24 Local class field theory

In this lecture we give a brief overview of local class field theory, following [1, Ch. 1]. Recall that a local field is a locally compact field whose topology is induced by a nontrivial absolute value (Definition 9.1). As we proved in Theorem 9.7, every local field is isomorphic to one of the following:

- $\mathbb{R}$ or $\mathbb{C}$ (archimedean);
- finite extension of $\mathbb{Q}_p$ (nonarchimedean, characteristic 0);
- finite extension of $\mathbb{F}_p((t))$ (nonarchimedean, characteristic p).

In the nonarchimedean cases, the ring of integers of a local field is a complete DVR with finite residue field.

The goal of local class field theory is to classify all finite abelian extensions of a given local field K. Rather than considering each finite abelian extension L/K individually, we will treat them all at once, by fixing once and for all a separable closure $K^{\rm sep}$ of K and working in the maximal abelian extension of K inside $K^{\rm sep}$.

Definition 24.1. Let K be field with separable closure $K^{\rm sep}$. The field

$$K^{\rm ab} := \bigcup_{\substack{L \subseteq K^{\rm sep} \\ L/K \text{ finite abelian}}} L$$

is the *maximal abelian extension of* K (in $K^{\rm sep}$).

The field $K^{\rm ab}$ contains the field $K^{\rm unr}$, the maximal unramified subextension of $K^{\rm sep}/K$; this is obvious in the archimedean case, and in the nonarchimedean case it follows from Theorem 10.1, which implies that $K^{\rm unr}$ is isomorphic to the algebraic closure of the residue field of K, which is abelian because the residue field is finite every algebraic extension of a finite field is pro-cyclic, and in particular, abelian. We thus have the following tower of field extensions

$$K \subseteq K^{\rm unr} \subseteq K^{\rm ab} \subseteq K^{\rm sep}.$$

The Galois group $\mathrm{Gal}(K^{\rm ab}/K)$ is the profinite group

$$\mathrm{Gal}(K^{\rm ab}/K) \simeq \varprojlim_{L} \mathrm{Gal}(L/K),$$

where the fields L range over finite abelian extensions of K in $K^{\rm sep}$ and are ordered by inclusion. As a profinite group (like all Galois groups), the group $\mathrm{Gal}(K^{\rm ab}/K)$ is a totally disconnected compact Hausdorff group (see Problem Set 11), and we have the Galois correspondence

$$\begin{aligned}
\{\text{subsextensions } L/K \text{ in } K^{\rm ab}/K\} &\longleftrightarrow \{\text{ closed subgroups } H \text{ in } \mathrm{Gal}(K^{\rm ab}/K)\}\\
L &\longmapsto \mathrm{Gal}(K^{\rm ab}/L)\\
(K^{\rm ab})^H &\longleftarrow\!\shortmid\ H,
\end{aligned}$$

in which abelian extensions L/K correspond to finite index open subgroups of $\mathrm{Gal}(K^{\rm ab}/K)$; note that since $\mathrm{Gal}(K^{\rm ab}/K)$ is abelian, every subgroup of $\mathrm{Gal}(K^{\rm ab}/K)$ is normal and it follows that every subextension of $K^{\rm ab}/K$ is Galois (and abelian).

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When K is a nonarchimedean local field its abelian extensions are easy to understand; either $K = \mathbb{R}$, in which case $\mathbb{C}$ is its only nontrivial abelian extension, or $K = \mathbb{C}$ and there are no nontrivial abelian extensions. Now suppose K is a nonarchimedean local field with ring of integers $\mathcal{O}_K$, maximal ideal $\mathfrak{p}$, and residue field $\mathbb{F}_\mathfrak{p} := \mathcal{O}_K/\mathfrak{p}$. If L/K is a finite unramified extension, where L has residue field $\mathbb{F}_\mathfrak{q} := \mathcal{O}_L/\mathfrak{q}$, then Theorem 10.1 gives us a canonical isomorphism

