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ACCOMPANYING SPACES TO A LINEAR TWO-DIMENSIONAL SPACE OF CONTINUOUS FUNCTIONS WITH A CONTINUOUS FIRST DERIVATIVE

JITKA KOJECKÁ

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Dedicated to Prof. Miroslav Laitoch on his 60th birthday

M. Laitoch defined in [2] the n-th accompanying equation (n-natural) to a 2^{nd} order linear differential equation

$$y'' = Q(t)\,y \tag{Q}$$

($Q < 0$ is a continuous function on its definition interval $i \subset E_1$) with a given basis $[\alpha, \beta]$, where α, β are given real constants, $\alpha^2 + \beta^2 \neq 0$. If u, v are two independent integrals of (Q), then the function

$$U = \frac{\alpha u + \beta u'}{\sqrt{\alpha^2 - \beta^2 Q}}, \qquad V = \frac{\alpha v + \beta v'}{\sqrt{\alpha^2 - \beta^2 Q}}$$

form a basis of the space of all integrals of the first accompanying equation to (Q).

The present paper investigates the properties of a linear two-dimensional space of continuous functions with the basis $(\varrho(\alpha u + \beta u'), \varrho(\alpha v + \beta v'))$, where (u, v) is the base of a linear two-dimensional space of continuous functions with a continuous first derivative, $\varrho > 0$ is a continuous function and α, β $(\alpha^2 + \beta^2 \neq 0)$ are given real constants. There are investigated zeros of functions and extremes of phases relative to this space and conditions are stated under which this space is of the 0^{th} class, i.e. it has no extreme points. It is referred to [3], [4] and [5] where the linear two-dimensional spaces of continuous functions are studied from the point of view of Academician O. Borůvka's theory on transformations of solutions of the 2^{nd} order linear differential equations and to [6], where the spaces of continuous functions with a continuous first derivative are considered. We continue to use the results of the works cited at the end of this article.

0. In all what follows we are dealing with functions from $C_1(i)$, $i \subset E_1$; $y' \in C_0(i)$ will always denote the derivative of the function $y \in C_1(i)$.

Remark 0.1. Three cases arise for the function $y \in C_1(i)$, $y \not\equiv$ constant on i, and its derivative (cf. definition 1.1 [3]):

1. y, y' are dependent on the interval i,
2. y, y' are independent on the interval i,
3. y, y' are neither dependent or independent on the interval i.

Theorem 0.1. Let $y \in C_1(i)$, $y \not\equiv$ constant on i. The functions y, y' are dependent on the interval i exactly if $y \equiv ke^{ct}$ on the interval i, $t \in i$, where k, c are nonzero constants.

Proof: I. Let y, y' be functions dependent on i. Then there exist real numbers a, b ($a^2 + b^2 \neq 0$) such that $ay + by' \equiv 0$ on i. If one of the numbers a, b were equal to zero, then with respect to the assumption $y \not\equiv$ constant, the other number would also be equal to zero, which would, however, contradict the assumption $a^2 + b^2 \neq 0$. Thus it holds in the whole interval i that $y' = cy$, where $c = -a/b$. Next, it must hold for all $t \in i$ that $y(t) \neq 0$. Namely, if there would exist a point $t_0 \in i$ such that $y(t_0) = 0$, then $y'(t_0) = 0$ would follow from equation $y'(t_0) = cy(t_0)$ and in view of this fact the equality $y' = cy$ on i may be satisfied by the functions $y \equiv 0$ and $y' \equiv 0$ only, which again conflicts with our assumption that $y \not\equiv$ constant. The function $y = ke^{ct}$, where $k \neq 0$ is a constant, is the solution of the equation

$$y'(t) = cy(t), \qquad t \in i.$$

II. If $y = ke^{ct}$, $t \in i$, where k, c are nonzero constants, we obtain $y' = kce^{ct}$. Then there exist numbers a, b, for instance $a = -c, b = 1$, and it holds $ay + by' \equiv 0$ on i, hence y and y' are dependent on i.

Corollary 0.1. Let $y \in C_1(i)$, $y \not\equiv$ constant on i. The functions y, y' are independent on i exactly if $y \not\equiv ke^{ct}$, $t \in j$, holds on every interval $j \subset i$, where k, c are nonzero constants.

Corollary 0.2. Let $y \in C_1(i)$, $y \not\equiv$ constant on i. The function y, y' are neither dependent or independent on i exactly if there exists an interval $j \subset i, j \neq i$, where $y \equiv ke^{ct}$, $t \in j$, and $y \not\equiv ke^{ct}$, $t \in i \setminus j$, on the interval $i \setminus j$, where k, c are nonzero constants.

1. Zeros of functions of an accompanying space $P\varrho[\alpha, \beta]$ to a space S

Let $u, v \in C_1(i)$ and (u, v) be a basis of a linear two-dimensional space S (cf. definition 1.2 [3]) whose range of definition is the interval $i \subset E_1$. Let (u', v') be a basis of the linear two-dimensional space S', where S' is the set of derivatives of all functions of the space S. By Theorem 1.2 [6] no function $y \in S$ is equal to a nonzero constant on any interval $j \subset i$.

Convention 1.1. We assume throughout that for every function $y \in S$ the functions y, y' are independent on interval i. The functions identically equal to zero will be excluded from our considerations.

Theorem 1.1. Let S be a space with a basis (u, v), $\varrho(t)$, $t \in i$, be a function continuous and positive on the interval i, α, β are given real constants, $\alpha^2 + \beta^2 \neq 0$. Then the set of all functions having the form $\varrho(t)\,(\alpha y(t) + \beta y'(t))$, where $t \in i$ and $y \in S$, form a two-dimensional space of continuous functions with a basis $(\varrho(\alpha u + \beta u'), \varrho(\alpha v + \beta v'))$ and with a definition interval i.

Proof: We show first that the functions $\varrho(\alpha u + \beta u')$ and $\varrho(\alpha v + \beta v')$ are independent on i. If they were not independent on i, then there would exist constants a, b $(a^2 + b^2 \neq 0)$ and the interval $j \subset i$ such that

$$a\varrho(\alpha u + \beta u') + b\varrho(\alpha v + \beta v') \equiv 0 \qquad \text{on } j,$$

hence

$$\alpha(au + bv) + \beta(au' + bv') \equiv 0 \qquad \text{on } j.$$

Because of the independence of functions u, v and because of the independence of each function from S and its derivative, the above equality is satisfied for $a = 0$, $b = 0$ only, whence it follows that $\varrho(\alpha u + \beta u')$ and $\varrho(\alpha v + \beta v')$ are independent on i.

Let y $(= c_1 u + c_2 v) \in S$ be an arbitrary function, c_1, c_2 be real constants. Then $\varrho(\alpha y + \beta y') = \varrho(\alpha(c_1 u + c_2 v) + \beta(c_1 u' + c_2 v')) = c_1(\varrho(\alpha u + \beta u')) + c_2(\varrho(\alpha v + \beta v'))$. The set of all functions $\varrho(\alpha y + \beta y')$ is thus a set of all linear combinations $c_1\varrho(\alpha u + \beta u') + c_2\varrho(\alpha v + \beta v')$ and by definition 1.2 [3] it is a linear two-dimensional space of continuous functions.

Corollary 1.1. The functions $\varrho(\alpha y_1 + \beta y_1')$ and $\varrho(\alpha y_2 + \beta y_2')$, where $y_1, y_2 \in S$, are independent (dependent) exactly if y_1, y_2 are independent (dependent).

Definition 1.1. *The space from Theorem 1.1 of all functions $\varrho(\alpha y + \beta y')$, where $y \in S$, will be called an accompanying space to the space S with respect to the number basis $[\alpha, \beta]$ with a weight ϱ and we denote it by $P\varrho[\alpha, \beta]$.*

Lemma 1.1. To every function $x \in P\varrho[\alpha, \beta]$ there exists exactly one function $y \in S$ or $y' \in S$ such that $x = \varrho(\alpha y + \beta y')$.

