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TE AMSTERDAM

PROCEEDINGS OF THE SECTION OF SCIENCES

VOLUME XXXI

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(Translated from: Verslag van de gewone vergaderingen der Afdeeling Natuurkunde, Vols. XXXVI and XXXVII).

CONTENTS

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PROCEEDINGS

VOLUME XXXI

No. 1

President: Prof. F. A. F. C. WENT
Secretary: Prof. B. BROUWER

(Translated from: "Verslag van de gewone vergaderingen der Afdeeling Natuurkunde", Vols. XXXVI and XXXVII)

CONTENTS

Physics. — *The spectrum of ionized Neon (Ne II)*. First communication. By T. L. DE BRUIN. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of May 28, 1927).

Introduction.

The spectrum, originating from the neutral atom, is called the arc spectrum. The energy levels or terms are of the form:

$$\underset{(arc)}{Term} = \frac{R}{(n+\triangle)^2}$$

where R = the RYDBERG constant (= 109737) n = the principal quantum number and $\triangle$ the so called quantum defect caused by the core of the atom. The spark spectrum has its origin in the ionized atom, the atom that has lost one electron. Now the energy levels have the form:

$$\underset{(ion)}{Term} = \frac{4R}{(n+\triangle)^2}$$

The arc spectrum of Neon (indicated by the symbol *Ne I*) is of great complexity, but by the fundamental work of PASCHEN [1]) now it is one of the best known spectra. MERTON [2]) has found, first, that the *Neon* atom under condensed discharges can emit another spectrum than the arc spectrum. This spectrum lies mainly in the blue-violet region. Careful new measurements of this spectrum have been made by L. BLOCH, E. BLOCH and G. DÉJARDIN [3]). This material we have used for the analysis of the *Ne II* spectrum. The analysis of the *F I* spectrum, given in former papers [4]) and the theory of complex spectra developed by RUSSELL, SAUNDERS, PAULI and specially by HEISENBERG and HUND, could be preliminary steps for this analysis.

Analogy between the Ne II atom and the F I atom.

In former papers in these Proceedings, I have given an analysis of the spectrum emitted by the neutral Fluorine atom (*F I*). The analysis has been briefly discussed in relation in the HEISENBERG—HUND theory of complex spectra, and a correlation with the scheme of terms predicted by the theory has been established. In that paper we noted already that the

1) PASCHEN, Ann. d. Phys., **60**, 405, 1919; **63**, 201, 1920.

2) MERTON, Proc. R. S. London, *A*, **89**, 447, 1914.

3) L. BLOCH, E. BLOCH, G. DÉJARDIN, Journ. d. Phys., **7**, No. 5. 129, 1926.

4) T. L. DE BRUIN, Versl. Kon. Akad. Amsterdam, **35**, Juni 1926, Dec. 1926; Zeitsch. f. Phys., **39**, 869, 1926.

spectra of the ionized rare gases should have the same structure as the neutral halogen atoms. The *Ne II* spectrum should have the same structure as the *F I* spectrum. These atoms have the same configurations two 1_1, two 2_1 and five 2_2 electrons. In the case of *Fluorine I* the charge of the core is $+9$ in the case of *Neon II* $+10$.

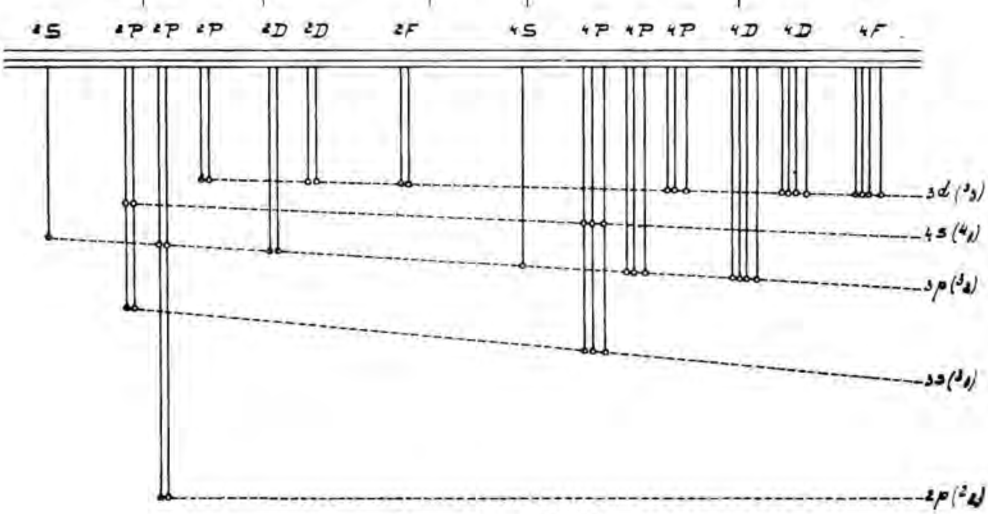
TABLE 1.

n_k	1_1	2_1	2_2	Nucleus
F I	2	2	5	+ 9
Ne II	2	2	5	+ 10

The theoretical termscheme for *Ne II*, deduced from the atom configuration, is quite the same as that for *F I* and the following terms are predicted by theory :

TABLE 2. Neon II (Deep terms).

Electron configuration								Basic term: ^3P	
								Terms	
n_k 1_1		2_1	2_2	3_1	3_2	3_3	4_1	Quartet	Doublet
2		2	5						P
2		2	4	1				P	P
2		2	4		1			D P S	D P S
2		2	4			1		F D P	F D P
2		2	4				1	P	P



Term scheme (*Ne II.*).

The ground term should be, as in the case of *F I*, a ^2P term with the term difference of the Röntgen *L*-doublet. This doublet we know for *Ne II* from the two series limits of the arc spectrum, which have the difference of 780 frequentie units. According to the formula of SOMMERFELD for the Röntgen *L*-doublet :

$$\triangle\nu_L = 0.365\,(Z - s)^4 \qquad Z = \text{atomic number}$$
$$s = \text{screening constant}$$

we have $\triangle\nu = 780$, $Z = 10$, $s = 3.20$. In the case of *F I* for $Z = 9$ and $s = 3.20$ we find the value $\triangle\nu = 412$. This value agrees satisfactory with the experimental value (407) detected from the spectrum [1]). The order of the ground doublet ^2P for *Ne II* will be : 33.4 Volt [2]) = ± 270.000, for *F I* approximately ± 16.7 Volt [3]) = ± 135.000.

The key for the analysis of the *F I* spectrum is given by the ^4P term with the term differences 160.0 and 274.6. For the analysis of the *Ne II* spectrum it was important to find the analogous term. In the case of *Ne II* this term has the differences 299.0 and 518.0. From this we can deduce the structure of the spectrum. The analoguous multiplets $^4(PP')$ are : (This multiplet in *F I* is found by me on basic of the ZEEMAN effects.)

TABLE 3.

Ne II	$^4P_1$	299.0	$^4P_2$	518.0	$^4P_3$
$^4P'_1$	(4) 3751.25 26650.2		(7) 3709.66 (*R*) [4]) 26949.0		—
182.5					
$^4P'_2$	(8) 3777.14 (*R?*) 26467.5		(7) 3734.94 26766.6		(9) 3664.05 (*R*) 27284.4
222.7					
$^4P'_3$	—		(8) 3766.28 (*R?*) 26543.8		(10) 3694.19 (*R*) 27061.8
F I	**$^4P_1$**	**159.9**	**$^4P_2$**	**274.6**	**$^4P_3$**
$^4P'_1$	(0) 7515.0 13303.0		(12) 7426.2 13462.1		—
102.1					
$^4P'_2$	(5) 7573.32 13200.6		(5) 7482.95 13360.0		(18) 7332.25 13634.5
122.7					
$^4P'_3$	—		(5) 7552.20 13237.5		(20) 7398.96 13511.7

1) BOWEN : Phys. Rev., Febr. 1927.

2) MOHLER : Science, April 1926.

3) Deduced from the termscheme.

4) R = reversed.

While this multiplet $^4(PP')$ in the case of *F I* lies far in the red, in the case of *Ne II* it lies far in the violet, as could be expected from the factor $4R$ in the expression for the term. The ratio for the ionisation potentials of *F I* and *Ne II* is approximately the same as for the frequentie numbers of the analogous multiplets above mentioned viz. 2.

Terms in the Ne II spectrum.

From the value of the groundterm ^2P in *Ne II* $\pm$ 33.4 Volt $= \pm$ 270000 we can deduce the order of the terms due to the binding in a 2_2 orbit to the ion. We have :

$$270.000 = \frac{4R}{n^2} \quad n = \text{effective quantum number}$$

with $n = 1.275$. The doublet terms arising from the binding of a 3_2 electron have approximately the value : $\frac{4R}{2.275^2} = \pm 82000$. The quartet terms arising from this binding lie somewhat deeper. (See figure.) The terms arising from the binding of a 3_3 electron have approximately the value : $\frac{4R}{3^2} = \pm 50.000$. Except the ground doublet we have also four groups of terms arising respectively from the binding of a 3_1, 3_2, 3_3 (4_1) electron and an electron with looser coupling.

In the *Ne II* spectrum I have found these four groups of terms. The deepest quartet terms are identified. The other terms are indicated with preliminary symbols. In another communication in these Proceedings we will give more details.

TABLE 4. Termtable *Ne II.*

Term-symbol	j	Term values (relative)	Term difference	Atom-configuration	Remarks
^4P	3	117000.0		$(s^2 p^4).3_1$ [1]	
			518.0		
	2	116482.0			
			299.0		
	1	116183.0			

[1]) The symbol $(s^2 p^4).3_1$ indicates that the atom configuration consists of the ion, having two s-electrons ($k = 1$) and four p-electrons ($k = 2$), and the binding of an 3_1 electron.

TABLE 4. Termtable *Ne II (Continued).*

Term-symbol	j	Term values (relative)	Term difference	Atom-configuration	Remarks
$^4P'$	3	89938.2	222.6	$(s^2 p^4).3_2$	
	2	89715.6	182.5		
	1	89533.1			
^4D	4	87022.7	337.8	$(s^2 p^4).3_2$	
	3	86684.9	249.7		
	2	86435.2	144.1		
	1	86291.1			
^4S	2	83178.0		$(s^2 p^4).3_2$	
T	2	81965.7			
$^4D'$	(4)	56994.0	81.8	$(s^2 p^4).3_3$	
	3	56912.2	106.0		
	2	56806.2	98.3		
	1	56707.9			
P_2	2(3)	55862.0			
xP	2	55657.4			
yP_3	3(1)	55333.7			
aP_2	2	55183.8			
bP_2	2	55141.4			
cP_3	3	55104.9			
dP_2	2	54959.4			
U	2	54411.5			
eP_3	3	54133.0			
fP_2	2	53755.6			
gP_1	1	53450.8			
nP_3	3	33287.2			
kP	(2,3)	33227.4			
mP_2	2	33141.4			
R_1	1	32621.4			
lP	(3)	32530.7			
oP_3	3	32306.2			
pP_3	3	23359.2			
qP_2	2	23327.5			
rP_1	1	23261.1			

TABLE 5.

	$^4P_{123} - {}^4P'_{123}$			
8	3777.14	26467.5	$^4P_1 - {}^4P'_2$	*R?*
8	3766.28	26543.8	$^4P_2 - {}^4P'_3$	*R?*
4	3751.25	26650.2	$^4P_1 - {}^4P'_1$	
7	3734.94	26766.6	$^4P_2 - {}^4P'_2$	
7	3709.66	26949.0	$^4P_2 - {}^4P'_1$	*R*
10	3694.19	27061.8	$^4P_3 - {}^4P'_3$	*R*
9	3664.05	27284.4	$^4P_3 - {}^4P'_2$	*R*
	$^4P_{123} - {}^4D_{1234}$			
2	3270.79	30564.9	$^4P_3 - {}^4D_2$	
5	3360.63	29747.8	$^4P_1 - {}^4D_2$	
7	3355.09	29796.9	$^4P_2 - {}^4D_3$	
5	3344.44	29941.8	$^4P_1 - {}^4D_1$	
10	3334.89	29977.3	$^4P_3 - {}^4D_4$	
5	3327.22	30046.5	$^4P_2 - {}^4D_2$	
4	3311.32	30190.8	$^4P_2 - {}^4D_1$	
8	3297.74	30315.1	$^4P_3 - {}^4D_3$	
	$^4P_{123} - S^4_2$			
4	3028.90	33005.7	$^4P_1 - {}^4S_2$	$^4P' - {}^4D'_1$
7	3001.72	33304.5	$^4P_2 - {}^4S_2$	
7	2955.77	33821.8	$^4P_3 - {}^4S_2$	
	$^4P'_{123} - {}^4D'_{123}$			
5 d.v.	3054.70	32726.9	$^4P'_1 - {}^4D'_2$	
6	3047.60	32803.2	$^4P'_2 - {}^4D'_3$	
3	3045.56	32825.1	$^4P'_1 - {}^4D'_1$	
3	3037.75	32909.5	$^4P'_2 - {}^4D'_2$	
5	3034.49	32944.9	$^4P'_3 - {}^4D'_4$	
4	3028.90	33005.7	$^4P'_2 - {}^4D'_1$	$^4P_1 - {}^4S_2$
4	3027.07	33025.6	$^4P'_2 - {}^4D'_1$	
3	3017.36	33131.9	$^4P'_3 - {}^4D'_2$	

TABLE 5 *(Continued)*.

	$^4D_{1234} - {^4D'}_{1234}$			
1	3390.56	29485.2	$^4D_1 - {^4D'}_2$	
1	3386.20	29523.2	$^4D_2 - {^4D'}_3$	
0	3379.34	29582.7	$^4D_1 - {^4D'}_1$	
2	3374.08	29629.2	$^4D_2 - {^4D'}_2$	
6	3367.25	29689.3	$^4D_3 - {^4D'}_4$	
1	3362.90	29727.7	$^4D_2 - {^4D'}_1$	
2	3357.89	29772.1	$^4D_3 - {^4D'}_3$	$^4S_2 - gP_1$
1	3345.88	29878.9	$^4D_3 - {^4D'}_2$	
4	3329.18	30028.8	$^4D_4 - {^4D'}_4$	
1	3320.28	30109.3	$^4D_4 - {^4D'}_3$	
1	2895.05	34531.8	${^4P'}_2 -$ } aP_2	(*Ne III ?*)
2	2910.44	34349.0	${^4P'}_1 -$ } aP_2	
3	3173.58	31501.1	$^4D_3 -$ } aP_2	
3	3213.77	31107.1	$^4D_1 -$ } aP_2	
3	3571.24	27993.5	$^4S_2 -$ } aP_2	
2	2872.96	34796.1	${^4P'}_3 -$ } bP_2	
0	2891.50	34574.0	${^4P'}_2 -$ } bP_2	$^4P' - dP_2$
1	2906.85	34391.4	${^4P'}_1 -$ } bP_2	
0	3169.35	31543.5	$^4D_3 -$ } bP_2	
4	3194.58	31294.0	$^4D_2 -$ } bP_2	
3	3209.39	31149.6	$^4D_1 -$ } bP_2	
3	3565.82	28036.1	$^4S_2 -$ } bP_2	
1	2869.93	34833.8	${^4P'}_3 -$ } cP_3	
1	2888.39	34611.2	${^4P'}_2 -$ } cP_3	
2	3132.20	31917.2	$^4D_4 -$ } cP_3	
2	3165.68	31579.6	$^4D_3 -$ } cP_3	
2	3190.86	31330.3	$^4D_2 -$ } cP_3	
4	3561.21	28072.4	$^4S_2 -$ } cP_3	
6	4428.56	22573.9	$lP -$ } cP_3	$nP_3 - P_2$

TABLE 5 (*Continued*).

1	2858.02	34979.0	$^4P'_3$—		
3 dr.	2876.41	34755.5	$^4P'_2$—		
0	2891.50	34574.0	$^4P'_1$—		$^{4'}P'_2-bP_2$
4	3118.00	32062.6	$^4D_4$—	dP_2	
2	3151.12	31725.6	$^4D_3$—		
3	3176.14	31475.8	$^4D_2$—		
2	3190.86	31330.1	$^4D_1$—		?
7	3542.89	28217.5	$^4S_2$—		?
4	2792.04	35805.8	$^4P'_3$—		
4	2809.51	35582.9	$^4P'_2$—	eP_3	
4	3039.62	32889.7	$^4D_4$—		?
3	2762.97	36182.3	$^4P'_3$—		
2	2780.05	35959.9	$^4P'_2$—		
3	2794.22	35777.6	$^4P'_1$—		
3	3059.15	32679.3	$^4D_2$—	fP_2'	
2	3035.95	32929.0	$^4D_3$—		qP_2-P_2
1	3072.68	32535.4	$^4D_1$—		
1	3397.84	29422.0	$^4S_1$—		
3	2756.68	36264.8	$^4P'_2$—		
1	2770.63	36082.2	$^4P'_1$—		
2	3030.82	32984.8	$^4D_2$—	gP_1	
2	3044.10	32840.9	$^4D_1$—		
1	3362.90	29727.7	$^4S_2$—		$^4D_2-{}^4D'_1$
3	4217.15	23706.0	$^4D'_4$—		
5	4231.61	23625.0	$^4D'_3$—		
0	4565.3	21898.2	aP_2—		
0	4574.4	21854.7	bP_2—	nP_3	?
1	4612.78	21672.8	dP_2—		
0	4795.7	20846.2	eP_3—		

TABLE 5 *(Continued)*.

0	4205.60	23771.1	$^4D'_3 -$		
1	4224.57	23664.4	$^4D'_2 -$		
0	4242.19	23566.8	$^4D'_1 -$		
2	4535.35	22042.8	$aP_2 -$	mP_2	
0	4544.1	22000.4	$bP_2 -$		
0	4849.4	20615.0	$fP_2 -$		
0	4922.3	20310.0	$gP_1 -$		
2	4062.89	24606.1	$^4D'_3 -$		
1	4080.44	24500.2	$^4D'_2 -$		
3	4369.79	22877.9	$aP_2 -$		
2	4377.95	22835.3	$bP_2 -$	oP_3	
2	4384.98	22798.3	$cP_3 -$		
4	4413.16	22653.1	$dP_2 -$		
2	4580.36	21826.2	$eP_3 -$		
0	2972.28	33634.4	$^4D'_4 -$		
0	2979.5	33552.9	$^4D'_3 -$	pP_3	
1	2988.93	33447.0	$^4D'_2 -$		$^4D'_1 - cP_1$
3	3141.38	31829.9	$aP_2 -$		?
0	2976.65	33585.0	$^4D'_3 -$		
0	2986.08	33478.9	$^4D'_2 -$	qP_2	
0	2994.90	33380.4	$^4D'_1 -$		
0	2980.2	33545.0	$^4D'_2 -$	rP_1	
1	2988.93	33447.0	$^4D'_1 -$		$aP_2 - pP_3$
0	4086.69	24462.7	$^4D'_4 -$		
0	4100.27	24381.7	$^4D'_3 -$		
0	4118.11	24276.1	$^4D'_2 -$		
1	4627.93	21601.9	$eP_3 -$	lP	
1	4322.66	23127.4	$xP_2 -$		
4	4413.16	22653.1	$aP_2 -$		$dP_2 - oP_3$
4	4421.38	22611.4	$bP_2 -$		

TABLE 5 *(Continued).*

1	2933.71	34076.6	$^4P'_3$ —		
1	2953.10	33852.8	$^4P'_2$ —		
2	3243.42	30822.9	$^4D_3$ —		
3	3269.84	30573.7	$^4D_2$ —		
2	3659.91	27315.3	$^4S_2$ —	P_2	
0	4399.95	22720.8	mP_2 —		
6	4428.56	22573.9	nP_3 —		cP_3 — lP
0	4244.17	23555.1	oP_3 —		
0	3075.73	32502.8	pP_3 —		
1	3072.68	32535.4	qP_2 —		
1	2916.22	34280.9	$^4P'_3$ —		
0	2935.29	34058.2	$^4P'_2$ —		
1	2951.15	33875.8	$^4P'_1$ —		
3 dr.	3248.16	30777.8	$^4D_2$ —		
2	3263.40	30634.1	$^4D_1$ —	xP_2	
1	3632.74	27520.6	$^4S_2$ —		
1	4439.95	22516.5	mP_2 —		
2	4468.86	22370.8	nP_3 —		
7	3829.75	26103.9	P_2 —		
4	3799.99	26308.4	xP_2 —		
9	3727.09	26823.0	bP_2 —	T	
2	3721.87	26860.6	cP_3 —		
3	3701.79	27006.3	dP_2 —		
0	3592.0	27831.7	eP_3 —	?	
1	3121.57	32025.9	$^4D_2$ —		
1	3135.79	31880.7	$^4D_1$ —		
0	3475.40	28765.5	$^4S_2$ —		
4	3628.09	27554.8	T_2 —		
1	4732.5	21124.6	nP_3 —	U	
0	4719.2	21184.1	kP —		
0	4699.6	21272.5	mP_2 —		
5	4569.02	21880.4	lP_2 —		
4	4522.65	22104.7	oP_3 —		

TABLE 5 *(Continued)*.

1	4206.43	23766.4	$^4D'_4$—	
2	4220.89	23685.0	$^4D'_3$—	
2	4240.10	23577.7	$^4D'_2$—	
3	4257.82	23479.6	$^4D'_1$—	
2	4416.77	22634.6	P_2—	
4	4457.04	22430.1	xP_2—	kP
2	4553.21	21956.4	aP_2—	
0	4561.9	21914.6	bP_2—	
0	4600.16	21732.3	dP_2—	
0	4781.9	20906.4	eP_3—	
1	3154.80	31688.6	$^4D_4$—	
3	3188.73	31351.4	$^4D_3$—	
5	3214.33	31101.7	$^4D_2$—	
0	3590.47	27843.6	$^4S_2$—	
4	3753.79	26632.2	T—	yP_3
1	4534.55	22046.7	nP_3—	
1	4384.08	22803.4	lP_3—	
2	4341.42	23027.4	oP—	
0	3123.2 ?	32009.2	qP—	
3	4133.65	24184.9	$^4D'_2$—	
3	4150.65	24085.7	$^4D'_1$—	
2	4339.76	23036.2	xP_2—	
4	4430.91	22562.4	aP_2—	
3	4439.30	22519.8	bP_2—	
2	4446.43	22483.7	cP_3—	R_1
1	4475.34	22338.4	dP_2—	
2	4588.11	21789.3	U—	
0	4647.4	21511.4	eP_3—	
0	4730.1	21135.3	fP_2—	

Comparison with the F I spectrum.

For determination of more exact values of terms it will be necessary to know a series relation. However, it will be interesting to compare the order of the analogous terms of which the theoretical interpretation is sure.

In both spectra the $^4P'$ term lies in contrary to the experimental rule of HUND deeper than the ^4D term. The interval ratio in the $^4P'$ terms has a value which not agrees with the value called for by LANDE's intervalrule.

TABLE 6.

F I				*Ne II*		Interval ratio		
Term	*j*	Term-value (relative)	Term-differ-ence	Termvalue (relative)	Term-difference	LANDÉ	*F I*	*Ne II*
^2P	2	135320	407	±270000	[780]			
	1	134913		269220				
		(16.6 Volt)		(33.4 Volt)				
^4P	3	58617.0	274.7	117000.0	518.0			
	2	58342.3		116482.0		1.67	1.72	1.73
	1	58182.3	160.0	116193.0	299.0			
$^4P'$	3	45104.8	122.9	89938.2	222.6			
	2	44981.9		89715.6		1.67	1.19	1.21
	1	44879.2	102.7	89533.1	182.5			
^4D	4	44035.4	176.6	87022.7	337.8			
	3	43858.8		86484.9		2.33	2.12	2.34
			144.5		249.7			
	2	43714.3		86435.2		1.67	1.73	1.73
	1	43630.9	83.4	86291.1	144.1			
^4S	2	42595.0		83178.0				

Summary.

A number of 180 lines of the ionized *Neon (Ne II)* has been classified in a termscheme. The deepest quartet terms are identified. The *Ne II* spectrum has an analogous structure as the *F I* spectrum. The analogous terms have in *Ne II* approximately the double value as in *F I*.

Laboratory "Physica" of the University or Amsterdam.

May 1927.

Physics. — *On the liberation of electrons from a metal surface by positive ions.* By F. M. PENNING. (Communicated by Dr. G. HOLST.)

(Communicated at the meeting of October 29, 1927).

§ 1. The work to be described in the following pages was undertaken on the suggestion of Dr. HOLST, in order to get more information about the part played by the positive ions in a gas discharge. According to the theory of TOWNSEND [1]) new electrons are formed in the bulk of the gas as a consequence of the ionising action of the positive ions; according to another view it is the bombardment of the cathode by positive ions of rather high velocity which gives rise to the liberation of these electrons. Several years ago a new theory was developed by HOLST and OOSTERHUIS [2]). This theory supposes that positive ions, which strike without any appreciable velocity a suitable metal surface, are able to extract electrons from this surface (of course besides those which are required for neutralisation). The condition for this happening is $V_i > 2\varphi$ (V_i = ionising potential of the gas, φ = work function of RICHARDSON). This theory was suggested by the phenomenon of the "negative striations" which HOLST and OOSTERHUIS observed in Neon of about 10 mm pressure at low currents (about 1 μA) [3]). From the facts that these striations had a sharp boundary towards the cathode side and that the layer immediately before the cathode was absolutely dark, it was concluded that the electrons were liberated from the cathode surface and not from atoms in the bulk of the gas [4]). Moreover, the velocity of the positive ions in this case must be very small, as a consequence of the small value of the mean free path.

In this connection we shall mention an analogous phenomenon which was observed in this laboratory by G. HERTZ [5]). In a tube as shown in fig. 1. with a heating filament K and an anode A a Neon low voltage arc is burning (pressure about 1 cm). When the plate P is held at a negative potential with respect to A, it attracts positive ions (which can diffuse through the gauze). Now between P and A striations are observed of the same kind as the above-said. Obviously these striations are caused by electrons which are liberated from P by positive ions and which excite the neon atoms on their way to A. The first layer is observed at a potential

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1) J. S. TOWNSEND, Electricity in gases, 1915.

2) G. HOLST and E. OOSTERHUIS, Physica **1**, 78, 1921; Comptes Rendus **175**, 577, 1922; Phil. Mag. **46**, 1117, 1923.

3) l.c.

4) This may be concluded also from the fact that the material of the cathode has great influence on the sparking potential, see G. HOLST and E. OOSTERHUIS, Versl. Kon. Ak. v. Wet. Amsterdam **29**, 849, 1920.

5) Unpublished.

difference of about 20 V between P and A. Under these circumstances the potential difference along the distance of a mean free path for a pos. ion is only 0.01 V; clearly in this case there can be no question of a liberation of electrons by the bombardment of high velocity ions.

It seems possible, however, that in the experiment just described the electrons were liberated by radiation. In order to investigate this point, another discharge tube was used as designed in fig. 2. Instead of one plate there were two plates P_1 and P_2 mounted in the same way within two glass tubes G_1 and G_2. The radiation from the main discharge may

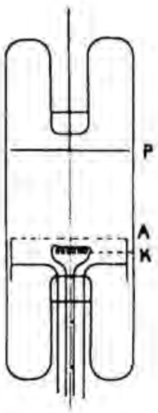

Fig. 1.

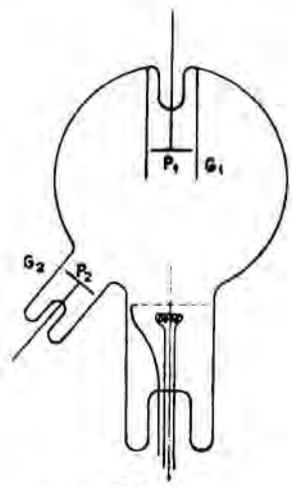

Fig. 2.

reach directly P_1 but it cannot reach P_2. However, the striations show themselves in about the same manner before P_1, as before P_2; therefore we may conclude that the radiation cannot have much influence.

§ 2. We have tried to determine experimentally how many positive ions (especially neon ions [1])) of zero velocity are required to liberate one electron from a metal surface. We should, however, mention immediately that up to the present only preliminary results have been obtained, as a consequence of the difficulties of these experiments.

For the first experiments a tube was used of the type shown in fig. 1, but with a gauze G and a plate M instead of the plate P alone (fig. 3). In order to get a larger supply of ions, the anode A was placed under the cathode K. When the gauze is brought at a negative potential with respect to K, it will be struck only by pos. ions; on the other hand, when M is brought at a positive potential with respect to A, it will gather only electrons. In order that no electrons from the main discharge may reach M, the holes of the gauze should be small [2]); smallness of the holes on the

[1]) Unless otherwise mentioned the following experiments are made with neon.

[2]) The gauze used had 24 threads of 0.15 mm diameter at 1 cm in each direction.

other hand gives rise to the inconvenience that only a fraction of the electrons from G reaches M.

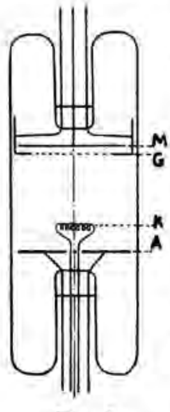

Fig. 3.

As our intention was to study the action of positive ions with low velocity, the values of the potentials in the tube should be chosen as small as possible. Therefore the gas-pressure should be high (several mm) and in that case the electrons, liberated from G, may ionise the gas atoms between M and G, in case the potential of M is raised. Therefore the current to M, plotted as a function of the potential of M, does not show saturation; distinct kinks are observed at the points where the potential has increased with V_i.

The electron current becomes more constant when the gas pressure is decreased, but then we must give up at the same time the advantage of the low potentials. When, moreover, the distance between M and G is taken as small as possible (about 1 mm), the current to M becomes nearly constant with increasing potential. The number of electrons, reaching M, is then 3 to 5 % of the number of positive ions which go to G (at pressures between 0.2 and 0.02 mm and pos. ion currents from 0.2 to 20 mA). Besides, the variation of the potential of G (with respect to the cathode) between —20 and —80 V had little influence on this percentage; so it is obvious that the number of liberated electrons is affected only to a small extent by the velocity of the positive ions.

It is not right, however, to conclude from these measurements that from 20 to 30 ions are required to extract one electron from the metal; on the contrary a smaller number will be sufficient, because the electrons, liberated from G, will move partly also in the direction to A.

§ 3. For measurements of the same kind we used also a positive column in neon of low pressure (0.02 mm). The tube is shown in fig. 4. The discharge (some hundred mA) is maintained between the oxide cathode K and the anode A. The

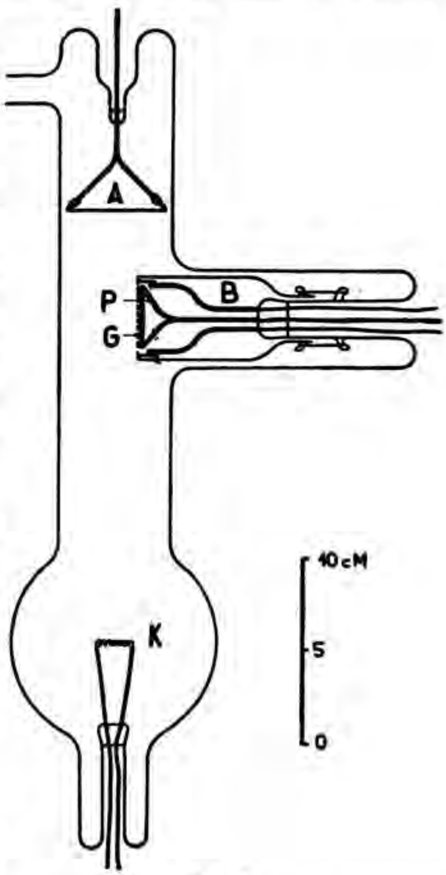

Fig. 4.

positive column contains a collector which, when brought at a negative potential, will gather pos. ions. In order to measure the number of electrons a, which is liberated on the average by one positive ion, we use again the method described in § 2. A side tube contains a gauze G (copper) and a plate P; the current to both these electrodes is measured. To protect the backside of the plate P from electrons, this plate is enclosed in a glasstube B, which is fastened to the glasswall. The results for a few series of measurements are shown in fig. 5 [1]). The potentials are given with respect

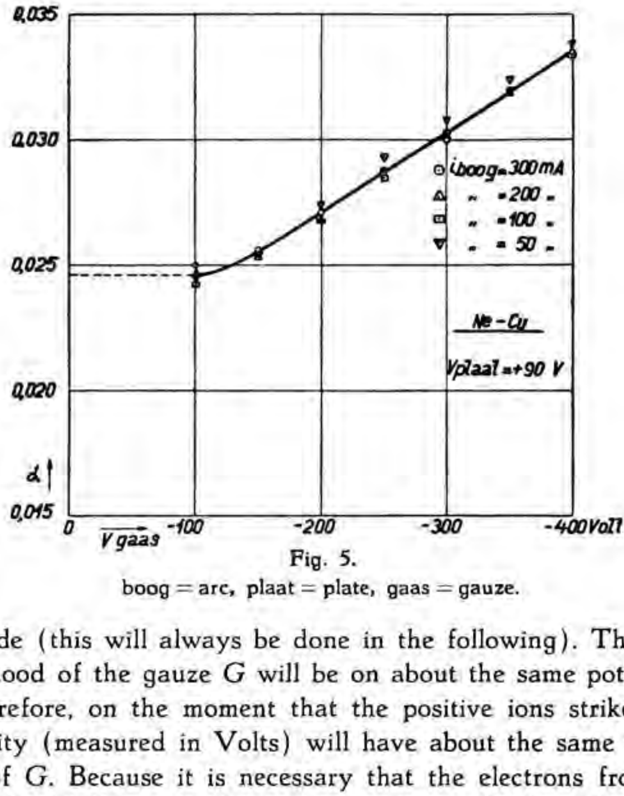

Fig. 5.

boog = arc, plaat = plate, gaas = gauze.

to the anode (this will always be done in the following). The gas in the neighbourhood of the gauze G will be on about the same potential as the anode, therefore, on the moment that the positive ions strike the gauze, their velocity (measured in Volts) will have about the same value as the potential of G. Because it is necessary that the electrons from the main discharge should be stopped, the gauze cannot be brought on a much higher potential as —100 V (the potential of the cathode is in this measurement about —60 V). The probable trend of the curve above —100 V is dotted in fig. 5. The value of a for a velocity zero (a_0) is found to be about 2 % (even if the curve through the measured points should be extrapolated in a different way). As the gauze in this tube was of the same construction as that of § 2, these measurements are open to the same objection as mentioned there. From the measurements of this and the foregoing § we may conclude: 20 to 50 pos. ions (of zero velocity) at the utmost are required to liberate one electron from a *Cu*-surface.

The same tube was used also for measurements with argon filling and

1) The values of the electron currents are in this case from 0.1 to 0.01 mA.

with Mg instead of Cu as electrode material. As should be expected in connection with the values of V_i and φ, the value of α was for $Ar <$ for Ne and for $Mg >$ for Cu.

§ 4. To meet several drawbacks of the described measurements, afterwards another method was followed. It was desirable to study the action on the pos. ions in a space of very low pressure, where the collisions between positive ions (or secundary electrons) and atoms could be neglected. When the gas pressure in the main discharge is lowered under 0.02 mm the arc potential is rising rapidly. This should be prevented, as we shall study the action of pos. ions of low velocity. Therefore it is necessary to bring the pos. ions in a room where the gas is pumped away and which is communicating only through a narrow canal with the main discharge. After experiments with tubes of somewhat other constructions the tube of fig. 6 gave the best results (tube II) [1]).

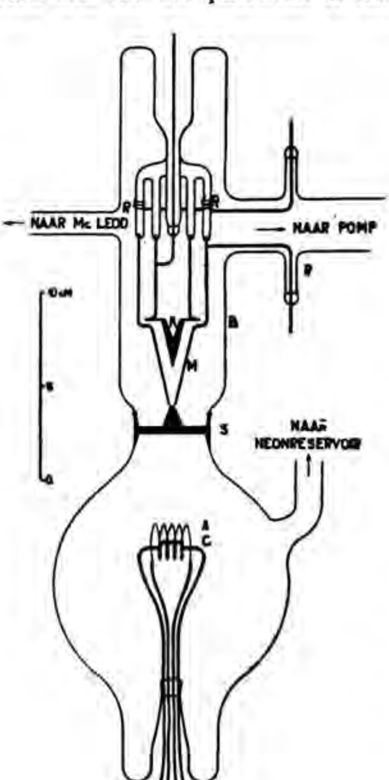

Fig. 6.

The cathode C is an oxide filament, the anode a large tungsten spiral which could be degassed by glowing. In the sidewall of the bulb containing the main discharge, a chromic iron piece S is molten and to this again the tube B with the collector system. This system contains an iron cone K which is used as a collector for the pos. ions; the cone is surrounded by a mantle M which collects the secondary electrons. S is perforated by a canal of 1 mm diameter and 15 mm length. B is connected by a short, wide tube with a diffusion pump which keeps the gas-pressure on a value of about 0.0002 mm (with a pressure of 0.02 mm in the bulb). The pressure at both sides can be measured with a Mc-Leod gage. As in these experiments the gas is rapidly disappearing by metalsputtering, the bulb is connected with a large reservoir of 10 l capacity; in this way many measurements may be made before it is necessary to admit fresh gas. The discharge tube is

[1]) In the tube I, used before, the anode was mounted at a much larger distance from the cathode, so that a positive column occurred. The chromic iron piece S was molten in the side wall of the tube, containing the positive column. The canal in S was 20 mm. ong in this tube. The arrangement of tube II was chosen afterwards to lower the arc potential.

separated by tubes in liquid air from parts of the apparatus containing mercury or stopcocks.

As only a small number of pos. ions is reaching the cone K the currents are very small (unto $< 10^{-9}$ A); therefore the electric insulation should be very careful. The leads-in of M and K are molten in the glass, at a large distance of each other. Moreover, the leads-in of M and the glass-bars in the tube which support M are surrounded by conducting guard rings R which are helt on the same potential as M. In the same way the points of support of the connecting wires, of the galvanometer and so on are protected from leakage.

Before the measurements the tube was pumped at a temperature of about 400° C. Afterwards the metal parts were degassed; the iron cone could be glown in vacuum by means of high frequency currents.

As it is impossible to measure directly with this arrangement the effect of positive ions with zero velocity, it was tried to measure the percentage of liberated electrons (a) as a function of the velocity of the pos. ions, and to extrapolate to a velocity 0. The measurements were made in the following way: the canal in S and the mantle M were helt on the same potential, e.g. —100 V with respect to the anode (the arc potential is always < 100 V). The pos. ions arrive in the part of low pressure with a velocity between 0 and 100 V. Between S and K a potential difference of 50—900 V is put, which accelerates the pos. ions; as the pos. ions make nearly no collisions in this part of the tube (mean free path about 50 cm) they arrive at the cone with a velocity varying from 0—100 to 900—1000 V. The liberated electrons are moving towards M; indeed, although the end of the canal is at the same potential as M, the direction of the field is such that the percentage of the electrons which is going to S may be neglected. Moreover as a consequence of the relative positions of K and M there is a strong electric field at the surface of K, so that we may expect a rapid saturation of the electron current. As a matter of fact with this collector system saturation was reached already at a potential difference of about 20 V between K and M, whereas with a system used before, this was only the case at 200 V. To show this, in Fig. 7 are plotted the current to the mantle M and the cone K for tube I (collector system the same as in tube II [1])) with the following potentials:

Canal (= kanaal): —150 V.
Cone (= kegel): —500 V.
Mantle (= mantel): varied.

(All these potentials are measured as before with respect to the anode.) An electron current in the direction from cone to mantle is taken as negative for both electrodes; so in Fig. 7 for potentials above —500 V pos. ions are going to the cone and electrons to the mantle. In case

[1]) Compare the footnote on p. 18.

$V_{mantle} = V_{canal}$ the positive ions which energe from the canal will not be retarded ; when the mantle is made more negative, the ion current to the

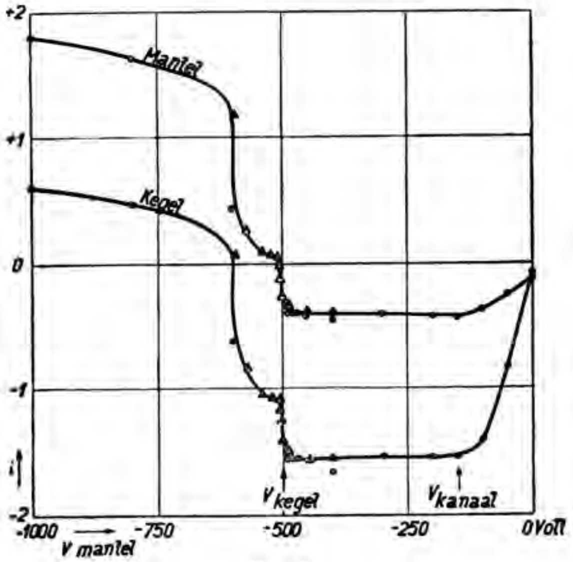

Fig. 7.

cone remains constant ; on the contrary, when the mantle becomes positive with respect to the canal the ions will be partly stopped and the current to the cone becomes smaller. For potentials of the mantle between —150 and —480 V the electron current and the pos. ion current remain nearly constant, showing a good saturation. When $V_{mantle} = V_{cone}$ the current of the mantle becomes positive instead of negative, as should be expected. The small pos. ion current to the mantle at first increases slowly but then again very rapidly and when $V_{mantle} = -1000$ V the mantle and the cone have interchanged their functions. The value of a which is deduced from the measurements at —1000 V is larger as that found in case the positive ions are collected by the cone, but in the first case the velocity of the ions is twice as large.

Concerning the final measurements the following may be remarked. The currents were measured with a galvanometer (sensibility 0.3×10^{-10} A); in order to use the same instrument for both currents, a throw-over switch was applied, consisting of mercury cups in paraffine ; to avoid leaking currents each mercury cup was surrounded by a conducting ring helt at the same potential, in the way mentioned at the description of the tube II. With the aid of such precautions we succeeded in reducing the leaking currents to a negligeable amount. The electrodes were connected to the switch in such a way that the galvanometer deflections had the same direction for both currents, and further by suitable shunts, the order of magnitude of the deflections was made the same, in order to diminish the influence of the zero point of the galvanometer.

§ 5. The results of a few series of measurements with the tubes I and II are shown in Fig. 8. Every series was measured in the direction of increasing and of decreasing voltages (the corresponding points are designed as circles and triangles). As ordinate is plotted a : the number of

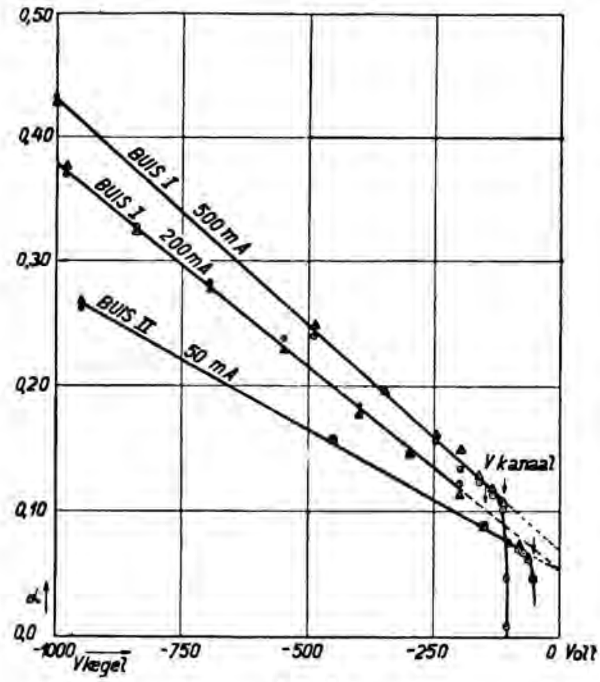

Fig. 8.
Buis = tube. Kegel = cone. Kanaal = canal.

electrons liberated on the average by one pos. ion. As the current to the cone consists of the sum of the pos. ions which reach it and the electrons which leave it :

$$\alpha = \frac{| i_{\text{mantle}} |}{| i_{\text{cone}} | - | i_{\text{mantle}} |}.$$

As abscissa is plotted the potential of the cone ; this gives at the same time, within the limits already mentioned higher up, the velocity of the pos. ions (measured in Volts). The potential of the canal should of course be lower than the lowest point of the cathode ; the mantle is on the same potential as the canal. As soon as the potential difference between the cone and the mantle becomes lower than about 20 V, the mantle will no longer collect all the liberated electrons (see Fig. 7) ; the values, found for a, are then too low. Consequently, the curves in Fig. 8 are then deviating from the original direction ; when $V_{\text{mantle}} = V_{\text{cone}}$ they go down steeply. There is, however, no reason to suppose that the real value of a also should vary suddenly ; on the contrary always when in consequence of a lower potential difference between the cathode and the anode, a lower value could be chosen for V_{canal}, the curve bent off at a higher value of V_{mantle}. It is, therefore, very probable that we may extrapolate the curves of Fig. 8 in the manner as shown by the dotted lines and that for iron a_0 (the value of a for pos. ions with zero velocity) is about 0.06 [1]).

The values of a_0 in Fig. 8 are diverging between 0.05 and 0.07. As a matter of fact in other series (measured partly with tubes of a somewhat other type) even more differing values were found, viz. unto 0.12 ; likewise the value of a for larger ion velocities was not constant ; the value for 1000 V (a_{1000}) varied, e.g. with tube I in the course of the measurements (which extended over 19 days) between 0.44 and 0.27. Probably the condition of the metal surface has a great influence, as is apparent also in photoelectric researches [2]). We intend to make further researches with regard to this point. Anyhow it could be stated that immediately after the glowing of the metal a_{1000} was smaller viz. 0.23—0.30 ; afterwards the value was increasing again gradually.

Though the value of a is not yet known with sufficient accuracy, we may conclude from the above-mentioned experiments (also in connection with the results of § 2 and § 3) : Neon ions which reach without velocity a copper or iron surface are able to liberate electrons from this surface, and the value of a_0 is about 5 to 10 %.

§ 6. Finally, we shall try to answer the question how far the obtained results are in agreement with those of other authors. Many experiments of this type have been done with positive ions which are emitted in vacuum by heated aluminium phosphate and similar substances. We also tried at first in this way to get some information about the liberating of electrons by positive ions. The result was, that ions of this type with zero velocity did not cause any measurable electron emission from a Ni-surface [3]). At a velocity of 100 V, in our experiments a was still < 0.01. However, we have in this case ions of *Na* [4]) or other alkali metals with small V_i, and one should expect that an extra electron can be liberated

[1]) Recently, by means of another apparatus, the measurements could be extended unto ion velocities of less than 7 Volts. In this case also the value of $\varkappa$ was still a few percents, see Physica **8**, 13, 1928. (*Note added in the translation*).

[2]) See e.g. LEE A. DU BRIDGE, Phys. Rev. **29**, 451, 1927. This writer measured the total photoelectric current from a Pt surface. This was glown several times at 1250° C.; the constant limit value was reached only after 50, sometimes only after 150 hours glowing; after 20 hours glowing e.g. the measured value was still $2^1/_2 \times$ the limit value.

[3]) The same result was obtained by N. CAMPBELL, Phil. Mag. **29**, 783, 1915; W. L. CHENEY, Phys. Rev. (2) **10**, 325, 1917; A. L. KLEIN, Phys. Rev. (2) **26**, 800, 1925; W. F. JACKSON, Phys. Rev. (2) **28**, 524, 1927. Compare also: E. BADAREU, Phys. Zts. **25**, 137, 1924. The numerical values for $\varkappa$ at higher velocities show very large differences.

[4]) See e.g. O. W. RICHARDSON, The emission of electricity from hot bodies, 1921. By measuring the current in a diode (with a hot anode, covered with aluminium phosphate) as a function of the potential, we could show that the ions in the above-said experiments were Na-ions.

only by an ion when $V_i > \varphi$. Now this condition is not fulfilled for the ions used and for metals as *Fe*.

With gas ions (H_2) experiments have been done by BAERWALD. This physicist also used a tube which was devided into two parts by a canal. This canal was used as a cathode of a glow discharge; therefore the canalrays which entered the low pressure part of the tube always had a rather large velocity. As collector metal brass was used. In order to compare our method with that of BAERWALD [1]), we performed also some measurements on hydrogen. A tube of type I was used, but with a somewhat different collector-system. However, we found with ions of 1000 V velocity a value of a which was only 10 % of that found by BAERWALD. This must perhaps partly be ascribed to the different manner in which the pos. ions are formed (molecule- and atom ions ?); also we should make allowance for the circumstance that BAERWALD's apparatus could not be pumped at higher temperature and that it could not be degassed; finally perhaps brass gives in this respect quite other results as *Cu* or *Fe*.

It is a pleasure for the writer to thank Mr. J. C. DOUZE for his assistance in the measurements and in the constructing of the tubes.

Natuurkundig Laboratorium der
N.V. Philips' Gloeilampenfabrieken.
Eindhoven, October 28th, 1928.

[1]) H. BAERWALD, Ann. d. Phys. **65**, 167, 1921.

Anatomy. — *A cutaneous branch of the facial nerve in a teleost.* By Dr. C. J. VAN DER HORST. (Central Institute of Brain Research. Amsterdam.) (Communicated by Dr. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of December 17, 1927).

The branchial nerves of lower vertebrates are built up from several components viz., motor fibers, gustatory fibers, fibers for the general sensibility of the mucous membrane in mouth and throat, and fibers for the general sensibility of the skin (HERRICK, JOHNSTON).

Concerning different nerves (e.g. facial, vagus, etc.), the contribution of the different sensory components is very variable. Thus the trigeminus consists for the greater part of fibers for general sensibility of the skin, a smaller part finds its endings in the epithelium of the mouth cavity, whereas gustatory fibers are absent.

In this point the trigeminus differs considerably from the three following branchial nerves: facialis, glossopharyngeus and vagus. These three nerves consist for the greater part of gustatory fibers and fibers of general sensibility to the mucous membrane, whereas only very few fibers, or none at all, have free endings in the skin of the head.

As far as is known a cutaneous branch of the vagus is always present, though it is sometimes so small that it is hardly visible. In some cases (*Prionotus* after HERRICK) this vagus branch is very large.

Cutaneous fibers seem only seldom to be present in the glossopharyngeus. EWART found them in selachians, HOUSER mentions them in *Mustelus canis*. On the other hand NORRIS and HUGHES did not find them in *Acanthias*. According to COLE cutaneous fibers of the glossopharyngeus occur in *Chimaera* and according to JOHNSTON in *Petromyzon* and *Acipenser*.

More often than in the glossopharyngeus, cutaneous fibers are present in the facial nerve. According to JOHNSTON, the ramus hyomandibularis VII of *Petromyzon dorsatus* contains a considerable number of cutaneous fibers that join centrally the spinal trigeminus tract. Whereas the cutaneous branches of the glossopharyngeus and vagus have their endings on both the dorsal and ventral aspects of the head, those of the facialis in *Petromyzon* are limited to the ventral side.

KAPPERS was able to demonstrate a cutaneous branch of the facialis, at least the central part of it, in *Heptanchus* and *Hexanchus*. Compared with the spinal fifth, the descending facialis in these sharks is only a very small bundle, which soon after its entrance, joins the trigeminus. In *Chlamydoselache* MERRIT HAWKES also found some small branches of the facialis going to the skin of the ventro-lateral side of the head. On the other hand cutaneous fibers are wanting in *Acanthias*, according to NORRIS and HUGHES, in the roots as well as in the peripheral branches of the facialis.

In *Chondrostei* NORRIS found no trace of a somatosensory element in the facialis roots. ALLIS described a cutaneous branch in *Polyodon*, but NORRIS is of the opinion that it is a lateralis branch. And yet the truncus hyomandibularis receives cutaneous fibers, but by way of an anastomosis with the vagus. This connection has already been found before by STANNIUS in *Acipenser*. ALLIS showed its presence in *Polyodon*. NORRIS could state its somato-sensory character without doubt in *Polyodon* and *Acipenser*. In *Scaphirhynchus* this was less clear.

In Amphibians a connection exists between the vagus and the facialis in the same way. In *Siren lacertina*, however, this anastomosis is very thin (DRÜNER); sometimes it may be wanting altogether. NORRIS was able to show that in this animal, near the exit of the motor facialis, cutaneous fibers split off from the very superficially located spinal trigeminus and go out with the facialis. The ganglion of these cutaneous fibers is situated at the ventral side of the motor facialis. RÖTHIG also demonstrated such a cutaneous branch belonging to the facialis in *Megalobatrachus* and *Bufo*.

In teleosts the truncus hyomandibularis contains cutaneous fibers, as was demonstrated by RUTKIEWICZ in *Ameiurus* and by HERRICK in *Ameiurus*, *Menidia* and *Gadus*, but these cutaneous fibers unite only peripherally with the facialis. They leave the brain with the spinal trigeminus and their cellbodies form a part of the ganglion Gasseri. A real cutaneous branch of the facialis has not been shown in teleosts before the present finding.

Somato-sensory fibers were found by KINGSBURY in *Amia*; this was confirmed by NORRIS. The latter author could point out the presence of these fibers in even a greater number in *Lepidosteus*. This somato-sensory bundle consists almost entirely of fibers without myelin sheath. They split off from the spinal trigeminus which runs quite near the outer surface of the oblongata and their ganglion is situated at the caudal and ventral side of the whole complex of the trigeminus and facialis ganglia. The fibers run in the truncus hyomandibularis to the periphery.

As mentioned above a cutaneous branch of the facialis has not been found in teleosts before, but in *Albula* such a branch is present. Although the fibers are unmyelinated for the greater part, they form such an apparent and well defined bundle that it is the most striking one in the medulla oblongata, especially in the region of the vagus, and this bundle of the descending facialis can be traced from its entrance to its central ending without any difficulty.

In the intracerebral path, the cutaneous branch of the facialis in *Albula* differs considerably from the cutaneous branches of the other above mentioned animals. In the latter the somato-sensory facialis joins the descending trigeminus directly after its entrance. In *Amia* and *Lepidosteus* the trigeminus bundle runs near the very surface of the oblongata, according to NORRIS, and the same holds true in *Siren lacertina*. But in *Albula*, as in all other teleosts, the descending trigeminus is situated deeper in the oblongata at the dorsal side of the secundary gustatory tract.

In selachians, the trigeminus bundle has the same position and thus the cutaneous branch of the facialis of *Hexanchus* and *Heptanchus* penetrates deeply in the oblongata to join the trigeminus, as was shown by KAPPERS. On the other hand, in *Albula* the descending facialis remains near the periphery and a union with the trigeminus does not occur. Only near the end nucleus behind the calamus scriptorius does the trigeminus approach the surface and the bundles run alongside each other.

The cutaneous branches of the vagus, which are very apparent in *Albula*, no longer unite with the descending trigeminus in this teleost, though in others they do. In *Albula* these branches unite with the descending facialis. A cutaneous branch of the glossopharyngeus is not present.

From its entrance into the oblongata, the spinal facialis runs caudad in the ventral part of the lateralis nucleus near the outer surface. With the diminishing size of the lobus liniae lateralis, the descending facialis approaches the dorsal side of the oblongata more and more. In the vagus-region the spinal trigeminus also comes near the surface. These two descending bundles are separated here by the entering sensory vagus roots.

The cutaneous fibers of the vagus join the facialis directly at its ventral side. A little behind the calamus scriptorius after the entrance of the last sensory vagus roots, the descending trigeminus takes a position at the medial side of the facialis and vagus bundles.

Whereas the visceral commissural nucleus is rather small, the somatic one is large. At its lateral side and hence caudad, this somatic commissural nucleus forms one mass with the endnucleus of the descending trigeminus, facialis and vagus bundles and further on with the somato-sensory area of the spinal cord. Around this nucleus, the three descending bundles are arranged in such a way that the facialis is at its dorsal, the vagus at its lateral and the trigeminus at its ventral side. So the fibers penetrate the nucleus from all sides (fig. 1).

The dorsal spinal roots split up in two parts at their entrance. Of the first root smaller numbers of fibers curve dorsad and, running near the surface, join the vagus and the facialis. Greater numbers of fibers enter horizontally and join the trigeminus. Of the second spinal root, most of the fibers bend dorsad at their entrance.

Frontally, about at the level of the first dorsal spinal root, the somato-sensory nucleus is so large that it occupies nearly half the spinal cord (fig. 1). At the level of the second root, where nearly all facialis and vagus fibers seem to have disappeared, the nucleus is considerably smaller. And yet still further caudad the somato-sensory area remains very large in comparison with other teleosts. Here the nucleus exhibits the peculiarity that it does not form a more or less compact mass as is usually the case in teleosts, but has the shape of a rather thin curved lamella (fig. 2). Thus it assumes a form similar to the lower olive and other, especially sensory, areas that enlarge their surface by making folds. These lamellations not situated at the outer surface of the central nervous system are called by KAPPERS „inner cortical structures".

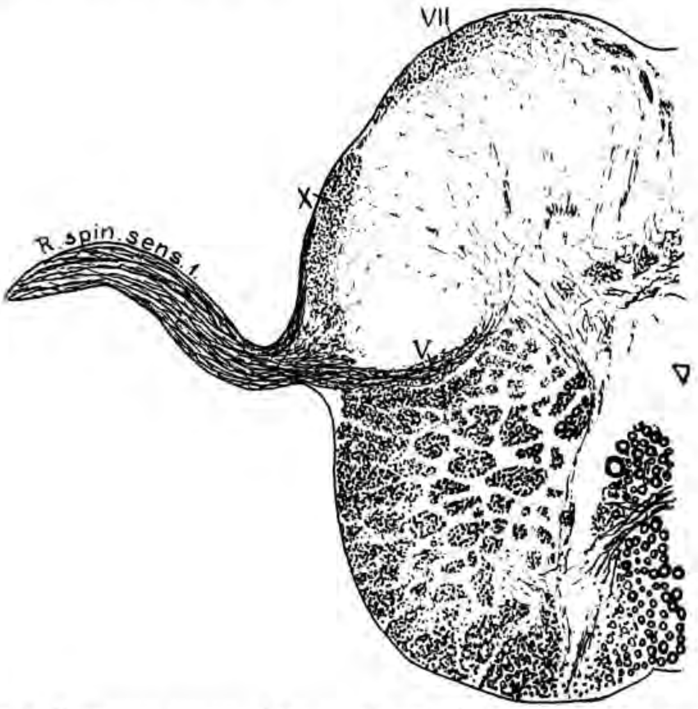

Fig. 1. Transverse section of the spinal cord of *Albula vulpes* at the level of the first sensory spinal root.

At its entrance into the medulla oblongata, the cutaneous branch of the facialis is situated rather at a distance from the gustatory branch (fig. 3). The latter runs more dorsally at the fronto-medial side of the nerveus lateralis anterior. Ventral to the lateralis is the entrance of the nervus octavus and here the fibers of this nerve surround the cutaneous branch of the facial from all sides. At its entrance, frontal to the nervus octavus, the descending facialis is in close relation with the motor seventh. The fibers of the latter surround those of the former almost entirely. According to NORRIS, in *Amia* and *Lepidosteus* also, the cutaneous branch of the facialis enters the oblongata in close proximity to the motor part and in *Siren* the motor root surrounds the cutaneous branch much in the same way as in *Albula*.

Somewhat nearer the periphery, these two facialis roots join the nervus lateralis anterior at its ventral side. The lightly coloured fibers of the cutaneous branch are separated from the lateralis by the heavily myelinated and thus, by the WEIGERT-method, darkly coloured motor fibers.

To this complex, the gustatory bundle of the facialis is added and, more frontally, also the trigeminus. The ganglion of the cutaneous root of the

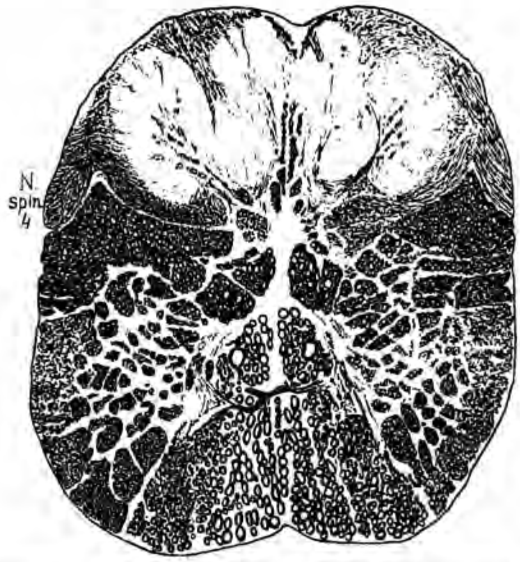

Fig. 2. Transverse section of the spinal cord of *Albula vulpes* at the level of the fourth spinal nerve.

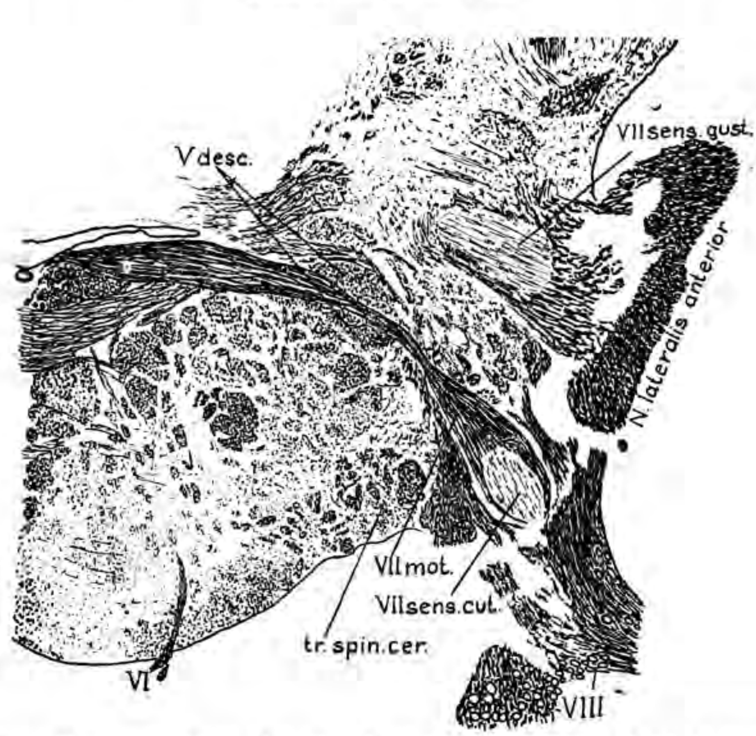

Fig. 3. Transverse section of a part of the medulla oblongata of *Albula vulpes* near the entrance of the N. facialis.

facialis is situated wholly within the skull cavity in contrast to the other ganglia of this complex. The cellbodies are much smaller than those of the lateralis. The cutaneous branch of the facialis goes to the periphery in the truncus hyomandibularis, as is always the case if such a branch is present.

I have studied various other teleosts in search of a cutaneous branch of the facialis. Nowhere have I found it. In *Megalops*, like *Albula* a very primitive teleost, it certainly is absent. But it is possible that a cutaneous facialis is present in the *Mormyridae*. The tractus spino-cerebellaris dorsalis that is highly developed in this family of fishes runs quite near the surface, as is also shown in the figures of STENDELL. So this tract in *Mormyridae* does not follow the ordinary way in close contact with the descending trigeminus, but runs nearer the surface and in the position where a descending facialis could be expected. It is possible that this tract contains facial fibers ; my material, however, was not sufficient to decide this and neither do the figures of STENDELL. In *Gadus* and notably in *Lota*, a cutaneous facialis is absent, though the sensibility of the skin of the head is highly developed in these fishes, and in relation with this, the cutaneous branches of the trigeminus and vagus are very large. Also in other teleosts the truncus hyomandibularis contains fibers for general sensibility of the skin that arise from the ganglion Gasseri, according to HERRICK, and so are trigeminus fibers that only join the truncus peripherally. It is possible that the cutaneous fibers going out in *Albula*, *Amia* and *Lepidosteus* with the motor facialis, leave the central nervous system in other fishes farther frontally, together with the cutaneous fibers of the trigeminus.

No doubt the sensibility of the skin of the head is highly developed in *Albula*, as has already been indicated by the extension of the somato-sensory area in the frontal part of the spinal cord. This extension is not caused by a hypertrophy of the dorsal spinal roots as in *Trigla* and *Prionotus* ; on the contrary these roots are rather small in *Albula*. And yet the descending trigeminus is not especially large in *Albula* ; in *Gadidae* it is much larger.

This makes it apparent that the same cutaneous fibers that enter with the trigeminus in nearly all teleosts, reach the central nervous system in *Albula*, *Amia* and *Lepidosteus* with the motor facialis. This shows some similarity with the phenomenon in motor roots that ADDENS calls "central anastomosis". The peculiarity of *Albula*, in contradistinction to *Amia* and *Lepidosteus*, is that the descending facialis does not join the spinal trigeminus, but remains as a well defined, separate bundle along its whole course.

According to GILL, *Albula* is met with especially near the coast, where it looks for its foods in the shallow water. The food consists of molluscs, especially lamellibranchiates, that live in the mud or sand and therefore are probably found by feeling. When the fish is looking for its food in the shallow water it has the head downwards and puts the tail out of the water. This habit may give an explanation for the highly developed sensibility of the head in *Albula*.

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(Communicated at the meeting of October 29, 1927).

§ 1. *Introduction, material and method.*

By BLAAUW the influence has been examined of various temperatures during flower-formation on the whole hyacinth. For this purpose the *late* variety Queen of the Blues was taken (BLAAUW, 1924).

Next however it was important to know how other varieties would behave. For this purpose we now examined the varieties l'Innocence and la Victoire *flowering early* in culture. In this case the bulbs were exposed only to those five temperatures for which most optima were found in Queen of the Blues, while we took care that sufficient room was left for possible deviating optima in these early hyacinths. Besides the results of these temperatures were only traced for the most important organs. This examination however is more extensive in so far, that the bulbs to be planted were not only exposed to an after-treatment in 13° C. after a preliminary treatment of 8 weeks in 23°, 25.5° and 28° C., but after a preliminary treatment of 3,5 and ca 8 weeks in 17°, 20°, 23°, 25.5° and 28° C.

To get an insight in the foundation and object of this experiment, for drawings, more detailed description and for comparison, see BLAAUW 1924.

At the beginning of July 1926 there were destined for this examination 210 + 640 bulbs of l'Innocence and 640 bulbs of la Victoire. Both varieties were so-called biennial of scooped material (= 3 year old) and selected for a circumference of 12—13 cms. The Queen of the Blues previously examined was a year younger and of a circumference of 7.5—9 cms. This should be specially borne in mind in judging flower-cluster-formation and number of flowers.

Of the 210 bulbs l'Innocence 10 bulbs were fixed on July 5, 1926 in alcohol 96 %, in order to trace the condition of the vegetation-point on the day on which the experiment was started. 200 bulbs were put in the temperatures 17°, 20°, 23°, 25.5° and 28°, viz. 40 in each temperature. Of these 40 bulbs groups of 10 bulbs were left in these temperatures for 3, 5, ca 8 and 12 weeks and next fixed, in order to trace in which temperature after a definite period the development had progressed farthest.

The 640 bulbs of l'Innocence and la Victoire were treated as has been explained with respect to the above 200 l'Innocence, but they were not fixed: after 3,5 and ca 8 weeks they were transferred from the above temperatures to 13° and 17° and planted after 12 weeks simultaneously with the groups of 20 which had been kept in 17° to 28° throughout the 12 weeks. For each of the 32 different combinations produced in this way 20 bulbs were taken. Since however the 20 bulbs l'Innocence which after 8 weeks should have gone from 17° to 13°, were kept in 17° bij mistake, this combination is left out; accordingly in the subjoined tables we find two groups with the treatment 17° for 12 weeks mentioned for l'Innocence.

§ 2. *Results of the temperature-treatment for the leaves.*

Number of foliage-leaves formed.

1. *At the beginning of the experiment.* The number of young foliage-leaflets on July 5, 1926 was 54 for the 10 bulbs examined. Of these 10 bulbs 7 had 5 leaflets, 2 bulbs 6 leaflets (in one of these bulbs one leaflet was not entirely split off) and 1 bulb 7 leaflets (one of which was not entirely split off).

2. *After the fixations.* After 12 weeks 200 bulbs had been treated in 20 various modes (see tab. 1) and fixed. The number of leaves in these 200 bulbs averagely amounted to 54.7 per 10, when those which are not entirely split off are also counted. Therefore some increase is again to be noted, just as in Queen of the Blues, but also with respect to l'Innocence we mean to be justified in drawing the conclusion that this slight *increase* is *of an accidental nature* and not due to the exposures to these temperatures.

TABLE 1. *l'Innocence.*

Number of young foliage-leaflets per 10 bulbs, when they have been in 17°—28° C., from 3 to 12 weeks.

	After 3 weeks	5 weeks	$7^1/_2$ weeks	12 weeks
17°	55	56	53	54
20°	53	53	54	54
23°	54	56	53	56
25.5°	54	53	57	58
28°	54	55	56	56

Is not this sufficiently clear from the fact that 54 is found in 25.5° after 3 weeks, 53 after 5 weeks, after $7\frac{1}{2}$ weeks 57 and after 12 weeks 58. This would have to signify an increase, but this is impossible, for after 5 weeks the leaf-formation has long been finished and the flower-cluster-formation is in full swing.

Number of shooting leaves in spring.

In § 1 it has been said, that 640 bulbs of l'Innocence and of la Victoire were planted out after the various exposures. The tables 2 and 2a give their number of leaves, which in spring, when they had been pushed from the bulbs, could be counted. The recorded numbers have been counted on groups of 20 bulbs and converted per 10.

From these tables it appears, that (just as in Queen of the Blues) not all leaves grow out in lower temperatures.

TABLE 2. *l'Innocence.*
Number of assimilating leaves in the spring of 1927 per 10 bulbs.

	3 weeks		5 weeks		7 1/2 weeks		12 weeks
Preliminary exposure 17°	34.0	—	35.5	—	—	—	46.5 and 47.0
20°	39.5	48.0	47.5	52.5	51.1	50.0	54.0
23°	51.0	52.5	53.2	53.7	53.5	54.5	54.5
25.5°	46.5	54.2	53.0	54.5	55.5	53.5	53.0
28°	52.0	55.0	54.0	56.5	54.5	56.5	55.5
After-exposure to:	13°	17°	13°	17°	13°	17°	—

TABLE 2a. *la Victoire*, as table 2.

	3 weeks		5 weeks		7 1/2 weeks		12 weeks
Preliminary exposure 17°	56.5	—	61.6	—	61.5	—	64.0
20°	59.5	60.0	61.5	62.6	61.0	62.0	62.5
23°	61.5	59.0	62.0	59.5	61.1	60.5	60.5
25.5°	60.5	61.0	60.0	60.5	61.5	62.3	62.0
28°	62.0	62.0	61.5	61.5	62.5	62.5	64.0
After-exposure to:	13°	17°	13°	17°	13°	17°	—

In order to make all leaves grow out 13° is nearly always less favourable than 17°. Further it may be observed that the higher the temperature of the preliminary exposure, the shorter it need last to make all the leaves shoot. In l'Innocene 20° is necessary for 12 weeks, 25.5° and 28° but for 3 weeks. l'Innocence however can have a lower temperature than Queen of the Blues. All or nearly all the leaves already grow out in l'Innocence in 20° for 12 weeks, whereas in Queen of the Blues this does not occur before 25.5° for 8 weeks.

La Victoire (table 2*a*) deviates still more from Queen of the Blues; here there are no great differences to be indicated in the chosen temperatures. The complete number 64, which probably averagely amounts to 61 or 62, is here (with 64) already attained in 17° for at most 12 weeks, whereas the number 56.5 for the lowest temperature, viz. 3 weeks in 17° and next in 13°, compared with the two other varieties, deviates but little from the complete number. So in this respect la Victoire can stand 13° much better than l'Innocence.

Length of the foliage-leaves.

1. Table 3 gives the length of the young outer foliage-leaf measured at the peeled bulbs of the *fixed material*. The optima are printed fat. For Queen of the Blues and l'Innocence they are found in nearly the same treatment. The principal differences are, that for l'Innocence an optimum

TABLE 3. *l'Innocence*.
Average lenghts in mms of the outer foliage-leaves, still in embryonic condition ($n = 10$).

	Beginning July 5, '26	After 3 weeks	5 weeks	$7^1/_2$ weeks	12 weeks
17°	3.69	4.72	6.45	10.30	**20.83**
20°	3.69	5.24	6.86	9.79	18.97
23°	3.69	5.23	**7.02**	**10.82**	**18.12**
25.5°	3.69	**5.47**	6.51	9.47	15.51
28°	3.69	5.19	6.31	9.09	13.30

lies at ± 8 weeks 23°, whilst for Queen of the Blues this lies at ± 8 weeks 20° and further that for Queen of the Blues the leaves in 3 weeks 23°, 25.5° and 28° are equally long. Accordingly there is a striking correspondence. Here it is again corroborated, that according as the organ-*enlargement* (especially in September) is acting a more important part, as compared with the organ-*formation*, the optimum is shifted to a lower temperature.

2. Of the *planted* bulbs the tables 4 and 4*a* give the average length of the outer leaf above the ground at the point of time, that the leaves of the greater part of the bulbs already showed above ground.

The measuring was rather inaccurate, especially on account of the rough surface of the soil, so that only the great differences are of value.

TABLE 4. *l'Innocence.*

Average lengths of the outer foliage-leaves above ground in mms per bulb after removal of the cover (peat-litter) on January 28, 1927 (× = under the surface or on the same level).

Preliminary exposure	3 weeks		5 weeks		7½ weeks		12 weeks
17°	15.5	—	17.3	—	—	—	3.0 and 5.8
20°	16.0	×	13.3	1.3	4.3	6.0	2.8
23°	27.8	20.8	26.0	7.3	10.5	4.0	2.0
25.5°	**59.0**	28.8	25.5	21.3	6.0	4.5	2.3
28°	36.3	25.0	31.0	26.8	22.5	23.5	6.0
After-exposure to:	13°	17°	13°	17°	13°	17°	—

TABLE 4*a*. *la Victoire*, as table 4.

Preliminary exposure	3 weeks		5 weeks		7½ weeks		12 weeks
17°	4.0	—	3.8	—	1.5	—	×
20	10.3	2.8	12.8	×	2.3	×	×
23°	**31.3**	1.3	13.0	1.8	6.3	×	×
25.5°	**23.8**	1.5	19.0	0.8	6.5	5.3	×
28°	**31.8**	2.8	19.3	5.3	7.8	5.0	×
After-exposure to:	13°	17°	13°	17°	13°	17°	—

From the tables 4 and 4*a* this conclusion may be drawn: for the stretching of the leaves up to this time the after-treatment in 13° is better than in 17°. Further it may be stated, that l'Innocence 3 weeks 25.5° and next 13° surpasses the other treatments favourably. Here too it appears, that the optimum which is found after 12 weeks 17° with the leaflets in embryonic condition (table 3) is shifted when reckoned over a longer period (during the stretching).

For la Victoire (table 4*a*) we find a corresponding optimum in January in 3 weeks 23°, 25.5° and 28° with after-treatment in 13°.

Number of sheath-leaves in l'Innocence.

On July 5 each of the 10 bulbs examined had 2 sheath-leaves. In the bulbs examined after exposure (fixed material) there occurred in each

TABLE 5. *l'Innocence.*

Number of sheath-leaves per 10 bulbs. (The number at the beginning of the experiment was 20 per 10 bulbs.

	After 3 weeks	5 weeks	$7^1/_2$ weeks	12 weeks
17°	19	17	19	16
20°	17	19	19	18
23°	19	19	19	16
25.5°	17	19	16	18
28°	19	17	17	17

group of 10 at least 1 bulb with one sheath-leaf (table 5). Of the 200 bulbs 155 bulbs had two sheath-leaves, 44 bulbs one sheath-leaf and 1 bulb three sheath-leaves (this latter in 3 weeks 17°). That a certain exposure can influence this figure cannot be concluded from this.

§ 3. *The direct effect of 17°—28° during 3 to 12 weeks on the development of the floral whorls in l'Innocence.*

About the 210 fixed bulbs the results as to number of foliage-leaves and sheath-leaves and length of the outmost foliage-leaf have already been communicated in § 2. In the present § the other observations on these bulbs are mentioned, viz. in what stage of development the vegetation-point that is to yield the flower-cluster for the following spring, was found in consequence of the 20 temperature-exposures.

In order to enable us to represent the progress of the formation of the floral whorls BLAAUW (1920 and 1924) adopted 10 stages of development. Stage I is still a simple vegetation-point, in stage X this has grown into a cluster, the lowest flowers of which are quite complete. In the variety l'Innocence the stages can in the main be distinguished in the same way, as in the previously examined Queen of the Blues. What differences there are, will be discussed afterwards.

The stages of development in question have been drawn from nature and lithographed by Mr. VAN TONGEREN. The objects, stained with a strong aqueous solution of iodine and iodide of potassium were examined (also during the drawing) through a binocular microscope. The magnification has been given with the illustrations (45—50 ×). What has been drawn, was partly selected from the material of this research, partly from other l'Innocence-material of equal age. In determining the stages the lowest flowers were examined as these are the first to develop. The number of flowers, their size and shape do not act a part in this. Neither does the origin and development of the new vegetation-point, that (normally) will pass on to flower-formation the next summer.

In the plates denotes: VP, vegetation-point; L, foliage-leaf; LL, scar foliage-leaf; L*, rudimentary foliage-leaf; NVP, new vegetation-point; BR, bract; BLP, flower-primordium; NPH 1, first leaf of the new phyllome-series; S, bractlet; T I and T II, tepals of the outer and the inner whorl; M I and M II stamens of the outer and the inner whorl; VD, carpel. When of the NVP and of the first sheath-leaf of the NVP the swelling is not yet visible (not any or extremely little external differentiation), but the spot where these organs will arise, is already visible (of the NVP through strong staining by the aqueous solution of iodine and iodide of potassium), this was designated (NVP) and (NPH 1).

The subsequent stages in l'Innocence are:

I. The vegetation-point VP is still splitting off leaves (L) and is still low (fig. 1).

II. The vegetation-point has finished splitting off leaves and is raised, while no differentiation is visible but a weak indication (NVP) or a slight swelling NVP of the new vegetation-point (figs. 2, 3 and 4).

III. Besides the new vegetation-point which gets clearer now some crescentshaped prominences BR are visible (figs. 5 and 6).

IV. In the lower primordia we now see the bract (as a roundish prominence) and the primordium of the proper flower BLP (likewise as a roundish prominence) separated from each other by a furrow (figs. 7 and 8).

V. The three outer tepals T I are to be distinguished as three independent primordia (fig. 9).

VI. The three inner tepals T II id. (figs. 10 and 11).

VII. The three outer stamens M I id.

VIII. The three inner stamens M II id. (fig. 12).

IX. The three carpels VD id. (fig. 13).

X. The three carpels are raised, while the margins of each carpel are turned in (fig. 14).

Besides the above description the plates and stages need the following explanation of a more secondary nature:

Figures 1 and 2 have been drawn in the same magnification and also in the same position, so that the height of the two vegetation-points (fig. 1 and fig. 2) are comparable in the drawings. Figs. 3 and 4 give the growing-point seen from above.

Figure 4 has been drawn from the same object as fig. 3, but turned about 180 degrees.

