} = 67 + \frac{4}{60} + \frac{55}{60}$. His procedure is based on a figure in which by a process of exhaustion he completes the square by using the well-known fourth proposition of Euclid's Book II, to wit, if a straight line is cut at random, the square of the total is equal to the squares of the segments and twice the rectangle contained by the segments —see [16]. For a picture of this geometrical process, see Figure 4[7].

The fact of the universality of this mathematical problem about approximating square and cubic roots is reflected, indeed, in its ubiquity in different cultures, apart from the Western one, as it occurs, for instance, in the Chinese and Hindu civilizations. In this way, from the Chinese culture, we can

par excès que celui par défaut; or ce sont 125 et 64. Et ce par quoi l'un excède est 25 unités, ce par quoi l'autre est en défaut est 36 unités; et fais les 5 par les 36: il en résulte 180; plus les 100: il en résulte 280; et applique les 180 aux 280: il en résulte 9/14; ajoute-les au côté du plus petit cube, c'est-à-dire au 4: il en résulte 4 unités et 9/14. Autant que cela sera le côté cubique des 100 unités à très peu près." /We are going to say how to take the cubic side for 100 units. Take the cube closer to 100, as well by excess as by default; they are 125 and 64. So, the excess is of 25 units, and the defect of 36 units; multiply 5 by 36: it outcomes 180; to add 100: it is 280; and apply (divide) 180 to 280: it outcomes 9/14; add them to the side corresponding to the less cube, that is, to 4: it outcomes 4 units and 9/14. Therefore, this will be the cubic side for the 100 units, more or less. [Our translation]

[6] For example, consult A.Rome (ed.), *Commentaires de Pappus et de Théon d'Alexandrie sur l'Almageste. Texte établi et annoté. Tome II. Théon d' Alexandrie, Commentaire sur les livres 1 et 2 de l'Almageste.* (Studi e Testi 72). Città del Vaticano: Biblioteca Apostolica Vaticana, 1936.

[7] See also [8, pp.204-205].



Ἵνα δὲ καὶ ἐπί τινος τῶν ἐν τῇ Συντάξει παρακειμένων ἀριθμῶν ὑπ' ὄψιν ἡμῖν γένηται ἡ τῆς κατὰ μέρος ἀφαιρέσεως διάκρισις, ποιησόμεθα τὴν ἀπόδειξιν ἐπὶ τοῦ ͵δφ ἀριθμοῦ, οὗ τὴν πλευρὰν ἐξέθετο μοιρῶν ξζ δ νε. ἐκκείσθω χωρίον τετράγωνον τὸ ΑΒΓΔ, δυνάμει μόνον ῥητόν, οὗ τὸ ἐμβαδὸν ἔστω μοιρῶν ͵δφ, καὶ δέον ἔστω τὴν σύνεγγυς αὐτοῦ

τετραγωνικὴν πλευρὰν ἐπιλογίσασθαι. ἐπεὶ οὖν ὁ

Figure 4: The process for squaring 4500, image from [30, p.56, v.I]

mention Liu Hui (c.220–c.280), who commented the treatise Computational Prescriptions in Nine Chapters (*Jiuzhang suanshu*) based upon the ancient Chinese tradition collected in more ancient manuscripts. In Chapter 4 (*shao guang*), we find methods for calculating square and cubic roots. For more information about the treatise and Liu Hui, consult [9, pp. 133-153] and [5, Chapter 7], and for a textbook on the Chinese Mathematics, see [20].

Concerning Hindu tradition we can cite the *Sulvasutra* ('the rule of chords'), where we find for instance the construction of a square having double area than another given square of side L, under the rule $L+\frac{L}{3}+\frac{L}{12}-\frac{L}{34\cdot 4\cdot 3}$ for the side of the new figure (for an interpretation of this formula, consult [10, p.189 and ss.]). From Figure 5 we see that $1+\frac{1}{3}+\frac{1}{12}$ is an approximation by excess to $\sqrt{2}$; to obtain $1+\frac{1}{3}+\frac{1}{12}-\frac{1}{34\cdot 12}$, a modern explanation may be to realize that $\frac{577}{34\cdot 12}$ is close to $\sqrt{2}$ (it is a convergent of the continued fraction associate to $\sqrt{2}$), thus making $1+\frac{1}{3}+\frac{1}{12}-x=\frac{577}{34\cdot 12}$ yields to $x=\frac{1}{34\cdot 12}$.

Let us notice that to Hindu tradition is credited to have contributed with the computation of square and cube roots by dividing the number in periods and to arrange the process into columns and lines (see [23, p. 8]). Moreover, according to [7, pp. 174-175], in the *Lilavati* of Bhaskara (1114–1185) we can appreciate the high level of perfection that the art of computation had reached among Hindus and, despite the Greek writers on root extraction, the importance of the transmission of Hindu methods to the Arabs and by them to the Europeans.

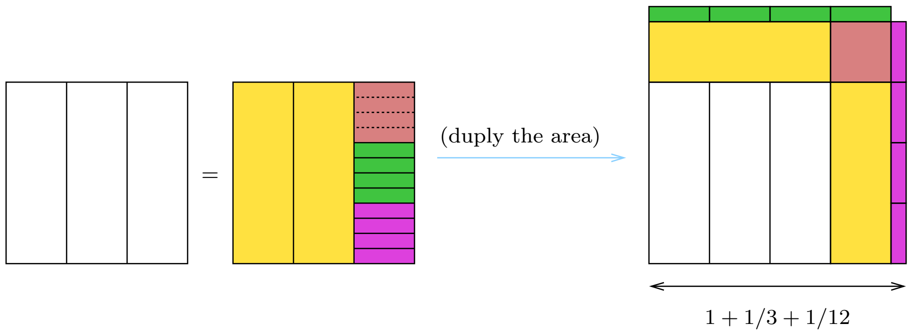


Figure 5: Image elaborated from [10]

In relation, thus, with the Arab tradition, the Arabic mathematicians also handled different numerical approximations for the resolution of square, cubic and affected equations. In fact, it is said that they extended the concept of square and cubic roots to general powers ([7, p. 175]). For square roots, apart from the Hero's rule $\sqrt{a^2+b} \approx a + \frac{b}{2a}$, used among others by Al-Battani (c.858–929) ([23]), other rules were also applied, as $\sqrt{a^2+b} \approx a + \frac{b}{2a+1}$ in the case of Al-Karaji (c.953–c.1029) ([23]) or Al-Nasawi (c.1011–c.1075) (see [29]); concerning cubic roots $\sqrt[3]{a^3+r}$, the same Al-Nasawi uses $a + \frac{r}{3a^2+1}$ to give $154 + \frac{32}{71149}$ for approximating $\sqrt[3]{3652296}$, even the formula $a + \frac{r}{3a^2+3a+1}$ is considered according to [29].

