Some preparatory problems for the exam

* denotes a bit more dificult problem.

1 What are conjugation classes in $\mathbb{H}^* = \mathbb{H} - \{0\}$?

2 * Show that $SO(4)$ and $(SU(2) \times SU(2))/\mathbb{Z}_2$ are isomorphic as Lie groups. (Hint: use quaternions)

3 Give an example of three nonisomorphic Lie algebras of the dimension three.

4 Let N be the group of upper-triangular matrices with 1's at the diagonal. Is the exp onto?

5 * Let G be a connected compact Lie group. Show that exp for G is onto.

6 * Show by using only elementary tools (eg. not invoking weights, Schur functors etc.) that $\Lambda^2(\mathbb{K}^n)$ is an irreducible representation of $GL_n(\mathbb{K})$ for any field $\mathbb{K}$.

7 * Do the same for $Sym^2(\mathbb{K}^n)$ provided that $char(\mathbb{K}) \neq 2$. Find a nontrivial subrepresentation in $Sym^2((\mathbb{Z}_2)^2)$.

8 In the Lie algebra $End(\mathbb{C}[[x]])$ we have the element x (multiplication by x) and $\frac{\partial}{\partial x}$. Compute $exp^{-1}(exp(x)exp(\frac{\partial}{\partial x}))$ in terms of the commutator.

9 What is the Killing form for the Lie algebra of 2×2 upper-triangular matrices?

10 Decompose $gl_2(\mathbb{C}) = End(\mathbb{C}^2)$ into orthogonal sum of spaces on which the Killing form is either positive definite or negative definite or zero.

11 For any representation V of a complex reductive Lie group relate V/G to V^G. (Here $V/G = V/W$ where W is the space spanned by the vectors $gv - v$ for $g \in G$.) Give a counterexample when G is not reductive.

12 Let G be a compact connected Lie group. Is it true that any abelian subgroup of G is contained in a maximal torus?

13 Let V be any representation of $SL_2(\mathbb{C})$, and let d_i denote the dimension of the space of the weight i (with respect to some choice of the maximal torus). Show that for nonnegative i we have $d_i \geq d_{i+2}$.

14 Consider the representation of $SL_3(\mathbb{C})$: $\bigwedge^2 \mathbb{C}^3 \otimes Sym^2\mathbb{C}^3$. Decompose it into irreducible components.

15 Consider the group $PSL_3(\mathbb{C}) = GL_3(\mathbb{C})/center$. Which Young diagrams correspond to the representations of $PSL_3(\mathbb{C})$.

16 From the Weyl character formula for $SL_3(\mathbb{C})$ compute the dimensions of the weight spaces of the irreducible representation associated to the partition (4,2,0).

17 Decompose into irreducible components the representation of $SL_5(\mathbb{C})$ which is the tensor product of $\bigwedge^4 \mathbb{C}^5 \otimes Sym^3(\mathbb{C}^5)$.

18 Decompose into irreducible components the representation of $SL_4(\mathbb{C})$ which is the tensor product of $\bigwedge^3 \mathbb{C}^4 \otimes sl_4(\mathbb{C})$.

19* Show that $K_{\lambda\mu} > 0$ ($K_{\lambda\mu}$ stands for the Kostka number) iff λ dominates μ (ie. for all i, $\sum_{j<i} \mu_j \leq \sum_j \lambda_j$).

20 Let λ, μ, ν be Young diagrams with at most two rows. Show that $N_{\lambda\mu\nu} \leq 1$ ($N_{\lambda\mu\nu}$ is a multiplicity of S_ν in $S_\lambda \otimes S_\mu$).

21* Derive from the formula

$$\dim(\Gamma_\lambda) = \Pi_{\alpha \in R_+} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}$$

explicit formulas for dimensions of irreducible representations of $sl_n(\mathbb{C})$, $so_n(\mathbb{C})$.

22* Show that $\det[x_j^{n+1-i} - x_j^{-(n+1-i)}] = \mathrm{VDM}(x_1 + x_1^{-1}, \ldots, x_n + x_n^{-1})\Pi_i(x_i - x_i^{-1})$.

23 Let $G \to H$ be a map of connected Lie groups. Is it possible define a map of Weyl groups?

24 Show that $N_{SL_2(\mathbb{C})}(\mathbb{C}^*)$ is not a semidirect product of $\mathbb{C}^*$ and $\mathbb{Z}_2$.

25 Suppose that a map of reductive Lie groups $f : G \to H$ is an isomorphism on maximal tori. Show that f is mono. Does it have to be epi? Can one get rid of the assumption of the reductivity?

26 Let $V = W \oplus W^*$ with the usual hyperbolic quadratic form Q. Let I be the 2-sided ideal generated by W. Describe I and $C(Q)/I$ as $Spin(2n)$ representations.

27 Let $G = Sp(2)$ with the natural representation in $\mathbb{C}^4$. Compute the weights for $\bigwedge^2(\bigwedge^2 \mathbb{C}^4)$ and decompose it into irreducible representations.

28 Let $G = Sp(2)$ with the natural representation in $\mathbb{C}^4$. Find at least two non proportional vectors killed by all the positive roots in $\bigwedge^2(Sym^2\mathbb{C}^4)$.

29 Let $G = SO(4)$ with the natural representation in $\mathbb{R}^4$. Consider the representation $\bigwedge^2 \mathbb{R}^4$. Is it irreducible? And what about the complexification?

30 Compute (without using root and weight lattices) centers of $SO_n(\mathbb{C})$ and $Spin_n(\mathbb{C})$ (Hint: use the form $-I$ for $Spin_n(\mathbb{C})$).

31 Compute Γ_W/Γ_R for so_n (thus computing $C(Spin_n(\mathbb{C}))$ again), and find the lattice of weights of complex representations of $SO_n(\mathbb{C})$.

32* Let G be the universal covering of $SL_2(\mathbb{R})$. Is it true that exp for G is $1-1$?

33* Prove that there is no faithful real representation of the universal covering of $SL_2(\mathbb{R})$.