In fig. 3 the last fully formed foliage-leaf has been slightly lifted to show how this leaf covers the vegetation-point. This innermost foliage-leaf namely frequently presses upon the growing-point or the developing flower-cluster so heavily, that a distinct ridge or dent is visible on it. This therefore has no actual signification with respect to the formation of organs; this should be borne in mind, because this may give rise to errors in judging the object. In figures 2, 3, 4, 5 and 10 the dent thus originated has been marked with a ×.

The leaflet last formed (L* in figures 2 and 5) has often not even been finished and is arrested as a rudimentary organ in the shape of a scale. This rudimentary leaflet may be followed as far as stage X and further, but it is no more visible in the illustrations after stage III.

In stage IV— (which approaches stage IV very closely) we see (fig. 7) the first indication of a sheath-leaf in the new vegetation-point; in the figure (NPH 1). In the following fully drawn clusters the further development may be traced.

In stage IV (fig. 8) in the lowest flowers we see the first differentiation of a bractlet, indicated with S, the further development of which may likewise be traced in some flowers in figures 9 and 10. See on this bractlet: BLAAUW 1920, Summary § 3.

In stage VI the entire flower-cluster has been given (fig. 10), but the first (lowest) flower, indicated BLP 1 of this same flower-cluster has been opened and drawn in the right hand top corner of the plate (fig. 11), in order to indicate the feature of stage VI, i.e. the differentiated condition of the inner tepals. Of the following stages but one of the flowers has been drawn after being opened. This latter is necessary, because after stage V the tepals begin to overlap the further inward parts of the flower. This opening sometimes causes the turned down tepals to be torn loose at the inside of the base, which is shown bij the scars in the drawings.

The differences with Queen of the Blues with regard to the stages.

Ad stage II. In Queen of the Blues stage II was considered the condition in which the vegetation-point is raised, whilst no further differentiation is visible. In l'Innocence it was observed, that the new vegetation-point may already be visible in the axil of the last leaflet (by cell-divisions at that place or by some swelling) fully split off (binocular; staining with iodine and iodide of potassium), when the vegetation-point has finished splitting off leaves and before there is any indication of a bract.

Characteristic of stage II is only the rise of the vegetation-point, not the appearance of the new vegetation-point, because also in judging the later stages we have left this out of account.

Ad stages III and IV. In Queen of the Blues stage III has been called the condition, in which the 1st or the 1st and 2nd flower-primordia are visible as weak swellings. Besides the new growing-point at the base is not mentioned before this.

In l'Innocence we first observed a crescent-shaped roundish prominence of the bract (= stage III), while the part above this which is to give rise to the proper flower-primordium does not show any prominence. Not until after this there appears above the bract a prominence of the proper flower. At first this is lower than the bract (= III+ or IV—), but it will soon be higher (= IV). Further the bract lags considerably behind in growth.

In limiting the stages we can also mention as a criterion for stage III the rise of a single crescent-shaped prominence. Then it is left undecided for the present, whether this prominence is the bract or a primordium which will yet be differentiated into a bract and a flower.

Ad stage VII. For Queen of the Blues (and also for l'Innocence) stage VII is reached, when the three outer stamens are visible as independent organs. But by that time in the examined material of l'Innocence there was always something to be seen of the inner whorl; for that reason the stage is called VII+ or VIII—. Now it is possible, that not all the stages and transitions from one stage to another could be observed in this material, the number of different treatments being limited here. Yet the impression remains that the stages VII and VIII, when they do not fairly coincide, succeed very quickly, i.e. in this way, that the inner whorl commences forming, when the outer is not yet finished.

The progress of the flower-cluster-formation.

This has been synoptically represented in a "curve of dots" (fig. 15). (See BLAAUW 1924 and LUYTEN, JOUSTRA, BLAAUW 1925). Each dot indicates a bulb and is placed with the stage of development (ordinate), in which the vegetation-point or the lowest flower were, while the treatment is indicated on the abscis. A +stage was put in the upper and a —stage in the lower half of a square. In the advanced stages IX the anthers could be measured under the binocular; their length was ca 0.2 mm. As smallest measure in stage X ca. 2.5 mms. was found. A sub-division of stage X or rather the development after stage IX has been rendered according to the length of the anthers. It is necessary therefore to take into account, that on the ordinate above and below the heavy line values have been plotted which were obtained in different ways.

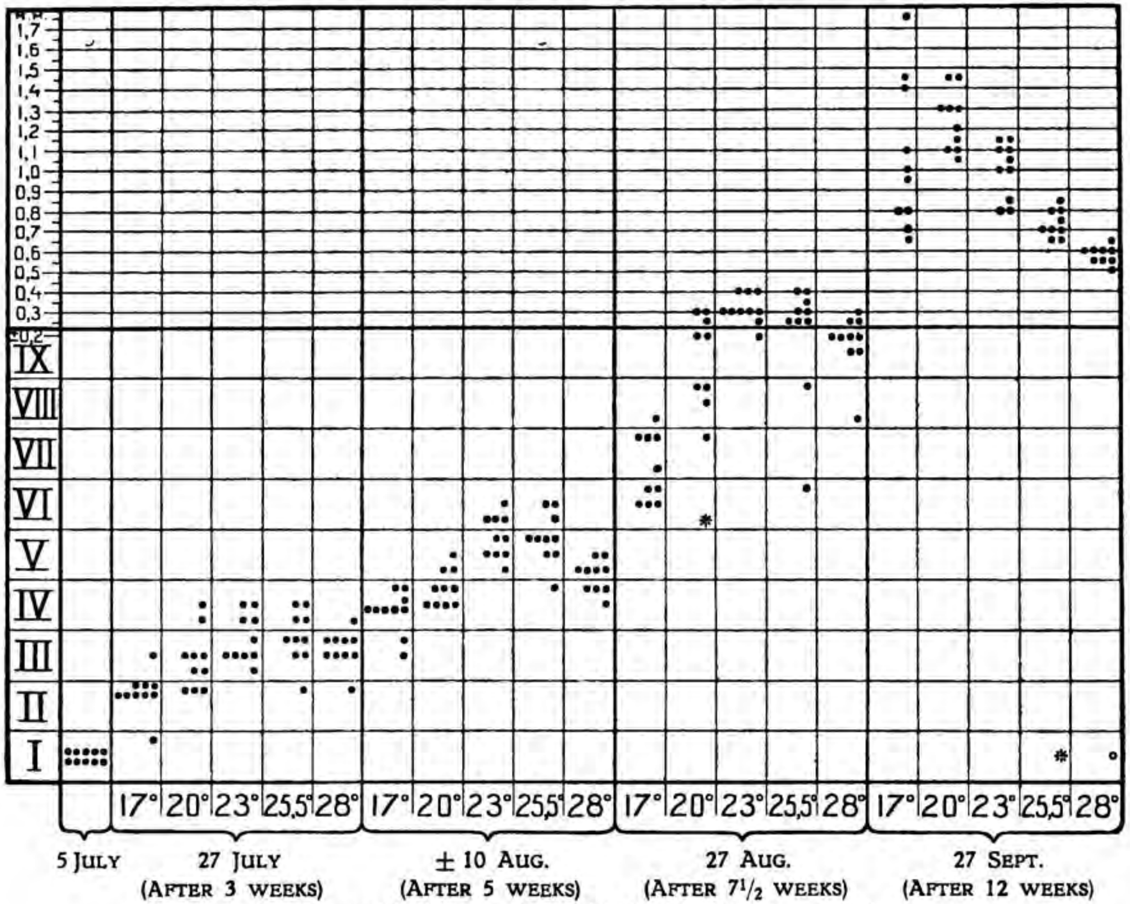
Here a discussion follows of the Fig. 15 in question:

1. *On July 5* at the beginning of the experiment all of the 10 examined bulbs were in stage I. It was however the end of the period of splitting off leaves as appears from tables 1 and 2.

2. *After 3 weeks* 25.5° has progressed farthest, though 23° and 28° are falling behind but little.

3. *After 5 weeks* the flower-cluster is further developed than in whatever temperature after 3 weeks. 23° and 25.5° are farthest. It was

Fig. 15. The progressing flower-cluster-formation.



(* = diseased bulb, o = abnormal bulb. For the rest see text).

remarkable in this case, that in 25.5° in two bulbs the new vegetation-point at the foot of the cluster had already passed on to flower-formation. In both bulbs this had already attained stage IV.

4. *After 7½ weeks* 23° and 25.5° are farthest and besides, especially in 23° very uniformly developed. The development of the 10 bulbs in 20° is particularly unequal. One of the bulbs in 20° was diseased, but reached stage VI after all (indicated with * in fig. 15).

5. *After 12 weeks* 17° is farthest, but (just as in Queen of the Blues) most unequal. In this case anthers were measured of a length of 0.65 to 1.75 mms. Of a failure of the flower-cluster, as in Queen of the Blues, there is no question in l'Innocence in 17°. But the Queen of the Blues examined at the time was a year younger with a circumference of 7.5—9 cms and accordingly already therefore possessing slighter flowering-capacities. Averagely most advanced is 20°. Most uniform in development are the bulbs in 28° (anthers of a length of 0.5 to 0.65 mm) and in 25.5° (from 0.65 to 0.85 mm.); 25.5° is the farthest of these. In 25.5° there was a diseased bulb, which had not progressed at all in its development. In 28° there was a similar bulb likewise entirely lagging behind in development, which was abnormal in a different way. Here too in 25.5° the new vegetation-point in one of the bulbs had proceeded to flower-formation (stage V+). This was not the bulb that had not developed at all.

§ 4. *The total effect on the flower-cluster in l'Innocence and la Victoire.*

In order to trace the total effect of the various exposures the bulbs were planted out after 12 weeks (on Sept. 27, 1926) in a cistern with fixed groundwaterlevel of 60 cms. (described in BLAAUW 1922). The total effect on number and length of the leaves has already been rendered in § 2 (together with the direct effect); in this § the total effect on the flower-clusters follows. See the tables 6 and 6*a*. At the top (between the double and the heavy horizontal lines) the exposure has been given. Next the number of planted bulbs follows. 20 bulbs were taken per treatment, but in some cases, where the figure 19 is found, one has dropped out. Next there is added to the dates the number of flower-clusters, which started flowering on these dates. Then the number of flowering and non-flowering bulbs and the average number of flowers per cluster; at the foot of these the mean error of this average. The signification of the * we find at the close of this §. Mind, that in table 6 two groups are found with an exposure of 12 weeks 17°.

Beginning of flowering.

On March 17, 1927 the first flowers of la Victoire and on March 19 those of l'Innocence came into bloom. The number of clusters which came

TABLE 6. Flowering of *l'Innocence* (see § 4).

For........	3 weeks								
in........	17°	20°		23°		25.5°		28°	
next in.....	13°	13°	17°	13°	17°	13°	17°	13°	17°
Number of planted bulbs	20	20	20	20	20	20	19	20	20
Beginning of flowering on: 19 Mrch						1	1		
21 „		1		13	8	10	9	10	16
23 „	5	5	4	6	10	2	8	8	3
25 „	2	1	12		2		1		
27 „			2						
29 „			1						
31 „									
2 April									
4 „			1						
Number of flowering bulbs (= n)	7	7	20	19	20	13	19	18	19
Number of non-flowering bulbs	13	13		1		7		2	1
Number of flowers per cluster (= M)	9.3*	8.6*	10.1	9.9*	9.7(*)	**10.7***	9.1	8.2*	10.5
$m = \pm$	0.9	0.9	0.3	0.5	0.5	0.4	0.5	0.4	0.4

For........	5 weeks								
in........	17°	20°		23°		25.5°		28°	
next in.....	13°	13°	17°	13°	17°	13°	17°	13°	17°
Number of planted bulbs	19	20	20	19	20	20	20	20	20
Beginning of flowering on: 19 Mrch				$7_{(8)}$		$7_{(11)}$		6	3
21 „	3	9		12	9	13	19	14	15
23 „	3	5	3		10		1		
25 „	3	3	10		1				1
27 „									
29 „			6						
31 „									
2 April									
4 „									
Number of flowering bulbs (= n)	9	17	19	19	20	20	20	20	19
Number of non-flowering bulbs	10	3	1						1
Number of flowers per cluster (= M)	10.6*	11.4*	10.4*	10.6*	10.8*	**11.6***	**11.5**	11.2*	9.7
$m = \pm$	0.6	0.4	0.4	0.5	0.5	0.4	0.4	0.2	0.3

For........	7 1/2 weeks								
in........	17°	20°		23°		25.5°		28°	
next in.....	17°	13°	17°	13°	17°	13°	17°	13°	17°
Number of planted bulbs	20	19	20	20	20	20	20	20	20
Beginning of flowering on: 19 Mrch				1					
21 „		2		10	3	9	5	16	6
23 „	2	10	8	8	16	10	15	3	11
25 „	4	6	8	1				1	3
27 „	6	1	2		1				
29 „	5		2						
31 „	3								
2 April									
4 „									
Number of flowering bulbs (= n)	20	19	20	20	20	19	20	20	20
Number of non-flowering bulbs						1			
Number of flowers per cluster (= M)	10.0*	10.4*	10.0*	11.1*	11.0	**11.3**	**11.9**	10.9*	10.1
$m = \pm$	0.5	0.6	0.5	0.4	0.3	0.3	0.6	0.5	0.5

For........	12 weeks				
in........	17°	20°	23°	25.5°	28°
next in.....	—	—	—	—	—
Number of planted bulbs	20	20	20	20	20
Beginning of flowering on: 19 Mrch					
21 „					
23 „	1				
25 „	4	9	6		
27 „	9	7	11	5	
29 „	3	4	2	12	10
31 „			1	1	4
2 April	1			2	4
4 „					2
Number of flowering bulbs (= n)	18	20	20	20	20
Number of non-flowering bulbs	2				
Number of flowers per cluster (= M)	9.5*	10.1*	10.9*	**11.2***	11.1*
$m = \pm$	0.4	0.4	0.3	0.4	0.4

TABLE 6a. Flowering of *la Victoire* (see § 4).

For.........	3 weeks									5 weeks									7½ weeks									12 weeks				
in.........	17°	20°		23°		25.5°		28°		17°	20°		23°		25.5°		28°		17°	20°		23°		25.5°		28°		17°	20°	23°	25.5°	28°
next in......	13°	13°	17°	13°	17°	13°	17°	13°	17°	13°	13°	17°	13°	17°	13°	17°	13°	17°	13°	13	17°	13°	17°	13°	17°	13°	17°	—	—	—	—	—
Number of planted bulbs	20	19	20	20	19	20	20	20	20	19	19	19	20	20	20	19	20	20	20	20	20	19	19	20	20	20	20	20	20	19	20	20
Beginning of flowering on: 17 Mrch				2		3							1		6		1									1						
19 „		3		7		5		2			3		13		14		14	6				11		3		6						
21 „	3	5		8	5	9	7	11	5	4	11		6	12		15	5	11	3	13		5	2	17	16	12	13					
23 „	1	7	2	2	6	3	10	5	11	3				6		4		2	9	5	4	3	11		4	1	5					
25 „	1	2	7		5		3		2	6	4	7		2				1	5	2	11		6				2					
27 „	3	1	4		3			2	2	5	1	11							2		5							3	2			
29 „	6		7									1																13	16	10	12	16
31 „	1																											4	2	6	6	4
2 April																														2	1	
4 „																														1	1	
Number of flowering bulbs (= n)	15	18	20	19	19	20	20	20	20	18	19	19	20	20	20	19	20	20	19	20	20	19	19	20	20	20	20	20	20	19	20	20
Number of non-flowering bulbs	5	1		1						1									1													
Number of flowers per cluster = M	14.9*	19.3*	24.6	19.4*	23.8	18.3*	20.4	13.9*	19.3	21.3*	23.4*	24.4	22.3*	23.7	20.5*	20.3	19.5	18.1	24.9(*)	27.0	24.2	21.8	22.1	20.0	22.0	19.4	17.8	23.9	25.4	24.1	20.8	19.5
$m = \pm$	1.5	1.1	0.9	1.1	1.2	0.5	1.1	0.8	0.5	1.5	2.1	0.7	0.8	1.0	0.6	0.6	1.4	0.7	0.7	1.3	0.9	0.7	0.7	0.6	0.9	0.6	0.5	1.1	1.8	1.1	0.7	0.9

into bloom on these days (and further every other day) have been given in the tables.

l'Innocence flowers earliest in an exposure of 5 weeks to 25.5°, 23° and 28° with an after-treatment to 13°. In 23° and 25.5° the figures 8 and 11 are placed between brackets. We doubted namely in these cases whether we should note 7 or 8 clusters for each in 23° and 11 clusters in 25.5°, but at last we decided on 7 for each. Yet it may appear from this, that 25.5° is a little in advance.

La Victoire has a conspicuous start in an exposure of 5 weeks to 25.5° with an after-exposure to 13°.

Finally it is evident that for a rapid flowering the after-exposure to 13° is more favourable than to 17°.

When we compare the celerrimum for *flower-formation* in fig. 15 with the rate of *coming into bloom* in table 6, we see parallel "curves". After 3 and 5 weeks and 7½ weeks 23° and 25.5° and also 28° are the first to flower in each case, while the flower-cluster-formation is likewise most advanced in these exposures. After an exposure of 12 weeks to the same temperature the flowers from 17° are the first to bloom, from 28° the last. The rate of flower-formation in these temperatures is parallel. Throughout the winter this celerrimum was maintained.

Accordingly a celerrimal effect is obtained by exposure first for some weeks to a high temperature (25.5°, 23° to 28°), next to a low temp. (17°, still better: 13°). If however in the later weeks the temperature is kept high, this retards so much, that permanently 17° and 20° are more rapid than permanently 23°—28°.

It is a striking fact, that this effect in these early varieties is achieved in the same temperatures as in the late variety Queen of the Blues. Especially with a view to treatment for early flowering it was important to control this point once more in well-known early varieties.

Number of flowering and non-flowering bulbs.

It has appeared from fig. 15, that in l'Innocence every growing-point has proceeded to flower-formation in the chosen temperatures. From table 6 it is to be seen, that it may happen, that the flower does not develop in the lower temperatures, especially in an after-treatment to 13°, preceded by too short a preliminary exposure to but 20° or 17°. After 5 weeks to and above 23° nearly every flower-cluster succeeds, both in the after-exposure to 13° and to 17°.

In la Victoire (table 6*a*) fewer bulbs have fallen behind in the chosen temperatures, from which it may be concluded, that this variety can endure a lower temperature.

The Queen of the Blues previously examined was a year younger and already for that reason the number of flowering bulbs is smaller (see § 6).

Number of flowers per cluster.

The number of flower-clusters (= n) varies from 7 to 20 in the two varieties; it usually amounts to 19 or 20 (tabs. 6 and 6*a*). The mean error of the average number of flowers per cluster was calculated according to JOHANNSEN with the formulas:

$$m = \pm \frac{\sigma}{\sqrt{n}} \text{ and } \sigma = \pm \sqrt{\frac{\Sigma\, p\, a^2}{n} - b^2}.$$

For l'Innocence the mean error is fairly uniformly low. In 3 weeks 17° and 20° and next in 13° it is greatest (number of flower-clusters = 7). For la Victoire the mean errors differ more and are greater on account of the deviations of the variants of the mean value being greater and more irregular, owing to the great number of flowers that occurs on the clusters in this variety.

In what temperature must the preliminary exposure take place and after what time? Taking the mean error (M ± 3 m) into consideration, we can but point out differences between the greatest and the smallest numbers in l'Innocence. The ever recurring optimum at ca 25.5° after 3, 5, ca 8 and 12 weeks is however striking. Even clearer is a similar optimum in la Victoire, but here it is found in 20° (after 3 and 5 weeks also in 23°). In 23°, 25.5° and 28° the numbers get smaller and smaller. The preliminary treatment must last longer in the two varieties than 3 weeks, though this is not so necessary, when the above-mentioned optima are applied.

If we consider whether the after-treatment to 17° or to 13° is necessary, we must not give preference to either of them in l'Innocence. For only where the slightest number of flowers occurs (3 w. 28° + 9 w. 13°) there is, taking the mean error into account, a real difference to be stated between the treatment in 17° and in 13°. In la Victoire this is different: When the preliminary exposure to 20°—28° lasts but 3 weeks, 17° is better, because in that case a greater number of flowers remains rudimentary in 13°. When the preliminary exposure to 20°—28° lasts 5 weeks, the number of rudimentary flowers is smaller and when an after-exposure to 13° or to 17° follows no real difference in number can be pointed out. Accordingly here (with the mean error) it may be more or less traced in the table, where the number of rudimentary flowers is great.

Where these rudimentary flowers occur in la Victoire the following indications are made: In the tables 6 and 6*a* sub "number of flowers per cluster" an asterisk or an asterisk between brackets has been placed. In the case of la Victoire this means, that in those cases the clusters bore a number of apical flowers (from 1 to 5) which showed aberrations due to the treatment (as in Queen of the Blues, BLAAUW 1924, first part § 8). With (*) is indicated, that it was only distinctly observed on one of the clusters. Only those flowers were taken into consideration, which could be clearly distinguished with the naked eye. It occurs in 3 and 5 weeks only then, when an after-treatment to 13° took place. In 5 weeks 28° and next 13° it does no more occur ; it was observed still however at a cluster in 7½ weeks 17° and next 13°.

Especially when many apical flowers fall behind, this goes together with a thinner common flower-stalk and smaller and paler flowers.

Though therefore in the after-treatment to 13° the celerrima are found, they are no optima here in the sense that the flower-cluster is best developed and richest in flowers.

In the case of l'Innocence * and (*) indicate, that the tepals in those cases were slightly greenish at the apex. Generally this does not occur in the after-treatment to 17°, but it does occur when the bulbs have been kept in the same temperature for 12 weeks.

§ 5. *Increase in weight of the bulbs.*

On Sept. 27, 1926 (after the treatment in the various temperatures) the sets of bulbs were weighed and next planted in a cistern with equal soil and ground-water-level, as has already been described in § 4. From this date to the date of lifting (July 4, 1927) they have been, though all in an equal measure, exposed to various influences of the weather. The differences, we find in the tables 7 and 7*a* therefore are due to the different exposures in the previous summer, while after that for months together the climate has influenced them all equally.

The small figures in the tables denote the weight per bulb (an average of 20 bulbs) on Sept. 27, 1926 and on July 5, 1927 (for l'Innocence) and on July 6, 1927 (for la Victoire). So after lifting they were weighed on two subsequent days, which has been recorded, because the loss of weight in one day is already worth mentioning. After each brace the difference, i.e. the average increase in weight, has been given in large figures.

TABLE 7. *l'Innocence.*

Increase in weight per bulb in grams from Sept. 27, 1926 (date of planting) to July 4, 1927 (date of lifting)

		3 weeks		5 weeks		7½ weeks		12 weeks
Preliminary exposure	17°	47.1 / 28.4 } 19.7	—	48.1 / 28.5 } 19.6	—	—	—	52.6 / 27.7 } 24.9 and 50.3 / 27.7 } 22.6
	20°	51.8 / 28 1 } 23.7	54.5 / 27.7 } 27.8	50.4 / 28.4 } 22.0	50.0 / 27.4 } 22.6	50.1 / 27 6 } 22.5	49.8 / 27.7 } 22.1	51 9 / 27.9 } 24.0
	23°	49.4 / 28.3 } 21.1	54.2 / 28.2 } 26 0	49.7 / 28.1 } 21.6	51.7 / 28.2 } 23.5	52.0 / 28.3 } 23.7	54.7 / 27.8 } 26.9	55.1 / 28.6 } 26.5
	25.5°	46.9 / 28.5 } 18.4	54.1 / 27.4 } 26.7	51.6 / 28.2 } 23.4	58 0 / 28.0 } **30.0**	55.3 / 27.9 } 27.4	53.6 / 28.0 } 25.6	53.7 / 28.4 } 25.3
	28°	57.9 / 27.7 } **30.2**	58.3 / 27.7 } **30.6**	52.3 / 28.1 } 24.2	56.3 / 28.0 } 238.	53 5 / 27.8 } 25.7	54.3 / 27 9 } 26.4	56.4 / 28.6 } 27.8
After-exposure to:		13°	17°	13°	17°	13°	17°	—

TABLE 7a. *la Victoire*, as table 7.

Preliminary exposure	3 weeks		5 weeks		7½ weeks		12 weeks
17°	37.1 / 25.5 } 11.6	—	40.8 / 25.7 } 15.1	—	42.4 / 25.7 } 16.7	—	46.0 / 25.8 } 20.2
20°	43.3 / 25.4 } 17.9	45.6 / 25.6 } 20.0	45.2 / 25.4 } 19.8	46.0 / 25.4 } 20.6	41.7 / 25.4 } 16.3	45.0 / 25.8 } 19.2	46.1 / 25.6 } 20.5
23°	43.7 / 25.4 } 18.3	46.7 / 25.6 } 21.1	44.6 / 25.2 } 19.4	44.1 / 25.4 } 18.7	46.4 / 25.4 } 21.0	45.9 / 25.8 } 20.4	46.5 / 26.2 } 20.3
25.5°	44.6 / 25.2 } 19.4	47.2 / 25.5 } 21.7	47.4 / 25.2 } 22.2	51.3 / 25.5 } 25.8	49.3 / 25.3 } 24.0	48.3 / 25.5 } 22.8	49.1 / 26.4 } 22.7
28°	46.8 / 25.0 } 21.8	48.5 / 25.4 } 23.1	46.5 / 25.1 } 21.4	48.7 / 25.1 } 23.6	45.8 / 25.1 } 20.7	48.4 / 25.3 } 23.1	49.5 / 26.2 } 23.3
After-exposure to:	13°	17°	13°	17°	13°	17°	—

BLAAUW (1924) points out, that little attention must be paid to slight differences in weight. In this case this is evident from the 2 sets of 20 bulbs which (one by mistake) were both left in 17° for 12 weeks. Here there is a difference of 2.3 grms, so that therefore no value must be attached to 10 % increase in weight.

On comparing table 7 with 7*a*, it strikes us at once, that the increase in weight in l'Innocence is in every case greater than in la Victoire. On our starting the experiment the bulbs were equally large for the two varieties; positively the more vigorous leaves of l'Innocence with larger assimilating surface than those of la Victoire act an important part here.

Nearly everywhere the after-treatment in 17° appears to be more favourable than in 13°, especially in l'Innocence. An exception to this is for instance 7½ weeks 25.5° + 4½ w. 13°; yet this difference is smaller than 10 %, so that it is a question whether it must be attached any value to. But it is pointed out here, because in Queen of the Blues (BLAAUW 1924, tab. 32) there was also stated greater increase in weight in this case.

Taking the preliminary exposures also into consideration the following exposures are optimal for l'Innocence: 5 w. 25.5° + 7 w. 17° and 3 w. 28° + 9 w. 13° or 17°. For la Victoire 5 w. 25.5° + 7 w. 17°. For Queen of the Blues (BLAAUW 1924, tab. 8) there is an optimum at *eight* weeks 25.5° + 7 w. 17°. In these early varieties therefore optima are found in the higher temperatures (25.5° and 28°), during a shorter period (3 and 5 weeks).

§ 6. *Summary in connection with the application. Comparison with Queen of the Blues.*

W i t h r e s p e c t t o t h e g r o w i n g o u t o f a l l l e a v e s i n t h e f i e l d i n s p r i n g it has appeared, that the higher the temperature of the preliminary exposure, the shorter it should last. For l'Innocence (tab. 2) this temperature may be lower, and needs last shorter than for Queen of the Blues (BLAAUW 1924, tabs. 2 and 24). For la Victoire (tab. 2*a*) this temperature may be lower yet and the preliminary exposure shorter than for l'Innocence. As to the combination of optima (BLAAUW 1924, §§ 1 and 9) this causes no difficulties, as all following optima for this are found as a rule at so high a temperature and for so long a period, that all leaves shoot.

For the stretching of the foliage till January (judged from the outmost foliage-leaf) for l'Innocence (tab. 4) a treatment in 25.5° for 3 weeks and next 13° is optimal. For la Victoire (table 4*a*) this may also be 23° and 28°. The after-exposure to 17° is not favourable in this case, especially not for la Victoire. For Queen of the Blues (BLAAUW 1924, tab. 4 and § 11) this lies in a higher temperature, which is to be applied for a longer period.

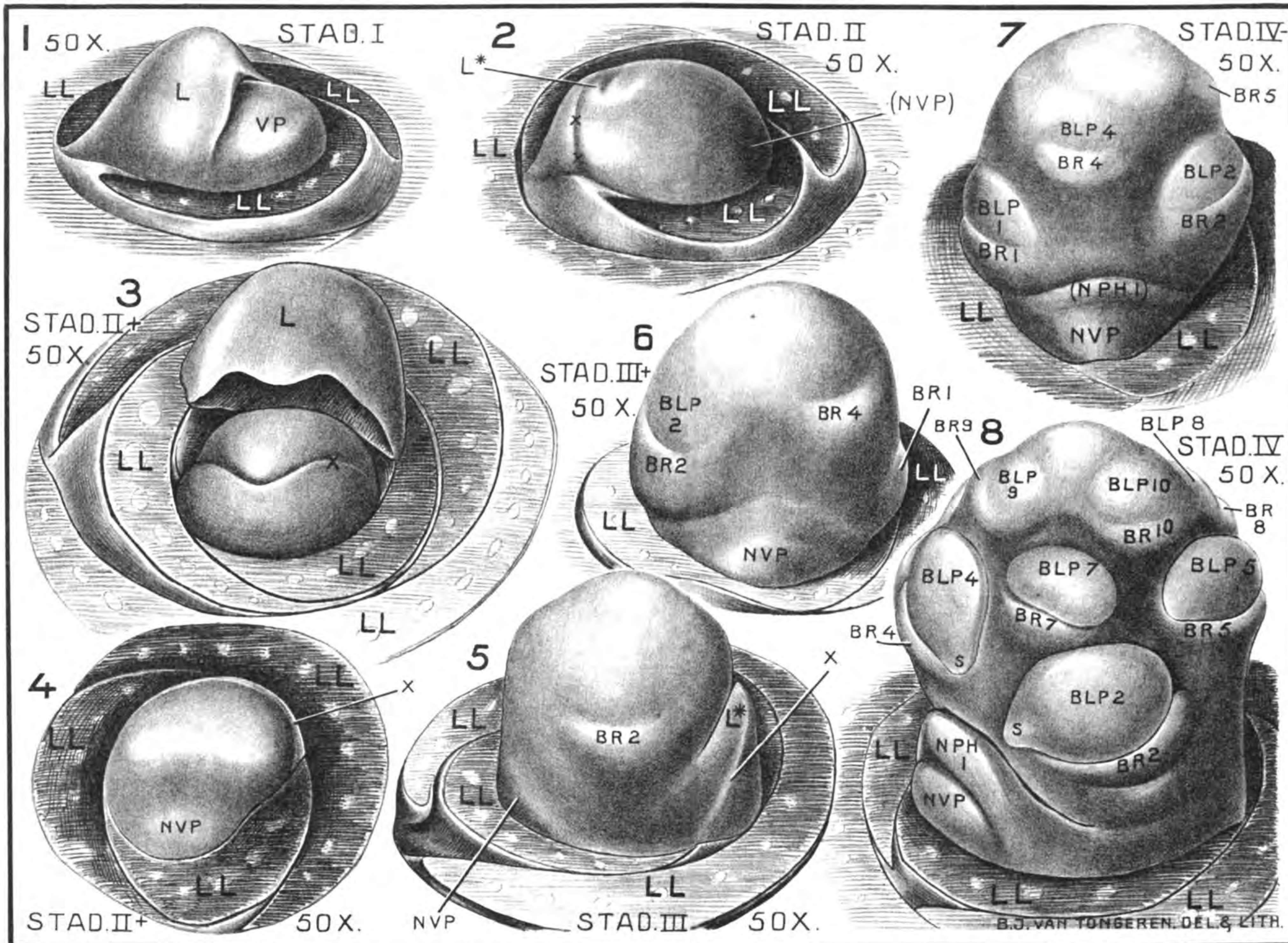
For celerrimal flowering in l'Innocence and la Victoire 25.5° is necessary for 5 weeks and then 13° till the planting-period (tabs. 6 and 6*a*). In l'Innocence the lower surfaces of the tips of the tepals were slightly green in this case, which does not do much harm to the beautiful appearance of the flower. In la Victoire however the apical flowers had remained rudimentary in 13°. This however was stated after a stay for months together outdoors in the field. For a celerrimal flowering (forcing) indoors the results may be more favourable and at any rate an after-treatment in 17° would cause too great a delay i.e. no celerrimal effect. In Queen of the Blues (according to experiments not yet published) the celerrimum of flowering is also found at 25.5° or 28° for 5 weeks, followed by an after-treatment in 13°.

Non-flowering bulbs (tabs. 6 and 6*a*) occur in lower temperatures than the above-mentioned optima, so that — with a bulb-circumference of 12 to 13 cms — we need not take this into account for the combination of the optima.

Queen of the Blues of 7.5—9 cms (BLAAUW 1924, tab. 14) is much closer to the limit of flowering-ability; those of 12—13 cms (ibid. tab. 23) are perfectly parallel, as far as the exposure was identical, with these early varieties. The Queens of 7.5—9 cms must not be compared in this respect with the older bulbs of the early varieties, because these have a more vigorous flowering-capacity and consequently yield a greater number of flowering bulbs.

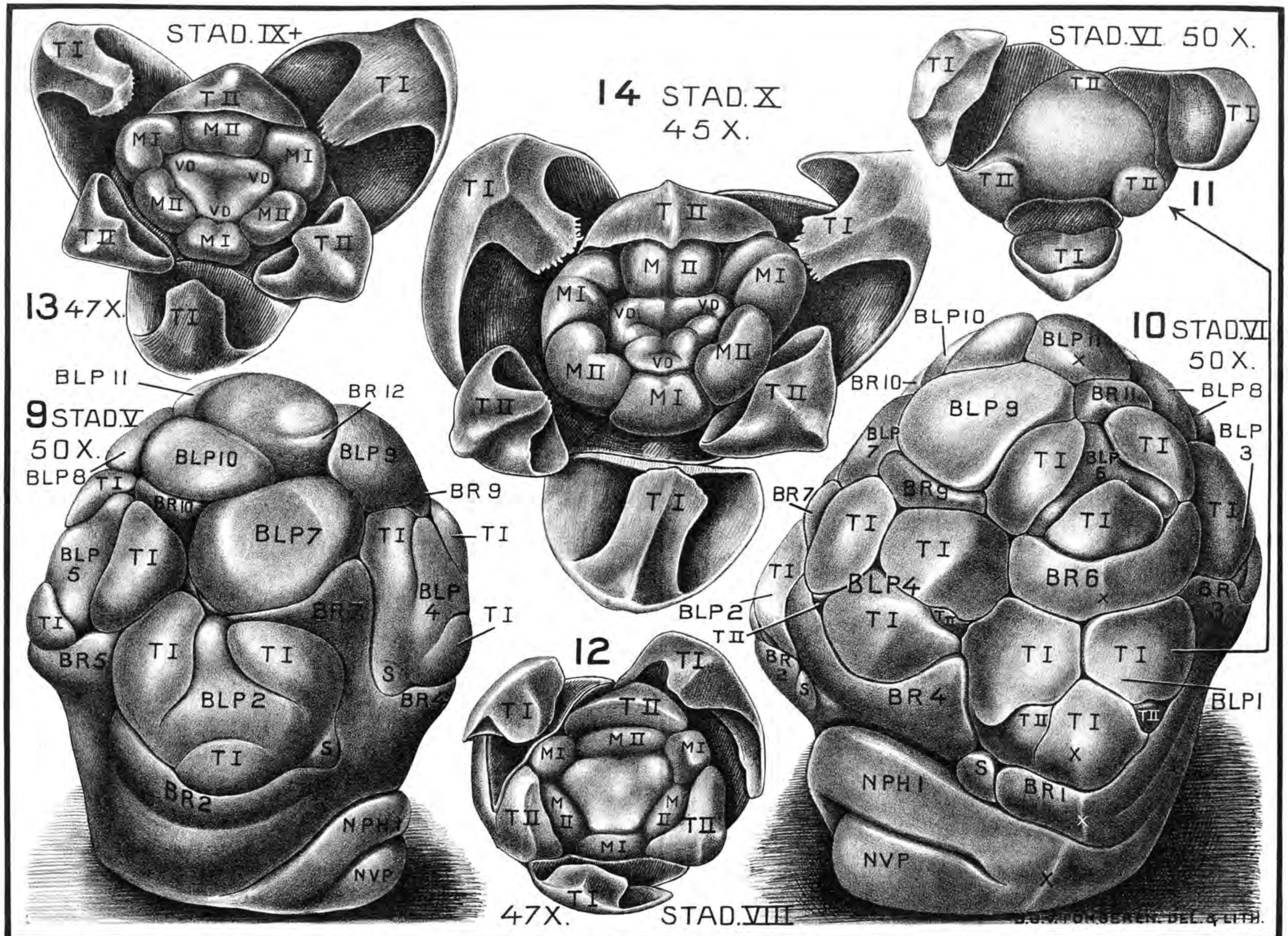
Of la Victoire and l'Innocence it may be said, that with a short preliminary exposure in this respect the former can endure a lower temperature better than l'Innocence.

With respect to the greatest number of flowers per cluster for l'Innocence (tab. 6) ca 25.5° for 5 weeks or 7½ weeks and next 13° or 17° is optimal. In the same way the Queen of the Blues of



DRUK v. P.J. MULDER & Zn LEIDEN (HOLLAND)

HYACINTHUS ORIENTALIS. (L'INNOCENCE)



HYACINTHUS ORIENTALIS. (L'INNOCENCE)

DRUK v P.J. MULDER & Zn LEIDEN (HOLLAND)

12—13 cms (BLAAUW 1924, tab. 23) showed an optimum at 8 w. 25.5° + 4 w. 17°. It should be mentioned here, that in order to be compared, the bulbs must be of equal age, as is evident from comparison of tables 19 and 23 in BLAAUW 1924. For la Victoire (tab. 6*a*) a preliminary exposure to 20° for a long period (up to 12 weeks) is favourable, but for the combination of the optima, especially with a view to increase in weight, we shall have to raise the temperature (23°, still better: 25.5°), but a still higher temperature is markedly detrimental to the number of flowers.

Optima for the increase in weight (tabs. 7 and 7*a*) for the two varieties lie at 25.5° for 5 weeks and next 17° (besides for l'Innocence at 28° for 3 weeks and next 13° or 17°). An after-treatment in 17° is desirable for those. On comparing with Queen of the Blues (see BLAAUW 1924, tabs. 8 and 26), we are inclined to say that these early varieties should be exposed to the high temperatures for a shorter period than this Queen.

Final conclusion. For an optimal field-culture ca 25.5° for *eight weeks* and next 17° has appeared to be the best treatment for the late variety Queen of the Blues. The early varieties l'Innocence and la Victoire we had probably better expose to 25.5° *for a shorter period* (e.g. 5 weeks) and next to 17°.

This is also proved (with l'Innocence) by the fact that the flower-cluster after 8 weeks in 23° and 25.5° has already progressed so far, as is evident from the curve of dots (fig. 15).

We emphasize that we refer to healthy bulbs. For severely diseased sets quite a different treatment may be desirable as a cure.

Wageningen, October 1927. *Laboratory for Plant-physiological Research.*

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Physics. — *The influence of pressure on the electrical conductivity of gold up to 1000 atmospheres.* By A. MICHELS and P. GEELS. (18th communication of the "VAN DER WAALS-Fund.") (Communicated by Prof. J. D. VAN DER WAALS.)

(Communicated at the meeting of October 29, 1927).

These measurements complete an investigation on the influence of pressure on the conductivity of gold, the preliminary results of which were given in the 17th communication [1]).

It must be repeated that the material used in this investigation was gold-wire in a hard drawn state so that the influence of tempering might afterwards be investigated.

The method of measurement.

The bridge circuit depicted in figure 1, which has been described by F. E. SMITH [2]), was used in place of the WHEATSTONE bridge used in the preliminary investigation.

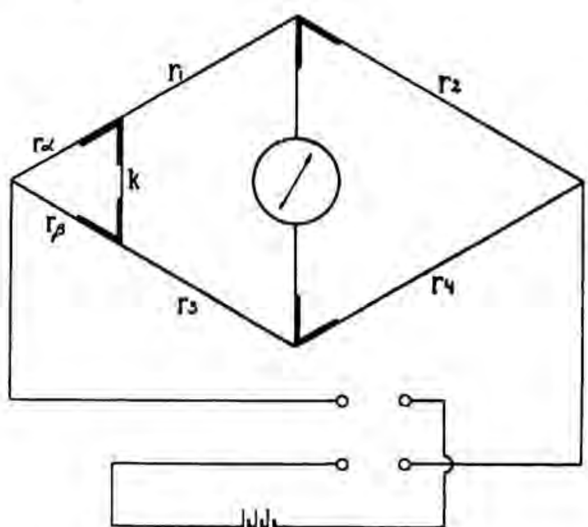

Fig. 1.

The peculiarity of this circuit, like that of the THOMPSON-bridge, is that

1) These Proceedings **30**, (1927) 47.

2) Phil. Magazine 1912. Bd. **24**. p 562.

the influence of the resistances of the leads is reduced to below the accuracy of observation.

As it is hoped to return to the advantages and disadvantages of this method, as well as to the best choice of the resistances, in a later communication, it will be sufficient to state here that :

$$\frac{r_\alpha}{r_\beta}=\frac{r_1}{r_3}=\frac{r_2}{r_4},$$

r_α and r_β being made of the order of 1000 ohms, whilst k was kept as small as possible ; r_1 was the measuring wire, and variations in the temperature were compensated by a gold wire r_3, which was not put under pressure (l.c.).

r_2 and r_4 are the coils of standard resistances of 1000 ohms with a calibrated variable resistance introduced into one of each pair of leads for regulation and for the compensation of the pressure effect.

Following the usual method, the coils of the standard resistances were commutated in each measurement in order to compensate for any irregularities in the resistances of the two coils. (This appeared necessary as deviations of the order of 5×10^{-6} were obtained, due to differences in the temperature coefficients.) The deflection of the galvanometer was observed by commutating the bridge current, the effects of thermal currents being thus eliminated.

The sensitivity.

The sensitivity of the circuit was sufficient to cause an alteration of resistance of 4×10^{-7}, when an ordinary MOLL galvanometer and a measuring current of 3 milliamps was used. This corresponds to a temperature difference of the measuring wire of 10^{-4} degrees.

Calculation showed that the increase in temperature of the gold wire resulting from the above measuring current, would not give an error greater than this.

The accuracy of the pressure measurements up to 250 atmospheres was greater than that of the resistance measurements.

As the new pressure balance for higher pressures had not been completed, the old, although improved, high pressure balance of the "VAN DER WAALS Fund" had to be used for 250—1000 atmospheres, and an accuracy cannot be claimed of more than 1 in 1000.

The observations were made in two series, one from 0—250 atms., with pressure intervals of 50 atms., and the other from 0—1000 atms. with pressure intervals of 200 atms. A few points overlap, and provide a good connection between the two series.

The measuring wire was once heated for a considerable time at 40° between two sets of measurements from 0—250 atmospheres.

In order to economise space all the results are not communicated.

The results of the measurements.

The tables below give :

Table I a series of measurements up to 250 atms. before the tempering at 40° ;

Table II a series after the tempering ;

Table III the results over the pressure range 0—1000 atms.

The tables are subdivided, and give under :

a. the total change of resistance per original Ohm ;

b. the change of resistance per Ohm per kilogram, per sq. centimetre for the various pressure intervals.

TABLE I*a*.

$$-\frac{\triangle w}{w} \times 10^6.$$

Pressure in Kg/cm²	51.5	101.5	151.5	201.5	251.5
16.95°	258^{6}	503^{6}	742^{5}	977^{1}	1206^{1}
28.46°	252^{5}	493^{4}	729^{1}	959^{2}	1185^{5}

TABLE I*b*.

$$-\frac{\triangle w}{\triangle p \, . \, w} \times 10^6.$$

Pressure in Kg/cm²	0—51.5	51.5—101.5	101.5—151.5	151.5—201.5	201.5—251.5
16.95°	5.02	4.90	4.78	4.69	4.58
28.46°	4.90	4.82	4.71	4.60	4.53

TABLE II*a*.

$$-\frac{\triangle w}{w} \, . \, 10^6$$

Pressure in Kg/cm²	51.5	101.5	151.5	201.5	251.5
18.75°	255^{6}	497^{9}	734^{2}	963^{6}	1183^{6}
18.87°	255^{5}	497^{3}	733^{8}	964^{2}	1183^{6}
28.5°	251^{8}	487^{1}	713^{2}	929^{0}	1136^{4}
28.2°	251^{3}	489^{0}	71 ^{8}	930^{8}	1136^{9}
40.0°	244^{8}	469^{0}	677^{7}	878^{6}	1071^{0}

TABLE II*b*.

$$-\frac{\triangle w}{\triangle p\, w} \cdot 10^6$$

Pressure in Kg/cm²	0—51.5	51.5—101.5	101.5—151.5	151.5—201.5	201.5—251.5
18.75°	4.96	4.85	4.73	4.59	4.40
18.87°	4.96	4.84	4.73	4.61	4.39
28.5°	4.89	4.71	4.52	4.32	4.15
28.2°	4.88	4.75	4.52	4.32	4.12
40.0°	4.75	4.48	4.17	4.02	3.85

TABLE III*a*.

$$-\frac{\triangle w}{w} \cdot 10^6$$

Pressure in Kg/cm²	209.4	408.3	607.1	806.0	1004.8
20.9°	994^8	1819^1	2506^9	3118^4	3699^5
28.3°	968^2	1763^1	2424^7	3029^6	3609^3
39.1°	917^8	1668^0	2298^0	2894^3	3473^3

TABLE III*b*.

$$-\frac{\triangle w}{\triangle p\, w} \cdot 10^6$$

Pressure in Kg/cm²	0—209.4	209.4—408.3	408.3—607.1	607.1—806.0	806.0—1004.8
20.9°	4.75	4.14	3.46	3.07	2.92
28.3°	4.62	4.00	3.33	3.04	2.92
39.1°	4.38	3.77	3.17	3.00	2.91

Discussion of the results.

1⁰. A deviation of 1.3×10^{-6} from a mean was obtained in one case in the measurements below 250 atmospheres, the remainder of the measurements showing a much better agreement.

As was expected, the measurements with the large pressure-balance did not give such good results, the maximum deviation from the mean amounting to 4×10^{-6}.

The deviations correspond to a temperature variation of $^1/_{3000}$° for the lower pressures, and $^1/_{1000}$° for the higher.

2⁰. The tempering of hard drawn wire for a considerable time at 40° appears to have a very noticeable influence on the pressure effect.

In connection with this it must be mentioned that, although the wire had previously been compressed to 1300 atms. with the result that the hysterisis phenomena did not appear in the normal measurements, these phenomena reappeared after the tempering, but were diminished after first raising the pressure.

It thus appears almost impossible to compare the results of work on the pressure influence on metal resistances, unless the history of the wire is given accurately. All that is known of the temperature treatment of the gold wire used by BRIDGMAN is, that it was tempered between 0° and 140°—150°. BRIDGMAN, therefore, used a more strongly tempered metal, and this is in agreement with the pressure coefficient found by him (between 0—1000 atms. mean 3.1×10^{-6} at 250), for the latter deviates considerably from the present measurements in the same direction as does the coefficient found after tempering to 40°. In view of the tempering mentioned, it is doubtful whether BRIDGMAN determined his coefficients with sufficient reproducibility. Also, it is noteworthy that BRIDGMAN does not give a single intermediate determination between 0 and 1000 atmospheres, and that he has only attempted to determine the curve by a sort of extra-interpolation method, whilst the present measurements indicate, that the curve is most important in the region of lower pressures.

Again, BRIDGMAN did not observe any noticeable hysterisis phenomena such as is mentioned above.

A graphical representation of the present results is given in Fig. 2, and

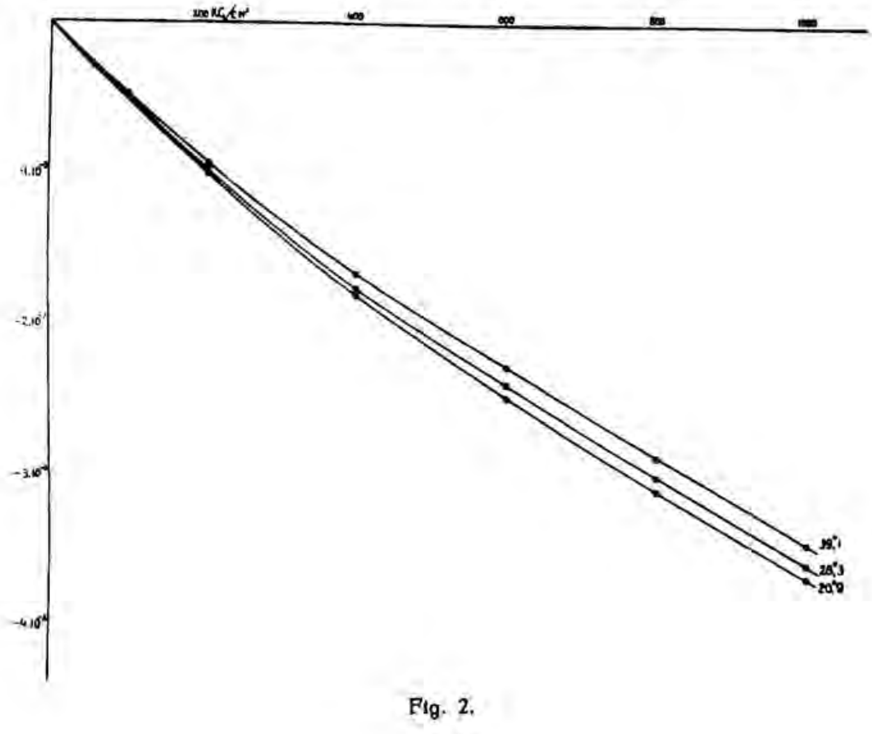

Fig. 2.

the deviation from the linear equation of the pressure effect, i.e. the deviations from the straight line drawn between the first and last points of Fig. 2, are plotted in Fig. 3.

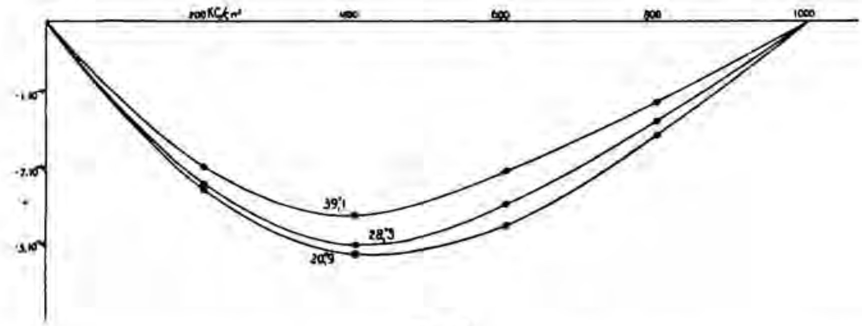

Fig. 3.

As BRIDGMAN has also published a graph similar to Fig. 3 it is possible to compare his and the present results.

Geology. — *On the age of Alkaline rocks from the island of Timor.* By Prof. H. A. BROUWER.

(Communicated at the meeting of November 26, 1927).

Among the Permian sediments of the island of Timor which consist principally of tuffs, marls with tuffaceous material, marls, limestones and effusive rocks were also found locally conglomerates with pebbles of eruptive material. The pebbles from these sediments were not yet examined microscopically and their description is not included in my paper on the rocks of East-Netherlands-Timor [1]). Permian conglomerates with pebbles of eruptive material have been described also from Djambi. TOBLER [2]) describes from there a thick series, consisting of porphyries and quartz-porphyries, keratophyres and quartzkeratophyres with tuffaceous sandstones and thick beds of conglomerate in which the pebbles consist of the same volcanic rocks.

The Timor conglomerate, of which the pebbles have been studied, has been found in the neighbourhood of the path Soefa-Maubesi, where this path passes the grass-grown hills which can be followed from the mountain Somohole in a northerly to north-easterly direction. The locality is near and to the N. E. of the most northern of these hills, where the path turns round it. Tuffaceous material and lapilli are found in many of the highly fossiliferous Permian sediments of this region while several eruptive rocks have been already described [1]). These latter rocks are often interstratifed with the Permian, they are partly effusive rocks, partly perhaps intrusive sheets. The top of the mountain Somohole consists for instance of medium-grained augite-amphibole-diorite and farther to the north quartz-bearing augite-amphibole-diorite have been found. The latter rock has a peculiar chemical composition, as I already have shown (loc. cit. p. 23) ; it differs from the normal quartz-diorites by its high Na_2O content and its low Al_2O_3 content, characteristics which point to a transition to alkaline rocks.

Gabbros, diabases and porphyrites have also been described from this region.

Typical alkaline rocks are found in adjacent regions. Of the alkali-rhyolites, alkalitrachytes and keratophyres, which have been described in my previous papers [3]), an alkalirhyolite is found in the Fatu Forfaik to

[1]) H. A. BROUWER. Gesteenten van Oost-Nederlandsch Timor. Jaarboek v. h. Mijnwezen in Ned. Oost-Indië. Verhand. 1916. I.

[2]) A. TOBLER. Djambi Verslag. Jaarboek v. h. Mijnwezen in Ned.-Indië. Verhand. 1919. III.

[3]) H. A. BROUWER. Neue Funde von Gesteinen der Alkalireihe auf Timor. Centralbl. f. Min. Geol. und Pal. 1913, p. 570; 1914, p. 741.

the south-west of Sufa. Between the bivouac Sufa and the grass-grown hills to the north of the Somohole we found a brecciated alkalirhyolite with aegirine in the groundmass; alkalitrachytes and keratophyres have been found at different places in East-Netherlands-Timor. Many alkaline rocks are found in close connection with Permian sediments, we only mention the camptonites of the Noil Tonini, the alkalirhyolite of the Fatu Forfaik near Sufa, which rests upon Permian crinoidal limestone and the alkalitrachyte in the valley of the Noil Musa, which is found with Permian rocks.

Constituents of the Permian conglomerate.

Several pebbles of which some are up to several centimeters in diameter were studied under the microscope. Partly they are typical effusive rocks, partly they are medium-grained or porphyritic with a holocrystalline groundmass. Many of the pebbles are strongly weathered and their identification is difficult, because the dark minerals and sometimes also the feldspar are entirely decomposed. It is probable that more alkaline rocks are found among them but they cannot be identified with certainty without a chemical analysis, because the typical minerals cannot be determined.

Quartz (alkali) syenite porphyry.

This rock consists of phenocrysts of microperthitic feldspar and of acid plagioclase (principally albite) in a groundmass consisting of the same feldspars, rather much quartz, much ore (and ore-bearing substance), also some calcite. The original dark minerals of the rock is transformed in secondary minerals. The character of the feldspar shows however sufficiently the high alkali-content of the rock, so that it can be brought with great probability to the quartz-bearing alkali-syeniteporphyries.

(Alkali) Syenite.

It is a fine- to medium-grained rock, consisting mainly of microperthitic feldspar with decomposed dark minerals, numerous small crystals of ore and apatite. The original composition of the dark minerals cannot be identified, they are completely altered to calcite and a substance rich in iron. Their forms sometimes have a resemblance to those of pyroxene, which perhaps contained the aegirine-molecule in connection with the large quantity of ferrugineous weathering-products.

(Alkali) Trachytes.

A great many of the pebbles, which were studied microscopically, are reddish to brownish rocks, which are rich in secondary minerals, principally calcite and also ferruginous weathering-products. The phenocrysts are sometimes abundant but they are always entirely altered, the feldspar generally into calcite, the original dark minerals into calcite with much of a ferruginous substance. In the groundmass, which is also rich in secondary products, numerous small feldspar-laths can often be recognized. Their extinction-angle is always small, orthoclase and acid plagioclase cannot be distinguished. Typical trachytic structures are characteristic for several of the rocks. It is possible, that andesitic rocks occur among those, which are strongly altered.

Quartzalkalitrachytes.

These are the only pebbles of the Permian conglomerates, in which the dark minerals can still be determined. They are confined to the groundmass, but certainly belong to aegirine-augite or aegirine, so that there can be no doubt, that these pebbles belong to alkaline rocks.

The phenocrysts are orthoclase or anorthoclase, they are partially altered, calcite is the principal secondary mineral. The groundmass is rich in ferruginous weathering-products and consists of feldspar, rather much quartz, ore and small elongated crystals, which, when they are not decomposed, show a distinct greenish colour. They are optically negative, have a small extinction-angle and belong to aegirine-augite or aegirine.

Although these rocks are altered rather strongly, the unaltered character of a part of the aegirine in the groundmass makes their determination possible with certainty, while the unaltered parts of the other rocks allow a probable determination only.

The pebbles of alkaline rock in the Permian conglomerates prove, that these rocks are not younger than Permian. In connection with the abundance of effusive material in the Permian sediments of the adjacent regions, while pebbles of eruptive rocks are the only ones, which have been found, and in connection with the fact that similar alkaline rocks have been found closely connected with Permian sediments, the conclusion that the alkaline rocks are of Permian age seems to be warranted.

Botany. — *The influence of growth-promoting substances on decapitated flower-stalks of Bellis perennis.* By INA E. UYLDERT. (Communicated by Prof. F. A. F. C. WENT.)

(Communicated at the meeting of November 26, 1927)

The fact is well-known that the growth of flower-stalks is greatly retarded by the removal of the flower-buds. Sometimes the growth may even come to a complete standstill.

SÖDING (1926) showed that this retardation does not take place when the flower-bud is fixed again on the stalk by means of gelatine.

CHOLODNY (1924) removed the tops of Lupinus hypocotyledons. In also removing the central portion of the organ, he obtained hollow cylinders incapable of geotropic curvature. If the coleoptile tips of Zea mays were placed in these stumps, the curvature could be induced again. In this way he stimulated the organs to resume their growth.

In view of these experiments it seemed feasible to induce growth in decapitated flower-stalks by means of growth-substances from other plants. The experiments described in this paper were carried out with the decapitated flower-stalks of Bellis perennis and growth-substances prepared from the coleoptiles of Avena sativa by Dr. F. W. WENT and Mr. H. E. DOLK.

Intact plants of Bellis were planted in zinc boxes. The boxes were placed in a glass-covered glass basin and placed in the green house. In this way they were kept in a moist atmosphere at a temperature of about 20° C. Large flowerbuds were cut off with a rasor, 5 to 7 millimeters below the bud. The remaining stalk was marked with India ink 15 millimeters below the wound. Three series of growth measurements were carried out.

A. By shoving a tightly fitting glass capillary over the stumps, the buds could be fitted on the stump with a drop of 15 % gelatine.

B. By sealing the capillary with wax on one side, and filling them with agar which contained the growth-substances one could observe the influence of foreign growth-substances.

C. Pure agar was used as control.

The increase in length was measured with a millimeter scale ; tenth of millimeters were estimated.

In series A the flower-buds were replaced.

In series B agar with growth-substances from Avena was put on the

TABLE I (flowerbuds, cut off with 7 millimeters of flower-stalk)

	A. flowerbuds increase in length in millimeters in 24 hours			B. agar with growth substances increase in length in millimeters in 24 hours			C. Control increase in length in millimeters in 24 hours	
1	2.6		1	0.7		1	0.2	
2	0.5		2	0.4		2	0.4	
3	1.1	average	3	1.3	average	3	0.2	average
4	0.7	1.36	4	1.5	1.09	4	0.5	0.39
5	0.8		5	1.2		5	0.6	
6	2.8		6	1.5		6	0.5	
7	1.0		7	1.0		7	0.3	

The increase in length of a 15 millimeter zone was measured.

flower-stalks. The quantity of growth-substances amounted to 3500 tip-minute [1]) per flower-stalk.

Serie C were the control experiment without flower-buds or growth substances.

This table shows that the flower-stalks in column B have grown considerably more than in column C and as much or a little less as in column B.

TABLE II (flowerbuds cut off with 5 millimeters of flower-stalk)

	A. flowerbuds increase in length in millimeters in 24 hours			B. agar with growth substances increase in length in millimeters in 24 hours			C. Pure agar increase in length in millimeters in 24 hours			D. Control increase in length in millimeters in 24 hours	
1	1.8		1	5.0		1	0.2		1	0.1	
2	2.2		2	1.8		2	0.7		2	1.1	
3	1.8	average	3	1.3	average	3	0.4	average	3	0.9	average
4	1.0	1.43	4	1.6	1.93	4	0.2	0.40	4	0.6	0.65
5	1.2		5	1.1		5	0.5		5	0.7	
6	0.6		6	0.8		6	0.4		6	0.5	

The increase in length of a 15 millimeter zone was measured.

[1]) One "tip-minute" is the amount of growth-promoting substances, which diffuses out of the coleoptile-tip into an agar disc in one minute. 600 tip-minute may mean therefore the amount diffusing out of 6 tips in 100 minutes or out of 12 tips in 50 minutes.

In series A flower-buds were replaced.

In series B agar was put on with growth-substances, 2000 tip-minute per flower-stalk.

In series C pure agar, without growth-substances, was put on.

D is another control in which the buds were simply cut off.

Then again pure agar was put on 12 flower-stalks (A) of 6 flower-
The increase in length of a zone of 15 millimeters in 24 hours amounted
stalks (B) the flower-bud was simply cut off.
to 0.38 millimeters at A and 0.33 millimeters at B as an average.

This shows that in harmony with table II, pure agar has no influence on the growth.

Therefore the result of this experiment is, that flower-buds as well as growth-substances from Avena accelerate the growth considerably.

The table shows the remarkable fact, that the increase in length of B 1 was exceedingly great. It may be observed that the increase in length with normal plants measured over the same zone, in the same time, amounts to 3—10 millimeters. It therefore appeared possible to make use of such experiments to demonstrate the presence of growth-substances. For, at the beginning of the experiment, it was unknown to me, in which agar the growth-substances were present; I received the information after the measurements were completed.

The growth-substances present in the coleoptiles of Avena and the flower-buds exert an analogous influence on the flower-stalks of Bellis perennis; they both accelerate the growth.

Botanical Laboratory.

Utrecht, November 1927.

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Chemistry. — *The Essential Oil from Gastrochilus panduratum Ridl.* By A. J. ULTÉE. (Communicated by P. VAN ROMBURGH.)

(Communicated at the meeting of December 17, 1927).

The rhizome of this plant, which belongs to the Zingiberaceae family, is characterised by the presence of an essential oil possessing a fresh, pleasant aroma, resembling to some extent the smell of Esdragon and Basicilum oil.

As this rhizome constitutes an ingredient of an Indian rice dish, it is obtainable at the so-called "pasars" (under the Malayan name *Temoe koentji*) but only in small amounts, so that considerable patience was required to collect sufficient material for the examination of this hitherto uninvestigated oil.

The yield of the essential oil is also very small, and varied from only 0.06 % (by weight) in the young fresh rhizomes, to 0.32 % in older material, which however contained much less water.

The specific gravity of the freshly distilled oil varied according to the different methods of extraction, from 0.8636 at 25° (young material) to 0.8731 at 31°. If the oil is allowed to remain exposed to the air, its consistency and specific gravity rapidly increase, a result that would almost certainly be caused by the oxidation of an olefinic terpene present in the oil. After some months preservation in a bottle which was repeatedly opened, the specific gravity had already risen to fully 0.9.

The oil was of a faint yellow colour and showed a dextro-rotation ; in a tube 10 cm. long, different preparations showed rotations from +10°30′ at 27° (young material) to +12°56′ at 32°.

The oil had a neutral reaction ; the presence of esters was determined by boiling the oil with alcoholic potash and subsequent back titration with sulphuric acid.

2.2460 grams oil required 1.20 cm^3 0.5 n alcoholic potash.
1.7950 „ „ „ 0.99 „ „ „ „

If one assumes that the methyl ester of cinnamic acid (see below) is the only ester present, it is thus contained in the oil to an extent of 4.32 and 4.46 % respectively.

Since this ester has been isolated by previous investigators, amongst others VAN ROMBURGH, from several members of the Zingiberaceae, the oil was examined first for this ester. Methyl cinnamate is characterised by its extremely slow rate of distillation when distilled with steam.

100 cm^3 of the oil were thus steam distilled until the residue (10 cm^3) had about the same specific gravity as water. Upon fractionation of this

residue, the temperature rose at once to 220°, the greater part (3 cm^3) distilling between 250—270°. A thick brown residue remained in the flask and could not be further distilled without undergoing decomposition.

After repeated distillation, a small amount of distillate was obtained which boiled between 255—260° and which solidified upon cooling. It melted at ± 33°.

These properties agree with those of methyl cinnamate, so that it is practically certain that this compound is present in the oil.

The oil that had passed over during the steam distillation, was distilled at ordinary pressure, it distilled mainly between 160—220°, the greater part being collected between 180—200°. The specific gravity of the portion that first collected amounted to 0.8553 at 28°.

A resinous residue remained in the flask, and probably represented the oxidation product of an olefinic terpene, consequently during the further examination of the oil, the distillations were carried out under reduced pressure.

The fraction that boiled between 180—200° possessed a distinct smell of cineol, and since this compound is known to occur in various members of the Zingiberaceae, the oil was examined for it. The oil was shaken out with a 50 % solution of resorcinol, separated from the unabsorbed portion and steam distilled. In this way I obtained a liquid that floated upon the aqueous distillate, which after drying was distilled from sodium.

The liquid thus obtained had the following properties :

1. Boiling point : 176—177° ;
2. Specific gravity at 25° : 0.9226 ;
3. The addition product formed with iodol melted at ± 112°.

These properties show that cineol is undoubtedly present.

In order to obtain an idea as to the amount actually present, 5 cm^3 of the original oil were shaken out with the resorcinol solution, whereby 1.6 cm^3 were absorbed, corresponding to a cineol content of 32 %.

Since in this determination, oxygenated compounds, and thus methyl cinnamate, can influence the result, the figure obtained is probably on the high side.

In the fraction distilling over below 200°, and thus probably on the low side, 26.4 % of cineol was found.

During the vacuum distillation (35 mm) of the original oil, at 120° some crystals collected in the condenser. The fraction distilling between 120—140° was now distilled once more at ordinary pressure, up to 215°, and after strongly cooling the distillate, crystallisation took place. After suction filtration, pressing between filter paper, and sublimation, the crystals, which possessed a strong camphoraceous smell, melted at 175°.

When mixed with pure camphor, the melting point rose to 177°.

When thrown upon water, the crystals showed the typical phenomenon of camphor.

The occurrence of camphor in this essential oil is thus proven. The content is however very small and probably amounts to not more than a few percent.

The lower boiling fractions from the vacuum distillation, which had been freed from cineol by shaking out with resorcinol solution, and which contained the supposed olefinic terpene, were repeatedly distilled from sodium in an atmosphere of carbon dioxide in order to prevent resinification, this however, could not be completely avoided.

Finally, a very mobile oil having the following constants was obtained :

1. Boiling point at 13 mm 62—64° ;
2. Specific gravity at 17° 0.8253 ;
3. Refractive index $n_D^{12} = 1.4843$;
4. Optical rotation $\alpha_D^{12} = 11°.42'$.

The boiling point at ordinary pressure (in an atmosphere of carbon dioxide) was 174° at 753 mm, but it altered quickly.

A boiling test carried out in conformity with the method employed by ENKLAAR (see thesis) for ocimene, gave the following results :

At the commencement	the boiling point was	174°
After 2 minutes	,, ,, ,, ,,	175
,, 3 ,,	,, ,, ,, ,,	176
,, 5 ,,	,, ,, ,, ,,	177.5
,, 10 ,,	,, ,, ,, ,,	178.5
,, 25 ,,	,, ,, ,, ,,	180
,, 60 ,,	,, ,, ,, ,,	183

The course followed is thus different from that of ocimene.

By distillation in vacuum (at 13 mm to 80°) a liquid having a specific gravity of 0.8349 at 24° was obtained.

The olefinic terpene was reduced with sodium and alcohol according to the method of ENKLAAR. After steam distillation it was distilled at ordinary pressure from sodium. A fraction was obtained that boiled between 168—169° at 754 mm. *It possessed a smell similar to that of hydromyrcene.*

Optical rotation : $\alpha_D^{12} = 12°54'$.

Specific gravity at 14° 0.805. Refractive index $n_D^{12} = 1.4553$.

The olefinic terpene (in cooled acetic acid) was treated with bromine in acetic acid solution. Hydrogen bromide was evolved and the liquid became coloured, but no solid compound was obtained.

In conclusion it may be mentioned, that the lower fractions obtained from the distillation of the olefinic terpene had a turpentine like smell and a higher specific gravity, so it is probable that another terpene is also present in this essential oil.

Anatomy. — *The influence of the cephalization coefficient and body size upon the form of the forebrain in mammals.* By C. U. ARIËNS KAPPERS.

(Communicated at the meeting of October 24, 1927).

In the following pages I give some figures concerning the general form of the forebrain in less and more cephalized mammals of the same orders, and some data concerning the influence of the body size upon this form in different varieties of the same species, or in different, but equally cephalized species of the same family.

As only the forebrain is concerned here, my indices should not be confused with those obtained by measuring the length-width index of the endocranial cavity, since in this cavity the cerebellum also is included.

Whereas in the anthropoids (also in several other monkeys) and in man, the length-width index of the endocranial cavity corresponds with that of the forebrain, which in these animals and in man covers the cerebellum entirely, it is different with lower mammals, where the forebrain covers the cerebellum only partly or not at all (Fig. 1) and its length therefore is not the same as that of the endocranial cavity.

As a standard for the greatest length of the forebrain I did not take the greatest horizontal measurement (*b*—*b'*) but the greatest possible distance between the front- and hindpole (*a*—*a'* in Fig. 1), which in most animals is an inclining line. The accompanying figure explains my intention. I did this because it gives a better expression of the size of the brain and its increase in length.

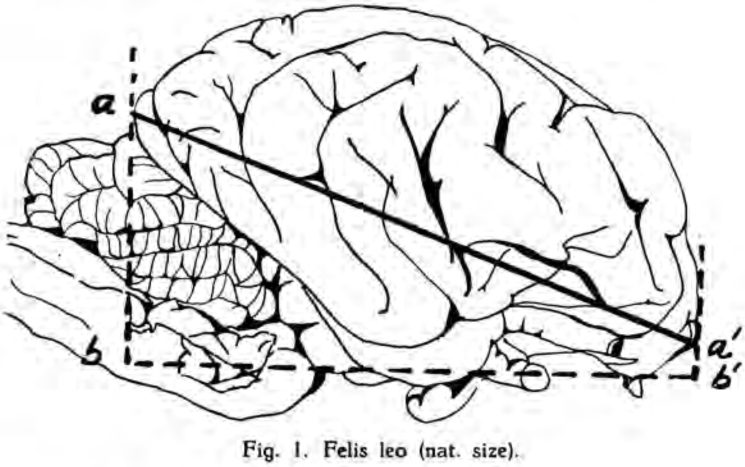

Fig. 1. Felis leo (nat. size).

In the following tables are given the greatest length and width of the forebrain and the index resulting therefrom in more and less cephalized species of the same orders and in some differently cephalized species of the same families. In these animals consequently, the cephalization coefficient *k* of DUBOIS' [1]) formula, $E = kP^r$, is obviously different.

I noted also the total weight of the brain up to 2—3 mm beyond the calamus. *The material used was weighed without meninges* [2]) after preservation in formaline 10 % [3]). In foot notes the brainweight found by others [4]) is mentioned.

MARSUPIALS.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Metachirus opossum (S.Am.)	2.7 cm	1.9 cm	70.4	3.75 gr.
Macropus cervinus	5.— „	4.6 „	92.—	61.— „
Macropus robustus	5.4 „	5.— „	92.6	64.— „
Phascolomys latifrons	4.2 „	4.4 „	105.—	37,5 „

Among marsupials (table 1), according to DUBOIS, a difference in the cephalization coefficient occurs between the North American (Virginian) opposum [5]) and the Macropodidae, that of the latter being about $1\frac{1}{2} \times$ as large as that of the opossum. This apparently corresponds with a greater forebrain index, a distinct brachencephaly in the Macropodidae.

[1]) DUBOIS. The proportion of the weight of the brain to the size of the body in mammals. Verhandelingen der Kon. Akad. v. Wetenschappen, Vol. 5, N⁰. 10, 1897 (the same in Bulletin de la Société d'Anthropologie de Paris 1897 and Arch. f. Anthropologie Bnd. 25, 1898).

DUBOIS. On the significance of the large cranial capacity of Homo neandertalensis. These Proceedings Vol. 23, 1921,

DUBOIS. Phylogenetic and ontogenetic increase of the volume of the brain in Vertebrata. These Proc. Vol. 25, 1922,

DUBOIS. The brainquantity of specialized genera of mammals. These Proc. Vol. 27, 1924.

[2]) This may explain the slightly smaller weights of some of my specimens in comparison with those found by others.

[3]) According to FLATAU the brainweight increases about 1 % after preservation (during 15 months) in 10 % formaline. (Beitrag zur technischen Bearbeitung des Zentral-Nervensystems, Anat. Anzeiger, Bnd. 13, 1897).

[4]) Specially the following: KOHLBRUGGE, Zool. Ergebn. einer Reise in Niederl. Ost Indien, Leyden 1891, p. 139 and Natuurk. Tijdschr. v. Ned. Indië, Deel 55, p. 261, Batavia 1896, Zoogdieren v. d. Tengger. Further: Zeitschr. f. Morph. und Anthrop., p. 43—55, Bnd. 2, 1890, and Monathschr. f. Psychiatrie und Neurologie, 1901.

WEBER, Vorstudien über das Hirngewicht der Säugethiere, Festschr. f. Carl Gegenbaur, Leipzig 1896 and these Proceedings 1896.

ZIEHEN, Anatomie des Zentralnervensystems, Fischer, Jena, 1899.

SPITZKA, Brainweight of Animals etc. Journ. of Comp. Neurol., Vol. 13, 1903.

[5]) Didelphis virginiana has a brainweight (ZIEHEN) of 3.9—4.5 gr.

The remarkable brachencephaly of the Wombat is not a result of a greater cephalization compared with the Macropodidae, but is due to flatness of the front part of its skull, which gave it the epithet "latifrons".

In Rodents with various cephalization coefficients the following proportions were found :

RODENTIA.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Cavia cobaya	2.24 cm	2.10 cm	93.7	4.3 gr.
Dasyprocta aguti	3.44 „	3.35 „	97.4	17.5 „ [1]
Hydrochoerus capybara	5.69 „	5.69 „	100.—	58.— „ [2]
Dolichotis patagonica	3.8 „	4.55 „	120.—	31.5 „

Here also, the increase of brachencephaly in animals with a greater cephalization coefficient is obvious, since in Hydrochoerus and Dolichotis this coefficient is more than twice as large as in Cavia aperea, the wild form of Cavia cobaya.

The same is observed if we compare the poorly cephalized artiodactyle, Sus scrofa, with the more cephalized artiodactyles, the giraffa and dromedary, and with the perissodactyle horse. The cephalization coefficients of the giraffa, dromedary and horse are twice as large as that of Sus. It is obvious that the last three have also a higher index. Upon the very high index of the dromedary, the youth of this specimen also has influence (compare page 76).

UNGULATES.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Sus scrofa dom.	7.28 cm	5.68 cm	78	112.— gr. [3]
Bos taurus	10.6 „	10.09 „	95.2	403.— „ [4]
Camelopard. giraffa	13.5 „	11.96 „	88.6	665.— „ [5]
Camelus dromed. (young sp.)	9.8 „	10.— „	102.—	478.— „ [6]
Equus caballus	10.79 „	10.17 „	94.2	549.— „ [7]

[1] SPITZKA found variations of 16—21 gr. WEBER found 20 gr.

[2] A specimen weighed by WEBER (apparently larger) showed 75 gr.

[3] ROGNER (quoted from ZIEHEN) found variations from 111—120 gr.

[4] BISCHOF found variations from 400—500 gr.

[5] WEBER's largest specimen had a brainweight of 680 gr.

[6] LAPICQUE weighed a dromedary cerebrum of 650 gr., ZIEHEN one of 655 gr.

[7] WEBER found 615 gr., SPITZKA only 519.5 gr. COLIN 593—640 gr.

The brachencephaly of the *elephant* is striking, especially if we compare this animal with its nearest relative among living ungulates, *Procavia dorsalis*, but also in comparison to the Tapir. This is again in accordance

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Procavia dorsalis	3.92 cm	3.1 cm	79.—	19.7 gr. [1]
Tapirus indicus	8.— „	7.7 „	96.2	204.9 „ [2]
Elephas indicus	23.— „	28.5 „	124.—	5474.— „ [3]

with the greater cephalization of the elephant which, according to DUBOIS, is about 4 × greater than its nearest living relative, the ungulate Procavia, and also more than the tapir.

If we now study the Carnivora, Viverridae, Mustelidae, Procyonidae and Ursidae and among the Cetacea an Odontocete and a Mysticete, it is obvious that here again the more cephalized forms are more brachencephalic. So the Viverridae generally have the smallest cephalization coefficient.

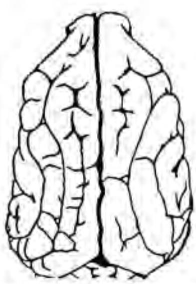

Fig. 2. Paradoxurus musanga. adult (nat. size)

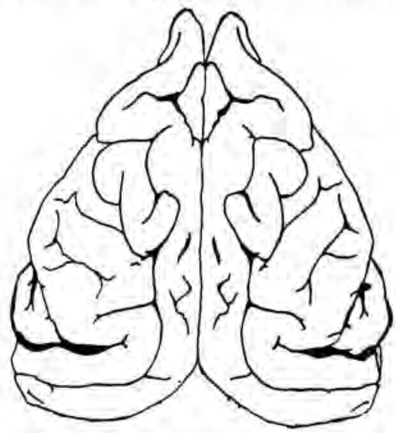

Fig. 3. Lutra vulg. adult, (nat. size)

CARNIVORES.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Mungos mungo	3.4 cm	2.5 cm	73.5	12.— gr. [4]
Paradoxurus musanga	4.2 „	3.2 „	76.2	23.5 „ [5]
Arctitis binturong	5.— „	3.9 „	78.—	37.— „

[1]) WEBER found with related Hyrax capensis variations of 19.2 gr. to 21 gr.

[2]) WEBER found 265 gr.

[3]) The heaviest specimen of CRISP had a brainweight of 5430 gr.

[4]) Herpestes mungo by WEBER: 10.9 gr.

[5]) WEBER: 22 gr.

CARNIVORES (*Continued*).

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Putorius putorius [1]	3.1 cm	2.3 cm	74.2	7.7 gr. [2]
Meles taxus	5.46 „	4.5 „	82.4	45.— „
Lutra vulgaris	5.8 „	5.— „	86.2	45.6 „ [3]
Nasua rufa	5.2 cm	3.71 cm	72.0	29.9 gr. [4]
Procyon cancrivorus	7.— „	5.7 „	81.4	93.— „
Ursus arctos (young spec.)	9.—cm	8.— cm	88.8	234.— gr. [5]
Ursus malayanus	9.4 „	8.5 „	90.4	313.— „ [6]
Ursus maritimus (young sp.)	11.— „	10.1 „	92.—	424.— „ [7]

CETACEA.

