Multi-linear Algebra
Notes for 18.101

1 Linear algebra

To read these notes you will need some background in linear algebra. In particular you'll need to be familiar with the material in § 1–2 of Munkres and § 1 of Spivak. In this section we will discuss a couple of items which are frequently, but not always, covered in linear algebra courses, but which we'll need for our "crash course" in multilinear algebra in § 2–6.

The quotient spaces of a vector space

Let V be a vector space and W a vector subspace of V. A W-*coset* is a set of the form

$$v + W = \{v + w\,,\; w \in W\}\,.$$

It is easy to check that if $v_1 - v_2 \in W$, the cosets, $v_1 + W$ and $v_2 + W$, coincide while if $v_1 - v_2 \notin W$, they are disjoint. Thus the W-cosets decompose V into a *disjoint* collection of subsets of V. We will denote this collection of sets by V/W.

One defines a vector addition operation on V/W by defining the sum of two cosets, $v_1 + W$ and $v_2 + W$ to be the coset

$$v_1 + v_2 + W \tag{1.1}$$

and one defines a scalar multiplication operation by defining the scalar multiple of $v + W$ by λ to be the coset

$$\lambda v + W\,. \tag{1.2}$$

It is easy to see that these operations are well defined. For instance, suppose $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$. Then $v_1 - v_1'$ and $v_2 - v_2'$ are in W; so $(v_1 + v_2) - (v_1' + v_2')$ is in W and hence $v_1 + v_2 + W = v_1' + v_2' + W$.

These operations make V/W into a vector space, and one calls this space the *quotient space* of V by W.

We define a mapping

$$\pi : V \to V/W \tag{1.3}$$

by setting $\pi(v) = v + W$. It's clear from (1.1) and (1.2) that π is a linear mapping. Moreover, for every coset, $v + W$, $\pi(v) = v + W$; so the mapping, π, is onto. Also note that the zero vector in the vector space, V/W, is the zero coset, $0 + W = W$. Hence v is in the kernel of π if $v + W = W$, i.e., $v \in W$. In other words the kernel of π is W.

In the definition above, V and W don't have to be finite dimensional, but if they are, then one can show

$$\dim V/W = \dim V - \dim W\,. \tag{1.4}$$

(A proof of this is sketched in exercises 1–3.)

The dual space of a vector space

We'll denote by V^* the set of all linear functions, $\ell : V \to \mathbb{R}$. If ℓ_1 and ℓ_2 are linear functions, their sum, $\ell_1 + \ell_2$, is linear, and if ℓ is a linear function and λ is a real number, the function, $\lambda\ell$, is linear. Hence V^* is a vector space. One calls this space the *dual space* of V.

Suppose V is n-dimensional, and let $e_1, \ldots, e_n$ be a basis of V. Then every vector, $v \in V$, can be written uniquely as a sum

$$v = c_1 v_1 + \cdots + c_n v_n \quad c_i \in \mathbb{R}\,.$$

Let

$$e_i^*(v) = c_i\,. \tag{1.5}$$

If $v = c_1 e_1 + \cdots + c_n e_n$ and $v' = c_1' e_1 + \cdots + c_n' e_n$ then $v + v' = (c_1 + c_1')e_1 + \cdots + (c_n + c_n')e_n$, so

$$e_i^*(v + v') = c_i + c_i' = e_i^*(v) + e_i^*(v')\,.$$

This shows that $e_i^*(v)$ is a linear function of v and hence $e_i^* \in V^*$.

Claim: $e_i^*(v)$ is a linear function of v and hence $e_i^* \in V^*$.

Proof. First of all note that by (1.5)

$$e_i^*(e_j) = \begin{cases} 1\,, & i = j \\ 0\,, & i \neq j \end{cases}\,. \tag{1.6}$$

If $\ell \in V^*$ let $\lambda_i = \ell(e_i)$ and let $\ell' = \sum \lambda_i e_i^*$. Then by (1.6)

$$\ell'(e_j) = \sum \lambda_i e_i^*(e_j) = \lambda_j = \ell(e_j)\,, \tag{1.7}$$

i.e., ℓ and ℓ' take identical values on the basis vectors, e_j. Hence $\ell = \ell'$.

Suppose next that $\sum \lambda_i e_i^* = 0$. Then by (1.6), with $\ell' = 0$; $\lambda_j = 0$. Hence the e_j^*'s are linearly independent.

□

Let V and W be vector spaces and

$$A : V \to W \tag{1.8}$$

a linear map. Given $\ell \in W^*$ the composition, $\ell \circ A$, of A with the linear map, $\ell : W \to \mathbb{R}$, is linear, and hence is an element of V^*. We will denote this element by $A^*\ell$, and we will denote by

$$A^* : W^* \to V^*$$

the map, $\ell \to A^*\ell$. It's clear from the definition that

$$A^*(\ell_1 + \ell_2) = A^*\ell_1 + A^*\ell_2$$

and that

$$A^*\lambda\ell = \lambda A^*\ell\,,$$

i.e., that A^* is linear.

Definition. A^* is the transpose of the mapping A.

We will conclude this section by giving a matrix description of A^*. Let $e_1, \dots, e_n$ be a basis of V and $f_1, \dots, f_m$ a basis of W; let $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_m^*$ be the dual bases of V^* and W^*. Suppose A is defined in terms of $e_1, \dots, e_n$ and $f_1, \dots, f_m$ by the $m \times n$ matrix, $[a_{i,j}]$, i.e., suppose

$$Ae_j = \sum a_{i,j} f_i\,.$$

Claim. A^* is defined, in terms of $f_1^*, \dots, f_m^*$ and $e_1^*, \dots, e_n^*$ by the transpose matrix, $[a_{j,i}]$.

Proof. Let

$$A^* f_i^* = \sum c_{j,i} e_j^*\,.$$

Then

$$A^* f_i^*(e_j) = \sum_k c_{k,i} e_k^*(e_j) = c_{j,i}$$

by (1.5). On the other hand

$$\begin{aligned} A^* f_i^*(e_j) &= f_i^*(Ae_j) \\ &= \sum_k a_{j,k} f_i^*(f_k) = a_{j,i}\,, \end{aligned}$$

so $a_{j,i} = c_{j,i}$.

□

Exercises.

1. Let V be an n-dimensional vector space and W a k-dimensional subspace. Show that there exists a basis, $e_1, \ldots, e_n$ of V with the property that $e_1, \ldots, e_k$ is a basis of W. *Hint:* Induction on $n-k$. To start the induction suppose that $n-k=1$. Let $e_1, \ldots, e_{n-1}$ be a basis of W and e_n any vector in $V-W$.

2. In exercise 1 show that the vector $f_i = \pi(e_{k+i})$, $i = 1, \ldots, n-k$ are a basis of V/W. Conclude that the dimension of V/W is $n-k$.

3. In exercise 1 let U be the linear span of the vectors, e_{k+i}, $i = 1, \ldots, n-k$.

   Show that the map

   $$U \to V/W\,, \quad u \to \pi(u)\,, \tag{1.9}$$

   is a vector space isomorphism, i.e., show that it maps U bijectively onto V/W.[1]

4. Let U, V and W be vertex spaces and let $A : V \to W$ and $B : U \to V$ be linear mappings. Show that $(AB)^* = B^*A^*$.

[1]This exercise shows that the notion of "quotient space", which can be somewhat daunting when one first encounters it, is in essence no more complicated than the notion of "subspace".