Proof: Let y and $\bar{y}$ be two functions of the space S for which $\varrho(\alpha y + \beta y') = x = \varrho(\alpha\bar{y} + \beta\bar{y}')$. Then $\alpha(y - \bar{y}) + \beta(y' - \bar{y}') \equiv 0$ and – because of the independence of each function $y \in S$ and its derivative – the above relation is satisfied for $y \equiv \bar{y}$ only and thus also $y' \equiv \bar{y}'$.

Theorem 1.2. The mapping of S on the space $P\varrho[\alpha, \beta]$ defined by the operator $D. = \varrho\left(\alpha. + \beta\frac{\mathrm{d}}{\mathrm{d}t}.\right)$ is an isomorphism of S onto $P\varrho[\alpha, \beta]$.

Proof: By definition $P\varrho[\alpha, \beta]$ we have $DS = P\varrho[\alpha, \beta]$. With respect to Lemma 1.1 the mapping D is schlicht and it holds for $y_1, y_2 \in S$:

$$D(y_1 + y_2) = \varrho(\alpha(y_1 + y_2) + \beta(y_1' + y_2')) =$$
$$= \varrho(\alpha y_1 + \beta y_1') + \varrho(\alpha y_2 + \beta y_2') = Dy_1 + Dy_2,$$
$$D(cy_1) = \varrho(\alpha cy_1 + \beta cy_1') = c\varrho(\alpha y_1 + \beta y_1') = cDy_1.$$

Remark 1.1. With reference to Lemma 1.1 [6] we can in analogy prove that the mapping of the space S' onto the space $P\varrho[\alpha, \beta]$ defined by the operator $D'. = \varrho(\alpha \int . \,\mathrm{d}t + \beta.)$ is an isomorphism S' onto $P\varrho[\alpha, \beta]$.

Convention 1.2. Since in the main zeros of functions of the space $P\varrho[\alpha, \beta]$ investigated throughout this paper, we shall assume $\alpha \neq 0$ and $\beta \neq 0$. If there namely were $\alpha = 0$ or $\beta = 0$, we would investigate in fact the zeros of functions of the space S or S', which is the content of [5] and [6].

Convention 1.3. Let (u, v) be a basis of the space S, then in all what follows w will stand for the Wronskian of functions u, v, i.e.

$$w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v.$$

Lemma 1.2. Let $t_0 \in i$ and let for the function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ hold $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$. Then there arises exactly one of the possibilities for the function y ($\in S$):

1° $y(t_0) = 0$, $y'(t_0) = 0$,

2° $y(t_0) \neq 0$, $y'(t_0) \neq 0$ and $\dfrac{y'(t_0)}{y(t_0)} = -\dfrac{\alpha}{\beta}$.

Proof: The assertion follows directly from the equation $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$ and from the condition $\alpha \neq 0$ and $\beta \neq 0$.

Definition 1.2. If $y(t_0) = 0$ and $y'(t_0) = 0$ holds for the function $y \in S$ and the point $t_0 \in i$, then we say that t_0 is the zero of the function $\varrho(\alpha y + \beta y')$ of the type 1. If $y(t_0) \neq 0$ and $\dfrac{y'(t_0)}{y(t_0)} = -\dfrac{\alpha}{\beta}$ holds for the function $y \in S$ and the point $t_0 \in i$, we say that t_0 is the zero of the function $\varrho(\alpha y + \beta y')$ of the type 2.

Lemma 1.3. Let $t_0 \in i$. Then there exists a function $y \in S$ such that t_0 is the zero of the function $\varrho(\alpha y + \beta y')$ of the type 1 exactly if $w(t_0) = 0$.

Proof: The assertion follows from Theorem 1.7 [6].

Theorem 1.3. Let $t_0 \in i$ be a singular point of the space $P\varrho[\alpha, \beta]$. Then $w(t_0) = 0$.

Proof: Any two independent functions $\varrho(\alpha u + \beta u')$, $\varrho(\alpha v + \beta v')$ of the space $P\varrho[\alpha, \beta]$ have at t_0 a zero value, thus

$$\alpha u(t_0) + \beta u'(t_0) = 0$$
$$\alpha v(t_0) + \beta v'(t_0) = 0.$$

With respect of the assumption $\alpha \neq 0$, $\beta \neq 0$ the above system of equations has a zero determinant, i.e.

$$0 = u(t_0)\, v'(t_0) - u'(t_0)\, v(t_0) = w(t_0).$$

Corollary 1.2. Let $t_0 \in i$ and $w(t_0) \neq 0$. Then t_0 is a regular point of the space $P\varrho[\alpha, \beta]$.

Theorem 1.4. Let $t_0 \in i$, $w(t_0) = 0$ and let $y \in S$ exist such that t_0 is a zero of the function $\varrho(\alpha y + \beta y')$ of the type 2. Then t_0 is a singular point of the space $P\varrho[\alpha, \beta]$.

Proof: From the assumption $w(t_0) = 0$ now follows by 1.7 [6] that there exists a function $y_1 \in S$ such that $y_1(t_0) = 0$ and $y_1'(t_0) = 0$. Since $y_1(t_0) = 0$ and $y(t_0) \neq 0$ are y_1, y independent and by Corollary 1.1 the functions $\varrho(\alpha y + \beta y')$ and $\varrho(\alpha y_1 + \beta y_1')$ are also independent. According to Theorem 1.3 [3] t_0 is a singular point of the space $P\varrho[\alpha, \beta]$.

Corollary 1.3. Let the assumptions of Theorem 1.4 be satisfied. Then there exists a function $y_1 \in S$ independent on y such that t_0 is a zero of the function $\varrho(\alpha y_1 + \beta y_1')$ of the type 1.

Theorem 1.5. Let $t_0 \in i$ be a regular point of the spaces S and S'. The point t_0 is a singular point of the space $P\varrho[\alpha, \beta]$ exactly if there exist independent functions $y_1, y_2 \in S$ such that t_0 is a zero of the function $\varrho(\alpha y_1 + \beta y_1')$ of the type 1 and t_0 is a zero of the function $\varrho(\alpha y_2 + \beta y_2')$ of the type 2.

Proof: I. Let t_0 be a singular point of $P\varrho[\alpha, \beta]$. Then by Theorem 1.3 $w(t_0) = 0$ and thus $y_1 \in S$ such that t_0 is a zero of the function $\varrho(\alpha y_1 + \beta y_1')$ of the type 1. If the function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ had at t_0 a zero of the type 1, then there would $y(t_0) = 0$ and this is because of the assumption on regularity of S possible only then, if y_1, y are dependent. Hence it follows for a function $y_2 \in S$ independent on y_1 that the function $\varrho(\alpha y_2 + \beta y_2')$ contains a zero of the type 2 at t_0.

II. If t_0 is a zero of the function $\varrho(\alpha y_1 + \beta y_1') \in P\varrho[\alpha, \beta]$ of the type 1 and the function $\varrho(\alpha y_2 + \beta y_2') \in P\varrho[\alpha, \beta]$ of the type 2, then the assertion follows from Theorem 1.4 and from Corollary 1.3.

Corollary 1.4. Let $t_0 \in i$ be a singular point of the space $P\varrho[\alpha, \beta]$. Then there arises exactly one of the possibilities:

1° there exist functions y_1, $y_2 \in S$ such that t_0 is a zero of the function $\varrho(\alpha y_1 + \beta y_1')$ of the type 1 and the function $\varrho(\alpha y_2 + \beta y_2')$ of the type 2. Then t_0 is a regular point of the spaces S and S'.