Balaenoptera sulfurea	22.—cm	29.8 cm	135	5676.— gr. [8]
Phocaena communis	7.8 „	12.4 „	160	389.— „ [9]

Only Paradoxurus and Arctitis have (DUBOIS) a $1\frac{1}{4}\times$ larger cephalization coefficient than the Mustelid Putorius, and among Mustelids Meles and Lutra, have the greatest cephalization coefficient. My table shows that these two are also the most brachencephalic while Putorius is less brachencephalic than the Viverridae.

The largest cephalization coefficient of all carnivora, however, occurs in the Ursidae whose cephalization coefficient on the average is twice as large as that of the Viverridae (DUBOIS l.c. tertio p. 322) and $1\frac{1}{2}\times$ as large as that of the Nasuae. In accordance with this it appears that the Ursidae are also the most brachencephalic.

It is remarkable that Procyon cancrivorus (the crab washbear), which in the zoological system is more related to the Nasuae, also belonging to

[1]) With the small Grison and Mephitis I found broader brains (80 and 82), notwithstanding their small cephalization coefficient. I ascribe this to the smaller size of these animals (Cf. p. 8 and 9).

[2]) DUBOIS: 7.8 gr.

[3]) SPITZKA found with Lutra 39 gr.; HUSCHKE (citated after ZIEHEN) 42—51 gr.

[4]) SPITZKA found variations from 29 to 41 gr.

[5]) SPITZKA found with Ursus Americanus variations from 192 to 248 gr.; WEBER found a cerebrum of Ursus arctos weighing 407 gr.

[6]) WEBER found 325 gr.

[7]) Ursus maritimus may, according to WEBER, reach a fresh brainweight of 530 gr.

[8]) WEBER's specimen of Balaenoptera sibbaldi had a brainweight of 7000 gr.

[9]) ZIEHEN found an average of 512 gr.

the Procyonidae, has a larger cephalization coefficient and accordingly a larger forebrain index than the Nasuae.

Among the Cetacea, the Odontocetes have a greater cephalization coefficient than the much bigger Mysticete Balaenoptera (DUBOIS). And again this corresponds with a greater brachencephaly in Odontocetes.

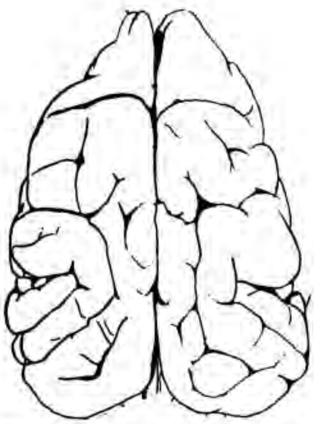

Fig. 4. Nasua rufa, adult (nat. size)

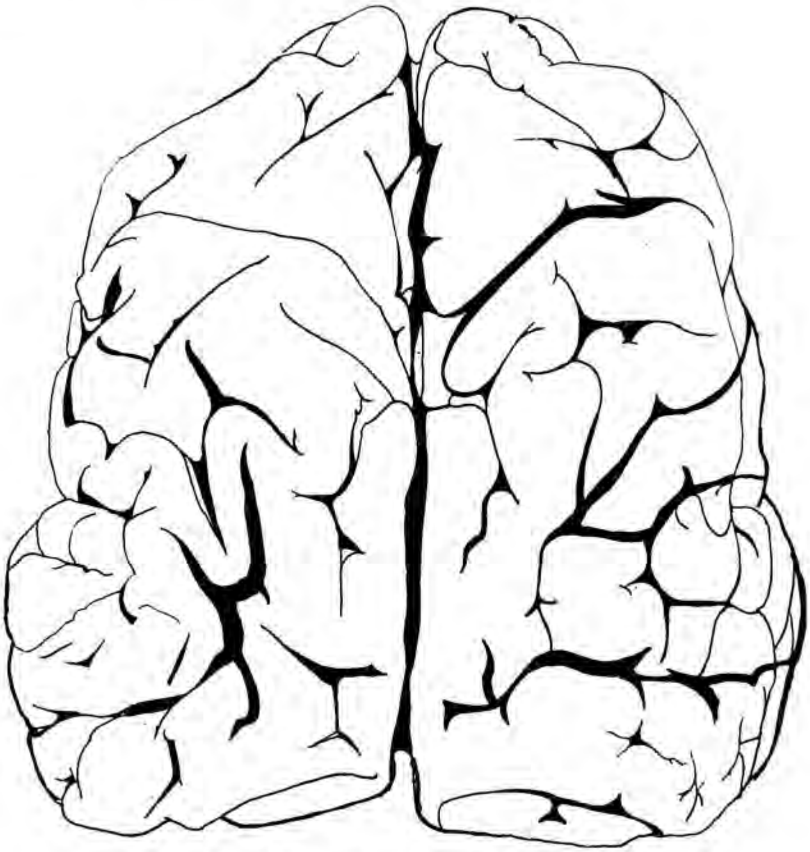

Fig. 5. Ursus maritimus, adult (nat. size)

PROSIMIAE.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Lemur mongoz	4.53 cm	3.8 cm	83.8	23.5 gr.[1]
Chiromys madagascariensis	4.7 "	4.5 "	95.8	34.6 gr.[2]

Of the Prosimiae, Chiromys has a cephalization coefficient twice as large as the Lemurs. Correspondingly its forebrain is more brachencephalic.

[1]) WEBER found variations of 21—28 gr.

[2]) WEBER found a brainweight of 43 gr.

WESTERN APES

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Callithrix jacchus	3.5 cm	2.46cm	70.3	10.— gr. [1]
Cebus apella	7.— „	5.5 „	78.5	69.— „ [2]

In the Platyrrhine apes, the coefficient of Cebus is four times as large as that of Callithrix. Accordingly we see that its forebrain index is less dolichencephalic than that of Callithrix, although the latter has a smaller body and consequently one might expect a larger index in Callithrix (see p. 72, 73, 74).

In Katarrhine monkeys and in Anthropoids I found the following indices.

EASTERN APES.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Cercopithecus pygerythrus	6.8 cm	5.5 cm	80.9	76.— gr.
Macacus nemestrinus	7.5 „	6.— „	80.—	103.8 „ [3]
Symphalangus synd. (small spec..)	7.6 cm	6.—cm	80.—	107.— gr. [4]
ANTHROPOIDS.				
Pan (Troglodytes) Schweinf.	10.8 „	9.1 „	84.2	374.— „ [5]
Simia sat. (small spec.)	9.8 „	8.6 „	87.7	295.— „ [6]

The cephalization coefficient of these Anthropoids appears to be twice as large as that of the Katarrhine monkeys (DUBOIS) and accordingly they are more brachencephalic [7]). The Gibbon (Symphalangus syndac-

[1]) SPITZKA found variations of 7—9 gr. in Jacchus vulg.

[2]) For Cebus capucinus WEBER also found 69 gr.

[3]) SPITZKA found in Mac. nemestrinus an average brainweight of 110 gr.

[4]) KOHLBRUGGE found variations from 102 to 130 gr.

[5]) The Troglodytes of WEBER varied from 340—348 gr. One specimen of SPITZKA weighed 380 gr.; that of MÖLLER 391 gr.; one of MARSHALL 412 gr.

[6]) The statements of WEBER vary from 306—339 gr. DENIKER and BOULART reported a cerebrum of 400 gr., just as MILNE EDWARDS. R. FICK, one of even 440 gr. (quoted from ZIEHEN)

[7]) The index of the Gorilla brain and the endocranium varies very strongly. Two young cerebra of Mad[elle] COUPIN gave me an average of 84.4, two endocranial casts received from ELLIOT SMITH 79,2. BOLK (these Proceedings 1925) found variations between 80.6 and 85.9 (an average of 83.25) and besides on specimen with an index of 72.2. HARRIS, having investigated the greatest number of endocrania, found variations from 72.1 to 86.8. (American Journal of physical anthropology, 1926)

tylus), which usually is considered as a transition between Cercopitheci and Anthropoids, stands closer to the first in this respect.

From these comparisons it appears that generally the more highly cephalized species of an order tend to brachencephaly. Their forebrain has a rounder instead of a longer form. Deviations of this rule are discussed p. 78—80. This tendency must be explained by the fact that if a brain increases more than in the usual proportions with the body, its volume increases more than its surface, the latter being limited by the skull. Consequently it approaches the form of the globe, since a globe contains the greatest volume with the smallest surface.

As the more highly cephalized species in my tables are also generally larger animals, one might be inclined to consider body size as a factor, causing brachencephaly. That this is not so, however, clearly appears from the following tables, in which smaller and larger, but equally cephalized representatives of the *same species*, or of the *same family*, are compared, so that the influence of body size can be traced separately.

From these tables it appears that the larger Canidae, Felidae, Nasuae, Pinnipedia and Odontoceti are *less brachencephalic* than the smaller ones, so that the brachencephaly cannot be the result of a larger body. The same appears with the Giraffidae, Antilopes, Cervidae (excepted Cephalobus and Capreolus) and Equidae. And again this is observed in monkeys, if we compare equally cephalized species of different size.

In the following groups the cephalization coefficient k is about the same. The species differ only in size. It is obvious that in nearly all cases an *increase in body size* is accompanied with a *decrease of*

Influence of body size.

CANIS. [1]

Animal	Forebr. length	Forebr. width	Forebr. index.	Total brainweight
Griffon	5.6 cm	5.05 cm	90.2	55.2 gr.
Spaniel	7.1 „	5.75 „	81.—	90.— „
St. Bernard	7.4 „	5.70 „	77.—	97.— „ [2]
German dog	7.7 „	5.70 „	74.—	97.5 „

[1]) The weights which I found for these dogs are nearly all lower than those found by CORNEVIN (Études sur le poids de l'encéphale dans les diverses races des espèces domestiques. Journ. de Médicine vétérinaire et de Zootechnie, Année 1899 p. 248) and RÜDINGER, Ueber die Hirne verschiedener Hunderassen (Verhandl. der Anat. Gesellsch. in Strassburg 1894 p. 173). I presume that they made the weighings with the meninges (see note 2, p. 66).

[2]) A very large specimen examined by WEBER had even a brain weight of 123 gr.

FELIDAE.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Felis bengal. minuta	3.7 cm	3.5 cm	94.6	23.5 gr. [1]
Felis domestica	4.2 „	3.9 „	93.—	34.— „ [2]
Felis nebulosa	5.2 „	4.8 „	92.3	46.— „
Felis pardalis	5.7 „	5.18 „	91.—	72.— „
Felis concolor	7.4 „	6.25 „	84.5	149.5 „ [3]
Felis leo (small spec.)	8.3 „	7.3 „	88.—	195.— „ [4]
Felis tigris	8.8 „	7.4 „	84.1	202.— „ [5]

NASUA.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Nasua narica	4.6 cm	3.6 cm	78.3	23.3 gr.
Nasua rufa	5.2 „	3.71	72.—	29.9 „

PINNIPEDIA.

Phoca vitulina (small spec.)	7.1 cm	8.3 cm	116.9	205.— gr [6]
Otaria jubata	9.7 „	10.1 „	104.—	268.— „ [7]
Zalophus californianianus	10.1 „	10.1 „	100.—	347.— „ [8]

ODONTOCETI.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Phocaena communis	7.8 cm	12.4 cm	160	389.— gr. [9]
Tursio tursiops	12.85 „	16.85 „	132	± 1350.— „ [10]

[1]) WEBER: 23.6 gr.

[2]) WEBER found an average of 31 gr.

[3]) A specimen of WEBER had a brainweight of 137.5 gr.

[4]) WEBER found as average brainweight 216 gr.

[5]) WEBER: 246 gr.

[6]) WEBER found variations of 242—290 gr.

[7]) These measures have been made on the drawing of MURIE (who gives also this weight) in his descriptive anatomy of the sealion (Otaria jubata). Transactions of the zool. Society of London. Vol. 8, 1874.

[8]) SPITZKA found with this animal 335 gr. brainweight, WEBER 347—399 gr.

[9]) ZIEHEN found an average brainweight of 512 gr.

[10]) WEBER weighed a cerebrum of 1886 gr. It is remarkable that my specimen of Balaenoptera sulfurea (a Mysticete) has a somewhat larger index than Tursio. Possibly the heavy cerebrum of my Balaenoptera is a little broadened by lying on its basis.

ANTILOPES.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Gazella dorcas	6.1 cm	5.5 cm	90.1	78.— gr.
Gazella soemmer.	7.5 „	6.7 „	89.3	128.5 „
Cephalophus maxwelli	*5.35 „*	*4.7 „*	*87.8*	*47.— „* [1])
Damaliscus albifrons	9.12 „	7.87 „	86.3	254.5 „
Taurotragus oryx liv.	12.2 „	10.3 „	84.4	437.— „
GIRAFFIDAE.				
Ocapia Johnstoni	10.6 cm	10.—cm	94.—	450 gr. [2])
Camelopard. giraffa	13 5 „	11.96 „	86.6	665 „

brachencephally or an increase of *dolichencephali.* This corresponds with the phenomenon found by KLATT [3]) that larger animals of the same family have longer skulls.

Consequently the greater brachencephaly in more cephalized species is a result of their greater cephalization only.

The elongation of the brain in larger animals of the same species is a phenomenon analogous to what is seen in onto-

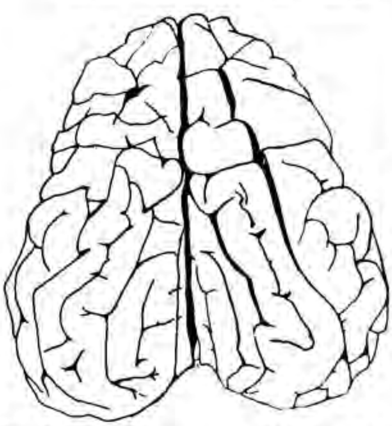

Fig. 6. Cervulus muntjac adult (nat. size).

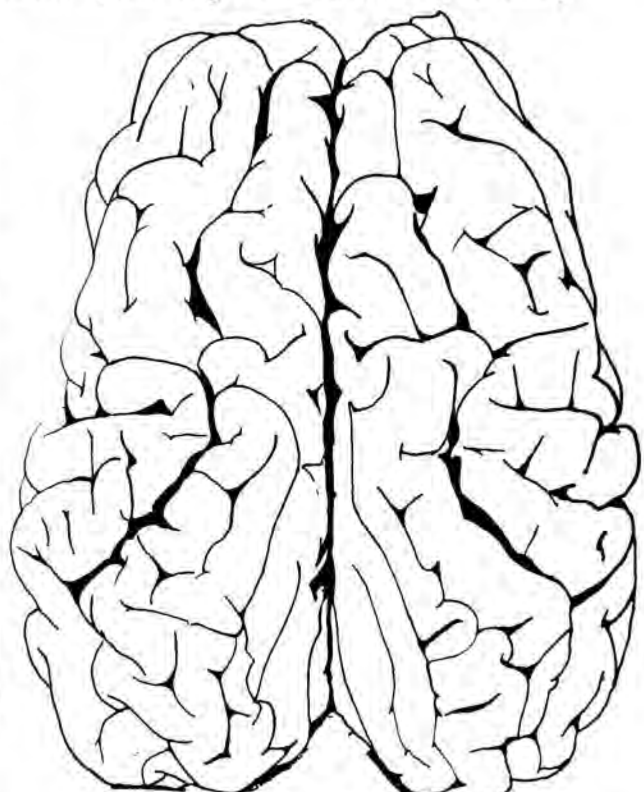

Fig. 7. Rucervus duvauc. adult (nat. size).

[1]) WEBER's heaviest cerebrum weighed 41.1 gr.

[2]) After the statement of BLACK: Journ. of Comp. Neurol. Vol. **25**, 1915.

[3]) KLATT. Ueber den Einflusz der Gesamtgrösze auf das Schädelbild nebst Bemerkungen über die Vorgeschichte der Haustiere. Archiv für Entwicklungsmechanik. Bnd **36**, 1913.

CERVIDAE.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Cervulus muntjac (small sp.)	5.23 cm	4.49 cm	85.8	44.— gr.
Rusa porcinus	8.2 „	7.— „	85.4	143.— „ [1]
Dama dama	8.22 „	6.99 „	85.0	167.— „
Rusa hipp. molucc.	8.34 „	7.1 „	85.1	186.— „
Capreolus caprae	*6.79 „*	*5.70 „*	*83.9*	*85.— „* [2]
Rucervus eldi	9.65 „	7.4 „	79.3	202.— „
Rucervus duvauceli	10.7 „	8.3 „	77.5	297.— „
EQUIDAE.				
Equus asinus	9.2 cm	8.74 cm	95.—	334.7 gr. [3]
Equus caballus	10.79 „	10.17 „	94.2	549.— „ [4]
SIMIAE.				
Cercopithecus cynosurus	5.9 cm	4.8 cm	81.3	52.— gr. [5]
Cercopithecus pygerythrus	6.8 „	5.5 „	80.9	76.— „
Macacus rhesus	6.8 cm	5.57 cm	82.—	76.5 gr. [6]
Macacus nemestrinus	7.5 „	6.— „	80.—	103.8 „ [7]

genetic development, as appears from the ontogenetic tables (comp. also fig. 8 with 3 and 9 and 10).

Fig. 8. Lutra vulg. neon. nat. size). Comp. this figure with fig. 3.

The same observation is made by MANOUVRIER and by M^elle^ COUPIN for the Gorilla and by the latter also for the Chimpanzee and Cercopithecus [8]). I have mentioned elsewhere that the same occurs in man [9]).

The fact that the length of the brain generally increases more than the width, in the individual development as well as in the larger representatives

1) WEBER: 142 gr.

2) WEBER found in Cervus capreolus 98 gr. HUSCHKE (quoted from ZIEHEN) 94 gr.

3) COLIN found with Equus asinus (quoted from ZIEHEN) 385 gr.

4) WEBER: 615 gr., COLIN: 593—640 gr.; SPITZKA 519.5 gr.

5) SPITZKA: 68.5 gr., WEBER: 70.5 gr.

6) SPITZKA found with Macacus rhesus an average brainweight of 80 gr. with variations of 71 to 98 gr.

7) SPITZKA found with Macacus nemestrinus an average brainweight of 110 gr. with variations of 84 to 128 gr.

8) F. COUPIN. Le développement comparé du cerveau chez l'homme et chez les singes. Revue scientifique 1926 p. 9 and fig. 2. It is remarkable that Mad[elle] COUPIN could not state this with Semnopithecus, similarly I did not find it in Bos taurus.

9) ARIËNS KAPPERS, Indices for the anthropology of the brain applied to Chinese, dolicho- and brachycephalic Dutch, foetuses and neonati. Proceedings Kon. Akademie. Amsterdam, Vol. **30**, 1926.

of the same species or family cannot be explained only by abscence of necessity of the brain in larger animals to tend to globular form. This appears from the fact that among Cetacea, the brain which in smaller forms is flattened antero-posteriorly, in larger forms approaches the globular form because the index decreases in the direction of hundred.

CARNIVORES.

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Lutra vulg. neon.	2.6 cm	2.7 cm	104.—	6.6 gr.
„ „ adult.	5.8 „	5.— „	86.2	45.6 „
Felis dom. neon.	2.—cm	2.1 cm	105.—	4.15 gr.
„ „ adult.	4.2 „	3.9 „	93.—	34.— „

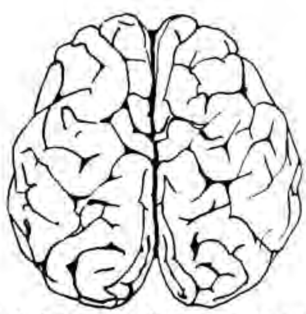

Fig. 9. Sus scrofa dom. neon. (nat. size)

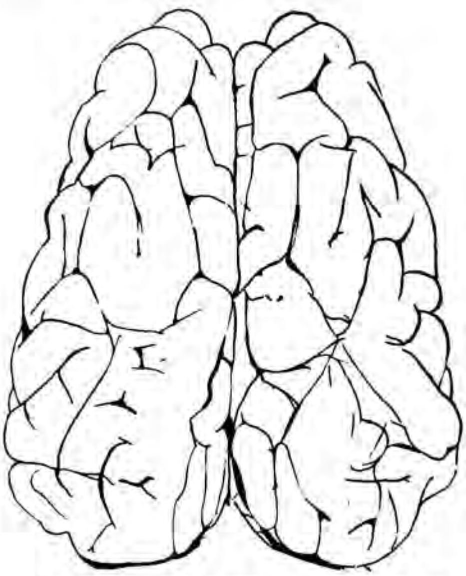

Fig. 10. Sus scrofa dom. (± nat. size) adult.

UNGULATES.

Sus scrofa neon.	3.5 cm	3.2 cm	91.4	14.— gr.
„ „ adult.	7.9 „	6.4 „	81.—	112.— „
Rusa porcinus neon.	5.4 cm	5.2 cm	96.3	54 gr.
„ „ adult.	8.2 „	7.— „	85.4	143 „
Camelop. giraf. neon.	10.2 cm	10.9 cm	108.2	466 gr.
„ „ adult.	13.5 „	11.96 „	88.6	665 „

Moreover as the extension of the whole skull (not only the endocranium) takes place especially in antero-posterior direction, we must consider the possibility that the *skull* itself in larger animals of the same family is relatively more lengthened, anteriorly because the insertion of teeth requires more place and posteriorly by the influence of the stronger cervical muscles and ligaments. On the other hand the skull narrows relatively in the larger specimens by the pressure of the larger masticatory muscles.

There is still another question that I want to discuss.

DUBOIS [1]) (l.c.) and LAPICQUE [2]) have shown that the exponent r in the formula $E = kP^r$ between adults of the same species (e.g. various dogs) is smaller ($^5/_{18}$) than in animals of different species ($^5/_9$).

DUBOIS (1922) called the first exponent "ontogenetic", the second, "phylogenetic", presuming that in larger varieties of the *same* species, as in ontogenetic (especially postembryonic) development, the increase of the brain is due merely to an *enlargement of the neurones* [3]). This neurone enlargement according to CONKLIN and G. LEVI, follows an exponent of about $^5/_{18}$ with respect to the body weight.

On the other hand, in more *cephalized* animals of various size in the larger species the enlargement of the neurones is accompanied by a *numeric increase of neurones*, the latter fact according to this anthor being responsible for the higher exponent $^5/_9$.

Is there anything in my tables that may confirm this ? Does the increase of the brain in larger varieties of the *same* species show more conformity with the enlargement of the brain during postembryonic development than it does in larger but not closely related species ?

Comparing the forebrain indices of the different varieties of the species Canis famil., we see that the difference in brainweight of only 42.3 gr. is accompanied by an increase of 18.2 units in the forebrain index.

Comparing members of the family of the Felidae (which is still very homogeneous) we find, with a maximal difference in the brain weight of 78.5 gr., a much smaller index difference, viz. 10.5. In the less homogeneous family of the Cervidae we find with a weight difference of 253 gr. an index difference below 10, (8.3), and in the still less homogeneous group of Antilopes with a still higher weight difference (359 gr.) an even smaller index difference (5.3).

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[1]) DUBOIS. Ueber die Abhängigkeit des Hirngewichts von der Körpergrösze beim Menschen. Archiv für Anthropologie Band **25**, Heft **4**, 1898.

[2]) LAPICQUE. Sur la relation du poids de l'encéphale au poids du corps. Comptes rendus de la Société de Biologie, Paris 1898.

LAPICQUE. Le poids encéphalique en fonction du poids corporel entre individus d'une même espèce. Bulletin de la Société d'Anthropologie de Paris, 1907.

[3]) This is true for the Norwegian rat and tame albino rat (*Sugita*) compare DONALDSON: The significance of the brainweight. Transactions of the annual meeting of the American neurological Association, Philadelphia, 1924. — Archives of Neurology and Psychiatry, March 1925, p. 385.

If now we consider the index differences during postembryonic development, we find them very large, all above 10; Lutra 15.8, Felis 12, Rusa 10.7, Sus 10.4. The lowest of these still comes in the group of the very homogeneous Felidae. All other index differences are higher and approach varieties of one species (cf. Canis familiaris).

In other words the elongation of the brain in larger animals shows a greater conformity with the ontogenetic elongation if a group is more homogeneous and its elongation resembles the ontogenetic elongation in varieties of the *same* species. This appaiently is in conformity with DUBOIS' supposition.

That the increase of the brain in fullgrown varieties of the same species or of closely related species of one family is only based upon cell enlargement (though this is surely the most important factor) is perhaps too strongly expressed, the more so since we know that some, though not much cell division also occurs during postembryonic development (AGDUHR) [1]).

In regard to the question of ontogenetic brain-body-weight relations I refer to the recent investigations of ANTHONY and COUPIN [2]), on what they call the "indice de valeur cérébrale": $\frac{PE}{PE'}$, in which PE is the real brain weight of a foetus and PE' the calculated brain weight which that foetus would have *if* it were an adult of this size.

We have seen that in animals with a higher cephalization coefficient the greater brachencephaly is a consequence of the greater degree of development of the brain compared to the body, in consequence of which the brain tends to the largest volume with the smallest surface (skull). In very primitive forms, however, particularly the most primitive recent species, the forebrain is very short, and consequently brachencephaly occurs, here also, although now it is not a consequence of the greater volume, but of the shortness of the forebrain.

So Ornithorrhynchus and Echidna have an index of 111.5 and 127.6.

Among Marsupialia the primitive Caenolestes and Orolestes are conspicuous for the shortness of their forebrain, in comparison to the Opossum. In Caenolestes obscurus the forebrain index (measured after a drawing of C. J. HERRICK) [3]) is 125.5; in Orolestes inca (after the drawing of Miss OBENCHAIN) [4]) 137.4. Orycteropus, which is also a primitive animal, has, after a drawing of SONNTAG and WOOLLARD [5]) a

[1]) AGDUHR. Is the postembryonic growth of the nervous system due only to an increase in number of the neurones? Proceedings Royal Acad. Amsterdam Vol. **27**, 1919.

[2]) ANTHONY et COUPIN. Introduction à l'étude du développement pondéral de l'encéphale. l'Indice de valeur cérébrale au cours de l'évolution individuelle. Zagreb. 1925—1926.

[3]) The brain of Caenolestes. Publications of the Fieldmuseum, Zool. Vol. **14**, 1925.

[4]) The brains of the South American Marsupials Caenolestes and Orolestes, Publications of the Fieldmuseum, Zool. Vol. **14**, 1925.

[5]) A monograph on Orycteropus afer: II Nervous system, sense organs and hairs. Proceedings of the Zool. Society of London 1925. WEBER counts this animal among the Ungulates.

PRIMITIVE ANIMALS COMPARED WITH NON-PRIMITIVE (—).

Animal	Forebr. length	Forebr. width	Forebr. index	Total brainweight
Monotremes.				
Ornithorrhynchus	2.6 cm	2.9 cm	111.5	11.— gr.
Echidna	4.— „	5.1 „	127.6	19.— „
Marsupials.				
Caenolestes obscurus	0.954 cm	1.2 cm	125.5	—
Orolestes inca.	0.914 „	1.24 „	137.4	—
(Macropus rob.)	5.4 „	5.— „	(92.6)	(64.— gr.)
Orycteropus	4.9 cm	5.4 cm	110.—	—
Ungulates.				
Perissodactyla				
Rhinoceros unic.	12.— „	13.3 „	111.—	864.— gr.
(Equus caballus)	10.79 „	10.17 „	(94.2)	(549.— „)
Artiodactyla				
Hippopotamus	10.3 „	11.— „	107.—	582.— „
Hippopot.; Garrod	—	—	111.6	—
(Sus scrofa dom.)	7.9 „	6.4 „	(81.—)	(138.5 „
Tragulus mem.	3.27 „	3.3 „	101.—	18.5 „
(Cervulus muntjac, small sp.)	5.23 „	4.49 „	(85.8)	(44.— „)
Sirenia.				
Halicore; Dexler [1]	6.9 cm	7.14 cm	103.5	282 gr.
Manatus; Garrod	6.9 „	7.3 „	106.—	?
Carnivores.				
Viverra civetta	4.5 cm	4.0 cm	88.—	40.5 gr.
(Paradoxurus musanga)	4.2 „	3.1 „	(73.8)	(23.5 „)

very large forebrain index, viz. 110. The same is observed in Tragulus memmina (total brainweight 18.5 gr.) [2]), which is more primitive than the Antilopes and also has a smaller cephalization coefficient and yet shows a higher index than the latter, viz. 101.

While in the above named cases the small size of these animals might be considered as a factor in the increase of brachencephaly, this is not the

[1]) Das Hirn von Halicore dugong. Morph. Jahrb. Bnd. **45**, 19.

[2]) WEBER 17.1 gr.

case with the Rhinoceros, Hippopotamus and the Sirenia. Yet Rhinoceros sumatrensis in GARROD's drawing [1]) has an index of 119. OWEN's [2]) Rhinoceros unicornis has an index of 111 with a brain weight of 864 gr. A Hippopotamus of my collection has an index of 107 [3]) with a brain weight of 582 gr. (WEBER).

Halicore, calculated after the drawing of DEXLER, has an index of 103.5; Manatus, after that of GARROD [4]), 106.

Among the Viverridae, the most primitive and poorly cephalized Viverra civetta has an index of 88.8, while its relatives Mungos and Paradoxurus, though smaller, have an index of 73.5 and 76.2 respectively.

As stated above, however, the brachencephaly of the forebrain in these primitive animals has a very different character from the brachencephaly in highly cephalized animals, being the result of the shortness of the brain in the first case, whereas in highly cephalized animals it is due to greater width. The latter is a result of increase, the former, of lack of development of the forebrain.

CONCLUSIONS.

1. Higher cephalization within a certain group of animals causes brachencephaly.

2. Larger animals are usually more dolichencephalic than similarly cephalized small animals of the same group [5]).

3. The forebrain of primitive now living [6]) mammals is usually strongly brachencephalic, but this brachencephaly is of an entirely different character.

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[1]) See GARROD. Transactions of the Zool. Society of London. Vol. **10**, 1877—1879 The brain of the Sumatra Rhinoceros.

[2]) OWEN. On the Anatomy of the Indian Rhinoceros (unicornis) Ibidem vol. **4**, 1862.

[3]) GARROD's specimen had even a forebrain index of 111.6, after his drawings in the transactions of the Zool. Society of London Vol. **11**, 1885.

[4]) GARROD, Ibidem, Vol. **10**, 1877 (see also MURIE, ibidem Vol. **11**, 1879).

[5]) For the skull, this phenomenon is often stated in man. See KOHLBRUGGE: Tijdschrift v. h. Kon. Nederl. Aardrijksk. Genootschap 2de Serie. Deel **28**, 1911, p. 785.

[6]) Concerning extinct species, little may be said in this respect, because as a rule only data about the endocranium are available. But these animals are generally less cephalized (DUBOIS) than their present relatives.

Physics. — *On the change of the dielectric constant of liquid helium with the temperature. Provisional measurements.* By M. WOLFKE and W. H. KEESOM. (Comm. N⁰. 190a from the Physical Laboratory at Leiden.)

(Communicated at the meeting of December 17, 1927).

§ 1. *Introduction.* The dielectric constant of liquid helium has been measured at the temperature of the boiling-point under normal pressure by M. WOLFKE and H. KAMERLINGH ONNES [1]). Now it was of the greatest importance to get to know the course of the dielectric constant of liquid helium with the temperature, particularly in connection with the peculiar density-maximum, found by KAMERLINGH ONNES [2]) and more closely studied by KAMERLINGH ONNES and BOKS [3]) and which until now has not yet found an explanation.

For these measurements by which very small differences in the dielectric constant had to be accurately determined, the method, applied to former ones, was no more sufficient. For that reason a new method for measuring was developed for this purpose by one of us at the Physical Laboratory of the Technical Institute at Warsaw. This method and the corresponding arrangement are described in the following report.

§ 2. *Measuring-method. Apparatus.* The measuring-method is literally a compensation-method by which the changes of the dielectric constant in the measuring-condenser are compensated by measurable changes of the capacity of a cylinder-condenser which can be very accurately adjusted by a micrometer-screw. This cylinder-condenser we shall call in future „micro-condenser". This microcondenser is connected parallel to the measuring-condenser which contains liquid helium. The capacity of both these condensers together has been compared with a constant capacity by means of a method depending on the beats of two high-frequent electro-magnetic oscillation circuits.

The diagram of connections is represented in Fig. 1. It consists of two mutually independent oscillation circuits according to MEISSNER: 1 and 2. The oscillation circuits are fed by a joint anode-battery of 160 V in the constant-current connection [4]). In order to avoid a mutual influence of

[1]) These Proc., **27**, 621, 1924. Comm. Leiden, N⁰. 171*b*.

[2]) These Proc., March 1911, p. 1093. Comm. Leiden N⁰. 119.

[3]) Reports and Communications presented by H. KAMERLINGH ONNES to the 4th International Congress of Refrigeration, London 1924. Comm. Leiden N⁰. 170*a*.

[4]) Comp. J. ZENNECK und H. RUKOP, Lehrbuch der drahtlosen Telegraphie, 1925, p. 607, fig. 547.

both the circuits, the high-frequency currents are arrested by both the choking-coils D_1 and D_2.

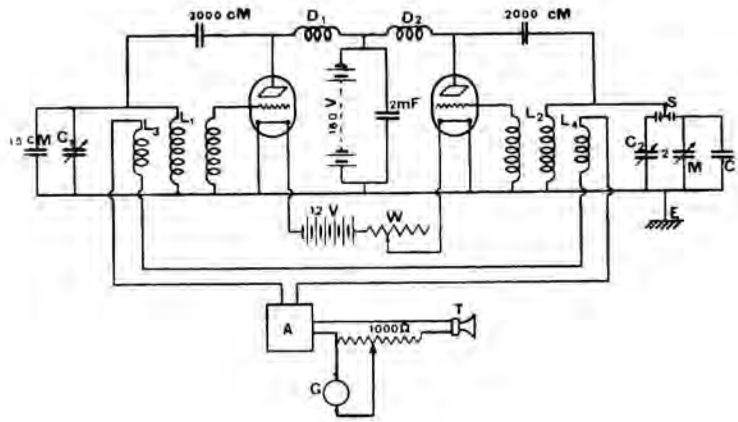

Fig. 1.

The oscillation circuit 1 consists of the self-inductions L_1 and the variable precision-condenser C_1 with a capacity of about 250 cm. Parallel to this a constant capacity of 45 cm has still been connected. In the oscillation circuit 2 is found, beside the self-induction L_2 identical to L_1, a system of condensers: the measuring condenser C which is found in the cryostat (see further), the microcondenser M connected parallel to it (fig. 2) and the normal variable condenser C_2. By changing the plugs in S either both the condensers C and M or the variable condenser C_2 can be switched in the oscillation circuit 2. Both the oscillation circuits have a joint earth-point E. For generating the oscillations the Philips-lamps type E proved to be the most suitable. The heating-current was produced by the same battery of accumulators of 12 V with in series the resistance W.

The oscillations of both the circuits 1 and 2 are superposed by means of the self-inductions L_3 and L_4. The beats caused in this way are amplified by the amplificator A. The latter consists of a detector-lamp, two low-frequency amplificator-lamps in the usual transformator-circuit and of an output-transformator. The thus amplified beats can be observed either in the radio-telephone T or in the string-galvanometer G. The string-galvanometer G has been joined to the telephone circuit in potentiometer connection. With some practice the beats can be made so slow that the swinging to and fro of the wire in the string-galvanometer can be followed.

The place of the observer is at about 4 m from the apparatus. The two variable condensers C_1 and C_2 are adjusted by the observer by means of thin cords and pulleys, the microcondenser by means of a 4 m long stick of bamboo. In this way the oscillation state during measuring is not disturbed by the movements of the observer. Further all precautions have been taken, as the earthing of the metal covers of the condensers etc. In order to avoid over-oscillations the self-inductions L_1 and L_3 on the one side and L_2 and L_4 on the other side have been coupled very loosely to the concerning regenerative-coils.

The normal variable condenser C_2, capacity about 300 cm, from the firm SPINDLER and HOYER at Göttingen has been calibrated by the P.T.R. in Berlin. This condenser has a scale divided in degrees, with nonius; the change of capacity is 1.35 cm for each degree. In order to improve the exactness of reading, a dish of 14 cm diameter with a division in degrees and nonius, was attached to the fine-regulating axis; the turning of this dish over 1 degree answers to a change of capacity to the amount of 0.17 cm.

The *microcondenser M* was made in the workshop of the Physical Laboratory I of the Technical Institute in Warsaw. Figure 2 shows this condenser in section. Within an iron cylinder *A* is a brass cylinder *B* which is isolated by an amber stop *C* and a fitting of ebonite *C'*. Exactly in the axis of this brass cylinder *B* is an iron cylinder *D* which can be exactly adjusted by means of the micrometer-screw *E*.

The micrometer-screw has been made very accurately and has a pitch of 0.5 mm; it possesses a head *G* of which the circumference is divided in 100 parts. A scale *F* put at one side serves for the reading of the number of complete rotations. The entire change of capacity of the micro-condenser by screwing in the cylinder *D* was accurately measured in the arrangement itself by means of the normal variable condenser described above; the mean of 10 different measurings was 5.2 cm. This agrees up to 1 % with the value which was calculated from the measurements of the condenser. At the zero-point in the beginning the movable cylinder *D* is already 10 mm in the isolated cylinder *B*, so that the boundary disturbances of the electric field do not come into account. In the final state the cylinder *D* is only 40 mm in the cylinder *B*, so that it is still 20 mm removed from the bottom. Through this the proportionality between the change of capacity and the number of rotations of the micrometer-screw *E* is amply secured. To the change of capacity of 5.2 cm correspond 60 rotations of the micrometer-screw each of 100 divisions at a rotation, so 6000 divisions in all, hence to each division of the head *G* of the micrometer-screw a change of capacity corresponds of $8.7 \cdot 10^{-4}$.

The cryostat with the measuring-condenser agrees substantially with that one, which has been used with the first measurings of the dielectric constant of liquid helium [1]), and has been represented in Fig. 3. The condenser now consists of 6 concentric brass cylinders with a capacity of about 175 cm. The movable contact, which was formerly found at *K*, has been replaced by a junction directly soldered on.

Further above the glass vessel *g*, silvered over inside, in which is the

[1]) These Proc. **27**, p. 622; Comm. Leiden N⁰. 171*b*. p. 11, fig. 3.

measuring condenser C, a reservoir A was placed which was also filled with helium in order to keep the surface of helium, notwithstanding the volume-contraction at the cooling, continually above the upper part of the condenser-vessel g. The condenser C itself hangs by the leads.

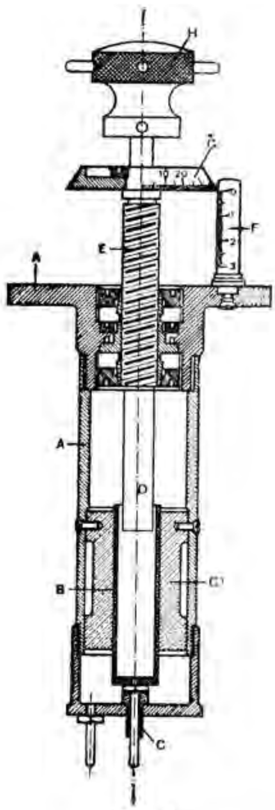

Fig. 2.

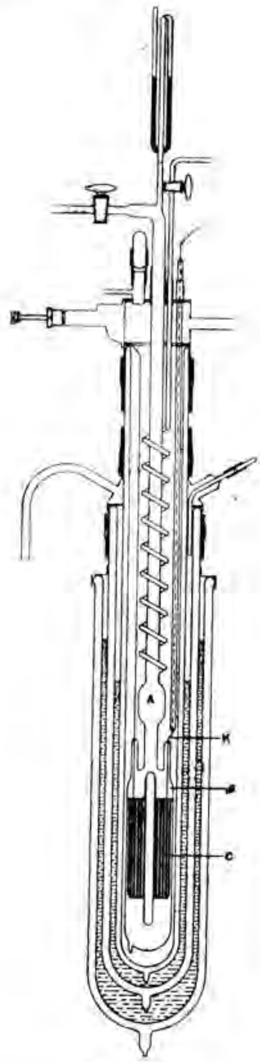

Fig. 3.

The *measurements* were made in the following way. As has already been said, they depend on the comparison of the unknown capacity with that of a normal-condenser, the small changes of the unknown capacity being compensated by means of the micro-condenser. The course of a measuring is as follows : In the oscillation circuit 2 (Fig. 1) the measuring-condensor C with the microcondensor M is switched in ; then by turning the condensor C_1 in the oscillation circuit 1 the beats are made slower till the wire in the string-galvanometer G remains quiet; this point is accurately adjusted. Then without changing the condenser C_1 in the oscillation circuit 1 the normal-condenser C_2 is connected with the oscillation circuit 2 by means of the plug S and regulated so long until the wire of the string-galvanometer remains quiet again. With this, attention should be paid that the adjusting is always brought about through a last turning in the same direction as well by the condenser C_1 as also by C_2. The capacity of the two condensers C and M is then equal to that of the normal-condenser C_2. However with this the capacities of the leads have not been reckoned with, so that this method only allows to measure capacity-differences, not the absolute values of the capacities. Small changes of the capacity of the measuring-condenser C are measured by compensating with constant adjusting of the normal-condenser C_2 at a definite initial-capacity of the measuring-condenser, each capacity-change of the latter by means of the microcondenser M and then to read scale and head from this.

The *accuracy of adjusting the normal-condenser* was determined in this way that at a constant capacity of the measuring-condenser and a constant position of the microcondenser the normal-condenser was adjusted several times and the position of the dish was read. From that followed as mean error of adjusting of the normal-condenser a value which corresponds to a capacity of $\pm$ 0.03 cm.

In suchlike way the *adjusting accuracy of the microcondenser* M was also determined. The mean deviation was $\pm$ 6 divisions of the head which corresponds to a capacity of about $\pm 5.10^{-3}$ cm. With a capacity of about 175 cm of the measuring condenser this adjusting-accuracy allows a capacity-change of about 3.10^{-3} % still to be measured, so that the accuracy of the fourth decimal in the relative value of the dielectric constant is completely secured.

At all measurements the wave-length was 600 m. Before the beginning of a measurement some time was waited after the putting into work of the apparatus, until the oscillation circuits were stable. This was the case after about an hour.

§ 3. *Capacity of the measuring-condenser.* As has already been said the here described measuring-method was worked out specially for the accurate measurement of very small capacity-changes and it is not well fit for the accurate determining of the absolute values of capacities. For that reason an indirect method was applied for measuring the absolute value of the capacity of the measuring-condenser. With this method the capacity was determined from the difference of the capacities in the vacuum and in liquid helium boiling under atmospheric pressure and from the formerly (Comm. Leiden N⁰. 171*b*) measured value of the dielectric constant of liquid helium at the boiling-point.

If we call C_0 the capacity of the measuring-condenser in a vacuum at the normal boiling-point of helium, 4.21° K., C the capacity in liquid helium at the same temperature, K_0 the dielectric constant of liquid helium at that temperature, $\triangle C$ the difference of the two capacities, measured and read on the normal condenser, then we see that :

$$\triangle C = C - C_0, \text{ and } C = KC_0, \text{ with } K_0 = 1.048,$$

from which follows :

$$C_0 = \frac{\triangle C}{K_0 - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

First the measurement was made in a vacuum. The glass vessel g in the cryostat (Fig. 3) which contains the measuring-condenser, was evacuated and filled with helium-gas under a pressure of some mm of mercury. The cryostat was filled with liquid helium, so that the vessel g was quite immersed. The gasfilling in g has no measurable influence on the value of the capacity; its purpose was to secure the exchange of temperature between the liquid helium in the cryostat and the measuring-condenser. The temperature in the cryostat was deduced from the vapour-pressure of the helium by means of the formula given by KAMERLINGH ONNES and WEBER [1]).

We found $\triangle C = 8.36$ cm from which by means of (1) follows for the capacity of the measuring-condenser empty at the boiling-point of helium :

$$C_0 = 174 \text{ cm.}$$

The formerly measured value of the dielectric constant of liquid helium at the boiling-point (see Comm. Leiden N⁰. 171*b*) is exact up to 0.1 %. Accordingly K_0—1 is exact up to about 2 %. Since the readings on the normal-condenser are much more exact we can consider the value of C_0 exact as up to 2 %. The accuracy of this measurement is of secondary importance to the proper purpose of the research, treated in this Communication, since the relative values are only required in investigating the variation of the dielectric constant of the liquid helium with the temperature.

§ 4. *Change of the dielectric constant of liquid helium with the temperature.* For these measurements the glassvessel with the measuring-condenser in the cryostat was filled with liquid helium, while it was immersed itself in liquid helium.

The *temperature* in the cryostat was adjusted in the usual way by reducing the vapour-pressure of the helium in the cryostat till the point desired was reached; its value was deduced from the vapour-pressure (see § 3). As initial temperature served the normal boiling-point of helium.

The changes of the dielectric constant at different temperatures were measured as follows. The normal-condenser was accurately adjusted at the initial-temperature of the liquid helium and then changed no more during the entire measurement. At each further temperature the adjusting was only

[1]) These Proc., Sept. 1915, p. 493. Comm. Leiden N⁰. 147*b*.

performed with the aid of the microcondenser. The changes of capacity were then read on the latter.

From the position of the microcondenser the value of the dielectric constant at a definite temperature can be calculated in the following way: If we call C the capacity of the measuring-condenser filled with liquid helium at the initial temperature and if $\triangle C$ be the increase of this capacity by the passing on to the new temperature, then the dielectric constant K at this new temperature follows from :

$$K = \frac{C + \Delta C}{C_0} = K_0 \left(1 + \frac{\Delta C}{C} \right), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $C = K_0 C_0 = 182.4$.

With that the capacity C_0 of the empty condenser has been accepted as constant for all temperatures in liquid helium. This may be done without objection as the relative capacity-changes in consequence of the thermal expansion in this temperature-region should make themselves perceivable only in the 6th decimal of the dielectric constant.

The first measurements were made on the 11th of June 1927. The results obtained are given in table I. The 4th column contains the change in the position of the microcondenser counted from that one at the initial temperature, the 5th the relative value of the dielectric constant calculated according to (2), the 6th that one of the dielectric constant itself, calculated from the value formerly measured at the normal boiling-point of helium.

TABLE I.

N^0.	p mm Hg	T ^oK.	Micro-condenser.	$\frac{K}{K_0}$.	K
1	766.5	4.21	0	1	1.0480
2	82.9	2.64	1735	1.00824	1.0566
3	69.6	2.55	1920	1.00912	1.0576
4	60.1	2.48	1988	1.00944	1.0579
5	49.8	2.39	2054	1.00976	1.0582
6	38.1	2.28	2211	1.01050	1.0590
7	25.8	2.12	1948	1.00925	1.0577
8	13.8	1.90	2013	1.00956	1.0580

With the values of K from table I and the densities of the liquid helium according to KAMERLINGH ONNES and BOKS [1]) the molecular electric

[1]) H. KAMERLINGH ONNES and J. D. A. BOKS, l. c., p. 1 note 3.

polarization can be calculated by means of the formula of CLAUSIUS—MOSOTTI. These values are united in table II.

After attempts of a repetition of this measurements had failed on the 17th of June and on the 12th of July, both times through the breaking of the helium-glass in the cryostat, a second series of measurements was made on the 19th of July. The obtained results are given in table III.

TABLE II.

T °K.	ρ	$\frac{K-1}{K+2} \cdot \frac{1}{\rho}$
4.21	0.1251	0.1259
2.64	0.1443	0.1283
2.55	0.1451	0.1299
2.48	0.1454	0.1304
2.39	0.1459	0.1305
2.28	0.1462	0.1320
2.12	0.1458	0.1295
1.90	0.1455	0.1305

TABLE III.

N°.	Time		p mm Hg	T °K.	Micro-condenser.	$\frac{K}{K_0}$	K
1	14 u	26 m	753.0	4.19	0	1.00000	1.0480
2	15	25	48.8	2.39	1933	1.00922	1.0577
3		39	40.1	2.30	2024	1.00966	1.0581
4		59	34.1	2.24	2014	1.00961	1.0581
5	16	09	30.2	2.19	1975	1.00942	1.0579
6		22	24.9	2.11	1937	1.00924	1.0577
7		34	19.9	2.03	1974	1.00942	1.0579
8		46	10.1	1.80	1921	1.00917	1.0576
9		58	8.8	1.75	1898	1.00906	1.0575
10	17	11	31.1	2.20	2027	1.00967	1.0581
11		19	36.2	2.26	2041	1.00974	1.0582 [1]

[1] During this measurement the helium in the cryostat had sunk so low, that the temperature of the condenser began to become uncertain.

During these measurements the circumstances were less favourable than with the preceeding series. Besides that we apparently did not wait long enough until the temperature should have completely regulated itself, a possible source of errors had also arisen because the leads outside the cryostat were possibly in connection with the earth through a precipitated moisture, through which an undefined most probably also variable resistance was switched parallel to the measuring-condenser.

As it appears from the graph (Fig. 4) the two measuring-series point to a discontinuity, a jump, in the course of the dielectric constant with the temperature.

Since this result however principally depends only on one series of

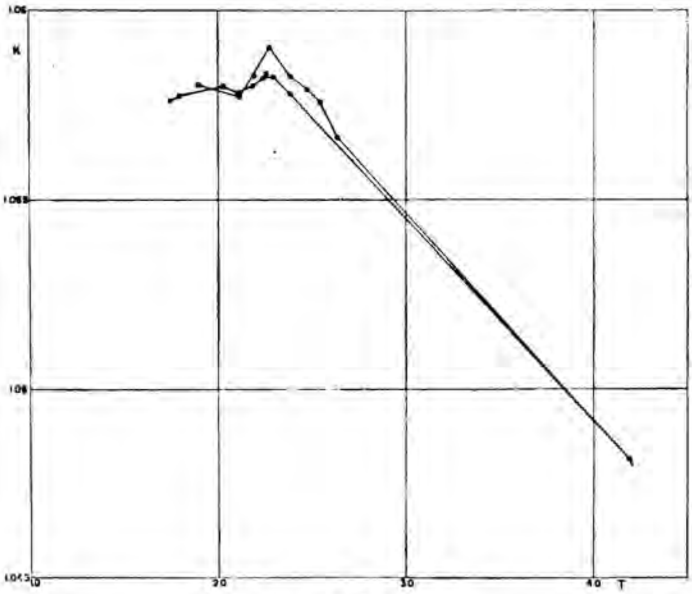

Fig. 4.

⊙ Measurements of June 11, 1927.

⊡ Measurements of July 19, 1927.

measurements (the first) we decided to wait for a new series of measurements under improved circumstances before passing on to publication. However, now that in the mean time in different ways (compare the following communication) the appearance of an abrupt change in the liquid helium has been asserted we were of opinion not to put off the publication any longer. Meanwhile we consider the results given here as being only provisional, also because there is some possible doubt about the accurate equalness of temperature of the helium in the condenser with that of the helium in the cryostat.

Physics. — *Two different liquid states of helium.* By W. H. KEESOM and M. WOLFKE. (Communication N⁰. 190*b* from the Physical Laboratory at Leiden.)

(Communicated at the meeting of December 17, 1927).

§ 1. When measuring the dielectric constant of liquid helium between the boiling-point and 1.9° K. on June 11th last, we observed [1]) that at a temperature almost corresponding with that one at which KAMERLINGH ONNES had found a maximum in the density curve, the dielectric constant shows a sudden jump or at least a jump made in a very small temperature-region. The thought suggested itself that at that temperature the liquid helium transforms into an other phase, liquid as well. If we call the liquid, stable at the higher temperatures "liquid helium I", the liquid, stable at the lower temperatures "liquid helium II", then the dielectric constant of liquid helium I should be greater than that of liquid helium II. Seeing that this result was only founded on one series of measurements we wished to repeat the measurements in order to get more security. Two attempts for that purpose failed (on June 17th and July 12th). At a repetition on July 19th the circumstances were less favourable, so that in our opinion the results, though they *did* point in the same direction as those of June 11th, did not yet give a sufficient affirmation. So we decided to wait for new measurements.

§ 2. Meanwhile our attention had been drawn by the following facts:

a. The results of the density-measurements of KAMERLINGH ONNES and BOKS [2]) may be associated as well, if not better (see Fig. 1, which we must compare with Fig. 5 of Comm. No. 170*b*) with the admission of a jump in the density at 38 mm helium-pressure (to the amount of about $1^0/_{00}$, the density of liquid helium II smaller than that of liquid helium I) than with that of a smooth maximum at that place.

b. In the paper of DANA and KAMERLINGH ONNES [3]) on the specific heat of liquid helium results are only mentioned at temperatures higher than the one expressed above. However, they have also experimented round that temperature, but then they found values, which did not agree with those measured at higher temperatures. For a part of the experiments this will be due to condensation of the helium-vapour, as DANA and KAMERLINGH ONNES point out. With some experiments leading

[1]) These Proc., p. 81. Comm. Leiden N⁰. 190a.

[2]) Comm. Leiden N⁰. 170*b*.

[3]) These Proc. **29**, 1061, 1926. Comm. Leiden N⁰. 179*d*.

to deviating results a suchlike condensation would not have been expected according to the data, which are kept in the archives of the laboratory.

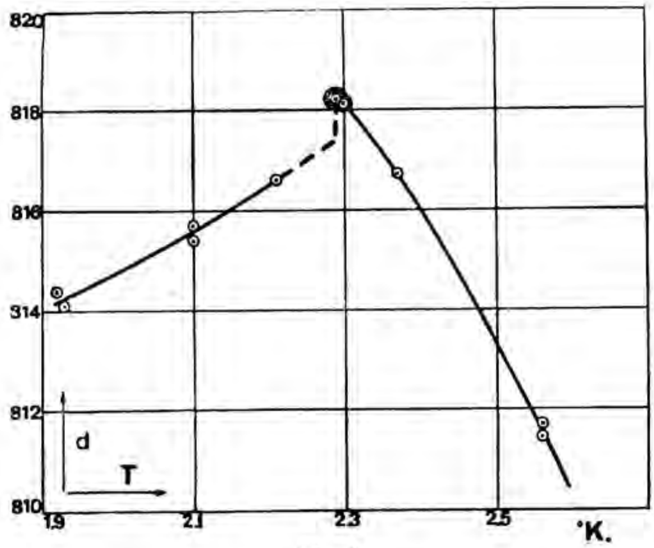

Fig. 1.

In fact from three of those experiments we could deduce nearly corresponding values for the transformation heat with a mean value:

liquid helium I = liquid helium II + 0.13 cal./gram.

c. The results of the measurements of DANA and KAMERLINGH ONNES [1]) concerning the heat of vaporization of liquid helium clearly point to a jump in the heat of vaporization (see Fig. 3 there). They already say that it may be possible that those results should indicate "that near the maximum density something happens to the helium, which within a small temperature range takes place perhaps even discontinuously" [2]). According to those results at the transformation point the heat of vaporization of liquid helium II would be greater than that of liquid helium I and that to an amount which corresponds in order of magnitude with the value of the transformation heat given under *b*.

d. Measurements of VAN URK, KEESOM and KAMERLINGH ONNES [3]) concerning the surface tension of helium seem to indicate clearly a jump in the value of the surface tension between 33 and 39 mm mercury-pressure to the amount of about 3 % (see Fig. 2 there) and in such a way that the surface tension of liquid helium II is smaller than that of liquid helium I.

[1]) These Proc. **29**, 1051, 1926. Comm. Leiden N⁰. 179*c*.

[2]) These Proc. **29**, 1057 note 2, 1926. Comm. Leiden N⁰. 179*c*, p. 31 note 1. The authors continue: "The change of density of the liquid also indicates something of the same kind".

[3]) These Proc. **28**, 958, 1925. Comm. Leiden N⁰. 179*a*.

§ 3. While the repetition, under improved circumstances, of the measurements of the dielectric constant mentioned in § 1, had to be put off on account of the absence of one of us from Leiden [1]), the provisional value of the transformation-heat mentioned in § 2 *b* raised the surmise that the transformation of the liquid helium should show itself in the cooling- or heating-curve, when a quantity of liquid helium was being cooled regularly (by gradual lowering of the pressure) resp. heated itself in consequence of the supply of heat by conduction and radiation.

This experiment was carried out on November 18th last. The course of the pressure on a mercury-manometer was thereby followed with the eye. With the cooling as well as with the heating a short halt of the manometer was stated each time at 39 mm pressure.

Besides a stirrer, with which was continually stirred during the experiment, the helium-bath contained a constantin resistance. The intention was to register photographically by means of this the course of the temperature with the time. This, however, did not succeed on that day partly because the heating just like the cooling passed off sooner than we had expected, partly because the influence of a change in temperature of the leading wires to the constantin-resistance (Diesselhorst-galvanometer, Thomsonbridge connection) did not appear to be sufficiently eliminated.

A repetition took place on Nov. 29th. Seven experiments were made: 4 coolings and 3 heatings.

The course of the mercury-manometer showed again the same phenomenon as on Nov. 18th. Especially the course by heating was very characteristic. The manometer beginning from about 5 mm at first rose comparatively slowly, then stood still for a moment (always at the same point) and after that suddenly began to rise quickly. At one of these experiments, at which the manometer stood still (it was estimated during 4 sec.) the pressure was read 38 mm.

During each of those experiments the deflection of the galvanometer in the Thomsonbridge connection could be registrated [2]). On each of the thus obtained resistance-time-registrations, which can be almost considered as temperature-time-curves, the transformation-point can be clearly recognized. Visually nothing was to be seen of the forming of a second phase in the liquid helium. This need not astonish us as the refraction-indices of the two phases will differ very little [3]).

From the fact that at lower temperatures the liquid is evidently being moved as easily by the stirrer as at higher it also appears that the viscosity of the helium II must be small. On December 7th measurements were

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1) We hope that this repetition will soon take place.

2) We heartily thank Mr. J. N. VAN DEN ENDE, phil. cand., for his assistance in these experiments.

3) Herewith should be compared what has been observed at the experiments concerning the solidification of helium: these Proc. 29, 1136, 1926, Comm. Leiden N⁰. 184*b*.

made concerning the resistance of the constantin wire mentioned above, particularly near the transformation-point, the pressure of the helium being kept constant during each measurement. We shall possibly revert to these experiments later. After that we had the opportunity to register still 3 cooling- resp. heating-curves in the way mentioned above. During these registrations the temperature changed considerably slowlier than at the preceeding experiments. Though the then obtained curves show peculiarities, which we cannot yet quite interpret, still each of them shows very clearly that at a pressure of about 38 mm something very peculiar takes place.

On December 16th measurements were made by Dr. SOPHUS WEBER, Mr. NØRGAARD and one of us on the vapour-pressure of helium particularly also near the point in question here. This time the temperature was measured with a heliumthermometer of which the pressure was read by means of a hot-wire manometer. Later on the results of these measurements will be communicated. Fig. 2 gives the successive readings of the galvanometer in the hot-wire manometer connection, when after a series of these measurements the heliumbath heated itself. The transformation-point is very clearly indicated in this heating-curve.

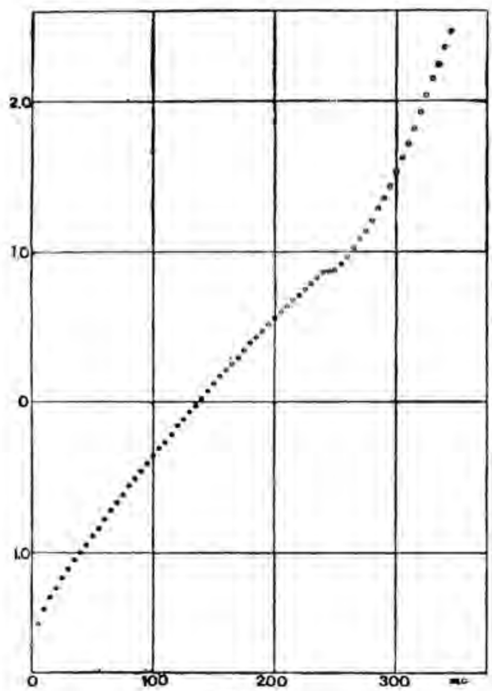

Fig. 2.

§ 4. Through the preceeding experiments we think the fact to be established that at a presure of about 38 mm a very peculiar change appears in the liquid helium, which in any case takes place within a very small temperature-region (0.08 of a degree). Though we have not been able to observe visually the two phases side by side, we yet think it most probable that we have to do here with two different states of liquid helium, which transform into each other at the temperature corresponding to the pressure mentioned. Of those two phases the liquid helium II (stable at the lower temperatures) compared with liquid helium I has :

a smaller density,
a greater heat of vaporization,
a smaller surface-tension,

while the transformation liquid helium II → liquid helium I takes place with an absorption of heat, of which the amount can be valued for the present at 0.13 cal./gram.

We put the temperature, at which the transformation takes place, for the present at 2.3° K. [1]).

It is remarkable that this transition takes place at a temperature, which, roughly speaking, may be considered to correspond with those at which other substances have their melting-point.

So helium has a triplepoint liquid I-liquid II-vapour, which until now has only been found with a number of complicately composed substances, which have a mesomorphic (liquid crystalline) phase. Whether the latter is also the case with helium, will have to appear from further experiments.

Likewise whether and, if so, in which point the transformation curve liquid I-liquid II meets the melting-curve.

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[1]) It is evident that the deduction of the temperature from the value of the vapour-pressure especially near this point, shall want a reconsideration.

Botany. — *Notes on Pteridophyta from Djambi, Sumatra.* By O. POSTHUMUS. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of January 28, 1928).

The following remarks are based on the Pteridophyta of a collection, which I made in the interior of the residency of Djambi (Sumatra), from July to November 1925. The plants are partly collected near Bangko, situated at the junction of the S. Mesoemai with the S. Merangin, where the latter ceases to be navigeable; partly in the surroundings of the camp Selemboekoe, about 30 K.M. west of Bangko; only one specimen near Sarolangoen and two near Paoe, both situated at the S. Tembesi.

The landscape can be described as a plateau, with but a slight relief, in which, in consequense of a recent upheavel, the rivers have made deep ravines; palaeozoic strata, which are covered by tertiary and quaternary deposits of vulcanic origin, come to the surface in the deepest parts.

Originally this region was covered with primary forest, continuous with that of the Barisan mountains, which are situated more westward. A considerable portion, however, in the more accessible parts, has been destroyed by the natives for their "ladangs" (not irrigated rice fields). These are left to themselves after one or two harvests have been obtained; then secundary jungle appears, or, especially during the last years, rubber plantations are planted by the natives.

The climate is aequatorial; the rainfall is rather high and almost equally distributed throughout the year. No well pronounced dry season occurs and the humidity of the air, especially in the shadow of the primary forest, is high and does not seem to be liable to much variation. This is shown in the fern-vegetation of the primary forest by the presence of Hymenophyllaceae, which are found cn trees, but also on the soil, on steep, shadowy slopes, or on bigger boulders in the bed of the smaller rivers; they were never found in the secundary vegetation.

The high degree of humidity is especially favourable to reboisement. Some months after the rice-fields have been left to themselves, the young trees (bloekar) already attain a considerable height. Alang-alang (Imperata cylindrica Beauv.) is found only on the roads, on some clearings and in older native rubber plantations, apparently only if the vegetation is periodically burnt down. If left to itself, the (secundary) forest will come up soon.

In the following list the Ferns are arranged after the system of the Index Filicum of C. CHRISTENSEN, whose nomenclature is also followed;

the distribution of the species is mentioned as far as known to me, from the literature, which is quoted only in some cases in the discussion, and from material of the Buitenzorg herbarium.

HYMENOPHYLLACEAE.

Trichomanes bipunctatum Poiret, Encycl., VIII, p. 69, 1808.

In primary forest, covering the rocks, on shadowy humid places; near S. Merangin, opposite to M. Djangkang, about 170 M. above sea-level: N⁰. 618, 27 VII, 1925.

Tropical Asia and Africa, Polynesia.

Trichomanes hispidulum Mettenius; Kuhn, Linnaea, XXXV, p. 389, 1868.

In primary forest, on the soil, in humid shadowy places; near camp Selemboekoe, about 180 M. above sea-level: N⁰. 613, 26 VII, 1925.

Malacca, Borneo.

Trichomanes humile Forster, Prodrome, p. 84, 1780.

In primary forest, covering steep, humid, shadowy, rocky slopes; near S. Merangin, opposite to M. Djangkang, about 140 M. above sea-level: N⁰. 625, 27 VII, 1925.

Malay Archipelago, Formosa, N. Zealand, Polynesia.

Trichomanes singaporianum (v. d. Bosch) van Alderwerelt van Rosenburgh, Bull. Jard. Bot. de Buitenzorg, (2) XX, p. 25, 1915.

In primary forest, on moist shadowy places, on the soil or on the base of old trees; near the camp Selemboekoe about 180 M. above sea-level: N⁰. 612, 26 VII, 1925.

Malacca, Mergui islands.

Trichomanes sumatranum van Alderwerelt van Rosenburgh, Bull. dept. agric. Indes. néerl., XVIII, p. 4, 1908.

In primary forest, in shadowy, humid, places, frequent on the boulders in small rivers; M. Karing, about 140 M. above sea-level: N⁰. 604, 25 VII, 1925; S. Karing, about 140 M. above sea-level: N⁰. 764, 24 VIII, 1925.

Sumatra, Java.

Hymenophyllum holochilum (v. d. Bosch) C. Christensen, Index Filicum, p. 36, 1905.

In the primary forest, on old trees; S. Selemboekoe, near the camp, about 180 M. above sea-level: N⁰. 785, 25 VIII, 1925.

Sumatra, Malacca, Banca, Riouw, Lingga-Archipelago, Java, Borneo, Philippines, New Guinea.

Hymenophyllum subrotundum van Alderwerelt van Rosenburgh, Bull. Jardin Bot. de Buitenzorg (2) XX, p. 19, 1915.

In primary forest, epiphytic, in humid, shadowy places, forming tufts (together with Polypodium inconspicuum Bl.) ; S. Selemboekoe, near the camp, about 180 M. above sea-level : N⁰. 786, 25 VIII, 1925 ; S. Mengkarang, about 200 M. above sea-level : N⁰. 1082 bis, IX, 1925.

Sumatra.

CYATHEACEAE.

Cibotium barometz J. Smith, London Journal of Botany, I, p. 437, 1842.

Rather short three-fern ; in the primary forest on steep riverbanks ; S. Merangin opposite to M. Djangkang, about 140 M. above sea-level : N⁰. 751, 21 VIII, 1925.

China, Hongkong, Formosa, Sumatra, Banca, Malacca, Borneo, Moluccas.

Cyathea moluccana R. Brown ; Desvaux, Prodrome, p. 322, 1827.

Small tree-fern, about ½ M. high ; leaves about 3 M. long ; occurs scattered in secundary forest (about 12 years old) ; near Bangko, about 60 M. above sea-level : N⁰. 481, 10 VII, 1925.

Sumatra, Malacca, Riouw, Lingga-Archipelago, Borneo, Moluccas.

Alsophila latebrosa Wallich ; Presl, Tent. Pterid, p. 62, 1836.

In native rubber plantations, which are about 8 years old ; near Bangko, about 60 M. above sea-level : N⁰. 568, 18 VII, 1925.

British India, Malay Archipelago.

DIPTERIDACEAE.

Dipteris conjugata Reinwardt, Sylloge plantarum II, p. 3, 1824.

In primary forest, on steep stony slopes of the river, in shadowy or rather open places ; S. Mengkarang between doesoen Baroe and M. Koekoen, about 200 M. sea-level : N⁰. 892, 19 IX, 1925.

Malacca, Malay islands, Polynesia.

POLYPODIACEAE.

Dryopteris calcarata (Blume) O. Kuntze, Revisio generum plant., II, p. 812, 1891.

In primary forest, on and between the boulders in the shadowy bed of small rivers ; S. Karing, about 140 M. above sea-level : N⁰. 692, 15 VIII, 1925.

Br. India, South China, Malay Archipelago, Polynesia.

Dryopteris Dayi (Beddome) C. Christensen, Index Filicum, p. 260, 1905.
In primary forest, on the soil, in rather shadowy spots: near S. Ketidoeran Siamang, about 160 M. above sea-level: N⁰. 908, 15 X, 1925.
Malacca.

Dryopteris didymosora (Parish) C. Christensen, Index Filicum, p. 262, 1905.
In native rubberplantations (about 10 years old), in which the undergrowth is now and then destroyed; scattered, in rather open places; near the camp Selemboekoe, about 180 M. above sea-level: N⁰. 736, 19 VIII, 1925.
Northern India, Assam, Malacca, Borneo, Amboina, China.

Dryopteris malayensis C. Christensen, Monogr. Dryopteris, I, p. 171, 1913.
In primary forest, on the soil, in shadowy places; near doesoen Baroe (S. Merangin), about 200 M. above sea-level: N⁰. 855, 4 IX, 1925
Malacca, Malay islands, Philippines.

Dryopteris salicifolia (Wallich) C. Christensen, Index Filicum, p. 290, 1905.
In primary forest, growing on rocky slopes on rather bright places near the river; M. Karing, about 140 M. above sea-level: N⁰. 655, 3 VIII, 1925.
Sumatra, Malacca, Borneo.

Dryopteris sarawakensis (Baker) van Alderwerelt van Rosenburgh, Malayan Ferns, p. 200, 1909.
In primary forest, in the shadow on steep riverbanks, and between the boulders in the river-bed; M. Karing, about 140 M. above sea-level: N⁰. 602, 25 VII, 1925.
Borneo.

Dryopteris truncata (Poiret) O. Kuntze, Rev. gen. Plant., II, p. 814, 1891.
In secundary forest, on rather moist, shadowy places; near Bangko, about 60 M. above sea-level: N⁰. 475, 10 VII, 1925.
Br. India, Malay Archipelago, Trop. Australia, Polynesia, Madagascar.

Dryopteris unita (Linn.) O. Kuntze, Rev. gen. Plant. II, p. 811, 1891.
Along roads, on open fields and in older native rubberplantations, in which the undergrowth is destroyed now and then; along the road from Limboer to Bangko, about 50 M. above sea-level: N⁰. 543, 17 VII, 1925.
Br. India, Malay Archipelago, Polynesia, Madagascar.

Dryopteris urophylla (Wallich) C. Christensen, Index Filicum, p. 299, 1905.
In native rubberplantations, scattered, on rather humid places; near Bangko, about 60 M. above sea-level: N^0. 474, 10 VII, 1925.
Br. India, Malay Islands, Polynesia, Madagascar.

Dryopteris verruculosa van Alderwerelt van Rosenburgh, Bulletin Jardin Bot. de Buitenzorg (2) X, p. 12, 1913.
In primary forest; near doesoen Baroe (S. Merangin), about 200 M. above sea-level: N^0. 856, IX, 1925.
Eastern-Java, Sumatra.

Didymochlaena truncatula (Swartz) J. Smith, Journal of Botany, IV, p. 169, 1841.
In primary forest; near Soengai Manau, about 400 M. above sea-level: N^0. 946, X, 1925.
All tropical countries.

Aspidium angulatum (Willdenow) J. Smith in Mettenius, Ann. Lugd. Bat., I., p. 239, 1864.
In primary forest; scattered on shadowy spots; near road to S. Manau, between S. Karing and S. Selemboekoe, about 180 M. above sea-level: N^0. 845, 31 VIII, 1925.
Malacca, Borneo, Moluccas, New Guinea.

Aspidium nebulosum (Baker) C. Christensen, Index Filicum, p. 84, 1905.
In primary forest: near doesoen Baroe (S. Merangin) about 200 M. above sea-level: N^0. 851, IX, 1925.
Sumatra, Banca.

Aspidium singaporianum Wallich; Hooker et Greville, Icones Filicum, pl. 26, 1827.
In primary forest; scattered, on shadowy spots; S. Merangin, opposite to M. Djangkang; about 140 M. above sea-level: N^{0}639, 27 VII, 1925.
Sumatra, Banca, Malacca, Borneo.

Aspidium vastum Blume, Enumeratio plant. Jav., p. 142, 1828.
In primary forest, scattered on shadowy spots; near doesoen Baroe (S. Merangin); about 200 M. above sea-level: N^0. 853, IX, 1925.
Br. India, Birma, Malay Archipelago.

Polybotrya appendiculata (Willdenow) J. Smith, Journal of Botany, IV, p. 150, 1841.
In primary forest, at shadowy, humid, steep, rocky slopes; M. Karing, about 140 M. above sea-level: N^0. 606, 25 VII, 1925.
Tropical Asia.

Nephrolepis cordifolia (Linnaeus) Presl, Tentamen Pteridographiae, p. 179, 1836.

In primary forest, hanging down at steep, rather exposed, rocky slopes; S. Merangin opposite to M. Djangkang, about 140 M. above sea-level: N°. 623, 27 VII, 1925.

Tropical countries.

Humata angustata J. Smith, Journal of Botany, III, p. 416, 1841.

In primary forest; with a long rhizome creeping on trees on rather dry spots; near M. Karing, about 160 M. above sea-level: N°. 895, 6 IX, 1925.

Sumatra, Banca, Riouw, Lingga-Archipelago, Malacca, Borneo.

Humata heterophylla (Smith) Desvaux, Prodrome, p. 323, 1825.

In primary forest; creeping with a long rhizome on trees, scattered on rather dry places; between S. Karing and S. Selemboekoe, about 180 M. above sea-level: N°. 844, 31 VII, 1925.

Malay Archipelago.

Humata repens (L. Fils) Diels, in Engler-Prantl, Nat. Pflanzenfam., I, abt. IV, p. 209, 1899; var. *minor* Nees.

In primary forest, scattered, on trees; near S. Ketidoeran Siamang, about 140 M. above sea-level: N°. 761, 24 VIII, 1925.

S. China, Malay Archipelago, trop. Australia.

Davallia triphylla Hooker, Species Filicum, I, p. 162, pl. 46 A, 1846.

In higher parts of the primary forest, climbing, with a long rhizome in groups; near the road to Soengai Manau on the watershed between S. Karing and S. Selemboekoe, about 180 M. above sea-level: N°. 843, 31 VIII, 1925.

Malacca.

Tapeinidium gracile van Alderwerelt van Rosenburgh, Malayan Ferns, p. 315, 1909.

In primary forest, on humid shadowy slopes; near the camp Selemboekoe, about 180 M. above sea-level: N°. 778, 25 VIII, 1925.

Sumatra, Borneo, Java, Philippines, New Guinea.

Schizoloma ensifolium (Swartz) J. Smith, Journal of Botany, III, p. 414, 1841.

In native rubber-plantations, which are about 7 years old and have but little undergrowth, scattered; near Bangko, about 60 M. above sea-level: N°. 461, 9 VII, 1925.

Tropical Africa, Asia and Australia; Polynesia.

Lindsaya decomposita Willdenow, Species plantarum, V., p. 425, 1810. (L. davallioides Blume, Enumeratio plant. Jav. p. 218, 1828.)

In primary forest, in shadowy places, on the soil or on the base of old trees; between the camp Selemboekoe and S. Karing, about 180 M. above sea-level: N⁰. 825, 29 VIII, 1925; S. Merangin, opposite to M. Karing; N⁰. 680, 9 VII, 1925.

Trop. Asia, Australia, Polynesia.

Lindsaya lancea (Linnaeus) Beddome, Ferns Brit. Ind., Suppl. p. 6, 1876.

In primary forest, on the soil or on the base of old trees near the camp Selemboekoe, about 180 M. above sea-level: N⁰. 616, 26 VII, 1925.

Trop. America, Ceylon, Malacca, Sumatra, Java, Borneo.

Lindsaya scandens Hooker, Species Filicum, 1, p. 205, pl. 63 B, 1846.

In primary forest, in humid, shadowy places, on the soil or on the base of old trees; near the road to S. Manau, about 200 M. above sea-level: N⁰. 668, 5 VIII, 1925.

Malay Archipelago.

Diplazium aequibasale (Baker) C. Christensen, Index Filicum, p. 227, 1905.

In primary forest, on shadowy, moist places; near creeks, between the rocks; S. Karing, about 140 M. above sea-level: N⁰. 695, 15 VIII, 1925; S. Karing, near the S. Merangin: N⁰. 607, 25 VII, 1925.

Borneo.

Diplazium bantamense Blume, Enumeratio, p. 191, 1828.

In primary forest, in shadow, on the soil; near the S. Merangin, opposite to M. Djangkang, about 140 M. above sea-level: N⁰. 638, 27 VII, 1925.

Tropical Asia, Japan.

Diplazium confertum (Baker) C. Christensen, Index Filicum, p. 230, 1925.

In primary forest, on the soil, sometimes at steep, shadowy banks of the creeks; S. Selemboekoe, about 150 M. above sea-level: N⁰. 783, 25 VIII, 1925; S. Boekit Tinggi, near doesoen Baroe, about 140 M. above sea-level: N⁰. 664, 4 VIII, 1925.

Sumatra, Borneo, Celebes.

Asplenium cymbifolium Christ, Bulletin Herbier Boissier (2) VII, p. 999, 1906.

In primary forest in shadowy places; the rhizome vertically climbing with short internodes, leaves with broad bases, between which are numerous roots; forming a thick mass; near the camp Selemboekoe, above 180 M. above sea-level: N⁰. 615, 26 VII, 1925.

Philippines, Singkep.

Asplenium Nidus Linnaeus, Spec. plant, II, p. 1079, 1753.

In primary forest in the shadow, scattered, on trees; in the axils of the radially arranged leaves with a mass of roots, between which humus is formed; near camp Selemboekoe, about 180 M. above sea-level: N⁰. 640, 27 VII, 1925; near doesoen Baroe (S. Merangin), about 200 M. above sea-level: N⁰. 1084, IX, 1925.

Tropical Africa, Asia and Australia.

Asplenium tenerum Forster Prodromus, p. 80, 1786.

In the primary forest, on trees, in shadowy spots; B. Mangkok, near the road to S. Manau, about 200 M. above sea-level: N⁰. 669, 5 VII, 1925.

Trop. Africa, Asia, Polynesia.

Blechnum Finlaysonianum Wallich; Hooker et Greville, Icones Filicum pl. 275, 1831.

In native rubber plantations, scattered in rather light spots; near Bangko, about 60 M. above sea-level: N⁰. 468, 9 VII, 1925.

Sumatra, Malacca, Borneo, Java, New Guinea.

Blechnum orientale Linnaeus, Spec. plant., ed. II, p. 1530, 1753.

Between bushes, on left ladangs (dry rice-fields), in newly cleared forest, in native rubber plantations (about 7 years old), scattered; near Bangko, about 60 M. above sea-level: N⁰. 467, 9 VII, 1925; N⁰. 522, 15 VII, 1925.

Tropical Asia, Australia and Polynesia.

Stenochlaena aculeata (Blume) Kunze, Bot. Zeitung, p. 142, 1848.

In the primary forest, in rather humid places; rhizome vertically climbing, on trees; leaves horizontal, adpressed; these specimens belong to the form inermis, the rhizome bearing no spines; the rhizome of N⁰. 902 is thicker than that of the other specimens; S. Merangin, opposite to M. Karing, about 150 M. above sea-level: N⁰. 681, 9 VIII, 1925; near the S. Karing, about 180 M. above sea-level: N⁰. 743, 19 VIII, 1925; S. Selemboekoe, about 150 M. above sea-level: N⁰. 898, 6 IX, 1925; near the camp, about 180 M. above sea-level: N⁰. 902, 903, 26 IX, 1925.

Tropical Asia.

Stenochlaena palustris (Burm.) Beddome, Ferns of Br. India, Suppl. p. 26, 1876.