Following with this quick tracing on the approximation of n-th roots, let us mention to Leonardo of Pisa (c.1170–c.1250), commonly known as Fibonacci, who in Chapter XIV of his *Liber Abaci* shows us how to proceed with the extraction of square and cubic roots (for an English translation of *Liber Abaci*, see [27]). We emphasize that this procedure is based on the developments, from a geometrical point of view, of the square and cube of a binomial $a + b$. Leonardo of Pisa also deals with square and cubic roots in Chapters II and V, respectively, of *De Practica Geometria* (for an English translation of this work, consult [17]). In Chapter II we find, for instance, $884 + \frac{32}{2\cdot 884} = 884 + \frac{16}{884}$ as the approach for $\sqrt{781488} = \sqrt{884^2 + 32}$; and in Chapter V we find again $a + \frac{r}{3a(a+1)+1}$ when, for instance, he approaches $\sqrt[3]{900} = \sqrt[3]{9^3 + 171}$ by $9 + \frac{171}{271}$ or approximately $9 + \frac{2}{3}$. We can say that Fibonacci was aware of the Indo-Arabic numbering and of the calculation methods associated with it, see [5]. Let us notice that the formula $a + \frac{r}{3a(a+1)}$ was also employed for the cubic root, for instance we can mention the Spanish mathematician Juan de Ortega (1480–1568)[8].

As an instance of a new technique, let us finish with a comment to the work of Rafael Bombelli (1526-1572), who in his *Algebra* proposes a

[8]In his book "Tratado subtilissimo de Arismetica y de Geometria", Juan Cromberger: Sevilla, 1537, we have, e.g., $\sqrt[3]{18889} \approx 26 + \frac{1313}{2106}$.

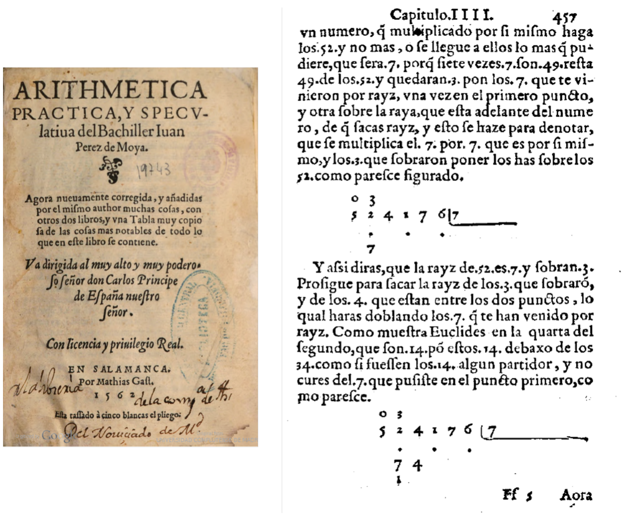
recursive process to approximate $\sqrt{13}$, which currently we identify with the generation of a continued fraction. Indeed, if in modern notation we put $\sqrt{13} = \sqrt{3^2+4} = 3+x$, then $13 = 9+6x+x^2$ or $x(x+6) = 4$, that is, $x = \frac{4}{6+x}$. If we repeat the argument, we have $\sqrt{13} = 3+x = 3+\frac{4}{6+\frac{4}{6+\frac{4}{6+\cdots}}}$. The reader can consult a detailed explanation of Bombelli's method in [26]. For $\sqrt{2}$, we obtain $\sqrt{2} = 1+\frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}$, and its first convergents are $\{1; \frac{3}{2}; \frac{7}{5}; \frac{17}{12}; \frac{41}{29}; \frac{99}{70}; \frac{239}{169}; \frac{577}{408}\}$.

3 The Viète's procedure

During the 16-th century we know different texts devoted to Arithmetic in which we find the solution of square roots, both in exact and approximated manners, still following the guideline of *Elements*, and in general the extraction of roots of bigger powers by a numerical procedure based on that we actually we denominate Newton's binomial. In this way, let us cite to Pérez de Moya (c.1512–1596), and his book *Arithmetica practica y speculativa*, 1562; when Pérez de Moya expounds how to compute the square root of 524176, that is, 724, after dividing 524176 in binary groups $(52 \cdot 41 \cdot 76)$, he explains: "And you will say that the root of 52 is 7 and it remains 3. Continue in order to extract the root of that 3 remaining and of those 4 placed between the two marked points, you will do that by doubling the number 7 obtained previously as the root. How it is shown in the fourth proposition of the Book II of the Elements, they are 14 and put these 14 under 34 as if 14 be a divisor, and do not worry about 7 of the first point,..."[9].

To better understand the rule, firstly realize that from the division of the number into pairs of figures, we know that the integer part of the root has three digits, $abc = 100a+10b+c$. Once we know that the number is placed between 700 and 800 (because $7^2 < 54 < 8^2$, so $a = 7$), for obtaining the second digit b in our current notation we would proceed as follows: $(700+10b)^2 = 524176$ or $14000b + 100b^2 = 524176 - 490000$, that is, $14000b(1+\frac{100b}{14000}) = 34176$; then, $b \leq \frac{34176}{14000}$, and as an approximation we take b as the integer part of $\frac{34176}{14000}$ or, even, the integer part of $\frac{34}{14}$, that is, $b = 2$; since $730^2 > 524176$, at the moment we have $720 < \sqrt{524176} < 730$. To obtain the third digit, take into account the development $524176 = (700+20+c)^2 = 490000 + 28400 + 1440c + c^2$, or $5776 = 34176 - 28400 = 1440c(1+\frac{c}{1440})$; now $c \approx \frac{5776}{1440} \approx \frac{577}{144}$, that is, we do the attempt $c = 4$, and we finally obtain the integer root 724.

[9]"Y asi diras, que la rayz de 52 es 7 y sobran 3. Prosigue para sacar la rayz de los 3 que sobraron, y de los 4 que están entre los dos puntos, lo qual haras doblando los 7 que te han venido por rayz. Como muestra Euclides en la quarta del segundo, que son 14, pon estos 14 debaxo de los 34 como si fuessen los 14 algun partidor, y no cures del 7 que pusiste en el puncto primero,..." (page 457).



ARITHMETICA PRACTICA, Y SPECV-latiua del Bachiller Iuan Perez de Moya.

Agora nueuamente corregida, y añadidas por el miſmo author muchas coſas, con otros dos libros, y vna Tabla muy copioſa de las coſas mas notables de todo lo que en eſte libro ſe contiene.

Va dirigida al muy alto y muy poderoſo ſeñor don Carlos Principe de Eſpaña nueſtro ſeñor.

Con licencia y priuilegio Real.

EN SALAMANCA. Por Mathias Gaſt. 1562

Eſta taſſado à cinco blancas el pliego.

Capitulo.IIII. 457

vn numero, q̃ multiplicado por ſi miſmo haga los.52.y no mas, o ſe llegue a ellos lo mas q̃ pudiere, que ſera.7. porq̃ ſiete vezes.7.ſon.49.reſta 49.de los.52.y quedaran.3. pon los.7. que te vinieron por rayz, vna vez en el primero puncto, y otra ſobre la raya, que eſta adelante del numero, de q̃ ſacas rayz, y eſto ſe haze para denotar, que ſe multiplica el. 7. por. 7. que es por ſi miſmo, y los.3.que ſobraron poner los has ſobre los 52.como pareſce figurado.

0 3
5 2 4 1 7 6 | 7
7

Y aſsi diras, que la rayz de.52.es.7.y ſobran.3. Proſigue para ſacar la rayz de los.3.que ſobraró, y de los. 4. que eſtan entre los dos punctos, lo qual haras doblando los.7. q̃ te han venido por rayz. Como mueſtra Euclides en la quarta del ſegundo, que ſon.14.pó eſtos.14. debaxo de los 34.como ſi fueſſen los.14. algun partidor, y no cures del.7.que puſiſte en el puncto primero, como pareſce.