2° t_0 is a zero of any function of the space $P\varrho[\alpha, \beta]$ of the type 1. Then t_0 is a singular point of the spaces S and S'.

Theorem 1.6. Let $t_0 \in i$ be a singular point of the space S (S'). Then it holds that either t_0 is a regular point of the spaces S' and $P\varrho[\alpha, \beta]$ (S and $P\varrho[\alpha, \beta]$) or t_0 is a singular point of the spaces S' and $P\varrho[\alpha, \beta]$ (S and $P\varrho[\alpha, \beta]$).

Proof: Let t_0 be a singular point of the space S. Then it holds for any function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ that $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = \varrho(t_0)\beta y'(t_0)$ whence the assertion follows. Entirely analogous is the proof for t_0 being a singular point of S'.

Theorem 1.7. Let $t_0 \in i$ be a regular point of the space $P\varrho[\alpha, \beta]$ and let a function $y \in S$ exist such that t_0 is a zero of the function $\varrho(\alpha y + \beta y')$ of the type 2. Then $w(t_0) \neq 0$.

Proof: Let the function $y_1 \in S$, $y_1(t_0) \neq 0$, be independent on y. Then it follows from

$$\frac{y'(t_0)}{y(t_0)} = -\frac{\alpha}{\beta} \quad \text{and} \quad \frac{y_1'(t_0)}{y_1(t_0)} \neq -\frac{\alpha}{\beta},$$

that $y'(t_0)\, y_1(t_0) - y_1'(t_0)\, y(t_0) \neq 0$ and because of Lemma 1.2 [6] $w(t_0) \neq 0$.

Corollary 1.5. Let the assertions of Theorem 1.7 be satisfied. Then no function of the space $P\varrho[\alpha, \beta]$ has a zero of the type 1 at t_0.

Theorem 1.8. Let $t_0 \in i$ be a regular point of the spaces S and S'. Then for any function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$, for which $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$ holds, t_0 is a zero of the type 1 exactly if t_0 is a regular point of the space $P\varrho[\alpha, \beta]$ and $w(t_0) = 0$.

Proof: I. Let any function of the space $P\varrho[\alpha, \beta]$, having a zero value at t_0, have a zero of the type 1 at t_0. Then by Theorem 1.7 [6] $w(t_0) = 0$ and because of Theorem 1.5 t_0 is a regular point of $P\varrho[\alpha, \beta]$.

II. If t_0 is a regular point of the space $P\varrho[\alpha, \beta]$ and $w(t_0) = 0$, then with respect to Theorem 1.7 [6] there exists a function of the space $P\varrho[\alpha, \beta]$ such that t_0 is its zero of the type 1 and by Theorem 1.5 no function of the space $P\varrho[\alpha, \beta]$ has a zero of the type 2 at t_0.

Theorem 1.9. Let $t_0 \in i$ be a regular point of the spaces S and S'. The point t_0 is a regular point of the space $P\varrho[\alpha, \beta]$ exactly if there holds one of the assertions below:

1° $w(t_0) = 0$ and for any function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ having a zero value at t_0, t_0 is a zero of the type 1.

2° $w(t_0) \neq 0$ (said otherwise: for any function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ having a zero value at t_0, t_0 is a zero of the type 2).

Proof: The assertion is the corollary of the previous theorems.

Theorem 1.10. Let $t_0 \in i$ be a regular point of the spaces $S, S', P\varrho[\alpha, \beta]$ and $w(t_0) = 0$. Then there exist real constants $\lambda, \mu, \lambda \neq 0, \mu \neq 0$ such that t_0 is a singular point of the accompanying space $Pv[\lambda, \mu]$ to the space S, where $v > 0$ is a function continuous on the interval i.

Proof: According to Theorem 1.9, for any function $y \in S$ for which $y(t_0) \neq 0$ $\frac{y'(t_0)}{y(t_0)} \neq -\frac{\alpha}{\beta}$. Let us write $\frac{y'(t_0)}{y(t_0)} = -\frac{\lambda}{\mu}$. Then t_0 is a zero of the function $v(\lambda y + \mu y')$ of the type 2 and by Theorem 1.4 t_0 is a singular point of the space $Pv[\lambda, \mu]$.

2. Extreme points of the space $P\varrho[\alpha, \beta]$

Lemma 2.1. Let $t_0 \in i$ and $y \in C_1(i)$ be such that $y(t_0) = 0$. Let t_0 not be a limit point of zeros either of the function y nor y'. Then there exists $\delta > 0$ such that for $t \in (t_0 - \delta, t_0)$ $\frac{y'(t)}{y(t)} < 0$ and for $t \in (t_0, t_0 + \delta)$ $\frac{y'(t)}{y(t)} > 0$.

Proof: With respect to the assumptions of our Lemma there exists $\delta > 0$ such that $y(t) \neq 0$ holds for $t \in (t_0 - \delta, t_0 + \delta)$, $t \neq t_0$ and likewise $y'(t) \neq 0$. Let for $t \in (t_0 - \delta, t_0)$ hold:

1. $y(t) < 0$, then y is increasing and thus $y'(t) > 0$,
2. $y(t) > 0$, then y is decreasing and thus $y'(t) < 0$,

whence it follows that $\frac{y'(t)}{y(t)} < 0$ for $t \in (t_0 - \delta, t_0)$.

Let for $t \in (t_0, t_0 + \delta)$ hold:

1. $y(t) < 0$, then y is decreasing and thus $y'(t) < 0$,
2. $y(t) > 0$, then y is increasing and thus $y'(t) > 0$,

whence it follows that $\frac{y'(t)}{y(t)} > 0$ for $t \in (t_0, t_0 + \delta)$. •

Theorem 2.1. Let $t_0 \in i$ and $y \in C_1(i)$ such that $y(t_0) = 0$ and $y'(t_0) \neq 0$. Then it holds:

$$\lim_{t \to t_0+} \frac{y'(t)}{y(t)} = +\infty \qquad \text{and} \qquad \lim_{t \to t_0-} \frac{y'(t)}{y(t)} = -\infty.$$

Proof: The assertion follows with respect to Lemma 2.1 from the assumption of $y'(t_0) \neq 0$.

Corollary 2.1. Let $t_1, t_2 \in i$, $t_1 < t_2$, be the neighbouring zeros of the function $y \in C_1(i)$ and let $y'(t_1) \neq 0$ and $y'(t_2) \neq 0$ hold. Then the function $\frac{y'}{y}$ maps the interval (t_1, t_2) onto the interval $(-\infty, +\infty)$.

Theorem 2.2. Let $t_0 \in i$ and $y \in C_1(i)$ such that $y(t_0) = 0$ and $y'(t_0) = 0$. Let t_0 not be a limit point of zeros either of the function y nor y' and let t_0 not be a limit point of extremes of the function y'. Then

$$\lim_{t \to t_0+} \frac{y'(t)}{y(t)} = +\infty \quad \text{and} \quad \lim_{t \to t_0-} \frac{y'(t)}{y(t)} = -\infty.$$