$$\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p}) = \langle x \mapsto x^{\#\mathbb{F}_\mathfrak{p}} \rangle,$$

between the Galois group of L/K and the Galois group of the residue field extension $\mathbb{F}_\mathfrak{q}/\mathbb{F}_\mathfrak{p}$, which is the cyclic group generated by the Frobenius automorphism $x \to x^{\#\mathbb{F}_\mathfrak{p}}$. We henceforth use $\mathrm{Frob}_{L/K} \in \mathrm{Gal}(L/K)$ to denote the inverse image of the Frobenius automorphism under this isomorphism (note that this only makes sense when L/K is unramified).

24.1 Local Artin reciprocity

Local class field theory is based on the existence of a continuous homomorphism

$$\theta_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

known as the *local Artin homomorphism* (or *local reciprocity map*), as defined by the following theorem (which we will not prove but would like to understand).

Theorem 24.2 (Local Artin Reciprocity)**.** *Let K be a local field. There is a unique continuous homomorphism*

$$\theta_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

with the property that for each finite extension L/K in K^{ab} the homomorphism

$$\theta_{L/K} \colon K^\times \to \mathrm{Gal}(L/K)$$

obtained by composing θ_K with quotient map $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathrm{Gal}(L/K)$ satisfies:

- *if K is nonarchimedean and L/K is unramified then $\theta_{L/K}(\pi) = \mathrm{Frob}_{L/K}$ for every uniformizer π of $\mathcal{O}_K$;*
- *$\theta_{L/K}$ is surjective with kernel $\mathrm{N}_{L/K}(L^\times)$, inducing $K^\times/\mathrm{N}_{L/K}(L^\times) \simeq \mathrm{Gal}(L/K)$.*

It is worth contrasting the simplicity of local Artin reciprocity with the more complicated global version that we saw in Lecture 21, involving quotients of a ray class group $\mathrm{Cl}_K^\mathfrak{m}$ of modulus $\mathfrak{m}$ by a Takagi group $T_{L/K}^\mathfrak{m} := \mathcal{P}_K^\mathfrak{m}\mathrm{N}_{L/K}(\mathcal{I}_L^\mathfrak{m})$:

- there is no modulus $\mathfrak{m}$ to worry about (not even a power of $\mathfrak{p}$); working in K^{ab} lets us treat all class fields at once;
- there are no class groups involved; the class group of a local field is necessarily trivial (since $\mathcal{O}_K$ is a PID), so we instead just take quotients of $K^\times$;
- the Takagi group is replaced by the *norm group* $\mathrm{N}_{L/K}(L^\times) \subseteq K^\times$.
- we have the topology of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ and $K^\times$ to work with; this simplifies both the statement and proofs of local class field theory.
- the local Artin homomorphism is not an isomorphism, but after taking profinite completions it becomes one; this will be discussed below.

24.2 Norm groups

Definition 24.3. Let K be a local field. A *norm group* (of K) is a subgroup of the form

$$\mathrm{N}(L^\times) := \mathrm{N}_{L/K}(L^\times) \subseteq K^\times,$$

for some finite abelian extension L/K.

Remark 24.4. In fact, if L/K is any finite extension (not necessarily abelian, not necessarily Galois), then $\mathrm{N}(L^\times) = \mathrm{N}(F^\times)$, where F is the maximal abelian extension of K in L; this is the Norm Limitation Theorem (see [1, Theorem III.3.5]). So we could have defined norm groups more generally. This is not relevant to classifying the abelian extension of K, but it demonstrates a key limitation of local class field theory (which extends to global class field theory): subgroups of $K^\times$ contain no information about nonabelian extensions of K.