0 3
5 2 4 1 7 6 | 7
7 4
1

Ff 5 Aora

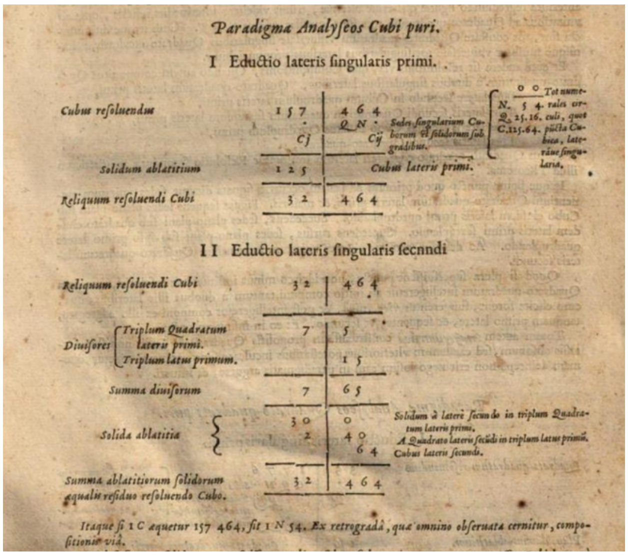
Figure 6: A fragment of $\sqrt{524176}$ in *Arithmetica practica y speculativa* by Pérez de Moya

All these ideas and procedures will crystallize masterfully in the work *De Numerosa Potestatum* (1600/1646), from François Viète.

The first edition of this work was in charge of Marino Ghetaldi (1568–1626) in 1600 (see [31]). After, in 1646 a new version appeared, inside the posthumous book *Opera Mathematica* of Viète, edited by Frans van Schooten (1615–1660) (see [32]), also in Latin[10]. The work is divided into three parts: *De Numerosa Potestatum Purarum Resolutione* ([31, ff. 2-6], [32, pp. 163-172]), where Viète solved numerically the so-called *pure* equations, $x^n = a$ in our actual notation; *De Numerosa Potestatum Adfectarum Resolutione* ([31, ff. 7-34], [32, pp. 173-223]), devoted to the numerical resolution of (polynomial) equations having other terms apart from the biggest degree term and the independent coefficient (this kind of equations are called *affected* equations in the Viète's terminology), and in that part Viète extended the method applied for the extraction of pure roots; finally, the last part, titled *Consectarium Generale ad Analysim Potestatum Adfectarum, Et praeceptorum quae ad eam pertinent, recollectio* ([31, ff. 34-36], [32, pp. 224-228]), contains a summary of general consequences about the analysis developed in the previous part as well as an enumeration of precepts to be considered for the resolution of equations. For more information about this work, the reader is referred to the comments appearing in [33] and the study developed by J. Stedall in [28] and, more recently, by A. Mellado in [22, Cap. 7].

Concerning pure equations, Viète proposes the resolution of five equations, Problems I-V, from the square root to the sixth root of a positive integer number. In our current notation, they are $x^2 = 2916; x^3 = 157464; x^4 =$

[10]We have to mention that there exists an English translation, [33], although incomplete.


Paradigma Analyseos Cubi puri.

I Eductio lateris singularis primi.

Cubus resoluendus	157	464	Sedes singularium Cuborum vel solidorum sub gradibus.
Solidum ablatitium	125		Cubus lateris primi.
Reliquum resoluendi Cubi	32	464	

N. 5 4. Q. 25.16. C.125.64. Tot numerales circuli, quot puncta Cubica, lateráue singularia.

II Eductio lateris singularis secnndi

Reliquum resoluendi Cubi	32	464	
Diuisores: Triplum Quadratum lateris primi.	7	5	
Diuisores: Triplum latus primum.		15	
Summa diuisorum	7	65	
Solida ablatitia	30	0	Solidum à latere secundo in triplum Quadratum lateris primi.
	2	40	A Quadrato lateris secūdi in triplum latus primū.
		64	Cubus lateris secundi.
Summa ablatitiorum solidorum æqualis residuo resoluendo Cubo.	32	464	

Itaque si 1 C æquetur 157 464, fit 1 N 54. Ex retrogradâ, quæ omnino obseruata cernitur, compositionis viâ.

Figure 7: Extraction of the cubic root of 157464, [31]

331776; $x^5 = 7962624$; $x^6 = 191102976$, which correspond in the *cosist* notation given by Viète to IQ æquari $29,16$; IC æquari $157,464$; IQQ æquari $331,776$; IQC æquari $7,962,624$; ICC æquari $191,102,976$, respectively.

The resolution of these equations (all of them having an exact root) is carried out under very ingenious tables, original from Viète, on which he relies so that, supported in the development of the power of a binomial, he can obtain their solutions. (For instance, in Figure 7 we have the table to obtain the cubic root of 157464, [31, p.4]; see also [33, pp.317-320] for a detailed explanation).

Respect to the numerical approach of n-th roots, in the precept 9 preceding the resolution of pure equations, Viète proposes to approach by above and below in the case of irrational roots, and explains:

> But if, although they are less, there is no dot for a power left over, it is clear that the root of the term to be resolved is irrational. To the collective root, therefore, add a fraction the numerator of which is the remainder of the term to be solved and [the denominator of which is] the same as what the divisors would be if another point were added to the term to be resolved. This fraction, added to the sum of the individual roots, makes a root for the power to be resolved larger than the true root. If, in the second power, the denominator is increased by 1, it makes a root smaller than the true one. The root lies implicitly between these divisors. It may be elicited as closely as you wish by, say, adding zeros to that which is to be solved and continuing the work. And

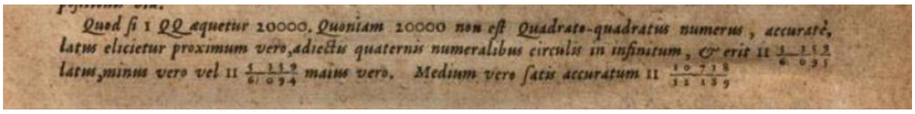
> this is necessarily within the limits of the tenth, otherwise the operation has not been carried out correctly[11].
> Translation from [33, pp. 313-314].