Proof: With respect to the assumptions of the Theorem there exists $\delta > 0$ such that $y(t) \neq 0$, $y'(t) \neq 0$ holds for $t \in (t_0, t_0 + \delta)$ and the functions $y(t)$ and $y'(t)$ are strictly monotone in the interval $(t_0, t_0 + \delta)$. Let us restrict ourselves to the case that $y(t) > 0, y'(t) > 0$ for $(t_0, t_0 + \delta)$ and let us investigate $\lim_{t \to t_0+} \frac{y'(t)}{y(t)}$. The functions $y(t)$, $y'(t)$ are thus increasing on the interval $\langle t_0, t_0 + \delta \rangle$ and the function y is strictly convex. Let h be a number, $0 < h < \delta$, then by the mean value theorem there exists $t \in (t_0, t_0 + \delta)$ such that

$$y(t_0 + h) - y(t_0) = y'(t)\, h. \tag{2.1}$$

Because of y' being increasing on the interval $\langle t_0, t_0 + \delta \rangle$, the point t from (2.1) is uniquely determined and it is obviously the function h. Let $t = T(h)$ for $h \in (0, \delta)$ and $T(0) = t_0$. Then for $h \in (0, \delta)$ we have

$$T(h) = (y')^{-1} \left(\frac{y(t_0 + h) - y(t_0)}{h} \right)$$

and

$$\lim_{h \to 0+} T(h) = (y')^{-1} \left(\lim_{h \to 0+} \frac{y(t_0 + h) - y(t_0)}{h} \right) = (y')^{-1} (y'(t_0)) = t_0.$$

The function $T(h)$ is thus continuous on the interval $\langle 0, \delta)$. For any $h \in (0, \delta)$ we have

$$\frac{y(t_0 + h)}{y(T(h))} > 1$$

and thus

$$\frac{y(t_0 + h)}{y(T(h))\, h} > \frac{1}{h}.$$

Since $\lim_{h \to 0+} \frac{1}{h} = +\infty$ we have $\lim_{h \to 0+} \frac{y(t_0 + h)}{y(T(h))\, h} = +\infty$. In applying (2.1) we obtain

$$+\infty = \lim_{h \to 0+} \frac{y(t_0 + h) - y(t_0)}{y(T(h))\, h} = \lim_{h \to 0+} \frac{y'(T(h))\, h}{y(T(h))\, h} = \lim_{h \to 0+} \frac{y'(T(h))}{y(T(h))}. \tag{2.2}$$

Let us now show that the function $T(h)$ is schlicht – increasing on the interval $\langle 0, \delta)$. If it namely were for $h_1 < h_2$ $T(h_1) \geqq T(h_2)$, i.e. $t_1 \geqq t_2$, where $t_1 = T(h_1)$, $t_2 = T(h_2) \in (t_0, t_0 + \delta)$ and since $y'(t_1) \geqq y'(t_2)$, then it would be:

$$\frac{y(t_0 + h_1)}{h_1} \geqq \frac{y(t_0 + h_2)}{h_2},$$

which is impossible with respect to y being strictly convex on the interval $(t_0, t_0 + \delta)$. Thus there exists an inverse function $T^{-1}(t)$ and we have

$$\lim_{t \to t_0} T^{-1}(t) = T^{-1}(t_0) = 0.$$

Inserting $h = T^{-1}(t)$ into (2.2) we get

$$+\infty = \lim_{h \to 0+} \frac{y'(T(h))}{y(T(h))} = \lim_{t \to t_0+} \frac{y'(t)}{y(t)},$$

which was to be proved.

In case of $y(t) < 0$ and $y'(t) < 0$ for $t \in (t_0, t_0 + \delta)$ let us denote $z(t) = -y(t)$ and $z'(t) = -y'(t)$ then for all $t \in (t_0, t_0 + \delta)$ $\frac{z'(t)}{z(t)} = \frac{y'(t)}{y(t)}$ which are the conditions of the previous case.

The assertion $\lim_{t \to t_0-} \frac{y'(t)}{y(t)} = -\infty$ is to be proved in analogy to $\lim_{t \to t_0+} \frac{y'(t)}{y(t)} = +\infty$.

Corollary 2.2. Let the assumptions of Theorem 2.1 or Theorem 2.2 be satisfied. Then

$$\lim_{t \to t_0} \frac{y(t)}{y'(t)} = 0$$

holds.

Theorem 2.3. Let $t_1, t_2 \in i$, $t_1 < t_2$, be the neighbouring zeros of the function $y \in C_1(i)$. Let next the sequence of zeros of the function y' from the interval (t_1, t_2) not have any limit point t_1 or t_2 and in case of $y'(t_1) = 0$ or $y'(t_2) = 0$, let t_1 or t_2 not be a limit point of extremes of the function y' from the interval (t_1, t_2). Then the function $\frac{y'}{y}$ maps the interval (t_1, t_2) onto the interval $(-\infty, +\infty)$.

Proof: The assertion follows from Theorems 2.1 and 2.2.

Convention 2.1. We shall concern ourselves in what follows with regular spaces of a certain type on i only. It means two independent functions of the space S or S' or $P\varrho[\alpha, \beta]$ have no zeros in commun and no function of the space S or S' or $P\varrho[\alpha, \beta]$ has not any limit point of zeros inside the definition interval i. For short we shall call the zero $t_0 \in i$ of the function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ of type 1 or type 2 the zero of type 1 or type 2. We shall exclude from our considerations the zeros of type 1 which are the limit points of extremes of the function from the space S' having at these points a zero value, i.e. we assume that for any $t_0 \in i$ there exist the limits $\lim_{t \to t_0+} \frac{y'(t)}{y(t)}$ and $\lim_{t \to t_0-} \frac{y'(t)}{y(t)}$, where $y \in S$.

Lemma 2.2. Let $t_1, t_2 \in i$, $t_1 < t_2$, be the neighbouring zeros of the function

$\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$. Then the function y has in the interval (t_1, t_2) one zero at most.

Proof: Let us assume there exist at least two zeros of the function y in the interval (t_1, t_2). Let us denote by $t_3, t_4 \in (t_1, t_2)$, $t_3 < t_4$, the neighbouring zeros of the function y. Then by Theorem 2.3 the function $\frac{y'}{y}$ assumes the value $-\frac{\alpha}{\beta}$ on the interval (t_3, t_4) and by Lemma 1.2 there exist in (t_1, t_2) a zero of the function $\varrho(\alpha y + \beta y')$, contrary to our assumption.

Theorem 2.4. Let $t_1, t_2 \in i$, $t_1 < t_2$, be the neighbouring zeros of the function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$. Let the function y have in the interval (t_1, t_2) exactly one zero, then t_1 and t_2 are the zeros of type 2.

Proof: Let us denote by $t_0 \in (t_1, t_2)$ the zero of the function y. If the point t_1 were the zero of type 1, then by Theorem 2.2 $\lim_{t \to t_1+} \frac{y'(t)}{y(t)} = +\infty$ and since by Theorem 2.1 $\lim_{t \to t_0-} \frac{y'(t)}{y(t)} = -\infty$, the function $\frac{y'}{y}$ would assume the value $-\frac{\alpha}{\beta}$ on the interval (t_1, t_0) and by Lemma 1.2 the zero of the function $\varrho(\alpha y + \beta y')$ would be in the interval (t_1, t_0). This, however, contradicts our assumption. The proof for the point t_2 proceeds similarly.

Corollary 2.3. Let $t_1, t_2 \in i$, $t_1 \neq t_2$, be the neighbouring conjugate points of the space $P\varrho[\alpha, \beta]$. Then t_1 and t_2 are not the zeros of type 1 simultaneously.

Remark 2.1. In the following Lemma 2.3, Theorem 2.5 and in its Corollary 2.4 the assumption $S \subset C_1(i)$ is not necessary and $S \subset C_0(i)$ suffices. This assertion is true for any two-dimensional regular space of continuous functions of a certain type on its definition interval.

Lemma 2.3. Let (u, v) be a basis of the space S. Let $t_1, t_2 \in i$, $t_1 < t_2$, be the neighbouring zeros of the function u. Let $v \neq 0$ in the interval (t_1, t_2) or let v have at least two zeros in the interval (t_1, t_2). Then at least one extreme point of the space S lies in the interval (t_1, t_2).

Proof: The assertion follows from Theorem 5 [5].

Theorem 2.5. The point $t_0 \in i$ is an extreme point of the space S exactly if the function $y \in S$, for which $y(t_0) = 0$, does not change the sign at t_0.