Local Artin reciprocity tells us that in order to understand the finite abelian extensions of a local field K, we just need to understand its norm groups. In fact, Theorem 24.2 already tells us a quite a lot: in particular, the isomorphism $K^\times/\mathrm{N}(L^\times) \simeq \mathrm{Gal}(L/K)$ implies that $[K^\times : \mathrm{N}(L^\times)] = [L : K]$ is finite. Moreover, there is an order-reversing isomorphism between the lattice of norm groups in $K^\times$ and the lattice of finite abelian extensions of K; this is essentially the Galois correspondence with Galois groups replaced by norm groups.

Corollary 24.5. *The map $L \mapsto \mathrm{N}(L^\times)$ defines an inclusion reversing bijection between the finite abelian extensions L/K in K^{ab} and the norm groups in $K^\times$ that satisfies*

$$(a)\ \mathrm{N}((L_1L_2)^\times) = \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times) \qquad \textit{and} \qquad (b)\ \mathrm{N}((L_1 \cap L_2)^\times) = \mathrm{N}(L_1^\times)\mathrm{N}(L_2^\times).$$

Moreover, every subgroup of $K^\times$ that contains a norm group is a norm group.

Here we write L_1L_2 for the compositum of L_1 and L_2 inside K^{ab}.

Proof. We first note that if $L_1 \subseteq L_2$ are two extensions of K then transitivity of the field norm (Corollary 4.46) implies

$$\mathrm{N}_{L_2/K} = \mathrm{N}_{L_1/K} \circ \mathrm{N}_{L_2/L_1}$$

and therefore $\mathrm{N}(L_2^\times) \subseteq \mathrm{N}(L_1^\times)$; thus the map $L \mapsto \mathrm{N}(L^\times)$ reverses inclusions.

This immediately implies $\mathrm{N}((L_1L_2)^\times) \subseteq \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times)$, since $L_1, L_2 \subseteq L_1L_2$. For the reverse inclusion, we note that every $x \in \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times)$ lies in the kernel of $\theta_{L_1/K}$ and $\theta_{L_2/K}$ (by Theorem 24.2), hence in the kernel of $\theta_{L_1L_2/K}$ (because $\mathrm{Gal}(L_1L_2/K) \hookrightarrow \mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K)$), and is therefore an element of $\mathrm{N}((L_1L_2)^\times)$ (by Theorem 24.2 again). This proves (a).

We now show that that $L \mapsto \mathrm{N}(L^\times)$ is a bijection; it is surjective by definition, so we just need to show it is injective. If $\mathrm{N}(L_2^\times) \subseteq \mathrm{N}(L_1^\times)$ then by (a) we have

$$\mathrm{N}((L_1L_2)^\times) = \mathrm{N}(L_1^\times) \cap \mathrm{N}(L_2^\times) = \mathrm{N}(L_2^\times),$$

and Theorem 24.2 then implies $\mathrm{Gal}(L_1L_2/K) \simeq \mathrm{Gal}(L_2/K)$, and therefore $L_1 \subseteq L_2$. If $\mathrm{N}(L_1^\times) = \mathrm{N}(L_2^\times)$ we can apply this argument in both directions to obtain $L_1 = L_2$, which shows that the map $L \mapsto \mathrm{N}(L^\times)$ is injective.

We next prove that every subgroup of $K^\times$ that contains a norm group is a norm group. Suppose $I = \mathrm{N}(L^\times)$ is a norm group lying in a subgroup $H \subseteq K^\times$, for some finite abelian extension L/K, and let $F := L^{\theta_{L/K}(H)}$. Theorem 24.2 implies that the diagram

$$\begin{array}{ccc} K^\times & \xrightarrow{\theta_{L/K}} & \mathrm{Gal}(L/K) \\ \Big\downarrow \| & & \Big\downarrow \mathrm{res} \\ K^\times & \xrightarrow{\theta_{F/K}} & \mathrm{Gal}(F/K) \end{array}$$

commutes, and we note that $\mathrm{Gal}(L/F) = \theta_{L/K}(H)$ is precisely the kernel of the map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(F/K) \simeq \mathrm{Gal}(L/K)/\mathrm{Gal}(L/F)$ induced by restriction. It follows that

$$H = \theta_{L/K}^{-1}(\mathrm{Gal}(L/F)) = \mathrm{N}(F^\times)$$

is a norm group as claimed.