That is, in actual notation, the following approximation is proposed for $\sqrt[n]{A^n + R}$, namely: Integer root + Rest/(Divisor+1)< Solution < Integer root + Rest/Divisor, or

$$A + \frac{R}{d+1} < Solution < A + \frac{R}{d},$$

where the divisor d is 2A (square root), $3A^2{+}3A$ (cubic root), $4A^3{+}6A^2{+}4A$ (fourth root), etc. Let us say that the precept 9 is correct for the approximation of square roots, but it fails to be true in some cases whenever the index of the searched root is greater than or equal to 3. To see it, consider for instance the computation of $\sqrt[4]{20000}$. This root appears at the end of Problema III in [31]. Viète itself provides us the corresponding estimations, and gives $11{+}\frac{5359}{6095}$ and $11{+}\frac{5359}{6094}$, because in this example we have $20000 = 11^4{+}5359 = A^4 + R$, and the divisor is $d = (A+1)^4 - A^4 - 1 = 4A^3 + 6A^2 + 4A = 6094$. When checking the values, we find $11 + \frac{5359}{6095} < 11 + \frac{5359}{6094} < \sqrt[4]{20000}$, that is, the two approximations are in fact by defect. If we look at [32], in this edition the precept 9 remains invariable with respect to the Ghetaldi's edition, but now when estimating the value of $\sqrt[4]{20000}$, we appreciate a slight difference, because now we see that the text changes and, instead of the procedure of the former edition, now quaternary groups of zeros are added to the integer part in order to extract as many decimal digits as desired, in particular we have $11 + \frac{8917}{10000}$ and $11 + \frac{8918}{10000}$, see Figure 8 (notice that, although now the strategy is correct -we suppose that by truncating at some place the chains of quaternary groups-, however we also have two approximations by below of $\sqrt[4]{20000}$). At no time is noticed and justified the slight change in the new edition. Nor the question in [33] is mentioned, in fact, it is omitted ("Problems III and IV, here omitted, deal with the extraction of the roots of pure fourth and fifth powers", [33, p.320]). Moreover, as far as we know, no mention to this fact has been signaled in the literature. With this note, we have tried to fill this lack.

[11]"Quod si dum cedunt non supersit aliquod addictum Potestati punctum, argumentum est magnitudinis resolvendæ latus esse irrationale. Collecto itaque lateri adiungitur fragmentum cuius numerator est numerus è magnitudine resolutâ reliquus. Divisores iidem, qui essent si aliquod punctum Potestati addictum superesset resolvendum, & tale fragmentum singularium laterum summæ adiunctum facit latus Potestatis resolutæ maius vero. Et si denominatori addatur unitas, facit latus minus vero. In divisoribus enim inest implicitè latus, quod alioqui proximè esset eliciendum, ut pote productâ per numerales circulos eâ quæ resolvitur, Potestate, & continuato opere. At illud constat necesse est intra denarii metam, alioquin ritè non fuit operatum".


Quod si 1 QQ æquetur 20000. Quoniam 20000 non est Quadrato-quadratus numerus, accuratè, latus elicietur proximum vero, adiectis quaternis numeralibus circulis in infinitum, & erit 11 $\frac{3\ 152}{6\ 093}$ latus, minus vero vel 11 $\frac{5\ 359}{61\ 094}$ maius vero. Medium vero satis accuratum 11 $\frac{10\ 718}{32\ 139}$

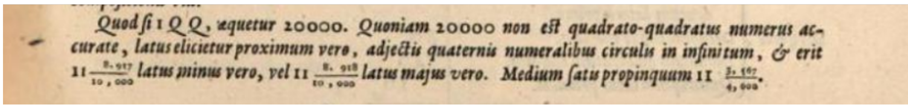

Quod si 1 Q Q, æquetur 20000. Quoniam 20000 non est quadrato-quadratus numerus accurate, latus elicietur proximum vero, adjectis quaternis numeralibus circulis in infinitum, & erit 11 $\frac{8,917}{10,000}$ latus minus vero, vel 11 $\frac{8,918}{10,000}$ latus majus vero. Medium satis propinquum 11 $\frac{3,567}{4,000}$.

Figure 8: Extraction of the cubic root of 20000, [31] (above), and [32] (below)

4 A question arises

We look for an approximation of

$$\sqrt[n]{A^n + R},$$

where $A \geq 1$ is an integer number, $0 < R \leq (A+1)^n - A^n - 1 =: d$ (notice that we assume $R > 0$, that is, the root is irrational, and that for $\widetilde{R} = (A+1)^n - A^n$, we in fact evaluate $\sqrt[n]{A^n + \widetilde{R}} = \sqrt[n]{(A+1)^n}$; and recall that d is called the *divisor* of the approximation).

Then we consider the two following approximations of $\sqrt[n]{A^n + R}$,

$$AD(n, A, R) := A + \frac{R}{d+1} \tag{1}$$

and

$$AE(n, A, R) := A + \frac{R}{d}. \tag{2}$$

It is clear that $AD(n, A, R) < AE(n, A, R)$. Then, we propose the problem of determining whether these approximations are or not approximations by default or excess, respectively, of $\sqrt[n]{A^n + R}$.

For the case by default, it is a simple task to show that

$$AD(n, A, R) < \sqrt[n]{A^n + R}, \tag{3}$$

or equivalently $\left(A + \frac{R}{d+1}\right)^n < A^n + R$, merely use the Newton's formula for the power of a binomial and notice that $R < d + 1 = (A+1)^n - A^n$.

For the case of square roots, $n = 2$, indeed $AD(2, A, R)$ and $AE(2, A, R)$ are true approximations by default and excess, respectively (for $\sqrt{A^2 + R} < AE(2, A, R)$ consider that $d = 2A$ and square to obtain $A^2 + R < \left(A + \frac{R}{2A}\right)^2$). Therefore

$$A + \frac{R}{2A+1} < \sqrt{A^2 + R} < A + \frac{R}{2A}.$$

At the moment, we know that in general $AD(n, A, R)$ always provides a default approximation for all $n \geq 2$. Also, $AE(n, A, R)$ gives an approximation by excess when $n = 2$, that is, the precept 9 in [31] is appropriate when we look for the approximation of the square root of $A^n + R$.

Next, fixing A and $n \geq 3$, we are interested in knowing when the value $AE(n, A, R)$ generates a true approximation by excess of $\sqrt[n]{A^n + R}$. The next section is devoted to this task, for the cases $3 \leq n \leq 7$ since they can be treated in a direct way (the case $n \geq 8$ is more involved, see Theorem 4 below, and will be commented at the end jointly with the conclusions of this paper).

5 The solution for cases $3 \leq n \leq 7$

According to the results presented in Section 4, we assume $n \geq 3$. When we try to find the integer values R, with $0 < R \leq d$, such that $A + \frac{R}{d}$ is an approximation by excess of $\sqrt[n]{A^n + R}$ we have to analyze whether inequality

$$\left(A + \frac{R}{d}\right)^n > A^n + R \tag{4}$$

holds or not. To stress the dependence of d on the values n and A, sometimes we will write $d = d(n, A)$.

We are interested in computing the amount of integer values $R, 0 < R \leq d$, for which (4) is true. Then, it makes sense to define

$$\nu = \nu(n, A) = \text{Card}\left\{R : 0 < R \leq d(n, A), AE(n, A, R) > \sqrt[n]{A^n + R}\right\}. \tag{5}$$

Putting $s = d - R$, the inequality (4) is converted to

$$1 + s + \left(A + 1 - \frac{s}{d}\right)^n > (A + 1)^n, \tag{6}$$

therefore

$$\nu(n, A) = \text{Card}\left\{s \in \mathbb{Z} : 0 \leq s < d(n, A), (6) \text{ is true}\right\}. \tag{7}$$

Fix $n \geq 3, A \geq 1$, and hence $d = d(n, A) = (A + 1)^n - A^n - 1$. We define the function $f_{n,A}(x) = f(x, n, A)$, $f_{n,A} : [0, d] \to \mathbb{R}$ as

$$f_{n,A}(x) = 1 + x + \left(A + 1 - \frac{x}{d}\right)^n - (A + 1)^n. \tag{8}$$

For simplicity of notation, we will write $f_{n,A}$ by f when no confusion can arise.