Proof: According to Lemma 1 [5] every point $t_0 \in i$ is the zero of a function from the space S. Hence, let $y(t_0) = 0$ hold for $y \in S$. With respect to the regularity of the space S $y_1(t_0) \neq 0$ holds for any function $y_1 \in S$, independent on y.

I. Let t_0 be an extreme point of the space S. Then by Theorem 3.2 [3] $\lim_{t \to t_0} \frac{y_1(t)}{y(t)} = +\infty$ or $\lim_{t \to t_0} \frac{y_1(t)}{y(t)} = -\infty$. Since $y_1(t_0) \neq 0$, there exists a neighbourhood of the point t_0 at which y_1 is positive or negative so that, for the above limits to be valid, y cannot change its sign at t_0.

II. Let the function $y \in S$ not change the sign at its zero t_0. Taking the function $y_1 \in S$ independent on y, we obtain $\lim_{t \to t_0} \frac{y_1(t)}{y(t)}$ being equal to $+\infty$ or $-\infty$ and by Theorem 3.2 [3] t_0 is an extreme point of the space S.

Corollary 2.4. The point $t_0 \in i$ is an ordinary point of the space S exactly if the function $y \in S$, for which $y(t_0) = 0$, does not change its sign at t_0.

Theorem 2.6. Let there exist a neighbourhood $U(t_0)$ of the point $t_0 \in i$ such that $w(t_0) = 0$ and $w(t) \neq 0$ holds for all $t \in U(t_0)$, $t \neq t_0$. Then

a) if w changes its sign at t_0, then t_0 is an extreme point of the space S.

b) if w does not change its sign at t_0, then t_0 is an extreme point of the space S'.

Proof: By Theorem 1.10 [6] the first phase $A(t)$, $t \in i$, of the basis (u, v) from the space S has the continuous first derivative

$$A'(t) = \frac{-w(t)}{u^2(t) + v^2(t)}.$$

For $A(t)$ to have an extreme at t_0 it is necessary that w changes its sign at t_0. Further, by Lemma 1.4 [6] the point t_0, where $w(t_0) = 0$, is either an extreme point of the space S or an extreme point of the space S' – thus if w does not change its sign at t_0, then t_0 is an extreme point of the space S'.

Theorem 2.7. Let $w(t_0) = 0$, where $t_0 \in i$. The point t_0 is an extreme point of the space $P\varrho[\alpha, \beta]$ exactly if t_0 is an extreme point of the space S'.

Proof: Assuming that $w(t_0) = 0$ then, by Lemma 1.3, there exists the function $y \in S$ such that $y(t_0) = 0$ and $y'(t_0) = 0$.

I. Let t_0 be an extreme point of the space $P\varrho[\alpha, \beta]$. Then it follows, by Lemma 2.5, that for the function $\varrho(\alpha y + \beta y')$ there exists $\delta_1 > 0$ such that for $t \in (t_0 - \delta_1, t_0 + \delta_1)$, $t \neq t_0$, we have $\varrho(t)(\alpha y(t) + \beta y'(t)) > 0$ or $\varrho(t)(\alpha y(t) + \beta y'(t)) < 0$. It suffices to assume next $\varrho(t)(\alpha y(t) + \beta y'(t)) > 0$, thus $\alpha y(t) + \beta y'(t) > 0$. By Lemma 2.1 there exists $\delta_2 > 0$ such that for $t \in (t_0 - \delta_2, t_0)$ either $y'(t) < 0$ and $y(t) > 0$ or $y'(t) > 0$ and $y(t) < 0$ and for $t \in (t_0, t_0 + \delta_2)$ there is either $y'(t) > 0$ and $y(t) > 0$ or $y'(t) < 0$ and $y(t) < 0$. Let us take $\delta = \min(\delta_1, \delta_2)$.

1. Let $\beta > 0$. Then it follows from $\alpha y + \beta y' > 0$ that $y' > -\frac{\alpha}{\beta} y$. Since $\lim_{t \to t_0 -} \frac{y'(t)}{y(t)} = -\infty$, the function $\frac{y'}{y}$ cannot be lower limited on the interval $(t_0 - \delta, t_0)$, it must hold there $y < 0$ and consequently $y' > 0$. Next it holds $\lim_{t \to t_0 +} \frac{y'(t)}{y(t)} = +\infty$, thus the function $\frac{y'}{y}$ cannot be upper limited on the interval $(t_0, t_0 + \delta)$ and it must hold there $y > 0$ and consequently $y' > 0$. Herefrom we

see that the function y changes the sign in its zero t_0 and thus t_0 is by Corollary 2.4 an ordinary point of the space S and the function y' does not change the sign in its zero t_0 and thus t_0 is by Theorem 2.5 the extreme point of the space S'.

2. Let $\beta < 0$. Then it follows from $\alpha y + \beta y' > 0$ that $y' < -\frac{\alpha}{\beta} y$ and in analogy with part 1. we get $y(t) > 0$, $y'(t) < 0$ for $t \in (t_0 - \delta, t_0)$ and $y(t) < 0$, $y'(t) < 0$ for $t \in (t_0, t_0 + \delta)$. With respect to Corollary 2.4 and to Theorem 2.5 t_0 is again an ordinary point of the space S and an extreme point of the space S'.

II. Let t_0 be an extreme point of the space S'. Then by Theorem 2.5 there exists for the function y' that $\delta_1 > 0$ such that $y'(t) > 0$ or $y'(t) < 0$ for $t \in (t_0 - \delta_1, t_0 + \delta_1)$, $t \neq t_0$. Next it suffices to assume that $y'(t) > 0$. With respect to Lemma 2.1 there exists $\delta_2 > 0$ such that $y(t) < 0$ for $t \in (t_0 - \delta_2, t_0)$ and $y(t) > 0$ for $t \in (t_0, t_0 + \delta_2)$. Let us take $\delta = \min(\delta_1, \delta_2)$.

1. Let $\beta > 0$. Since $\lim_{t \to t_0 -} \frac{y'(t)}{y(t)} = -\infty$, there exists $\delta_3 > 0$ such that the inequality $\frac{y'}{y} < -\frac{\alpha}{\beta}$ is satisfied on the interval $(t_0 - \delta_3, t_0) \subset (t_0 - \delta, t_0)$ whence it follows that $\alpha y + \beta y' < 0$ on the interval $(t_0 - \delta, t_0)$ and consequently the function $\varrho(t)(\alpha y(t) + \beta y'(t)) < 0$ for $t \in (t_0 - \delta, t_0)$. Since $\lim_{t \to t_0 +} \frac{y'(t)}{y(t)} = +\infty$, there exists $\delta_4 > 0$ such that the inequality $\frac{y'}{y} > -\frac{\alpha}{\beta}$ is satisfied on the interval $(t_0, t_0 + \delta_4) \subset (t_0, t_0 + \delta)$ whence it follows that $\alpha y + \beta y' < 0$ on the interval $(t_0, t_0 + \delta)$, hence the function $\varrho(t)(\alpha y(t) + \beta y'(t)) < 0$ for $t \in (t_0, t_0 + \delta)$. Since $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$, we get by Theorem 2.5 that t_0 is an extreme point of the space $P\varrho[\alpha, \beta]$.

2. Let $\beta < 0$. Then proceeding analogous as in part 1. we get $\alpha y(t) + \beta y'(t) > 0$ for $t \in (t_0 - \delta, t_0)$ and $\alpha y(t) + \beta y'(t) > 0$ for $t \in (t_0, t_0 + \delta)$. The function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ does not change the sign at its zero t_0 so that by Theorem 2.5 t_0 is an extreme point of the space $P\varrho[\alpha, \beta]$.