It remains only to prove (b). The field $L_1 \cap L_2$ is the largest extension of K that lies in both L_1 and L_2, and it follows from the inclusion reversion bijection $L \mapsto \mathrm{N}(L^\times)$ that $\mathrm{N}(L_1^\times)\mathrm{N}(L_2^\times)$ is the smallest subgroup of $K^\times$ containing both $\mathrm{N}(L_1^\times)$ and $\mathrm{N}(L_2^\times)$, and it must be a norm group $\mathrm{N}(F^\times)$, since it contains a norm group. The field F must be the largest field that lies in both L_1 and L_2, hence $F = L_1 \cap L_2$. $\square$

Lemma 24.6. *Let L/K be an extension of local fields. If $\mathrm{N}(L^\times)$ has finite index in $K^\times$ then it is open.*

Proof. The lemma is clear if L and K are archimedean, so we assume otherwise. Suppose $[K^\times : \mathrm{N}(L^\times)] < \infty$. The unit group $\mathcal{O}_L^\times$ is compact, so $\mathrm{N}(\mathcal{O}_L^\times)$ is compact and therefore closed in the Hausdorff space $K^\times$. For any $\alpha \in L$ we have

$$\alpha \in \mathcal{O}_L^\times \iff |\alpha| = 1 \iff |\mathrm{N}_{L/K}(\alpha)| = 1 \iff \alpha \in \mathcal{O}_K^\times,$$

and therefore

$$\mathrm{N}(\mathcal{O}_L^\times) = \mathrm{N}(L^\times) \cap \mathcal{O}_K^\times.$$

It follows that $\mathrm{N}(\mathcal{O}_L^\times)$ is equal to the kernel of the map $\mathcal{O}_K^\times \hookrightarrow K^\times \twoheadrightarrow K^\times/\mathrm{N}(L^\times)$ and therefore $[\mathcal{O}_K^\times : \mathrm{N}(\mathcal{O}_K^\times)] \le [K^\times : \mathrm{N}(L^\times)] < \infty$. Thus $\mathrm{N}(\mathcal{O}_L^\times)$ is a closed subgroup of finite index in $\mathcal{O}_K^\times$, hence open (its complement is a finite union of closed cosets, hence closed), and $\mathcal{O}_K^\times$ is open in $K^\times$, so $\mathrm{N}(\mathcal{O}_L^\times)$ is open in $K^\times$.[1] $\square$

Remark 24.7. If K is a local field of characteristic zero then one can show that in fact every finite index subgroup of K is open, but this is not true in positive characteristic.

24.3 The main theorems of local class field theory

It follows from local Artin reciprocity that all norm groups have finite index; Lemma 24.6 then implies that all norm groups are finite index open subgroups of $K^\times$. The other key result of local class field theory states is that the converse also holds.

Theorem 24.8 (Local Existence Theorem)**.** *Let K be a local field. For every finite index open subgroup H of $K^\times$ there is a unique finite abelian extension L/K inside K^{ab} for which $H = \mathrm{N}_{L/K}(L^\times)$.*

The local Artin homomorphism $\theta_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is not an isomorphism; indeed, it cannot be, because $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is compact but $K^\times$ is not. However, the local existence theorem implies that after taking profinite completions it becomes one. We can thus summarize all of local class field theory in the following theorem.

[1] Recall that in a nonarchimedean local field, $|K^\times|$ is discrete in $\mathbb{R}_{>0}$ and we can always pick $\epsilon > 0$ so that $\mathcal{O}_K^\times = \{x \in K^\times : 1-\epsilon < |x| < 1+\epsilon\}$, which is clearly open in the metric topology induced by $|\ |$.