In secundary jungle, along roads or in rather open places, climbing between shrubs, sometimes rather frequent; near the road from the camp Selemboekoe to Bangko, about 100 M. above sea-level: N⁰. 729, 18 VIII, 1925; near doesoen Baroe (S. Merangin) about 200 M. above sea-level: N⁰. 861, IX, 1925.

Tropical Asia, Australia, Polynesia.

Syngramma valleculata (Baker) C. Christensen, Index Filicum p. 340, 1905.

In primary forest, in groups, on the soil or on the base of old trees; near camp Selemboekoe, about 180 M. above sea-level: N⁰. 617. 26 VI, 1925.

Borneo.

Ceropteris calomelanos (Linnaeus) Underwood, Bulletin Torrey Bot. Club, XXIX, p. 632, 1902.

In open places, along the roads, on clearings in the forest, on sand-banks in the rivers; near Bangko, about 60 M. above sea-level: N⁰. 514, 14 VII, 1925.

All tropical countries.

Cheilanthes tenuifolia (Burmeister) Swartz, Synopsis, p. 129, 332, 1806.

In scattered groups between alang-alang (Imperata cylindrica Beauv.) and along the roads; between Bangko and the camp Selemboekoe, about 100 M. above sea-level: N⁰. 719, 18 VII, 1925.

Tropical Asia, Australia, Polynesia, New Zealand.

Pteridium aquilinum (Linnaeus) Kuhn; von Decken, Reisen, III, Bot. p. 11, 1879.

Along the roads and on clearings in the forest; near Bangko, about 60 M. above sea-level: N⁰. 516, 14 VII, 1925.

All tropical and temperate countries.

Vittaria pumila Mettenius; Kuhn Linnaeus, XXXVI, p. 65, 1869.

In primary forest, epiphytic; near Bangko, about 60 M. above sea-level: N⁰. 588*a*, 20 VII, 1925.

Borneo.

Vittaria scolopendrina (Bory) Thwaites, Enum. Plant. Zeyl. p. 381, 1864.

In primary forest, on trees, in scattered groups; S. Selemboekoe, near the camp, about 180 M. above sea-level: N⁰. 767, 25 VIII, 1925.

Tropical Africa and Australia, Polynesia.

Vittaria zosterifolia Willdenow, Species Plantarum, V, p. 406, 1810.

On the nodes of bamboos, growing near the S. Mesoemai, near Bangko, about 60 M. above sea-level: N⁰. 532, 15 VIII, 1925.

Madagascar, tropical Asia, Polynesia.

Taenites blechnoides (Willdenow) Swartz, Synopsis, p. 24, 220, 1806.

In native rubber-plantations, scattered; near Bangko, about 60 M. above sea-level: N⁰. 453, 9 VII, 1925.

Tropical Asia, Polynesia.

Hymenolepis spicata (L. Fils) Presl, Epim. Bot., p. 159, 1849.
In primary forest; epiphytic, near M. Ketidoeran Siamang, about 140 M. above sea-level: N⁰. 759, 25 VIII, 1925.
Madagascar, tropical Asia, Australia and Polynesia.

Polypodium heterocarpum (Blume) Mettenius, Fil. Hort. Lips, p. 37, pl. 25, fig. 24—25, 1856.
In primary forest, on trees; near M. Ketidoeran Siamang, about 140 M. above sea-level: N⁰. 760, 25 VIII, 1925.
Sumatra, Malacca, Banca, Riouw-Lingga islands, Java, Borneo, Celebes.

Polypodium inconspicuum Blume, Enumeratio, p. 130, 1828.
In primary forest, epiphytic, on humid shadowy places; forming tufts (together with Hymenophyllum subrotundum v. A. v. R.); S. Mengkarang, about 200 M. above sea-level: N⁰. 1082, IX, 1925.
Malay Archipelago, Philippines.

Polypodium nigrescens Blume, Enumeratio, p. 126, 1828.
In primary forest, both in rather open and shadowy places; epiphytic; near S. Mesoemai, near Bangko, about 60 M. above sea-level: N⁰. 534, 15 VII, 1925; S. Merangin, opposite to M. Djangkang, about 140 M. above sea-level; N⁰. 630, 27 VII, 1925.
Tropical Asia and Australia, Polynesia.

Polypodium phymatodes Linn., Mantissa, p. 306, 1771.
In primary forest, epiphytic; near doesoen Baroe, about 200 M. above sea-level: N⁰. 888, IX, 1925.
Tropical Africa, Asia, Australia, Polynesia.

Polypodium revolutum (J. Smith) C. Christensen, Index Filicum, p. 559, 1906.
Epiphytic, in groups on the nodes of bamboos; bamboo bushes on a sandbank, S. Merangin, near M. Titi Meranti; N⁰. 679, 9 VIII, 1925.
Sumatra, Java, Celebes, Philippines, N. Caledonia.

Polypodium Whitfordi Copeland, Phil. Journ. of Science, I, Suppl. V. p. 256, pl. 4 B, 1916.
In primary forest, on trees; S. Ketidoeran Siamang, about 140 M. above sea-level: N⁰. 762, 24 VIII, 1925.
Philippines (Luzon).

Loxogramma Blumeanum Presl, Tentamen Pterid., p. 215, 1836.
In primary forest, on trees; near M. Karing, about 140 M. above sea-level: N⁰. 906, 13 IX, 1925.
Tropical Africa and Asia.

Loxogramma involutum Presl, Tentamen Pterid., p. 215, 1836. (Polypodium scolopendrinum (Bory) C. Christensen, Index Filicum, p. 562, 1906.)

In primary forest, epiphytic, in rather shadowy places; near doesoen Baroe (S. Merangin), about 200 M. above sea-level: N⁰. 884, IX 1925; N⁰. 1083, IX, 1925.

Tropical and subtropical Asia.

Cyclophorus acrostichoides (Desvaux) Presl. Epim. bot. p. 130, 1849.

On the nodes of the stems of bamboos, at the S. Mesoemai, near Bangko, about 60 M. above sea-level: N⁰. 533, 15 VII, 1925.

Ceylon, Malay Archipelago, Queensland, Polynesia.

Cyclophorus angustatus (Swartz) Desvaux, Berliner Magazin, V, p. 300, 1811.

In primary forest, climbing, with a long rhizome on the stems; near Bangko, about 60 M. above sea-level: N⁰. 588, 20 VII, 1925; near S. Karing, about 140 M. above sea-level: N⁰. 610, 25 VII, 1925; S. Selemboekoe, near the camp, about 180 M. above sea-level: N⁰. 781, 25 VIII, 1925.

British India, Sumatra, Malacca, Riouw-Lingga islands, Banca, Borneo, Polynesia.

Elaphoglossum Beccarianum (Baker) C. Christensen, Index Filicum, p. 303, 1925.

In primary forest, on trees, in shadowy places; near M. Ketidoeran Siamang, about 130 M. above sea-level: N⁰. 909, 24 X, 1925.

Borneo.

PARKERIACEAE.

Ceratopteris thalictroides (Linn.) Brongniart, Bull. Soc. Phil., Paris, p. 186, 1821.

Water-fern; Batang Soengai: N⁰. 934, IX, 1925.

All tropical countries.

GLEICHENIACEAE.

Gleichenia linearis (Burm.) Clarke, Trans. Linn. Soc., II, Bot. I, p. 428, 1880.

Along the roads, between shrubs, on clearing in the forest; near Bangko, about 60 M. above sea-level: N⁰. 517, 14 VII, 1925; near camp S. Selemboekoe, about 180 M. above sea-level: N⁰. 765, 25 VIII, 1925.

Trop. countries.

Schizaeaceae.

Lygodium circinnatum (Burm.) Swartz., Synopsis Filicum, p. 153, 1806.
Along the roads and between shrubs, near the camp Selemboekoe, about 160 M. above sea-level: N⁰. 727, 18 VIII, 1925; near M. Karing, about 140 M. above sea-level: N⁰. 603, 25 VII, 1925; near doesoen Baroe (S. Merangin) about 200 M. above sea-level: N⁰. 860, IX, 1925.
Tropical Asia, Queensland.

Lygodium flexuosum (Linn.) Swartz, Schraders Journ., p. 106, 1801.
Between alang-alang (Imperata cylindrica Beauv.) on open places; along the road to Bangko, about 180 M. above sea-level: N⁰. 717, 718, 18 VIII, 1925.
Tropical Asia, Queensland.

Lygodium salicifolium Presl, Suppl. Tent. Pterid., p. 102, 1845.
In native rubber-plantations, near Bangko, about 60 M. above sea-level: N⁰. 464, 9 VII 1925.
Tropical Asia.

Marattiaceae.

Angiopteris evecta Hoffmann, Commentatio Soc. Reg. Gött., XII, p. 29, pl. 5, 1796.
Between shrubs (secundary jungle) near the river-banks; S. Tembesi near Sarolangoen, about 40 M. above sea-level: without number, 1 VII, 1925.
Tropical Asia.

Ophioglossaceae.

Helminthostachys zeylanica (Linn.) Hooker, Gen. Filicum, pl. 47, 1840.
Near doesoen Baroe (S. Merangin), about 200 M., above sea-level: N⁰. 867, IX, 1925.
Tropical Asia and Australia.

Lycopodiaceae.

Lycopodium cernuum Linnaeus, Species plantarum, I, p. 1103, 1753.
In native rubber-plantations, about 7 years old, without undergrowth, scattered in rather shadowy places; near Bangko, about 60 M. above sea-level: N⁰. 466, 7 VII, 1925.
Tropical and subtropical countries.

Lycopodium Phlegmaria Linnaeus, Species plantarum, I, p. 1101, 1753.
At the margin of the primary forest, epiphytic, pendulous; near the road to S. Manau near the camp Selemboekoe; about 180 M. above sea-level: N⁰. 796, 26 VIII, 1925.
Tropical Asia and Australia.

Lycopodium pinifolium Blume, Enumeratio, p. 264, 1828.

In primary forest, epiphytic, pendulous: near M. Ketidoeran Siamang, 140 M. above sea-level: N⁰. 910, 24 X, 1925.

Malay islands, New Guinea.

PSILOTACEAE.

Psilotum complanatum Swartz, Synopsis, p. 414, pl. 4, fig. 5, 1806.

In primary forest, hanging down from below a nest-fern, Asplenium Nidus: S. Karing, about 140 M. above sea-level: N⁰. 821, 28 VIII, 1925.

Tropical and subtropical countries.

SELAGINELLACEAE.

Selaginella alopecuroides Baker, Handbook Fern Allies, p. 77, 1887.

On shadowy places, near the water side; S. Lesing, near Paoe, about 30 M. above sea-level: N⁰. 1006, X, 1925.

Borneo.

Selaginella atroviridis Spring, Monographie, II, p. 124, 1848.

In primary forest, on the soil or on the rocky slopes of small rivers, in shadowy places: M. Karing, about 140 M. above sea-level: N⁰. 644, 1 VIII, 1925; path to Ds. Baroe (S. Merangin), about 180 M. above sea-level: N⁰. 666, 4 VIII, 1925.

Tropical Asia.

Selaginella convolvens van Alderwerelt van Rosenburgh, Bulletin Jard. Bot., Buitenzorg XI, p. 23, 1913.

In primary forest, op humid rocky slopes, in shadow; M. Karing, about 140 M. above sea-level: N⁰. 605, 25 VII, 1925.

Tropical Asia.

Selaginella involvens Hieronymus, Hedvigia, L. p. 2, 1911.

In primary forest, op humid rocky slopes, in shadow; M. Karing, about 140 M. above sea-level: N⁰. 609, 25 VII, 1925.

Tropical Asia.

Selaginella phanotricha Baker, Handbook Fern Allies, p. 109, 1887.

In primary forest on the soil or on the base of old trees in shadow; S. Selemboekoe, about 180 M. above sea-level: N⁰. 772, 25 VIII, 1925.

Borneo.

Selaginella plana Hieronymus in Engler-Prantl, Nat. Pflanzenfam. I, abt. IV, p. 704, 1900.

In primary forest, on rather open, rocky places; also in native rubber-plantations; near Bangko, about 60 M. above sea-level: N⁰. 458, 9 VII, 1925; S. Merangin, opposite to M. Djangkong, about 140 M. above sea-level: N⁰. 619, 27 VII, 1925.

Tropical Asia.

Selaginella Willdenowii Baker, Handbook Fern Allies, p. 93, 1887.

On shadowy places, near the water side: S. Lesing, near Paoe, about 30 M. above sea-level: N^{0}. 1009, X, 1925.

Tropical Asia.

The collection which is described on the preceding pages, represents the fern-flora of a relative small district only, in the hilly part east of the Barisan Mountains. Though therefore the conclusions derived from its study have not general value for the fern flore of Sumatra as a whole, still some remarks may be made here.

The analysis of the vegetation of a rather small district, in which the conditions are not differing very much, has, to a certain degree, some advantages to the study of the flora of a country, like Sumatra, as a whole. The circumstances governing the distribution of plants, find also their expression in the relation of the flora of a small district with that of other countries.

From the 74 species of Ferns, the majority (two thirds) is found throughout the Malay Archipelago in the plains and the lower regions of the mountains; not only on Sumatra, Malacca, Borneo, Celebes and the Moluccas, but also on Java; many of them are also known from British India, China and N. Australia, some of them occur in Madagascar and Polynesia, a number even in nearly all tropical countries.

From the remaining 22 species the known distribution is the following:

Sumatra, Malacca, Borneo, Celebes and the Moluccas:

Cibotium Barometz (Linn.) J. Smith (also known from Formosa, Hongkong, China).

Cyathea moluccana R. Brown (also known from Assam).

Dryopteris didymosora (Parish) C. Christ. (new for Sumatra; also known from Assam Br. India and S. China).

Aspidium angulatum (Willd.) J. Smith (known from New Guinea).

Cyclophorus angustatus (Swartz) Desvaux (also known from Br. India and Polynesia).

Cibotium barometz has not been mentioned from Java in the literature, neither were specimens from not-cultivated plants from this island present in the Buitenzorg herbarium; it has been collected, however by TEYSMANN on the mount Radjabasa in the S. E. part of Sumatra, on the point nearest to Java (specimen in the Buitenzorg herbarium). *Cyclophorus angustatus* has been mentioned with doubt from Java by RACIBORSKY [1]); this statement has never been confirmed.

Sumatra and the Philippines:

Asplenium cymbifolium Christ. (new to Sumatra, already known from Singkep) [2]).

Polypodium Whitfordii Copeland (new to Sumatra).

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1) RACIBORSKI, Pteridophyta von Buitenzorg, 1898, p. 100.

2) VAN ALDERWERELT VAN ROSENBURGH, Bull. Jard. Bot. de Buitenzorg (3) V, 1922, p. 184.

Sumatra, Malacca, Borneo:
Trichomanes hispidulum Mettenius (new to Sumatra).
Dryopteris salicifolia (Wall.) C. Chr.
Aspidium singaporianum Wallich.
Humata angustata J. Smith.

Sumatra, and Borneo (N.-W. and N. Borneo):
Dryopteris sarawakensis (Baker) v. A. v. R. (new to Sumatra).
Diplazium aequibasale (Baker) C. Chr. (new to Sumatra).
Diplazium confertum (Baker) C. Chr. (also known from Celebes) [1]).
Syngramma valleculata (Baker) C. Chr. (new to Sumatra).
Vittaria pumila Mett. (new to Sumatra).
Elaphoglossum Beccarianum (Baker) C. Christ. (new to Sumatra).

Sumatra and Malacca:
Trichomanes singaporianum (v. d. Bosch) v. A. v. R. (new to Sumatra).
Dryopteris Dayi (Beddome) C. Chr. (new to Sumatra).
Davallia triphylla Hooker (new to Sumatra).

From Sumatra only are known:
Hymenophyllum subrotundum v. A. v. R.
Aspidium nebulosum (Baker) C. Chr. (also from Banca).

Thus we see a rather high number of species which are found rather far eastward, but appear to be absent on Java; on the contrary only two species of the collection are known from Java and Sumatra, only:
Trichomanes sumatranum v. A. v. R.
Dryopteris verruculosa v. A. v. R.
These species are both rare on Java; *Trichomanes sumatranum* has been found once near Buitenzorg [2]); *Dryopteris verruculosa* once near Srigontjo at the south coast, S. of Bantoer (Malang) [3]); the specimens of the later species differ slightly from those gathered in Bencoolen (Lebong Tandai) and in Djambi.

From the Ferns, 20 species occurred in secundary vegetation; except *Cyathea moluccana* and *Dryopteris didymosora* they all belonged to the species, which Sumatra has in common with Java, and which are found also throughout the Malay Archipelago, sometimes even far without; even *Cyathea moluccana* and *Dryopteris didymosora*, which have not been found on Java, have a rather large distribution, from British India and Assam to the Moluccas.

[1]) C. CHRISTENSEN, Svensk. bot. Tidskr., XVI, 1922, p. 93.
[2]) VAN ALDERWERELT VAN ROSENBURGH, Bull. Jardin Bot. de Buitenzorg (3) V, 1922, p. 226.
[3]) VAN ALDERWERELT VAN ROSENBURGH, Bull. Jard. Bot. Buitenzorg, XI, 1913, p. 12; id., (3) II, 1920, p. 150.

In addition to the above remarks concerning the secundary vegetation, the following can be said concerning the species, which are found nearly throughout the whole Archipelago, also on Java.

When comparing the flora of Java and Sumatra, one is struck by the fact, already mentioned by MIQUEL [1]), that a number of plants, which in Java are known at rather high altitudes only, occur in Sumatra also in the plains. This is already stated for the Ferns bij RACIBORSKY [2]), who remarks that the lowest localities of many species are found on Sumatra at a lower level than on Western Java; on Mid Java (Tegal) they are found at a still higher altitude.

This is also demonstrated for some of the ferns mentioned above; the bulk of the species which Sumatra has in common with Java occurs there in the plains too; but a number of them are known in Java only from localities situated at a higher altitude than those on Sumatra; from these forms the following list is given; the numbers in brackets placed behind the names indicate the altitude of the lowest localities on Java of the species, which are known to me.

Hymenophyllum holochilum (v. d. Bosch) C. Christ. (900 M.).
Trichomanes sumatranum v. A. v. R. (750 M.).
Dipteris conjugata Reinwardt (900 M.).
Dryopteris calcarata (Blume) O. K. (500 M.).
Didymochlaena truncatula (Swartz) J. Sm. (700 M.).
Tapeinidium gracile v. A. v. R. (900 M.).
Lindsaya decomposita Willd. (800 M.).
Lindsaya lancea (Linn.) Bedd. (600 M.).
Lindsaya scandens Hooker (800 M.).
Diplazium bantamense Blume (1000 M.).
Asplenium tenerum Forst. (500 M.).
Blechnum Finlaysonianum Wallich (1000 M.).
Polypodium heterocarpum (Blume) Mett. (900 M.).
Polypodium inconspicuum Blume (1650 M.).
Polypodium revolutum (J. Sm.) C. Christ. (450 M.).

This difference is probably chiefly due to the fact, that the rainfall is here more equally distributed throughout the year than on Java [3]). There generally the plains have a more pronounced dry season than most of the mountainous parts. If the above supposition be true, we might expect these forms to occur on Java also in those parts of the plains or in hilly districts where the rain is equally distributed in the way: in the remote western parts of the Preanger Residency (near the Wynkoopsbaai), in the Eastern

[1]) MIQUEL, Flora van Sumatra, 1860, p. 38.

[2]) RACIBORSKI, Farne von Tegal, 1899, p. 235.

[3]) The importance of this fact, more especially for the culture of tea on Java, has been demonstrated by BACKER and VAN SLOOTEN, Theeonkruiden, 1924, p. 13; map on p. 12.

part of the Preanger, near the South coast (W. of Tjidoelang), and on the southern slope of the Smeroe. The flora of these regions, however, is hardly known.

From the above mentioned species only two, *Blechnum Finlaysonianum* and *Polypodium revolutum* occur, in Djambi, in the secundary vegetation. They belong, however, to the group, which contains nearly all ferns which may occur in the secundary vegetation. This phenomen can be explained by the supposition that the lesser resistance against drought is here the limiting factor. Then the species can not live on Java in the temporary rather dry plains in the secundary vegetation, where usually rather much resistance is necessary, especially in the young plants. The extremes of these conditions are for the plants more important than the total amount of the rainfall during longer periods.

The fern-flora of Java is rather well known, relatively much better than that of Borneo and Sumatra. The bulk of the species, mentioned above, are found throughout the whole Archipelago and even far without. We see that from the remaining species Sumatra has only some rare ones in common with Java; the other ones are not found on Java, but, however, have a rather large extension in eastward direction.

The same phenomenon is even valid for genera e.g. for *Dipteris, Matonia, Syngramma, Lecanopteris* and some families of higher plants. The species, which Java and Sumatra have in common, are found throughout the Archipelago; but those species with a more restricted distribution are found in Sumatra and Malacca, Borneo only.

The conclusions got by the analysis of this collection of ferns gathered in a rather small district in the hilly part of the lowlands of Mid Sumatra, are in accord with those derived from the study of the distribution both of several groups of higher plants and of the fauna. Though Sumatra and Java are separated only by the rather narrow straits of Sunda, their flora and fauna appear to be much more different than the flora and fauna of Sumatra and Malacca, even more different than the flora and fauna of Sumatra and Borneo.

For the discussion of this fascinating subject the reader may be referred to the paper of Dr. H. J. LAM [1]), who has checked the conclusions, derived from the distribution of the fauna with those of some groups of plants especially the Sapotaceae. The above conclusions are in accord with his. Also there the Sunda straits appears to be a boundary of primary importance; even more than the "line of Wallace". The line through the straits of Sunda is identical with the western part of the line of "Van Kampen", that is the eastern boundary of the area of many animals, among which most large mammals [2]). The supposition that this boundary really is the straits of Sunda and not, for instance, is situated some hundred

[1]) Annales Jard. Bot. Buitenzorg, XXXVII, p. 33—49, 16 pl.

[2]) l.c., p. 42.

miles towards the north, is supported by the occurrence of some plants in the southern Lampongs which have not been found in Java. From the literature may be cited here: *Anisoptera marginata* Korthals (from Sumatra, Malacca and Borneo) found near Tandjoeng Karang, near Telok Betoeng [1]); from the Ferns the occurrence of *Cibotium barometz* J. Sm. on the Radjabasa, mentioned above. Further collecting in these districts will doubtless give valuable information concerning these questions.

[1]) VAN SLOOTEN, Bull. Jardin Bot. de Buitenzorg (3) VIII, 1926, p. 7.

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Chemistry. — *Osmosis of ternary liquids*. Experimental part I. By Prof. F. A. H. SCHREINEMAKERS and Mr. B. C. VAN BALEN WALTER.

(Communicated at the meeting of January 28, 1928).

In previous communications, to be continued later on, one of us[1]) has discussed the osmosis of two ternary liquids in osmotic contact with one another with the aid of a membrane permeable for each of the three substances. We are now going to discuss a group of the systems examined.

We take the liquids L_1 and L'_1 of which the first contains only $Na_2\,CO_3 + H_2O$ and the second $Na\,Cl + H_2O$ only. In fig. 1 we draw the $Na\,Cl$-amount of these liquids on the horizontal axis and on the vertical axis their $Na_2\,CO_3$ amount; then L_1 is represented by a point 1 on the vertical axis and L'_1 by a point 1′ on the horizontal axis.

If we bring both liquids in osmotic contact with one another, then we have at the moment $t=0$ viz. at the beginning of the osmosis, the system:

$$L_1 \mid L'_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This travels along an osmosis-path, as has been discussed before; in order to determine the form of this path, we at intervals took away a little of both liquids to analyse them; consequently we did not determine the theoretical path, but an experimental path[2]) of the system.

As may be seen in the tables I—V, the membrane in the systems I—IV, consisted of a pig's bladder; this had first been degreased with aether; the membrane of system V was of parchment.

The paths of the systems I and V have been represented schematically in fig. 1 by the curves I and V; with the aid of the tables they can be accurately drawn. As the starting-points 1 of both paths coincide approximately on the Y-axis, they have in fig. 1 been drawn as coinciding.

Each path consists of the two conjugated branches[3]) 1 . e and 1′. e. Consequently the left side liquid of system (1) travels along the branch 1 . e and the right side liquid travels along the branch 1′. e in a direction, which has been indicated by the arrows.

[1]) F. A. H. SCHREINEMAKERS. Osmosis of ternary liquids. General considerations I, II, III and IV. These Proceedings **30**, 761. In future they will be quoted as: Gen. I, Gen. II, etc.

[2]) Gen. I.

[3]) Gen. I.

The paths of the systems II, III and IV have not been drawn in fig. 1; approximately the points 1 of those paths coincide with point 1 in fig. 1; the points 1′ of paths II and III are situated between those of I and V; the point 1′ of path IV coincides approximately with that of V.

Although, therefore, the paths IV and V approximately have the same starting-points 1 and 1′, yet their paths do not coincide; we shall refer to this later on.

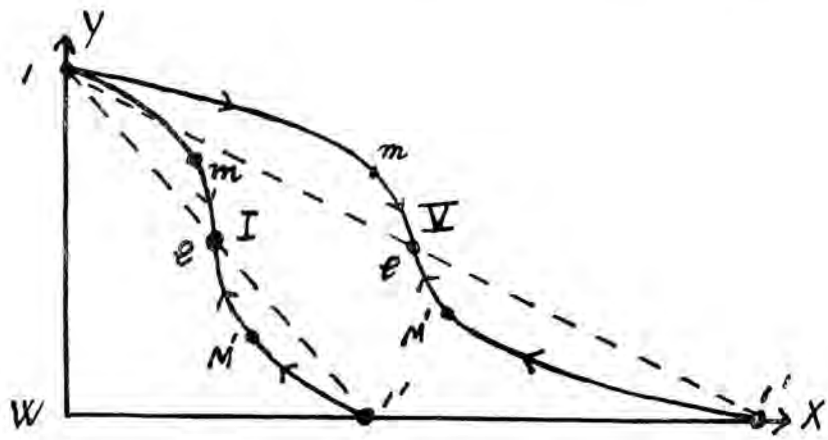

Fig. 1.

In the first column of the tables I—V we find the number of the successive determinations; sub t we find the time, viz. the number of hours after the beginning of the osmosis. In the third and fourth columns we find sub X the *NaCl*-amount and sub Y the Na_2CO_3-amount of the left side liquid; in the fifth and sixth columns we find the same for the right side liquid. The concentrations have been indicated in procents of weight. As the amount of water follows at once from the X- and Y-amounts, this has been indicated only for the systems I and II in the tables I^a and II^a. We shall refer later on to the meaning of the other columns.

In the schemes I—V the apparent and the real osmosis [1]) of the systems have been represented. The real osmosis, that is to say the direction in which the substances diffuse through the membrane, has been represented by the horizontal arrows; the apparent osmosis viz. the changes in concentration of each of the three substances, has been indicated by the vertical arrows. As we shall first discuss the apparent osmosis, the reader had better neglect the horizontal arrows to begin with.

It appears from the form of path I in fig. 1 and from table I that

[1]) Gen. II and III.

the X-amount (so $NaCl$) of the left side liquid is smaller than that of the right side liquid during all the osmosis; besides we see that the X-amount of the left side liquid continuously increases and that the X-amount of the right side liquid continuously decreases. This has been represented in scheme I sub X by the symbol:

$$\uparrow < \downarrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As this is also the case with the other systems, as is apparent from the tables, we may say, therefore:

the X-amount of those systems changes during all the osmosis normally-normally.

We can also represent the change of the X-amount in an $X.t$-diagram (Gen. II); in order to do this we draw the time t of the osmosis on the horizontal axis and the X-amount of both liquids on the vertical axis.

If with the aid of table I we draw this diagram for system I, we get a figure of the same type as fig. 2 (Gen. II); in this the $X.t$-path of the left side liquid has been represented by the curve 1 . e, which starts from point O and that of the right side liquid by curve 1'. e. In order to facilitate a general view of the subject (just as in the corresponding other cases) the path of the left side liquid has here been fully drawn and that of the right side liquid has been dotted. Both curves go towards the point e, situated at infinite distance, which indicates the W-amount of the final liquid e.

From the course of these curves it appears also that the X-amount changes normally-normally on both sides of the membrane; in the figure this has been indicated by the letters N.

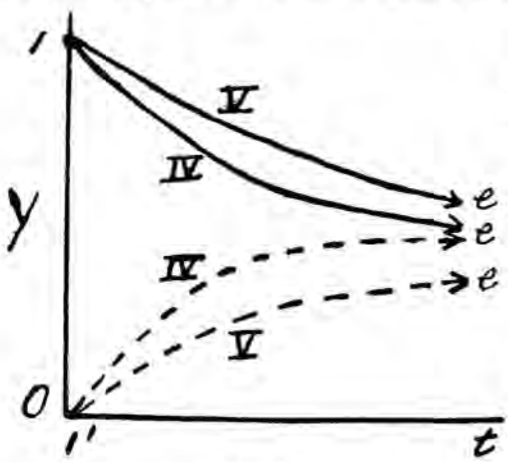

Fig. 2.

It is easy to see that also the $X.t$-diagrams of the other systems may be represented schematically by fig. 2 (Gen. II).

We now consider the Y-amount (Na_2CO_3) of system I. It appears from fig. 1 and table I that during all the osmosis this Y-amount is larger on the left side of the membrane than on the right side; we also see that the Y-amount on the left side decreases continuously and that it increases on the right side. This has been represented in scheme I sub Y by the symbol:

$$\downarrow > \uparrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As this is also the case for the other systems, we may say therefore:

the Y-amount of these systems changes: normally-normally during all the osmosis.

In fig. 2 the Yt-diagram of the systems IV and V has been represented schematically; the correct form of these paths may be easily drawn with the aid of the tables IV and V. The Yt-paths of the other systems have a corresponding form, viz. ascending when starting form point 1′ and descending from point 1. As is apparent from the tables the beginning-points 1 of the paths coincide approximately.

The symbol (3) also follows at once from this diagram. Formerly we have already discussed the Yt-diagram of fig. 3 (Gen. II); in this the path of the left side liquid has a maximum, which does not occur in fig. 2.

We now consider the W-amount of system I; for this purpose we use table I^a in which the W-amount of this system has been given. We see that the W-amount of the left side liquid begins by decreasing till the determination noted with a † and increases afterwards. Consequently a minimum W-amount is situated in the vicinity of 3, which we shall call m; this is situated between 2 and 3 or between 3 and 4; therefore, certainly between 2 and 4.

The W-amount of the right side liquid increases till the determination noted with † and decreases afterwards. Consequently the maximum W-amount, which we shall call M' is situated somewhere between 2 and 4 (2′ and 4′).

For clearness' sake we begin by imagining that m and M' coincide with determination 3. We then see that on the left side of the membrane from 1 to 3 the W-amount is smaller than on the right side and that it yet continues to decrease on the left and increase on the right. So for part 1.3 (and 1′.3′) of the path the symbol:

$$^*\downarrow < \uparrow^* \qquad (4)$$

obtains, which we also find sub W in scheme I. During the osmosis from 1 to 3 the W-amount of system I changes, therefore, anormally-anormally.

It appears from the determinations 3 to 7 that the W-amount increases on the left side of the membrane and that it decreases on the right side; as the W-amount is smaller on the left side than on the right side, the symbol:

$$\uparrow < \downarrow \qquad (5)$$

will follow, which we also find in scheme I for the parts 3.5 and 5.7, consequently for part 3.7.

From (4) and (5) follows: during the osmosis the W-amount of system I changes on the part:

1—3 of its path: anormally—anormally
3—7 „ „ „ : normally—normally.

The above has been deduced, assuming that the points m and M' coincide with 3; as this is generally not the case of course, there are transitions between the symbols (4) and (5). Although we are able to deduce them at once from tab. I^a, we shall do this with the aid of the $W.t$-diagram.

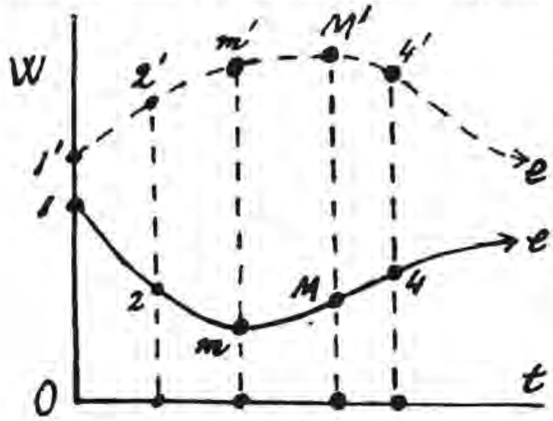

Fig. 3.

We see from table I^a that the path of the left side liquid can be represented schematically by curve 1.2.4.e and that of the right side liquid by curve 1'.2'.4'.e of fig. 3 [1]); the minimum, point m is situated somewhere between 2 and 4; the maximum, point M' between 2' and 4'. In this figure we have assumed that during the osmosis first the left side liquid reaches its minimum m and afterwards the right side liquid its maximum M'. We see from this:

on 1 . m (1' . m') is valid: the symbol $*\!\downarrow < \uparrow\!*$
on $M\,e$ (M' . e) „ „ „ „ $\uparrow < \downarrow$.

In point m the W-amount of the left side liquid is a minimum; so it remains constant during an infinitely small time dt; in point m', however, it increases in that time dt; we represent this by:

$$| < \uparrow\!* \;.$$

As the W-amount on the left side of the membrane does not change, the left arrow has been substituted by a dash.

On part mM of the path the W-amount of the liquid increases; on part $m'M'$ the W-amount also increases; consequently we find the symbol:

$$\uparrow < \uparrow\!*$$

so that the W-amount now changes normally-anormally.

For the point $M\,(M')$ we find:

$$\uparrow < |$$

Consequently we get the successive symbols:

$$*\!\downarrow < \uparrow\!* \;, \quad | < \uparrow\!* \;, \quad \uparrow < \uparrow\!* \;, \quad \uparrow < | \;, \quad \uparrow < \downarrow \quad . \;. \quad (6)$$

from which we see how the first passes into the last.

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[1]) In drawing this figure we may imagine all concentrations given in tab. II^a reduced by a definite number e.g. 90; naturally the course and mutual position of the paths are not changed in this way.

If we assume that during the osmosis the maximum M' is found first on the right side of the membrane and afterwards the minimum m on the left side of the membrane, then we find:

$$^*{\downarrow} < {\uparrow}^* \;.\quad ^*{\downarrow} < | \;.\quad ^*{\downarrow} < {\downarrow} \;.\quad | < {\downarrow} \;.\quad {\uparrow} < {\downarrow} \quad \ldots \quad (7)$$

Consequently we find: the W-amount of system I changes:

a. at the beginning of the osmosis: anormally-anormally.

b. afterwards: normally-normally.

During the transition from a to b it changes according to (6), so normally-anormally, or according to (7), so anormally-normally.

In the schemes I—V only the symbols resulting from the determinations have been given. The reader will have no trouble in seeing where transitions will occur and how they are to be found; we shall refer later on to some special cases.

In table II[a] we find the W-amount of the liquids of system II; it appears from this that during the osmosis from 1 to 5 the W-amount on the left side is larger than that on the right side and that it decreases on the left and increases on the right.

Consequently for this part 1.5 obtains the symbol:

$${\downarrow} > {\uparrow} \quad \ldots\ldots\ldots \quad (8)$$

Therefore, the W-amount changes normally-normally. Although this symbol is valid for part 1.5 of the path and not for all points between 5 and 6, yet it has been indicated in scheme II for the whole of part 1.6; presently we shall see that between 5 and 6 a transition occurs to the next symbol.

From 6 to 9 the W-amount on the left side of the membrane is smaller than that on the right side. If we limit again ourselves to the values found, we see that in the determinations noted with a † the W-amount is a minimum on the left side and a maximum on the right side. Consequently for part 6.8 of the path obtains the symbol:

$$^*{\downarrow} < {\uparrow}^* \quad \ldots\ldots\ldots \quad (9)$$

and for part 8 . 9:

$${\uparrow} < {\downarrow} \quad \ldots\ldots\ldots \quad (10)$$

which pass into one another either according to (6) or according to (7).

We now consider the transition between the symbols (8) and (9). It appears from table II[a] that between 5 and 6 the liquids get the same W-amount; this has been indicated by the horizontal line $q\,q'$; we shall call those points of the path also q and q'. As in a time dt the W-amount of the left side liquid decreases and that of the right side liquid increases, we consequently have the symbol:

$$^*{\downarrow} = {\uparrow}^* \quad \ldots\ldots\ldots \quad (11)$$

So we have two liquids with the same W-amount, which, however, changes again during the osmosis, viz. first according to (9) and (10) again to become the same in point e.

From the preceeding discussions it follows: the W-amount of system II changes:

a. in the beginning of the osmosis: normally-normally.

b. afterwards: anormally-anormally.

c. at last again: normally-normally.

Between *a* and *b* is situated a transition according to symbol (11) and between *b* and *c* according to (6) or (7).

The above may also be easily deduced from the $W.t$-diagram; if we draw this with the aid of table II^a, we see that the path of the left side liquid may be represented schematically by curve 1.2.3.4.e and that of the right side liquid by curve 1.'2.'3.'4.'e of fig. 4 (Gen. II) [1]). The point of intersection of the curves represents the point q and q' where both liquids have the same W-amount; in this figure the minimum is situated on the left side of the maximum, but of course it can also be situated on its right.

From tables III—V we are able to deduce the W-amount of these systems, their schemes, the transitions occurring in them and their $W.t$-diagrams. These diagrams belong to the type of fig. 4 (Gen. II), although the positions of the point of intersection, the minimum and the maximum may vary greatly with respect to one another.

Comparing fig. 3 with fig. 4 (Gen. II), we see that the latter can pass into the former when the point of intersection drops out. We now find:

the W-amount of the systems II—V changes normally-normally as well at the beginning as at the end of the osmosis; between them a series of transitions occurs which the reader can deduce for each separate case.

Previously we have seen (Gen. II) that we may also deduce the change of the W-amount of a system from its path in fig. 1.

It is possible namely to draw a tangent in each of these paths in a point m of branch 1 . e and in a point M' of branch 1'. e, which cuts equal parts off the X- and Y-axis and consequently runs parallel to the side XY.

So the W-amount decreases on branch 1 . m . e from 1 to m and afterwards it increases from m to e; on branch $1'M'e$ it increases from 1' to M' and afterwards it decreases from M' to e. Consequently the W-amount is a minimum in m and a maximum in M'.

[1]) It is clear that the figures 2, 3 etc. here do not represent the corresponding determinations of table IIa.

Through point e of each of the paths II—V (of wich only V has been drawn) we may draw a line, which runs parallel to side XY and intersects the path in two points; these points represent the conjugated liquids q and q' with the same W-amount.

Of course we can also draw a line through point e of path I, running parallel to XY; as, however, this line does not intersect the path no points q and q' will be found here.

With the aid of these data the reader can now deduce the change of the W-amount from these paths also.

Tables IV and V show that at the beginning of the osmosis both liquids of system IV have approximately the same composition as those of system V; so the points 1 ond 1′ of both paths coincide approximately in fig. 1. This, however, is not the case with the paths themselves; this is indeed evident, as in IV a pig's bladder and in V a piece of parchment had served as membrane.

We also see from the tables that the liquids in system IV change their composition much more quickly than those of V.

If after an osmosis of 77 hours we compare e.g. the X-, Y- and W-amounts of the left side liquid of system IV with those of system V, we find:

the X-amount of IV has increased with: 10.306 %
" " " " V " " " 4.805 %

the Y-amount in system:

IV has decreased with $8.552 - 6.666 = 1.886$ %
V " " " $8.435 - 7.950 = 0.485$ %

the W-amount in system:

IV has decreased with $91.448 - 83.028 = 8.420$ %
V " " " $91.565 - 87.245 = 4.320$ %.

One gets a clearer impression of these differences by drawing the Xt-, Yt- and Wt-diagrams of these systems; for the Yt-diagram we then get fig. 2 in which the paths have been drawn schematically. From this we see that the Y-amount of the left side liquid (curve $1 . e$) decreases more quickly in system IV than in V and that the Y-amount of the right side liquid (curve $1' . e$) increases more quickly in system IV than in V.

It is clear that these phenomena will always occur when different membranes are used. As, however, all pig's bladders etc. also differ the one from the other, differences are to be expected here too, and very large they will sometimes be, as we shall discuss later on.

TABLE I. A pig's bladder. Procent of weight.
$X = NaCl$ $Y = Na_2CO_3$ $W = H_2O$

	t	*X*	*Y*	*λ*	*Y*	*X*	*Y*	*W*		
1	0	0	8.654	8.477	0					
2	24	1.195	7.901	7.285	0.7281	58.1	33.9	8.0	VI	←
3	72	2.681	6.724	5.770	1.905	47.8	35.3	16.9	„	„
4	139	3.651	5.680	4.819	2.961	21.6	28.7	49.7	IV	→
5	193	3.958	5.145	4.526	3.483	10.2	27.3	62.5	„	„
6	241	4.045	4.839	4.399	3.780					
7	285	4.143	4.657	4.333	3.969					

TABLE Ia.

	W	*W*
1	91.346	91.523
2	90.904	91.987
3	+ 90.595	92.325 +
4	90.669	92.210
5	90.897	91.991
6	91.115	91.821
7	91.200	91.698

SCHEME I.

	X	*Y*	*W*
1—3	↑<↓	↓>↑	*↓<↑*
	←	→	←
3—5	„	„	↑<↓
			→*
0 5—7	↑<↓	↓>↑	↑<↓

TABLE II. A pig's bladder. Procent of weight.
$X = NaCl$ $Y = Na_2CO_3$ $W = water$

	t	*X*	*Y*	*X*	*Y*	*X*	*Y*	*W*		
1	0	0	8.637	17.584	0					
2	7	0.9376	8.435	16.771	0.2155	30.8	9.9	59.3	IV	→
3	20	2.624	8.006	15.109	0.6123	60.4	13.6	26.0	VII	←
4	44	4.954	7.278	12.843	1.273	68.9	18.2	12.9	„	„
5	68	6.401	6.685	11.428	1.816	66.5	22.1	11.4	VI	„
6	92	7.317	6.160	10.472	2.298	32.1	10.2	57.7	„	„
7	118	7.839	5.738	9.870	2.732	21.8	7.3	70.9	„	„
8	165	8.397	5.198	9.272	3.315	36.5	38.9	24.6	IV	→
9	214	8.601	4.856	9.072	3.728	8.2!	5.5!	86.3!	I	→

TABLE IIa.

	W	
1	91.363	82.416
2	90.627	83.014
3	89.370	84.279
4	87.768	85.884
5	86.914	86.756
	q	q′
6	86.523	87.230
7	86.423	87.398
8	+ 86.405	87.413 +
9	86.543	87.200

SCHEME II.

		X	Y	W
	1—4	↑<↓ ←	↓>↑ →	↓>↑ →
	4—6	”	”	↓>↑ → *
	6—7	”	”	* ↓<↑ * →
	7—8	”	”	* ↓<↑ * → *
0	8—9	↑<↓ * → *	”	↑<↓ → *

TABLE III. A pig's bladder. Procent of weight.
$Y = NaCl$ $Y = Na_2CO_3$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	8.559	21.026	0					
2	6	0.7965	8.438	20.162	0.1461	42.9	7.7	49.3	IV	→
3	15	2.128	8.228	18.834	0.386	40.3	8.0	51.7	”	”
4	29	3 881	7.877	17.126	0.718	53.4	10.1	36.5	VII	←
5	53	6.322	7.285	14.836	1.271	44.7	10.8	44.5	”	”
6	78	7.893	6.746	13.394	1.728	77.1	21.0	1.9	VI	”
7	120	9.286	5.947	11.883	2.425	26.7	5.3	68.0	”	”
8	169	10.026	5.304	11.287	3.044	4.5	10.1	85.4	I	→
9	217	10.256	4.908	11.023	3.464	12.0	2.0	86.0	”	←

SCHEME III.

		X	Y	W
	1—5	↑<↓ ←	↓>↑ →	↓>↑ →
	5—7	”	”	↓>↑ ← *
0	7—8	↑<↓ → *	”	* ↓<↓ →
0	8—9	↑<↓ ←	↓>↑ ← *	↑<↓ ←

TABLE IV. A pig's bladder. Procent of weight.
$X = NaCl$ $Y = Na_2CO_3$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	8.552	26.773	0					
2	$5^1/_2$	1.029	8.427	25.802	0.166	30.8	7.0	62.2	IV	→
3	13	2.389	8.225	24.405	0.358	56.4	7.3	36.3	VII	←
4	26	4.979	7.878	22.139	0.701	46.3	7.3	46.4	IV	→
5	50	7.958	7.281	19.204	1.233	55.2	9.4	35.3	VII	←
6	77	10.306	6.666	17.038	1.770	63.1	14.1	22.8	„	„
7	120	12.183	5.902	15.276	2.466	67.1	23.6	9.3	„	„
8	167	12.952	5.302	14.456	3.043	25.2	6.1	68.7	VI	„
9	241	13.415	4.772	14.062	3.612	12.7	5.6	81.7	I	→

SCHEME IV.

	X	Y	W
1—7	↑<↓ ←	↓>↑ →	↓>↑ →
7—8	„	„	*↓<↑* ←
0 8—9	↑<↓ →*	„	↑<↓ →*

TABLE V. Parchment. Procent of weight.
$X = NaCl$ $Y = Na_2CO_3$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	8.435	26.580	0					
2	$4^1/_2$	0.245	8.402	26.250	0.0514	40.2	7.1	52.7	IV	→
3	26	1.578	8.273	24.758	0.2493	36.6	5.8	57.6	„	„
4	27	4.805	7.950	21.585	0.699	34.6	5.8	59.6	„	„
5	15	7.312	7.620	19.347	1.070	35.8	7.2	57.0	„	„
6	174	9.188	7.255	17.627	1.408	48.9	9.7	41.4	„	„
7	242	10.913	6.742	16.116	1.850	53.8	15.2	31.0	VII	←
8	317	11.986	6.262	15.106	2.285	71.0	26.9	2.1	„	„
9	410	12.503	5.758	14.373	2.817	18.7	0.4	80.9	VI	„
10	509	12.922	5.332	14.002	3.320	11.3	7.0	81.7	I	→

SCHEME V.

	X	*Y*	*W*
1—8	↑<↓ ←	↓>↑ →	↓>↑ →
8—9	"	"	*↓<↑* →
0 9—10	↑<↓ →*	"	↑<↓ →*

Leiden, Lab. of inorg. Chemistry. *(To be continued).*

Mathematics. — *Ueber eine Abstandsformel in der Theorie der orthogonalen Differentialinvarianten der Kurven des R_n.* By Prof. Dr. R. WEITZENBÖCK.

(Communicated at the meeting of January 28, 1928.)

Wir leiten hier eine besonders einfache Formel her für die Abstände der oskulierenden linearen Räume einer regulären Kurve des R_n vom Koordinatenanfangspunkte.

Sind $y_i = y_i(t)$ $(1 = 1, 2, \ldots, n)$ die Cartesischen Koordinaten des Punktes y einer regulären Kurve C des R_n, dann ist in y der oskulierende R_d festgelegt durch die Matrix

$$\left\| \begin{matrix} y_1\, y_2 \ldots\ldots y_n & 1 \\ y'_1\, y'_2 \ldots\ldots y'_n & 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \\ y_1^{(d)} y_2^{(d)} \ldots\ldots y_n^{(d)} & 0 \end{matrix} \right\| \qquad y_i^{(h)} = \frac{d^h y_i}{dt^h} \quad \ldots \ldots \quad (1)$$

Machen wir von homogenen rechtwinkligen Koordinaten Gebrauch mit $y_{n+1} = 1$, so sind die homogenen Punktkoordinaten dieses oskulierenden R_d gegeben durch die $(d+1)$-reihigen Determinanten der Matrix (1):

$$a_{i_1 i_2 \ldots i_{d+1}} = (yy'y'' \ldots y^{(d)})_{i_1 \ldots i_{d+1}} = (012 \ldots d)_{i_1 \ldots i_{d+1}} \quad . \quad . \quad (2)$$

wobei die Indizes i_ν jetzt die Zahlen $1, 2, \ldots, n+1$ durchlaufen.

Den Abstand p_d des oskulierenden $R_d(a)$ von $O(x_i = 0)$ finden wir wie folgt. Wir projizieren O auf $R_d(a)$ indem wir durch O einen zu $R_d(a)$ total-senkrechten $R_{n-d}(p)$ legen und den Schnittpunkt P dieses $R_{n-d}(p)$ mit dem $R_d(a)$ ermitteln.

Die homogenen Raumkoordinaten $p'_{k_1 \ldots k_d}$ des projizierenden R_{n-d} sind die d-reihigen Determinanten der Matrix, die aus (1) erhalten wird, wenn wir die erste Zeile weglassen. Bezeichnen wir dann die Reihe der $n+1$ Grössen $y_1^{(h)} : y_2^{(h)} : \ldots : y_n^{(h)} : 0$ mit $y^{(h)} \mid$ oder kürzer mit $h \mid$, so haben wir:

$$p'_{k_1 \ldots k_d} = (1 \mid 2 \mid \ldots d \mid)_{k_1 \ldots k_d} \quad \ldots \ldots \quad (3)$$

wobei die Indizes k_ν wieder die Zahlen $1, 2, \ldots, n+1$ durchlaufen.

Die Gleichung des Punktes P in homogenen Koordinaten wird dann:

$$(u'P) = (u'p'^d a'^{n-d}) = 0. \quad \ldots \ldots \quad (4)$$

Hieraus finden wir:

$$(P|P) = (P|\,p'^{d}\,\alpha'^{n-d}) = (n-d)!\,(\alpha|P)\,(\alpha p')^{d} =$$

$$= d!\,(n-d)!\,(\alpha\,|\,q'^{d}\,\beta'^{n-d})\,(\alpha\,|\,1)\,(\alpha\,|\,2)\ldots(\alpha\,|\,d)$$

Hier formen wir so um, dass alle $d+1$ Reihen $\alpha\,|$ in den Klammerfaktor hineinkommen; hierbei führen $(\beta'\,1), \ldots, (\beta'\,d)$ nach (2) auf Null, sodass bleibt:

$$(P\,|\,P) = \frac{1}{d+1}\,d!\,(n-d)!\,(\alpha\,|^{\,d+1}\,\beta'^{n-d})\,(q'1)\,(q'2)\ldots(q'd)$$

$$(P\,|\,P) = \frac{1}{d+1}\,d!\,[(n-d)!]^{2}\,(\beta\,|\,\alpha)^{d+1}\,(q'1)\,(q'2)\ldots(q'd) \quad . \quad . \quad (5)$$

Setzen wir jetzt

$$M_{rs} = y_1^{(r)}\,y_1^{(s)} + y_2^{(r)}\,y_2^{(s)} + \ldots + y_n^{(r)}\,y_n^{(s)}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

dann haben wir nach (2):

$$(\beta\,|\,\alpha)^{d+1} = (d+1)!\,\sum_{1}^{n} \beta_{i_1 \ldots i_{d+1}}\,\alpha_{i_1 \ldots i_{d+1}} =$$

$$= (d+1)!\,\sum_{1}^{n} (012\ldots d)_{i_1 \ldots i_{d+1}}\,(012\ldots d)_{i_1 \ldots i_{d+1}}$$

$$(\beta\,|\,\alpha)^{d+1} = (d+1)!\,\triangle_{012\ldots d} = (d+1)!\begin{vmatrix} M_{00} & M_{01} & \ldots & M_{0d} \\ M_{10} & M_{11} & \ldots & M_{1d} \\ . & . & . & . \\ M_{d0} & M_{d1} & \ldots & M_{dd} \end{vmatrix}. \quad (7)$$

Und analog nach (3):

$$(q'\,1)\,(q'\,2)\ldots(q'\,d) = (q'\,|\,1)\,(q'\,|\,2)\ldots(q'\,|\,d) =$$

$$= \sum_{1}^{n} q'_{k_1 \ldots k_d}\,(1\,|\,2\,|\ldots d\,|\,)_{k_1 \ldots k_d} = \sum_{1}^{n} (1\,|\,2\,|\ldots d\,|\,)_{k_1 \ldots k_d}\,(1\,|\,2\,|\ldots d\,|\,)_{k_1 \ldots k_d} =$$

$$= \triangle_{12\ldots d}.$$

Also wird:

$$(P\,|\,P) = d!^{2}\,[(n-d)!]^{2}\,\triangle_{012\ldots d}\,\triangle_{12\ldots d}. \quad . \quad . \quad . \quad . \quad (8)$$

Bedeutet ferner l' die Grössenreihe $0:0:\ldots:0:1$, so ist das Quadrat p_d^2 des Abstandes $p_d = OP$ gegeben durch

$$p_d^2 = \frac{(P\,|\,P)}{(l'P)^2} = \frac{P_1^2 + P_2^2 + \ldots + P_n^2}{P_{n+1}^2}.$$

Nach (4) haben wir:

$$\left.\begin{aligned}(l'P) = (l'p'^d\, a'^{n-d}) = (n-d)!\,(al')\,(ap')^d = \\ = d!\,(n-d)!\sum_1^{n+1} a_{i_1 \dots i_{d+1}}\,(l'1|2|\dots d|)_{i_1 \dots i_{d+1}} = \\ = \sum_1^{n+1} (l'1|2|\dots d|)_{i_1 \dots i_{d+1}}\,(012\dots d)_{i_1 \dots i_{d+1}} = d!\,(n-d)!\,\triangle_{12\dots d}\end{aligned}\right\} \quad (9)$$

Somit kommt schliesslich:

$$p_d^2 = \frac{\triangle_{012\dots d}}{\triangle_{12\dots d}} \quad \dots\dots\dots \quad (10)$$

Es ist also z. B. der Abstand des Punktes y, der Tangente, der Schmiegungsebene, u.s.f. gegeben durch:

$$p_0^2 = M_{00}\ ,\quad p_1^2 = \frac{\begin{vmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{vmatrix}}{M_{11}}\ ,\quad p_2^2 = \frac{\begin{vmatrix} M_{00} & M_{01} & M_{02} \\ M_{10} & M_{11} & M_{12} \\ M_{20} & M_{21} & M_{22} \end{vmatrix}}{\begin{vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{vmatrix}}\ . \quad (11)$$

Mathematics. — *Ueber Verallgemeinerungen des Satzes von* DESARGUES. By Prof. Dr. R. WEITZENBÖCK.

(Communicated at the meeting of January 28, 1928.)

Sind a, b, c und α, β, γ zwei Dreiecke der projektiven Ebene mit den Seiten a', b', c' bzw. α', β', γ' und ist $f = (a\,b\,c) \neq 0$, $\varphi = (\alpha\,\beta\,\gamma) \neq 0$, dann besagt der Satz von DESARGUES: Gehen die drei Geraden $a\alpha$, $b\beta$ und $c\gamma$ durch einen Punkt, dann liegen die drei Punkte $a'\alpha'$, $b'\beta'$, $c'\gamma'$ auf einer Geraden und umgekehrt.

Dieser Satz ist durch J. V. PONCELET, M. CHASLES, O. HERMES, F. SCHUR und andere auf verschiedene Arten auf zwei Tetraeder des dreidimensionalen Raumes verallgemeinert worden[1]). Die Frage nach einem analogen Satze bei drei Tetraedern wurde von W. FR. MEYER[2]) behandelt. Wir beweisen hier, dass es bei drei beliebigen Tetraedern eine derartige Verallgemeinerung nicht gibt.

Ferners geben wir zwei einfache Verallgemeinerungen für den projektiven Raum von vier und von fünf Dimensionen und schliessen mit einigen Bemerkungen über die möglichen Verallgemeinerungen bei n Dimensionen.

§ 1.

Der Satz von DESARGUES kann analytisch am einfachsten nach E. HUNYADI[3]) wie folgt bewiesen worden. Dass $a\alpha$, $b\beta$, $c\gamma$ durch einen Punkt gehen, wird ausgedrückt durch

$$K = ((a\alpha)\ (b\beta)\ (c\gamma)) = 0 \quad \ldots\ldots\ldots \quad (1)$$

Dies ist wegen $((a\alpha)\,u'v') = (au')(\alpha v') - (av')(\alpha u')$ gleichbedeutend mit:

$$K = \begin{vmatrix} (ab\beta) & (\alpha b\beta) \\ (ac\gamma) & (\alpha c\gamma) \end{vmatrix} = \begin{vmatrix} (c'\beta) & -(\gamma' b) \\ -(b'\beta) & (\beta' c) \end{vmatrix} = 0. \quad \ldots\ldots \quad (2)$$

Die zu K duale Invariante ist $K' = ((a'\alpha')\ (b'\beta')\ (c'\gamma'))$.

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[1]) F. SCHUR, Math. Ann. **19** (1882), p. 429—432. Ausführliche Litteratur hierüber findet man bei A. BARUCH, Rend. di Palermo **44** (1920), p. 261—300. Hiezu noch: E. STUDY, Marburger Ber. (1900), p. 78; G. KOHN, Jahresber. d. Deutschen Math. Ver. **22** (1913), p. 343 und Wiener Ber. **127** (1918) p. 2073 und 2088. Vgl. auch den Art. III C 8 von K. ZINDLER der Math. Encykl., Nr. 2. (1922).

[2]) W. FR. MEYER, Archiv der Math. u. Phys. **1** (1901), p. 372.

[3]) E. HUNYADI, Crelle **139** (1879), p. 79.

Setzen wir hier $a'_l = (bc)_{ik}$ und $\alpha'_l = (\beta\gamma)_{ik}$, so entsteht:

$$K = (abc) \,.\, (\alpha\beta\gamma) \,.\, \begin{vmatrix} (\beta' c) & -(b'\gamma) \\ -(\gamma' b) & (c'\beta) \end{vmatrix} = f \,.\, \varphi \,.\, K. \quad \ldots \quad (3)$$

Also verschwinden K und K' gleichzeitig.

Dieser Beweis setzt auch den Inhalt des Satzes in deutliches Licht: Eine Invariante der drei Verbindingslinien der Eckpunkte der beiden Dreiecke verschwindet gleichzeitig mit der dualen, aus den Schnittpunkten der Seiten aufgebauten Invariante. Diese Fassung führt unmittelbar auf die Frage nach analogen Sätzen bei mehr als zwei Dimensionen.

Im dreidimensionalen Raume geben zunächst zwei Tetraeder $abcd$ und $\alpha\beta\gamma\delta$ mit $f = (abcd) \neq 0$ und $\varphi = (\alpha\beta\gamma\delta) \neq 0$ die vier Verbindungslinien $a\alpha$, $b\beta$, $c\gamma$, $d\delta$. Hier haben bereits 2 Geraden, z. B. $a\alpha$ und $b\beta$ eine relatieve Invariante:

$$A_{12} = \Sigma\, (a\alpha)_{12}\, (b\beta)_{34} = (a\alpha b\beta). \quad \ldots \quad (4)$$

Sind a', b', c', d' und α', β', γ', δ' die Seiten der Tetraeder, ist also

$$a'_1 = +(bcd)_{234},\ a'_2 = -(bcd)_{134},\ a'_3 = +(bcd)_{124},\ a'_4 = -(bcd)_{123}$$
$$b'_1 = -(acd)_{234},\ b'_2 = +(acd)_{134},\ b'_3 = -(acd)_{124},\ b'_4 = +(acd)_{123}$$

etc., so ist mit A_{12} dual: $A'_{12} = (a'\alpha' b'\beta')$. Nun ist aber:

$$(a'b')_{ik} = f \,.\, (cd)_{rs} \text{ und } (\alpha'\beta')_{rs} = \varphi \,.\, (\gamma\delta)_{ik};$$

also wird:

$$A'_{12} = f \,.\, \varphi \,.\, A_{34} \quad \ldots \quad (5)$$

was unmittelbar geometrisch zu deuten ist.

Aus (5) leitet man weiters ab, dass auch für die Invariante

$$K = \begin{vmatrix} O & A_{12} & A_{13} & A_{14} \\ A_{21} & O & A_{23} & A_{24} \\ A_{31} & A_{32} & O & A_{34} \\ A_{41} & A_{42} & A_{43} & O \end{vmatrix} \quad (A_{ik} = A_{ki}) \quad \ldots \quad (6)$$

eine analoge Gleichung gilt: $K' = f^4 \,.\, \varphi^4 \,.\, K$, woraus diejenigen verallgemeinerungen des DESARGUES'schen Satzes gewonnen werden, die sich um die hyperboloidische Lage der zwei Tetraeder gruppieren.

Bei d r e i Tetraedern $ABCD$, $abcd$ und $\alpha\beta\gamma\delta$ hat schon W. FR. MEYER darauf hingewiesen, dass die naheliegende Verallgemeinerung: Gehen die vier Ebenen $Aa\alpha, \ldots$ durch einen Punkt, dann liegen die vier Punkte $A'a'\alpha', \ldots$ in einer Ebene, nicht allgemein richtig ist.

B e w e i s. Dass $Aa\alpha$, $Bb\beta, \ldots$ durch einen Punkt gehen, wird ausgedrückt durch das Verschwinden der Invariante

$$K = (\,(Aa\alpha)\,(Bb\beta)\,(C c\gamma)\,(Dd\delta)\,) = \begin{vmatrix} (ABb\beta) & (abB\beta) & (\alpha\beta bB) \\ (ACc\gamma) & (acC\gamma) & (\alpha\gamma cC) \\ (ADd\delta) & (adD\delta) & (\alpha\delta dD) \end{vmatrix} \quad . \ . \quad (7)$$

Dual hiezu wird die Bedingung, dass die vier Punkte $A'a'\alpha', \ldots$ in einer Ebene liegen, gegeben durch das Verschwinden der Invariante

$$K' = ((A'a'\alpha')(B'b'\beta')(C'c'\gamma')(D'd'\delta')) = \begin{vmatrix} (A'B'b'\beta') & (a'b'B'\beta') & (\alpha'\beta'b'B') \\ (A'C'c'\gamma') & (a'c'C'\gamma') & (\alpha'\gamma'c'C') \\ (A'D'd'\delta') & (a'd'D'\delta') & (\alpha'\delta'd'D') \end{vmatrix}.$$

Drücken wir hier A', a' und α' durch BCD, bcd und $\beta\gamma\delta$ aus, so entsteht, wenn wir die Abkürzung

$$(u'v')_{(xy)} = (u'x)(v'y) - (u'y)(v'x)$$

verwenden:

$$K' = F \cdot f \cdot \varphi \cdot \begin{vmatrix} (b'\beta')_{(CD)} & (\beta'B')_{(cd)} & (B'b')_{(\gamma\delta)} \\ (c'\gamma')_{(DB)} & (\gamma'C')_{(db)} & (C'c')_{(\delta\beta)} \\ (d'\delta')_{(BC)} & (\delta'D')_{(bc)} & (D'd')_{(\beta\gamma)} \end{vmatrix}.$$

Wäre nun obiger Satz richtig, so müssten K' und K gleichzeitig verschwinden. Dass dies nicht der Fall ist, lässt sich durch folgendes Beispiel zeigen. Wir wählen $ABCD$ als Koordinatensimplex. Dann wird $F = 1$ und nach (7) haben wir:

$$K = \begin{vmatrix} b_3\beta_4 - b_4\beta_3 & -(ab\beta)_{134} & (\alpha\beta b)_{134} \\ c_4\gamma_2 - c_2\gamma_4 & (ac\gamma)_{124} & -(\alpha\gamma c)_{124} \\ d_2\delta_3 - d_3\delta_2 & -(ad\delta)_{123} & (\alpha\delta d)_{123} \end{vmatrix}.$$

Dagegen kommt statt K':

$$K' = f \cdot \varphi \cdot \begin{vmatrix} -(acd)_{124}\beta'_4 - (acd)_{123}\beta'_3 & (\beta'c)d_2 - (\beta'd)c_2 & (a\delta cd)\gamma_2 - (a\gamma cd)\delta_2 \\ -(abd)_{123}\gamma'_2 + (abd)_{134}\gamma'_4 & (\gamma'd)b_3 - (\gamma'b)d_3 & (ab\beta d)\delta_3 - (ab\delta d)\beta_3 \\ +(abc)_{134}\delta'_3 + (abc)_{124}\delta'_2 & (\delta'b)c_4 - (\delta'c)b_4 & (abc\gamma)\beta_4 - (abc\beta)\gamma_4 \end{vmatrix}.$$

Nun spezialisieren wir wie folgt:

$$f = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & 0 & b_4 \\ c_1 & 0 & c_3 & c_4 \\ d_1 & d_2 & 0 & d_4 \end{vmatrix} \neq 0, \quad \varphi = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & 0 & 0 \\ \gamma_1 & 0 & \gamma_3 & \gamma_4 \\ 0 & 0 & 0 & \delta_4 \end{vmatrix} \neq 0.$$

Mit diesen Werten wird $K = 0$, dagegen

$$K' = -f \cdot \varphi \, (abc\beta) \cdot \gamma_4 \cdot \alpha_2\delta_4 \, (\gamma c)_{13} \cdot a_3 \, (bd)_{12} \cdot \delta_4 \alpha_3 \beta_1 \neq 0.$$

§ 3.

Im projektiven R_4 gehen wir aus von zwei Simplexen $abcde$ und $\alpha\beta\gamma\delta\varepsilon$. Die 5 Geraden $a\alpha, \ldots, e\varepsilon$ haben unter anderem die folgende Invariante:

$$K = \Sigma \, (c\gamma)_{ik} \, (a\alpha b\beta)_i \, (d\delta e\varepsilon)_k = (a\alpha b\beta c)(d\delta e\varepsilon\gamma) - (a\alpha b\beta\gamma)(d\delta e\varepsilon c) \quad . \quad (8)$$

Ihr Verschwinden sagt aus, dass die Schnittebene der beiden R_3 $a\alpha b\beta$ und $d\delta e\varepsilon$ von der Geraden $c\gamma$ getroffen wird.

Dual zu K haben wir:

$$K' = (a'\alpha'b'\beta'c')(d'\delta'e'\varepsilon'\gamma') - (a'\alpha'b'\beta'\gamma')(d'\delta'e'\varepsilon'c');$$

Hier ist wegen

$$(a'b'c')_{ikl} = f^2 \,.\, (de)_{rs}, \; (\alpha'\beta')_{ik} = \varphi \,.\, (\gamma\delta\varepsilon)_{rst}$$

der erste Klammerfaktor von K' auch gleich $f^2 . \varphi . (d\delta\, e\varepsilon\, \gamma)$ und analog die übrigen Faktoren. Dies gibt

$$K' = f^3 \,.\, \varphi^3 \,.\, K, \quad \ldots \ldots \ldots \quad (9)$$

was wieder leicht geometrisch zu deuten ist.

Nimmt man in (8) $a\alpha$ statt $d\delta$, geht also aus von der Invariante

$$J = \Sigma\, (c\gamma)_{ik}\, (a\alpha b\beta)_i\, (a\alpha e\varepsilon)_k = (a\alpha b\gamma c)\, (a\alpha e\varepsilon\gamma) - (a\alpha b\beta\gamma)\, (a\alpha e\varepsilon c), \quad (10)$$

dann erhält man für die zu J duale Invariante J':

$$J' = \Sigma\, (c'\gamma')_{ik}\, (a'\alpha' b'\beta')_i\, (a'\alpha' e'\varepsilon')_k =$$
$$= f^3 \,.\, \varphi^3 \,.\, \Sigma\, (c\gamma)_{ik} (b\beta d\delta)_i\, (d\delta e\varepsilon)_k = f^3 \,.\, \varphi^3 \,.\, J_1, \text{ wo}$$
$$J_1 = (b\beta d\delta c)\, (d\delta e\varepsilon\gamma) - (b\beta d\delta\gamma)\, (d\delta e\varepsilon c) \quad \ldots \ldots \quad (11)$$

eine von J verschiedene Invariante der fünf Geraden $a\alpha, \ldots, e\varepsilon$ ist. Es führt also J' auf $J_1 \neq J$. Wir hatten übrigens schon bei (5) § 2 ein derartiges Ergebnis.

Ein weiteres Beispiel für dieses Verhalten von J haben wir bei $n = 6$, also im projektiven Raum R_5 von 5 Dimensionen. Hier haben bereits drei Verbindungslinien $a\alpha$, $b\beta$ und $c\gamma$ eine Invariante

$$A_{123} = (a\alpha b\beta c\gamma). \quad \ldots \ldots \ldots \quad (12)$$

Für $A'_{123} = (a'\alpha' b'\beta' c'\gamma')$ finden wir:

$$A'_{123} = f^2 \,.\, \varphi^2 \,.\, (e\varepsilon f\eta g\zeta) = f^2 \,.\, \varphi^2 \,.\, A_{456}. \quad \ldots \ldots \quad (13)$$

Hier ist $A'_{123} = 0$ die Bedingung dafür, dass die drei Geraden $a\alpha$, $b\beta$ und $c\gamma$ einem linearen R_4 angehören.

§ 4.

Im allgemeinen Falle gehen wir bei $n - 1$ Dimensionen aus von zwei Simplexen $abc \ldots lm$ und $\alpha\beta\gamma \ldots \lambda\mu$ und haben n Verbindungslinien $a\alpha, \ldots, m\mu$.

Sei $K(h_1, h_2, \ldots, h_n)$ eine ganze rationale projektive Invariante dieser Geraden, vom Grade h_1 in den $(a\alpha)_{ik}$, vom Grade h_2 in den $(b\beta)_{ik}, .\, K$ ist dann eine Summe von Termen, jeder Term ist ein Produkt von $\frac{2s}{n}$ Klammerfaktoren $(a\alpha b\beta c\gamma d\delta \ldots \varepsilon\eta \ldots)$, wobei $s = h_1 + h_2 + \ldots + h_n$.

Gehen wir zur dualen Invariante K' über, so ergibt das Zurückgehen zu den Reihen $a, b, c, \ldots \alpha, \beta, \gamma, \ldots$ die Gleichung

$$K'(h_1, h_2, \ldots, h_n) = (f \,.\, \varphi)^{\frac{s}{n}(n-2)} \,.\, K_1 \left(\frac{2s}{n} - h_1, \frac{2s}{n} - h_2, \ldots, \frac{2s}{n} - h_n \right). \quad (14)$$

wo die positiven Zahlen $\frac{2s}{n} - h_1 \ldots$ die Graden von K_1 in $(a\alpha)_{ik}, \ldots$ angeben.

(14) *stellt die umfassendste Verallgemeinerung des* DESARGUEUS'*schen Satzes auf zwei Simplexe des* R_{n-1} *dar.*

Es ergibt sich noch die Frage: wann ist K_1 mit K identisch?

Eine notwendige Bedingung ist leicht anzugeben. Aus (14) entnimmt man nämlich für diesen Fall:

$$\frac{2s}{n} - h_i = h_i, \text{ also } h_1 = h_2 = \ldots = h_n = \frac{s}{n}.$$

Es muss also K von gleichem Grade in allen Geraden $a\alpha$, $b\beta, \ldots$ sein. Ob dies für $K_1 = K$ auch hinreicht, dürfte wohl erst dann zu entscheiden sein, wenn man über die Struktur von K Näheres weis.

Mathematics. — *Ueber eine Configuration von zehn Geraden im projektiven R_3.* By Prof. Dr. R. WEITZENBÖCK.

(Communicated at the meeting of January 28, 1828.)

Zu vier Geraden des Raumes gibt es zwei, eine oder unendlich viele Transversalen, d.h. Geraden, die alle vier schneiden. Wir betrachten im Folgenden den Fall, dass vier Geraden genau eine Transversale besitzen oder, wie wir sagen wollen, ein "einfach-singuläres" Quadrupel bilden.

Aus fünf Geraden K_1, K_2, K_4, K_4 und K_5 lassen sich fünf Quadrupel Q_i bilden. *Wir wollen nachweisen, dass es Figuren von fünf Geraden gibt, bei denen jedes Quadrupel Q_i einfach-singulär ist.* Man erhält so ein zweites Quintupel $L_1, L_2, \ldots, L_5$ von Geraden, wobei L_i die Transversale von Q_i ist. Die Beziehung zwischen den Quintupeln K und L ist reziprok und alle 10 Geraden, von denen höchstens nur 6 reell sein können, gehören einem linearen Komplexe an.

§ 1.

Sind a_{ik}, b_{ik}, c_{ik}, d_{ik} und e_{ik} die homogenen Punktkoordinaten von fünf Geraden, ist also z.B.

$$K_1 = \Sigma\, \pi'_{ik}\, a_{ik} = \Sigma\, \pi_{rs}\, a_{ik} = \tfrac{1}{2}\, (\pi' a)^2 = \tfrac{1}{2}\, (\pi\, a')^2 = 0$$

die Gleichung der Geraden a_{ik}, so haben wir 10 relative Invarianten $A_{ik} = A_{ki}$, wo z.B. $A_{12} = \Sigma\, a'_{ik}\, b_{ik} = \Sigma\, a_{rs}\, b_{ik} = a_{12}\, b_{34} + a_{13}\, b_{42} + a_{14}\, b_{23} + a_{34}\, b_{12} + a_{42}\, b_{13} + a_{23}\, b_{14}$ ist.

Die Fläche zweiter Ordnung F_{123}, die durch K_1, K_2 und K_3 bestimmt wird, hat in Linienkoordinaten die Gleichung:

$$\Pi_{123} = \begin{vmatrix} 0 & A_{12} & A_{13} & K_1 \\ A_{21} & 0 & A_{23} & K_2 \\ A_{31} & A_{32} & 0 & K_3 \\ K_1 & K_2 & K_3 & 0 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hieraus findet man als Bedingung, dass K_4 diese F_{123} berührt:

$$A^{55} = \begin{vmatrix} 0 & A_{12} & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Ferner wird das Produkt der beiden Transversalen zu K_1, K_2, K_3 und K_4 gegeben durch:

$$\Pi_{1234} = \begin{vmatrix} 0 & A_{12} & A_{13} & A_{14} & K_1 \\ A_{21} & 0 & A_{23} & A_{24} & K_2 \\ A_{31} & A_{32} & 0 & A_{34} & K_3 \\ A_{41} & A_{42} & A_{43} & 0 & K_4 \\ K_1 & K_2 & K_3 & K_4 & 0 \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Das Quadrupel $Q_5 = (K_1, K_2, K_3, K_4)$ ist einfach-singulär, wenn $A^{55} = 0$, $\Pi_{1234} \equiv\!\!/\!\!\equiv 0$. Bei $\Pi_{1234} \equiv 0$ ist Q_5 hyperboloidisch.

Ist Q_5 einfach-singulär, so wird Π_{1234} ein Quadrat und die Transversale L_5 kann dann dargestellt werden durch

$$\begin{vmatrix} 0 & A_{12} & A_{13} & A_{14} & K_1 \\ A_{21} & 0 & A_{23} & A_{24} & K_2 \\ A_{31} & A_{32} & 0 & A_{34} & K_3 \\ A_{41} & A_{42} & A_{43} & 0 & K_4 \\ A_{51} & A_{52} & A_{53} & A_{54} & 0 \end{vmatrix} = 0.$$

Nennen wir $A \neq 0$ die fünfreihige Determinante $|A_{ik}|$ und A^{ik} die Minoren von A_{ik}, so lautet diese Gleichung: $\Sigma K_i A^{i5} = K_i A^{i5} = 0$. Allgemein haben wir also, wenn alle fünf Quadrupel Q_i einfach-singulär sein sollen:

$$A^{ii} = 0 \quad , \quad L_i = \sum_{\lambda=1}^{\lambda=5} K_\lambda A^{\lambda i} = K_\lambda A^{\lambda i} = 0 \quad . \quad . \quad . \quad . \quad (4)$$

Hieraus folgt wegen $A \neq 0$ leicht die Reziprozität der Beziehung zwischen den beiden Quintupeln K und L. Die simultane Invariante $(L_i\, L_k)$ wird nämlich nach (4): $A_{\lambda\mu}\, A^{\lambda i}\, A^{\mu k} = A \,.\, \delta^i_\mu\, A^{\mu k} = A \,.\, A^{ik}$. Nennen wir also L'_i die Gerade, die aus den L so entsteht wie L_i aus den K, so wird:

$$L'_i = L_\lambda \,.\, A^4 \,.\, A^3 \,.\, A_{\lambda i} = A^7 \,.\, K_\mu\, A^{\mu\lambda}\, A_{\lambda i} = A^8 \,.\, K_i \,.$$

Bei $A \neq 0$ ist also L'_i mit K_i identisch, während $A = 0$ den Ausnahmefall gibt, wo alle K eine einzige Transversale L besitzen.

§ 2.

Wir konstruieren ein Quintupel K mit einfach-singulären Quadrupeln Q_i wie folgt. Auf F_{123} nehmen wir zwei Punkte P_{45} und P_{54} so an, dass ihre Verbindungslinie keine Erzeugende von F_{123} ist. In diesen Punkten legen wir Tangenten K_4 bzw. K_5 an F_{123}. Dann sind die beiden Quadrupel Q_4 und Q_5 bereits einfach-singulär: $A^{55} = 0$, $A^{44} = 0$. Hierauf versuchen wir die restlichen drei Gleichungen $A^{ii} = 0$ zu befriedigen.

Wir legen das Koordinatentetraeder 1234 so, dass $13 = K_1$ und $24 = K_2$ wird, dass P_{45} auf 12 und P_{54} auf 34 zu liegen kommt. (Den trivialen

Fall, dass alle fünf Geraden K eine einzige Transversale besitzen, lassen wir beiseite).