It is straightforward to check that $f(0) = 1$ and $f(d) = 0$. Moreover, it is a simple exercise to check that $f_{n,A}(x)$ is a *convex* map, attaining its minimum value at a point $s_m \in (0, d)$, in fact $s_m = d\left(A + 1 - \sqrt[n-1]{\frac{d}{n}}\right)$. Figure 9 shows us the general aspect of the graph of $f_{n,A}$: is strictly decreasing in the interval $(0, s_m)$, having a zero ζ in such a zone, and it strictly increases in (s_m, d).

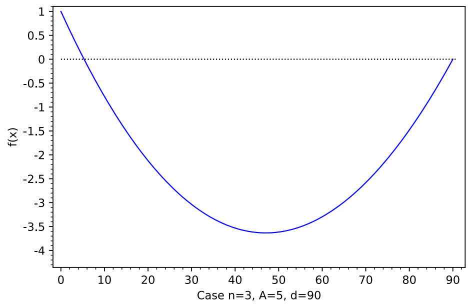


Figure 9: One example for $f_{n,A}(x)$

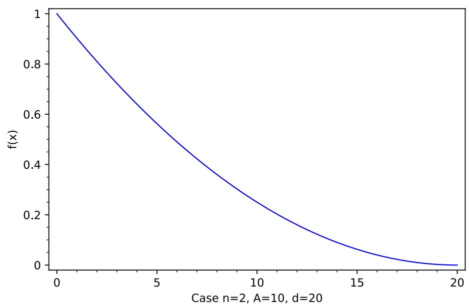


Figure 10: For $n = 2$, $A + \frac{R}{d}$ approximates by excess the exact value of $\sqrt{A^2 + R}$

Observe that if we extend f to the case $n = 2$, the corresponding map $f_{2,A}$ is non negative, which confirms the already known fact, seen in Section 4, stating that all the approximations $A + \frac{R}{d}$ are by excess (see also Figure 10).

In the sequel, it will be fruitful to apply the well-known Descartes' rule of signs[12].

[12] In his original text, see [13, La Géométrie, Livre III, p.373], R. Descartes (1596–1650) states the rule as follows: "On connoist aussy de cecy combien il peut y avoir de vrayes racines, & combien de fausses en chasque Equation. A sçavoir il y en peut avoir autant de vrayes, que les signes $+$ & $-$ s'y trouvent de fois estre changés; autant de fausses qu'il s'y trouve de fois deux signes $+$, ou deux signes $-$ qui s'entresuivent". / We know how many true (positive) roots, and how many false (negative) roots, can exist in every equation. To wit, it can have so many true roots as changes of signs $+$ and $-$ occur; and it can have so many false roots as the times that two signs $+$ or two signs $-$ appear consecutively [our translation]. Descartes does not provide any proof of this fact, and he limits himself to give an example, the equation $x^4 - 4x^3 - 19x^2 + 106x - 120 = 0$ whose positive roots ('vrayes racines') are $x = 2, 3, 4$, and $x = -5$ is the unique negative root ('fausses racines'); let us notice that the above equation is constructed by multiplication of the equations $x - 2 = 0$, $x - 3 = 0$, $x - 4 = 0$, and $x + 5 = 0$. For a short proof, the reader is referred to [34]; for a historical account of this result, and its first completely rigorous proof, we suggest the reading of [4].

Theorem. *(Descartes' rule of signs) Let* $P(x) = a_0x^{b_0} + a_1x^{b_1} + \ldots + a_nx^{b_n}$ *be a polynomial with nonzero real coefficients* a_j, $0 \leq j \leq n$, *and where the* b_j *are integers satisfying* $0 \leq b_0 < b_1 < \ldots < b_n$. *Then, the number of positive real zeros of* $P(x)$ *(counted with multiplicities) is either equal to the number of variations in signs in the sequence of the coefficients* $a_0, a_1, \ldots, a_n$ *or less than that by an even whole number.*

Corollary 1. *Under the hypothesis of the above Descartes' rule of signs, if the number of variations in signs of* $a_0, a_1, \ldots, a_n$ *is exactly* 1, *then* $P(x)$ *has a unique positive root.*

To obtain $\nu(n, A)$ we adopt the direct strategy of finding values x_1, x_2 close enough such that $f_{n,A}(x_1) > 0$ and $f_{n,A}(x_2) < 0$.

5.1 Cubic, fourth and fifth roots

In the case of cubic roots ($n = 3$), for $A \geq 1, d = 3A^2 + 3A, 0 < R \leq d$, we only need to compute $f_{3,A}(A)$ and $f_{3,A}(A + \frac{1}{2})$ (remember (8)):

$$\begin{aligned} f_{3,A}(A) &= \frac{(3A+2)(3A+4)}{27(A+1)^3} > 0, \\ f_{3,A}(A+\frac{1}{2}) &= -\frac{(6A^2+4A-1)(6A^3+14A^2+8A-1)}{216A^3(A+1)^3} < 0. \end{aligned}$$

We deduce that the root ζ of $f_{3,A}(x) = 0$ belongs to the interval $(A, A + \frac{1}{2})$. So, by the properties of $f_{3,A}$ presented at the beginning of this Section, $f_{3,A}(s) > 0$ for the integers $s = 0, 1, \ldots, A$, that is, $AE(3, A, R) > \sqrt[3]{A^3 + R}$ for $d - A \leq R \leq d$, and consequently, by (5),

$$\nu(3, A) = A + 1.$$

Example 2. *We approach* $\sqrt[3]{990} = \sqrt[3]{A^3 + R} = \sqrt[3]{9^3 + 261}$ *by* $A + \frac{R}{d} = 9 + \frac{261}{270} = 9 + \frac{29}{30}$. *We obtain an approximation by excess since* $\left(9 + \frac{29}{30}\right)^3 > 990$. *In this case,* $d - A = 3 \cdot 9^2 + 3 \cdot 9 - 9 = 261 \leq R = 261 \leq d = 270$.

On the other hand, $A + \frac{\widetilde{R}}{d} = 9 + \frac{260}{270} = 9 + \frac{26}{27}$ *is less than* $\sqrt[3]{989} = \sqrt[3]{A^3 + \widetilde{R}} = \sqrt[3]{9^3 + 260}$. *Now, we obtain a default approximation, which is in accordance with* $0 < \widetilde{R} = 260 < d - A = 261$. □

For the case of fourth roots, now we obtain

$$\nu(4, A) = \left\lfloor \frac{2A}{3} \right\rfloor + 1 = \left\lfloor \frac{2A+3}{3} \right\rfloor.$$

Similarly to the cubic case, this is an immediate consequence of the values

$$f_{4,A}\left(\frac{2A}{3}\right) > 0, \quad \text{and} \quad f_{4,A}\left(\frac{2A+1}{3}\right) < 0$$

(easily checked by a personal computer) and of the properties of the convex function $f_{4,A}$.