Theorem 24.9 (Main Theorem of Local Class Field Theory). *Let K be a local field. The local Artin homomorphism induces a canonical isomorphism*

$$\widehat{\theta}_K\colon \widehat{K^\times} \xrightarrow{\sim} \operatorname{Gal}(K^{\mathrm{ab}}/K)$$

of profinite groups.

Proof. We first note that $\operatorname{Gal}(K^{\mathrm{ab}}/K)$ is a profinite group under the Krull topology, isomorphic to the inverse limit

$$\operatorname{Gal}(K^{\mathrm{ab}}/K) \simeq \varprojlim_L \operatorname{Gal}(L/K), \tag{1}$$

where L ranges over the finite abelian extensions of K (in K^{sep}); see Theorem 23.22. It follows from Lemma 24.6, the local existence theorem (Theorem 24.8), and the definition of the profinite completion, that

$$\widehat{K^\times} \simeq \varprojlim_L K^\times/\mathrm{N}(L^\times), \tag{2}$$

where L ranges over finite abelian extensions of K (in K^{sep}). By local Artin reciprocity (Theorem 24.2), for every finite abelian extension L/K we have an isomorphism

$$\theta_{L/K}\colon K^\times/\mathrm{N}(L^\times) \xrightarrow{\sim} \operatorname{Gal}(L/K),$$

and these isomorphisms commute with the inclusions among the finite abelian extensions of K. We thus have an isomorphism of the inverse systems appearing in (1) and (2); the isomorphism is canonical because the Artin map is unique and the isomorphisms in (1) and (2) are both canonical. $\square$

In view of Theorem 24.9, we would like to better understand the profinite group $\widehat{K^\times}$. If K is archimedean then $\widehat{K^\times}$ is either trivial or the group of order 2, so let us assume that K is nonarchimedean. If we pick a uniformizer π for the maximal ideal $\mathfrak{p} \in \mathcal{O}_K$, then we can uniquely write each $x \in K^\times$ in the form $u\pi^{v(x)}$, with $u \in \mathcal{O}_K^\times$ and $v(x) \in \mathbb{Z}$, and this defines an isomorphism

$$\begin{aligned} K^\times &\xrightarrow{\sim} \mathcal{O}_K^\times \times \mathbb{Z} \\ x &\longmapsto (x/\pi^{v(x)}, v(x)). \end{aligned}$$

Taking profinite completions (which commutes with products), we obtain an isomorphism

$$\widehat{K^\times} \simeq \mathcal{O}_K^\times \times \widehat{\mathbb{Z}},$$

since the unit group

$$\mathcal{O}_K^\times \simeq \mathbb{F}_\mathfrak{p}^\times \times \mathcal{O}_K \simeq \mathbb{F}_\mathfrak{p}^\times \times \varprojlim_n \mathcal{O}_K/\mathfrak{p}^n$$

is already profinite (hence isomorphic to its profinite completion, by Corollary 23.19). Note that the isomorphism $\widehat{K^\times} \simeq \mathcal{O}_K^\times \times \widehat{\mathbb{Z}}$ is *not* canonical; it depends on our choice of π, and there are uncountably many π to choose from.

Taking profinite completions gives a canonical commutative diagram of exact sequences of topological groups:

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}^\times \arrow[r] \arrow[d, "\wr"] & K^\times \arrow[r, "v"] \arrow[d, hook, "\theta_K"] & \mathbb{Z} \arrow[r] \arrow[d, hook, "\phi"] & 0 \\
1 \arrow[r] & \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) \arrow[r] & \mathrm{Gal}(K^{\mathrm{ab}}/K) \arrow[r, "\mathrm{res}"] & \mathrm{Gal}(K^{\mathrm{unr}}/K) \arrow[r] & 1
\end{tikzcd}$$

in which the map ϕ on the right is given by

$$\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}} \simeq \mathrm{Gal}(\overline{\mathbb{F}}_{\mathfrak{p}}/\mathbb{F}_{\mathfrak{p}}) \simeq \mathrm{Gal}(K^{\mathrm{unr}}/K)$$