Corollary 2.5. Let $w(t_0) = 0$, $t_0 \in i$, and t_0 is an extreme point of the space S'. Then t_0 is an extreme point of any accompanying space $Pv[\lambda, \mu]$ to the space S, where $\lambda, \mu \neq 0$ are arbitrary numbers and $v > 0$ is a function continuous on the interval i.

Theorem 2.8. Let $t_0 \in i$ and $w(t_0) \neq 0$. Then t_0 is an extreme point of the space $P\varrho[\alpha, \beta]$ if and only if it holds for the function $y \in S$, for which $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$, that $\frac{y'}{y}$ has an extreme at t_0.

Proof: I. Let t_0 be an extreme point of the space $P\varrho[\alpha, \beta]$. Then it holds for the function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ having the zero value at t_0 that $y(t_0) \neq 0$ and

$y'(t_0) \neq 0$, hence it exists $\delta_1 > 0$ such that $y(t) \neq 0$ for $t \in (t_0 - \delta_1, t_0 + \delta_1)$ and it holds further $\delta_2 > 0$ such that $\varrho(t)(\alpha y(t) + \beta y'(t)) > 0$ or $\varrho(t)(\alpha y(t) + \beta y'(t)) < 0$ for $t \in (t_0 - \delta_2, t_0 + \delta_2)$, $t \neq t_0$. Let us take $\delta = \min(\delta_1, \delta_2)$ and we can next assume that $\varrho(t)(\alpha y(t) + \beta y'(t)) > 0$ for $t \in (t_0 - \delta, t_0 + \delta)$, $t \neq t_0$.

1. Let $\beta > 0$. From the relation $\varrho(\alpha y + \beta y') > 0$ on the interval $(t_0 - \delta, t_0) \cup \cup (t_0, t_0 + \delta)$ we get: if $y > 0$ on the interval $(t_0 - \delta, t_0 + \delta)$, then it holds $\frac{y'(t)}{y(t)} > -\frac{\alpha}{\beta}$ for $t \in (t_0 - \delta, t_0) \cup (t_0, t_0 + \delta)$ and since $\frac{y'(t_0)}{y(t_0)} = -\frac{\alpha}{\beta}$ holds, $\frac{y'}{y}$ has its minimum at t_0; if $y < 0$ holds on the interval $(t_0 - \delta, t_0 + \delta)$, then the function $\frac{y'}{y}$ has its maximum at t_0.

2. Let $\beta < 0$. The proof proceeds analogous to that of part 1. and we get that $\frac{y'}{y}$ has at t_0 for $y > 0$ on $(t_0 - \delta, t_0 + \delta)$ its maximum and for $y < 0$ its minimum.

II. Let the function $\frac{y'}{y}$, where $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$, have its extrem at the point t_0; it suffices to assume that it has the maximum. Thus, there exists $\delta > 0$ such that $y(t) \neq 0$ and $\frac{y'(t)}{y(t)} < -\frac{\alpha}{\beta}$ for $t \in (t_0 - \delta, t_0 + \delta)$, $t \neq t_0$, where by Lemma 1.2 $\frac{y'(t_0)}{y(t_0)} = -\frac{\alpha}{\beta}$.

1. Let $\beta > 0$. If $y > 0$ $(y < 0)$ on the interval $(t_0 - \delta, t_0 + \delta)$ then $\varrho(t) \times \times (\alpha y(t) + \beta y'(t)) < 0$ $(\varrho(t)(\alpha y(t) + \beta y'(t)) > 0)$ foı $t \in (t_0 - \delta, t_0 + \delta)$, $t \neq t_0$, and thus the function $\varrho(\alpha y + \beta y')$ does not change the sign at its zero t_0. By Theorem 2.5 t_0 is an extreme point of the space $P\varrho[\alpha, \beta]$.

2. Let $\beta < 0$. If $y > 0$ $(y < 0)$ on the interval $(t_0 - \delta, t_0 + \delta)$ is for $t \in \in (t_0 - \delta, t_0 + \delta)$, $t \neq t_0$, $\varrho(t)(\alpha y(t) + \beta y'(t)) > 0$ $(\varrho(t)(\alpha y(t) + \beta y'(t)) < 0)$ and thus by Theorem 2.5 t_0 is an extreme point of the space $P\varrho[\alpha, \beta]$.

Theorem 2.9. Let $t_0 \in i$ and $w(t_0) \neq 0$. Let a function $y \in S$, $y(t_0) \neq 0$, exist such that the function $\frac{y'}{y}$ has the extreme at t_0. Then there exist real constants λ, $\mu \neq 0$ such that t_0 is an extreme point of the accompanying space $Pv[\lambda, \mu]$ to the space S, where $v > 0$ is a function continuous on i.

Proof: Since $y(t_0) \neq 0$ the function $\frac{y'}{y}$ has its finite value at t_0. Let us denote it $-\frac{\lambda}{\mu}$, where $\mu \neq 0$. By Theorem 2.8 t_0 is an extreme point of the space $Pv[\lambda, \mu]$.

Theorem 2.10. Let $t_1, t_2 \in i$, $t_1 < t_2$, be the neighbouring zeros of the function $\varrho(\alpha y + \beta y') \in P\varrho[\alpha, \beta]$ and for all $t \in \langle t_1, t_2 \rangle$ let $y(t) \neq 0$ and $w(t) \neq 0$. Then there lies at least one extreme point $P\varrho[\alpha, \beta]$ in the interval (t_1, t_2).

Proof: Assuming $y \neq 0$ on the interval $\langle t_1, t_2 \rangle$ it follows that $\frac{y'}{y}$ is continuous on $\langle t_1, t_2 \rangle$, hence also limited on $\langle t_1, t_2 \rangle$. Assuming $w \neq 0$ on the interval $\langle t_1, t_2 \rangle$ it follows that every point on $\langle t_1, t_2 \rangle$ is a zero of type 2 and for every function $y_1 \in S$ independent on y $\frac{y_1'(t)}{y_1(t)} \neq \frac{y'(t)}{y(t)}$ for all $t \in \langle t_1, t_2 \rangle$ where $y_1(t) \neq 0$. Let $t_0 \in (t_1, t_2)$ be a zero of the function $\varrho(\alpha y_2 + \beta y_2') \in P\varrho[\alpha, \beta]$. If $\varrho(\alpha y_2 + \beta y_2')$ does not change its sign at t_0, then, by Theorem 2.5, t_0 is an extreme point of the space $P\varrho[\alpha, \beta]$ and there is nothing more do prove. Thus let the function $\varrho(\alpha y_2 + \beta y_2')$ change its sign at t_0. Then, because of the fact that for every $t \in \langle t_1, t_2 \rangle$ $\frac{y_2'(t)}{y_2(t)} \neq \neq \frac{y'(t)}{y(t)}$, there must exist at least one point $T \neq t_0$, $T \in (t_1, t_2)$, such that $\varrho(T)(\alpha y_2(T) + \beta y_2'(T)) = 0$. By Lemma 2.3 at least one extreme point of the space $P\varrho[\alpha, \beta]$ exists in the interval (t_1, t_2).

Theorem 2.11. Let $t_0, t_1 \in i$, $t_0 < t_1$ $(t_0 > t_1)$, be neighbouring conjugate points of the space $P\varrho[\alpha, \beta]$ and let $w(t_0) = 0$ and $w(t) \neq 0$ for $t \in (t_0, t_1\rangle$ $(t \in \langle t_1, t_0))$. Let $\frac{y'(t_0)}{y(t_0)} < -\frac{\alpha}{\beta} \left(\frac{y'(t_0)}{y(t_0)} > -\frac{\alpha}{\beta}\right)$ hold for the function $y \in S$, where $y(t_0) \neq 0$. Then at least one extreme point of the space $P\varrho[\alpha, \beta]$ lies in the interval (t_0, t_1) $((t_1, t_0))$.