Example 3. *Let us revisit the Viète's example* $\sqrt[4]{20000} = \sqrt[4]{11^4 + 5359}$. *In this case, the approximation* $11 + \frac{5359}{6094}$ *is by default as* $d = 6094$, $A = 11$, $R = 5359$, *and* $\lfloor \frac{2A+3}{3} \rfloor = 8$. □

For quintic roots, $n = 5$, we follow the same strategy of the previous cases and obtain that

$$\nu(5, A) = \left\lfloor \frac{A}{2} \right\rfloor + 1 = \left\lfloor \frac{A+2}{2} \right\rfloor.$$

To see it, with the help of a mathematical software,

$$\begin{aligned} f_{5,A}\left(\frac{A}{2}\right) = {} & \frac{1}{100000(A+1)^5(A^2+A+1)^5} \times (10A^3 + 20A^2 + 20A + 9) \\ & \times \big(5000A^{10} + 36000A^9 + 127000A^8 + 288500A^7 + 464600A^6 \\ & + 552400A^5 + 490750A^4 + 321860A^3 \\ & + 149620A^2 + 44770A + 6561\big) > 0, \end{aligned}$$

and

$$\begin{aligned} f_{5,A}\left(\frac{A}{2} + \frac{1}{2}\right) = {} & -\frac{1}{100000(A+1)^5(A^2+A+1)^5} \\ & \times (10A^3 + 10A^2 + 10A - 1) \\ & \times \big(10000A^{11} + 35000A^{10} + 79000A^9 + 111000A^8 \\ & + 115500A^7 + 81400A^6 + 41300A^5 + 11250A^4 \\ & + 1390A^3 - 510A^2 + 40A - 1\big) < 0. \end{aligned}$$

Thus, the zero ζ of $f_{5,A}(x) = 0$ lies in the open interval $(\frac{A}{2}, \frac{A}{2} + \frac{1}{2})$. Since A is a positive integer, we deduce that $f_{5,A}(s) > 0$ only for $s = 0, 1, \ldots, \lfloor \frac{A}{2} \rfloor$. This means that $AE(n, A, R) > \sqrt[5]{A^5 + R}$ whenever $R = d, d-1, \ldots, d - \lfloor \frac{A}{2} \rfloor$. □

5.2 Sixth and seventh roots

We compute now the value of $\nu(6, A)$, that is, the cardinal of values $R \in \{1, \ldots, d(6, A)\}$ such that $AE(6, A, R) > \sqrt[6]{A^6 + R}$. We will obtain

$$\nu(6, A) = \left\lfloor \frac{2A-1}{5} \right\rfloor + 1 = \left\lfloor \frac{2A+4}{5} \right\rfloor.$$

To prove it, we do a discussion on the values of A.

(i) Let $A = 1$. Then $d = d(n, A) = (A+1)^n - A^n - 1 = 2^6 - 1 - 1 = 62$. We have

$$f_{6,1}(1) = 2 + \left(2 - \frac{1}{62}\right)^6 - 2^6 = -62 + \left(\frac{123}{62}\right)^6 = -\frac{58788614519}{56800235584} < 0.$$

By the properties of $f_{n,A}$, it follows that we have a unique approximation by excess when $s = 0$, or equivalently $R = d$. Notice that $\lfloor \frac{2A+4}{5} \rfloor = \lfloor \frac{6}{5} \rfloor = 1$ in this case.

(ii) Let $A = 2$. Now, $d = d(n, A) = 3^6 - 2^6 - 1 = 664$. Again

$$f_{6,2}(1) = 2 + \left(3 - \frac{1}{664}\right)^6 - 3^6 = \frac{-165436579422164 31}{85705457236443136} < 0.$$

So, our unique approximation by excess corresponds to $s = 0$, or $R = d$. As in the previous case, $\lfloor \frac{2A+4}{5} \rfloor = 1$.

(iii) Let $A = 3$. We have $d = 4^6 - 3^6 - 1 = 3366$. By a direct computation (check it) we find that

$$f_{6,3}\left(\frac{6}{5}\right) > 0, \quad \text{but} \quad f_{6,3}(2) < 0,$$

so the unique values giving us an approximation by excess are $s = 0$ and $s = 1$, that is, $R \in \{d, d-1\}$. Notice that $\lfloor \frac{2A+4}{5} \rfloor = 2$.

(iv) Let $A \geq 4$. We claim that $f_{6,A}\left(\frac{2A}{5}\right) < 0$. With the help of a computer, we obtain $f_{6,A}\left(\frac{2A}{5}\right) =: \frac{N_6(A)}{D_6(A)}$, where $D_6(A) = 15625d^6$ and

$$\begin{aligned} N_6(A) = {} & \left(30A^4 + 75A^3 + 100A^2 + 75A + 28\right) \\ & \times (-4050000A^{19} - 32400000A^{18} - 103005000A^{17} - 51975000A^{16} \\ & + 912481875A^{15} + 4658816250A^{14} + 13800777375A^{13} \\ & + 29961127500A^{12}51357591250 + A^{11} + 71924369500A^{10} \\ & + 83733302450A^9 + 81679154000A^8 + 66842104875A^7 \\ & + 45673566250A^6 + 25767455655A^5 + 11766939100A^4 \\ & + 4208223600A^3 + 1113483200A^2 + 195334080A \\ & + 17210368). \end{aligned}$$

Define $Q_6(A) := \frac{N_6(A)}{30A^4+75A^3+100A^2+75A+28}$, a polynomial of degree 19. Then $Q_6(3) > 0$ and $Q_6(4) < 0$. From the Descartes' rule of signs and its Corollary 1, we conclude that $Q_6(A)$ has a unique positive real root, located at the open interval $(3, 4)$ and, since $Q_6(A) \to -\infty$ when A tends to infinity, $Q_6(A) < 0$ for all $A \geq 4$. Thus, $N_6(A) < 0$ for all $A \geq 4$. Therefore, $f_{6,A}\left(\frac{2A}{5}\right) < 0$ for all $A \geq 4$. This ends the claim.

On the other hand, we get $f_{6,A}(\frac{2A-1}{5}) > 0$, because

$$f_{6,A}\left(\frac{2A-1}{5}\right) = \frac{1}{15625A^6(6A^4+15A^3+20A^2+15A+6)^6} \times (30A^5+75A^4+100A^3+75A^2+28A+1) \times (8100000A^{24}+121500000A^{23}+875070000A^{22} + 4074840000A^{21}+\cdots+282630A^3+8395A^2 + 140A+1) > 0.$$

(all the coefficients of the last polynomial in A are positive).

Now we are in a position to finish Case (iv), $A \geq 4$: since $f_{6,A}(\frac{2A-1}{5}) > 0$ and $f_{6,A}(\frac{2A}{5}) < 0$, we conclude that $f_{6,A}(s) > 0$ for the integer values $s = 0, 1, \ldots, \lfloor \frac{2A-1}{5} \rfloor$, realize that A is a positive integer.

With our discussion of Cases (i)-(iv), we finish the case of sixth roots. □

Finally, for seventh roots we are going to show that

$$\nu(7,A) = \left\lfloor \frac{2A+n-2}{n-1} \right\rfloor = \left\lfloor \frac{2A+5}{6} \right\rfloor.$$

Similarly to sixth roots, we distinguish several cases.

(i) $A = 1$. Then $f_{7,1}(1) < 0$ so $f_{7,1}(s) > 0$ only for the integer $s = 0$. Thus, $\nu(7,1) = 1 = \lfloor \frac{2A+5}{6} \rfloor = \lfloor \frac{7}{6} \rfloor = 1$.