and sends 1 to the sequence of Frobenius elements $(\mathrm{Frob}_{L/K})$ in the profinite group

$$\mathrm{Gal}(K^{\mathrm{unr}}/K) \simeq \varprojlim_{L} \mathrm{Gal}(L/K) \subseteq \prod_{L} \mathrm{Gal}(L/K),$$

where L ranges over finite unramified extensions of K; here we are using the canonical isomorphisms $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(\mathbb{F}_{\mathfrak{q}}/\mathbb{F}_{\mathfrak{p}})$ given by Theorem 10.1.

The group $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) \simeq \mathcal{O}^\times$ corresponds to the inertia subgroup of $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}})$. The top sequence splits (but not canonically), hence so does the bottom, and we have

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) \simeq \mathrm{Gal}(K^{\mathrm{unr}}/K) \times \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}}) \simeq \mathcal{O}^\times \times \widehat{\mathbb{Z}}.$$

The unit group $\mathcal{O}^\times$ contains the descending chain of multiplicative subgroups

$$\mathcal{O}^\times \supseteq 1 + \mathfrak{p} \supseteq 1 + \mathfrak{p}^2 \supseteq 1 + \mathfrak{p}^3 \supseteq \cdots$$

which are mapped isomorphically to a descending chain of *higher ramification groups* in the inertia group $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{unr}})$.[2]

For each choice of a uniformizer $\pi \in \mathcal{O}_K$ we get a decomposition $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$ corresponding to $K^\times = \mathcal{O}_K \pi^{\mathbb{Z}}$. The field K_π is the subfield of K^{ab} fixed by $\theta_K(\pi) \in \mathrm{Gal}(K^{\mathrm{ab}}/K)$. Equivalently,

$$K_\pi = \bigcup_{\substack{\text{finite abelian} \\ \text{totally ramified} L/K \\ \text{for which } \pi \in \mathrm{N}(L^\times)}} L.$$

Example 24.10. Let $K = \mathbb{Q}_p$ and pick $\pi = p$. The decomposition $K = K_\pi K^{\mathrm{unr}}$ is

$$\mathbb{Q}_p^{\mathrm{ab}} = \bigcup_{n} \mathbb{Q}_p(\zeta_{p^n}) \cdot \bigcup_{m \perp p} \mathbb{Q}_p(\zeta_m),$$

where the first union on the RHS is fixed by $\theta_K(p)$ and the second is fixed by $\theta_K(\mathcal{O}_K^\times)$.

Constructing the local Artin homomorphism is the difficult part of local class field theory and we will not prove it in this course (but see 18.786). However, assuming the local existence theorem, it is easy to show that, if it exists, the local Artin homomorphism is unique.

Proposition 24.11. *Let K be a local field and assume every finite index open subgroup of $K^\times$ is a norm group. There is at most one homomorphism $\theta\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ of topological groups that has the properties given in Theorem 24.2.*

[2]This sequence corresponds to the "upper numbering" of the ramification groups in [2, Ch. IV].

Proof. Let $\mathfrak{p} = \langle \pi \rangle$ be the maximal ideal of $\mathcal{O}_K$, and for each integer $n \geq 0$ let $K_{\pi,n}/K$ be the finite abelian extension for which $\mathrm{N}(K_{\pi,n}^\times)$ is the norm group corresponding to the finite index subgroup $(1+\mathfrak{p}^n)\langle\pi\rangle$ of $K^\times \simeq \mathcal{O}_K^\times\langle\pi\rangle$. Suppose $\theta\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is a continuous homomorphism as in Theorem 24.2. Then $K_\pi = \cup K_{\pi,n}$, and $\theta(\pi)$ fixes K_π, since $\pi \in \mathrm{N}(K_{\pi,n}) = \ker\theta_{K_{\pi,n}/K}$ for all $n \geq 0$. We also know that $\theta_{L/K}(\pi) = \mathrm{Frob}_{L/K}$ for all finite unramified extensions L/K, which uniquely determines the action of $\theta(\pi)$ on K^{unr}, and hence on $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$.