Proof: Let $t_0, t_1 \in i$, $t_0 < t_1$, are the neighbouring zeros of the function $\varrho(\alpha x + \beta x') \in P\varrho[\alpha, \beta]$. Then t_0 is a zero of type 1, t_1 is a zero of type 2, and by Theorem 2.4 and Lemma 2.2, we get $x \neq 0$ on the interval $(t_0, t_1\rangle$. By Lemma 1 [5] there exists to every point $T_1 \in (t_0, t_1) \subset i$ a function $\varrho(\alpha y_1 + \beta y_1') \in P\varrho[\alpha, \beta]$ such that $\varrho(T_1)(\alpha y_1(T_1) + \beta y_1'(T_1)) = 0$. If $\varrho(\alpha y_1 + \beta y_1')$ does not change its sign at T_1, then by Theorem 2.5 T_1 is an extreme point of the space $P\varrho[\alpha, \beta]$. Let $\varrho(\alpha y_1 + \beta y_1')$ change its sign at T_1. Then, with respect to the $\frac{y_1'(t_0)}{y_1(t_0)} < -\frac{\alpha}{\beta}$ and $w \neq 0$ on $(t_0, t_1\rangle$, there exists at least one point $T_2 \in (t_0, t_1)$, $T_2 \neq T_1$, such that $\varrho(T_2)(\alpha y_1(T_2) + \beta y_1'(T_2)) = 0$. By Lemma 2.3 at least one extreme point of the space $P\varrho[\alpha, \beta]$ lies in the interval (t_0, t_1).

Completely analogous proceeds the proof for $t_0 > t_1$ and $\frac{y'(t_0)}{y(t_0)} > -\frac{\alpha}{\beta}$.

Corollary 2.6. Let $t_0 \in i$ and $w(t_0) = 0$. Let next $w(t) \neq 0$ for all $t \in i$, $t \neq t_0$. Then there exist the constants $\lambda, \mu \neq 0$ such that at least one extreme point of the accompanying space $Pv[\lambda, \mu]$ to the space S lies in the interval i, being different from t_0, where $v > 0$ is a function continuous on i.

Convention 2.2. The two-dimensional space of continuous functions whose definition interval does not contain any extreme point will be called a space of the 0^{th} class on its definition domain.

Theorem 2.12. Let $w \neq 0$ on the interval $i = (a, b)$. The space $P\varrho[\alpha, \beta]$ is the space of the 0^{th} class on the interval (a, b) if and only if every function $\frac{y'}{y}$, where $y \in S$, takes on the value $-\frac{\alpha}{\beta}$ exactly once between the neighbouring zeros of the function y and, so far the smallest zero $t_1 \in (a, b)$ or the greatest zero $t_2 \in (a, b)$ of the function y exists, then $\frac{y'}{y}$ takes on the value $-\frac{\alpha}{\beta}$ once at most in the interval (a, t_1) or (t_2, b).

Proof: With respect to Lemma 1.2 and to Corollary 2.1, the assertion follows from Theorems 3, 4, 5 [5].

Theorem 2.13. Let $w \neq 0$ on the interval i. Every space $Pv[\lambda, \mu]$, where λ, μ $(\lambda^2 + \mu^2 \neq 0)$ are arbitrary constants and $v > 0$ is a function continuous on i, is a space of the 0^{th} class on i if and only if every function $\frac{y'}{y}$, $y \in S$, is monotone on every interval $j \subset i$ where it is defined.

Proof: The assertion follows from Theorem 2.9.

Remark 2.2. If the function $\frac{y'}{y}$ is monotone in $j \subset i$, then in view of Corollary 2.1, it is obviously decreasing.

Remark 2.3. Evidently, the set of all integrals of the 2nd order differential equation of the Jacobi type

$$y'' = Q(t)\, y, \qquad (Q)$$

where $Q(t)$, $t \in i$, is a continuous function on the interval i, forms a two-dimensional space of continuous functions with a definition interval i (in the sense of definition 1.2 [3]). By Theorem 1.16 [6] the set of derivatives of all integrals of (Q) forms a two-dimensional space with a definition interval i if and only if $Q \not\equiv 0$ on every interval $j \subset i$.

Lemma 2.4. Let Q be continuous on i and $Q \not\equiv 0$ on every interval $j \subset i$. Then there exists a solution u of (Q) for which u, u' are dependent on i exactly if $Q \equiv k$ on i, where $k > 0$ is a constant.

Proof: I. Let u, u' be dependent on i. Then by Theorem 0.1 $u = c_1 e^{ct}$, $t \in i$, where c_1, c are nonzero constants. Let for the solution v of (Q) hold that u, v are independent on i. Then we obtain for v from the equation for differentiation of the Wronskian of the functions u, v the equation

$$v'' = c^2 v$$

whence it follows that $Q \equiv c^2$ on i.

II. The function $e^{\sqrt{kt}}$, $t \in i$, is the solution of the differential equation

$$y'' = ky$$

on the interval i, where $k > 0$ is a constant. The assertion follows directly from this with respect to Theorem 0.1.

Theorem 2.14. Let Q be continuous on i and $Q \not\equiv 0$ on every interval $j \subset i$. It holds for every solution y of (Q) that y, y' are independent on i exactly if $Q \not\equiv k$ on every interval $j \subset i$, where $k > 0$ is a constant.

Proof: The assertion follows from Lemma 2.4 with respect to Corollary 0.1.

Remark 2.4. M. Laitoch defined in [2] the first accompanying equation

$$y'' = Q_1 y \qquad (Q_1)$$

corresponding to a basis $[\alpha, \beta]$ of the equation (Q), where $Q < 0$ is a continuous function on i, and α, β are arbitrary constants satisfying the condition $\alpha^2 + \beta^2 \neq 0$, the carrier being of the form

$$Q_1 = Q + \frac{\alpha\beta Q'}{\alpha^2 - \beta^2 Q} + \sqrt{\alpha^2 - \beta^2 Q}\left(\frac{1}{\sqrt{\alpha^2 - \beta^2 Q}}\right)''.$$

By appealing to Theorem 4.1 [2] it holds that if u is a solution of (Q), then the solution

$$u_1 = \frac{\alpha u + \beta u'}{\sqrt{\alpha^2 - \beta^2 Q}}$$

is a solution of (Q_1).

Theorem 2.15. The set of all integrals of the first accompanying equation (Q_1) corresponding to the $[\alpha, \beta]$ of (Q) forms a two-dimensional accompanying space $P_\varrho[\alpha, \beta]$ to the space S of the integrals of the equation (Q), where

$$\varrho = \frac{1}{\sqrt{\alpha^2 - \beta^2 Q}}.$$

Proof: Because of the assumption $Q < 0$ on i, it holds (by Theorem 2.14) for every solution of (Q) that y, y' are independent on i. Obviously $\alpha^2 - \beta^2 Q > 0$ and consequently ϱ is continuous and positive on i. Let u, v be two independent solution of (Q). Then, with respect to Theorem 4.1 [2]

$$u_1 = \varrho(\alpha u + \beta u'), \qquad v_1 = \varrho(\alpha v + \beta v')$$

are two independent solution of (Q_1) whence (with respect to Theorem 1.1) the assertion follows.

REFERENCES

[1] O. Borůvka: *Lineare Differentialtransformationen 2. Ordnung*. VEB Deutscher Verlag der Wissenschaften. Berlin 1967.

[2] M. Laitoch: *L'équation associée dans la théorie des transformations des équations différentielles du second ordre*. Acta Univ. Palackianae 12 (1963) Olomouc.

[3] K. Stach: *Die allgemeinen Eigenschaften der Kummerschen Transformationen zweidimensionaler Räume von stetigen Funktionen*. Publ. Fac. Sci. UJEP Brno, 478, 1966.