F_{123} hat dann die Gleichung $x_1 x_4 - x_2 x_3 = 0$ und die Erzeugendenschaar, welcher K_1, K_2 und K_3 angehören, wird gegeben durch:

$$x_2 - \lambda x_1 = 0 \quad , \quad x_4 - \lambda x_3 = 0.$$

$\lambda = 0$ gibt K_1, $\lambda = \infty$ gibt K_2 und K_3 sei festgelegt durch $\lambda = r$. K_3 schneidet dann 12 im Punkte $P_{35} = 1 : r : 0 : 0$ und 34 im Punkte $P_{34} = 0 : 0 : 1 : r$. P_{45} und P_{54} wollen wir auf 12 bzw. auf 34 festlegen durch zwei von r verschiedene Zahlen p und q, sodass:

$$P_{45} = 1 : p : 0 : 0 \quad \text{und} \quad P_{54} = 0 : 0 : 1 : q.$$

K_4 muss F_{123} berühren, muss also in der durch 12 und die Erzeugende $\lambda = p$ bestimmten Ebene liegen; man kann also K_4 festlegen durch:

$$K_4 = \left\| \begin{matrix} 1 & p & 0 & 0 \\ 1 & p' & \varrho & \varrho p \end{matrix} \right\| \quad \text{mit} \quad p' \neq p \quad , \quad \varrho \neq 0.$$

Analog bei K_5:

$$K_5 = \left\| \begin{matrix} 0 & 0 & 1 & q \\ 1 & q & \sigma & \sigma q' \end{matrix} \right\| \quad \text{mit} \quad q' \neq q \quad , \quad \sigma \neq 0$$

Auf diese Weise erhält man als Koordinaten $\pi_{12} : \pi_{13} : \pi_{14} : \pi_{34} : \pi_{42} : \pi_{23}$ der Geraden K_i:

$$\left. \begin{array}{lcccccc} K_1 \ldots & 0 & : 1 & : 0 & : 0 & : 0 & : 0 \\ K_2 \ldots & 0 & : 0 & : 0 & : 0 & : 1 & : 0 \\ K_3 \ldots & 0 & : 1 & : r & : 0 & : -r^2 & : r \\ K_4 \ldots & p'-p & : \varrho & : \varrho q & : 0 & : -p^2\varrho & : p\varrho \\ K_5 \ldots & 0 & : -1 & : -q & : \sigma(q'-q) & : q^2 & : -q \end{array} \right\} \quad . \quad . \quad (5)$$

Hieraus findet man:

$$|A_{ik}| = \left| \begin{array}{ccccc} 0 & 1 & -r^2 & -p^2\varrho & q^2 \\ & 0 & 1 & \varrho & -1 \\ & & 0 & -\varrho(p-r)^2 & (q-r)^2 \\ & & & 0 & \sigma(p'-p)(q'-q) + \varrho(p-q)^2 \\ & & & & 0 \end{array} \right| , \quad . \quad . \quad (6)$$

$$A^{11} = \sigma(p'-p)(q'-q)\,[\sigma(p'-p)(q'-q) - 4\,\varrho(p-r)(q-r)]$$
$$A^{22} = r^2\sigma(p'-p)(q'-q)\,[r^2\sigma(p'-p)(q'-q) - 4\,pq\,\varrho(p-r)(q-r)]$$
$$A^{33} = \sigma(p'-p)(q'-q)\,[\sigma(p'-p)(q'-q) - 4\,pq\,\varrho].$$

Setzen wir $A^{ii} = 0$, so folgt leicht: $r^2 = pq$, $r = p + q$, $p^3 - q^q = 0$. Wegen $p \neq q$ gibt dies, wenn ε eine primitieve dritte Einheitswurzel bedeutet ($\varepsilon^2 + \varepsilon + 1 = 0$):

$$p = -r\varepsilon \qquad q = -r\varepsilon^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

wozu dann noch für p', q', ϱ und σ die Beziehung kommt:

$$\sigma\,(p'+\varepsilon r)\,(q'+\varepsilon^2 r)=4\,\varrho r^2 \quad . \quad . \quad . \quad . \quad (8)$$

Nach (5) liegt K_4 fest, wenn $\frac{p'-p}{\varrho}=\frac{p'+\varepsilon r}{\varrho}$ gegeben ist und K_5 ist durch $\sigma\,(q'-q)=\sigma\,(q'+\varepsilon^2 r)$ fixiert. Nun ist aber nach (8): $\frac{p'+\varepsilon r}{\varrho}=$ $=\frac{4r^2}{\sigma\,(q'+\varepsilon^2 r)}$; setzen wir also $s=\sigma\,(q'+\varepsilon^2 r)$, so werden K_4 und K_5 durch s festgelegt. Statt (5) kommt dann:

$$\left.\begin{array}{l}K_1\ldots\ 0\ :\ 1\ :\ 0\ :\ 0\ :\ 0\ :\ 0\\ K_2\ldots\ 0\ :\ 0\ :\ 0\ :\ 0\ :\ 1\ :\ 0\\ K_3\ldots\ 0\ :\ 1\ :\ r\ :\ 0\ :-r^2\ :\ r\\ K_4\ldots\frac{4r^2}{s}:\ 1\ :-\varepsilon r\ :\ 0\ :-\varepsilon^2 r^2\ :-\varepsilon r\\ K_5\ldots\ 0\ :-1\ :\ \varepsilon^2 r\ :\ s\ :\ \varepsilon r^2\ :\ \varepsilon^2 r\end{array}\right\} \quad . \quad . \quad (9)$$

und die A_{ik} werden jetzt:

$$|A_{ik}|=\left|\begin{array}{ccccc}0 & 1 & -r^2 & -\varepsilon^2 r^2 & \varepsilon r^2\\ & 0 & 1 & 1 & -1\\ & & 0 & -\varepsilon r^2 & \varepsilon^2 r^2\\ & & & 0 & r^2\\ & & & & 0\end{array}\right|, \quad . \quad . \quad . \quad . \quad (10)$$

Mit diesen Werten finden wir:

$$|A^{ik}|=\left|\begin{array}{ccccc}0 & 8r^6 & -8r^4 & -8\varepsilon r^4 & 8\varepsilon^2 r^4\\ & 0 & 8r^6 & 8r^6 & -8r^6\\ & & 0 & -8\varepsilon^2 r^4 & 8\varepsilon r^4\\ & & & 0 & 8r^4\\ & & & & 0\end{array}\right|, A=32r^6 \quad . \quad . \quad (11)$$

und also nach (4):

$$\left.\begin{array}{l}L_1=8r^4\,.\,[\qquad\qquad r^2\,.\,K_2\ -1\,.\,K_3\ -\varepsilon\,.\,K_4\ +\varepsilon^2\,.\,K_5]\\ L_2=8r^4\,.\,[\ \ r^2\,.\,K_1\qquad\qquad +r^2\,.\,K_3\ +r^2\,.\,K_4\ -r^2\,.\,K_5]\\ L_3=8r^4\,.\,[-1\,.\,K_1\ +r^2\,.\,K_2\qquad\qquad -\varepsilon^2\,.\,K_4\ +\varepsilon\,.\,K_5]\\ L_4=8r^4\,.\,[-\varepsilon\,.\,K_1\ +r^2\,.\,K_2\ -\varepsilon^2\,.\,K_3\qquad\qquad +1\,.\,K_5]\\ L_5=8r^4\,.\,[\ \ \varepsilon^2\,.\,K_1\ -r^2\,.\,K_2\ +\varepsilon\,.\,K_3\ +1\,.\,K_4\qquad\qquad]\end{array}\right\} \quad (12)$$

Um allgemeine Gleichungen zu erhalten, kann man r^2 und ε durch GRASSMANN'sche Doppelverhältnisse ausdrücken. Nach (10) haben wir nämlich:

$$\frac{A_{12}\,A_{34}}{A_{23}\,A_{14}}=-\frac{\varepsilon}{r^2}\quad,\quad \frac{A_{14}\,A_{23}}{A_{12}\,A_{34}}=\varepsilon \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hieraus sind ε und r^2 durch absolute Invarianten auszudrücken und können in (12) eingesetzt werden.

Nach (12) und (9) ergeben sich für die homogenen Punktkoordinaten der Geraden L_i die Werte:

$$\left.\begin{array}{lcccccccccccc} L_1 \ldots & -\frac{4\varepsilon r^2}{s} & : & 0 & : & -2r & : & \varepsilon^2 s & : & 4r^2 & : & -2r \\ L_2 \ldots & \frac{4r^2}{s} & : & 4 & : & 2r & : & -s & : & 0 & : & 2r \\ L_3 \ldots & -\frac{4\varepsilon^2 r^2}{s} & : & 0 & : & 2r & : & \varepsilon s & : & 0 & : & 2r \\ L_4 \ldots & 0 & : & 0 & : & 0 & : & 1 & : & 0 & : & 0 \\ L_5 \ldots & 1 & : & 0 & : & 0 & : & 0 & : & 0 & : & 0 \end{array}\right\} . \quad (14)$$

Die zehn Geraden K und L schneiden sich in 20 Punkten $P_{ik} = (K_i L_k), i \neq k$. Wir erhalten eine in sich duale Configuration mit der Signatur

$$(20_2^7 \ , \ 10_4^4 \ , \ 20_7^2).$$

Schliesslich stellen wir noch die Koordinaten der 20 Punkte P_{ik} (bezogen auf das Grundtetraeder $1 = P_{15}$, $2 = P_{25}$, $3 = P_{14}$, $4 = P_{24}$) in nachfolgender Tabelle zusammen:

	K_1	K_2	K_3	K_4	K_5
L_1	★	$0 : 2\varepsilon r : 0 : s$	$2\varepsilon r : 2\varepsilon r^2 : -s : -sr$	$2r : 2r^2(1-\varepsilon^2) : -\varepsilon^2 s : sr$	$2\varepsilon r : -2r^2 : -s : sr(\varepsilon-1)$
L_2	$2r : 0 : -s : 0$	★	$2r : 2r^2 : s : sr$	$2r(1-\varepsilon^2) : 2r^2 : -\varepsilon^2 s : sr$	$-2\varepsilon r : 2\varepsilon r^2 : s(1-\varepsilon^2) : sr$
L_3	$2\varepsilon^2 r : 0 : s : 0$	$0 : 2\varepsilon^2 r : 0 : -s$	★	$2\varepsilon^2 r : 2r^2 : s : -\varepsilon sr$	$2\varepsilon^2 r : -2\varepsilon r^2 : s : \varepsilon^2 sr$
L_4	$0 : 0 : 1 : 0$	$0 : 0 : 0 : 1$	$0 : 0 : 1 : r$	★	$0 : 0 : 1 : -\varepsilon^2 r$
L_5	$1 : 0 : 0 : 0$	$0 : 1 : 0 : 0$	$1 : r : 0 : 0$	$1 : -\varepsilon r : 0 : 0$	★

(15)

Auf eine ausführliche Untersuchung der sehr zahlreichen Eigenschaften der in obigem nachgewiesenen Configuration will ich bei späterer Gelegenheit eingehen. Hier sei nur noch Folgendes angeführt. Nach (10) haben wir:

$$U = A_{12} A_{34} = -\varepsilon r^2, \ V = A_{13} A_{42} = -r^2, \ W = A_{14} A_{23} = -\varepsilon^2 r^2 . \quad (16)$$

woraus $U + V + W = 0$ und $VW + WU + UV = 0$ folgen. Es gehört also z.B. K_4 dem durch K_1, K_2 und K_3 bestimmten VOSSE'schen Komplexe an und K_1, K_2, K_3 und K_4 bilden ein sogenanntes KLUYVER'sches Quadrupel [1]).

[1]) Hiezu: H. MOHRMANN, Mathem. Zeitschr. 2 (1918) p. 27—51 und 5 (1919) p. 268—283.

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CONTENTS

Mathematics. — *Invarianten der Integranden vielfacher Integrale in der Variationsrechnung.* I. By Prof. L. KOSCHMIEDER in Brünn. (Communicated by Prof. R. WEITZENBÖCK).

(Communicated at the meeting of October 29, 1927).

Durch das folgende hoffe ich erstens den Gegenstand zweier meiner vorhergehenden Mitteilungen [1] [2]) weiter zu fördern: die Ermittelung von Invarianten der Funktion F (14) im Zusammenhange mit der *ersten* Variation des Integrals I (15), dessen Integrand die Grösse F ist als Funktion von $n+1$ Abhängigen und deren ersten Ableitungen nach n Unabhängigen. Ich werde hier auf die damals erzielten Ergebnise zurückgreifen und die dort verwendeten Bezeichnungen benutzen, soweit das zweckmässig ist.

Zweitens aber und hauptsächlich will ich unter dem gleichen Gesichtspunkte die *zweite* Variation in die Betrachtung einbeziehen. Dabei folge ich dem Wege, den A. L. UNDERHILL [3]) bei *einfachen* Integralen eingeschlagen hat; es scheint mir sehr bemerkenswert, dass die von UNDERHILL für $n=1$ durchgeführte, schon in diesem Sonderfalle ziemlich verwickelte Rechnung sich, wie ich zeigen werde, auf *n-fache* Integrale (15) verallgemeinern lässt.

Ich habe a.a.O. [1] [2]) invariantentheoretische Begriffe nur in dem Umfange herangezogen, wie es zur Aufstellung zweier besonderer Invarianten, der „mittleren extremalen Flächenkrümmung" und des „Rauminhaltes", erforderlich war. Im Hinblick auf die folgenden sowohl wie auch auf künftige Untersuchungen ist es angebracht, mit einer allgemeinen invariantentheoretischen Erörterung zu beginnen (I). Diese kann man mit Rücksicht auf die für $n=1$ von UNDERHILL [4]) gegebenen einschlägigen Entwickelungen kurz halten, auch wenn man sie soweit wie möglich sogleich auf eine Funktion G (3) bezieht, die von umfassenderer Art ist als F. Den Anfang (§ 1) bilden Bemerkungen über Punkttransformation. Es ist vorteilhaft, ausser der ersten Reihe von Veränderlichen y eine zweite (§ 2) in Betracht zu ziehen, deren Mitglieder η sich kogredient zu den Werten y transformieren. Wie man unter geeigneten Umständen von einer die Grössen η enthaltenden Invariante zu einer andern übergehen kann, in der nur Grössen y auftreten, zeigt § 3 im Sonderfalle (14). In § 4 kommt die Wirkung des Differentiations- und Variations-

[1]) Math. Zeitschr. **24** (1926), S. 181—190.

[2]) Math. Annalen **94** (1925), S. 252—261.

[3]) Invariants of the Function $F(x, y, x', y')$ in the Calculus of Variations, Trans. Amer. Math. Soc. **9** (1908), S. 316—338.

[4]) A. a. O. [3]), S. 317—326, 331.

verfahrens auf die Invarianz zur Sprache. Entsprechende Fragen bei der Parametertransformation sind Gegenstand des § 5.

Diesem allgemeinen Teile folgt die eingangs angekündigte Anwendung (II, III) auf den besonderen Fall (14), (15). Bei der Betrachtung der ersten Variation (II, 1) ergibt sich (§ 6) die a.a.O. [1]), [2]) nicht erörterte Invarianz der Transversalität und der WEIERSTRASSschen E-Funktion; diejenige des durch sein Verschwinden die Extremalen des Integrals I kennzeichnenden Ausdrucks W wird einfacher hergeleitet als a.a.O. [1]), § 3. Um aus W die absolute Invariante S zu erhalten, zieht man (§ 7) die Grössen $\Phi_{\alpha\beta}$ heran; es wird die Übereinstimmung ihrer Determinante Φ mit der von TH. DE DONDER [5]) eingeführten Funktion F_1 dargetan und so die Beziehung zu einer von ihm a.a.O. [5]) angegebenen Invariante hergestellt. In § 8 weise ich auf das Integral I als Anlass zu einer Massbestimmung im $n+1$-stufigen Raume hin.

Der nächste Abschnitt (II, 2), der die zweite Variation betrifft, ist das Kernstück dieser Abhandlung. Ich habe die auf diesen Gegenstand bezüglichen, im bisherigen Schrifttum vorliegenden Ergebnisse in einer kürzlich erschienenen Arbeit [6]) aufgezählt; dort leite ich ferner die Formel für $\delta^2 I$ in Bezug auf eine Extremale her. *Ohne* diese Einschränkung scheint diese Rechnung bei *beliebiger* [7]) Variation noch nicht durchgeführt zu sein. Diese Lücke möchte ich hier ausfüllen: Mit Benutzung der a.a.O. [6]) beschafften Rechenhilfsmittel bringe ich in § 9 die Grösse $\delta^2 F$ auf solche Gestalt, wie sie im Falle *einfacher* Integrale durch die Transformation von UNDERHILL [8]) zustandekommt. Diese ist die Verallgemeinerung der bekannten Transformation von WEIERSTRASS [9]), insofern als sie sich nicht wie diese nur auf eine Extremale, sondern auf eine beliebige Kurve bezieht.

Nunmehr werden (§ 10—§ 13) die mit der zweiten Variation zusammenhängenden Invarianten Ψ, U, Ψ_0, U_0 gewonnen, die für $n=1$ bez. in die von UNDERHILL [10]) gefundenen Invarianten Φ, K, Φ_0, K_0 übergehen. In § 10 spalten wir von dem Ausdrucke $\delta^2 F$ die Invariante Ψ ab. Sie enthält die Variationen δx_i; eine von diesen Veränderlichen der zweiten Reihe freie Invariante U (§ 11) ergibt sich aus Ψ nach dem Verfahren des § 3. Zur Verallgemeinerung der Invariante K von UNDERHILL gelangt auch DE DONDER, aber auf anderem als dem eben geschilderten Wege [11]).

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[5]) C. R. Acad. sc. Paris **155** (1912), S. 1005.

[6]) Revista Matemática Hispano-Americana (2) **1** (1926), S. 129—146.

[7]) Unter Zugrundelegung einer *Normal*variation ist die zweite Variation eines Doppelintegrals von A. KNESER berechnet: Lehrbuch der Variationsrechnung (2. Aufl. 1925), S. 339—348.

[8]) A. a. O. [3]), § 6, § 7.

[9]) Vgl. z. B. O. BOLZA, Vorlesungen über Variationsrechnung (1909), S. 224—227.

[10]) A. a. O. [3]), §§ 8, 9, 12, 13.

[11]) A. a. O. [5]). — Zur Herleitung des Zusammenhanges der dort angegebenen Determinante von Linearformen mit unserem Ausdrucke (110) scheint längere Rechnung erforderlich.

Ψ, U sind *Punkt*invarianten. Die von mir weiterhin ermittelten, bei Punkt- *und* Parametertransformation invarianten Funktionen Ψ_0, U_0 dürften hier zum ersten Male angegeben sein. Man erhält die erstere (§ 12), indem man die Invariante S variiert und von δS invariante Bestandteile absprengt. Aus Ψ_0 geht schliesslich (§ 13) die nur Veränderliche der ersten Reihe enthaltende Invariante U_0 wiederum dadurch hervor, dass man die in Ψ_0 auftretenden Variationen δx_i nach § 3, § 5 durch kogrediente Grössen ersetzt. Mit Hilfe von U bzw. U_0 lässt sich bei einer Extremale die zweite Variation $\delta^2 I$ in einfacher Weise ausdrücken.

Im letzten Teile (III) befasse ich mich mit einer etwas anders gerichteten Fragestellung, die auf M. FUJIWARA zurückgeht. Dieser hat im Falle $n = 2$, indem er die BELTRAMIschen Differentiatoren in Bezug auf eine während der Rechnung auftretende invariante quadratische Differentialform benutzte, für eine festberandete Extremale eine Zerlegung der Grösse $\delta^2 I$ in drei parameterinvariante Integrale gefunden [12]; ich habe a. a. O. [6]) dieses Ergebnis, gleichfalls unter der Annahme $W = 0$, auf n-fache Integrale übertragen. Mit Hilfe mehrerer dort bewiesener Formeln leite ich hier allgemeiner bei einer *beliebigen* Überfläche eine entsprechende Zerspaltung der zweiten Variation in parameterinvariante Summanden her.

I. *Invariantentheoretisches.*

Die Operationen, die uns zuerst beschäftigen, beziehen sich auf die Bestimmungszahlen x_i ($i = 1, 2, \ldots, N$) eines Punktes in einem Bereiche X_N eines N-stufigen Raumes. Es handelt sich um die Gruppe der analytischen *Punkttransformationen* [13])

$$x_i = \varphi_i\,(x'_1, x'_2, \ldots, x'_N), \quad \ldots \ldots \ldots \quad (1)$$

die in dem Bereiche X'_N die von Null verschiedenen Funktionaldeterminanten $\partial(x_1, \ldots, x_N)/\partial(x'_1, \ldots, x'_N) = D$ besitzen und zwischen X'_N und seinem Bilde X_N eine umkehrbar eindeutige Beziehung vermitteln.

In X_N betrachten wir einen Raum X_n von n Stufen ($n < N$)

$$x_i = \varphi_i\,(u_1, u_2, \ldots, u_n); \quad \ldots \ldots \ldots \quad (2)$$

die rechten Seiten in (2) seien analytische Funktionen der in einem Bereiche O_n unabhängig veränderlichen Parameter u_α ($\alpha = 1, 2, \ldots, n$). Es sei eine Funktion

$$\mathbf{G} = \mathbf{G}\,(x_i, x_{i,\alpha_1}, \ldots, x_{i,\alpha_1 \ldots \alpha_q}) \quad \ldots \ldots \ldots \quad (3)$$

gegeben, die die Abhängigen (2) und deren Ableitungen [14]) nach den

[12]) Tokyô Sûgaku-Buturigakkwai Kizi (2) **6** (1911/12), S. 123—127.

[13]) Vgl. für $N = 2$ BOLZA [9]), S. 343. — Man hat es in (1) und durchweg im folgenden mit reellen Funktionen reeller Veränderlicher zu tun.

[14]) Wir bezeichnen diese wie a. a. O. [15]) durch griechische Zeiger. Wo es ohne Einbusse an Deutlichkeit möglich ist, sparen wir lateinische und griechische Zeiger und schreiben eine τ-te Ableitung kurz $\mathfrak{d}^\tau x$.

Unabhängigen u_α bis zur q-ten Ordnung [15]) enthält und in ihren Argumenten analytisch ist. Die *Grundfunktion* G dient als Integrand des über O_n zu erstreckenden Grundintegrals

$$J = \int^{(n)} G du, \quad du = du_1 \, du_2 \ldots du_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Die zweite Gruppe, die uns hier angeht, besteht aus den analytischen *Parametertransformationen* [16])

$$u_\alpha = \mathfrak{u}_\alpha (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

die in dem Bereiche $\bar{O}_n$ mit positiven Funktionaldeterminanten $\mathfrak{D}$ [1]), (12) versehen sind und diesen ein-eindeutig auf O_n abbilden. Die Funktion G verwandle sich unter der Wirkung von (5) nach der Formel

$$G(x, \bar{\mathfrak{d}} x, \ldots, \bar{\mathfrak{d}}^q x) = \mathfrak{D} G(x, \mathfrak{d} x, \ldots, \mathfrak{d}^q x) \;\; ^{17)}; \quad . \quad . \quad . \quad . \quad (6)$$

dann ist das Integral (4) parameterinvariant.

§ 1. *Erweiterte Punkttransformationen.*

Die Gruppe der Transformationen (1) *erweitern* wir wie folgt. Wir sehen die aus (2) zu berechnenden Grössen $\mathfrak{d}^{\mathfrak{r}} x$ ($\mathfrak{r} = 1, 2, \ldots, r$) als neue zu den x hinzutretende Veränderliche an, die sich unter dem Einflusse von (1) nach bestimmten Gesetzen verwandeln:

$$\left. \begin{aligned} x_{i,\alpha} &= \frac{\partial x_i}{\partial x'_k} x'_{k,\alpha}, \\ x_{i,\alpha\beta} &= \frac{\partial^2 x_i}{\partial x'_k \, \partial x'_l} x'_{k,\alpha} x'_{l,\beta} + \frac{\partial x_i}{\partial x'_k} x'_{k,\alpha\beta}, \\ & \ldots \ldots \ldots \ldots \ldots \;\; ^{18)} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Das Ergebnis der Einsetzung der Werte (1), (7) in G (3) werde mit G' bezeichnet,

$$G(x, \mathfrak{d} x, \ldots, \mathfrak{d}^q x) = G'(x', \mathfrak{d} x', \ldots, \mathfrak{d}^q x') \quad . \quad . \quad . \quad . \quad (8)$$

Den Grössen (1), (7) fügen wir die Funktion G und ihre Teilableitungen nach irgend welchen ihrer Argumente (3) bis zur p-ten Ordnung $G^{(\mathfrak{p})}$ ($\mathfrak{p} = 1, 2, \ldots, p$) als weitere Veränderliche hinzu, deren Verwandlungseigenschaften bei dem Wechsel (1) sich durch geeignete Differentiation von (8) ergeben, z.B.

$$\frac{\partial G}{\partial x_{i,\alpha_1 \ldots \alpha_q}} = \frac{\partial G'}{\partial x'_{k,\alpha_1 \ldots \alpha_q}} \frac{\partial x'_k}{\partial x_i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\ldots \ldots \ldots \ldots$$

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15) Besondere von solchen Integranden herrührende Invarianten habe ich unlängst angegeben, Math. Zeitschr. **25** (1926), S. 74—86.

16) Vgl. für $n = 2$ BOLZA [9]), S. 664.

17) $\bar{\mathfrak{d}} x, \ldots$ bedeuten wie a. a. O. [15]), S. 77, die Ableitungen der x_i nach den $\bar{u}_\alpha$.

18) Lateinische Zeiger laufen von 1 bis N, griechische von 1 bis n. Ueber die Fort-

Die Erweiterungen (7), (8), (9) der Transformationen (1) zusammen mit diesen selbst bilden dann diejenige unendliche kontinuierliche Gruppe $\mathfrak{Y}$, auf welche es hier ankommt.

Es liege nun irgend eine Funktion $g(x, \mathfrak{d}x, \ldots, \mathfrak{d}^r x, G, G^{(1)}, \ldots, G^{(p)})$ vor. Mit g' bezeichnen wir kurz den Ausdruck, der aus den Veränderlichen $x', \ldots, G'^{(p)}$ nach demselben Gesetze gebildet wird wie g aus $x, \ldots, G^{(p)}$,

$$g' = g(x', \mathfrak{d}x', \ldots, \mathfrak{d}^r x', G', G'^{(1)}, \ldots, G'^{(p)}) \quad . \quad . \quad . \quad (10)$$

Besitzt im besonderen die Funktion g bei allen Transformationen von $\mathfrak{Y}$ die Eigenschaft

$$g' = D^a g, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

so heisst g eine *Invariante* dieser Gruppe vom Gewichte a.

Die Funktion G selbst ist hiernach gemäss (8) eine absolute Invariante, d.h. eine solche vom Gewichte 0.

§ 2. *Einbeziehung kogredienter Veränderlicher.*

Ausser der ersten Reihe Y der von (3) herrührenden Veränderlichen des § 1 tritt in der Variationsrechnung des Integrals (4) eine zweite Reihe η von Veränderlichen $\xi_{i;\alpha_1\ldots\alpha_r}$ auf, die sich unter der Wirkung von (1) kogredient zu den die Reihe y bildenden Grössen $x_{i,\alpha_1\ldots\alpha_r}$ transformieren,

$$\left.\begin{aligned} \xi_{i;\alpha} &= \frac{\partial x_i}{\partial x'_k}\,\xi'_{k;\alpha}, \\ \xi_{i;\alpha\beta} &= \frac{\partial^2 x_i}{\partial x'_k\,\partial x'_l}\,\xi'_{k;\alpha}\,\xi'_{l;\beta} + \frac{\partial x_i}{\partial x'_k}\,\xi'_{k;\alpha\beta}, \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

Indem man die Operationen von $\mathfrak{Y}$ um die Transformationen (12) vermehrt, erweitert man $\mathfrak{Y}$ zu einer neuen Gruppe H. Eine Funktion h der Veränderlichen der beiden Reihen Y und η nennen wir eine Invariante vom Gewichte a gegenüber H, wenn bei allen Transformationen von H

$$h(x'_i, x'_{i,\alpha}, \ldots, \xi'_{i;\alpha}, \xi'_{i;\alpha\beta}, \ldots) = D^a\, h(x_i, x_{i,\alpha}, \ldots, \xi_{i;\alpha}, \xi_{i;\alpha\beta}, \ldots)$$

oder nach Art der Bezeichnung (11) kurz

$$h' = D^a h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

ist.

§ 3. *Übergang von einer Invariante h zu einer Invariante g.*

Bei dem im zweiten Teile angewandten Verfahren zur Ermittelung von Invarianten stellen sich mehrfach zunächst solche von der Art h ein.

Da uns hauptsächlich an Invarianten g liegt, so sind Übergangsmöglichkeiten von Invarianten h zu solchen der Art g erwünscht. UNDERHILL[19]) hat darüber im Falle $n=1$, $N=2$, $q=1$ einen Satz aufgestellt; wir beschränken uns hier darauf, diesen auf die Funktion

$$F = G(x_i, x_{i,\alpha}), \quad i = 1, 2, \ldots, n+1 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

als Integranden des statt (4) später ausschliesslich betrachteten Integrals

$$I = \int^{(n)} F\, du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

zu verallgemeinern. Es handelt sich also bei (14) in den Bezeichnungen (3) um den Sonderfall der Grundfunktion, in dem n beliebig, $N=n+1$ und $q=1$ ist.

Wir benutzen die von J. RADON[20]) und G. VIVANTI[21]) aus der Parameterinvarianz des Integrals I [vgl. (6)] gezogene Folgerung, dass die Grössen $x_{i,\alpha}$ in den Integranden F nur eingehen verbunden zu den Determinanten θ_k [22]), die man bis auf den Faktor $(-1)^{k+1}$ aus der Funktionalmatrix $\| x_{i,\alpha} \|$ dadurch erhält, dass man in ihr die k-te Zeile (der $x_{k,\alpha}$) streicht. Genauer ist

$$F = \Gamma(x_i, \theta_i) \quad {}^{23)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

eine Funktion der x_i und θ_i, die in den θ_i positiv-homogen von erster Stufe ist; daher besteht die Beziehung

$$\theta_i \frac{\partial F}{\partial \theta_i} = F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Was die Veränderlichen (12) betrifft, so fügen wir den Grössen $\xi_{i;\alpha}$ solche ξ_i hinzu, die sich bei der Verwandlung (1) kogredient zu den Differentialen dx_i umsetzen, also nach den Formeln

$$\xi_i = \frac{\partial x_i}{\partial x'_k} \xi'_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Nach diesen Vorbemerkungen beweisen wir folgenden

Satz 1. Aus einer Invariante h vom Gewichte a, die von den Veränderlichen der zweiten Reihe nur die ξ_i enthält und in diesen homogen von b-ter Stufe ist, erhält man eine Invariante g vom Gewichte $a+b$, wenn man die ξ_i bezüglich durch die Grössen $\partial F / \partial \theta_i$ ersetzt. [24])

lassung des Summenzeichens treffen wir bei beiden Arten von Zeigern die in der Tensorrechnung übliche Vereinbarung; Ausnahmen von dieser machen wir kenntlich.

[19]) A. a. O.[3]), S. 323.

[20]) Monatsh. f. Math. u. Phys. **22** (1911), S. 55.

[21]) Rend. Circ. mat. Palermo **33** (1912), S. 271.

[22]) Wir schreiben θ_k, nicht wie a. a. O.[1]) (S. 181) Δ_k, weil wir den Buchstaben Δ unten (III) in anderem Sinne verwenden.

[23]) Es hat kein Bedenken, das Zeichen F auch nach Einführung der Argumente θ_i beizubehalten; man verstehe $\partial F/\partial\theta_i = \partial\Gamma/\partial\theta_i$ u.s.w. Uebrigens bedienen wir uns, wo es angebracht scheint, der $x_{i,\alpha}$: $\partial F/\partial x_{i,\alpha} = \partial G/\partial x_{i,\alpha}$ u.s.w.

[24]) DE DONDERS Schritt von den linearen Formen DM, $\delta x_i \cdot \partial F/\partial x_{i,\alpha}$ zu deren Deter-

Aus der Invarianz (8) der Grundfunktion

$$F' = F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

und der Transformation[1]), (19′) der Determinanten θ_i

$$D\theta'_k = \frac{\partial x_i}{\partial x'_k}\,\theta_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

ergibt sich nämlich, dass sich die Ausdrücke $\partial F/\partial\theta_i$ so ändern:

$$D\,\frac{\partial F}{\partial\theta_i} = \frac{\partial x_i}{\partial x'_k}\,\frac{\partial F'}{\partial\theta'_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Diese Formeln nehmen die Gestalt (18) an, wenn man $D.\partial F/\partial\theta_i = \xi_i$, $\partial F'/\partial\theta'_k = \xi'_k$ setzt. Nun ist nach Voraussetzung

$$h\,(x'_i, x'_{i,\alpha}, \ldots, \xi'_i) = D^a\,h\,(x_i, x_{i,\alpha}, \ldots, \xi_i),$$

sofern nur (18) gilt; daher ist im besonderen

$$h\left(x'_i, x'_{i,\alpha}, \ldots, \frac{\partial F'}{\partial\theta'_i}\right) = D^a\,h\left(x_i, x_{i,\alpha}, \ldots, D\,\frac{\partial F}{\partial\theta_i}\right).$$

Hieraus folgt die Behauptung mit Rücksicht auf die Homogeneität des Ausdrucks h in den ξ_i.

§ 4. *Differentiation und Variation als invarianzerhaltende Verfahren.*

Indem wir zu der allgemeineren Grundfunktion G (3) zurückkehren, unterwerfen wir eine Invariante der *Differentiation*, erstens nach den Parametern u_α. Treten ausser diesen weitere Veränderliche $\varepsilon_\mathfrak{a}$ ($\mathfrak{a} = 1, 2, \ldots, \mathfrak{k}$) auf, die unter sich und von den u_α unabhängig sind, so handelt es sich zweitens um das die Ableitung nach den $\varepsilon_\mathfrak{a}$ betreffende *δ-Verfahren* in seiner Wirkung auf Funktionen $\psi\,(u_\alpha, \varepsilon_\mathfrak{a})$. In der Variationsrechnung[25]) setzt man mit Bezug auf eine Funktion $\bar\chi\,(u_\alpha, \varepsilon_\mathfrak{a})$, die für $\varepsilon = 0$ [26]) in $\chi\,(u_\alpha)$ übergeht, die $\mathfrak{s}$-te Variation von χ ($\mathfrak{s} = 1, 2, \ldots$)

$$\delta^\mathfrak{s}\,\chi = \delta\,(\delta^{\mathfrak{s}-1}\,\chi) = \left(\sum_\mathfrak{a}^{1,\mathfrak{k}} d\varepsilon_\mathfrak{a}\,\frac{\partial}{\partial\varepsilon_\mathfrak{a}}\right)^{(\mathfrak{s})} \bar\chi\,\Bigg|^{\varepsilon=0} \quad . \quad . \quad . \quad . \quad (22)$$

Hier erklären wir, ähnlich wie UNDERHILL[27]) im Falle[3]), das $\mathfrak{s}$-malige δ-Verfahren durch die Formel

$$\delta^\mathfrak{s}\,\psi = \delta\,(\delta^{\mathfrak{s}-1}\,\psi) = \left(\sum_\mathfrak{a}^{1,\mathfrak{k}} d\varepsilon_\mathfrak{a}\,\frac{\partial}{\partial\varepsilon_\mathfrak{a}}\right)^{(\mathfrak{s})} \psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

minante a. a. O. [11]) [dort Erklärung des Zeichens D, wegen M vgl. u. (78)] kann als Ersetzung der δx_i in der ersten von ihnen durch kogrediente Grössen angesehen werden; denn die Determinanten der Matrix $\| \partial F/\partial x_{k,\alpha} \|$, die aus dieser durch Streichung der i-ten Zeile hervorgehen, haben die Werte $F^{n-1}(-1)^{i-1}\,.\,\partial F/\partial\theta_i$. Letzteres beweist im Falle $n=2$ G. VIVANTI [Elementi del calcolo delle variazioni (1923), S. 171]; bei beliebigem n leitet man es leicht durch eine ähnliche Rechnung her, wie wir sie später zum Beweise von (88) durchführen.

[25]) Vgl. KNESER [7]), S. 5/6, 130.

[26]) Wir schreiben kurz $\varepsilon = 0$ statt $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_\mathfrak{k} = 0$; ebenso später $u = 0$ statt $u_1 = u_2 = \ldots = u_n = 0$.

[27]) A. a. O. [3]), S. 326.

ausführlicher geschrieben

$$\delta^{\mathfrak{s}} \psi = \sum_{\mathfrak{a}_1, \ldots, \mathfrak{a}_{\mathfrak{s}}}^{1,\mathfrak{t}} d\varepsilon_{\mathfrak{a}_1} \ldots d\varepsilon_{\mathfrak{a}_{\mathfrak{s}}} \frac{\partial^{\mathfrak{s}} \psi}{\partial \varepsilon_{\mathfrak{a}_1} \ldots \partial \varepsilon_{\mathfrak{a}_{\mathfrak{s}}}} \quad \ldots \ldots \quad (24)$$

Der Kürze halber sei es gestattet, den Ausdruck $\delta^{\mathfrak{s}} \psi$ als die $\mathfrak{s}$-te [28] Variation von ψ zu bezeichnen. Das Zeichen der Variation ist mit dem der Differentiation nach den Parametern u_α und dem der Integration über den unvariierten Bereich O_n vertauschbar.

Man denke die x_i als Funktionen A_i der u_α und der $\varepsilon_{\mathfrak{a}}$ und bilde nach (24) die Variationen $\delta^{\mathfrak{s}} \mathfrak{d}^{\mathfrak{r}} x$; diese transformieren sich nach den Formeln

$$\left.\begin{aligned}
\delta x_i &= \frac{\partial x_i}{\partial x'_k} \delta x'_k, \\
\delta x_{i,\alpha} &= \frac{\partial^2 x_i}{\partial x'_k \partial x'_l} \delta x'_k x'_{l,\alpha} + \frac{\partial x_i}{\partial x'_k} \delta x'_{k,\alpha}, \\
& \ldots \ldots \ldots \ldots \ldots \\
\delta^2 x_i &= \frac{\partial^2 x_i}{\partial x'_k \partial x'_l} \delta x'_k \delta x'_l + \frac{\partial x_i}{\partial x'_k} \delta^2 x'_k, \\
& \ldots \ldots \ldots \ldots \ldots
\end{aligned}\right\} \quad \ldots \ldots \quad (25)$$

Die Variationen $\delta^{\mathfrak{s}} x_i$ verwandeln sich unter dem Einflusse von (1) kogredient zu den Differentialen $d^{\mathfrak{s}} x_i$. Durch Hinzufügung von (25) zu (1), (7), (8), (9), (12) geht aus der Gruppe H (§ 2) abermals eine neue Ξ hervor. Die Invarianz gegenüber Ξ erklären wir entsprechend (11) und (13); Invarianten der Gruppe Ξ heissen *Punktinvarianten.*

Indem man Ableitungen nach den $\varepsilon_{\mathfrak{a}}$ durch deutsche Zeiger andeutet,

$$x_{i,\mathfrak{a}} = \frac{\partial x_i}{\partial \varepsilon_{\mathfrak{a}}}, \quad x_{i,\alpha\mathfrak{a}} = \frac{\partial x_{i,\alpha}}{\partial \varepsilon_{\mathfrak{a}}}, \ldots, \quad x_{i,\mathfrak{a}\mathfrak{b}} = \frac{\partial^2 x_i}{\partial \varepsilon_{\mathfrak{a}} \partial \varepsilon_{\mathfrak{b}}}, \ldots$$

kann man den Inhalt von (25) auch in der Gestalt ausdrücken

$$\left.\begin{aligned}
x_{i,\mathfrak{a}} &= \frac{\partial x_i}{\partial x'_k} x'_{k,\mathfrak{a}}, \\
x_{i,\alpha\mathfrak{a}} &= \frac{\partial^2 x_i}{\partial x'_k \partial x'_l} x'_{k,\mathfrak{a}} x'_{l,\alpha} + \frac{\partial x_i}{\partial x'_k} x'_{k,\alpha\mathfrak{a}}, \\
& \ldots \ldots \ldots \ldots \ldots \\
x_{i,\mathfrak{a}\mathfrak{b}} &= \frac{\partial^2 x_i}{\partial x'_k \partial x'_l} x'_{k,\mathfrak{a}} x'_{l,\mathfrak{b}} + \frac{\partial x_i}{\partial x'_k} x'_{k,\mathfrak{a}\mathfrak{b}}, \\
& \ldots \ldots \ldots \ldots \ldots
\end{aligned}\right\} \quad \ldots \ldots \quad (26)$$

Es liege nun eine absolute Invariante $\mathfrak{h}(\delta^{\mathfrak{s}} \mathfrak{d}^{\mathfrak{r}} x)$ vor, $\mathfrak{r} = 0, 1, \ldots, r$; $\mathfrak{s} = 0, 1, \ldots, s$; wir bezeichnen ihre Argumente kurz mit z_L, $L = 1, 2, \ldots$ Auf die Differentiation von $\mathfrak{h}$ beziehen sich

Satz 2. Mit $\mathfrak{h}$ ist auch die Ableitung $\partial\mathfrak{h}/\partial u_\alpha$ absolut punktinvariant.

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[28]) Unter $\delta^{\mathfrak{s}}\psi$ für $\mathfrak{s} = 0$ verstehen wir ψ.

Satz 3. Mit $\mathfrak{h}$ ist auch die Variation $\delta\mathfrak{h}$ absolut punktinvariant.

Die Beweise beider Sätze führen wir gleichzeitig. Bei dem Ansatze $x'_i = \Lambda'_i(u_\alpha, \varepsilon_\mathfrak{a})$ stellt die vorausgesetzte Invarianz $\mathfrak{h} = \mathfrak{h}'$ auf Grund von (1), (7), (25) eine Identität in u *und* ε [29]) dar; man kann sie etwa nach u_α bzw. $\varepsilon_\mathfrak{a}$ differenzieren. Das Ergebnis

$$\frac{\partial\mathfrak{h}}{\partial u_\alpha} = \frac{\partial\mathfrak{h}'}{\partial u_\alpha}, \quad . \quad . \quad . \quad (27^a) \qquad\qquad \frac{\partial\mathfrak{h}}{\partial \varepsilon_\mathfrak{a}} = \frac{\partial\mathfrak{h}'}{\partial \varepsilon_\mathfrak{a}} \quad . \quad . \quad . \quad . \quad (27^b)$$

hat ausgeführt die Gestalt

$$\sum_L \frac{\partial\mathfrak{h}}{\partial z_L} z_{L,\alpha} = \sum_L \frac{\partial\mathfrak{h}'}{\partial z'_L} z'_{L,\alpha} \quad . \quad . \quad (28^a) \qquad\qquad \sum_L \frac{\partial\mathfrak{h}}{\partial z_L} z_{L,\mathfrak{a}} = \sum_L \frac{\partial\mathfrak{h}'}{\partial z'_L} z'_{L,\mathfrak{a}} \quad . \quad (28^b)$$

Die Beziehungen (28) sind zunächst Identitäten in u und ε; sie bestehen aber auch, wenn man die Gesamtheit Z' der Ableitungen $x'_{i,\alpha_1\ldots\alpha_\mathfrak{r}\,\mathfrak{a}_1\ldots\mathfrak{a}_\mathfrak{s}}$ [30]) in ihnen durch eine solche $\hat{Z}'$ von *beliebigen* Werten $\hat{x}'_{i,\alpha_1\ldots\alpha_\mathfrak{r}\,\mathfrak{a}_1\ldots\mathfrak{a}_\mathfrak{s}}$ [31]) ersetzt und zugleich Z durch diejenige Gesamtheit $\hat{Z}$ von Werten $\hat{x}_{i,\alpha_1\ldots\alpha_\mathfrak{r}\,\mathfrak{a}_1\ldots\mathfrak{a}_\mathfrak{s}}$, welche mit $\hat{Z}'$ durch die Formeln (1), (7), (26) zusammenhängt. Dabei treten in (28) an Stelle der $\delta^\mathfrak{s} x'_{i,\alpha_1\ldots\alpha_\mathfrak{r}}$ die Grössen

$$\delta^\mathfrak{s} \hat{x}'_{i,\alpha_1\ldots\alpha_\mathfrak{r}} = \sum_{\mathfrak{a}_1,\ldots,\mathfrak{a}_\mathfrak{s}}^{1,\mathfrak{f}} d\varepsilon_{\mathfrak{a}_1} \ldots d\varepsilon_{\mathfrak{a}_\mathfrak{s}} \hat{x}'_{i,\alpha_1\ldots\alpha_\mathfrak{r}\,\mathfrak{a}_1\ldots\mathfrak{a}_\mathfrak{s}} \quad . \quad . \quad . \quad . \quad (29)$$

und an Stelle der $\delta^\mathfrak{s} x_{i,\alpha_1\ldots\alpha_\mathfrak{r}}$ die nach Art von (29) aus den Mitgliedern von $\hat{Z}$ gebildeten Grössen $\delta^\mathfrak{s} \hat{x}_{i,\alpha_1\ldots\alpha_\mathfrak{r}}$.

Um zu zeigen, dass die Invarianzen (28) auch in diesem allgemeineren Sinne gelten, wählen wir die Λ'_i in der Form der Polynome

$$x'_i = \sum_{\tau}^{0,R} \sum \sum_{\mathfrak{t}}^{0,S} \sum \hat{x}'_{i,(\tau_1)\ldots(\tau_n)(\mathfrak{t}_1)\ldots(\mathfrak{t}_\mathfrak{f})} \frac{u_1^{\tau_1}\ldots u_n^{\tau_n}}{\tau_1!\ldots\tau_n!} \frac{\varepsilon_1^{\mathfrak{t}_1}\ldots\varepsilon_\mathfrak{f}^{\mathfrak{t}_\mathfrak{f}}}{\mathfrak{t}_1!\ldots\mathfrak{t}_\mathfrak{f}!} \quad . \quad . \quad (30)$$

Dabei ist die vierte bzw. zweite Summe über die Werte $\mathfrak{t}_\mathfrak{a} \geqq 0$ bzw. $\tau_\alpha \geqq 0$ so zu erstrecken, dass $\mathfrak{t}_1 + \ldots + \mathfrak{t}_\mathfrak{f} = \mathfrak{t}$ bzw. $\tau_1 + \ldots + \tau_n = \tau$ ist; S bzw. R bezeichnet in beiden Fällen (*a*), (*b*) den jeweils grössten Wert des Zeigers $\mathfrak{s}$ bzw. $\mathfrak{r}$ [30]); die Koeffizienten bedeuten die Mitglieder von $\hat{Z}'$, die den zu Z' gehörigen Ableitungen

$$\frac{\partial^{\tau+\mathfrak{t}} x'_i}{\partial u_1^{\tau_1}\ldots\partial u_n^{\tau_n}\,\partial\varepsilon_1^{\mathfrak{t}_1}\ldots\partial\varepsilon_\mathfrak{f}^{\mathfrak{t}_\mathfrak{f}}} = x'_{i,(\tau_1)\ldots(\tau_n)\,(\mathfrak{t}_1)\ldots(\mathfrak{t}_\mathfrak{f})}$$

entsprechen. Der Ansatz (30) liefert in der Tat

$$x'_{i,(\tau_1)\ldots(\tau_n)\,(\mathfrak{t}_1)\ldots(\mathfrak{t}_\mathfrak{f})}\Big|^{u=0,\ \varepsilon=0} = \hat{x}'_{i,(\tau_1)\ldots(\tau_n)\,(\mathfrak{t}_1)\ldots(\mathfrak{t}_\mathfrak{f})} ;$$

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29) Diese Folge der Erklärung (23) gibt dem Beweise der Aussage (29) bequeme Form.

30) Jetzt läuft im Falle (*a*) $\mathfrak{r}$ bis $r+1$, im Falle (*b*) $\mathfrak{s}$ bis $s+1$.

31) Wir setzen bei ihnen diejenigen Zeigersymmetrien voraus, welche bei den ihnen entsprechenden Ableitungen statthaben.

er bewirkt, dass Z' in $\hat{Z}'$, also $\delta^{\mathfrak{s}} x'_{i,\alpha_1 \ldots \alpha_{\mathfrak{r}}}$ nach (24), (29) in $\delta^{\mathfrak{s}} \hat{x}'_{i,\alpha_1 \ldots \alpha_{\mathfrak{r}}}$ übergeht. Aus der gemachten Annahme, dass $\hat{Z}$ und $\hat{Z}'$ durch (1), (7), (26) verknüpft sind, folgt dann, dass gleichzeitig aus Z die Gesamtheit $\hat{Z}$ wird und mithin die Grössen $\delta^{\mathfrak{s}} x_{i,\alpha_1 \ldots \alpha_{\mathfrak{r}}}$ wirklich die Werte $\delta^{\mathfrak{s}} \hat{x}_{i,\alpha_1 \ldots \alpha_{\mathfrak{r}}}$ erhalten. Die Gleichungen (27^b) ergeben gemäss (23.1) sogleich den Satz 3

$$\delta \mathfrak{h}' = \delta \mathfrak{h} \quad \ldots \ldots \ldots \ldots \quad (31)$$

§ 5. *Parametertransformationen.*

Bei den Parametertransformationen (5), von denen jetzt kurze Rede sein soll, setzen wir voraus, dass in den Ausdrücken $\mathfrak{u}_\alpha$ die $\varepsilon_{\mathfrak{a}}$ nicht vorkommen. Die u_α, $\bar{u}_\alpha$ werden nicht variiert; in der Schreibweise des § 4 ist

$$\delta u_\alpha = \delta \bar{u}_\alpha = 0 \quad \ldots \ldots \ldots \ldots \quad (32)$$

Ein Wechsel (5) lässt die x_i ungeändert,

$$x_i = \bar{x}_i \quad \ldots \ldots \ldots \ldots \quad (33)$$

Es sei $x_i = A_i\,(u_\alpha, \varepsilon_{\mathfrak{a}}) = \bar{A}_i\,(\bar{u}_\alpha, \varepsilon_{\mathfrak{a}})$; bedient man sich der Abkürzungen

$$x_{i,\bar{\lambda}} = \frac{\partial x_i}{\partial \bar{u}_\lambda}\ , \quad x_{i,\bar{\lambda}\bar{\mu}} = \frac{\partial^2 x_i}{\partial \bar{u}_\lambda\, \partial \bar{u}_\mu}, \ldots,$$

so lauten die Verwandlungsformeln der Ableitungen $\mathfrak{d}^{\mathfrak{r}} x$

$$\left.\begin{aligned} x_{i,\alpha} &= x_{i,\bar{\lambda}}\, \frac{\partial \bar{u}_\lambda}{\partial u_\alpha}, \\ x_{i,\alpha\beta} &= x_{i,\bar{\lambda}\bar{\mu}}\, \frac{\partial \bar{u}_\lambda}{\partial u_\alpha} \frac{\partial \bar{u}_\mu}{\partial u_\beta} + x_{i,\bar{\lambda}}\, \frac{\partial^2 \bar{u}_\lambda}{\partial u_\alpha\, \partial u_\beta}, \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots \quad (34)$$

Die durch (5) bedingten Transformationen (33), (34) bilden eine Gruppe $\mathfrak{N}$; man kann sie zu einer neuen $\mathfrak{B}$ *erweitern*, indem man wie in § 2 eine zweite Reihe (oder deren mehrere) von Veränderlichen $\mathfrak{v}_{i;\alpha_1 \ldots \alpha_{\mathfrak{r}}}$ [32]) einführt, die sich kogredient zu den $x_{i,\alpha_1 \ldots \alpha_{\mathfrak{r}}}$ d. h. so umsetzen, dass die Formeln (33), (34) gelten, wenn man in ihnen $x_{i,\alpha_1 \ldots \alpha_{\mathfrak{r}}}$, $x_{i,\bar{\alpha}_1 \ldots \bar{\alpha}_{\mathfrak{r}}}$ bezüglich durch $\mathfrak{v}_{i;\alpha_1 \ldots \alpha_{\mathfrak{r}}}$, $\mathfrak{v}_{i;\bar{\alpha}_1 \ldots \bar{\alpha}_{\mathfrak{r}}}$ ersetzt. Ist Π eine Funktion der Grössen $x_{i,\alpha_1 \ldots \alpha_{\mathfrak{r}}}$, $\mathfrak{v}_{i;\alpha_1 \ldots \alpha_{\mathfrak{q}}}$, so bedeute $\bar{\Pi}$ dieselbe Funktion der $x_{i,\bar{\alpha}_1 \ldots \bar{\alpha}_{\mathfrak{r}}}$, $\mathfrak{v}_{i,\bar{\alpha}_1 \ldots \bar{\alpha}_{\mathfrak{q}}}$. Π heisst eine *Parameterinvariante* vom Gewichte c, wenn bei allen Transformationen von $\mathfrak{B}$

$$\bar{\Pi} = \mathfrak{D}^c\, \Pi \quad \ldots \ldots \ldots \ldots \quad (35)$$

Veränderliche der Art $\mathfrak{v}_{i;\alpha_1 \ldots \alpha_{\mathfrak{r}}}$ sind die Variationen $\delta^{\mathfrak{s}} x_{i,\alpha_1 \ldots \alpha_{\mathfrak{r}}} = w_M$ [33]).

[32]) $\mathfrak{r}, \mathfrak{q} = 0, 1, \ldots$.

[33]) $\mathfrak{r} = 0, 1, \ldots, \rho$; $\mathfrak{s} = 0, 1, \ldots, \sigma$; $M = 1, 2, \ldots$.

da diese bei (5) genau dieselbe Regel befolgen wie die $x_{i,\alpha_1\ldots\alpha_r}$ selbst,

$$\delta x_i = \delta \bar{x}_i \quad , \quad \delta x_{i,\alpha} = \delta \bar{x}_{i,\bar{\lambda}} \frac{\partial \bar{u}_\lambda}{\partial u_\alpha}, \ldots, \delta^2 x_i = \delta^2 \bar{x}_i, \ldots \qquad (36)$$

$\mathfrak{P}$ sei eine Funktion der w_M; Funktionen dieser Veränderlichen betrifft der

Satz 4. Ist $\mathfrak{P}$ eine Parameterinvariante vom Gewichte c, so gilt dies auch von der Variation $\delta\mathfrak{P}$.

Man beweist ihn entsprechend wie oben den Satz 3, indem man hier von den Abhängigkeiten $x_i = A_i(\bar{u}_\alpha, \varepsilon_\Omega)$ ausgeht. Bei der Variation der Voraussetzung (35) für $\mathfrak{P}$ ist zu berücksichtigen, dass nach (32) $\delta\mathfrak{D} = 0$ ist. In den Formeln, die man jetzt statt der früheren (28^b) erhält, kann wiederum an Stelle der Gesamtheit $\bar{\mathfrak{w}}$ der Werte $\bar{w}_M$ eine *beliebige* $\hat{\bar{\mathfrak{w}}}$ treten, wenn gleichzeitig $\mathfrak{w}$ durch eine Gesamtheit $\hat{\mathfrak{w}}$ ersetzt wird, die mit $\hat{\bar{\mathfrak{w}}}$ durch (33), (34) zusammenhängt.

Auf Grund der Beziehungen (36) ergibt sich

Satz 5. Ist $\mathfrak{P}$ eine Parameterinvariante vom Gewichte c, so gilt dies auch von der Ableitung $\partial\mathfrak{P}/\partial\delta^{\mathfrak{s}} x_i$.

Beispiel einer Parameterinvariante vom Gewichte 1 ist nach (6) die Funktion G selbst. Wir heben unter den Invarianten Π diejenigen besonderen $\varkappa$ hervor, welche wie G *nur die Veränderlichen* $\mathfrak{d}^{\mathfrak{r}} x$ enthalten. Auf solche Invarianten der Funktion F (14) zielt folgende Ergänzung des Satzes 1 ab:

Satz 6. Wenn unter den Voraussetzungen des Satzes 1 die ξ_i absolut parameterinvariant sind und $h(\mathfrak{d}^{\mathfrak{r}} x_i, \xi_i)$ eine Parameterinvariante vom Gewichte c ist, so trifft das letztere auch auf die dort aus h gewonnene Punktinvariante g zu.

Die erwähnte Eigenschaft (6) der Grundfunktion

$$\bar{F} = \mathfrak{D} F \qquad (37)$$

führt nämlich in Verbindung mit der Transformation [1]), $\overline{(19)}$ der Determinanten θ_i

$$\bar{\theta}_i = \mathfrak{D}\theta_i \qquad (38)$$

zu der Invarianz

$$\frac{\partial \bar{F}}{\partial \bar{\theta}_i} = \frac{\partial F}{\partial \theta_i} \qquad (39)$$

Es können mithin in der für h gültigen Formel (35) die Grössen $\partial F/\partial\theta_i$ an die Stelle der ξ_i treten.

Chemistry. — *Physical purity and powder Röntgenogram.* By N. H. KOLKMEIJER. (Communicated bij Prof. ERNST COHEN).

(Communicated at the meeting of October 29, 1927).

Opinions differ as to the existence of two modifications of mercuric oxide, a red and a yellow one. ERNST COHEN [1]) concludes from the existence of a difference of potential between two electrodes Hg (HgO) KOH, of which HgO with one had the red form, with the other the yellow, that the two forms are allotropic modifications. On the other hand WILH. OSTWALD [2]) and also SCHICK [3]) are of opinion that the difference in colour is the result of a difference in size of the particles, and base this opinion on their experiments on velocity of solution and solubility of the two forms.

LEVI [4]) lately tried to solve this point of difference by taking a powder photo of the two forms with X-rays. From a comparison of the two exposures, he concludes that there can no longer be any doubt as to the crystallographic identity of the two forms of HgO.

Without taking sides in this point of controversy [5]) it appears to me that it is not inexpedient to point out that LEVI's conclusion is not justified by his investigation. When it is not an ascertained fact that the two forms, investigated by LEVI, consist of one modification for 100% — and in his paper not a word is said which shows that he has verified it — it is highly possible that the yellow colour was caused f.i. by the admixture of a certain percentage of the yellow modification with the red. And if this percentage is not high enough, the admixture does not mark its lines on the films in the normal time of exposure (which LEVI, judging from the reproduction of his films has not exceeded).

In order to make this clear we wish to call attention to the fact, that COHEN has more than once pointed out, that when different physical properties are determined, it is not always known whether the preparate consists for 100 % of one modification, and is therefore „physically pure". If the

[1]) ERNST COHEN, Z. phys. Chem. **34**, 69, 1900.

[2]) WILH. OSTWALD, Z. phys. Chem. **17**, 183, 1895; **18**, 159, 1895; **34**, 495, 1900.

[3]) K. SCHICK, Ibid. **42**, 155, 1903. See also Gay Lussac C.R. **16**, 309, 1843; G. FUSEYA J. Am. Chem. Soc. **42**, 368, 1920; R. VARET C.R. **120**, 622, 1895; Bull. Soc. chim. **[3]**, **13**, 677, 1895; HULETT Z. physik. Chem. **37**, 400, 1901.

[4]) G. R. LEVI, Gazz. chim. Ital. **54**, 709, 1924. See also R. FRICKE Z. anorg. und allgem. Chem. **166**, 244, 1927.

[5]) Prof. COHEN told me by word of mouth that he himself inclines to the opinion of OSTWALD and SCHICK.

substance is a labile modification, at the temperature and pressure of the experiment, it is very difficult to get a preparate which does not at least contain a very small percentage of the stable modification. If, on the other hand, the preparate is the stable modification, it is highly possible that it contains a few percents of the labile modification. And in either case the change of labile to stable modification can have a slow course. Each of the two forms of HgO, the red and the yellow, may therefore very well be a mixture of a yellow and a red modification, and if we wish to make reliable determinations of the properties of such substances it is necessary first to make sure of the physical purity. How this can be done by means of the X-rays will appear from what follows.

From the observations made by ANDREWS [1]) in which even a considerable percentage of an alien substance did not appear on the film it is evident that in such a mixture of two modifications of the same substance nothing need appear of the presence of one of them, if the percentage of the latter is not rather great. Even 20 % of Ni in a Ni-Fe alloy did not give lines on the diagram. In order to trace whether similar percentages would also remain invisible if we took two modifications of the same substance, I have, at the request of Prof. COHEN, made exposures of a mixture of 90 % of white with 10 % of gray tin, and of a mixture of 80 % of white with 20 % of gray tin. In volume procents 12 % and 24 % of gray tin were present respectively. It was a happy chance that we could use for it physically pure preparations, which had been prepared by DOUWES DEKKER [2]) for his experiments. With the help of Röntgen photos was demonstrated that the preparations used were indeed physically pure. For even if a preparate contains a small percentage of impurities, only the lines of the principal component are marked off on the film, and precisely as if the preparate had had a purity of 100 %. If we can deduce from the film the structure, we can, at the same time, determinê the density of the physically pure modification from measurements made on an impure preparate. If of a preparate, the density is measured with a pycnometer, and if this agrees with the value which was determined in the way mentioned above, we know for certain [3]) that we have a physically pure preparate before us. In the preparates of DOUWES DEKKER the densities of the gray tin, determined with a pycnometer and a Röntgenogram, were 5.765 and 5.764 ; of the white tin 7.285 and 7.29.

In figure 1 we have the prints of our exposures, placed one over the other, of 100 % of gray, 80 % of white with 20 % of gray, 90 % of white with 10 % of gray, and 100 % of white tin [4]). On the one of 20 %

[1]) M. R. ANDREWS, Phys. Rev. (2), 17, 261, 1921.

[2]) K. DOUWES DEKKER, Diss. Utrecht 1927.

[3]) If, at least, the difference in density of the two modifications is not too small. (Cf. A. L. TH. MOESVELD, Chem. Weekbl., 24, 485, 1927).

[4]) During the exposure the temperature was continually about 18°. The temperature of the transitionpoint gray $\rightleftarrows$ white tin is 13°.

of gray tin we see clearly the strongest line of gray tin (indicated with an arrow.) On the print of 10 % of gray tin we can still see a trace of that

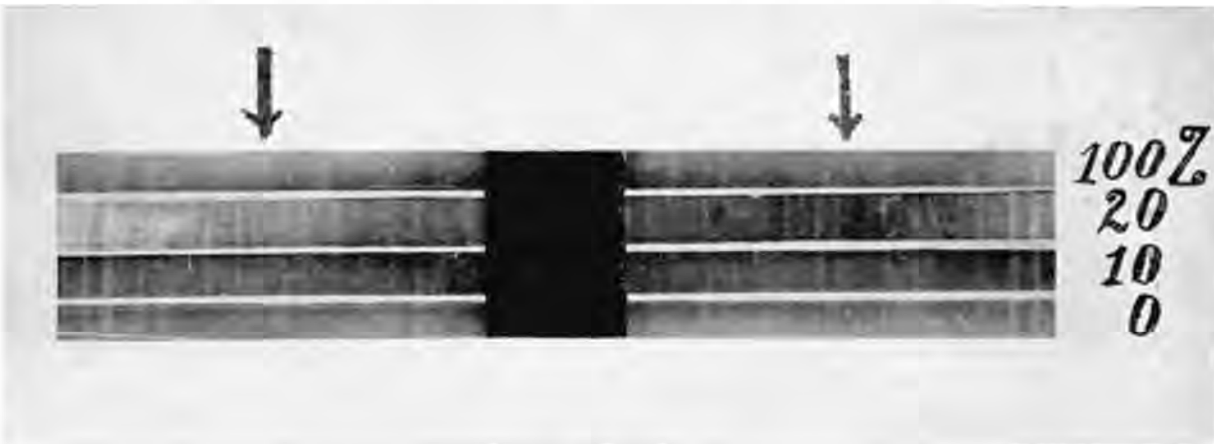

Fig. 1.

line, so indistinct however, that it would, no doubt, have not been observed, if we had not been informed beforehand of the composition of the preparate. This proves that a pollution with as much as 10 % of another modification might have remained unnoticed in LEVI's experiments. Moreover it is very striking that in LEVI's table some rather distinct lines (21 and 41) are given for the yellow substance, which were not found in the red. LEVI does give an explanation for it; but his line of argument does not seem to me very sound. As regards this fact, there is a striking agreement with figure 1 above.

Meanwhile it must be observed that when the time of exposure is ten times longer — apart from the absorption of the rays in the preparate — the lines of the 10 vol. % of admixture ought to be seen as well as the lines of 100 % of admixture with a normal time of exposure. It might happen that, on account of the long exposure the lines of the 90 % of main product would be so much over-exposed, that the lines of the admixture would be rendered invisible. Yet it does not seem impossible to trace sometimes small percentages of a new modification by means of the way indicated. It even seems to be possible to make quantitative determinations with it. Experiments on this subject will be made in this laboratory. We shall also try to make this method serve for quantitative analysis, when the components have a different chemical composition [1]). Qualitatively some results have already been obtained [2]).

In cases, in which the densities of the two modifications do not differ

[1]) I.a. L. VEGARD in the case of mixed crystals.

[2]) HULL could distinguish among other things from the photos between the cases that the powder was a mixture of NaCl and KF, and that it was a mixture of NaF and KCl.

very much, so that the other methods for tracing modifications (dilatometer) leave us in the lurch, the Röntgenometric method, given above, may be of great importance. Yet we must not forget that the radiation itself might act as a catalyst, which accelerates the transformation of the labile into the stable modification. This chance, however, is not very great, and perhaps we might also reduce it to a smaller compass by using many greatly different wavelengths.

VAN 'T HOFF-*Laboratory.*

Utrecht, October 1927.

Anatomy. — *The Proportion of cerebellar to total brain weight in Mammals.* By IRMARITA KELLERS PUTNAM [1]) M.D. (Communicated by C. U. ARIËNS KAPPERS).

(Communicated at the meeting of December 17, 1927).

There are few data available upon the proportion between the weight of the cerebellum and that of the whole brain in mammals. Perhaps this is partly because there are few large collections of mammalian brains in existence. Moreover figures which do exist in the literature coming from different sources are not comparable as the cerebella have been removed in no standard fashion and the weighing of the brain has been done under varying conditions and generally with the meninges, at the least with the pia mater and parts of the arachnoid [2]).

Material and Methods.

A hundred mammalian brains of the collection at the Central Institute for Brain Research, Amsterdam, were used for this investigation. The body weights and the ages of the animals were not noted. The collection included old and young animals, but no foetal brains nor brains of new born animals were included. The brains mentioned below had all been preserved, without meninges, for varying lengths of time in 4 % formaldehyde (10 % formaline). Brains in 10 % formaline (4 % formaldehyde), according to FLATAU (1897), the first month of preservation increase from one to two per cent of their original weight.

Between the first and fifth months they loose some of this initial increase, so that at the fifth month they are only one per cent heavier than their original weight and remain so for the following months (FLATAU's observations covered 15 months). Most of the brains in this series were preserved for a still longer period. As however in the last ten months of a fifteen months preservation the then obtained increase of 1 % does not change, there is little reason to believe that it should increase later.

There is a slight percentage difference between weight changes in brain and spinal cord during similar preservation. Whether there is a difference between the change of the cerebellar weight and the rest of the brain we do not know. We can assume however that this is very slight.

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[1]) Holder of the Vassar Alumnae Fellowship for graduate study, 1925—1926.

[2]) Weighing the brain with the pia (and part of the arachnoid) includes a source of errors, as this tissue may keep a fairly great deal of the preservation fluid.

In alcohol on the other hand brains loose large amounts of weight and the loss is continuous, so that at the end of fifteen months a brain has lost thirty four per cent of its original weight. We have listed only one brain preserved in this way (cf. p. 162).

The hypophysis, which was often allowed to remain in the sella turcica, when the brain was removed from the skull in the zoological garden, was dissected off those brains where it was still attached in order to equalize conditions. *All meningeal tissue was carefully removed* [1]).

The cerebellar peduncles were severed just above the emergence of the seventh and eigth nerves tangential to the brain stem, care being observed to leave the posterior corpora quadrigemina and the fourth nerve intact. The brain stem was cut 3 mm below the calamus scriptorius. Pains were taken to perform these manipulations as nearly as possible in the same way, as it is obvious that each one of them involves a source of error.

Of some of the brains only one half was available the other half being cut in microscopical sections. In these cases the *hemisection* is mentioned in my tables. They were only used for this statistic if it appeared that the hemisection was accurately made, and thus did not influence the percentage relation.

Weighing was done on a chemical balance sensitive to a milligram. The brains were removed from formaldehyde, dried with a soft towel (so that there was no more draining of fluid) and weighed directly, exposed to the air during the process. Very little time was required for this, since preliminary weighings had been made on the previous day to facilitate the final weighing.

It was found that one individual could so standardize the amount of drying and that there was less variation between two weighings of the same brain on successive days by this method than by either of the methods described in the next paragraph. Also it was found that the variations for small brains were less than for large brains which was the reverse with the other two methods. This is important because very slight changes in the weights of the small brains cause large percentage variations, while the reverse is the case for the large brains. However, these variations also constitute a source of error.

Dr. RICHARD S. LYMAN of Rochester University, New-York to whom I am indebted for much help, determined during his sojourn in the Central Institute for Brain Research the rate of loss of moisture when brains were allowed to dry in the open air, and found that a constant state of dryness was not reached at any point. As was to be expected the rate of loss was proportional to the surface area so that the cerebellum lost weight more rapidly than the rest of the brain and small brains more rapidly than large ones. An attempt was also made to bring the brains to a constant state of moisture by keeping them in a moist chamber.

While the daily variations of these brains were less than the hourly variations which Dr. LYMAN obtained, they were still greater than the variations obtained by the method

[1]) This may explain that the brain weight of several animals is less than the figures mentioned in the current litterature.

described in the paragraph above, which was therefore chosen for the series. Comparative results of the three methods applied to a brain of a young *Nasua narica* (preserved in formaline 10 %) are recorded below to establish the justification of this choice. (Table I).

I. Brain exposed to air during five hours (Determinations by R. S. LYMAN).

Moment of weighing	Cerebellar w.	Cerebral w.	Total brain w.	Cerebel. perc.
Aug. 13th 11.35 AM.	2.83 gr.	20.73 gr.	23.56 gr.	12.00 %
" " 12.00 M.	2.79 "	20.59 "	23.38 "	11.93 "
" " 12.30 PM.	2.75 "	20.46 "	23.21 "	11.84 "
" " 1.00 "	2.73 "	20.37 "	23.10 "	11.82 "
" " 1.30 "	2.70 "	20.20 "	22.96 "	11.76 "
" " 2.30 "	2.65 "	20.09 "	22.74 "	11.64 "
" " 3.30 "	2.60 "	19.93 "	22.53 "	11.53 "
" " 4.30 "	2.55 "	19.77 "	22.32 "	11.42 "
II. Brain kept in moist chamber.				
Sept. 5th	2.78 gr.	20.22 gr.	23.00 gr.	12.20 %
" 6th 11.00 AM.	2.745 "	20.155 "	22.90 "	12.0 "
" " 4.00 PM.	2.74 "	20.04 "	22.78 "	12.05 "
" 7th	2.72 "	20.00 "	22.72 "	11.99 "
" 12th	2.67 "	19.72 "	22.39 "	11.96 "
" 14th	2.64 "	19.58 "	22.22 "	11.88 "
III. Brain removed from formaldehyde softly dried and weighed directly				
Sept. 15th	2.73 gr.	20.53 gr.	23.26 gr.	11.73 %
" 16th	2.74 "	20.57 "	23.31 "	11.70 "
" 17th	2.73 "	20.54 "	23.27 "	11.73 "

The method of weighing under water might be still more accurate. However, this method was not tried as it was thought too elaborate for this purpose. The temperature of the water influencing the specific gravity of the particular water used, movements in the water must all be carefully controlled, if this method is to be as accurate practically for our purpose as it is theoretically.

The results of the weighings are given in the accompanying table, which shows the weight of the cerebellum, that of the cerebral hemispheres and stem, the total brain weight and the proportion between the weight of the cerebellum and that of the total brain, expressed as a percentage of the latter.

The classification of the animals is essentially that of OSBORN (1910). The primates however, are listed according ELLIOT (1913).

ORDER RODENTIA.

	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Sciuridae.				
Pteromys nitidus	1.39 gr.	7.50 gr.	8.89 gr.	15.62 %
Cynomys ludovicianus	1.5 "	9.95 "	11.45 "	13.10 "
Echinosciurus aureogaster	.96 "	5.24 "	6.20 "	15.45 "
Heterosciurus notatus	.65 "	3.63 "	4.28 "	15.18 "
Leporidae.				
Lepus cuniculus	.94 "	6.07 "	7.03 "	13.4 "
Hystricidae.				
Hystrix cristata	2.9 "	16.3 "	19.2 "	15.00 "
Hystrix javanica (hemisect.)	1.39 "	8.23 "	9.62 "	14.45 "
Coendu prehensilis	2.57 "	16.26 "	18.83 "	13.62 "
Chinchillidae.				
Lagostomus trichodactylus	1.9 "	13.48 "	15.38 "	12.35 "
Dasyproctidae.				
Dasyprocta aguti	2.32 "	17.58 "	19.90 "	11.65 "
Dasyprocta aguti	2.08 "	13.73 "	15.81 "	13.12 "
Octodontidae.				
Myopotamus coïpu	1.13 "	10.33 "	11.46 "	9.80 "

ORDER EDENTATA.

Myrmecophagidae.				
Myrmecophaga jubata	8.67 gr.	43.4 gr.	52.07 gr.	16.52 %
Bradipodidae.				
Choloepus didactylus	5.27 "	25.68 "	30.95 "	13.84 "
Choloepus didactylus	4.5 "	24.31 "	28.81 "	15.55 "
Choloepus didactylus (hemisection)	2.3 "	12.16 "	14.46 "	15.90 "
Dasypodidae.				
Dasypus villosus	2.19 "	12.86 "	15.05 "	14.53 "

ORDER UNGULATA.

Sub Order Artiodactyla.	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Dicotylidae.				
Dicotyles labiatus	7.67 gr.	58.98 gr.	66.65 gr.	11.55 %
Camelidae.				
Camelus dromedarius	57.5 „	420.— „	477.5 „	12.05 „
Auchenia glama	20.23 „	128.85 „	149.08 „	13.60 „
Giraffidae.				
Camelopardalis giraffa (young)	53.3 „	433.— „	486.3 „	10.90 „
Cervidae.				
Cariacus nemoralis	13 7 „	114.4 „	128.1 „	10.70 „
Alces machlis	27.8 „	232.5 „	260.3 „	10.65 „
Cervulus muntjac (young spec.)	4.45 „	39.5 „	43.95 „	10.12 „
Rusa hippelaphus	19.37 „	116.15 „	185.52 „	10.4 „
Rusa hippelaphus	19.7 „	166.2 „	185.52 „	10.6 „
Rucervus Eldi	21.9 „	179.7 „	201.6 „	10.86 „
Dama dama	14.9 „	146.5 „	161.4 „	9.22 „
Ovidae.				
Ovis tragelaphus (small spec., hemisect.)	12.1 „	107.15 „	119.25 „	10.1 „
Ovis tragelaphus	19 „	173.— „	192.— „	9.87 „
Capra hircus (small spec.)	10.35 „	73.6 „	83.95 „	12.32 „
Antilopes.				
Antilope cervicapra (hemisect.)	5.35 „	44.5 „	49.85 „	10.75 „
Antilope borea	6.3 „	53.95 „	60.25 „	10.45 „
Oreas Livingstoni	37.2 „	395.4 „	432.6 „	8.65 „
Catoblepas gnu (small spec., hemis.)	7.26 „	66.3 „	73.56 „	9.86 „
Anoa depressicornis	18.7 „	163.3 „	182.— „	10.26 „

Sub Order Perissodactyla.

Equidae.				
Equus caballus	57 gr.	446 gr.	503 gr.	11.3 %
Equus asinus	36.5 „	298.2 „	334.7 „	10.92 „
Tapiridae.				
Tapirus indicus	27.77 „	213.12 „	240.89 „	13.00 „

ORDER PROBOSCIDAE.

	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Elephas indicus (small specim.)	923.— gr.	2816.—gr.	3739.—gr.	24.68 %

ORDER ODONCETI.

Delphinidae.				
Phocaena phocaena (hemisection)	28.85 gr.	157.86 gr.	186.71 gr.	15.49 %
Phocaena phocaena	58.— „	332.— „	390.— „	15.— „

ORDER CARNIVORA.

Ursidae.				
Ursus arctos (young sp.)	36.— gr.	196.— gr.	232.— gr.	16.30 %
Ursus maritimus	68.8 „	365.5 „	434.3 „	15.75 „
Heliarctos malayanus	40.7 „	211.65 „	252.35 „	16.10 „
Mustelidae.				
Lutra vulgaris (hemisection)	2.05 „	22.— „	24.05 „	8.38 „
Lutra vulgaris (hemisection)	1.94 „	16.3 „	18.24 „	10.69 „
Putorius putorius	0.61 „	4.77 „	5.38 „	11.30 „
Mustela erminea (hemisection)	0.3 „	2.27 „	2.57 „	11.66 „
Mustela erminea (hemisection)	0.22 „	1.84 „	2.06 „	10.65 „
Mustela foina (hemisection)	0.40 „	2.81 „	3.21 „	12.60 „
Meles taxus	5.39 „	38.42 „	43.81 „	12.29 „
Viverridae.				
Paradoxurus musanga (hemisection)	1.22 „	7.35 „	8.57 „	14.28 „
Arctitis binturong (hemisection)	2.17 „	12.63 „	14.80 „	14.65 „
Herpestes griseus	1.18 „	9.38 „	10.56 „	11.17 „
Canidae.				
Canis familiaris	4.55 „	46.75 „	51.30 „	8.9 „
Canis familiaris	7.— „	66.93 „	73.93 „	9.6 „
Black and Tan (hemisection)	3.82 „	33.70 „	37.50 „	10.18 „
Retriever (hemisection)	4.24 „	42.12 „	46.36 „	9.15 „
Dachshund	5.62 „	62.81 „	68.43 „	8.25 „
Spaniel	8.57 „	81.7 „	90.27 „	9.50 „
Airdale Terrier	7.18 „	72.65 „	79.83 „	9.17 „

	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Wolf Hound (hemisection)	3.81 gr.	35.97 gr.	39.78 gr.	9.60 %
Shepherd Dog (hemisection)	3.18 „	32.7 „	35.88 „	8.85 „
Gordon Setter (hemisection)	3.80 „	41.80 „	45.60 „	8.46 „
Collie (hemisection)	4.2 „	35.66 „	39.26 „	10.62 „
German Dog	10.2 „	87.42 „	97.62 „	10.4 „
Irish Setter	7.94 „	73.62 „	81.56 „	9.70 „
Boxer (hemisection)	3.28 „	32.72 „	36.— „	9.02 „
Canis lupus	5.67 „	58.43 „	64.10 „	8.85 „
Vulpus lupus opus (hemisection)	1.64 „	12.11 „	13.75 „	11.90 „
Felidae.				
Zibethailurus pardalis (hemisection)	2.57 „	22.39 „	24.96 „	10.30 „
Zibethailurus pardalis	6.98 „	44.51 „	51.49 „	13.56 „
Felis leo	19.65 „	167.24 „	186.92 „	10.52 „
Felis leo (hemisection)	8.78 „	86.95 „	95.73 „	9.28 „
Felis concolor (hemisection)	7.53 „	48.11 „	55.64 „	13.05 „
Felis concolor (hemisection)	5.78 „	45.11 „	50.89 „	11.36 „
Felis macrosceles nebulosa	6.13 „	39.03 „	45.16 „	13.52 „
Lynx lynx	5.78 „	36.31 „	42.09 „	13.70 „
Phocidae.				
Phoca vitulina (small spec.)	26.10 „	143.40 „	169.50 „	15.4 „
Phoca vitulina	29.5 „	179.— „	208.50 „	14.1 „

ORDER PRIMATES.

Prosimiae.				
Daubentonidae.				
Cheiromys madagascariensis (sm. spec.)	3.38 gr.	22.23 gr.	25.61 gr.	13.18 %
Lemuridae.				
Lemur catta	1.38 „	9.03 „	10.41 „	13.35 „
Lemur macaco (hemisection)	1.42 „	9.— „	10.42 „	13.60 „
Lemur mongoz (hemisection)	1.40 „	9.17 „	10.57 „	13.23 „

	Wt. cereb.	Wt. cerebr.	Wt. together	% cereb.
Simiae.				
Callitrichidae.				
Callithrix pygmaea	4.66 gr.	37.61 gr.	42.27 gr.	11.— %
Hapale jacchus	.25 „	2.6 „	2.85 „	8.75 „
Oedipomidas oedipus / Oedipomidas oedipus (average of two hemisections)	.8 „	6.85 „	7.65 „	10.375„
Cebidae.				
Mycetes laniger	.55 „	6.03 „	6.58 „	8.36 „
Chrysothrix sciureus	3.76 „	28.30 „	32.06 „	11.70 „
Nyctipithecus trivirgatus	.38 „	3.06 „	3.44 „	11.06 „
Ateles ater (hemisect.)	4.15 „	34.— „	38.15 „	10.86 „
Lagothrix lagotricha	9.7 „	76.58 „	86.28 „	11.22 „
Cebus hypoleucus (hemisect.)	2.32 „	23.2 „	25.52 „	9.09 } 10.88
Cebus hypoleucus (hemisect.)	2.6 „	17.92 „	20.52 „	12.68 }
Cebus fatuellus / Cebus fatuellus (average of two hemisections)	2.78 „	25.10 „	27.88 „	9.95 „
Lasiopygidae.				
Cynocephalus hamadryas	10.15 „	84.50 „	94.65 „	10.70 „
Cynocephalus porcarius (hemisection)	7.8 „	67.88 „	75.68 „	10.30 „
Macacus rhesus	7.04 „	70.6 „	77.64 „	9.05 „
Macacus rhesus	6.07 „	65.3 „	71.37 „	8.65 „
Cercocebus fuliginosus	1.94 „	16.3 „	18.24 „	10.62 „
Mona mona (hemisection)	2.39 „	24.8 „	27.19 „	8.80 „
Inuus inuus	7.03 „	72.53 „	79.56 „	8.80 „
Cercopithecus callitrichus	5.42 „	50.26 „	55.68 „	9.70 „
Erythrocebus patas	6.93 „	64.76 „	71.69 „	9.66 „
Semnopithecus entellus (hemisection)	4.17 „	38.61 „	42.78 „	9.75 „
Anthropoidae.				
Simia satyrus	30.8 „	191.4 „	222.2 „	13.86 „
Simia satyrus (hemisection)	14.11 „	92.32 „	106.43 „	13.25 „
Troglodytes niger (hemisect.)	18.1 „	102.5 „	120.6 „	15.— } 14.05
Troglodytes niger (hemisect, alcohol)	17.27 „	114.19 „	131.46 „	13.11 }

Discussion.