Now suppose $\theta'\colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is another continuous homomorphism satisfying the properties in Theorem 24.2. By the argument above we must have $\theta'(\pi) = \theta(\pi)$ for every uniformizer π of $\mathcal{O}_K$, and $K^\times$ is generated by its subset of uniformizers: if we fix one uniformizer π, every $x \in K^\times$ can be written as $u\pi^n = (u\pi)\pi^{n-1}$ for some $u \in \mathcal{O}_K^\times$ and $n \in \mathbb{Z}$, and $u\pi$ is another uniformizer). It follows that $\theta(x) = \theta'(x)$ for all $x \in K^\times$ and therefore $\theta = \theta'$ is unique. □

Remark 24.12. One approach to proving local class field theory uses the theory of formal groups due to Lubin and Tate to explicitly construct the fields $K_\pi = \cup K_{\pi,n}$ appearing in the proof of Proposition 24.11, along with a continuous homomorphism $\theta_\pi\colon \mathcal{O}_K^\times \to \mathrm{Gal}(K_\pi/K)$ that extends uniquely to a continuous homomorphism $\theta\colon K^\times \to \mathrm{Gal}(K_\pi K^{\mathrm{unr}}/K)$. One then shows that $K^{\mathrm{ab}} = K_\pi K^{\mathrm{unr}}$ (using the Hasse-Arf Theorem), and that θ does not depend on the choice of π; see [1, §I.2-4] for details.

24.4 Finite abelian extensions

Local class field theory gives us canonical bijections between the following sets:

(1) finite-index open subgroups of $K^\times$ (all of which are necessarily normal);

(2) open subgroups of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ (which are necessarily normal and of finite index);

(3) finite abelian extensions of K in K^{sep} (which necessarily lie in K^{ab}).

The bijection from (1) to (2) is induced by the isomorphism $\widehat{K^\times} \simeq \mathrm{Gal}(K^{\mathrm{ab}}/K)$ of Theorem 24.9 and is inclusion preserving. The bijection from (2) to (3) follows from Galois theory (for infinite extensions), and is inclusion reversing, while the bijection from (3) to (1) is via the map $L \mapsto \mathrm{N}(L^\times)$, which is also inclusion reversing.

Example 24.13. Let p be an odd prime. What are the $\mathbb{Z}/p\mathbb{Z}$-extensions of $\mathbb{Q}_p$? That is, what are the fields $L/\mathbb{Q}_p$ for which $\mathrm{Gal}(L/\mathbb{Q}_p) \simeq \mathbb{Z}/p\mathbb{Z}$? We have

$$\begin{aligned}\mathbb{Q}_p^\times &\simeq \mathbb{Z}_p^\times \times p^{\mathbb{Z}}\\ &\simeq (1+p\mathbb{Z}_p) \times \mathbb{F}_p^\times \times \mathbb{Z}\\ &\simeq \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}.\end{aligned}$$

Taking quotients by p-th powers kills the middle factor, thus

$$\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times p} \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z},$$

and $\mathbb{Q}_p^{\times p}$ has index p^2 in $\mathbb{Q}_p^\times$. There are $p+1$ subgroups of $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$, and each corresponds to an open subgroup of $\mathbb{Q}_p^\times$ of index p. Under the bijection above, these correspond to $p+1$ extensions L/K with $\mathrm{Gal}(L/K) \simeq \mathbb{Z}/p\mathbb{Z}$, exactly one of which is unramified (by Theorem 10.1).

References

[1] J.S. Milne, *Class field theory*, version 4.02, 2013.

[2] Jean-Pierre Serre, *Local fields*, Springer, 1979.

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