[4] K. Stach: *Die vollständigen Kummerschen Transformationen zweidimensionaler Räume von stetigen Funktionen*. Archivum Math. UJEP Brno, 1967, T. 3, F. 3.

[5] S. Trávníček, J. Kojecká-Hodinářová: *Einige Bemerkungen über die Nullstellen im zweidimensionalen Raum stetiger Funktionen*. Acta Univ. Palackianae 33 (1972) Olomouc.

[6] J. Kojecká: *Lineare zweidimensionale Räume von stetigen Funktionen mit stetigen ersten Ableitungen*. Acta Univ. Palackianae 45 (1974) Olomouc.

Souhrn

PRŮVODNÍ PROSTORY K LINEÁRNÍMU DVOJROZMĚRNÉMU PROSTORU SPOJITÝCH FUNKCÍ SE SPOJITOU PRVNÍ DERIVACÍ

JITKA KOJECKÁ

Nechť $S \subset C_1(i)$ je dvojrozměrný prostor spojitých funkcí a nechť pro každou funkci $y \in S$ platí, že funkce y a y', kde y' je derivace funkce y, jsou nezávislé na intervalu i. Pak množina všech funkcí tvaru $\varrho(\alpha y + \beta y')$, $y \in S$, kde $\varrho > 0$ je spojitá funkce na intervalu i a α, β jsou dané reálné konstanty ($\alpha^2 + \beta^2 \neq 0$) tvoří dvojrozměrný prostor spojitých funkcí s definičním intervalem i. Nazýváme ho průvodní prostor k prostoru S vzhledem k číselné bázi $[\alpha, \beta]$ s váhou ϱ a značíme ho $P\varrho[\alpha, \beta]$.

V části 1 jsou zkoumány nulové body funkcí prostoru $P\varrho[\alpha, \beta]$ a singulárnost a regulárnost prostoru $P\varrho[\alpha, \beta]$.

V části 2 je předpokládáno, že prostory S, S' (množina derivací všech funkcí prostoru S) a $P\varrho[\alpha, \beta]$ jsou regulární prostory určitého typu na intervalu i (viz [3]). Jsou vyšetřovány extrémní body prostoru $P\varrho[\alpha, \beta]$ (tj. body intervalu i, ve kterých má fáze prostoru $P\varrho[\alpha, \beta]$ extrém). Nechť w je wronskián funkcí báze (u, v) prostoru S, pak dostáváme tyto výsledky:

Věta 2.7. Nechť $w(t_0) = 0$, kde $t_0 \in i$. Bod t_0 je extrémní bod prostoru $P\varrho[\alpha, \beta]$ právě tehdy, když je t_0 extrémní bod prostoru S'.

Věta 2.8. Buď $t_0 \in i$ a $w(t_0) \neq 0$. Bod t_0 je extrémní bod prostoru $P\varrho[\alpha, \beta]$

právě tehdy, když pro funkci $y \in S$, pro kterou je $\varrho(t_0)(\alpha y(t_0) + \beta y'(t_0)) = 0$, platí, že funkce $\frac{y'}{y}$ má v t_0 extrém.

Dále je uvedeno, za jakých předpokladů nemá prostor $P\varrho[\alpha, \beta]$ extrémní body, tj. $P\varrho[\alpha, \beta]$ je nulté třídy na intervalu i (věta 2.12) a za jakých předpokladů je každý průvodní prostor $Pv[\lambda, \mu]$ k prostoru S nulté třídy na intervalu i (věta 2.13).

V závěru práce je ukázána souvislost průvodního prostoru $P\varrho[\alpha, \beta]$ a prostoru všech integrálů první průvodní rovnice $y'' = Q_1 y$ při bázi $[\alpha, \beta]$ k diferenciální rovnici $y'' = Qy$, kde $Q < 0$ je funkce spojitá na intervalu i (viz [2]). Platí následující:

Věta 2.15. Množina všech integrálů první průvodní rovnice (Q_1) při bázi $[\alpha, \beta]$ k rovnici (Q) tvoří dvojrozměrný průvodní prostor $P\varrho[\alpha, \beta]$ k prostoru S integrálů rovnice (Q), kde

$$\varrho = \frac{1}{\sqrt{\alpha^2 - \beta^2 Q}}.$$

Резюме

СОПРОВОЖДАЮЩИЕ ПРОСТРАНСТВА К ЛИНЕЙНОМУ ДВУХРАЗМЕРНОМУ ПРОСТРАНСТВУ НЕПРЕРЫВНЫХ ФУНКЦИЙ С НЕПРЕРЫВНОЙ ПРОИЗВОДНОЙ ПЕРВОГО ПОРЯДКА

ЙИТКА КОЙЕЦКА

Пусть $S \subset C_1(i)$ есть двухразмерное пространство непрерывных функций и пусть для каждой функции $y \in S$ имеет место, что y и y' независимы на интервале i. Тогда множество функции вида $\varrho(\alpha y + \beta y')$, $y \in S$, где $\varrho > 0$ есть непрерывная функция на интервале i и α, β данные вещественные постоянные, образует двухразмерное пространство непрерывных функций, определенных на интервале i. Это пространство называем сопровождающим к пространству S по отношении к базису $[\alpha, \beta]$ с весом ϱ и обозначаем $P\varrho[\alpha, \beta]$.

В первой части исследуются нулевые точки функций пространства $P\varrho[\alpha, \beta]$ и сингулярность и регулярность пространства $P\varrho[\alpha, \beta]$.

Во второй части предполагается, что пространства S, S' (множество производных функций пространства S) и $P\varrho[\alpha, \beta]$ регулярные пространства определенного типа на интервале i (см. [3]). Исследуются экстремальные точки пространства $P_{\varrho}[\alpha, \beta]$ т.е. точки интервала i, в которых имеет фаза пространства $P_{\varrho}[\alpha, \beta]$ экстрем. Пусть w есть определитель Вронского функций базиса (u, v) пространства S, тогда получаем следующие теоремы:

Теорема 2.7. Пусть $w(t_0) = 0$, где $t_0 \in i$. Точка t_0 экстремальная точка пространства $P_{\varrho}[\alpha, \beta]$ тогда и только тогда, когда t_0 является экстремальной точкой пространства S'.

Теорема 2.8. Пусть $t_0 \in i$ и $w(t_0) \neq 0$. Точка t_0 есть экстремальная точка пространства $P_{\varrho}[\alpha, \beta]$ тогда и только тогда, когда для функции $y \in S$, удовлетворяющей равенству $\varrho(t_0)(\alpha y(t_0) + \beta y(t_0)) = 0$ имеет место, что $\frac{y'}{y}$ достигает в точке t_0 экстремальное значение.

Далее приводятся условия, при которых пространство $P_{\varrho}[\alpha, \beta]$ не имеет экстремальные точки, т.е. $P_{\varrho}[\alpha, \beta]$ нулевого класса на интервале i (теорема 2.12).

В заключение работы показана зависимость сопровождающего пространства $P_{\varrho}[\alpha, \beta]$ и пространства всех интегралов первого сопровождающего уравнения $y'' = Q_1 y$ с базисом $[\alpha, \beta]$ уравнения $y'' = Qy$, где $Q < 0$ является непрерывной функцией на интервале i (см. [2]).

Имеет место следующее:

Теорема 2.5. Множество всех интегралов первого сопровождающего уравнения (Q_1) с базисом $[\alpha, \beta]$ к уравнению (Q) образует двухразмерное пространство $P_{\varrho}[\alpha, \beta]$ к пространству S интегралов уравнения (Q), где

$$\varrho = \frac{1}{\sqrt{\alpha^2 - \beta^2 Q}}.$$