(ii) $A = 2$. Again $f_{7,2}(1) < 0$ and $\nu(7,2) = 1 = \lfloor \frac{9}{6} \rfloor = \lfloor \frac{2A+5}{6} \rfloor$.

(iii) $A \geq 3$. Using a mathematical software, $f_{7,A}(\frac{2A}{6}) =: \frac{N_7(A)}{D_7(A)}$, with

$$D_7(A) = 1801088541A^{35} + 37822859361A^{34} + 403443833184A^{33} + \ldots + 403443833184A^2 + 37822859361A + 1801088541$$

(all the coefficients of $D_7(A)$ are positive) and

$$\begin{aligned} N_7(A) = &-600362847A^{34} - 10806531246A^{33} - 97258781214A^{32} \\ &- 577549058814A^{31} - 2514834199962A^{30} - 8420174695296A^{29} \\ &- 21944119719060A^{28} - 43061625563922A^{27} - 53191033284627A^{26} \\ &+ 13234913195994A^{25} + 303648512134590A^{24} \\ &+ 1063881793793574A^{23} + 2599443322376040A^{22} \\ &+ 5157799213711410A^{21} + 8755187169969450A^{20} \\ &+ 13031632337209290A^{19} + 17242848176742405A^{18} \\ &+ 20448216949700610A^{17} + 21842457224825790A^{16} \\ &+ 21075654616272900A^{15} + 18392229344961864A^{14} \\ &+ 14515293924381222A^{13} + 10346477812600482A^{12} \end{aligned}$$

$$+ 6644210201266050A^{11} + 3829261973222007A^{10}$$
$$+ 1970068132488204A^{9} + 898255956563028A^{8}$$
$$+ 359478031796604A^{7} + 124638816387666A^{6} + 36779824104114A^{5}$$
$$+ 9007142933166A^{4} + 1762931499924A^{3} + 259451944899A^{2}$$
$$+ 25624743060A + 1280000000.$$

By the Descartes' rule of signs, $N_7(A)$ has a unique positive real root lying in the interval $(2,3)$ since $N_7(2) > 0$ and $N_7(3) < 0$ (see Figure 11).

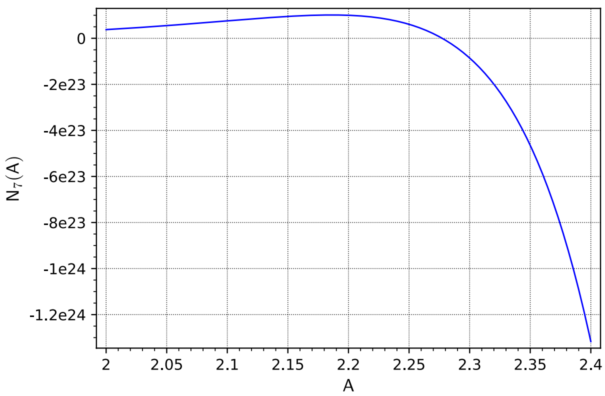


Figure 11: The graph of $N_7(A)$ near to its positive zero

Therefore, $f_{7,A}\left(\frac{2A}{6}\right) = \frac{N_7(A)}{D_7(A)} < 0$ for all $A \geq 3$.

On the other hand, also with the help of a mathematical software,

$$f_{7,A}\left(\frac{2A-1}{6}\right) =: \frac{P_7(A)}{Q_7(A)},$$

with

$$P_7(A) = 38423222208A^{41} + 1075850221824A^{40} + 14062899328128A^{39} + \cdots + 118984124A^{4} + 2524592A^{3} + +34776A^{2} + 280A + 1$$

and

$$Q_7(A) = 230539333248A^{42} + 4841325998208A^{41} + 51640810647552A^{40} + \cdots + 51640810647552A^{9} + 4841325998208A^{8} + 230539333248A^{7}$$

(all the non vanishing coefficients of $P_7(A)$ and $Q_7(A)$ are positive). Clearly, $f_{7,A}(\frac{2A-1}{6}) > 0$ for all $A \geq 1$. Since $\frac{2A-1}{6} \notin \mathbb{Z}$, we deduce that $f(s) > 0$ for the integer values $s = 0, 1, \ldots, \left\lfloor \frac{2A-1}{6} \right\rfloor$. So, $\nu(7, A) = \left\lfloor \frac{2A-1}{6} \right\rfloor + 1 = \left\lfloor \frac{2A+5}{6} \right\rfloor$. $\square$

6 The generalization

It is worth mentioning that we were able to find a general result for the value $\nu(n, A)$. Because its proof is very involved to be presented in this simple note, we only trace its main steps. We advance that, in general, the number $\nu(n, A)$ is either $\left\lfloor \frac{2A+n-2}{n-1} \right\rfloor$ or $\left\lfloor \frac{2A+n-2}{n-1} \right\rfloor - 1 = \left\lfloor \frac{2A-1}{n-1} \right\rfloor$.

In the first step we prove that for $n \geq 8$ and $A = 1$, it holds $\nu(n, 1) = 1$. This is due to the fact that, for $n \geq 8$,

$$f_{n,1}(1) = 2 + \left(2 - \frac{1}{2^n - 2}\right)^n - 2^n < 0.$$

Next, the crucial step is to demonstrate that for $n \geq 8$ and $A \geq 2$ we have

$$f_{n,A}\left(\frac{2A}{n-1}\right) < 0. \tag{9}$$

To reach it, we use that $1 - nx \leq (1-x)^n \leq 1 - nx + \frac{n(n-1)}{2}x^2$ for $x \in (0, 1)$ and $nx < 1$, and then taking $x = \frac{2A}{(A+1)(n-1)d}$ we obtain the inequality

$$\begin{aligned} & f_{n,A}\left(\frac{2A}{n-1}\right) \\ = \; & 1 + \frac{2A}{n-1} + \left(A + 1 - \frac{2A}{(n-1)d}\right)^n - (A+1)^n \\ = \; & 1 + \frac{2A}{n-1} - (A+1)^n \left(1 - \left(1 - \frac{2A}{(A+1)(n-1)d}\right)^n\right) \\ \leq \; & 1 + \frac{2A}{n-1} - (A+1)^n \left[\frac{2An}{(A+1)(n-1)d} - \frac{n(n-1)}{2}\left(\frac{2A}{(A+1)(n-1)d}\right)^2\right] \\ = \; & 1 + \frac{2A}{n-1} - \frac{2An(A+1)^{n-1}}{(n-1)d} + \frac{2A^2n(A+1)^{n-2}}{(n-1)d^2} \\ = \; & \frac{(n-1)d^2 + 2Ad^2 - 2An(A+1)^{n-1}d + 2A^2n(A+1)^{n-2}}{(n-1)d^2}. \end{aligned}$$

So, the proof of (9) is complete if we show that

$$gN(n, A) := (n-1)d^2 + 2Ad^2 - 2An(A+1)^{n-1}d + 2A^2n(A+1)^{n-2} < 0.$$

The proof of this last fact requires a laborious manipulation (and we omit it), as well as the final application of the Descarte's rule of signs.