The establishment of the percentage variations in individuals is a necessary prelimenary to any interpretation of figures involving the larger groups. Unfortunately this collection contains few duplicates. Where more than one individual of a species is present, the maximum difference between the percentage weights is 3.59 as may be seen in the following table.

III. Percentage variation between different individuals of the same species.

Species	Individual	Percentage	Differences in percentage
Ovis tragelaphus	I	10.10	0.23
	II	9.87	
Macacus rhesus	I	9.05	0.40
	II	8.65	
Simia satyrus	I	13.86	0.61
	II	13.25	
Mustela erminea	I	11.66	1.01
	II	10.65	
Felis leo	I	10.52	1.24
	II	9.28	
Phoca vitulina	I	15.4	1.30
	II	14.1	
Dasyprocta aguti	I	13.12	1.47
	II	11.65	
Felis concolor	I	13.05	1.69
	II	11.36	
Rusa hippelaphus	I	10.60	0.20
	II	10.40	
Troglodytes niger	I	15.—	1.89
	II	13.11	
Choloepus didactylus	I	15.90	2.06
	II	15.55	
	III	13.84	
Lutra vulgaris	I	10.69	2.31
	II	8.38	
Zibethailurus pardalis	I	13.56	3.26
	II	10.30	
Cebus hypoleucus	I	12.68	3.49
	II	9.08	

In connection with this percentage variation among individuals, it is profitable to consider the figures reported by ARIËNS KAPPERS (1926), who also found a considerable range of variation amongst man.

The percentages of cerebellar to total brainweight in the brains

of 25 Dutchmen were	8 % to 12.6 %	a range of 4.6 ;
of 22 Chinese	8.61 % to 12.22 %	a range of 3.61 ;
of 8 Japanese	9.51 % to 11.25 %	a range of 1.74.

It is very difficult to tell the cause of these variations and highly improbable that the cause in each case is the same.

For man WEISBACH (1867), who made a similar observation, believed the heavier specimens to have the greater cerebellar percentage.

The question wether body size has any influence on the cerebellar percentage of animals can be best controlled by comparing smaller and larger, though both adult representatives of the same species as enumerated in table III.

Of the animals mentioned there only of the two Mustela erminea, the two Zibethailurus pardalis and the two Felis concolor and two Simia satyrus, the largest specimens (according to the total brain weight) had larger cerebella.

On the other hand, however, of the two Dasyproctae, three Choloepus, two Ovis tragelaphus, two Lutrae, two Phocae and two Phocaenae, the specimen with the greatest (total brain) weight had a smaller percentage of cerebellum. From this no evidence can be obtained in favor of a constant influence of the bodysize (or total brainweight) in the percentage of the cerebellum.

Also KAPPERS could not confirm WEISBACH's opinion — that a larger weight should be constantly correlated with a larger cerebellum, although this occassionally occurs.

I have also made a comparison between the different representatives of the same order, suborder or genus wherever more than two specimens were available, just as I did in the cases of species. The advantage of this comparison is moreover that the differences in size are greater and more constant though certainly also other factors come in here (vide infra).

Here also it is evident that WEISBACH's thesis does not hold good as in the majority of cases the smaller genus has a higher percentage. So in the rodent suborder of Sciuridae the largest of all, Cynomus ludovicianus, has a cerebellar percentage of 13.10 %, whereas the average of the smaller Pteromys, Heterosciurus and Echinosciurus is more than 15.40 %. Amongst Antilopes [1]) the large Oreas Livingstoni has only 8.65 %, while all the others have about 10 % or more, the small Antilope cervicapra even 10.75 %, the highest percentage amongst this suborder.

The same is observed comparing the Camel (12.05 %) with the smaller Lama (13.60 %). The Giraffa, still larger than the Camel, has only 10.90 %.

Amongst the Ovidae the smaller Capra hircus has a higher percentage than the larger Ovis tragelaphus. In the suborder of the Perissodactyla the smaller Tapir has a higher percentage than Equus caballus and asinus.

[1]) Amongst the Cervidae the percentage varies so little that this suborder seems to be less fit for comparison. Its result would not be in favor of any rule in this respect.

Amongst the carnivorous suborder of the Ursidae we find that the largest representative Ursus maritimus has less cerebellum than both others, although in this whole suborder the percentage is very high (vide infra).

In the Mustelidae[1]) and Canidae no constant rule can be observed. It is however striking that of Canis lupus, Vulpes lupus and Vulpus lupus opus the latter, the smallest, again has the highest percentage. The same holds good if we compare the average figure of the two Felis leo, with the average figure of the two Felis pardalis and the average figure of the two Felis concolor, the average cerebellar percentage of the lion, which is the greatest animal of this family, being the smallest.

Amongst Prosimiae the smaller Lemurs have a slightly higher percentage than Chiromys.

So we see that if there is any rule, it is certainly not in favor of WEISBACH's conception but more likely in favor of the higher percentage in the smaller representants.

That, however, also this is not constant appears amongst others from the figure of Myrmecophaga compared to Choloepus and the figure of Cercocebus fuliginosus compared to those of Cynocephalus hamadryas and Anthropoids. From this results that other factors, than the size of the body excercize a considerable influence on this figure.

Among these factors are the different cephalization coefficient of different animals and some physiological differences that cannot be expressed in matter of cephalization.

Considering the percentages of cerebellar weight, we have to realize that this percentage may change as well by variation in the forebrain development as by variation in the weight of the cerebellum itself.

Variation in forebrain development will chiefly occur between orders and suborders where the cephalization index is very different, as this cephalization index largely depends on the forebrain, since this represents the greater mass of the encephalon.

So it may be explained that the average cerebellar % in man is only 10.5 % (KAPPERS) while in Anthropoids it is 13.72 %.

The question however arises if greater cephalization *necessarily* diminishes the cerebellar percentage. If for this we consider the different orders it appears that although in each order there are considerable variations (see below) some of the *highly* cephalized animals are conspicuous by their *large* cerebellar percentage.

Among these are the Proboscidae, Pinnipedii, Odontoceti and Edentata, compared to their next relatives. In the Proboscidae, Pinnipedii and Odontoceti this fact is the more striking as their cephalization also is very considerable (according to DUBOIS) compared to their next relatives.

We may conclude from this that in these animals the motor synergia has acquired an extraordinary precision. So in the Elephant the ability for complicated movements may be related to the large cerebellar per-

[1]) For the Viverridae see below.

centage. This animal possesses very precise independent monolateral movements of its extremities and a very fine adjustment of its trumpet.

In the Pinnepedii and Cetacea it is chiefly the swimming movement that involves a great cerebellar capacity. The agility of sealions is well known as also their equilibric acrobatics outside the water, often shown to the public. In Dolphins, who easily swim around a quickly moving steamer, the motile capacities are equally striking.

Still in both animals the cerebellar organization is very different. In the Pinnipedii, who greatly use their forelegs, the hemispheres of the cerebellum (the center of independent movements of the legs, BOLK) are increased. In the Cetacea, who have no extremities and where the strong tail is the sole moving agent, the pars floccularis is enlarged and the paraflocculus is enormous (BOLK) on account of its pontine connections (R. B. WILSON) and is chiefly responsable for the great size of the cerebellum. Now it is striking that in the whale, Balaenoptera sulfurea KAPPERS found a still higher percentage of cerebellum (18.95 %) than I did in Phocaena (15 %). Still the motile capacities of Balaenoptera are not nearly the same as those of Phocaena, as they hardly can follow a fast steamer. So I am inclined to believe that the higher cerebellar index in Balaenoptera is influenced by its cephalization index (which according to DUBOIS is only $^2/_3$ of that of the Odontoceti), as it is very probable that this smaller cephalization is largely due to the comparatively smaller forebrain in Balaeonoptera, which is also more dolichocephalous than the forebrain of Odontoceti, whose greater brachencephaly, according to KAPPERS (1927), is also a result of greater forebrain development.

As far as concerns the high cerebellar index in Edentates this fact may be due to the special character of their movements, which though being extremely slow, are highly complicated and require much independency of each extremity.

The accuracy of movements, even if slow, is of importance here as this involves a great deal of inhibition and synergia, which, as we know, are located in the cerebellum (TILNEY and RILEY). That Myrmecophaga has a still higher percentage than the other Edentates examined may be explaned by the fact that in addition to its extremities it has a very long snout, which is used as a sort of trumpet, for gathering food, as does the Elephant, and certainly has great proprioceptive capacities.

The influence of the character of the motitility of an animal is also seen in the Cervidae, which though being quick, have a great simplicity of gait, their extremities acting in a rather monotonous rythmic collaboration (alternation) of both sides. They have the lowest average of cerebellar percentage.

Among the order of Carnivora, there is a striking and fairly constant difference between the various suborders, the cerebellar percentage being the largest in the Ursidae, next come the Viverridae, then the Felidae, followed by the Mustelidae and finally the Canidae.

It is interesting however, that of the Carnivora the Ursidae also have the highest cephalization index, which might make us expect (cf. p. 165) that their cerebellar index should be smaller, as in the comparison of Odontoceti and Balaenoptera the cerebellar percentage is smaller in Odontoceti.

It is however well known that the capacities in finer adjustment of independent movements of the limbs and in conformity the proprioceptive instrument are very highly developed in bears. As they also have the greatest cephalization (according to DUBOIS Ursus malayanus holds a position amongst Carnivora as the anthropoids do amongst monkeys), we have here a case similar to that of the Elephant. As in the latter it is not improbable that in the cephalization of Ursus these proprioceptive functions, which are projected both on the cerebrum and on the cerebellum, but which are preponderant for the cerebellum, act the largest part.

On the other hand the relatively high cerebellar percentage of Viverridae is more likely due to the smaller development of their forebrain in comparison to other carnivora, the Viverridae having the lowest cephalization among Carnivora (DUBOIS).

This would also explain their relation to the Mustelidae, which as a rule are more cephalized [1]).

A very interesting phenomenon is offered by comparing Canidae and Felidae, two genus in which the coefficient of cephalization is practically the same.

Still we see that the Felidae have a higher average figure for the cerebellar percentage as dogs have. In none of the dogs the % mounts higher than 10¾ % whilst in the Felidae the average is far above this figure, rising even to 13.70 %.

This difference no doubt should be explained by the difference in motile abilities in both genus, those of the Felidae being doubtless much more developed than in dogs, specially as far as concerns finer adjustment of independent (unilateral) movements of each of the forelegs.

From the view point attained in this report it is an interesting fact that, just as the Elephants and the sealions (Pinnipedii), also the Ursidae and Felidae belong to those animals to whom tricks of equilibration can be best taught.

In the *Primates* the Prosimiae have a greater cerebellar % than the real monkeys. As their forebrain (which does not entirely cover the cerebellum, as it does in most monkeys) is relatively small, this might explain the higher cerebellar %, rather than a difference in motile abilities, that are great in both.

On the other hand the highest cerebellar percentage amongst all Primates is found in the Anthropoids, who at the same time are more cephalized, so that as in the Elephant, Ursidae, Felidae and Pinnipedii, this high cerebellar

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[1]) The smaller % in the latter does not necessarily include a smaller development of the cerebellum in relation to the body.

percentage can only be explained by their special cerebellar proprioceptive capacities.

From what is said above it appears that just as the Ursidae, Felidae and the Elephant and Odontoceti, so the Anthropoids, more than other monkeys should be considered as special cerebellar animals, — since in spite of their higher cephalization their cerebellar percentage is greater than that of their next relatives. On the other hand the fact that men have a cerebellar index smaller than Anthropoids should be ascribed by the greater development of the forebrain.

Summary.

1⁰. A comparison of the proportion between the weight of the cerebellum and the total brainweight in a series of mammals shows no constant correlation between the size of the animal and the proportionate weight of the cerebellum.

2⁰. Factors such as cephalization coefficient and capacities of adjustment of the extremeties, tail or trumpet have the greatest influence in the relative proportions.

3⁰. Animals, naturally endowed with *special* motile capacities, including those used for special motile tricks such as the elephant, sealions, cats, bears and anthropoids have the greatest cerebellar percentage.

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Chemistry. — *Osmosis of ternary liquids. Experimental part,* II. By F. A. H. SCHREINEMAKERS and B. C. VAN BALEN WALTER.

(Communicated at the meeting of February 25, 1928).

In the preceeding [1]) communication (Exp. I) we have discussed the apparent osmosis of the systems I—V consisting of $NaCl + Na_2CO_3 + H_2O$; now we shall discuss the apparent osmosis of systems, consisting of $Na_2S_2O_6 + BaS_2O_6 + H_2O$; we shall call these the systems VI—XII.

The left side liquid L_1 of these systems

$$L_1 \mid L'_1 \quad \ldots \ldots \ldots \ldots \quad (1)$$

only contains $H_2O + BaS_2O_6$ (dithionate of barium), the right side liquid L'_1 only $H_2O + Na_2S_2O_6$ (dithionate of natrium). If in figs. 1 and 2 we draw the amount of $Na_2S_2O_6$ on the X-axis and on the Y-axis the amount of BaS_2O_6 of these liquids, then L_1 is represented, therefore, by a point 1 on the Y-axis and L'_1 by a point 1′ on the X-axis.

It appears from the tables VI and VII that the paths VI and VII (fig. 1) have approximately the same points 1 and 1′; the same obtains

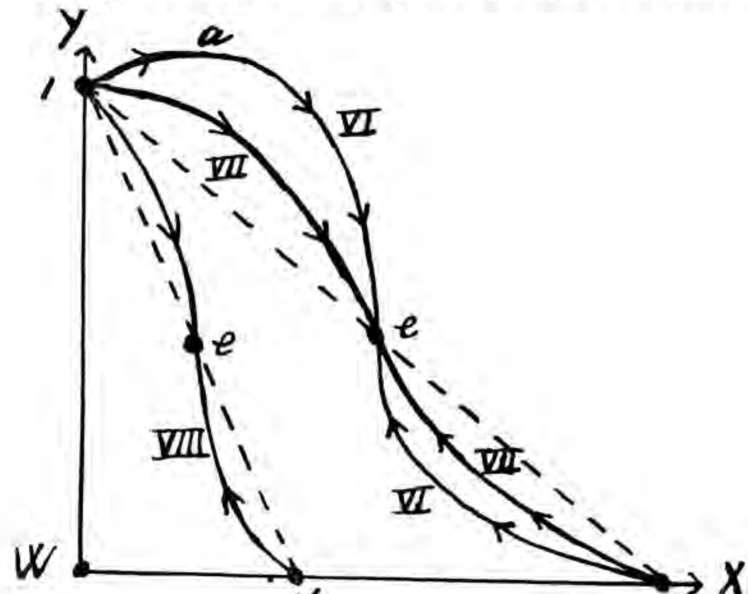


Fig. 1.

for the paths VIII—X, of which only path VIII has been drawn in

[1]) The communications with the "General consideration" are cited as: Gen. I, Gen. II etc.; those with the "Experimental part" as: Exp. I etc.

fig. 1; their points 1 also coincide approximately with those of the other paths in fig. 1; the points 1 and 1′ of the paths XI and XII in fig. 2 also coincide approximately.

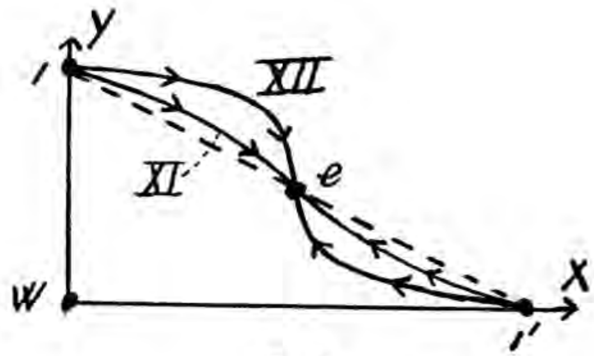


Fig. 2.

We shall first consider the X ($Na_2S_2O_6$)-amount of these systems. It appears from the tables and also from the form of the paths in fig. 1 and 2 that during the whole osmosis in each system the X-amount is smaller on the left side of the membrane than on the right side, that it increases continuously on the left and decreases continuously on the right. For all these systems VI—XII, therefore, obtains the symbol:

$$\uparrow < \downarrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Consequently during the whole osmosis the X-amount changes normally—normally.

So, the $Na_2S_2O_6$ in these systems behaves in the same way as the $NaCl$ in the systems I—V (Exp. I). This also appears when we draw the $X.t$-diagrams of these systems VI—XII with the aid of the tables; then we see that they can be represented by fig. 2 (Gen. II), which also schematically represents the Xt-diagrams of the systems I—V.

We now consider the Y (BaS_2O_6)-amount of the systems VII—XII; that of system VI we shall leave out of consideration for the present. From the tables and also from the figs. 1 and 2 it appears that the change can be represented by:

$$\downarrow > \uparrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The Y-amount of the systems VII—XII changes, therefore, normally—normally during the whole osmosis.

If we draw the Yt-diagrams of these systems with the aid of the tables, then we see that they can be represented by fig. 2 (Exp. I), which also schematically represent the $Y:t$-diagrams of the systems I—V.

Otherwise it is, however, with the Y-amount of system VI. It appears namely from table VI that this amount increases in the left side liquid, starting from 1, becomes a maximum in the determination noted with a +, and decreases afterwards; the Y-amount of the right side liquid, however, continuously increases. Consequently branch 1 . e of path VI consists of an ascending part 1 . a and a descending part a.e.; in point a the Y-amount is a maximum.

For part 1 . a (and 1′. a') of the path obtains, therefore:

$$*\uparrow > \uparrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and for the further part a . e (and a'. e):

$$\downarrow > \uparrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

So we find: the $Y(BaS_2O_6)$-amount of system VI changes:

on part 1 . a: anormally—normally
" " a . e: normally—normally

In point a between the symbols (4) and (5) occurs the transition:

$$| > \uparrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This expresses that during an infinitely small time dt the Y-amount of the left side liquid remains constant, while that of the right side liquid increases.

In scheme VI this transition-form has not been given neither in any of the other schemes, it appears from the table that it must be situated on part 3 . 4 of the path.

If, with the aid of table VI we draw the Y . t-diagram of this system, then we see that this can be represented schematically by fig. 3 (Gen. II); (of course the letter m and the ciphers 2, 3, etc. of this figure do not relate to the determinations in table VI).

In the systems I—V (Exp. I) and, as we shall see further, also in the systems VI, VII and XII, anormal changes of the W-amount occur; in all these systems I—XII, however, we only find one single example of anormal change af the concentration of an other substance; this is the change of the BaS_2O_6-amount in system VI. Later on we shall discuss systems, in which the concentration of $NaCl$, NH_4Cl and other substances changes anormally.

In order to discuss the W-amount of the systems, we divide them into three groups; first we take group VIII—X, of which only path VIII has been drawn.

If we deduce the W-amount of the liquids from the tables VIII—X, then it becomes apparent that: during the whole osmosis the W-amount on the left side of the membrane is smaller than on the right side; it increases continuously on the left side of the membrane and it decreases on the right.

This also appears from path VIII in fig. 1. If we draw an imaginary line through e parallel to the side XY, then we see that branch 1 . e

runs entirely above this line and branch 1'. e entirely below this line; we also see then that liquid 1 has a smaller W-amount and 1'a a larger W-amount than the final-liquid e; further it appears from the form of the path that there are no tangents, parallel to the side XY, so that neither a minimum-nor a maximum W-amount occurs.

So for these systems obtains what we also find given in the schemes VIII—X sub W viz. the symbol:

$$\uparrow < \downarrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Consequently during the whole osmosis the W-amount of the systems VIII—X changes normally—normally.

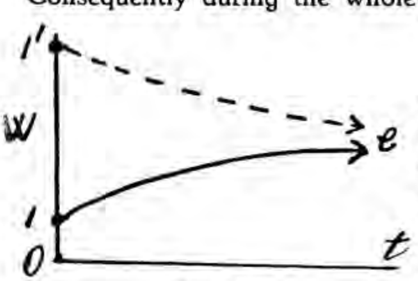


Fig. 3.

If we draw the W. t-diagrams of these systems, we see that they can be represented schematically by fig. 3. Consequently we have already found three different types of Wt-diagrams, viz. fig. 4 (Gen. II), fig. 3 (Exp. I) and fig. 3 in this communication.

We now take system XI; it appears from the path in fig. 2 or from table XI that for the change of the W-amount obtains the symbol:

$$\downarrow > \uparrow \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

So the W-amount of this system changes during the whole osmosis normally—normally.

We may also represent the Wt-diagram of this system schematically by fig. 3; then, however, we must interchange the points 1 and 1', dott the fully-drawn curve and draw the dotted curve in full.

It appears from figs. 1 and 2 that the paths VI, VII and XII have forms corresponding with path V in fig. 1 (Exp. I) and the paths II—IV which have not been drawn here.

If namely through point e a line is drawn parallel to the side XY, this will intersect the path in two points; these points represent the two liquids q and q' with the same W-amount. We can also draw a tangent, parallel to XY, in a point m of branch 1 . e and in a point M' of branch 1'. e; consequently the W-amount of the left side liquid is a minimum in m and that of the right side liquid is a maximum in M'.

If from the tables we deduce the W-amount of the liquids, then we are able to find the position of these points q, m and M'.

We now find that the W-change during the whole osmosis can be represented successively by the symbols:

$$\downarrow > \uparrow \qquad {}^*\!\downarrow < \uparrow^* \qquad \uparrow < \downarrow \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

We find them indicated sub W in the schemes VI, VII and XII. So we find:

the W-amount of the systems VI, VII and XII changes

a. in the beginning of the osmosis: normally—normally;

b. afterwards: anormally—anormally;

c. at last: normally—normally.

Between the first and the second symbol we have the transition;

$$\cdot\!\downarrow\; = \;\uparrow\!\cdot \qquad (10)$$

this occurs in the points q and q'. Between the second and the third symbol one of the transitions (6) or (7) (Exp. I) occurs. These transitions have not been given in the schemes.

Again the $W.t$-diagrams of these systems can be represented schematically by fig. 4 (Gen. II).

It appears from schemes I—XII:

the X-amount of all systems changes normally;

the Y-amount changes normally in all systems, except in VI where it also changes anormally;

the W-amount changes normally only in the systems VIII—XI, in the others it also changes anormally.

The paths XI and XII (fig. 2) have approximately the same points 1 and 1'. In XI a membrane of collodion, was used, in XII the membrane consisted of collodion in which a deposit of $Cu_2Fe(CN)_6$. Fig. 2 shows very clearly that both systems consequently travel along very different paths; we also see from the schemes that the W-amount in XI changes only normally, whereas in XII this occurs also anormally.

It also appears from the tables that the liquids in XI change their compositions much more quickly than in XII; we see e.g. that the X-amount of the left side liquid

of XI increases in 99 hours already with 4.243 %
of XII „ in 718 „ only „ 4.131 %

The Y-amount of the left side liquid decreases:

in XI in 144 hours already with 4.663—2.651 = 2.012 %
in XII in 1679 „ only „ 4.750—3.498 = 1.252 %.

The paths VI and VII (fig. 4) have also approximately the same points 1 and 1'. In VI a membrane of collodion was used, in which a deposit of $Cu_2Fe(CN)_6$, in VII the membrane consisted of collodion only. For this reason both systems travel along different paths; besides we see that the Y-amount in VI changes not only normally, but also anormally, while in VII it changes normally only.

It also appears from the tables that the liquids of VII change their compositions much more quickly than those of VI; we see e.g. that the X-amount of the left side liquid increases:

in VI in 1483 hours with 4.520 %
in VII in 200 „ already with 5.093 %

The paths VIII—X, of which in fig. 1 only VIII has been drawn, also have approximately the same points 1 and 1′, but differ otherwise. The membrane of VIII consisted of parchment, that of IX of collodion and that of X of collodion in which a deposit of $Cu_2Fe(CN)_6$. It appears from the tables that the liquids of IX (collodion) change their compositions quickest, those of X (collodion + deposit) are slowest. We see e.g. that the X-amount of the left side liquid increases:

in IX in 92 hours already with 2.176 %
in VIII in 194 „ only „ 1.988 %
in X in 2084 „ still only „ 1.634 %

In order to get a clearer view of the difference in rapidity with which the liquids of these and previous systems change their concentrations, we may use their Xt-, Yt- and Wt-diagrams, which the reader can draw with the aid of the tables.

TABLE VI. Collodion + deposit $Cu_2Fe(CN)_6$. Procents of weight.
$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	9.938	10.50	0					
2	43	0.383	9.970	9.743	0.268	11.1	6.8	82.1	IV	→
3	91	0.773	10.06 +	9.053	0.541	7.8	6.6	85.6	„	„
4	163	1.318	10.05	8.271	0.902	9.7	8.3	82.0	„	„
5	259	1.923	9.924	7.565	1.291	11.1	10.3	78.6	„	„
6	380	2.610	9.703	6.942	1.681	11.5	11.8	76.7	„	„
7	499	3.181	9.337	6.506	2.020	19.4	19.4	61.3	„	„
8	667	3.681	8.837	6.135	2.400	38.1	41.1	20.9	„	„
9	979	4.165	8.013	5.866	2.850	23.6	36.0	40.4	VI	←
10	1483	4.520	7.283	5.806	3.140					

SCHEME VI.

	X	Y	W
1.2	↑<↓ ←	*↑>↑ →	↓>↑ →
2.3	"	"	*↓<↑* →*
3.7	"	↓>↑ →	"
7.8	"	"	↑<↓ →*
8.9	"	"	↑<↓ ←
9.10	↑<↓	↓>↑	↑<↓

TABLE VII. Collodion. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	9.721	10.59	0					
2	24	1.018	8.954	9.523	0.792	43.9	29.0	27.1	VI	←
3	49	2.390	7.877	8.121	1.832	21.0	8.7	70.3	"	"
4	74	3.123	7.283	7.496	2.404	4.7	12.7	82.5	IV	→
5	101	3.461	6.982	7.086	2.755	25.0	15.3	59.7	VI	←
6	147	4.530	5.941	5.928	3.814	16.3	6.9	76.8	"	"
7	171	4.863	5.560	5.697	4.142	2.7	8.0	89.3	I	→
8	200	5.093	5.282	5.429	4.468	3.4	7.4	89.2	"	"

SCHEME VII.

	X	Y	W
1—3	↑<↓ ←	↓>↑ →	↓>↑ ←*
3—4	"	"	*↓<↑* →*
4—6	"	"	*↓<↑* ←
0 6—8	↑<↓ →*	"	↑<↓ →*

TABLE VIII. Parchment. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	9.933	4.484	0					
2	50	0.3744	9.397	4.132	0.4414	16.2	16.5	67.3	VI	←
3	121	0.8093	8.814	3.660	1.057	29.5	36.9	33.6	,,	,,
4	221	1.265	8.100	3.204	1.786	32.9	54.9	12.2	IV	→
5	360	1.678	7.247	2.739	2.641	10	12.7	77.3	VI	←
6	554	1.988	6.404	2.423	3.500	18.6	49.4	32.0	,,	,,

SCHEME VIII.

	X	Y	W
1—3	↑<↓ (←)	↓>↑ (→)	↑<↓ (←)
3—4	,,	,,	↑<↓ (→) *
4—6	,,	,,	↑<↓ (←)

TABLE IX. Collodion. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	9.745	4.485	0					
2	24	0.779	8.395	3.653	1.370	15.5	21.3	63.2	VI	←
3	50	1.570	6.880	2.877	2.957	6.9	21.7	71.4	IV	→
4	92	2.176	5.041	2.184	4.968	2.5	3.9	93.6	I	←

SCHEME IX.

		X	Y	W
	1—2	↑<↓ (←)	↓>↑ (→)	↑<↓ (←)
	2—3	,,	,,	↑<↓ (→) *
0	3—4	,,	↓>↑ (←) *	↑<↓ (←)

TABLE X. Collodion + deposit $Cu_2Fe(CN)_6$. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	*X*	*Y*	*X*	*Y*	*X*	*Y*	*W*		
1	0	0	9.982	4.535	0					
2	96	0.5429	9.380	4.018	0.5446	26.6	25.7	47.7	VI	←
3	144	0.9557	8.816	3.552	1.095	20.9	21.8	57.3	"	"
4	264	1.196	8.482	3.329	1.411	37.4	54.6	8.0	IV	→
5	601	1.411	8.079	3.124	1.750	15.1	12.0	72.9	VI	←
6	983	1.634	7.633	2.927	2.147	27.9	57.4	14.7	III	→

SCHEME X.

	X	*Y*	*W*
1—3	↑<↓ ←	↓>↑ →	↑<↓ ←
3—4	"	"	↑<↓ → *
4—6	"	"	↑<↓ ←

TABLE XI. Collodion. Procents of weight.

$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	*X*	*Y*	*X*	*Y*	*X*	*Y*	*W*		
1	0	0	4.663	10.57	0					
2	23	1.489	4.083	9.194	0.5199	42.7	13.2	44.1	VI	←
3	49	2.708	3.593	8.089	0.9475	32.6	9.3	58.1	"	"
4	71	3.617	3.165	7.362	1.257	10.5	0.1	89.4	"	"
5	99	4.243	2.925	6.714	1.539	1.7	5.1	93.2	IV	→
6	144	4.856	2.651	6.117	1.829	0.5	5.0	94.5	"	"

SCHEME XI.

	X	*Y*	*W*
1—4	↑<↓ ←	↓>↑ →	↓>↑ ← *
4—6	"	"	↓>↑ →

TABLE XII. Collodion + deposit $Cu_2Fe(CN)_6$. Procents of weight.
$X = Na_2S_2O_6$ $Y = BaS_2O_6$ $W = H_2O$

	t	X	Y	X	Y	X	Y	W		
1	0	0	4.750	10.49	0					
2	46	0.3769	4.657	9.642	0.2642	15.7	7.1	77.2	IV	→
3	94	0.8459	4.569	8.898	0.5223	6.2	5.3	88.5	„	„
4	214	1.817	4.443	7.607	0.9353	6.0	4.9	89.1	„	„
5	358	2.795	4.365	6.745	1.209	4.7	4.5	90.8	„	„
6	526	3.652	4.263	6.205	1.384	5.6	4.8	89.5	„	„
7	718	4.131	4.250	5.992	1.445	4.2	4.1	91.7	„	„
8	1006	4.639	4.080	5.839	1.521	6.5	7.1	86.4	„	„
9	1679	5.133	3.498	5.723	1.668	6.2	15.7	78.1	„	„

SCHEME XII.

	X	Y	W
1—6	$\underset{\longleftarrow}{\uparrow < \downarrow}$	$\underset{\longrightarrow}{\downarrow > \uparrow}$	$\underset{\longrightarrow}{\downarrow > \uparrow}$
6—8	„	„	$\underset{\longrightarrow *}{* \downarrow < \uparrow *}$
8—9	„	„	$\underset{\longrightarrow *}{\uparrow < \downarrow}$

(To be continued).

Leiden, Lab. of Inorg. Chemistry.

Geology. — *On a new Basis of Solution of the Caldera-Problem and some associated Phenomena.* By C. G. S. SANDBERG D.Sc. (Communicated by Prof. Dr. G. A. F. MOLENGRAAFF.)

(Communicated at the meeting of November 26, 1927).

The various attempts at solving the Caldera-problem are governed, generally, by the principle of seeking an explanation:

a. of the mode of causation of a large volcanic rim enclosing, partly or entirely, a more or less flat bottom (the caldera), the diameter of which is so disproportionately large in comparison with that of the presumed magmatic conduit, that the said enclosing rim cannot reasonably be admitted to represent the primary product of eruption of the said narrow conduit; and

b. of the phenomenon of the revival of volcanic activity at or near the places of former action, after a longer or shorter period of rest.

It is the intention of the writer to indicate here only a plausible solution for the problem formulated sub a., whilst accepting for the present as an empirically well established fact that mentioned sub *b.*

When considering the various studies of the caldera-problem, it appears, as far as I know, that they are governed by the following presumptions, viz.: that a volcanic cone, $b + c$, is (assumed to have been) built up round and above an eruption-centre *a*. which is, more or less arbitrarily, taken as the most likely one, and which is situated at the top of a volcanic conduit *p* (Fig. 1). Subsequently, part of the cone which would have been formed thus, i.e. the part marked *b*., is supposed to have been destroyed, leaving

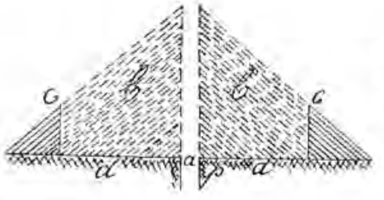


Fig. 1. Schematical section of caldera according to current conceptions. $a =$ centre of eruption; $p =$ Magmatic conduit or volcanic pipe; $d =$ caldera bottom; $b =$ vanished part of presumed original volcano; $c =$ rest of presumed original volcano, the caldera rim.

only a (caldera) rim c and a *grosso modo* flat bottom *d*. In other words, the problem as it was and is posed is based on the presumption that the eruption(s) from a magmatic conduit *p* with an eruption-centre *a* caused a volcanic cone $b + c$, and its solution was and is consequently governed by the search for a plausible explanation of the disappearance of the part *b* and the causal formation of the vertical enclosing wall and the *grosso modo* flat bottom *d*, the diameter of which so abnormally exceeds that of the presumed eruption-centre *a*. (V. HOCHSTETTER, STÜBEL, VERBEEK, DALY, DUTTON, CHAMBERLIN, WING EASTON, ESCHER a.o.).

R. A. DALY (1) expresses the current conception in the following words: "If the actually exposed "necks" of the world indicate the maximum size of central conduits, the vents beneath calderas must have cross-sections much smaller in area than the floor of the corresponding great depressions. The writer is in fact, inclined to make this the criterion for explosion craters from calderas."

The directions in which the solution of the problem have been sought may consequently be classified in the following catagories :

1. *The explosion theory*. The part *b* of the cone is assumed to have been blown away subsequent to an extraordinarily heavy explosion which is supposed to have emanated from the eruptive centre *a*, or from a point lower down the conduit *p*, part of the débris falling back into and so partly filling up the large opening thus caused, and so giving rise to a more or less flat floor.

2. *The subsidence theory*. Part *b* of the cone would have subsided along vertical or semi-vertical peripherical fault-planes, leaving only the rim *c*, subsequent to having been undermined below or above its base (somewhere near *a*) all round the conduit *p* or/and the conduits' upper extension, which we shall henceforth name the *cone-pipe* ;

3. *The re-fusion-backflow theory*. Some time after the erection of the cone $b + c$, by ejectamenta from *a*, a new eruption of incandescent gas-saturated magma would have liquefied the part *b*. Subsequently to a following heavy paroxysm, part of the re-fused cone would have been blown away, the rest flowing back to deeper regions through the conduit *p* and leaving only a rim *c* round an enlarged, *grosso modo* flat "depression" *d*.

As to the *last mentioned theory*, it may suffice to refer to WING EASTON's refutation of it (2), to which may be added that e.g. the facts observed during the Vesuvius eruption (3) established that such partial re-fusion of the cone may actually occur. They were realized when the rising magma reached its highest level in the cone-pipe. Yet instead of flowing back into the conduit the incandescent magma discharged laterally, breaking through the sides of the cone. The causal formation of a, *grosso modo*, vertical inner wall, a feature which cannot be causally connected with such backflow or discharge, as WING EASTON has pointed out, did not occur.

Again, with regard to the *explosion theory* it may suffice, to avoid repetition, to refer to WING EASTON's refutation, which may be summarized in his words : "The causation of a vertical inner-wall (of a caldera) subsequent to an explosion is certainly *possible*, locally ; yet it is not at all clear why this should necessarily occur, and then nearly always practically along the entire inner-circumference." (My translation from (2) p. 71.)

B. G. ESCHER's recent experiments (4), which only remotely touch the caldera-problem, have not weakened this conclusion in the least; besides no structure comparable with that of a caldera with its characteristic features was realized in the course of these experiments.

Let us now consider the *subsidence theory*. A very sharp distinction should always be maintained between caving in, gradual crumbling or sudden collapse *(écroulement)* of the enclosing wall, the rim, with subsequent lateral *enlargement* of a pre-existent "depression" and the causation of such a large space through *a.* subsidence (falling in) or *b.* down-throw of a superstructure, presumed to have been pre-existent.

Although different in principle, these phenomena have not always been rigorously differentiated by various students of the caldera-problem. (H. RECK, R. A. DALY and others).

Now as to the conception of caldera-formation by gradual crumbling or caving in of the rim-wall, there seems to be little doubt that WING EASTON's conclusion will be generally endorsed viz.: It seems natural that even a normal crater floor (the diameter of which should be identical with that of the orifice of the conduit) may come to exceed its theoretical dimensions through downfalling, explosion or perhaps re-fusion, but that would not transform such a crater into a caldera (the diameter of the Idjen-caldera is 16 km.; that of the Ringgit 21 km.). Besides they are characterized by very steep, often vertical inner walls (vide also v. WOLFF (5) *S*) which may attain a height of 1000 m (Rindjani) (Raoeng, Gendeng-Idjen, Tenger over 500 m). Very frequent also is the occurrence of a flat bottom sometimes called "sandsea" (Tenger, Slamet, Raoeng) [1])."

On the other hand we cannot possibly endorse WING EASTON's remark immediately following, viz.: "By these characteristics they (the calderas) are distinguished from normal craters", as will be shown below.

Since, therefore, we may discard the adequacy of the contention that caldera *formation* (not *enlargement*) may be caused by gradual crumbling or caving in of an encompassing wall, then the sole current theory which still remains to be tested is that of such causation as a product of subsidence *(effondrement)*, or of down-throw.

We have already mentioned that the caldera-structure is characterized by, among other things, a very steep, generally vertical or semi-vertical wall when not modified, secondarily, by denudation. We have also given some examples, which could be multiplied *ad libitum* (Kilauea, Barren Island Monte Somma, Knebelcaldera, Batoer, Fogo-Island etc. etc.)

This characteristic is not restricted even to terrestrial calderas, it is inherent also in lunar calderas.

The evident planetary nature of the said characteristic justifies the conclusion that it is primary and that it constitutes a genetic feature of the caldera-structure.

If this conclusion holds good it would exclude the admissibility of any

[1]) The double sound oe in Dutch is pronounced like the English oo in "poor".

of the current attempts at solving the problem, which one and all attribute the caldera-occurrence to a secondary origin.

Apart from this general conclusion, however, we shall analyse the theories sub. *a*. and *b*., beginning with WING EASTON's.

It seems quite impossible to admit now that, before extruding vertically, the gas-channels which WING EASTON requires to produce his cells of undermining and which would have emanated, *grosso modo*, from an eruption-point *a* (see Fig. 1) would develop per se in lateral direction to such an extent as would be necessary for the production of calderas with dimensions such as are mentioned by WING EASTON himself (see ante p. 181). On the other hand it is also clear that this *impasse* in his theory cannot be overcome by transplanting the assumed point of emanation of these super-heated gas-channels to a proportionately deeper level of the magmatic conduit, as this would virtually amount to enlarging the diameter of the said conduit. In whatever way we attempt to conceive this process of honeycombing the basis of an (assumed) volcanic superstructure, this causation of cells of undermining, it must seem extremely improbable that subsequent subsidence of the (presumed) superstructure would *eo ipso* cause the remaining portion (the rim of the caldera) to be bordered by vertical walls which appear very strongly to constitute or to have constituted one continuous vertical plane. (Fogo Island St. Paul, Barren Isl., Monte Somma, etc. etc. See also p. 181. W. EASTON).

Finally it is extremely doubtful whether well established instances of such a mode of caldera-formation can be furnished.

Now, as to the assumption of caldera-formation by downthrow, it must be admitted that F. A. PERRET's studies of the Vesuvius eruptions (3) showed, among other things, that a sudden subsidence (of part of the foot of a débris-cone) may cause an opening, which might happen to be cylindrical, and to be bordered by a very steep conical wall (l.c. p. 117—118). Such a phenomenon, however, could occur only where a void was pre-existent below the subsided mass, of a size at least equal to that of the volume of the subsided mass. Now, as long as it cannot be shown that such a void must necessarily exist (or be formed in course of time) below a volcanic cone, and that the part concerned of the superstructure of the (presumed) orginal volcano must needs subside therein (sometimes perhaps by jerks *(par saccades)* and along concentric planes), so long will the calderaproblem remain unexplained on the basis of this theory, and its *quasi* solution simply means a substitution of the problem for other questionable presumptions regarding those deeper parts of our globe, of which still less is known to us with any degree of certainty.

Moreover we would emphasize that, in respect of the downthrow established by PERRET (l.c. p. 117—18), *the diameter of the incandescent magma-column then in course of ascension, in other words the diameter of the conduit, was at least equal to and most probably even larger than that of the produced conical subsidence with its semi-vertical inner wall.*

PERRET's observation cannot be invoked, therefore, in support of the subsidence theory, since the latter is based on the premise that the diameter of the magmatic conduit is *much smaller* than that of the subsided area.

On the mechanism of volcanic-cone building.

If now we may reject as untenable such conceptions of the origin of calderas as imply a secondary cause for their occurrence, then the question arises which part of a volcanic structure would be characterized, *genetically*, by those, *grosso modo*, vertical inner walls which typify among others the caldera occurrence. In order to solve this question we will examine the mode of formation of, say, a strato-volcanic-cone; identical considerations apply to other types of volcanic cones, such as lava-domes *(Schildvulkane)*, among others. For convenience sake we shall take the form of the crater, which will generally be identical with that of the exit of the conduit at the base of the cone, to be circular, (though Askja is rectangular; Tjiremai is oval: the Barren Island volcano is circular; etc.) See Fig. 2.

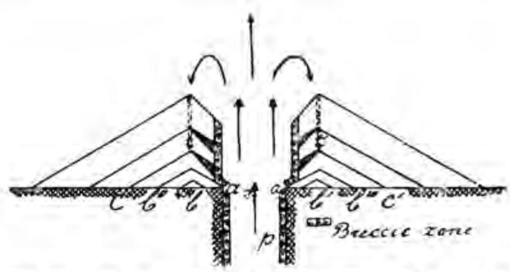


Fig. 2. Schematical vertical section of the mechanism of volcanic-cone formation. p = Volcanic-pipe (conduit); aa' = eruption-centre; bb' etc. = volcanic-cone.

We may readily conceive that a volcanic eruption emanating from a conduit p will cause the ejectamenta to accumulate round its opening aa' in the form of an encompassing cone, a rim, bb'. Under identical conditions for every part of the rim, the accumulating ejectamenta will roll or slide down under the influence of gravity and inter-friction until a state of equilibrium is reached, with its corresponding angle of inclination. Inner and outer-slope will thus be identical and in keeping with the nature of the ejected material. For strato-volcanoes this angle of slope is *grosso modo* from 30° to 40°, an inclination which is actually preserved along the outer slopes of strato-volcanoes.

In fact, supposing that the accumulation of ejectamenta and the subsequent development of the cone continues in height and corresponding base, it is clear that such may continue normally as indicated above, until its basis reaches the rim of the eruption-point a—a' (in section a—b'' and a'—b'''). Should the accumulation continue after this moment, then a new phase will have been inaugurated, as a normal readjustment of the ejected material rolling down towards the eruption channel *a—a'* will be hampered by the pressure of the outflowing, débris-charged gas-current; whilst that rolling down along the outer slope, *tc—t'c'*, of the cone will not be subjected to such a resistance. The former will consequently be dammed up. Hence the greater thickness of the accumulated débris-strata on the cone-pipe side, as LINCK had already demonstrated experimentally. Hence, again, the lesser inclination of the cone's apex towards the cone pipe as against the outer slope.

Considering now that a volcanic cone-pipe has been built up under the influence of the factors sketched above and that the gas-current extruding from the conduit, *p*, will be vertically directed, this direction being that of least resistance, it is clear why the encompassing wall of cone-pipes will always tend genetically towards the vertical. (Chaine des Puys, Auvergne, Vesuvius, Slamat etc.). This feature is, in other words, a causal effect of the mode of formation of volcanic cone-pipes. Whereas this feature is genetically inherent in no other part of a volcanic structure but its cone-pipe; and whereas the universality of its occurrence with calderas distinctly shows that this feature is a genetic quality of these structures also, the conclusion would seem justified that the vertical encompassing wall of calderas constitutes the wall (or its remnant) of the *then* volcanic pipe. In other words, *that the caldera-capacity represents in dimension and shape, the size and form of the magmatic conduits concerned and their respective extensions, the cone-pipes, of the then volcano or volcanoes.*

Before investigating whether this preliminary conclusion finds additional support from other equally well established phenomena, we will first study more closely this vertical encompassing wall of cone-pipes.

ESCHER's remarkable experiments (4) (Pl. 7) demonstrated the mode of development of such a channel in a homogeneous cover when pierced by a vertically directed gas-current. It is true that a volcanic cone does not represent the body of a pre-existent mass which covered the eruption-point and that strictly speaking, the cone-pipe is not as a rule pierced through a pre-existent cone; on the contrary the latter is built up all round the former. ESCHER's basis would consequently seem to be false. Yet in considering the mode of formation of volcanic cones as detailed above, it is clear that in its results there is no difference between the piercing of a pre-existent complex of strata by a gas-current and the keeping open of a channel by a débris-laden gas-current which would be covered by the ejected material but for the clearing action of the current. (The cone-pipe walls of lava-volcanoes *(Schildvulkane)* encompassed by more homogeneous material, yet formed in an identical way, are also vertical (5) p. 454 ff.).

If however the complex of strata through which such a gas-current is extruding should be heterogeneous, i.e. constituted of irregularly alternating more and less resistant strata (Kimberley pipes), then cavities may actually be hollowed out in the *grosso modo* vertical wall of an eruption channel (conduit or/and cone-pipe) by the greater eroding effect of the action of the current on strata of less resistance; which is clearly demonstrated in ESCHER's experiments (4) p. 70—71. Yet it should be noted that such deflections from the normal will only occur, theoretically, subsequent to much heavier explosions (gas-extrusions) than those which caused the erection of the cone-pipe. In every other case, in fact, the erosive effect of the ejectamenta-laden gas current, if any, may be neglected, as the material constituting the con-pipe arrived *in situ* at its state of equilibrium under the influence of gravity *and* that of the pressure of the gas-current. Now, as the former remains unchanged, it is clear that the state of equilibrium will remain unaffected also, unless the latter (gas-pressure) is considerably augmented (a considerable diminution of the gas-pressure may cause a local down-fall of material, as we shall see). Finally we would draw particular attention to a very remarkable observation made by PERRET (3) (p. 113) to the effect that a magma rising in a cone-pipe will tend to fill up any such deviations from the vertical in the wall of the cone-pipe by a process which PERRET describes as "plastic lining". Schematically we could represent the vertical section of a cone-pipe with its orifice, the crater, *during* a constant ejectamenta-charged eruption as given in Fig. 2. At the close of such an eruption the ejectamenta accumulated all round the crater will be in a state of labile equilibrium which may be disturbed by the slightest shock, an air-vibration even, as PERRET was able to establish. Thus the enormous quantities of latent energy accumulated in the masses surrounding the orifice, the crater, when rendered kinetic, will tend to establish a state of equilibrium, thereby sometimes causing huge avalanches to crash down to the bottom of the cone-pipe, and thus enlarging the *crater* and steepening its inwardly directed slope, to an angle greater than that corresponding with the angle of the slope of a normal débris-cone of the material converned, an angle which is preserved in that of the outer slope of a volcanic cone (3) (p. 99 ff and Fig. 63, p. 103).

Before closing our study of the eruption-channel, i.e. that of the magmatic conduit and its extension, the cone-pipe, we will point out that its mode of formation will tend to produce at least three principal zones of less resistance, to wit: one situated more or less centrally at or near the vertical axis of the eruption-channel; a second along the border of the said channel, which we will call the inner-peripherical zone; and a third more excentric still, which we will call the outer-peripherical zone. The existence of these zones is strikingly manifested in nature by the common occurrence of more recent points of eruption situated periphercally as well as centrally with respect to older channels (3) (p. 19, Fig. 3; (7) Kawah Ratoe of Tangkubang Prahoe; G. Tjiremai, a.o.). The phenomenon is also manifested by a marked tendency of primary fumaroles to occur centrally or else inner- or outer-peripherically and which is beautifully illustrated in the structure of the G. Pajang (Batoer-Complex), the remains of which disclose fine sections, vertical and horizontal. KEMMERLING established that within the, *grosso modo*, vertical cone-pipe, the solid central core is separated from its clastic mantle, the cone, by a zone of breccias enclosing the said solid core.

This remarkable phenomenon of the re-occurrence of younger eruption-points, centrally or/and peripherically situated in relation to an older one, which is as general an occurrence with volcanic cone-pipes as with calderas, strikingly discloses yet another close similarity between these two phenomena. A few examples, which may be multiplied ad libitum, of peripherically situated younger eruption-points of calderas are the Fogo Island, G. Batoer (with its G. Abang), the Knebel-caldera (Rudolf-crater and the S. E.-craters), the Piton de la Fournaise (Reunion) (9) (p. 263) with its peripherically arranged fumaroles; whilst more centrally situated younger eruption-points are exemplified in Fogo-Island (10) (p. 29, Fig. 32), Barren Island (11), G. Batoer (the active cone), Vesuvius, etc. etc.

Moreover we find both types abundantly represented among the lunar calderas.

On caldera-capacities considered as the original openings of older volcanic cone-pipes.

Having indicated the similarities between calderas and volcanic cone-pipes and having shown that at least one of the qualities they possess in common, that of the vertical inner wall, is a genetic feature of the latter and most probably also of the former, we will now examine:

1. How, if at all the genesis of calderas could be explained rationally on the basis of the assumption that the phenomenon differs from that of a volcanic cone-pipe in relative size alone;
2. Whether examples of caldera-formation in the manner assumed by us are known; and
3. Whether and how the actual remains of caldera-structures furnish a rational basis for reconstructing their history and mode of development.

Sub. 1. *Simple Calderas.* If the conception of the word *caldera* includes all those "depressions" of volcanic origin which are enclosed in toto or partly by a cone-shaped débris-mantle characterized by a steep, semi-vertical or vertical inner wall and a normal inclination of its outer slope, then the word would also include those volcanic "depressions" within which no younger eruption-point could be established. We shall call this kind of calderas *simple calderas*, the type of which is represented by, for instance, the Ngorongoro-caldera in East Africa, with a diameter of 17 to 22 km (personal communication of H. RECK), the Askja, of some 9 by 9.5 km [1]) and the Knebel-caldera (Island) of some 4.5 by 2.5 km in diameter.

Now, whereas the diameters of eruption-channels and their corresponding cone-pipes may vary from a few meters and even less (adventive craters, hornitos etc.) to 200 m (Vesuvius before 1906) (3), 400 m and 500 m

[1]) These dimensions, derived from his topographical map, do not seem to agree with H. RECK's estimates of its size, which he takes to be about 55 km² (p. 45).

(Vesuvius after 1906), 5.6 km (Kilauea) (5) (M. Loa, etc.), it seems extremely difficult to understand why the conception should be inadmissible, that the diameters of cone-pipes have been larger in the past than the average of those we know now, and, consequently, why the calderas above mentioned could not be the remnants of the cone-pipes of volcanoes. On the contrary, the very absence of smaller eruption-points within such calderas, although it could never be invoked as a direct proof, may plausibly be explained by such a conception, which would readily solve a haunting enigma.

In the case of the Askja-caldera, moreover, the above conclusion is strengthened by the structure and nature of the encompassing wall, the Dyngjufjöll, more especially by the typical peripherical arrangement of the smaller calderas, the so called "Lava Plateau", in the north east, and the Knebel-caldera in the south east of Askja, and again by a repetition of this mode of occurrence of younger eruptive channels round the periphery of the Knebel-caldera. In fact here we find the Rudolf-crater, the south eastern craters and southern fumaroles grouped in a way precisely similar again to that of secondary cones round older volcanic vents (see below) (6) (12).

THORODDSEN (12) (p. 198) considers rightly, in our opinion, the Dyngjufjöll as a remnant of an ancient strato-volcano. This view of the nature of the Dyngjufjöll finds support in its structure of alternating strata of lavas and clastic material and in their directions of slope ; yet, that the enclosed Askja-caldera would have been produced subsequently, and by down-throws of part of the upper structure along vertical fault-planes, we cannot admit.

Sub. 2. *Composite calderas.* Under this head we comprise such calderas as have or have had one or more indubitable eruption-points, situated more or less centrally or peripherically to the encompassing wall (the inner wall). (Barren Isl., Tenger, Vesuvius, etc.). This definition would virtually comprise also forms of calderas with eruption-points so excentrally situated that they actually occupy an outer-peripherical position. Such is the case e.g., as we previously pointed out, in the Askja- and Knebel-calderas which we discussed sub. 1 as *simple calderas.*

Now, if we admit that the caldera-space constitutes the space within the remnants of the cone-pipe of an older volcano, it implies that the composite-calderas would have been formed by a *process of filling up,* to a more or less degree, of such older cone-pipes by the products ejected by one or more succeeding younger eruptive cones.

The exact conception of the mechanism of such a process may best be realized by a description of such a caldera-formation which was actually witnessed and minutely registered in all its phases of development. It shall at the same time be an answer to the question posed sub. 2.

The example we shall choose is that of Vesuvius, which, with its Somma encompassing a still active eruption-point, represents the classical example of a composite-caldera. We choose this particular example because we know of no other strato-volcano, whose history has been so accurately registered during such a long period as has that of Vesuvius. Among the records we would specially mention those of the observations made by PERRET (3) of the eruptions of 1906 and 1913—'20. Their accuracy, minuteness and instructiveness and the vividness of their description permit one to follow the development of the phenomenon step by step and support, while severely testing, our contentions.

First of all we wish to place on record that both MERCALLI and PERRET (l.c. p. 14) seem convinced that the encompassing Somma-wall is nothing but the inner wall of the older Somma-volcano. DANA arrived at a similar conclusion with regard to the inner wall of the Kilauea-caldera [1]). Yet, so far as I know, neither of them nor any one since seems to have realized that these conclusions with respect to the caldera-walls of Kilauea and Vesuvius might contain the solution of the caldera-problem in general. Let us now study the history of the present eruptive channel of Vesuvius from the time immediately preceding its eruption of 1906.

The crater, which is the *orifice* of the cone-pipe, situated a little excentrally within the encircling Somma-wall, was then some 180 m in diameter, while that of the Somma-caldera measured some 3.5 km.

When we consider the observed phenomena bearing more specially on our subject, we find that the diameters of the crater and of the cone-pipe were enlarged respectively to 1000 m and 400 m by the 1906 eruption.

The passage from the lower-rim of the *crater* to the top of the cone-pipe-wall was sharp, not gradual (l.c. p. 98) ; the inclination of the crater was 45° and that of the cone-pipe consequently steeper ; the depth, from the top of the cone to the floor or the crater, measured 400 m so that the height of the steeper cone-pipe-wall above the crater-floor must have been some 100 m. These data characterize conditions actually existing in 1909, i.e. after three years of denudating activity. That the cone-pipe wall was perpendicular or nearly so, during and even years after the 1906 eruption, is testified by the various photos (3), and still more conclusively by the shape of the gas-column, 13 km in height, which was ejected through the cone-pipe during the eruption. In fact the apex of this inverted cone barely measured 20° so that its conduit must have had an inclination of at

[1]) In view of the greatness of the discharge in 1823, — so undermining, owing to its extent, as to drop abruptly to a depth of some hundreds of feet the floor of the crater leaving only a narrow shelf along the sides, — we reasonably conclude that at that time the lava-column beneath the floor was of as large area as the Kilauea pit itself, — or nearly seven and a half miles in circuit. We may also infer that, immediately before the discharge, wherever there was a lava-lake, the liquid top of the column was up to the floor of the crater, and elsewhere not far below it .... When the floor of the pit fell at the discharge in 1840 it was not thrown into hills and ridges, as it might have been had it dropped down its four hundred feet to solid rock in consequence of a lateral discharge of the lava beneath; on the contrary it kept its flat surface, thus showing that it probably followed down a liquid mass, that of the subsiding column of lava. (13) pp. 151—152.

least 80°. This phenomenon therefore furnishes a striking corroboration of our theoretical deductions from a study of the mechanism of cone-pipe-building which led us to the conclusion that the walls of volcanic cone-pipes will causally tend towards the vertical. In fact these extremely pointed inverted gas-cones, far from being an accidental mode of occurrence of this or previous Vesuvius-eruptions, are characteristic, as is well known, of certain types of gas-eruptions known as volcanian eruptions. This kind of eruptive manifestation, moreover, does not pertain to any specific type of volcano but may and often will occur in the course of any period of activity of any volcano.

PERRET does not describe the condition of the crater-floor immediately after the eruption of 1906 ; three years later, however, it appears that on it several débris-cones, derived from avalanches of crater- and cone-pipe-material, had accumulated against the steep wall of the cone-pipe. This condition remained practically unchanged until 1913. A new phase then announced its approach by an intensified activity of the magma.

Subsidences and magmatic absorption of parts of the crater floor and in particular of the foot of a débris-cone, and alternating formation and subsidence of eruptive conelets were the external signs of an incandescent magma rising in the conduit, the typical glare of its liquid surface manifesting itself on July 8 [1]).

It should be remembered that a conical depression (100 m in diameter and 20 m deep) with a semi-vertical wall was consequently formed in the foot of a (the south-western) débris-cone by subsidence. From its centre a volcanic conelet built itself up, its base closing over the lower part of the said depression. At the end of October lava began to flow out from the top of the conelet, gradually filling up first the enclosing depression and then the entire cone-pipe up to its junction with the lower rim of the crater.

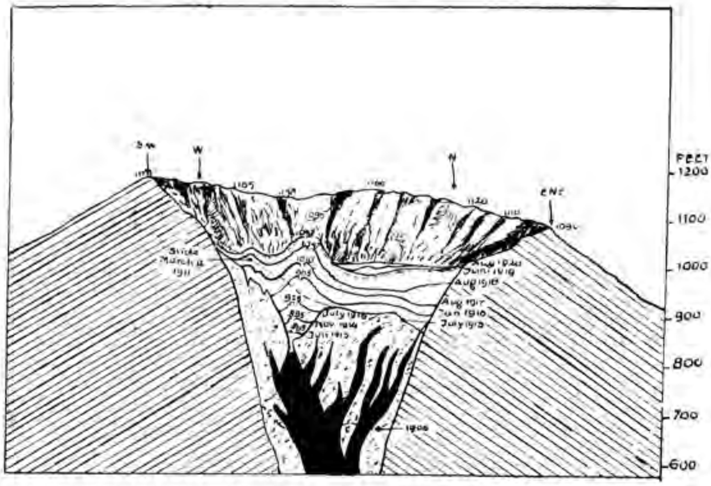
"The rising lava soon formed an eruptive conelet, and from this time "onward, up to the day of writing (1921), the entire course of events in "the external activity of this volcano has been characterized by an almost "continuous process of crater-filling activity (Fig. 61, our Fig. 3), through "superposition of material erupted explosively and effusively from the "eruptive conelet and adventitious vents."

Thus PERRET *(l.c. p. 119 ff.) summarizes his exact observations of a most remarkable phenomenon by the development of which the cone-pipe of Vesuvius of 1906 was converted into a miniature caldera, that of 1921, by a filling-up process identical, no doubt, with that which at one time (An. 79 b. C.?) converted the older Somma-pipe into the caldera-wall of the younger Vesuvius of that time.*

All the typical features characterizing terrestrial calderas equally

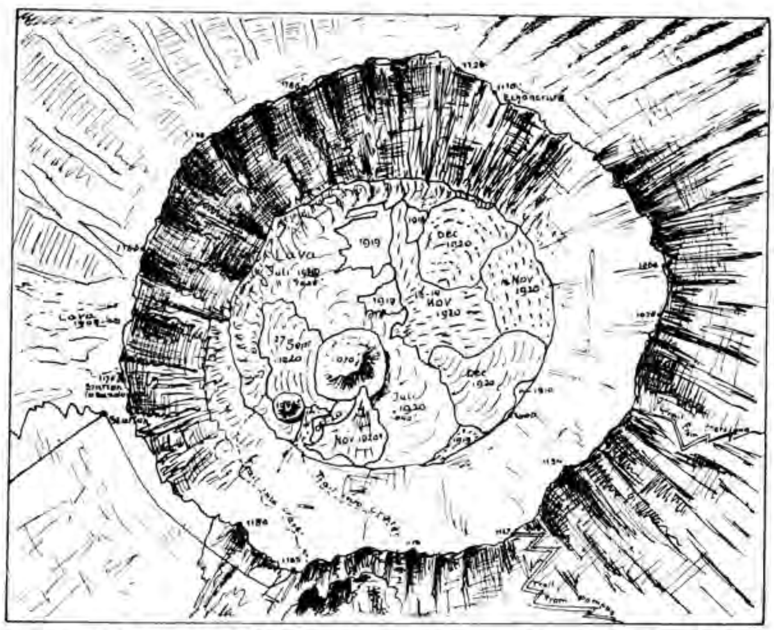
[1]) The records of the very accurate observations during the entire course of development of this phenomenon place it beyond any doubt that the ascent in the cone-pipe of the incandescent lava (magma) with its highly corrosive vapours and gasses occurred excentrally (semi-peripherically).

characterize the actual minute younger Vesuvius caldera; the abnormal proportions between the diameters of the younger eruption-channel and of



DRAWING BY MALLADRA.

Fig. 3. Schematical representation of the growth of the eruptive conelet and the resulting process of filling up the crater of Vesuvius. Composite caldera-formation. Drawing by Malladra: copied from (3) Fig. 61; section of Fig. 4.



FROM MALLADRA. MAP OF CRATER FLOOR ON DECEMBER 31, 1920.

Fig. 4. From Malladra. Map of crater floor of Vesuvius on December 31, 1920, showing a perfect composite caldera of fair size. Copy (3) Fig. 62.

its caldera are of an order identical with other, larger terrestrial composite calderas; the *grosso modo* vertical encompassing wall is present also, even though it is now almost entirely hidden from view by the filling ; the flat bottom, generally speaking, was caused and maintained by the extrusion of highly liquid lava ; finally, if the lava extrusions had been succeeded here by ejection of loose material (ashes, lapilli, etc.), the characteristic "sandsea" of Dutch East Indian and other volcanoes would certainly have been present in this case also. Thus this "sandsea"-phenomenon would likewise have found a ready explanation.

It is now necessary to test our contention, that the caldera space is nothing but a remnant of partly filled up older cone-pipes, on some other well known calderas.

On the probable mode of formation of certain calderas.

If the younger eruption-channel of Vesuvius had extruded centrally in the resent Vesuvius-caldera above described, just as it occurred in the G. Raoeng (7) (p. 58, Phot. 18), then we should have had a form of caldera comparable in every respect with that of Barren-Island and similar calderas.

The crater of the Slamat (7) (p. 35 ff and Phot. 8) presents a more composite caldera. "The crater proper (diameter over 400 m depth 228 m) is situated in the south western part of the cone and is enclosed on its north eastern side by some three crater-rims. Between the most northern and the central one a great plane or sandsea spreads out, strewn over with numerous bombs" (l.c. p. 38). Evidently we have here three inter-telescoping cone-pipes mutually tangent in the south west, the older ones of which are enclosing their successors (l.c. p. 39) in consequence of a successive displacement of eruption-channels in a south western direction. The result of such an occurrence was that the cones of the younger eruption-points could only develop individually in a north eastern direction and hence it is only in that direction that we find their remnants, namely the individualized encompassing walls and the sandsea (Fig. 5).

To nobody, we are sure, would the idea occur that the walls of this caldera, though a miniature of its kind, might be the product of down-throw, subsidence or magmatic fusion. Nobody could doubt that these walls are the remnants of older cone-pipes of which only the oldest was levelled a little or hollowed out, probably by the erosive and denudating action of (a) gas current(s). Yet, apart from its total size, this kind of caldera differs from the occurrences generally designated by that name, only in so far as the diameters of the older and younger cone-pipes differ but little here, which implies that the free development of the more recent was hampered. Still in essentials there is no difference whatever between this and other calderas.

The crater of the Sendoro (7) (p. 44 and Photos 10 and 11) presents a similar caldera within which "the remains of a younger cone encompassed by an older wall" is visible so that this crater again reproduces the live type of caldera with all the characteristic features.

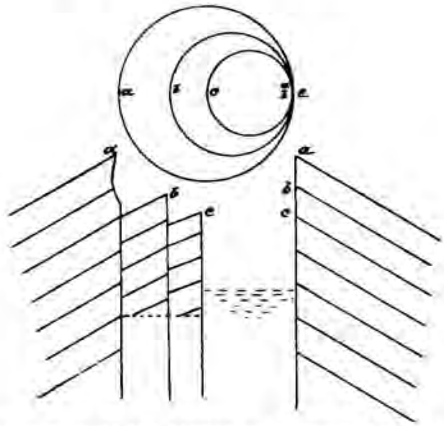


Fig. 5. Schematical section of the Slamat cone-pipe and crater shewing its caldera structure. Compiled by me after the description and aeroplane photos from (7)*pp* 35*ff* and Photos 8—9). aa = section of oldest cone-pipe; *bb* and *cc* of younger pipes with sandsea between *a* and *c* on the left.

It would not be difficult to give any number of other examples of typical caldera-craters (or crater-calderas) and it was in fact the comparative study of crater-types, especially the East Indian, and calderas, which led us to the conviction that these phenomena are essentially alike and differ only in the relative size of the eruptive channels.

Let us, finally, study the mighty caldera of the Batoer-complex, the Molengraaff-caldera on Bali, with its axes measuring 13.5 km in a N.W.—S.E. and 10 km in a S.W.—N.E. direction (8). Its floor lies at an altitude of 1000 m the highest points of its rim at 1745 and 2152 m and the lowest at 1267 and 1336 m respectively. Fig. 6, compiled after a topographical sketch by KEMMERLING, supplemented by my own observations, gives the main features of the caldera, a section of which is also given by KEMMERLING (8) [1]).

The vertical inner wall of this, the Molengraaff-caldera is very remarkable indeed. It closely follows the rim of the caldera between the points 1745 (G. Penoelisan), 1371 (W. side), 2152 (G. Abang), and 1270 except where it is interrupted by downfalls or hidden from view by débris-material, as is specially the case, e.g. E.-S.E. and N.-N.W. from the point 1371. Within this huge enclosure, which we will henceforth call the

1) This section is for its north western part incorrect, in so far as it does not show the vertical wall of the Molengraaff-caldera there, although parts of it may still be recognized here and there between the points 1745 and 1371, however much destroyed it be by avalanches. The vertical inner border of the lower plateau, at the south east side of the section, is most probably a remnant of the wall of the Molengraaff-caldera (the "outer-wall"), and KEMMERLING seems to concur with this interpretation of the nature of this feature (l.c. p. 61).

outer wall, in contradistinction to another, powerful yet smaller caldera-wall enclosed by it, we find the remnants of the latter, as another vertical,

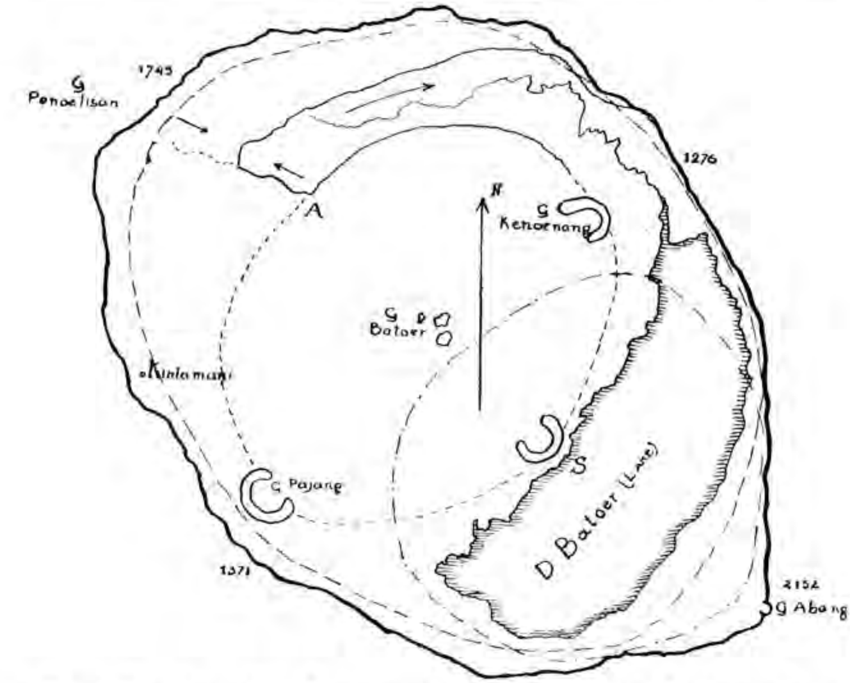


Fig. 6. Main lines of the Batoer-complex with its Molengraaff-caldera; after a topographical sketch (8) and personal observations.

roundish caldera-wall, the diameter of which must have measured some 7.5 km From a point A. marked on the sketch, eastward to a point close to G. Kenoenang, this steep *inner wall,* towering up vertically to a height varying between 200 and 300 m above the floor of the caldera, is scarcely modified. Its extension in the opposite direction, towards G. Pajang, is very mutilated and in parts almost destroyed, whilst that part which probably extended between G. Pajang, the eruption point marked S. and G. Kenoenang, has completely disappeared.

This steepness, *casu quo* verticality, of the inner-walls of the Molengraaff-caldera and its inner caldera, suggest that they probably constitute the remnants of the respective cone-pipes and their conduits.

Moreover, the said plateau, a sand-sea *optima forma,* points in the same direction, as it evidently constitutes a remnant of the cone of the inner caldera.

In fact it slopes outward and the general direction of its incline is towards the E. (vide its drainage) i.e. in exactly the same way as the rim of the Molengraaff-caldera between G. Penoelisan (1745) and the point 1276. Both the direction of these inclines and their parallelism may safely be taken as a result of the influence of current winds on the ejected material during the building processes of the respective cones. On the other hand, it would be inadmissible to interpret this concurrence of inclinations and the outward direction of the slope of the plateau, as the necessary results, i.e. causal effects of subsidence, downthrow, explosion or fusion.

Finally, we find peripherically arranged along the border of the „inner-caldera" first G. Pajang, than probably the eruption-point S. and finally G. Kenoenang, on the very zone which would have marked the zone of least resistance of the "inner volcano", namely on the zone situated between the solid core and its encompassing cone.

Since the erection of the inner-volcano above mentioned, it would seem that another eruptive channel forced its way, destroying or lowering not only the south eastern part of the *inner wall* but blowing away also part of the south eastern wall of the Molengraaff-caldera. Proof of this we find in the remaining half of G. Abang which is outer-peripherically situated in the *outer wall*, and which volcano was most probably halved in precisely the same way as was the *Piek of Rakata*, (on the outer periphery of Krakatau) during the well known Krakatau eruption of 1883.

This complex of Batoer-eruption-channels, localised within the encompassing *outer wall*, would thus have formed a twin-volcano similar to Tangkoeban Prahoe, which latter also must have been enclosed within an encompassing wall (14) (p. 732) the remnants of which may still be traced over a distance of some 15 km from G. Nanggarak (S. of Tjisaroea) over G. Lembang and Pr. Malang as far as Pr. Pangoekoesan (7) (p. 73). The present stage of activity in the history of the Molengraaff-caldera would have been inaugurated, then, by the eruptions of the peripherical eruption-points, G. Pajang, *S*, and G. Kenoenang, and finally by the eruptions of the centrally situated, still active G. Batoer, which is continuing to fill up the Molengraaff-caldera to the present day.

In the foregoing we have demonstrated by a few examples, which could be multiplied *ad libitum*, that the caldera phenomenon in its various aspects may be plausibly explained on the assumption that the vertical wall encompassing the caldera occurrence is nothing but the wall of an older eruption-channel, its cone-pipe or crater, or their remnants. Thus a composite caldera (see above) would be the remnant of an older eruptive channel filled up in part or *in toto* by younger eruption products, emanating from an inner, more or less considerably reduced, younger channel during a period of locally renewed magmatic activity.

The mechanism of such a filling-up process, resulting in the production of a miniature composite caldera, could be followed step by step during the Vesuvius-eruption of 1913—1922 (vide Figs. 3 and 4).

Should this conception of the nature of the caldera-phenomenon be correct, then we would have to conclude e.g. that the intensity of terrestrial volcanism has been dimishing over a large part of our globe at least since Tertiary times and perhaps since an earlier period. Whether such diminution dates from still earlier geological times, whether it comprises terrestrial volcanism in general, whether it is only of a local, relative or/and intermittant character, are questions the answer to which would require more extensive and detailed studies all over our globe.

It is my personal conviction that aeroplane-photography, if appropriately and systematically conducted, may render considerable service towards the solution of these questions and to that of magmatic activity in general. We do not consider it unlikely that in this manner the presence of terrestrial calderas with dimensions equal to or even surpassing those of the moon may be located.

On zones of secondary eruptions and on the causes of displacements (migration) of eruption-channels.

In the foregoing we have already touched upon (p. 185) another phenomenon pertaining to cone-pipes and calderas alike, which, consequently again points to an identity in substance of these volcanic phenomena. Considering moreover the universality of its character and the identical way of its mode of occurrence in the case of both cone-pipes and calderas, it constitutes a strong indication that the phenomenon may be a genetical feature, inherent in the formation of these volcanic structures.

We are alluding to the marked tendency of secondary eruption-channels to occur e.g. on the periphery of older eruption-channels. This phenomenon occurring universally, as is well known, in all kind of volcanoes (Hawaian volcanoes, Vesuvius, Bromo-Segoro-complex etc. etc.), pertains equally to calderas (Molengraaff-caldera, Fogo Island, Vesuvius-Somma-complex Askja-Knebel-group, etc. etc.).

Although secondary (i.e. younger) eruption-points may occur also in other places and from other causes, we will restrict ourselves here to the study of this particular mode of occurrence.

The occurrence of secondary eruption-points has hitherto been currently, explained, as a causal expression of the influence of fissures or faults radially or periclinally directed, whilst, inversely, the presence and *quasi* mode of occurrence of these secondary eruption-points is often advanced as the only vindication of the assumed existence of such fissures and faults (14) (7) (3) (5, p. 415, etc.). We do not wish to deny that eruptive occurrences may have been provoked by the presence of fissures and faults ; yet the idea, that the latter constitute the main (only ?) and primary cause of the former, we cannot admit unreservedly.

In fact, when considering the mode of arrangement of these secondary eruption-points with respect to the crater or caldera concerned, it soon becomes obvious that we may distinguish three main groups, i.e. : 1. That in which they are more or less centrally situated ; 2. that in which they are arranged along the inner side of the craters or calderas (inner-peripherical); 3. that in which they are arranged on or outside the cone-rim of craters or calderas (outer-peripherical).

The natural section of G. Pajang shows that the magmatic core, the filling of the cone-pipe, is separated from its clastic cone by a zone of breccias. PERRET (3) observed a similar occurrence at Vesuvius, a form of which he designated as the "plastic lining" of the cone-pipe wall. Which ever form the phenomenon may effect it may reasonably be expected that the cooling effect of the wall on a lava-column rising in its conduit (or cone-pipe) will tend to cause a zone of discontinuity between such column and its encompassing wall and that such tendency will be accentuated by the shrinking of the lava subsequent to its consolidation. Its formation is consequently genetically inherent in the mode of formation of a volcanic cone. (N. J. M. Taverne arrived at a similar conclusion in 1925).

The problem of the occurrence — so frequent — of secondary eruption-points and fumaroles arranged inner-peripherically, would thus find a ready solution in the presence of this zone. Moreover it would explain the peculiar and marked tendency of more recent eruptive and fumarolic action to (inner-) peripherical migrations, a phenomenon which is so common in all volcanic massives (see Fig. 7).

How are we to explain, however, a similar tendency of magmatic activity

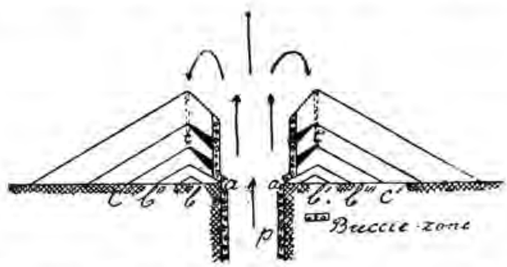


Fig. 7. Schematical vertical section of a volcanic cone shewing that the tendency of younger eruption-points (fumaroles etc.) to peripherical migration is a genetic quality, causally inherent in the mode of formation and subsequent structures of volcanic cones.

to that super-excentrical displacement of its successive sites which we have qualified as outer-peripherical?

When studying the structure of a volcanic cone more closely, we find that the zone of less resistance above described is bordered along its outer periphery by the clastic cone which, from this zone outwards, *is built up in anticlinal fashion.*

It is clear that the highly tensioned magmatic gasses and vapours, highly corrosive at that, will tend to discharge along this zone of less resistance and thus to penetrate into these clastic anticlinals, the roots of which are open along the said zone. Arrived at the tops *t*, *t'*, etc. of these anticlinals (which are again arranged, *grosso modo*, along a vertical concentric plane going through the top-rim) these vapours will meet with an extra-resistance when arriving up against the downward flank of the anticlinal. They will consequently collect at these several tops and their tension and corrosive capacity will tend to make them drill their way vertically through the covering masses, causing their discharge at last along and beyond the top-rim of the volcanic cone, i.e. *outer-peripherically*.

It is evident, therefore, that the outer-peripherical arrangement of secondary eruption-points and fumaroles, which pertains inherently to calderas and smaller volcanic cone-structures alike may be readily explained from the mode of formation of a volcanic cone surrounding its cone-pipe.

A striking confirmation of our contention as to the identity of cone-pipes and encompassing caldera-walls is again furnished by the Batoer-complex. A close study of the section of this caldera (8) discloses in its S.E. corner a remnant of a vertical inner wall, which is the inner border of a remarkable plateau situated at the foot of the bisected cone of G. Abang, and which in its turn constitutes part of the *outer wall*, i.e. of the Molengraaff caldera-wall.

In respect to this main-wall of the great Batoer-caldera, the bisected eruption channel of G. Abang [1]) is situated outer-peripherically whilst the plateau extends between a remnant of the original encompassing wall of the said Molengraaff-caldera and the bisected eruption channel. Hence we find that the slopes of the said plateau, which evidently represents a remnant of the old Batoer-cone, are directed inwards *and* outwards in anticlinal fashion.

It would seem that we have now sufficiently explained and founded our contention about the real nature of the caldera phenomenon, to which subject we intend to return shortly in a second communication.

Finally we have to offer our sincere thanks to both Dr. P. TESCH, director of the Geological Survey of the Netherlands at Haarlem, and to Mrs. E. A. VAN OOSTERZEE—BEELAERTS VAN BLOKLAND at the Hague for their kind assistance in the execution of the drawings which accompany the text.

The Hague, September 1927.

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Geology. — *On a new Basis of Solution of the Caldera-Problem and some associated Phenomena.* By C. G. S. SANDBERG D.Sc. (Second communication.) (Communicated by Prof. Dr. G. A. F. MOLENGRAAFF.)

(Communicated at the meeting of February 25, 1928).

In our previous communication (17) our consideration of the caldera-problem was based, more especially, on the mode of occurrence of the phenomena in strato-volcanoes, and on that of the migration of eruption points. Accordingly we tested our conclusions on well established facts in the history of Vesuvius, that volcano being the most carefully observed and described specimen of its type.

We now shall use the results of the studies of the classical type of lava-volcanoes *(Schildvulkane)*, the Hawaiïan and specially Kilauea, as testing material for our contention regarding the real nature of the caldera phenomenon, this lava-volcano, like Vesuvius, having been very closely observed during several decades.

Halemaumau, its still active eruptive centre, is situated in the south-western part of the great Kilauea-caldera, the vertical wall of which encompasses an area of some 5 by 3 km in diameter.

Inner-peripherically situated remnants of former eruption-channels of similar order may be located, S. of Volcano-House, at the N. E. corner of the caldera and in its south eastern part, the Sulphur Bank. Also, in 1888 (see WILKES and BRIGHAM's map [13] (Pl. IX)) two smaller specimen were situated on the edge of the then "black ledge" and two others occurred a little nearer to the north western part of the caldera-wall (Fig. 1 ; i.k. and l.m. respectively). These four eruption-points and the

Fig. 1.

lava-dome, to the S. of the last two mentioned, trend in a direction parallel to the north western part of the Kilauea-wall. Such concentric arrangement of eruption-points is repeated in the "black ledge" round Halemaumau (p.q.r.s.), where they occur *outer*-peripherically in relation to the present eruptive channel of Halemaumau and at the same time *inner*-peripherically in relation to a former, wider eruptive channel, the remnants of which surround the present one and the intervening black ledge.

Outer-peripherically arranged round the Kilauea caldera-wall we find Kilauea-Iki, a "depression" to the W. of it and Keanakakoi, and this phenomenon is repeated round Halemaumau by the mode of arrangement of the so-called "New Lake", a "depression" to the N. W. of it, and a similar one on the extreme S. W. of the old cone-pipe-wall of Halemaumau above referred to.

As further examples of main orifices surrounded by peripherically arranged secondary eruption-points we may mention Mauna Loa, Kea (19), the volcanoes of the Samoa Islands etc. etc. (20).

DANA's contention that the vertical wall encompassing the Kilauea caldera is nothing but a remnant of the cone-pipe-wall of the then Kilauea volcano is now accepted by JAGGAR, PENCK, ARN. HEIM, PERRET, among others, in so far that they agree that "Halemaumau is a volcano in a volcano", as ARN. HEIM expresses it (21). Curiously enough, however, the latter still qualifies this Kilauea-wall as that of an "Abbruchskrater", a "cratère d'effondrement", the down-throw of which would have resulted from undermining through fusion of its support (l.c. text to Pl. I). This contention seems in flagrant contradiction with established facts of Kilauean history and appears to be based, among other things, on an erroneous way of extrapolating observed phenomena of secondary moment.

In order, to make the gist of my argument clearer, I reproduce DANA's combined section of Kilauea, showing the successive changes in the form of its crater (cone-pipe) during the period 1823—1886 [13] (p. 127) (compare also his maps, sketches and descriptions). This section clearly shows e.g. how, in the course of time, the crater of Kilauea narrowed down to that of Halemaumau.

A study of DANA's section and descriptions clearly shows that ever since

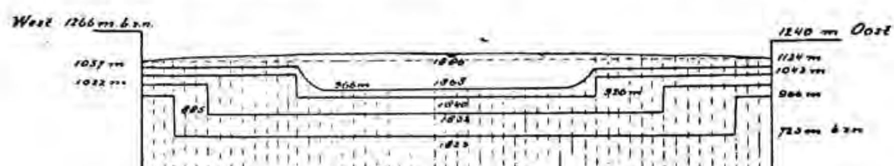

Fig. 2. Schematical combined section of the Kilauea-Caldera showing its mode of development subsequent to its principal eruptions during the period 1823—1886. After J. D. DANA. The broken vertical lines, added to the original by me, represent the directions of the dividing planes between successive accretions. Those between successive depositions, by overflows, have not been indicated to avoid crowding.

the 1823 eruption (and probably even before that of 1790) the absolute height above sea-level of the top of the caldera-wall of Kilauea, as well as its shape remained unchanged; i.e. during that period there occurred no elevation, subsidence (downthrow), or caving in of the caldera wall of sufficient importance to support the contention that the occurrence of this wall is of secondary origin, as is implied by current theories. In 1823 a narrow terrace extended (300 m below the western top-rim of the wall) all along the inner side of the caldera-wall of Kilauea. It was the black ledge of that time, the vertical inner wall of which enclosed the then crater-floor, the bottom of the "lower pit".

Now, how was this, and how were the younger terraces successively formed?

By down-throws or subsidences? By outblasts? or in some other way?

It is clear that a positive answer to these questions, if it could be given, might be decisive as to the origin of calderas.

We cannot affirm, on the evidence of direct observation at least, that the terrace of 1823 was actually formed in some other way, however much we may be justified in doing so on the basis of anology.

In fact, if this and subsequent terraces were formed by downthrow, collapse or out-blast, then any such process of presumed caldera-formation would necessarily have caused:

1. an outward displacement of the caldera-wall or (and) of that of the enclosed cone-pipe-wall; i.e. a permanent *enlargement* of either or both;

2. part of the former mass surrounding the eruption-channel, i.e. part of the (presumed) original cone and (or) its floor, to occupy a lower level after the supposed event than it did before (compare fig. 1 (17)).

Now, when considering how the Kilauea-caldera and its enclosed eruption-channel, the lower pit, fared after 1823, we find that no outward displacement of the Kilauea-wall worth mentioning has taken place, nor, we repeat, any material change of its altitude above sea-level, in spite of various eruptions since 1823. It is therefore most probable that the then terrace below the wall-rim was no more a product of subsidence, downthrow or outblast than are those of later formation. In fact, not only did no enlargements of the caldera space or of the magmatic conduit occur, subsequent to the eruptions between 1823 and 1886, *but the contrary actually happened, in so far that the capacity of the latter, that of the "lower pit", narrowed during each eruption until it was finally restricted to the dimensions of Halemaumau, the present eruptive channel.*

Correspondingly the surface of the black ledge, enclosing the channel, broadened and also assumed a **higher** *level than that of its predecessor, because of the deposition of fresh ejectamenta on its former surface.*

Thus, the surface of the caldera-floor acquired its maximum height and development subsequent to the eruption of 1886. It then extended as a slightly convex plane, over the entire space enclosed by the encompassing wall of Kilauea, at an altitude which was only 132 m below that of its western rim, a formation exactly like those of Askja, which is enclosed by Dyngjufjöll, and of Ngorongoro (East Africa). Consequently, if volcanic activity at Kilauea had ended then or had remained latent until the present

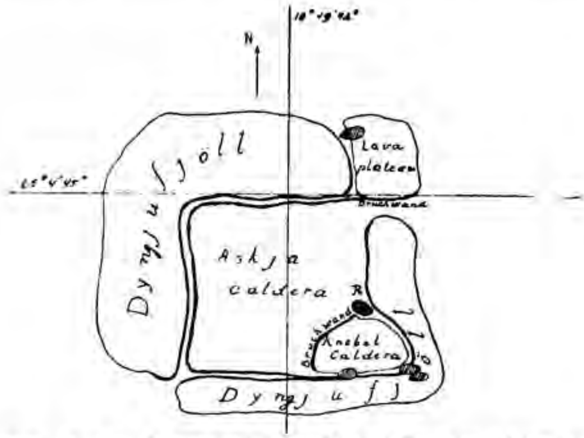

Fig. 3. Sketch of the Dyngjufjöll-Askja-Knebel-Complex. After H. RECK's topographical map [6], illustrating how the phenomenon of migration of younger eruption centres is repeated again and again.

Askja-caldera < Lava plateau; Knebel-caldera < Rudolf crater (R); S. E. craters (shaded); S. fumaroles (shaded)

day, whilst its previous history were unknown, (as is that of Ngorongoro, Askja-Knebel, and similar occurrences) then we should now search in vain, within the encompassing wall of the Kilauea-caldera, for the "central eruption channel" from which such a lava-floor emanated at one time.

From the foregoing it is clear that, in the course of time, the caldera of Kilauea went through a process of filling up identical with that of Vesuvius (17) (p. 1175—76). Far from being caused by downthrow, crumbling or explosions, it is this process which originated the successive terraces, black ledges, each younger one being broader and occupying a higher level than its predecessor, within the encompassure of the stable caldera wall. Why they will, inherently, be bordered, outwards *and* inwards, by *grosso modo* vertical walls, we have already explained (17).

Summarizing, we may conclude that the vertical wall encompassing the Kilauea-caldera is nothing but a top part of the cone pipe-wall of the old Kilauea-volcano, i.e. a product of primary origin. Further it is established that the successive terraces have been built up concomitantly with the narrowing-down process to which the magmatic conduit of Kilauea was subjected.

Still, there is no doubt that the phenomenon of subsidence and downthrow and temporary enlargement of the said conduit did take place at Kilauea at the close of various of its eruptive periods. And although of secondary importance to the problem of caldera-*formation*, it is very likely that a misappreciation of the real nature of these phenomena and a false extrapolation furnished considerable support to the subsidence theory. Hence it seems necessary to identify that real nature.

Our analysis of the mechanism of volcanic cone-building (17) showed that cone-pipes in course of erection are being built up, generally speaking, under the influence of gravity, inter-friction and pressure of the extruding magmatic column on the accumulating ejectamenta; i.e. that of a gas-current laden with fragments and particles of eruptive and other rock-material, or of a column of more or less plastic rock material highly charged with magmatic vapours and gasses. At the same time we showed that the material constituting the conepipe-wall would, *eo ipso*, arrive in a state of *labile* equilibrium the moment the intensity of the pressure of the said magmatic column changed. Hence the occurrence of subsidence phenomena, falling in *of the orifice* of the cone-pipe and its subsequent, though often temporary, enlargement which is wont to follow immediately on periods of eruption, e.g., of strato-volcanoes (Vesuvius and others, [3]).

It is clear that these phenomena of subsidence, collapse and temporary enlargements of the cone-pipes of *lava-volcanoes (Schildvulkane)*, which are also wont to follow their periods of eruption (22) (23) (24), are identical in origin and nature with those of strato-volcanoes. In fact, under the influence of the cooling and solidifying effect of the upper surface and of the sides of the cone pipe-walls, a rising lava-column will tend to "cake on" (accretion, plastic lining, etc.) and so cause a shore or black ledge, of zonal structure. It is clear, however, that the viscosity (solidification) of such a shore or black ledge, will decrease with increasing depth; i.e. the cone pipe-wall will tend to liquefaction more and more with both, depth and vicinity to the incandescent magmatic rock increasing. Now, this peripherical part of such a zonal shore or black ledge will remain in a state of equilibrium, *grosso modo*, so long as the surface of the lava-column is at or near the level of its surface. On the retreat of the lava-column, however, such parts of the newly formed black ledge [1]) as have not reached a sufficient degree of solidification will tend to subside or collapse and in doing so may drag with them, to a greater or less degree, more distant parts of the black ledge or cone-pipe.

As a matter of fact the phenomenon described above was witnessed by PERRET at Halemaumau. He wrote (22):

"But the shore [2]) — undercut and plastic in its foundations — soon yielded to gravity as the support of the lava column was withdrawn, and settled......, whereupon the unsupported black ledge began to give way in a series of majestic downfalls...... Although on a smaller scale, the collapse of these previous formations was comparable to that of Vesuvius after the eruption of 1906. At the west and north sides the rock of the wall was

[1]) The "shore" by PERRET (see note 2).

[2]) Recently accretioned black ledge (see note 1).

under a powerful stress and detachment was accompanied by a sharp report as, with a crushing roar, the avalanche of broken rock descended in a cloud of stony dust...... At the east edge, on the contrary, detachment was gradual, the dislocated masses of rock sliding downward with a long-drawn roll of thunder......" (See also (24) a.o. p. 219, Conclusions.) (Compare also the descriptions of subsidence phenomena at the Knebel-caldera subsequent to its great eruption, by WATT, JOHNSTRUP and others [6] [12].)

It is clear that these phenomena do not affect the problem of caldera-*formation* in the least and that they concern, exclusively, that of an *enlargement*, often only temporary and relatively minimal at that, of an existing magmatic conduit *by means of subsidence of part of a newly formed or even adjacent black ledge along their planes of zonal accretion.*

It would seem that our study of the caldera-problem has led us to another important result. In fact, it has shown that the filling-up process of a caldera-space, often of gigantic dimensions, is accomplished by lateral accretions from a conduit and by depositions on its surroundings of the products of its overflow, in such a manner that an eruptive massive causally ensues in which zones of less and greater resistance will alternate, irregularly yet systematically. The divisional planes between successive layers of accretion and deposition will constitute, *grosso modo*, a system of planes of least resistance.

Moreover, it is clear that a tendency to a certain mode of orientation will be inherent in these alternating zones of less and greater resistance and that such orientation will largely correspond with the physical state of the rock-material concerned, during its deposition or accretion. Should the rock have been extremely liquid (Kilauea and, in general, the basaltic group of rocks), then it may be expected that the zones will be directed in a vertical and horizontal sense corresponding respectively with the orientation of the accretions and depositions. For more viscous material, these directions may be expected to vary accordingly.

Furthermore it is equally clear from the foregoing that certain directions of trend will be imposed genetically on these zones of less and greater resistance e.g. one which is parallel, generally speaking, to the wall of the eruption channel concerned.

Now, should our conclusions be right, then it may be reasonably expected that gravitative, magmatic, and other tensions will naturally tend for relief, along the directions indicated above, in the shape of faultplanes, subsidences, zones of disruption, fissures, eruption-points, fumaroles, etc. etc. Finally it should be kept in mind that such layers of accretion and deposition, may acquire any shape, size, and thickness, ranging from a thin wafer to a big mass, and that consequently their dividing planes, *which are of primary origin*, may be extremely close together (foliation, schistosity) or very far apart (banked condition etc.).

It follows that when in such and similar eruptive massives foliation, divisional planes, fissures, faults, aligned eruption-points etc. are actually established, showing a pronounced and systematic orientation, such occurrence does not *eo ipso* justify the conclusion that these phenomena must have had a secondary origin i.e. were tectonically imposed after (or during) the consolidation of the massive, by the action of tangential or other forces.

As such massives may acquire huge dimensions — even supposing they cannot much surpass those we know already — it follows that our conclusions may fundamentally affect current principles on the origin and nature of the (tectonical) structure of eruptive masses and systems, *(Eruptiv Tektonik ;* (26)) as well as on the deductions based thereon. (Approximate areas of : Kilauea 14 km^2 ; Askja-Knebel 90 km^2 ; Ngorongoro 300 km^2 ; Ringgit 475 km^2 etc. etc.).

We will now put our conclusions to the test of facts which have been well established at Kilauea and in its neighbourhood.

An inter-comparison of several maps and sketches of Kilauea published since 1825 [13] [5] p. 463 ; (24) clearly demonstrates that zonal accretion and corresponding fissuring and faulting of the black ledge actually occurred and proceeded in a direction, *grosso modo,* parallel to the old Kilauea caldera-wall all round the still active remnant of its eruptive channel, Halemaumau.

Such orientation is proved by the trend of Lyman's ridge (e.f-g.h.), which indicates the inner border of DANA's black ledge of 1830 [13] (p. 85) ; by the system of contraction-planes with slight vertical displacements, which trend through the accreted black ledge of Kilauea, concentrical to the crater-wall of Halemaumau. We find our conclusions confirmed, not only by the mode of arrangement of outer-peripherically situated eruption channels round Kilauea as well as round Halemaumau (and also round Kilauea Iki, it would seem), but also by the arrangement of a series of smaller eruption points at or near the border of the black ledge of Kilauea (of 1840) (Fig. 1, *i, k, l, m*), and those within the outer wall of Halemaumau (*p, q, r, s.*). Last but not least, our conclusions are again supported by the directions of trend of the fault-planes going from W. by N. to E. in a wide semi-encompassing curve round the Kilauea-caldera, and comprising Kilauea-Iki and Keanakakoi.

By analogy with occurrences established in old Kilauea and round Halemaumau, it would seem justifiable to conclude that these systems of faults and fissures *round* Kilauea are either remnants of an older caldera-wall or else that they were originated in a black ledge of much larger dimensions, within which Kilauea and its smaller companions may have been the reduced eruption-channels of a larger volcano ; just as Halemaumau is the reduced remnant of old Kilauea.

We should therefore not be surprised to discover that, in and round caldera-areas such as Dygnjufjöll, Santorin and others, tectonic and eruptive phenomena happen to be arranged in perfect harmony of orientation with the directory lines of such areas ; *on the contrary we* *should expect such arrangement.* We therefore emphasize that the establishment of their existence in such areas does not justify to invoke the fact of their occurrence as an irrefutable proof of the contention that such and similar calderas were *generated*, directly or indirectly, by these tectonical or eruptive phenomena (subsidence or explosion).

Such phenomena were *generated*, and they subsequently matured, at least to a large degree, in the body of the pre-existing caldera, as a *causal result* of the way in which it was filled up.

Finally we wish to add a few words to our previous argument. Firstly, it may be mentioned that Kilauea, like Vesuvius, furnished a striking example of the effects of re-fusion of part of its cone and a subsequent flow-back of the re-fused material (Fusion theory of HOCHSTETTER and others). Halemaumau went through such a process when, subsequent to its eruption of 1868, part of the cone all round its eruptive channel, having been re-fused, was capped over completely by a relatively very thin, semi-solid crust. Now, after the subsidence (back flow) of the liquid lava, its thin roofing collapsed; yet the caldera thus formed was, again, not bordered by vertical walls (compare (17) p. 1166).

As instances of blasting-out phenomena, with subsequent cone-pipe enlargement of minor importance (explosion theory (17) (p. 1166) we would mention those which occurred in Vesuvius in 1913 and in Krakatau in 1883.

It would not be difficult to furnish further evidence in support of our contentions, from the eruptive histories of other volcanoes, such as Santorin, Pelée and others. We preferred, however, to restrict our tests principally to well established facts in the history of Vesuvius and Kilauea because they represent two classical and beautiful types of their kind. Moreover, we know no other volcanoes which have been the object of such continuous, detailed, and scientific studies as have these two.

The chosen testing material was therefore of the first order and corresponding demands were therefore made on the arguments by which we had to vindicate our contentions.

The Hague, December 1927.

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Figures placed between ( ) refer to the list below and those placed between [ ] to that at the end of (17).

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18. F. A. PERRET. Some Kilauean Ejectamenta. Am. Jour. Sci. Vol. XXXV. N⁰. 210 1913, pp. 611–618.

19. IM. FRIEDLAENDER. Ausbruch der Mauna Loa (Hawaii) in April 1926. Zeitschr. f. Vulkanologie. Bd. X. 1926, p. 49. Note especially, on its accompanying map, the mode of occurrence of secondary eruption points round the crater of Kea (NE of M. Loa).

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Mathematics. — *On infinitesimal deformations of V_m in V_n.* By J. A. SCHOUTEN. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of November 26, 1927).

Let the points x^ν of a geodesic line in V_n be subjected to a transformation:

$$'x^\nu = x^\nu + \varepsilon v^\nu , \quad \ldots \ldots \ldots \quad (1)$$

where v^ν is a field of contravariant vectors defined along the line, and ε a small constant. Higher powers of ε may be neglected. Then we can deduce the conditions to which v^ν must conform in order that the transformed line may also be geodesic. A differential equation of the second order is found, which for $V_n = R_3$ is due to JACOBI, and in the general case to LEVI-CIVITA [1]).

An analogous question arises for a minimal-V_m in a V_n. This also leads to a differential equation of the second order, due for $V_n = R_3$, $m = 2$, to SCHWARZ, and for general V_n and $m = 2$ to CARTAN [2]).

We may now seek in general to find the equations expressing the change of the fundamental quantities of a V_m in V_n, when the points of this V_m are subjected to a displacement εv^ν. By fundamental quantities we understand the fundamental tensors and the different curvature quantities. After this we can easily find the differential equations for v^ν for the case that the displacement εv^ν does not change certain given properties of the V_m. It is only necessary to substitute the identities, characterizing this property, into the general equations.

In this paper we first deduce the conditions for a geodesic V_m and for a minimal-V_m, they are immediate generalisations of results found by LEVI-CIVITA and CARTAN; after this we deduce the equations for the bending [3]) of a V_m in V_n and find some interesting conclusions for the special case $V_n = R_n$. We conclude with the transformation of a V_{n-1} in V_n that leaves the principal directions of the second fundamental tensor invariant and with the equivoluminar transformation of a V_m in V_n.

§ 1. *The fundamental quantities of the V_m.*

We use two coordinate systems: $x^\nu, \lambda, \mu, \nu = a_1, \ldots, a_n$ in V_n and $y^c, a, b, c, d = 1, \ldots, m$ in V_m. According to a known property we can avoid the use of the coordinate system y. But it is useful for the present investigation, as we will accept that the deformation εv^ν takes it along

[1]) Sur l'écart géodésique, Math. Ann. **97** (26), 291—320.

[2]) Sur l'écart géodésique et quelques notions connexes, Rend. Acc. Lincei (6a) **5** (27) 609—613.

[3]) Under bending we understand a flexion without tearing or stretching.

with it. Hence the quantities of the V_n carry Greek, those of the V_m Latin indices. Under transformations of the x the components of the quantities of the V_m do not change, being fixed with respect to the y. The same holds for the components of the quantities of the V_n under a transformation of the y. Furthermore we have quantities with both Greek and Latin indices, their components being changed as the components of a quantity of V_m in as much as the Greek indices are concerned, and as those of a quantity of V_n in as much as the Latin indices are concerned [1]).

The most important of these quantities is:

$$B^{\nu}_{a} = \frac{\partial x^{\nu}}{\partial x^{a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It can be shown very easily indeed that the B^{ν}_{a} behave like the components of a contravariant vector of the V_n under transformations of the x; and like components of a covariant vector under transformations of the y. With the aid of B^{ν}_{a} we deduce from the fundamental tensor $g_{\lambda\mu}$ of the V_n the fundamental tensor of the V_m:

$$g'_{ab} = B^{\lambda\mu}_{ab}\, g_{\lambda\mu} \quad ; \quad (B^{\lambda\mu}_{ab} = B^{\lambda}_{a}\, B^{\mu}_{b}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3$$

and with the aid of g'_{ab} we form the quantity:

$$B^{c}_{\lambda} = g'^{cb}\, B^{\mu}_{b}\, g_{\lambda\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It follows from (2) and (4) that

$$B^{c}_{\mu}\, B^{\mu}_{a} = B^{c}_{a} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

here we denote with B^{c}_{a} a set of m^2 numbers with value 1 when $a = c$, and 0 when $a \neq c$.

The quantities B^{ν}_{a} and B^{c}_{λ} allow us to define a unique correspondence between the quantities of the V_m and some of the quantites of the V_n and vice versa. We only need to show it for vectors. If

$$\overline{v}^{\nu} = B^{\nu}_{b}\, v^{b} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6^a)$$

corresponds to v^c, or, in covariant components:

$$\overline{v}_{\lambda} = g_{\lambda\nu}\, B^{\nu}_{b}\, g'^{bc}\, v_c = B^{c}_{\lambda}\, v_c \, , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6^b)$$

we have, on account of (5):

$$B^{c}_{\mu}\, \overline{v}^{\mu} = v^{c} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^a)$$

$$B^{\mu}_{a}\, \overline{v}_{\mu} = v_{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^b)$$

Hence it is equally possible to deduce v from $\overline{v}$ as $\overline{v}$ from v. We use this property to write v^{ν} and v_{λ} in stead of $\overline{v}^{\nu}$ and $\overline{v}_{\lambda}$; we therefore consider *them as another kind of components* of the vector v. We have reached

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[1]) Such quantities are already introduced, for the discussion of V_m in V_n, by E. BOMPIANI, Studi sugli spazi curvi, Atti Veneto **80** (20/21) 1113—1145, and more systematically by B. L. VAN DER WAERDEN, Differentialkovarianten von V_m in V_n, Abh. Math. Sem. Hamburg **5** (27) 153—160.

by proceeding in this way, that all vectors, and therefore all quantities, of the V_m can be considered as quantities of the V_n. On this property depends the mentioned possibility to discard the y totally.

We say that a vector v^ν, defined with respect to the V_n, *lies in the* V_m when $v^\nu = B^\nu_\lambda v^\lambda$. The geometrical meaning is clear: the direction of v^ν is tangent to V_m. It is obvious that a vector, not lying within the V_m, has no components with Latin indices. It is of course possible to form $B^c_\mu v^\mu$, but these are the components of the projection of v on V_m, not of v itself. In the same way we see that a quantity $P_{\lambda\mu\nu}$ has only then components $P_{ab\nu}$, if it "lies in the V_m with the indices λ and μ", that is to say, if $B^{\alpha\beta}_{\lambda\mu} P_{\alpha\beta\nu} = P_{\lambda\mu\nu}$.

A vectorfield v_λ, defined in V_n, has a covariant derivative

$$\nabla_\mu v_\lambda = \frac{\partial v_\lambda}{\partial x^\mu} - \Gamma^\nu_{\lambda\mu} v_\nu \quad ; \quad \Gamma^\nu_{\lambda\mu} = \left\{ \begin{matrix} \lambda\,\mu \\ \nu \end{matrix} \right\} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$\left\{ \begin{smallmatrix} \lambda\mu \\ \nu \end{smallmatrix} \right\}$ are the CHRISTOFFEL symbols belonging to the $g_{\lambda\mu}$. In the same way a vector field w_a, defined in V_m, has a covariant derivative, whose components with Latin indices are

$$\nabla'_b w_a = \frac{\partial w^a}{\partial y^b} - \Gamma'^c_{ab} w_c \quad ; \quad \Gamma'^c_{ab} = \left\{ \begin{matrix} a\,b \\ c \end{matrix} \right\}' . \quad . \quad . \quad . \quad . \quad (9)$$

and with Greek indices:

$$\nabla'_\mu w_\lambda = B^{ba}_{\lambda\mu} \frac{\partial w_a}{\partial y_b} - B^{ba}_{\mu\lambda} \Gamma'^c_{ab} w_c \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$\left\{ \begin{smallmatrix} ab \\ c \end{smallmatrix} \right\}'$ are the CHRISTOFFEL symbols belonging to the g'_{ab}. For a vector field u_λ of V_n, defined on V_m, the expression $\nabla_\mu u_\lambda$ has no meaning. But the expression

$$B^\alpha_\mu \nabla_\alpha u_\lambda = B^a_\mu B^\alpha_a \nabla_\alpha u_\lambda = B^a_\mu \frac{\partial u_\lambda}{\partial y^a} - B^\alpha_\mu \Gamma^\nu_{\lambda\alpha} u_\nu \quad . \quad . \quad . \quad (11)$$

has certainly a meaning, and represents another kind of covariant derivative. It can easily be proved, that for the case of u_λ lying in V_m:

$$\nabla'_b u_a = B^{\mu\lambda}_{ba} \nabla_\mu u_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In the same way we can build different kinds of derivatives for quantities of higher order of the V_n, defined on the V_m. One of the most frequently occurring quantities is $B^{\beta\alpha}_{\omega\mu} \nabla_\beta v_{\alpha\lambda}$, where $v_{\mu\lambda}$ is a field with index μ within V_m. For the $ba\lambda$-component of this derivative we easily find

$$B^{\beta\alpha}_{ba} \nabla_\beta v_{\alpha\lambda} = \frac{\partial v_{a\lambda}}{\partial y_b} - \Gamma'^c_{ab} v_{c\lambda} - B^\mu_b \Gamma^\nu_{\lambda\mu} v_{a\nu} \ ^{1)} \quad . \quad . \quad . \quad . \quad (13)$$

[1]) The factors B could be avoided in most cases by the introduction of new differentiation symbols for the different derivatives, e.g. $\overset{1}{\nabla}, \overset{2}{\nabla}$. If however in this way we want to come to a systematic notation applyable to all cases, the sign ∇ must indicate in which way the factors B affect the indices. This makes the notation less clear and more complicated than the notation used here as well as in Chapter III and IV of "Der Ricci Kalkül", SPRINGER 1924.

Together with the fundamental tensor the following fundamental quantities are the most important [1]).

1[st] *The curvature affinor.*

$$H_{\lambda\mu}^{\cdot\cdot\nu} = B_{\lambda\mu}^{\alpha\beta} \nabla_\alpha B_\beta^\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This quantity lies with its first two indices in the V_m. Hence it has also components with two Latin indices:

$$H_{ab}^{\cdot\cdot\nu} = B_{ab}^{\lambda\mu} H_{\lambda\mu}^{\cdot\cdot\nu} = B_{ab}^{\alpha\beta} \nabla_\alpha B_\beta^\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The vanishing of $H_{ab}^{\cdot\cdot\nu}$ is necessary and sufficient for V_m being geodesic. For $m = n - 1$ $H_{ab}^{\cdot\cdot\nu}$ passes into $-h_{ab} n^\nu$, where h_{ab} is the second fundamental tensor and n^ν the unit vector normal to V_{n-1}.

2[nd] *The mean curvature vector.*

$$D^\nu = \frac{1}{m} g'^{ab} H_{ab}^{\cdot\cdot\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Its vanishing is necessary and sufficient for V_m being a minimal manifold. For $m = n - 1$ we have $-hn^\nu = -h_{\cdot a}^{a} n^\nu$ in stead of mD^ν.

§ 2. *The fundamental equations.*

Under a deformation εv^ν these quantities are changed in the following way:

$$\text{I.} \quad \delta g'_{\lambda\mu} = 2\,\varepsilon\, B_{(\lambda}^{\alpha} C_{\mu)}^{\beta} \nabla_\alpha v_\beta \qquad \text{II.} \quad dg'_{ab} = 2\,\varepsilon\, B_{(ab)}^{\alpha\beta} \nabla_\alpha v_\beta$$

$$\text{III.} \quad \delta H_{\lambda\mu}^{\cdot\cdot\nu} = 2\,\varepsilon\, H_{\cdot(\lambda}^{\alpha\cdot\nu} C_{\mu)}^{\beta} \nabla_\alpha v_\beta - 2\,\varepsilon\, H_{\cdot(\lambda}^{\beta\cdot\nu} B_{\mu)}^{\alpha} \nabla_\alpha v_\beta - \varepsilon\, H_{\lambda\mu}^{\cdot\cdot\beta} g'^{\nu\alpha} \nabla_\alpha v^\beta -$$
$$- \varepsilon\, B_{\lambda\mu}^{\alpha\beta} C_\delta^\nu K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\delta} v^\gamma + \varepsilon\, C_\delta^\nu B_{\lambda\mu}^{\alpha\beta} \nabla_\alpha B_\beta^\gamma \nabla_\gamma v^\delta .$$

$$\text{IV.} \quad dH_{ab}^{\cdot\cdot\nu} = -\varepsilon\, H_{ab}^{\cdot\cdot\beta} g'^{\nu\alpha} \nabla_\alpha v_\beta - \varepsilon\, B_{ab}^{\alpha\beta} C_\delta^\nu K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\delta} v^\gamma +$$
$$+ \varepsilon\, C_\alpha^\nu B_{ab}^{\beta\gamma} \nabla_\beta B_\gamma^\delta \nabla_\delta v^\alpha - \varepsilon\, H_{ab}^{\cdot\cdot\alpha} \Gamma_{\alpha\beta}^\nu v^\beta .$$

$$\text{V.} \quad \delta D^\nu = -\varepsilon\, D^\beta g'^{\nu\alpha} \nabla_\alpha v_\beta - \frac{1}{m}\varepsilon\, C_\delta^\nu g'^{\alpha\beta} K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\delta} v^\gamma +$$
$$+ \frac{1}{m}\varepsilon\, C_\alpha^\nu g'^{\beta\gamma} \nabla_\beta B_\gamma^\delta \nabla_\delta v^\alpha - \frac{1}{m}\varepsilon\, H^{\alpha\beta\nu} \nabla_\alpha v_\beta .$$

$$\text{VI.} \quad dK'_{abcd} = -4\,\varepsilon\, B_{[a[c}^{\alpha\beta} H_{b]d]}^{\cdot\cdot\cdot\delta} K_{\gamma\alpha\beta\delta} v^\gamma + 4\,\varepsilon\, H_{[a[c}^{\cdot\cdot\alpha} B_{b]d]}^{\beta\delta} \nabla_\beta B_\delta^\gamma \nabla_\gamma v_\alpha$$
$$+ \varepsilon\, B_{abcd}^{\omega\mu\lambda\nu} \{K_{\alpha\mu\lambda\nu} \nabla_\omega v^\alpha + K_{\omega\alpha\lambda\nu} \nabla_\mu v^\alpha + K_{\omega\mu\alpha\nu} \nabla_\lambda v^\alpha + K_{\omega\mu\lambda\alpha} \nabla_\nu v^\alpha\}$$
$$+ \varepsilon\, v^\varepsilon B_{abcd}^{\alpha\beta\gamma\delta} \nabla_\varepsilon K_{\alpha\beta\gamma\delta}.$$

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[1]) Compare e.g. Chapter V of "Der Ricci Kalkül" (further on referred to as R.K.).

$$\text{VII. } dK'_{bc} = \varepsilon \left(- B^{\beta}_{c} H_{b}^{\cdot\,\alpha\delta} + m B^{\alpha\beta}_{bc} D^{\delta} - B^{\alpha}_{b} H_{c}^{\cdot\,\beta\delta} + g'^{\alpha\beta} H_{bc}^{\cdot\cdot\,\delta}\right) v^{\gamma} K_{\gamma\alpha\beta\delta} +$$

$$+ 4 \varepsilon g'^{ad} H_{[a[c}^{\cdot\,\cdot\,\alpha} B^{\beta\delta}_{b]d]} \nabla_{\beta} B^{\gamma}_{\delta} \nabla_{\gamma} v_{\alpha}$$

$$+ 2 \varepsilon B^{\mu\lambda}_{bc} g'^{\omega\nu} K_{\alpha(\lambda\mu)\nu} \nabla_{\omega} v^{\alpha} + 2 \varepsilon B^{\mu\lambda}_{bc} g'^{\omega\nu} K_{\alpha\omega\nu(\lambda} \nabla_{\mu)} v^{\alpha} -$$

$$- 2 \varepsilon K'_{abcd} g'^{a\lambda} g'^{d\mu} \nabla_{(\mu} v_{\lambda)} + \varepsilon v^{\varepsilon} g'^{\alpha\delta} B^{\beta\gamma}_{bc} \nabla_{\varepsilon} K_{\alpha\beta\gamma\delta}.$$

$$\text{VIII. } dK' = - 2 \varepsilon H^{\beta\alpha\delta} v^{\gamma} K_{\gamma\alpha\beta\delta} + 2 m \varepsilon g'^{\alpha\beta} D^{\delta} v^{\gamma} K_{\gamma\alpha\beta\delta} +$$

$$+ 2 \varepsilon H^{\beta\gamma\alpha} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v_{\alpha} - 2 m \varepsilon D^{\alpha} g'^{\beta\gamma} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v_{\alpha}$$

$$+ 4 \varepsilon K_{\alpha\mu\lambda\nu} g'^{\omega\nu} g'^{\mu\lambda} \nabla_{\omega} v^{\alpha} - 4 \varepsilon K'^{\alpha\beta} \nabla_{\alpha} v_{\beta} + \varepsilon v^{\varepsilon} g'^{\alpha\delta} g'^{\beta\gamma} \nabla_{\varepsilon} K_{\alpha\beta\gamma\delta}.$$

We obtain (I) starting from (1) and (2). From (I) equations (II) are deduced. For $m = n$ (II) passes into the well known equation for the variation of the fundamental tensor of the V_n under an infinitesimal transformation [1]. We obtain (III) and (IV) from (I) and (14); and (V) from (11). (VI—VIII) are deduced from (IV) and GAUSS' equation. For $m = n$ the quantities $H_{\mu\lambda}^{\cdot\cdot\,\nu}$ and D^{ν} vanish, and (VI) passes into the equation expressing the change of the curvature quantity under an infinitesimal transformation [2]. For $m = n - 1$ we have in stead of III, IV and V:

$$\text{III}'. \ \delta h_{\lambda\mu} = + 2 \varepsilon h^{\alpha}_{\cdot\,(\mu} C^{\beta}_{\lambda)} \nabla_{\alpha} v_{\beta} - 2 \varepsilon h^{\beta}_{\cdot\,(\mu} B^{\alpha}_{\lambda)} \nabla_{\alpha} v_{\beta} +$$

$$+ \varepsilon B^{\alpha\beta}_{\lambda\mu} K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta} n_{\delta} v^{\gamma} - \varepsilon n_{\alpha} B^{\beta\gamma}_{\lambda\mu} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v^{\alpha}.$$

$$\text{IV}'. \ dh_{ab} = \varepsilon B^{\alpha\beta}_{ab} K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta} n_{\delta} v^{\gamma} - \varepsilon n_{\alpha} B^{\beta\delta}_{ab} \nabla_{\beta} B^{\gamma}_{\delta} \nabla_{\gamma} v^{\alpha}.$$

$$\text{V}'. \ dh_{a}^{\cdot\,a} = \varepsilon K_{\alpha\beta} n^{\alpha} v^{\beta} - \varepsilon n_{\alpha} g'^{\beta\gamma} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v^{\alpha} - 2 \varepsilon h^{\alpha\beta} \nabla_{\alpha} v_{\beta}.$$

If we decompose in this case v^{ν} into a component w^{ν} in the V_{n-1} and another, ψn^{ν}, normal to the V_{n-1} (n^{ν} being unit vector) we find

$$B^{\alpha}_{\mu} \nabla_{\alpha} v_{\lambda} = \nabla'_{\mu} w_{\lambda} + \psi h_{\mu\lambda} - h^{\alpha}_{\mu} w_{\alpha} n_{\lambda} + n_{\lambda} \nabla'_{\mu} \psi \quad . \quad . \quad (17)$$

$$n^{\alpha} B^{\beta\gamma}_{\omega\mu} \nabla_{\beta} B^{\delta}_{\gamma} \nabla_{\delta} v_{\alpha} = - h_{\omega}^{\cdot\,\alpha} \nabla'_{\mu} w_{\alpha} - \nabla'_{\omega} h^{\alpha}_{\mu} w_{\alpha} - \psi h_{\omega}^{\cdot\,\alpha} h_{\mu\alpha} + \nabla'_{\omega} \nabla'_{\mu} \psi. \ (18)$$

The equation of KILLING $\nabla_{(\mu} v_{\lambda)} = 0$ [3] is characteristic for the rigid motions in V_n. It can indeed be shown without difficulty that in this case all diffferentials vanish.

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[1]) R.K., p. 209.

[2]) R.K., p. 208.

[3]) R.K., p. 212.

§ 3. *Geodesic V_m.*

Necessary and sufficient condition that a geodesic V_m remains geodesic under a deformation εv^ν, is, after (IV), that

$$C^\nu_\alpha B^{\beta\gamma}_{ab} \nabla_\beta B^\delta_\gamma \nabla_\delta v^\alpha - B^{\alpha\beta}_{ab} C^\nu_\delta K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\delta} v^\gamma = 0 \quad . \quad . \quad . \quad . \quad (19)$$

For $m = n - 1$ this equation passes into

$$n_\alpha B^{\beta\gamma}_{ab} \nabla_\beta B^\delta_\gamma \nabla_\delta v^\alpha - B^{\alpha\beta}_{ab} K_{\alpha\gamma\beta\delta} v^\gamma n^\delta = 0 \quad . \quad . \quad . \quad . \quad (20)$$

and for $m = 1$ into:

$$\frac{\delta^2}{ds^2} v^\delta - i^\alpha i^\beta K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\delta} v^\gamma = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

the equation of LEVI-CIVITA.

§ 4. *Minimal-V_m.*

Necessary and sufficient condition that the minimal property is not changed is

$$C^\nu_\alpha g'^{\beta\gamma} \nabla_\beta B^\delta_\gamma \nabla_\delta v^\alpha - C^\nu_\delta g'^{\alpha\beta} K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\delta} v^\gamma - H^{\alpha\beta\nu} \nabla_\alpha v_\beta = 0. \quad . \quad (22)$$

For $m = 2$ and $v^\nu \perp V_2$ this equation is equivalent to CARTAN's equation.

For $m = n - 1$ equation (22) passes into

$$n^\alpha g'^{\beta\gamma} \nabla_\beta B^\delta_\gamma \nabla_\delta v_\alpha - K_{\alpha\beta} n^\alpha v^\beta + h^{\alpha\beta} \nabla_\alpha v_\beta = 0 \quad . \quad . \quad . \quad (23)$$

If we take $v^\nu = \psi n^\nu$ and the unit vector $n^\nu \perp V_{n-1}$, we get

$$\nabla'^a \nabla'_a \psi - \psi K_{\alpha\beta} n^\alpha n^\beta + \psi h^{\alpha\beta} h_{\alpha\beta} = 0 \quad . \quad . \quad . \quad . \quad (24)$$

and, if in this case $V_n = R_3$, $m = 2$, we obtain the equation of SCHWARZ

$$\nabla'^a \nabla'_a \psi - 2 K'_0 \psi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

in wich K'_0 is the curvature of the V_2.

§ 5. *Bending.*

A V_m is bended when its metric is not changed under the deformation. Hence the necessary and sufficient condition is $dg'^{ab} = 0$, or, with respect to (II)

$$B^{\alpha\beta}_{ab} \nabla_{(\alpha} v_{\beta)} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

For $m = n - 1$ we get from (17)

$$\nabla'_{(a} w_{b)} \stackrel{*}{=} - \psi h_{ab} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If $V_n = R_n$ and the rank of h_{ab} is larger than 1, we can obtain from this differential equation an equation of the second order with $\nabla_{[a} w_{b]}$ as dependent variable, which does no longer contain the function ψ. If we write $\nabla'_{[a} w_{b]} = f_{ab}$, the integrability conditions of (27) are:

$$\tfrac{1}{2} K'^{\cdot\cdot\cdot d}_{cab} w_d - \nabla'_{[c} f_{a]b} = - (\nabla'_{[c} \psi) h_{a]b}, \quad . \quad . \quad . \quad . \quad (28)$$

For the deduction of the right member we have used CODAZZI's equation

$$\nabla'_{[c} h_{a]b} = 0 \quad \ldots \ldots \ldots \quad (29)$$

From (28) we obtain, making use of GAUSS' equation:

$$h_{b[c} h_{a]}^{\cdot d} w_d - \nabla'_{[c} f_{a]b} = -(\nabla'_{[c} \psi) h_{a]b} \quad \ldots \quad (30)$$

Transvection with h^{ab} gives

$$p_c^{\cdot a} h_a^{\cdot d} w_d + h^{ab} \nabla'_a f_{cb} = p_c^{\cdot a} \nabla'_a \psi \quad \ldots \ldots \quad (31)$$

in which we have used the abbreviation

$$h^{ab} h_{cb} - h^{db} h_{db} A_c^a = p_c^{\cdot a} \quad \ldots \ldots \ldots \quad (32)$$

The rank of p_{ab} is always n when the rank of h_{ab} is not equal to 1. Hence there exists an inverse tensor P_{ab} of p_{ab} and the introduction of P_{ab} into equation (31) gives the simpler formula

$$h_c^{\cdot a} w_a + P_c^{\cdot a} h^{db} \nabla'_d f_{ab} = \nabla'_c \psi \quad \ldots \ldots \ldots \quad (33)$$

from which we obtain by differentiation and alternation

$$\boxed{h_{[c}^{\cdot a} f_{d]a} = \nabla'_{[c} u_{d]}} \quad \ldots \ldots \ldots \quad (A)$$

in which we have used the abbreviation:

$$P_c^{\cdot a} h^{db} \nabla'_d f_{ab} = u_c \quad \ldots \ldots \ldots \quad (34)$$

If (33) is substituted into (17), we get, making use of (27) and (34)

$$B_\mu^\alpha \nabla_\alpha v_\lambda = f_{\mu\lambda} + u_\mu n_\lambda \quad \ldots \ldots \ldots \quad (35)$$

The integrability conditions of this equation are, so far as they are not a consequence of (A):

$$\boxed{\nabla'_a f_{bc} = 2\, h_{a[b} u_{c]}} \quad \ldots \ldots \ldots \quad (B)$$

It can easily be shown that the integrability conditions of (A) are a consequence of (B). Those of (B) are

$$\boxed{h_{[a[c} h_{b]}^{\cdot \alpha} f_{d]\alpha} = h_{[a[c} \nabla'_{b]} u_{d]}} \quad \ldots \ldots \ldots \quad (C)$$

and the integrability conditions of this equation are a consequence of (B). Hence the system (A), (B), (C) is complete [1]). If we compute for this case the right side of equation (VI), we see that (C) expresses the fact that $dK'_{abcd} = 0$.

Now we will first investigate under which conditions the bending is improper, that is to say, is only a pure motion of the R_n. Necessary and sufficient condition for this is that we can find a vector field in the R_n being equal to v_λ in every point of the V_m and being choosen in other points in such a way, that

$$\nabla_\mu v_\lambda = \nabla_{[\mu} v_{\lambda]} = F_{\mu\lambda} \quad \ldots \ldots \ldots \quad (36)$$

[1]) The equations (A), (B) and (C) are equivalent with a system deduced by SBRANA: Sulla deformazione infinitesima delle ipersuperficie, Ann. di Mat. (3) **15** ('08) 329—348.

is a constant bivector in R_n; that is to say that there exists in V_m a vector field p_λ, such that the bivector

$$F_{\mu\lambda} = f_{\mu\lambda} + 2 p_{[\mu} i_{\lambda]} \quad \ldots \ldots \quad (37)$$

is constant in R_n. This is however then and only then the case, when the system

$$\boxed{h_c^{\cdot a} f_{da} = \nabla_c p_d} \quad \ldots \ldots \quad (A_0)$$

$$\boxed{\nabla'_a f_{bc} = 2 h_{a[b} p_{c]}} \quad \ldots \ldots \quad (B_0)$$

admits a solution. It can be shown without difficulty that (A_0) and (B_0) also form a complete system, the equation corresponding tot (C) being here a consequence of (A_0). If a solution of (A_0, B_0) is found, we have for the corresponding motion

$$B_\mu^\alpha \nabla_\alpha v_\lambda = f_{\mu\lambda} + p_\mu i_\lambda \quad \ldots \ldots \quad (38)$$

Hence a solution of (A, B, C) is then and only then not a proper bending, if this solution also satisfies (A_0) for $u = p$.

Now the following theorem holds and can be proved easily by writing out the components with respect to the principal directions of h_{ab}:

Given the equation $h_{[a[c} k_{b]d]} = 0$, *in which* h_{ab} *is real and symmetrical and* k_{ab} *arbitrary. Then if* h_{ab} *has the rank 2,* k_{ab} *lies totally in the* R_2 *of* h_{ab}, *and if* h_{ab} *has a rank* > 2, k_{ab} *vanishes.*

Hence we deduce from (C) that a V_m in R_n admits only then proper bendings, if the rank of h_{ab} is 2 or less, a well-known property, first published by KILLING [1]). If the rank of h_{ab} is 2, we have the only case that the ∞^{n-3} directions of h_{ab} form, at each point, a *plane* R_{n-3} lying totally in the V_{n-1} and with the same tangent-R_{n-1} at each point. This was proved bij BOMPIANI [2]). Hence the V_{n-1} is thus built up by ∞^2 of such R_{n-3}. According to (B) $f_{\lambda\mu}$ is constant in each of these R_{n-3}. If the above mentioned theorem is applied to (C), we find that $h_b^{\cdot\alpha} f_{d\alpha} - \nabla'_b n_d$ lies totally in the R_2 of $h_{\lambda\mu}$; and this shows that also u_a in each of the R_{n-3} of V_{n-1} is constant. We have besides, from (IV′) and (35):

$$d\, h_{ab} = - h_a^{\cdot\alpha} f_{b\alpha} + \nabla'_a u_b \quad \ldots \ldots \quad (39)$$

Hence if y^1 and y^2 are chosen in such a way that the R_{n-3} become

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[1]) Die nichteuklidischen Raumformen in analytischer Behandlung, Leipz., 1885, p. 236 a.f.

[2]) Forma geometrica delle condizione per la deformabilità delle ipersuperficie, Rend. Acc. Lincei (5) **23**. I (14) 126—131. The first part is an immediate consequence of CODAZZI's equation, if written in orthogonal components with respect to the principal direction of h_{ab}, the second part follows from the geometrical meaning of $B_\mu^\alpha \nabla_\alpha n_\lambda$. Comp. STRUIK, Grundzüge der mehrdimensionalen Differentialgeometrie, SPRINGER 1922, p. 140. CARTAN, La déformation des hypersurfaces dans l'espace euclidéen réel à n dimensions, Bull. Soc. Math. de France **44** (16) 65—99, has a complete classification of all possible cases where a V_{n-1} is bended in a R_n.

the intersections of the systems of V_{n-2} belonging to y^1 and y^2, we see from this last equation that dh_{ab} always vanishes except for $n \leqq 2$ and $b \leqq 2$ simultaniously. Hence the plane R_{n-3} remain plane under bending. [1]).

Starting with a definite solution v_λ of (A, B, C) we can always obtain that $f_{\mu\lambda}$ and u_λ vanish at some point P, without essential change of the solution. We have only to determine the corresponding proper motion v'^λ, giving at P the same value for $f_{\mu\lambda}$ and u_λ. Then $v_\lambda - v'_\lambda$ is the desired solution not differing essentially from v_λ. This will be called the *reduction* of the bending with respect to P.

Let us give besides a short treatment of the case $m=2$, in which case $f_{\mu\lambda}$ passes into $\varphi I_{\mu\lambda}$; $I_{\mu\lambda}$ being the unit bivector of the V_2. The function φ is WEINGARTEN's "Verschiebungsfunktion" [2]). Then equations (A, B, C) become:

$$\boxed{\varphi\, h_{[c}^{\cdot\, a} I_{d]a} = \nabla'_{[c}\, u_{d]}} \quad \ldots\ldots \quad (A_1)$$

$$\boxed{\nabla'_a \varphi = {}^1/_2\, h_{ab}\, I^{bc}\, u_c} \quad \ldots\ldots \quad (B_1)$$

$$\boxed{0 = h_{[a\,[c}\, \nabla'_{b]}\, u_{d]}} \quad \ldots\ldots \quad (C_1)$$

From (A_1) and (B_1) we obtain:

$$\nabla'_a\, H^{ab} \nabla'_b \varphi = -{}^1/_8\, \varphi\, h^{a}_{\cdot a} \quad \ldots\ldots \quad (40)$$

in which H^{ab} is reciprocal to h_{ab}.

This equation comes in stead of (A) and is equivalent to the characteristic equation of WEINGARTEN [3]). It can easily be shown that every value of φ deduced from a solution of the characteristic equation satisfies (C_1) identically.

A remarkable case of proper bending is that in which the $(n-1)$-direction of each element of the V_{n-1} remains unchanged. The necessary and sufficient condition for this is that not only dg'_{ab}, but also $\delta g'_{\lambda\mu}$ vanishes. This occurs then and only then if, as we see from (I) and (II) $B^\alpha_\mu \nabla_\alpha v_\lambda$ lies totally in V_{n-1} and is at the same time alternating. Then the equations (A, B, C) pass into

$$\boxed{h_{[c}^{\cdot\, a} f_{d]a} = 0} \quad \ldots\ldots \quad (A_2)$$

$$\boxed{\nabla'_a f_{bc} = 0} \quad \ldots\ldots \quad (B_2)$$

$$\boxed{h_{[a\,[c}\, h_{b]}^{\cdot\,\alpha} f_{d]\alpha} = 0} \quad \ldots\ldots \quad (C_2)$$

[1]) BOMPIANI, Forma geometrica delle condizioni per la deformabilità delle superficie, Rend. Linc. 33 (14) 126—131.

[2]) Comp. e.g. BIANCHI-LUKAT, Vorlesungen über Differentialgeometrie, Leipzig, 1899, p. 289 a.f.

[3]) E.g. BIANCHI-LUKAT, l.c. p. 292, equation (7*).

It follows from (C_2) that, except for a scalar factor, f_{ab} is equal to the unit bivector in the R_2 of h_{ab}. It follows from (B_2), that this scalar factor is a constant and that the R_2 of h_{ab} is geodesically parallel (in V_{n-1}) at all points of V_{n-1}. If (A_2) is written out in orthogonal components with respect to the principal axes of h_{ab}, it appears that this equation is then and only then satisfied if the V_{n-1} is minimal. If we reduce the bending with respect to some point P, we see that there is essentially only one solution. Hence we have obtained the theorem, obtained by DARBOUX for the case of a V_2 in R_3 [1]):

Necessary and sufficient condition that a V_{n-1} in R_n, for which h_{ab} has the rank 2, can be subjected to a proper infinitesimal bending, with preservation of the $(n-1)$-direction of each element, is, that the V_{n-1} be a minimal-V_{n-1} and that the R_2 of h_{ab} be geodesically parallel in V_{n-1} at all points of V_{n-1}. If one such a bending is given, then any other can be obtained from it by the adjunction of a proper motion.

§ 7. *Infinitesimal deformations normal to V_{n-1} that keep the principal directions of h_{ab} invariant.*

Suppose $v^\nu \perp V_{n-1}$. Then we have from (IV′) and (18):

$$d\, h_{ab} = \varepsilon\, \psi\, B_{ab}^{\alpha\beta}\, K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta}\, n_\delta\, n^\gamma + \varepsilon\, \psi\, h_a^{\cdot c}\, h_{bc} - \varepsilon \nabla'_a \nabla'_b\, \psi \quad . \quad . \quad (41)$$

The tensor $h_a^{\cdot c}\, h_{bc}$ has the same principal directions as h_{ab}. Hence the necessary and sufficient condition that the principal directions of h_{ab} remain invariant, is:

$$\underset{a}{i^\alpha}\, \underset{b}{i^\beta} (\nabla'_\alpha \nabla'_\beta\, \psi - \psi\, K_{\gamma\alpha\beta}^{\cdot\cdot\cdot\,\delta}\, n_\delta\, n^\gamma) = 0; \quad a, b = 1, \ldots, n-1, \; a \neq b, \quad (42)$$

in which the i are unit vectors in the principal directions of h_{ab}. Such a transformation exists when we pass from one of the V_{n-1} of an n-uple orthogonal system to a neighbouring V_{n-1}. It can be shown indeed, that in this case one of the conditions for the existence of such a system is given by equation (42) [2]).

§ 8. *Infinitesimal transformations that keep invariant the the m-dimensional volume.*

The volume of the parallepiped with sides $dy^1, dy^2, \ldots, dy^m$ is, according to a well-known formula:

$$d\tau = dy^1 \ldots dy^m \sqrt{g'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

[1]) Leçons sur la théorie générale des surfaces, Paris, 1914, I. p. 383.

[2]) For literature compare SCHOUTEN and STRUIK, On n-uple orthogonal systems of V_{n-1} in V_n, These Proceedings 22, (1919), p. 594–605, 680–695.

A transformation is equivoluminar, when $d\tau$ remains invariant, or in consequence of (43) when $d\sqrt{g'}=0$. But

$$d\sqrt{g'}={}^1/_2\, g'^{-1/_2}\, dg'={}^1/_2 \sqrt{g'}\, g'^{ab}\, dg'_{ab}=\varepsilon \sqrt{g'} g'^{\mu\lambda} \nabla_\mu v_\lambda \quad . \quad (44)$$

This formula passes for $m=n-1$ into

$$d\sqrt{g'}=\varepsilon\,\psi \sqrt{g'}\,(\nabla'^a\, w_a+\psi\, h^a_{\cdot a}) \quad . \quad . \quad . \quad . \quad . \quad (45)$$

on account of (17). This shows that an infinitesimal transformation of a V_{n-1} in V_n perpendicular to this V_{n-1} is then and only then equivoluminar if the V_{n-1} is minimal. This theorem is due to BOMPIANI.[1] [2]).

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1) Studi sugli spazi curvi, Atti del R.I. Veneto 80. 2 (20/21) 1113—1145, p. 1141.

2) In a recent dissertation at the Massachusetts Institute of Technology, with title "Infinitesimal Deformation of Surfaces in Riemannian Space", W. F. Cheney investigates the bending of V_2 in V_n, especially for the cases $n=3$, $n=4$. In these cases he comes to equations, which for the case $V_n=R_n$ correspond to the equations (A_1, B_1) of our paper. An abstract of this dissertation is published in "Abstracts of Publications of the Massachusetts Institute of Technology", Vol. I (1928).

Physiology. — *Experimental contributions to the knowledge concerning the segmental innervation of the abdominal muscles in the dog.* (1st Communication.) *General statement of problem and I. The M. Rectus Abdominis* [1]). By Prof. G. VAN RIJNBERK and Miss L. KAISER.

(Communicated at the meeting of January 28, 1928).

GENERAL STATEMENT OF PROBLEM.

The muscles of the abdominal wall hold a somewhat special position among the muscles of the trunk in regard of problems of segmental anatomy. The structure of those muscles reminds us of the original myomery by the presence in many places of obvious rudiments of myosepta; besides the innervation being plainly metamerical in type. In the following communications we will give some contributions to a more exact knowledge of the actual facts concerning the segmental structure and radicular innervation (rhizomery) of those muscles and also of the mutual relation of those two.

I. RECTUS ABDOMINIS MUSCLE.

ANATOMICAL INTRODUCTION.

Gross Anatomy.

The M. Rectus Abdominis consists in dog in one long narrow layer of muscle that is inserted craniad with a relatively broad head to the 1st—5th ribs, a fascia intervening. Caudad the insertion consists of a narrow strip fixed to the pubal symphysis. The fibres take a parallel course, in cranial-caudal direction. The larger part of its course is intersected by fibrous partitions plainly, visible at the surface as white, somewhat sunken stripes, the so-called inscriptiones tendineae. Of those inscriptiones tendineae usually six can be counted [2]) limiting five muscle compartments about equal in length. We will designate the inscriptiones as I 1—6; the muscle compartments as M 1—5. The cranial and caudal part of the muscle are free from inscriptiones. In the caudal terminal part at the medial border

[1]) After research carried out in the Physiological Laboratory of the University of Amsterdam.

[2]) Those were easily found in the dogs investigated, contrarely to ELLENBERGER and BAUM who remark (Anatomie des Hundes, p. 162) „Man bemerkt an seinem Muskelbauche 3—6, jedoch nur ganz undeutliche, sehnige Inscriptionen".

often the beginning of an inscription can be found which does not penetrate the mass of the muscle at least cannot be traced further (17).

Peripheral Innervation.

The M. Rectus is innervated by a series of peripheral nerve branches, usually twelve in number. Of those seven go to the five compartments that are bordered by inscriptions. Of the others three spread in the unsegmented cranial part, and two in the unsegmented caudal division. Those twelve nerve branches belong to an equal number of spinal root pairs, viz. Thoracic 4 to Lumbar II.

EXPERIMENTAL PART.

The method as carried out by us consisted chiefly in stimulating the spinal roots concerned and the peripheral branches.

A. METHOD.

By opening the vertebral channel the spinal cord was exposed for at least twelve segments, between the third thoracic and third lumbar segment. In a few cases sixteen segments were exposed in order to test the entire territory that innervates the rectus muscle. In other cases a smaller number of segments caudal or cranial, was exposed. Thereupon to each root of one side (usually the left side ; twice — dog 5 and 6 — right side ; twice — dog 20 and 22 — at both sides) a conducting copper wire was fixed. Each of those wires ended in a mercury cup, numbered 1, 2 etc. and acted as unipolar stimulating electrode. A large metal plate covered with wash-leather was sewn under the skin of the neck and served as indifferent electrode. By immerging a copper rod in one of the mercury cups the secundary circuit of an inductorium could be completed and so each of the roots prepared could be stimulated at will. (Compare Fig. 1.) Now the animal was placed on

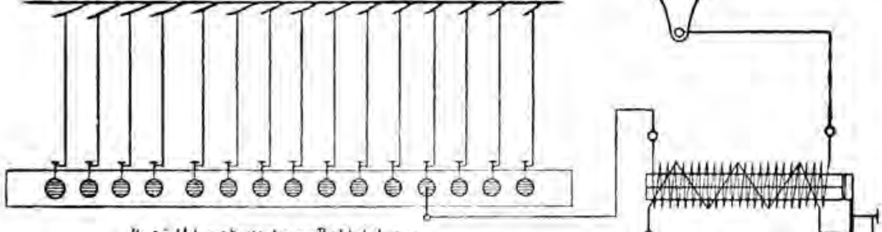

Fig. 1. Wiring diagram for stimulation of the spinal roots.

its back in order to expose the muscles of the abdominal wall. The M. rectus abdominis was freed as completely as possible from perimisial fascia, and the connections with the Mm. obliqui at the sides were carefully severed. Finally the Mm. obliqui were turned outward, exposing the M. transversus abdominis. On this muscle usually the 5 (6) most caudal peripheral branches that innervate the M. rectus take their course. The cranial branches take their origin from the intercostal nerves 4—9 or 10. Those branches cannot be exposed as easily, but it is not difficult to stimulate the same. Therefore a bipolar electrode is placed against the lower border of the rib concerned, and if necessary the ends of the electrode wire are run through the intercostal muscle in order to reach the nerve itself. The nerves that run across the M. transversus were stimulated at 2—3 cm distance from the border of the rectus muscle. The other nerve branches at a larger distance. The results of stimulation were described after simple observation and often registered photographically.

B. DESCRIPTION OF EXPERIMENTS.

Stimulation of a root partaking in the innervation of the rectus muscle always results in contraction of a small part of the muscle. Since the course of the fibres is parallel with the long axis of the muscle, the contracting part always shortens. Since the muscle is inserted into two relatively fixed points (symphysis and chest) contraction of a part of the muscle results in stretching the rest.

The contracting part increases clearly in bulk, usually the colour turns dark and the consistence becomes locally firmer. If various roots are stimulated one after another, it is very clear both visible and tangible that the contracting part changes its place.

1. *Experiments serving to determine the ventral spinal roots partaking in the innervation of the rectus muscle.*

In 15 dogs in which the caudal part of the muscle was investigated, (in two dogs at both sides) in 15 cases L II was found to be the most caudal root partaking in the innervation of the rectus muscle (Dog 1, 7, 10, 11, 13, 14, 16, 17, 18, 19, 20 and 22 R and L, and 23).

In one case the most caudal root was L III (Dog 8), and once it probably was L III (Dog 15). The inactivity of the next root was ascertained in ten out of the fifteen cases mentioned, and in the first of the exceptions.

In eight dogs the cranial part of the muscle was investigated, (two of which at both sides). In five cases the most cranial root partaking in the innervation appeared to be Th. 4 (Dog 13, 16, 19, 20, R and L), and in three dogs Th. 5 acted as such (Dog 11, 15 and 23), and in one Th. 6

(Dog 21). In three of the cases first mentioned it could be ascertained that Th. 3 was inactive.

In general a series of twelve roots, viz. Th. 4—L II appears to partake in the innervation.

2. *Experiments serving to determine the distribution of the twelve roots over the rectus muscle.*

a. The non-segmented cranial section (fig. 2).

In the innervation of the non-segmented cranial section (n-s c.s) usually three roots partake, viz. Th. 4, 5, 6. If but two roots innervate this part, stimulating Th. 5 results in contraction of the most cranial point, stimulating Th. 6 causing the caudal part to contract. (Dog 11 and 23.) In one case it was observed that Th. 6 and 7 innervated this section (Dog 21).

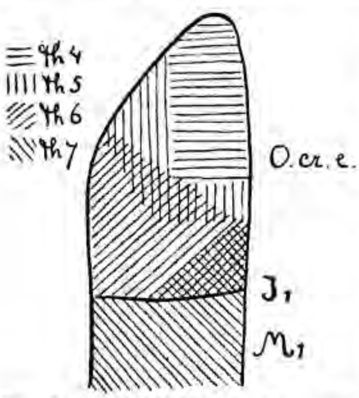

Fig. 2. Diagram of the rhizomery of the cranial non segmented section.

Sometimes Th. 7 partook in the innervation of this section by overlapping, usually supplying nerve fibres to the caudal lateral part. This happened, whether Th. 4 or Th. 5 consisted this most cranial root (Dog 13, 15, 19). In general overlapping was seen in this territory, least in the cranial lateral part, which received fibres from more than one root in one case only (Dog 19). It was a rule to find 5 and 6 overlapping, the only exception being Dog 11. Once it appeared possible that even Th. 8 had penetrated the n-s c. s.

b. The segmented medial part of the rectus muscle (fig. 3 and 4).

This medial part usually consists of five segments. Sometimes the most cranial inscription, bordering M 1 is incomplete. Some dogs showed but four segments in this part, bordered by five inscriptions. (Dog 18, 19, 21, 23).

In this territory contraction was obtained by stimulating Th. 7—Th. 13.

Th. 7. Causes as a rule contraction of M 1. In two cases (Dog 11 and 13) stimulating Th. 8 resulted also in contraction of a medial section of M 1. In Dog 20 at the left side Th. 7 left the innervation of the medial part of M 1 to Th. 8. Sometimes (Dog 13, 15, 19) Th. 7 exceeded M 1 and partook in the innervation of the n-s c.s.

Th. 8. This root has not a definite segment to innervate. In Dog 10 only M 1 was innervated independently by this root. In other cases it did not partake in the rectus innervation (Dog 7, 16) or had a small part in the innervation of M 1 or M 2 (resp. M 1 + M 2) which were innervated also by Th. 7 and 9.

Th. 9 innervates as a rule M 2 only.

Th. 10 innervates as a rule M 3 only. In one case (Dog 20) Th. 10 innervated at the right side M 3 together with Th. 11.

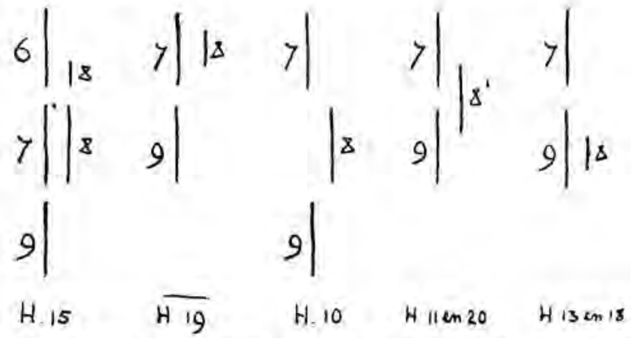

Fig. 3. Diagram of the relation of the roots Th 6, 7, 8, and 9.

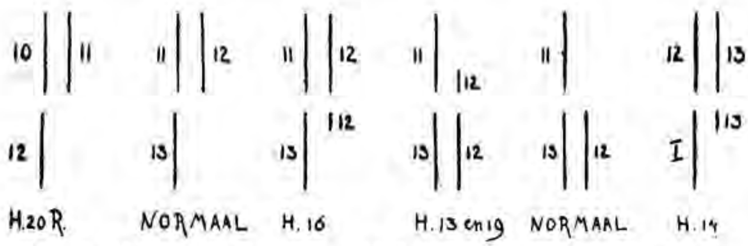

Fig. 4. Diagram of the relation of the roots Th 10, 11, 12, and 13.

Th. 11. Innervates regularly and exclusively M 4. But in half the cases Th. 12 partakes in the innervation of this segment.

Th. 12. Like Th. 8 this root does not possess a definite segment; it is true that with Th. 11 it partakes in the innervation of M 4 or with Th. 13 in M 5. Twice (Dog 13 and 19) this root innervated with Th. 13 the whole of M 5, and a small caudal part of M 4 at the same time, the remainder being innervated by Th. 11. Once (Dog 16) this root innervated the whole of M 4 together with Th. 11 and at the same time a small cranial section of M 5, the remainder being innervated by Th. 13. Twice (Dog 22 R and L) this root innervated the medial part of M 4 together with Th. 11 and the lateral part of M 5 together with Th. 13.

Th. 13. Stimulation of this root nearly always resulted in contraction of M 5. This segment may also receive fibres from Th. 12. In one case (Dog 14) stimulation of Th. 13 also caused contraction caudal to I 6, in a very small cranial part of the caudal non-segmented section of the rectus muscle.

c. The non-segmented caudal section (fig. 5).

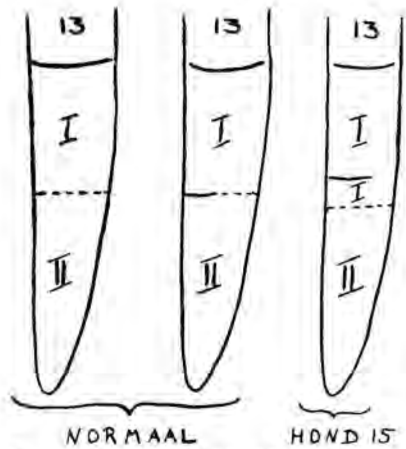

Fig. 5. Diagram of the rhizomery of the non-segmented caudal part, and of its relation to the fragmentary inscription (I 7).

The non-segmented caudal section (n-s ca.s.) as a rule is innervated by L I and L II. In one case only (Dog 8) it was L I, II and III. In one case it was uncertain whether L III partook in the innervation. Nearly always stimulating the most cranial of the roots results in contraction of a cranial section, stimulating the most caudal in contraction of the caudal part. If a larger or smaller rest of an inscription be present, the border of the two territories coincides with this inscription (I 7). Once (Dog 15) L I innervated a strip caudad to this inscription.

3. *Experiments concerning the question whether the peripheral nerve branches innervating the rectus muscle contain motor fibres from one or more ventral roots.*

In a large number of experiments we have stimulated the peripheral nerve branches and compared the effect to that obtained by stimulating the roots. Without exception it appeared that the results are identical. Stimulation of any peripheral branch innervating the rectus muscle has the same effect as stimulation of a single root. No possible interchange of fibres was observed. Each peripheral nerve branch innervating the rectus muscle consists therefore of fibres from one single root, therefore is purely uniradicular. The formation of plexus between the intercostal and lumbar nerves as often described by the anatomists, if really existing, cannot hold for the motor fibres innervating the rectus muscle.

4. *Experiments serving to discriminate more accurately the rhizomery and segmentation of the rectus muscle.*

The second chapter made it clear that at many points the territories innervated by various roots show overlapping. We want to look into this question once more.

In the non-segmented cranial section partial overlapping of Th. 5 and 6 is usual.

In the segmented medial part the territory of Th. 8 (at least if this root partakes in the innervation of the rectus muscle at all) and that of Th. 12 coincides almost always with that of the neighbouring root. For the rest each root innervates independently a single muscle segment, and respects

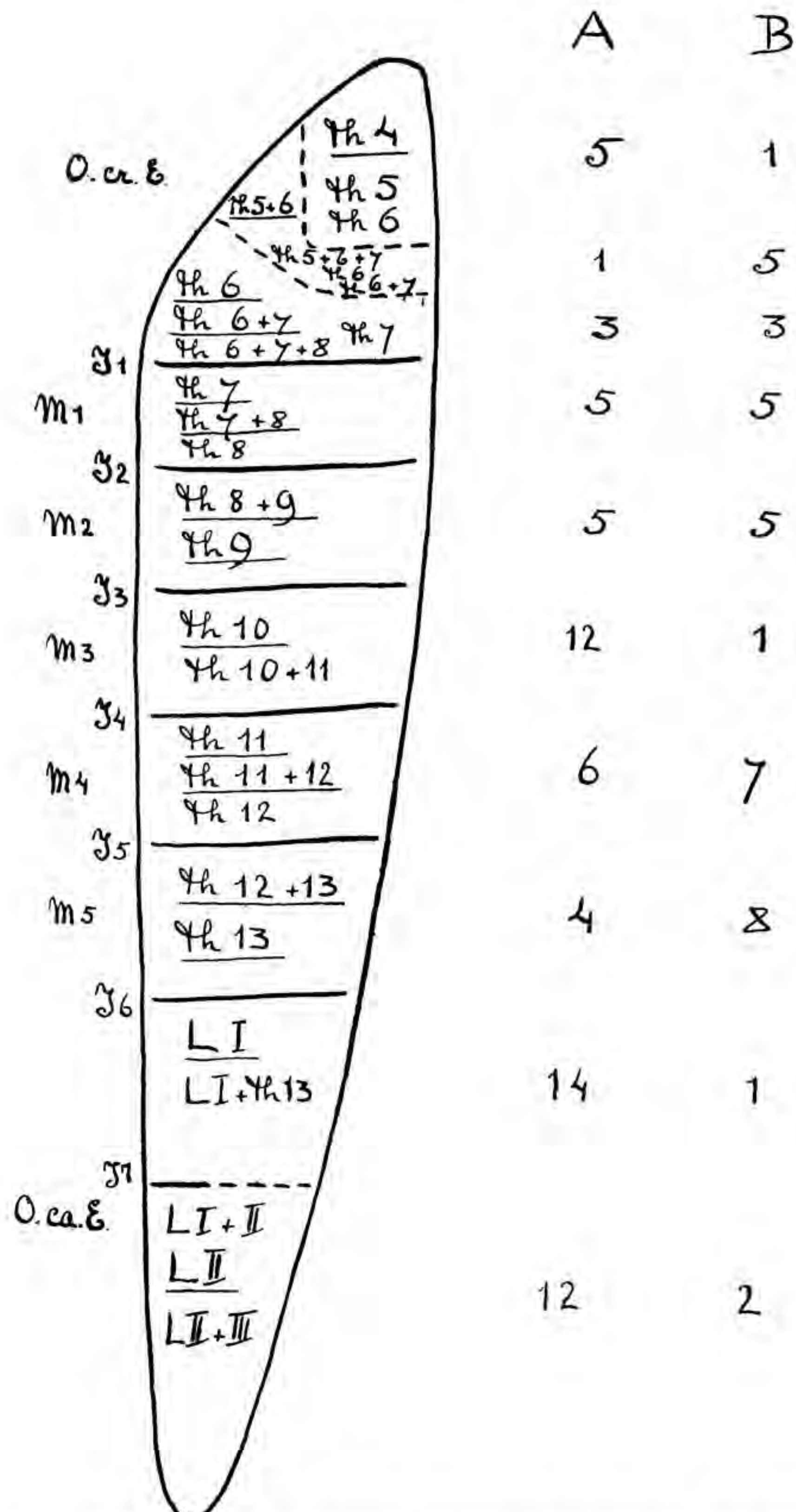

Fig. 6. Review of the relations between rhizomery and myomery in various case. Column A indicates for each segment the number of cases in which it was innervated by a single root, column B the number of cases in which fibres were supplied by two roots or more.

the segmental borders, with reservation to the exceptions as stated in chapter 2.

One remark must find here a place. The simple observation of the results of stimulating roots and nerves, even completed with photographical records is not sufficient to determine whether one or more muscle segments are in contraction. For in a dog tied on its back the insertions (high ribs and symphysis are practically, fixed points, between which the muscle is stretched. If a sect of the muscle contracts the rest must be stretched over the same distance. Suppose the distribution of a root is over two segments but in a different degree, then it is possible that stimulating this root shows a result in the segment that is the most innervated, and that the small contraction in the other segment dissappears by cause af the stretching of the whole. In order to obtain an insight in this question we have measured the change in length of the rectus segments during stimulation of the roots.

TABLE 1. Dog 7. Length of segments.

Segment	In rest	Segments stimulated:						
		L II	L I	Th 13	Th 12	Th 11	Th 10	Th 9
M 2	$5^1/_2$ cm.							**$3^1/_2$**
M 3	5			6	$5^1/_2$		**$2^1/_4$**	
M 4	$4^1/_2$			$4^1/_2$	**$2^1/_2$**	**$2^1/_2$**		
M 5	$4^1/_2$			**$2^1/_4$**	**$2^1/_2$**			
N-S Ca. S.	$13^1/_2$	**10**	**10**	14	$14^1/_4$			

TABLE 2. Dog 13. Length of segments.

Segment	In rest	Segments stimulated:					
		L II	L I	Th 13	Th 12	Th 11	Th 10
N-S. Cr. S.	5 cM.						
M 1	4	4.8	4.4				
M 2	4.4	4.5	4.5				**4.8**
M 3	4.4	4.5	4.5	4.4		3.9	**2.4**
M 4	3.2	3.9	4	**3.1**	**2.9**	**2.2**	3.5
M 5	3.2	3.9	3.5	**3**	**2**	3.5	3 +
N-S. Ca. S. a	4.9	4.7	**3.7**	4.9	5 +	5.4	5.2
N-S. Ca. S. b	8.3	**5.6**	8.5	8.5	8.8	8.8	8.4

Those two tables show clearly that stimulation of Th. 12 causes contraction of M 4 and M 5 as well. Therefore still another method was used by us.

This method consisted in stimulating a certain root, and then the corresponding nerve. Finally this nerve was severed, and again the root stimulated.

TABLE 3.

Roots investigated:	Th 10 (D. 14)	Th 11 (D. 15, 20)	Th 12 (D. 20)
Stimulation of root causes contraction of	M 3	M 4	M 4
Stimulation of nerve causes contraction of	M 3	M 4	M 4
Stimulation after cutting nerve results in (the rectus only being considered)	nothing	nothing	nothing

Now we have the non-segmented caudal section to consider.

This part was innervated in nearly all dogs by two roots, L I and L II; in one case (Dog 8), it were L I, L II and L III.

It be premised that contraction of the entire section never was caused by stimulation of a single root. Always stimulation of one single root resulted in a partial contraction of the non-segmented section. Stimulating the cranial root always caused the cranial part to contract; stimulation of the caudal root set up contractions in the caudal part. The non-segmented section therefore consists in two well defined territories, both innervated by separate roots. Compare fig. 5 and 6.

In case the non-segmented section consisted of two territories partly divided by an incomplete inscription, the border between the rhizomers was in at least nine cases identical with it. (Dog 13, 14, 16, 17, 18, 20 and 22 R and L.) In one case (Dog 15) the border between cranial and caudal rhizomer was much more caudal than the fragmentary inscription (fig. 5). Stimulation of the nerves corresponding with the roots here also has exactly the same effect as stimulation of the roots. This is important considering the fact that the nerve from L II (N. ileo-inguinalis) is connected with a branch of L I (ELLENBERGER and BAUM). Since this connection has a more central position than the spot stimulated by us, the fact that the effect of stimulation of root and of nerve was absolutely identical proves, that this connecting branches does not contain any motor fibres from L I to the rectus muscle. Furthermore stimulation of the root of L I remained without effect after cutting the peripheral nerve branch. This also proves that no motor fibres from L I reach the rectus muscle through a way other than the segmental homonym nerve.