In this way, if $2A \leq n - 1$ we can assure that $\nu(n, A) = 1$ (because $f_{n,A}(0) = 1 > 0$ and $f_{n,A}(1) < 0$ since $\frac{2A}{n-1} \leq 1$, recall the properties of the convex function $f_{n,A}$); at the same time, if $2A > n - 1$, we are able to prove that there exists $k = k(n, A)$, $k \in \left\{0, 1, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}$ such that

$$\frac{2A - (k+1)}{n-1} \leq s_0 < \frac{2A - k}{n-1},$$

where

$$s_0 = s_0(n, A) = \frac{d}{n(A+1)^n - d}$$

is precisely the value of the intersection between the tangent line to $f_{n,A}$ at the point $(0, 1)$ and the axis Ox (take into account that, by convexity, we have that $f_{n,A}(s_0) > 0$), see Figure 12.

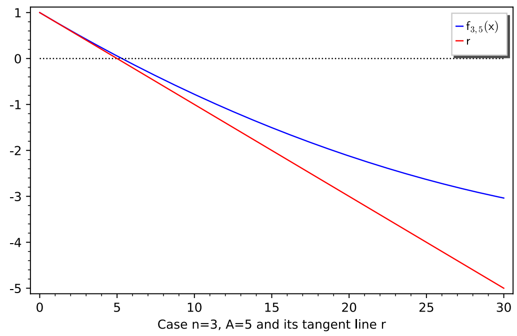


Figure 12: Function $f_{n,A}(x)$ and the tangent line passing by $(0, 1)$

Joining all the above ingredients, we advance the following result:

Theorem 4. *Suppose that n and A are positive integers, with $n \geq 8$, $A \geq 2$. Let $\nu(n, A)$ be the number given by (5) or (7); let $d(n, A) = (A+1)^n - A^n - 1$ and $f_{n,A}(x) = 1 + x + \left(A + 1 - \frac{x}{d(n,A)}\right)^n - (A + 1)^n$. Then:*

(a) If $2A \leq n - 1$, $\nu(n, A) = 1$.

(b) Assume that $2A > n - 1$.

(b1) If $n - 1 \nmid 2A - k$ for all $k = 1, 2, \dots, \lfloor \frac{n-1}{3} \rfloor - 1$,

$$\nu(n, A) = \left\lfloor \frac{2A + n - 2}{n - 1} \right\rfloor.$$

(b2) If $n - 1 \mid 2A - k$ for some $k = 1, 2, \dots, \lfloor \frac{n-1}{3} \rfloor - 1$, and we put $s_0 = \frac{d(n,A)}{n(A+1)^{n-1} - d(n,A)}$, then in turn:

(b2I) If $s_0 \geq \left\lfloor \frac{2A-k}{n-1} \right\rfloor$, $\nu(n, A) = \left\lfloor \frac{2A+n-2}{n-1} \right\rfloor$.

(b2II) If $s_0 < \left\lfloor \frac{2A-k}{n-1} \right\rfloor$ and $f_{n,A}(\frac{2A-k}{n-1}) > 0$, $\nu(n, A) = \left\lfloor \frac{2A+n-2}{n-1} \right\rfloor$.

(b2III) If $s_0 < \left\lfloor \frac{2A-k}{n-1} \right\rfloor$ and $f_{n,A}(\frac{2A-k}{n-1}) < 0$, $\nu(n, A) = \left\lfloor \frac{2A+n-2}{n-1} \right\rfloor - 1$.

7 Conclusions

In this note, we have presented a little piece of history relative to the numerical approach of n-th roots, in particular we have seen in the Viète's work *De Numerosa Potestatum* a numerical procedure (based upon the development of a binomial) strongly related with the extraction of square roots in the Babylonian or Greek tradition. According to De Morgan, [12], the Viète method was *improved by T. Harriot (1560–1621), and used by W. Oughtred (1574–1660), J. Wallis (1616-1703) as well as the generality of at least English writers* (sic), which demonstrates its spread in England; concerning the European continent, also several authors considered the method either in its original version, as Dechales (1621-1678) in his *Cursus seu Mundus Mathematicus*, first published in 1674, or trying to adapt it to an incipient algebraic language, as it occurs in the *Cursus Mathematicus* of P. Hérigone (1580–1643), published at Paris in six volumes during 1634 and 1642. Even, we know that famous mathematicians of the 17-th century referred to it in cases where was necessary its application, as G.W. Leibniz (1646-1716) who indicates its suitable use [13]. Anyway, following again [12], we can say that the method continued to be used up to the time at which the Newton approximation appeared at the last quarter of 17-th century. In this direction, let us also mention to T. Fantet de Lagny (1660–1734), who in [11, pp. 9-10] writes: “Il y a prés d'un siecle que le sçavant Mr. Viete a donné une methode generale pour l'extraction des racines de toutes les puissances pures, dans son livre *de numerosa potestatum resolutione...* Tous ceux qui ont écrit depuis sur l'extraction des racines, n'ont fait que copier Viete, & se copier les uns les autres, & on ne doit pas en estre surpris. Cette methode paroit si naturelle & si conforme à ce grand principe des Analystes, *que la resolution doit se faire par le voye opposée à la composition,* qu'il ne tombe pas mesme dans l'esprit qu'on en puisse inventer une meilleure”[14].

From the detailed reading of *De Numerosa Potestatum*, we have detected a lack in the different editions of this work, in relation with its precept 9 about approximation by excess of the n-roots, $n \geq 3$. In this direction, imagine the difficulty in Viète's time for computing by hand all the problems appearing in *De Numerosa Potestatum*, and compare it with the current easy treatment by

[13] Leibiniz writes “... per extractiones radicum appropinquatrices methodo Vietaea” -by Viéte's method of approximately root extraction- in [19, p. 780] for the root extraction operation, that is, he referred to the *De Numerosa Potestatum* as a reference for the root extraction.

[14] “It is almost a century since the learned Mr. Viete gave a general method for extracting the roots of all pure powers, in his book *de numerosa potestatum resolutione*. Anyone who has written about root extraction since then has only copied Viete, and copied each other, and we should not be surprised. This method seems so natural and so according to this great principle of the Analysts, *that the solution must be done by the way opposite to the composition,* that even our spirit does not think of inventing a better one” (our translation).

a personal computer, realize how actually the task is enormously simplified.

In turn, this study gave rise to the pure mathematical question of counting the values of remainders R for which $A+\frac{R}{d}$ is an approximation by excess of $\sqrt[n]{A^n+R}$, $0<R\leq d$, $d=(A+1)^n-A^n-1$. We have done a complete (and direct, basic) study of the value $\nu(n,A)$ in the cases $3\leq n\leq 7$, and we have advanced the general result in Theorem 4, whose proof was only slightly outlined.

Although, of course, the significance of the found main result in Theorem 4 is not transcendental, at least allows us to illustrate the importance and benefits of the study of History of Mathematics, given rise to questions and reflections through, in particular, the reading of primary sources. They can help us to elaborate materials for classroom, from a more human and didactic approach to the learning of mathematics. Realize that in this note we have employed a series of basic mathematics, and we think that materials for classroom could be implemented from it, dealing with different aspects, among others, for instance, the importance of the algebraization of Mathematics during 17-th century (compare the *cosist* notation in Viète with our modern notation), the handling of inequalities and combinatorial numbers, properties of convex functions and their drawing, calculus of extremal points, location of roots, even the student could develop an algorithmic procedure for testing the validity of Theorem 4.

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