5. *Experiments to compare the innervation of the rectus muscle right and left in the same dog.*

In one dog we have exposed both rectus muscles and the spinal cord from Th. 4 to L III, and applied stimulating electrodes to the roots at both sides in order to ascertain the results of stimulation of roots at both sides and to compared the effects.

The following facts were elicited.

a. Concerning the structure of the muscle.

The cranial apex was at both sides at the same level (fifth rib). The cranial inscription (I 1) was at both sides incomplete and situated at the right side somewhat more craniad than at the left. At the right side I 2 was about 2 cm more craniad than at the left. I 3 showed a smaller difference, but in the same direction; I 4 and the succeeding inscriptions were situated at the same level at both sides.

b. Concerning the rhizomeric division of the muscle.

Both muscles received their innervation through a series of roots from Th. 4 to L II. In the non-segmented sections no considerable differences between the rhizomeric divisions was found. But in the segmented medial part there existed at the level of Th. 10 to Th. 12 a marked difference. In the right muscle Th. 10 and 11 together innervated M 3, and Th. 12 M 4 and no other. But in the left muscle only Th. 10 innervated M 3, Th. 12 and 11 innervating M 4. At both sides Th. 9 innervated M2 and Th. 13 M 5.

Th. 8 also showed some pecularities. Both recti showed in this case a seperate strip of muscle, stronger developed at the right side, originating from the eight rib and inserting into I 3. At the right side Th. 8 partook only in the innervation of the rectus by supplying fibres to this strip; at the left side on the other hand a medial section of M 1 was innervated also by this root.

In two other dogs (22 and 23) the innervation of the rectus muscle was similar at both sides.

CONCLUSIONS.

1. The rectus abdominis muscle in dog consists of three parts:

a. a cranial section, not divided by inscriptions, usually innervated by three spinal roots (Th. 4, 5, 6);

b. a part, divided by six inscriptions and consisting of five sections usually innervated by seven spinal roots (Th. 7—13);

c. a caudal part not divided by inscriptions, innervated by two roots (L I and II).

2. The rectus abdominis muscle of the dog as a whole does not show any segmental shifting.

3. The most caudal root partaking in the innervation nearly always is the same. In the majority of cases this is L II.

4. The segmental variations that are to be found consist of a shortening of the muscle, the non-segmented cranial section receiving fibres from two in stead of three roots, and furthermore in the lacking of a segment in the medial part which contains in those cases four in stead of five segments. The non-segmented caudal part shows the least variation.

5. In the non-segmented cranial part the rhizomers are situated one after the other, except for a slight overlapping.

In the segmented medial part M 3 coincides usually with one rhizomeric division. All other sections may correspond with two rhizomers, and therefore originate from two united myomers.

The non-segmented caudal section consists of two seperate and succeeding parts.

6. The segmental nature is not quite pure in the territory, bordered by the inscriptions and apparently very clearly segmented. On the other hand, the cranial and caudal sections, (especially the latter) not divided by inscriptions, seem to show separation of myomers and of rhizomers.

7. Th. 4 is the highest root sending fibres to the most cranial apex of the rectus muscle. Th. 5 and 6 usually innervate the succeeding parts of the non-segmented cranial section.

Th. 10 innervates nearly always only M 3, no other roots partaking in the innervation of M 3.

Th. 7 and 9 innervate M 1 and 2 ; usually Th. 8 sends motor fibres in one of the segments or in both. An independent segment was innervated by Th. 8 in one case only (Dog 10).

Th. 11 and 13 innervate almost constantly M 4 and 5. As a rule Th. 12 sends motor fibres in one of those segments or in both. In one case only (Dog 20 R) Th. 12 innervated a segment (M 4) independently.

As a rule L I innervates the cranial, L II the caudal part of the non-segmented caudal section.

Botany. — *Some remarks concerning the remains, which have been described as fossil fern-stems and petioles.* By O. POSTHUMUS. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of February 25, 1928).

In this note some remarks are made concerning the classification of the fossils, which have been mentioned as remains of fern stems and petioles, either with their internal structure still visible or preserved as impressions or medullary casts only. A number of them certainly does not belong to the Ferns; their generic names are however, summarized too, because, in the literature, they have been mentioned as fern-remains; also some forms are named here, hitherto considered to be Spermophyta, but which probably are fern stems.

In some cases these fossils may be characteristic of the strata, in which they have been found; but often they are still more valuable, because furnishing data about the habit of the stem and the internal structure of extinguished members of the Vascular Cryptogams. The most striking impression is, that these data are widely scattered in the literature and very incomplete, but, when we realise how many interesting conclusions have already been drawn from the detailed study of the "*Inversicatenales*" by P. BERTRAND and others [1]), it becomes obvious how much may yet be expected from the rest. Especially a deeper insight into anatomical questions can be obtained only, when a more complete study of many of the fossil forms, which are mentioned in this enumeration, has been made. Not only in the interesting group of the *Dineuroidacea* and the *Clepsydropsidaceae*, characterised by the peculiar symetry of their leaves, which are grouped together by P. BERTRAND as *Inversicatenales*, as referred to above, results of fundamental interest have been obtained, but also in the *Osmundaceae* an almost continuous series of successive stages in development of their stelar structure has been found [2]). Under the names *Protopteris*, *Oncopteris* etc., stem-remains are known, which probably belong to the *Cyatheaceae*; slender rhizomes, named *Solenostelopteris* and *Fasciostelopteris*, are considered to belong to ferns of Polypodiaceous affinities.

Apart from better known groups, there is a number of rather obscure forms e.g. *Chelepteris*, *Cyatheopteris*, *Protopitys*, *Tietea*; it seems rather certain, that many of them will prove to be of utmost interest, especially in regard to their structure. When good specimens are available, a more

[1]) P. BERTRAND, Etude Zygopteridées 1909; see further; POSTHUMUS, Receuil trav. bot. néerl., XXI, 1924, p. 170—194.

[2]) POSTHUMUS, Receuil, trav. bot. néerl., XXI, 1924, p. 113—160.

detailed study will then probably prove to raise these obscure remains to the rank of material of primary importance in the arguing of fundamental questions of botany. This was already the case with the study of the fossil *Osmundaceae* by KIDSTON and GWYNNE—VAUGHAN.

From the remains 99 groups of generic rank have been distinguished; they are summarised in the following list according to their affinities; the number of "species" (in total 630) which have been distinguished is indicated behind the name.

I. STAUROPTERIDACEAE.

Stauropteris Binney (2).
Bensonites R. Scott (1).

II. DINEUROIDACEAE.

Dineuron Renault (2).
Diplolabis Renault (3).
Etapteris P. Bertrand (6).
Arpexylon Williamson (3).
Zygopteris (s. str.) Corda (1).
Metaclepsydropsis P. Bertrand (2).
Flichea Pelourde (1).
Aphyllum Unger (1).
Androphyllum Renault (–).
Androstachys Grand'Eury (2).

III. CLEPSYDROPSIDACEAE.

Clepsydropsis Unger (12).
Asteropteris Dawson (1).
Ankyropteris P. Bertrand (10).
Asterochlaena Corda (11).
Protoclepsydropsis Hirmer (1).
Botrychioxylon Scott (1).

IV. ANACHOROPTERIDACEAE.

Anachoropteris Corda (5).
Calopteris Corda (1).
Chorionepteris Corda (1).

V. BOTRYOPTERIDACEAE.

Botryopteris Renault (17).
Tubicaulis Cotta (10).
Grammatopteris Renault (2).
Selenochlaena Corda (2).
Protothamnopteris Beck (1).
Tracheotheca Oliver (–).

VI. PSARONIEAE.

Psaronius Cotta (107).
Tubiculites Grand'Eury (2).
Psaroniocaulon Grand'Eury (2).
Caulopteris (s. str.) Lindley et Hutton (47).
Stemmatopteris Corda (17).
Ptychopteris Corda (14).
Stipitopteris Grand'Eury (9).
Megaphytum Artis (38).
Zippea Schimper (3).
Cromyodendron Presl (1).
Psaropteris Schimper (–).
Xylopsaronius Pohlig (1).
Gyropteris Corda pars (1).
Rothenbergia Cotta (1).
Eksdalia Kidston (1).
Ilsaephytum Weiss (3).

VII. OSMUNDACEAE.

Thamnopteris Brongniart (4).
Zalesskya Kidston et Gwynne—Vaughan (3).
Bathypteris Eichwald (3).
Anomorrhoea Eichwald (1).
Osmundites Unger (18).
Paradoxopteris Hirmer (1).
Desmia von Eichwald (1).

VIII. CYATHEACEAE.

Alsophilina Dormitzer (3).
Oncopteris Dormitzer (2).
Protocyathea O. Feistmantel (2).
Protopteris Sternberg (25).
Rhizodendron Göppert (1).
Cibotiocaulis Ogura (1).
Cyathocaulis Ogura (1).
Cyathorhachis Ogura (1).
Rhizopterodendron Göppert (1).

IX. POLYPODIACEAE.

Solenostelopteris Kershaw (2).
Fasciostelopteris Stopes et Fuji (1).
Tempskya Corda (12).
Sedgwickia Göppert (1).

X. GLOSSOPTERIDACEAE.

Vertebraria Royle (12).
Clasteria Dana (1).
Blechnoxylon Etheridge (1).

XI. PTERIDOPHYTA incertae sedis.

Chelepteris Corda (7).
Lesangeana Mougéot (5).
Cottaea Göppert (2).
Cyatheopteris Schimper (2).
Lepidodendrites Fliche (1).
Diplocephalis Corda (1).
Mesoneuron Unger (2).
Protopitys Göppert (2).
Psammopteris von Eichwald (1).
Selenopteris Corda (2).
Silesiopteris Posthumus (1).
Gyropteris Corda pars (1).
Sphallopteris Corda (2).
Ptilorachis Corda (1).
Stereopteris Scott et Jeffry (1).
Tietea Solms—Laubach (1).

XII. SPERMOPHYTA incertae sedis.

Arctopodium Unger (2).
Hierogramma Unger (1).
Syncardia Unger (1).
Calamopteris Unger (2).
Calamosyrinx Unger (1).
Kalymma Unger (2).
Aulacopteris Grand'Eury (3).
Steleopteris Göppert (1).
Palaeopteris H. B. Geinitz (4).
Knorripteris Potonié (2).
Adelophyton Renault (1).
Megalorhachis Unger (1).
Periastron Unger (2).
Stephanida Unger (2).

XIII. COLLECTIVE GROUP.

Caulopteris (s.l.) Lindley et Hutton (82).
Filicites Schlotheim (2).
Rachiopteris Dawson (50).
Rhizomopteris Schimper (19).
Zygopteris Corda (s.l.) (24).

The first five families were formerly not taken separate, but considered to form one group only [3]), which has been named *Inversicatenales, Coenopterideae, Botryopterideae* or *Zygopterideae*. The internal structure of their stems and petioles, however, shows such considerable differences, that they may better be divided into the above-mentioned groups.

The *Psaronieae* contain the stems of *Psaronius* showing structure and a number of stems and petioles, the internal structure of which is unknown ; these are grouped into the other genera. The internal structure of Psaronius [4]) closely resembles that of some *Marattiaceae*, and it is mostly supposed that they belong to that family. It should be remarked, however, as has been done already by BOWER [5]), that the complicated structure of the vascular system is nearly as much analogous to that of some Polypodiaceae (e.g. *Pteris Kunzeana* Agardh [6]) or *Saccoloma demingense* (Sprengel) Prantl [7]) as to that of the *Marattiaceae*. Its simple leaf-trace resembles even more that of the *Polypodiaceae* ; the structure of the roots, however, is more alike that of the *Marattiaceae*, but triarch and tetrarch roots have been found in *Blechnum* and *Cyathea* [8]); the filamentous interstitial tissue of the root zone has no analogy in living Ferns. From the "genus" *Caulopteris* the upper palaeozoic specimens may be included in this group.

The next three families have also representants among the living Ferns. Only from the *Osmundaceae* structural stem remains have been found in Palaeozoic strata ; *Osmundites* is known to occur from the Jurassic to the Tertiary period, the other genera from Permian strata only. They show a beautiful series of successive degrees of complication of the structure of their vascular system.

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[3]) In the preface of part 12, Fossilium Catalogus, II, Plantae, an opinion is printed on p. 4, contrary to my views. We read there: „Der Name *Inversicatenales* ist nur als Sammelname zu betrachten. Es handelt sich hier eine Anzahl von Organen, von welchen mehrere zusammengehören können, oder zu anderen Gruppen, wie z.B. *Psaroniae* oder *Osmundaceae*, gerechnet werden müssen. Vorläufig jedoch müssen die einzelnen Formen noch getrennt aufgeführt werden".

In the original manuscript, however, I had written: „Der Name *Inversicatenales* ist nur ein Sammelname für einige Gruppen, welche untereinander und mit anderen Gruppen, z.B. mit den *Psaronieae* und *Osmundaceae* aequivalent sind". This opinion was in accordance with the view I had expressed before (Proc. Acad. of Science, Amsterdam 27, 1924, p. 835)· „But one should always bear in mind that it is a collective group only in wich a number of families is grouped together, the affinities of which are rather distant; and may, for instance, be compared with those of recent *Osmundaceae* and the *Gleicheniaceae*."

The discrepancy is due to the fact that I was prevented by absence to read the proofs personally, so that the editor of the Fossilium Catalogus was obliged to take care of it.

[4]) STENZEL, die Staarsteine, 1854; ZEILLER, Autun et Epinac, I, 1890, p. 178—272; RUDOLPH, die Psaronien und Marattiaceen, 1906.

[5]) BOWER, the Ferns, II, Cambridge. 1926, p. 260.

[6]) D. T. GWYNNE-VAUGHAN, Ann. of Bot., XVI, 1903, p. 702, fig. 14.

[7]) F. O. BOWER, Ann. of Bot. XXVII, 1913, p. 457.

[8]) DE BARY, Vergl. Anatomie, 1877, p. 377.

Of the *Cyatheacea* remains of fossil stems are known from the Jurassic. Cretaceous and Tertiary (*Caulopteris Laubeyi* Stenzel); their internal structure shows about the same degree of complication as in the recent species.

To the *Polypodiaceae* three genera, *Solenostelopteris* (a rhizome with a solenostele, much resembling that of a *Microlepia*), *Fasciostelopteris* (which shows some resemblance to the *Dennstaedtineae*) and *Tempskya*, are referred. The specimens of the latter genus seem to occur in Cretaceous and Tertiary strata only; their occurrence in older (Permian) strata has not been confirmed afterwards. The affinities of this genus, with its characteristic mass of roots, has not fully been made clear yet; *T. rossica* by the casual presence of two leaf-gaps, which are closer to each other than usual, may show an analogy to some species of *Cheilanthes* and *Pellaea* [9]). The stem has been compared with that of *Hemitelia crenulata* [10]).

The rhizome of *Glossopteris*, *Vertebraria* is also mentioned here: probably this plant, the affinities of which are not well known, belongs to the *Spermophyta*.

The eleventh group contains a number of forms, probably belonging to the *Pteridophyta*, but all of rather uncertain affinities. Some of them may, by investigation of more abundant material, prove to be of great interest. *Protopitys*, if not a group apart, shows some resemblance with *Megaphytum* and distichous species of *Psaronius*. *Sphallopteris* and *Chelepteris* possibly may belong to the *Osmundaceae*; *Tietea* is much different from any known Fern. The other forms are too badly known even to make suppositions concerning their affinities.

Another group of remains belongs to palaeozoic tribes of seedplants: *Arctopodium*, *Hierogramma* and *Syncardia* to *Cladoxylon*; *Calamopteris*, *Calamosyrinx* and *Kalymma* to *Medullosa*; *Palaeopteris* to *Cordaites*. From the other forms hardly anything can be said.

The last division contains some genera, instituted as collective groups. The Palaeozoic species of *Caulopteris* doubtless belong to the *Psaronieae*; the Mesozoic and Tertiary specimens partly to the *Cyatheaceae*, partly they are of unknown affinities. The genus *Zygopteris* has afterwards been divided by P. BERTRAND into a number of genera, grouped together by him in the *Inversicatenales*. They now form the bulk of the first five families (see note 3).

The other forms of this group are too badly known.

Concerning the nomenclature the following remarks may be made:

The combination *Botryopteris tridentata* has been made already before

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[9]) Ann. of Bot., XXVIII, 1914, p. 675 fig. 3, 7—10; fig. 4, 17—18; *(Cheilanthes)*; fig. 6, 10—19, 21—29 *(Pellaea)*.

[10]) SCHOUTE, Ann. Jard. Bot. Buitenzorg, (2) V, 1906, p. 198—207, pl. 18, 19.

by STOPES and WATSON [11]; at a former occasion [12] I had overlooked this fact.

The name *Botryopteris* for the genus of fossil plants has been given for the first time by RENAULT in 1875 [13]; in 1825 PRESL [14], however, had given already this name to a species, which appeared to belong to the genus *Helminthostachys (Ophioglossaceae)*. He used it once more for another plant in 1848 [15], this name has not been used any more by later authors. As this name had already been used for recent plants, POTONIÉ [16] distinguished the extinct genus by addition of the letter p: *p. Botryopteris;* also in *p. Callipteris* and other cases. This proposal has not been accepted by other authors.

The name *Thamnopteris* was first used by PRESL [17] for a subgenus of *Asplenium*. In 1849 he raised this group to a generic rank [18]; but most following authors continued to consider the group as a subgenus of *Asplenium* only. In 1849 BRONGNIART [19] gave the same name to a fossil stem, which afterwards appeared to belong to the *Osmundaceae*. It is not clear which of both names has been published first as a generic name; in its oldest sense, however, is not used as a generic distinction any more; the name for the fossil plants is still valid. As the rules of nomenclature for fossil plants have not yet been established, this question may be postponed until further consideration.

The name *Grammatopteris*, used at first for fossil plants by RENAULT in 1891 [20], has afterwards been used for recent ferns of the Dutch East-Indies [21]; the latter therefore have to be renamed.

In another case there is a great resemblance between a generic name of a fossil plant and a recent one. In 1818 DESFONTAINES [22] gave the name *Mezoneurum* (sometimes even written *Mezoneuron*) to a genus of the *Caesalpiniaceae*; in 1856 UNGER [23] used nearly the same name, *Mesoneuron*, to design a piece of a stem of unknown affinities, from the Upper Devonian or Lower Carboniferous of Thuringia. It will be best, also in this case, to postpone this question unto further consideration.

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[11]) STOPES AND WATSON, Trans. Roy. Soc., London, B, CC, 1908.

[12]) POSTHUMUS, Proceedings Roy. Ac., Amsterdam, 27, 1924, p. 836.

[13]) RENAULT, Ann. scienc. nat., Bot., (6) I, 1875, p. 223.

[14]) PRESL, Rel. Haenk., I, 1825, p. 76.

[15]) PRESL, Abh. kön. böhm. Ges. der Wiss., (5) V, 1848, p. 324.

[16]) POTONIE, Naturwiss. Wochenschr., XV, 1900, p. 420

[17]) PRESL, Tent. Pterid., 1836, p. 105.

[18]) PRESL, Epim. Bot., 1849, p. 68.

[19]) BRONGNIART, Tableau genres Végét. foss., 1849, p. 38.

[20]) RENAULT, Note Botryopteridées, 1891, p. 16.

[21]) v. ALDERWERELT VAN ROSENBURGH, Bull. Jard. Bot. de Buitenzorg, (3) IV, 1922, p. 318, pl. 15.

[22]) DESFONTAINES, Mém. Muséum, Paris, IV, 1818, p. 245, pl. 10, 11.

[23]) UNGER, Sandstein und Schieferflora, 1856, p. 172.

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Pasoeroean, November 1927.

Chemistry. — *The Reduction of α-Eleostearic Acid. (The Linoleic Acid 10.12 and the Oleic Acid 11.)* By J. BÖESEKEN and J. VAN KRIMPEN.

(Communicated at the meeting of January 28, 1928).

In a paper dealing with the α-eleostearic acid [1]) one of us in conjunction with Mr. J. HOOGLAND has communicated the preliminary results of the catalytic reduction of this acid, in which they came to the conclusion that the conjugated system of three double bonds was attached according to the principle of THIELE, so that first the octadecadeënoic 10—12 acid would be formed, and then the octadecenoic 11-acid.

For, on addition of one mol. of H_2 a compound, was obtained, which, judging from its behaviour towards the solution of WIJS, behaves as a substance with a conjugated system of double bonds, which on further reduction gave a *profuse yield* of the just-mentioned octadecenoic-11-acid (ester).

Now, however, the possibility is not excluded that on taking up the first molecule of hydrogen, one of the end-placed double bonds, either 9 or 13, was hydrogenated, so that the first obtained linoleic acid would be the octadecadeënoic 11 . 13 or the octadecadeënoic 9 . 11-acid, and that on further reduction this has given the octadecenoic-11-acid.

In order to get perfect certainty about the course of the reduction, it was, therefore, necessary, to separate the first-obtained acid, and to determine its constitution.

Besides, the octadecenoic -11-acid had not yet been obtained pure; it contained 5 % stearic acid, and the relatively high melting point 53° was possibly owing to this impurity.

Through improvement of the hydrogenation method, we have succeeded in obtaining the first reduction product of the α-eleostearic acid aethylester pure, and separating from this an acid with a conjugated system melting at 28.5°, and fixing the position of the double bonds in this acid by ozonisation at 10 and 12.

From this a good yield of *sebacic acid* and *caproic acid* was obtained, the middle part being converted to a syrup, the constitution of which we have not yet verified.

We point out here that this acid is isomeric with an acid [1]) that was obtained on distillation of ricinus-elaidic acid, and melts at 53°. This latter has been examined by Mr. W. C. SMIT, who, by determination of the refraction of the ester and from its behaviour towards the solution of

[1]) Recueil des trav. chim. **46**, 619 (1927).

WIJS, has derived that also this acid possesses a conjugated system. On ozonisation of this a good yield of *azelaic acid* and *heptoic acid* was obtained, while here too the middle two C-atoms have so far not yet yielded a definable compound in considerable quantities. The difference between this acid and that which was obtained on reduction of α-eleostearic acid is therefore confined to the position of the conjugated system, which in this latter acid lies one place further from the carbonyl group.

When the wood oil was hydrogenated with a third of the calculated quantity of hydrogen, the product, investigated according to BERTRAM's method [1]) appeared to contain no stearine. This means that this partial reduction takes place very selectively, as on one side the final product has not been reached, on the other side the about 8 % of oleine, which the wood-oil contains, have been left entirely intact.

Besides it appears from this that our initial product was free from stearic acid.

After the fixing of $^2/_3$ of the calculated quantity of hydrogen, stearine was, indeed, found, which might have been formed because the oleine was now attached, and the 11-acid is now further reduced.

As we stated above, this 11-acid separated already before was, at first, not obtained pure, it still contained 5 % stearic acid. In fact the acid once recrystallized from alcohol had a melting point of 52°—53°. If it was recrystallized from chloroform and if the first high-melting fractions were removed, the mother liquor appeared to contain a lower-melting compound, which, after repeated recrystallisations from the same solvent, melted constantly at 38°.5. We got the impression that this acid was the only component with one double bond, save the ordinary oleic acid present at the beginning.

By means of ozonisation and melting experiments the 11-oleic (elaidic) acid was found to be identical with the acid separated by S. H. BERTRAM from animal fats and called by him *vaccenic acid*. It is undoubtedly very remarkable that this is found in natural fats, and the question suggests itself whether there might be any connection between the reduction product of wood-oleic acid occurring in the vegetable kingdom, and this product of animal metabolism. The experimental details will be communicated elsewhere, but some constants way be given here.

a-Eleostearic acid. Melt. p. 45° (octodecatrienic-acid 9, 11, 13) was esterified and the aethylester was distilled in cathode-vacuum.

Boil. p. 169—170°.5, refraction at 15° of two separately obtained fractions 1.5043 and 1.5086.

HOOGLAND has also found in three fractions at 15° 1.5042, 1.5064, and 1.5080.

For the present it cannot be decided whether these relatively large differences still point to the presence of stereo isomers; the ester is made

[1]) Chem. Weekblad **24**, 226 (1927).

from α-eleostearic acid by the aid of alcohol and *hydrochloric acid*, so that an isomerisation is not excluded.

Octodecadienic acid-ethylester obtained by $^1/_3$ hydrogenation of the α-eleostearic acid ester.

Refraction: $n_D^{15°} = 1.4746$. Mol. Refr. $= 96.87$.
Calculated $= 95.28$, Exatation $= 1,59$ } It follows from this that there is
Iodine value $= 109$, instead of 180. } a conjugated system in this acid.
The free acid melts at $28°.5$; $n_D^{70°} = 1.4639$.

This ester, as well as the wood oil itself was hydrogenated for $^2/_3$ part, the product obtained was saponified, and the acids were recrystallized, the largest fraction melted at $38°.5$, showed no changes any more, yielded caproic acid and sebacic acid on ozonisation and presented no lowering of the melting-point with the *vaccenic acid* of BERTRAM (see above) $n_D^{70°} = 1.4432$.

Delft, January 1928.

Mathematics. — *A Group of Null Systems.* By Prof. JAN DE VRIES.

(Communicated at the meeting of Januari 28, 1928).

§ 1. In a null system (α, β, γ) a point has α nullplanes, a plane has β nullpoints and a line is γ times a nullray. In a paper "On Bilinear Null Systems" (These Proceedings **15**, 879) I have proved the existence of nullsystems $(1, 1, \gamma)$ for any value of γ and I have derived their properties. In a paper "Null Systems that are Defined by Two Linear Congruences of Rays" (These Proceedings **21**, 309) I have considered nullsystems $(1, pq, p+q)$ that are defined by two congruences of rays $[1, p]$ and $[1, q]$. Now I shall consider a group of nullsystems that are characterised by the symbol $(1, 2n, n+2)$.

§ 2. We shall consider as given the congruence $[k^3]$ of twisted cubics that pass through 5 points B_k and the congruence of rays $[1, n]$ formed by the lines t resting on the twisted curve α^n and on the line a which cuts α^n in $(n-1)$ points A_k.

Through a point M there pass one k^3 and one ray t; let r be the straight line that touches k^3 at M; we shall associate the plane rt to M as *nullplane* μ.

The points of contact R of the k^3 that touch a plane μ, lie on a conic ϱ [1]); its points of intersection with the n rays t in μ are the *nullpoints* of μ. Hence $\beta = 2n$.

The rays t that cut a line l, form a scroll of the degree $(n+1)$: the points of contact R of the tangents r that rest on l, lie on a cubic surface through l. Besides l the two surfaces have a curve of the order $(3n+2)$ in common that is formed by the nullpoints M of the planes μ through l. This curve cuts l in $(n+2)$ points M; hence $\gamma = n+2$ [2]).

§ 3. A point M for which r coincides with t, has a pencil of nullplanes and is, therefore, *singular* for the nullsystem. I shall indicate such a point by S; let the axis of the pencil (σ) of the nullplanes be indicated by s. The lines s form a scroll: this is the intersection of the congruence $[1, n]$ and the complex of the tangents r to the curves k^3.

[1]) The plane $B_1B_2B_3$ contains a pencil (k^2) of conics each of which forms a composite k^3 together with B_4B_5. Two of these k^2 touch the plane μ.

[2]) For $n = 1$ we find a null system $(1, 2, 3)$; I have treated its properties in these Proceedings **26**, 124.

As this is a complex of the degree *six* [1]), the lines s form a scroll of the degree $6(n+1)$. The *curve* λ^{3n+2} (§ 2) corresponding to l has, therefore, $6(n+1)$ points S in common with the *curve* (S) of the singular points.

In order to determine the order of (S) we shall consider the congruence $[r]$ of the tangents r that have their points of contact R in a plane φ. The points R of which the lines r meet in a point P, lie on a twisted curve of the order 7. A plane π contains two lines r; their points of contact are the points R in φ of the conics ϱ^2 in π (§ 2). The congruence $[7,2]$ of the lines r has $(2n+7)$ rays s in common with the $[1, n]$ of the rays t; accordingly the locus of the *singular points* is a curve of the *order* $(2n+7)$.

It contains the 5 base points B_k and the 10 points D each of which is the intersection of a plane $B_l B_m B_n$ and the line $B_p B_q$. Through each of these 15 points there passes one k^3 that touches a ray t. The plane $B_1 B_2 B_3$ contains four points D: they may be indicated by (12,345), (13,245), (23,145), and (45,123). Besides these four points and the three base points this plane contains $2n$ points S; they lie in pairs on the n rays t; for each of these is touched by two conics k^2 that are component parts of composite k^3.

§ 4. The nullpoints M of the planes μ that pass through a point P, lie on a surface $(P)^{n+3}$, for any ray through P is a nullray for $(n+2)$ of its points.

The surfaces $(P)^{n+3}$ and $(Q)^{n+3}$ have in common: the curve λ^{3n+2} defined by PQ, the curve $(S)^{2n+7}$, the curve α^n and the line a. Any point of α^n carries a pencil (t), hence a pencil (μ); any point of a is the vertex of a cone $(t)^n$, hence nullpoint of ∞^1 planes μ each of which contains n rays t. Accordingly a is an n-fold line and α^n is a single curve on (P). In fact $(n+3)^2 = (3n+2) + (2n+7) + n + n^2$.

The cubic surface of the conics ϱ^2 in planes through l (§ 2) has $3n$ points in common with α^n; each of these singular points has a nullplane through l. Analogously a contains three singular points that have a nullplane through l. Hence λ^{3n+2} has $3n$ *points* in common with α^n and it contains three *n-fold points* on a.

A surface $(O)^{n+3}$ has in common with λ^{3n+2}: the $2n$ nullpoints of the plane OPQ, the $6(n+1)$ points S that lie on λ (§ 3), the $3n$ points on α^n and the three n-fold points of λ. In fact $(n+3)(3n+2) = 2n + 6(n+1) + 3n + 3n^2$.

The 10 planes $B_k B_l B_m$ are *singular nullplanes*; their nullpoints lie on the n lines t in that plane; these are *singular nullrays*. Also the 10 lines $B_k B_l$ are *singular nullrays*, for they may be considered as lines r.

[1]) Each of the rays through a point P is cut in two points Q by a k^3. The locus (Q) is a surface of the 4th degree with conical point P. Any plane through P contains 6 tangents of (Q) that meet in P. The locus of the points of contact is a twisted curve of the order 7.

§ 5. If we replace the congruence [1,3] that has the curve α^3 and one of its chords a as directrices, by the [1,3] of the *bisecants* of α^3, we find another nullsystem (1, 6, 5) [1]). Analogous considerations lead to a curve $(S)^{13}$ that cuts the curve λ^{11} in 24 points. The curve α^3 is double on the surface $(P)^6$ and λ^{11} has 9 double points on α^3.

[1]) Also the planes of osculation of the curves k^3 form with their points of contact a nullsystem (1, 6,5). Cf. STURM, Die Lehre von den geometrischen Verwandtschaften, IV, 469.

Mathematics. — *Representation of a Bilinear Congruence of Twisted Cubics.* By Prof. JAN DE VRIES.

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1. Let (α^2) be a pencil of quadratic surfaces of which the base curve consists of the curve α^3 and the line c, and (β^2) a similar pencil with basis β^3 and c. The intersection of an α^2 and a β^2 consists of a cubic ϱ^3 and the line c that cuts it twice.

Any ϱ^3 has one of its bisecants m through the fixed point M; any ray m of the sheaf about M is a bisecant of one ϱ^3; its points of intersection with ϱ^3 are the points of the common pair of the involutions in which (α^2) and (β^2) cut m. The congruence $[\varrho^3]$ is called *bilinear*, because an arbitrary point carries *one* ϱ^3 and an arbitrary line is a chord of *one* ϱ^3. As the *image* of a ϱ^3 we shall consider the point of intersection R of m and a fixed plane Φ.

The curve μ^3, that passes through M, is represented on the conic σ^2 along which the cone μ^2 that projects μ^3 out of M, cuts the plane Φ.

2. The locus of the pairs of points on the rays m is a surface μ^4 with a conical point M; the cone of contact is μ^2. The intersection of μ^4 and μ^2 consists of the curve μ^3 and a figure of the fifth order. If P is a point of this figure, MP is a bisecant of the ϱ^3 through P, but at the same time of μ^3, hence of ∞^1 curves ϱ^3. Accordingly this figure consists of five lines that are *singular bisecants*.

Two of these lines may be indicated at once. The surface α^2 through M contains a scroll to which c belongs; the line a of this scroll through M is cut by any surface β^2 in two points of a ϱ^3 of the congruence; it is, therefore, a *singular bisecant*. Analogously M carries a line b that is cut by the pencil (α^2) in an involution of which any pair belongs to a ϱ^3.

On each of the singular bisecants s_1, s_2, s_3 that also pass through M, (α^2) and (β^2) define *the same* involution. The locus of the curves ϱ^3 that have s_k as bisecant, considered as product of the pencils (α^2) and (β^2), which have become projective, is a surface Σ^4_k.

The points of intersection S_1, S_2, S_3, A and B of the singular bisecants and Φ are *singular points* for the *representation*, as each of them represents ∞^1 curves ϱ^3.

3. The system of the curves ϱ^3 resting on a line l, is represented in the points of a rational curve λ. As l contains four points of Σ^4_k, S_k is a

quadruple point of λ; analogously A and B are *double points*. With σ^2 λ has only the 5 singular points in common, as l does not generally cut the curve μ^3. Consequently λ and σ^2 have $3 \times 4 + 2 \times 2$ or 16 points in common; λ is, therefore, a curve of the *order eight*. It has a third double point in the image of the ϱ^3 that has l as bisecant.

Two curves λ^8 have $8^2 - 3 \times 4^2 - 2 \times 2^2$ or 8 non-singular points in common. Accordingly on any *two lines* there rest *eight* curves ϱ^3 and the curves that are cut by l, form a surface Λ^8.

The surface β^2 through a point of α^3 contains all ϱ^3 that the pencil (α^2) has in common with β^2. Hence Λ^8 has the base curves α^3 and β^3 as *double curves*.

Two surfaces have 8 curves ϱ^3 and the double curves α^3 and β^3 in common. Hence c is a 16-fold line of the intersection; c is, therefore, a *quadruple line* and the curves ϱ^3 that cut c in a point C, form a surface Γ^4 [1]). Γ^4 has in common with Λ^8 4 ϱ^3, the curves α^3 and β^3 that must be counted twice, and the line c; this line is, accordingly, a *double line* of Γ^4.

Any curve ϱ^3 has 24 points in common with Λ^8; 8 of them lie in the points where it rests on c; the remaining 16 must lie on α^3 and β^3; consequently ϱ^3 cuts each of these base curves *four times*.

4. A surface Γ^4 has two pairs of points in common with s_k; accordingly the image curve γ of the system of curves ϱ^3 on Γ^4 has three double points S_k. Γ^4 has two points of α^3 and a pair of points of a ϱ^3 in common with the singular line a; hence γ passes through A and through B. It can only have singular points in common with σ^2 and is, therefore, a curve of the order *four*.

The curves $\gamma^4(A, B, S_k^2)$ and $\lambda^8(A^2, B^2, S_k^4)$ have four non-singular points in common; this proves again that c is cut in any of its points by four ϱ^3.

Two curves γ^4 have two non-singular points in common; through *any two* points of c there pass, therefore, *two* ϱ^3.

5. A plane through M cuts a surface α^2 along a conic which the pencil (β^2) cuts in a point C and a cubic involution. The pairs of this involution lie on the tangents of another conic; hence through M there pass two bisecants of curves ϱ^3 on α^2. Consequently the image curve of the system is a conic a^2.

a^2 can only have singular points in common with σ^2. In fact s_k and b are bisecants of ϱ^2 lying on α^2. Accordingly the *image curves* a^2 form a *pencil* with base points S_1, S_2, S_3 and B.

[1]) This surface is produced by the projective pencils (α^2) and (β^2) in which two homologous surfaces touch each other in the point C. Hence Γ^4 has a triple point in C and is a monoid.

Analogously the curves b^2 through S_k and A represent the systems that lie on the surfaces β^2.

The fourth point of intersection of an a^2 and a b^2 is the image of the ϱ^3 that is defined by the corresponding α^2 and β^2. Especially the figures $(S_1\, S_2, S_3\, A)$ and $(S_1\, S_2, S_3\, B)$ define the *system* of a *conic* δ^2 and a *line* d cutting it.

This proves that the planes δ of the conics δ^2 form a system with index 3. Accordingly through any point of α^3, β^3 or c there pass three δ^2 and α^3, β^3 and c are triple lines on the surface $\triangle$ formed by the curves δ^2.

6. A plane through c contains one line d that rests on α^3 and β^3 and, therefore, belongs to a composite figure ϱ^3.

α^3 and β^3 are projected out of a point C by two cubic cones that have five lines d in common besides the double generatrix c. Hence the locus of the lines d is a *scroll* $(d)^6$ with *quintuple line* c containing the curves α^3 and β^3.

Through a point of α^3 there pass three δ^2 and one d; accordingly on a β^2 there lie four figures (δ^2, d). The image curve of the system of the (δ^2, d) has, therefore, quadruple points in A and in B. It has a double point in S_1, for each of the planes $S_1\, S_2 M$ and $S_1\, S_3 M$ contains one δ^2. Consequently the *image curve* is a $\delta^7\,(A^4, B^4, S_k^2)$; for it can only have singular points in common with σ^2.

It has 16 points outside σ^2 in common with $\lambda^8\,(A^2, B^2, S_k^4)$; hence the figures (δ^2, d) form a surface of the 16th degree consisting of the scroll $(d)^6$ and a $\triangle^{10}$ with *triple* lines α^3, β^3 and c.

$\delta^7\,(A^4, B^4, S_k^2)$ and $\gamma^4\,(A, B, S_k^2)$ show again that a monoid Γ^4 contains eight figures (δ^2, d).

The surfaces $\triangle^{10}$ and $(d)^6$ have a figure of the order 27 in common besides α^3, β^3 and c; it consists of 9 ϱ^3 degenerated in three parts. In fact α^3 and β^3 have nine bisecants in common besides c and each of these belongs to a figure consisting of *three lines*.

7. The intersection of the surface μ^4 (§ 2) and a plane through M is a c^4 with double point M; six of its tangents r meet in M.

The curves ϱ^3 that have a tangent through M, form a surface O; the image curve of this system is an r^6 with double points in A, B and S_k, for any singular bisecant is touched by two ϱ^3.

From the number of non-singular points of intersection of $r^6\,(A^2, B^2, S_k^2)$ and $\gamma^4\,(A, B, S_k^2)$ and $a^2\,(B, S_k)$ it appears that O has the curves α^3 and β^3 as *quadruple* lines and the line c as *eightfold* line.

r^6 together with $\lambda^8\,(A^2, B^2, S_k^2)$ prove that the surface O is of the *sixteenth* degree.

8. The curves ϱ^3 that touch a plane ω, form a surface Ω. In the inter-

section of an α^2 and ω the surfaces β^2 form an involution I^3; as this has 4 double points, α^2 contains four ϱ^3 of the system Ω. Hence the line a is a chord of four of the ϱ^3 and A and B are quadruple points of the image curve.

On the intersection of ω and the surface Σ_k^4 the projective pencils (α^2) and (β^2) also form a system of point triples; as the genus of this curve is two, there are in this case 8 double points. Consequently S_k is eightfold on the image curve.

This has, therefore, 32 points in common with σ^2 and is an ω^{16} (A^4, B^4, S_k^8).

It has 16 points outside σ^2 in common with λ^8 (A^2, B^2, S_k^4); accordingly the curves that touch ω, form a surface Ω^{16}. On this α^3 and β^3 are *quadruple* curves; for ω^{16} has four non-singular points in common with $a^2 (B, S_k)$.

ω^{16} and $\gamma^4 (A, B, S_k^2)$ prove together that the line c is *eightfold*.

ω^{16} and $\delta^7 (A^4, B^4, S_k^2)$ prove, that Ω^{16} contains 32 figures ϱ^3 degenerated in two parts.

Botany. — *Zur Klärung des Xerophytenproblems.* By A. SEYBOLD.
(Communicated by Prof. Dr. F. A. F. C. WENT.)

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Die Wasserökonomie der xeromorphen Pflanzen, die in einer Umgebung zu leben vermögen, die sich durch grossen Dampfhunger kennzeichnet, ist in den letzten Jahren häufig untersucht worden. Alle Untersuchungen stimmten in dem Ergebnis überein, dass die SCHIMPER—WARMING'sche Xerophytentheorie der eingeschränkten Transpiration gegenüber den mesophytischen Pflanzen nicht zu recht bestünde und an ihre Stelle trat die Theorie der Dürreresistenz der Xerophyten, die hauptsächlich von dem Russen MAXIMOV begründet wurde. Der Hauptinhalt der Theorie stellt sich folgendermassen kurzgefasst dar: Xeromorphe Pflanzen vermögen mittels hoher Saugkräfte mit einem gut ausgebildeten Wurzelsystem und Reduktion der Transpirationsfläche dem grossen Dampfhunger der Atmosphäre ohne Schädigung zu widerstehen, eine Einschränkung der Transpiration pro Flächeneinheit weisen sie den Mesophyten gegenüber nicht auf, vielmehr kann ihre Transpiration flächenrelativ höher sein.

Demnach müsste die Xeromorphie mit anderen als transpirationsphysiologischen Verhältnissen in Zusammenhang gebracht werden. Die anatomisch-histologischen Strukturen xeromorpher Blätter hätten für die Wasserdampfabgabe keine Bedeutung, was allein vom physikalischen Standpunkte aus sehr sonderbar erscheint.

Im Rahmen einer eingehenden Transpirationsanalyse der Pflanze auf physikalischer Grundlage ergaben sich für das Xerophytenproblem neue Perspektiven, die zur Klärung dieser strittigen Frage beitragen dürften.

Die scharfe Scheidung, die SACHS für die Transpirationsanalyse macht: Die Transpiration aus Zellen und Geweben wird durch äussere und durch innere Ursachen und Bedingungen hervorgerufen und verändert, erweist sich auch hier äusserst fruchtbar. Vergleiche zwischen Meso- und Xerophyten können nur unter denselben äusseren Bedingungen einwandfrei angestellt werden. Der Oekologe, der die Pflanzen am natürlichen Standorte beobachtet, darf sich über diese fundamentale Forderung ebenso wenig hinwegsetzen wie der Laboratoriumsphysiologe. Dass eine xeromorphe Pflanze in dampfdruckarmer Luft flächenrelativ mehr transpirieren kann als eine mesophytische in ± stark dampfgesättigter, ist selbstverständlich. Dass aber die xeromorphen Blattstrukturen allein einem durch das xerophytische Klima physikalisch bedingtem Wasserdampfaustausch ohne Schädigung nachkommen können, wird kaum von der Hand zu weisen sein. Vom physikalischen Standpunkte aus muss eine Verdickung der Kutikula, ein starker Wachsüberzug, Haarbildungen u.s.w. die kutikuläre Transpiration erniederigen. Durch die Ausbildung solcher histologischer Elemente kommen aber die Stomata häufig in Lagen, denen ein relativ hoher Dampfdruck eigen sein muss. Das gilt für alle mehr oder weniger tief eingesenkten Stomata.

Werden Mesophyten mit Xerophyten verglichen, so muss das mit anderen Worten heissen: Wie reagieren die Transpirationssysteme der Mesophyten und die der Xerophyten auf mesophytisches und wie auf xerophytisches Klima? Dieser Umstand ist völlig ausser Acht geblieben und mit einer Umrechnung auf dasselbe Sättigungsdefizit ist ein einwandfreier Vergleich keineswegs statthaft, wie es beispielshalber MAXIMOV in seiner grundlegenden Arbeit getan hat. Die Verdunstung ist nicht schlechthin einer der äusseren Bedingungen (Temperatur, oder falscherweise dem Dampfdruckdefizit) proportional, sondern sie als mehrgliederige Funktion der physikalischen Zustände zu betrachten.

Pflanzen, die in xerophytischem Klima zu vegetieren vermögen, was ihnen durch xeromorphe Strukturen möglich ist, sind demnach prinzipiell von den Mesophyten unterschieden, die in einem ausgesprochen xerophytischen Klima nicht zu leben vermögen. Die mesophytische Pflanze hat eine relativ sehr hohe Kutikulartranspiration dem Xerophyten gegenüber und schon aus diesem Grunde wäre ihre Transpiration in xerophytischem Klima ungleich höher, wenn sie den starken Wasserverlust der Dürreresistenz decken könnte, was eine Frage der Leistungsfähigkeit des Wurzel- und Wasserleitungssystems ist.

Hier möge nur ein klimatischer Faktor in Betracht gezogen werden, der zweifelsohne eine der wichtigsten Funktionen der Wasserdampfbewegung bei den Xerophyten am natürlichen Standorte ist, die Luftbewegung und zwar mit geringem Dampfdruck, also landläufig gesprochen: trockener Wind. Da an anderer Stelle ein gut fundierter Beweis gegeben werden wird, dass die kutikuläre Transpiration in erster Linie durch den Wind eine Steigerung erfährt, den Gesetzmässigkeiten der Verdampfung relativ grosser Flächen folgend, begnügen wir uns hier mit den physiologischen Tatsachen, die aus einer Reihe von Experimenten gewonnen wurden. Vermöge der ungewöhnlich starken Kutikula ist die kutikuläre Transpiration gleich Null zu setzen, die Verdunstung findet also lediglich durch die Stomata statt. Nun ist aber von der grössten Bedeutung zu erfahren, dass die stomatäre Transpiration der Xerophyten durch den Wind überhaupt keine Steigerung erfährt, die täglichperiodische Spaltenapertur wird durch den Wind in keiner Weise beeinträchtigt, was für die Beurteilung des Gesamt-Gas-Austausches von Bedeutung ist. Die Assimilation erfährt somit durch eine Spaltenverengung, die bei starkem Wind eintreten könnte, keine Hemmung, da der CO_2 Diffusion keine Widerstandserhöhung auferlegt wird. Anders dagegen liegen die Verhältnisse bei den Mesophyten. Starker Wasserverlust bedingt Turgorerniederigung in den Schliesszellen der Stomata, d.h. die Spalten erfahren eine Verkleinerung. In zweierlei Hinsicht ist also das Verhalten der xeromorphen Systeme von Bedeutung. Die Transpiration wird im W i n d e nicht gesteigert, und die Spaltenapertur erleidet keine Veränderung. Damit wird die CO_2-Diffusion nicht gehemmt, rein der relativen Grösse der Gesamtporenfläche gemäss. Die Mesophyten aber, die im Winde ± rasch einen Spaltenschluss durch Turgorerniederigung eintreten lassen, um allzugrossem, schädlichem Wasserverlust vorzubeugen, erhöhen damit den Diffusionswiderstand für CO_2 im Sinne der Flächenverkleinerung. Damit ist eine physiologische Erklärung möglich, dass die Xerophyten in xerophytischem Klima zu leben vermögen, nicht aber die Mesophyten. Es frägt sich dann nur noch warum im allgemeinen die Xerophyten nicht ebenso häufig in mesophytischem Klima leben. Diese Frage nur nach dem Stande der Wasserbilanz zu beurteilen, wäre hinsichtlich der grossen Zahl der determinierenden Faktoren zu gewagt, doch können folgende Momente mit in Rechnung gesetzt werden. Die Geschwindigkeit des Wasserstromes wird bei Xerophyten im mesophytischen Klima, dem ein relativ hoher Dampfdruck eigen ist, verzögert, die Stoffwechselvorgänge werden also in erster Linie in Mitleidenschaft gezogen, die Wachstumsgeschwindigkeit aber korrelativ benachteiligt. Die klimatischen Faktoren wirken ohne Zweifel selektiv auf die zur Keimung kommenden, zufälligen Samenaggregate; mesophytische Pflanzen wachsen im mesophytischen Klima rascher gross als xerophytische. Damit soll keineswegs geleugnet werden, dass Xerophyten ganz und gar nicht in mesophytischem Klima leben könnten, ebensowenig, dass eine Pflanzenart nicht mit Standortsmodifikationen auf äussere Induktionen sich zu ändern vermöchte. Grundlegend erscheint uns aber die grosse Wirkung der Transpirationsforderung des t r o c k e n e n W i n d e s bei mesophytischen Strukturen, seine Inaktivität bei der Wasserbilanz der Xerophyten.

Die beigefügten Tabellen geben kontinuierliche Gewichtsverluste als Ausdruck des Transpirationsstromes in bestimmter Zeit wieder. Hier sei von relativen Berechnungen verschiedener Pflanzen abgesehen, da es nur darauf ankommt zu zeigen, dass die Transpiration im Winde bei dem xeromorphen N e r i u m O l e a n d e r keine Steigerung erfährt, wohl aber bei D a t u r a s u a v e o l e n s, die typisch mesomorph ist, infolge starker Kutikulartranspiration. Die physikalischen Aussenbedingungen sind während eines Versuches nahezu gleich, die Angaben sind den Tabellen beigefügt. Um aber wirklich unter sich vergleichbare Werte zu bekommen, sind die unter Wasser abgeschnittenen Sprosse, die in Wassergefässen abgedichtet standen, zum Teil zu gleicher Zeit dem Winde ausgesetzt worden, zum andern denselben Aussenbedingungen, dabei aber in ruhiger Luft verweilend. Intermittierend standen die Pflanzen in Wind und Ruhe. Die Gewichtsverluste die im Winde eintraten sind in den Tabellen fettgedruckt. Die Ursachen der Schwankungen sollen hier nicht diskutiert werden,

NERIUM OLEANDER.

Mittlere Temperatur 30°. Mittl. rel. Feuchtigk. 25 %. 2 Tageslichtlampen 500 Watt/220V.

Zeit	Pflanze 1	2	3	4	5
Gewichtsverlust in mg von:	Windgeschwindigkeit 1.7 m/sec.				
12—13h	**0.690**	**0.340**	0.510	0.765	0.440
13—14	**0.255**	**0.240**	0.255	0.280	0.230
14—15	0.215	0.210	**0.305**	**0.190**	0.230
15—16	0.290	0.370	**0.305**	**0.125**	0.260
16—17	**0.260**	0.370	0.345	0.135	**0.210**
	Windgeschwindigkeit 5.6 m/sec.				
17—18	**0.250**	0.380	0.295	0.160	**0.290**
18—19	0.280	0.290	0.330	0.150	0.345
19—22 Mittelw. pro Stunde	**0 303**	**0.247**	0.317	0.210	0.330
22—8 Mittelw. pro Stunde	0.137	0.087	**0.182**	0.245	**0.139**

DATURA SUAVEOLENS.

Mittlere Temperatur 23°. Mittlere Feuchtigk. 40 %.

Zeit	Pflanze 1	2	3	4	
	mg Gewichtsverlust				
12^{05}—12^{20}	0.270	0.190	0.145	0.080	
12^{20}—12^{35}	**0.600**	**0.370**	0.140	0.090	
12^{35}—12^{50}	0.290	0.190	**0.340**	**0.160**	
12^{50}—13^{05}	0.420	0.240	0.230	0.150	
13^{05}—13^{20}	**0.550**	**0.310**	0.220	0.140	Windges. 1.7 m/sec. Sonnenlicht
13^{20}—13^{35}	0.260	0.180	**0.400**	**0.210**	
13^{35}—13^{50}	**0.310**	**0.180**	0.280	0.110	
13^{50}—14^{05}	0.140	0.110	**0.340**	**0.160**	
14^{05}—14^{20}	0.150	0.110	0.170	0.110	
14^{20}—14^{35}	0.170	0.140	0.210	0.130	

die Werte veranschaulichen deutlich, dass die Transpiration bei Nerium vom Winde nicht beeinflusst wird, der Gang der Transpiration ist von ihm völlig unabhängig. Anders bei Datura. Im Winde wird die Verdunstung $\pm$ stark gefördert, wenngleich auch andere Faktoren bei der absoluten Grösse der Transpiration eine nicht untergeordnete Rolle spielen.

Absolute Vergleiche beider Pflanzen sind ohne umständliche Berechnungen nicht möglich. Um dies zu umgehen werden bei künftigen Untersuchungen, die vor allem die Maxima der Transpirationsleistungen bei verschiedenen Pflanzentypen klarlegen sollen, Meso- und Xerophyten nebeneinander im Experimente behandelt, soweit dies bei extremen Aussenbedingungen mit Mesophyten sich verwirklichen lässt. Derartige Versuche sind bereits in Angriff genommen, worüber später zu berichten sein wird.

Utrecht, Februari 1928.

KONINKLIJKE AKADEMIE VAN WETENSCHAPPEN
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(Translated from: "Verslag van de gewone vergaderingen der Afdeeling Natuurkunde", Vols. XXXVI and XXXVII)

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Mathematics. — *On the Limits of Holomorphic Functions.* By Prof. J. WOLFF. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of December 17, 1927).

FATOU's theorem: "if $f(z)$ is holomorphic and limited for $|z| < 1$, $\lim\limits_{\rho \to 1} f(\varrho e^{\varphi i})$ exists, except perhaps for values of φ the set of which has the measure zero", has been completed by the brothers RIESS in the following way (Stokholm Congress 1916): "If a is an arbitrary number this limit is equal to a only for a φ-set of the measure zero".

In the following note we shall derive this result without making use of integrals of LEBESGUE.

Without loss of generality we may assume $f(0) \neq a$. We suppose $|f| < M$. We have for $0 < \varrho < 1$

$$2\pi \lg |f(0) - a| \leqq \int_0^{2\pi} \lg |f(\varrho e^{\varphi i}) - a| \, d\varphi \quad . \quad . \quad . \quad . \quad (1)$$

Suppose $1 > \varepsilon > 0$ and let E_n be the set of intervals in which

$$\left| f \left\{ \left(1 - \frac{1}{n}\right) e^{\varphi i} \right\} - a \right| < \varepsilon.$$

If μE_n is the measure of E_n it follows from (1) and from $|f - a| < 2M$, that

$$2\pi \lg |f(0) - a| \leqq \mu E_n \,.\, \lg \varepsilon + 2\pi \lg (2M),$$

hence $\quad \mu E_n \leqq \dfrac{C}{\lg \frac{1}{\varepsilon}}, \ n = 1, 2, \ldots; \qquad C$ constant.

Now the values of φ for which $\lim\limits_{\rho \to 1} f(\varrho e^{\varphi i}) = a$, belong to the limes inferior of E_n for $n \to \infty$.

Hence the measure of the set of these values is at most $\dfrac{C}{\lg \frac{1}{\varepsilon}}$ and as ε may be chosen arbitrarily between 0 and 1, this measure is zero.

Utrecht, December 6, 1927.

Geology. — *Measurements on slickensides on planes of stratification in folded regions.* By W. NIEUWENKAMP. (Communicated by Prof. L. RUTTEN.)

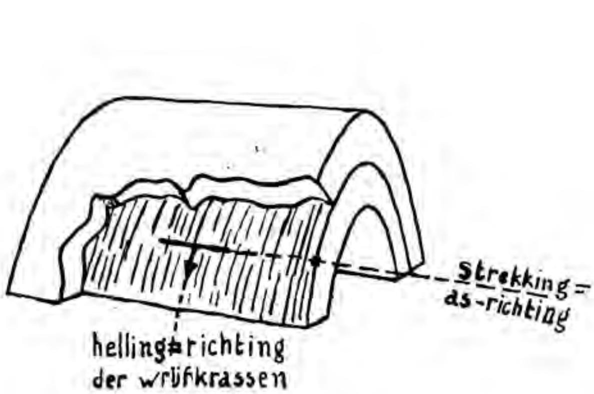
(Communicated at the meeting of January 28, 1928).

In folding the strata in the limbs of an anticline are pushed over each other. The younger strata push forward over the older ones toward the middle, toward the axis of the anticline. The truth of this statement is easily seen by every one who reproduces an anticline by means of a pack of cards. There is no reason why this movement should not in beds of suitable hardness, originate slickensides on the planes of stratification and as it is to be expected at right angles with the axis of the anticline the striae thereupon can furnish us an indication for the pitch of the latter. For when the axis of the fold is horizontal (parallel to the strike) the striae will follow the dip (Fig. A) when however the axis of the fold is inclined (when the fold has pitch) the striae will not follow the dip, deviating from it with an angle that increases with the amount of pitch (Fig. B).

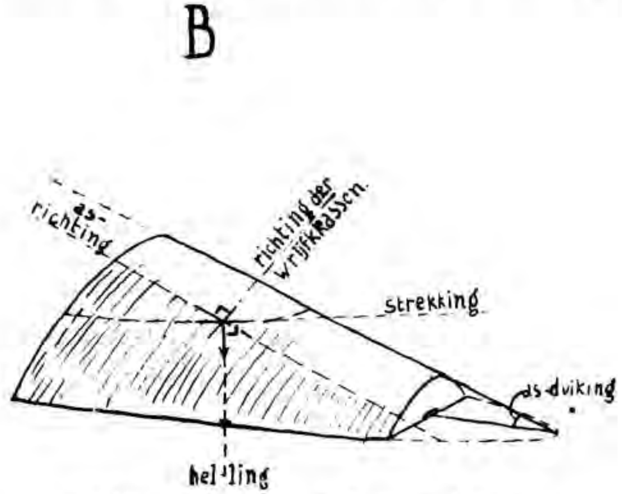
This would enable us to infer the pitch of the axis of folding from the direction of the striae in these slickensides. For if we draw on a plane of stratification (the axis runs parallel to the planes of stratification) a line at right angles with the striae, this line must be parallel to the axis of folding; so that by taking its inclination and azimuth we get pitch and direction of the latter.

In practice there are of course a number of causes that give to the striae

A



a direction differing from that required by this theory. Irregularities may engender a local deviation of the striae ; a fault just following the plane of



as-richting . . . . . . = direction of the axis,
richting der wrijfkrassen = direction of the striae,
strekking . . . . . . = strike,
helling . . . . = dip,
as-duiking . . . . . . = pitch.

stratification renders all our observations worthless for the object we have in view. Still it may be useful in our fieldwerk to have regard to the slickensides on the planes of stratification. An average of a number of observations of one and the same fold that do not differ too much, is surely to be relied upon.

This theory is to some extent verified by the students of geology of Utrecht, who have made in the Ardennes a fair number of observations of slickensides ; when several observations made in different parts of the same fold yielded the same result, this was always in complete accordance with the structural feature of the fold.

Mr. I. SWEMLE who worked out the geological structure of the broad strip of the Upper-Lower and Middle Devonian system between Ourthe and Lomme has on my request paid special regard to the slickensides on the planes of stratification, and has been so kind as to allow me to communicate some of his observations in this paper.

The first observation was made near the station of Jemelle in the sandstone of Co_1. The strata here strike N 65° E, and dip 75° S. The striae on the slickensides deviate eastward from the dip, the angle between the striae and the dip, measured here on four different slickensides, amounts to 9°. Hence a line on a plane of stratification at right angles with the striae has an azimuth of N 63° E. and dips westward at an angle of 9°.

South of Jemelle on the road to Forrières two slickensides have been measured with quite the same results on the southern limb of the same anticline. The strike of the strata is here N 85° E, with a dip of 30° S. Again the striae deviate east of the direction of the dip at an angle of 30° ; thus giving for the axis an azimuth N 60° E, pitch 14° W.

Five hundred m. to the south of the preceding locality the beds strike N 75° E dip 40° S. The striae again deviate in the same direction at an angle of 17°, thus giving for the azimuth of the axis N 62° E, pitch 11° W.

These three observations all tend to show that the pitch of this anticline is about 11° in a direction S 60° W. This result is in complete agreement with the geological map.

Physiology. — *Radiated Vitamin B and Automatin action.* By Prof. H. ZWAARDEMAKER.

(Communicated at the meeting of January 28, 1928).

The oldest known vitamin is that of EIJKMAN, which protects from Beriberi, and has afterwards been termed Vitamin B. Its physiological effect, however, is still unknown. By mere accident I have been in a position to get acquainted with a special action exerted by the anti-Beriberi vitamin, an action which it ever possesses in a small degree, but which it obtains in a large measure through corpuscular radiation.

While searching for chemical bodies that might serve as mother-substance of our automatins, interesting myself in substances soluble in water and in alcohol, but not in ether, I was enabled to test JANSEN and DONATH's crystalline form of anti-beriberi vitamin. At first I had only a substance adsorbed to clay at my disposal delivered in the form of tabloids, of which usually 4 per day were given to a beriberi-patient ; afterwards Prof. JANSEN sent me the approximately pure substance in ampoules of which 1 mgrm was injected intravenously in cases of acute beriberi. The ampoules contained hardly any potassium ; the tabloids contained much of it, but by shaking them with alcohol of 96 %, only little potassium is transferred to the solution. In our experiments the concentration did, therefore, not exceed 4 mgr per litre.

In the first place I will demonstrate an ordinary automatin-effect with real automatin.

An eel's heart arrested by deprival of potassium, resumes its beats

Fig. 1. An eel's heart brought to a standstill by deprival of potassium and automatin, recovers its pulsation through automatin from another heart.

through addition of the automatin derived from another radiated heart after a latency of 3½ hours. It will be seen that the frequency augments gradually, and has in this case reached its maximum after ¼ of an hour. This pulsation continues for 16 hours.

Next I will show the insignificant effect of non-radiated vitamin. Only separate beats, no quick rhythm appears.

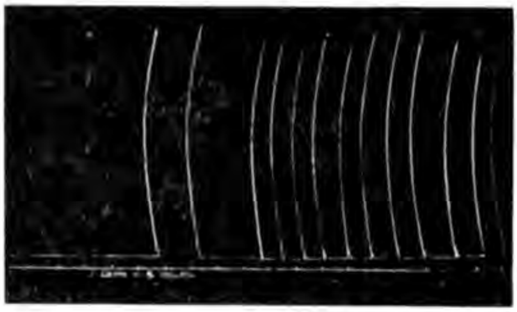


Fig. 2. Behaviour of a similar heart after adding non-radiated vitamine B to the circulating fluid.

This game is soon finished, after about two hours.

In the third place I will demonstrate the restoration of a test-heart by addition of radiated vitamin. After a latency of 1 hour intense contractions commence suddenly.

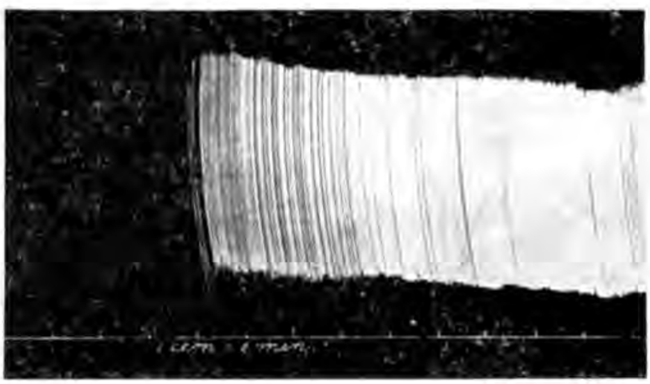


Fig. 3. Recovery of a similar heart after addition to the circulation of vitamin B that has been radiated beforehand during 15 hours.

The frequency gradually increases and reaches its maximum after 20 minutes. The regular beats continue for 13 hours.

Now it is clear that the form of the pulsations engendered by automatin are similar to those caused by vitamin B. In every experiment the latency depends on the time required for the preceding washing out of the potassium and automatin ; the frequency, on the other hand, depends on the concentration of the stimulating substance, whether this be automatin or vitamin B activated by radiation.

In this respect the time for activation through corpuscular radiation should not be too short. When using the weak rays of 5 mgr of Radio-element in enamel, 2 hours will be too short, 4 hours just sufficient, 2 $\times$ 12 hours better than 12 hours.

It will be easily understood that these experiments do not point to any connection between the vitamin B originating from the vegetable kingdom and the automatinogen of the animals. Otherwise we should rather be inclined to believe that vitamin ingested together with the food is transmitted to the blood and is stored up everywhere in the skeletal musculature. In that case it would then be exposed to the radiation of the tissue potassium and partially become automatin, which in its turn stimulates the various automatic organs.

But we know already that there are more substances, which — while remaining inactive if not radiated — are activated through corpuscular radiation to such an extent that they produce a pronounced automatin-effect, just as well as the automatinogen and the vitamin B; those substances belong to the compounds supposed by JANSEN and DONATH to be allied to their crystalline substance.

It is impossible that this number may be increased. This may caution us to be careful, but it does not affect the main point, viz. the fact that the vitamin B, used at Batavia for practical therapy against Beriberi, acquires automatin-action through radiation. This opens up a broad field for study, for now of course also the other beriberi symptoms have to be studied.

Up to now the number of experiments with vitamin B is about 30. In half of them the vitamin was radiated, the other half were made without radiation. We ever found the sharp contrast as described. In the present investigation I could avail myself of Dr. ZEEHUISEN's kind assistance. I also consider it as a pleasant duty to thank Professors EIJKMAN and JANSEN and Dr. DONATH for placing their preparations at my disposal.

Experimental Phonetics. — *Contributions to an experimental investigation of the Dutch language. II. oo, eu and ee followed by r or not* [1]).
By Miss L. KAISER. (Communicated by Prof. G. VAN RIJNBERK.)

(Communicated at the meeting of February 25, 1928).

Among vowels *oo* (*o*), *eu* (*ø*), and *ee* (*e*) take a peculiar position. This appears e.g. when we arrange the vowels in a so-called vowel-triangle: *aa* (*a*), oe (*u*), and *ie* (*i*) take the angles, *oo*, *eu* and *ee* being far away from them. The first mentioned are the so-called fundamental vowels, common to all primitive languages, the others are sounds that have only developed later. To pronounce oo, *eu* and *ee* the speech organs must assume a position that answers very definite requirements. To pronounce the so-called fundamental vowels, this is only the case to a less degree. In this connection it is worthwhile to compare the transition of *oo*, *eu* and *ee*, into the diphthongs *au* (*ɔu*), *ui* (*ʌy*) and *ei* (*εi*) in the careless pronunciation of the larger towns of Holland on the one hand as against the fact that in animal and instrumental sounds we may recognise *aa*, *oe* and *ie* often enough, but very seldom *oo*, *eu* and *ee*.

The influence of a following *r*-sound suffises to make these latter vowels lose their clearness. In Dutch in this case peculiar sounds are formed, which usually are compared or even identified with the short vowels: *o*, *u* and *i*. The vowels in *boor*, *beur* and *beer* are very different from those in *boot*, *beuk* and *beet*, and bear much resemblance to those in *bot*, *put* and *bit*, differing from these mainly or exclusively by their greater duration.

It is obvious that the way in which *r* itself is pronounced influences the phenomenon. To me it seems, that the altered vowels are most characteristic and constant before a lingual *r* (*r*), but that also with people who use a uvular *r* (*R*), this pronunciation is usual. Sometimes however, the vowels in that case are very much like the wide vowels o (ɔ), *u* (œ) and e (ε).

Concerning the articulation of *r* ZWAARDEMAKER [2]) stated, that in the pronunciation of *r* the muscles only fulfil the task of loosely raising the tongue, and that in the bottom of the mouth only then some tonus is noticeable, when a very strong relaxation has preceded. Hence after a vowel in which the bottom of the mouth is always tense, *r* must cause a relaxation. We may add that relaxation will take the more time and the more energy, as the vowel is pronounced with greater tension.

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[1]) From investigations made at the Physiological Laboratory of the Amsterdam University.

[2]) Nederl. Tijdschr. v. Geneesk. 1898. I. N⁰. 24.

As the characteristic tone of voiceless *r* STUMPF [1]) gives a^2—a^3 (870—1740 vibrations p.s.)

If we compare the accounts of different investigators concerning articulatory and acoustic qualities of *oo*, *eu* and *ee*, we find agreement on the following points. They are vowels pronounced with great tension in the speech organs, with a mid-position of the tongue (articulation for *oo* in the back-part of the mouth, for *eu* and *ee* in the front-part) and with a moderate opening of the jaws (varying little in the three sounds). The characteristic overtones are according to STUMPF [1]) :

oo formant 388—517
eu underformant 388—517 ; formant 1550—2324 ;
ee underformant 388—517 ; formant 1954—2764.

The few dates at hand concerning Dutch vowels (BOEKE, TER KUILE, BENJAMINS) agree very well with the above.

Experimental part.

A cinematographical record demonstrated that the outwardly visible differences between *oo*, *eu* and *ee*, followed by *r* or not, are very slight. In *oo* the opening of the mouth seems to be larger, in *ee* smaller when *r* follows. In *eu* no distinct difference could be noted. From records obtained by means of the labiograph of VON WILCZEWSKI the vertical distance between the lips was seen to be greater in *oo* and *eu* when *r* followed.

By means of ZWAARDEMAKER's apparatus to register speech movements (made of aluminium) records were made to compare the two series of sounds. As may be seen in fig. 1 the opening of the jaws was generally diminished when *r* followed. In *oo* and *eu* the difference was considerable, in *ee* small as in other vowels. This agrees well with EYKMAN's [2]) dates concerning the distance of the jaws.

Palatograms in which by means of E. A. MEYER's method also the profil of the tongue was investigated, showed a picture as in fig. 2.

In all three vowels the tongue moves in a straight way, when *r* follows. Only a very small part of the mouthhole remains free from the moving tongue. No *r* following, the tongue moves in a curved way, the so-called front resonance-chamber remaining free. This curved way is only possible with a strong tension in the tongue. The larger palatal surfaces touched by the tongue, observed when *r* is following, may be due in first place to the smaller distance between the jaws.

If we try to pronounce a pure *oo*, *eu* or *ee* before *r*, it appears to be impossible to make *r* follow the vowel immediately : between the two arises

[1]) C. STUMPF. Die Sprachlaute, 1926.

[2]) Onderzoekingen Physiol. Lab. te Utrecht. 5e reeks, II. 1901.

an *aa*-sound (compare English *poor*, *beer*, etc.). In the records made from such trials by means of ZWAARDEMAKER's apparatus, a great distance

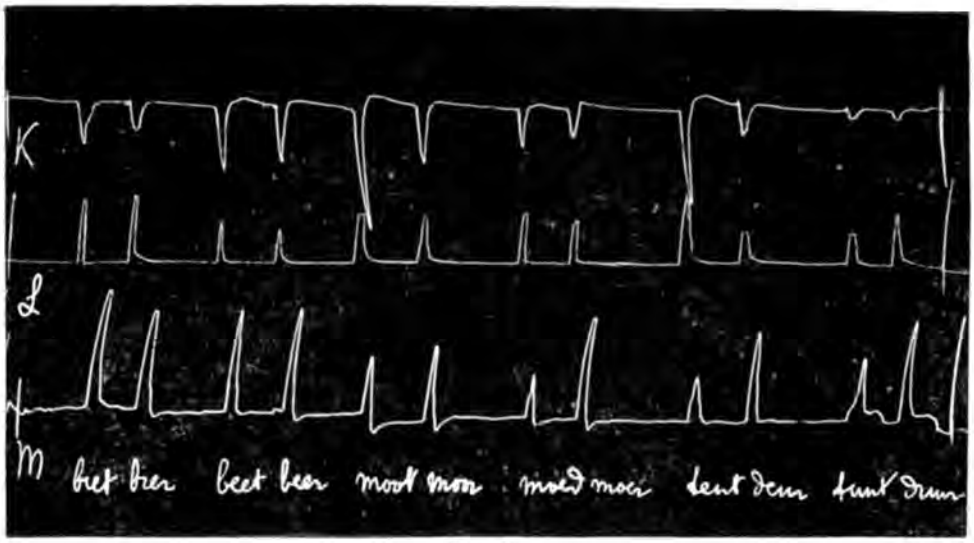


Fig. 1.

between the jaws and a strong relaxation of the bottom of the mouth before *r* were clearly perceptible.

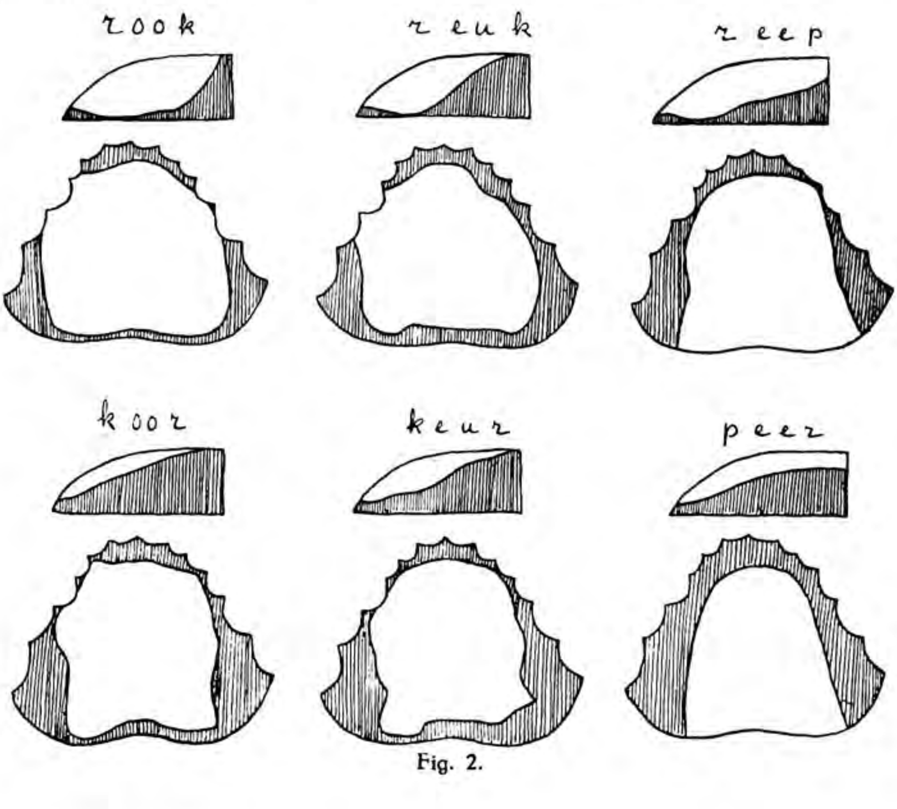


Fig. 2.

In deviating from the usual articulation it is possible, however, to pronounce a rather normal *oo*, *eu* or *ee* before *r*. To that effect the opening of the mouth must be actively or passively reduced in case of *oo*, and enlarged actively or passively in case of *eu* and *ee*.

A number of other influences make *oo*, *eu* and *ee* pass into the vowels that preceed *r*. Raising the point of the tongue, putting a piece of wood or wax in the front part of the mouth have almost the same effect. The other vowels remain practically unchanged thereby, as they do when followed by *r*.

To compare the acoustical qualities of the sounds, in the first place phonographical records were made, a lioretgraphe transcribing them ± 150 × enlarged on smoked paper. The greater part of these records were made from two subjects (B and K), both possessing a normal pronunciation with a lingual *r*. Fig. 3 shows some of these records. Accurate observing and simple measuring of these records showed the following: B spoke on a fundamental tone (pitch) of 170—190 p.s.; K on 200—225 p.s. The most important overtones were all within the limits indicated by STUMPF. For *oo* moreover a relatively strong tone of nearly 800 vibrations per second was found. This value agrees a.o. with that found by PAGET. In vowels not followed by *r* the second overtone was relatively strong especially in K, while in the vowels followed by *r* the importance of this overtone was of less importance, and became weak in B, while the amplitude of the third and to a less degree, that of the fourth overtone increased. The highest characteristic (formant) showed in *ee* a strong fall, under the influence of the following *r*, n.l. from 2300—2000.

For *eu*, where the highest characteristic was at 1600, a slight alteration in the same direction was noted. In *oo* the phenomenon was reversed, a feeble mounting of the overtone with frequency of 800 being noted. In all these cases, it was clearly visible that the highest characteristics towards the end of the vowel were transferred to a frequency of 1250, the formant of *r*. Records of the short vowels in *bot*, *dut* and *bit*, showed differences as well as striking similarity with both series. Determination of the first ten overtones by the method of HERMANN, which was applied in a few cases, affirmed the facts stated above.

The whispered vowels, too, were taken into account.

Also in whispering there is a marked difference between *oo*, *eu* and *ee* followed by *r* or not. Comparing also the sounds related to the investigated pairs of vowels with these latter, I came to the following series:

oe, *o* (ọ)[1]), *oo* (*r*), *o* (ɔ), the higher characteristic rising and the lower being indistinct;

uu, *eu*, *eu* (*r*), *u*, *œ*, the higher characteristic falling and the lower rising;

ie, *ee*, *ee* (*r*), *i*, *e* (ε), the higher characteristic falling and the lower rising.

1) KAISER. *The short o*. These Proceedings, Vol. 26, p. 745.

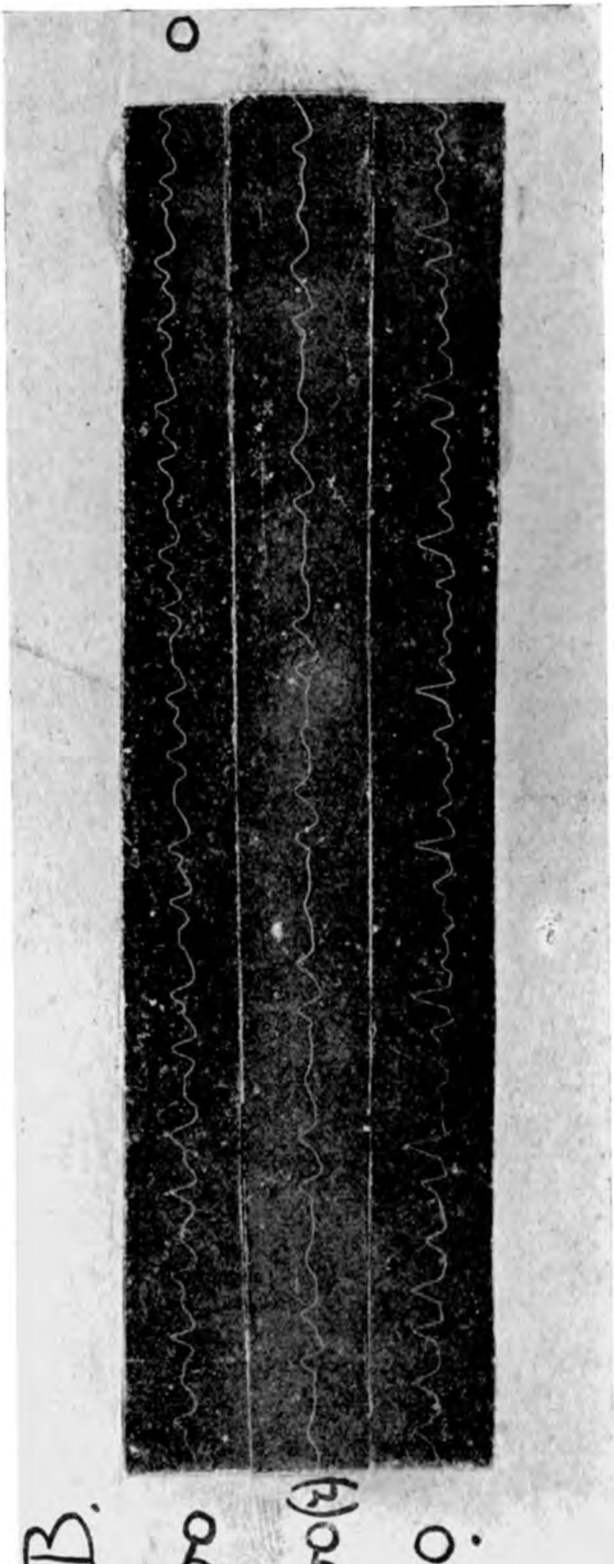


Fig. 3a.

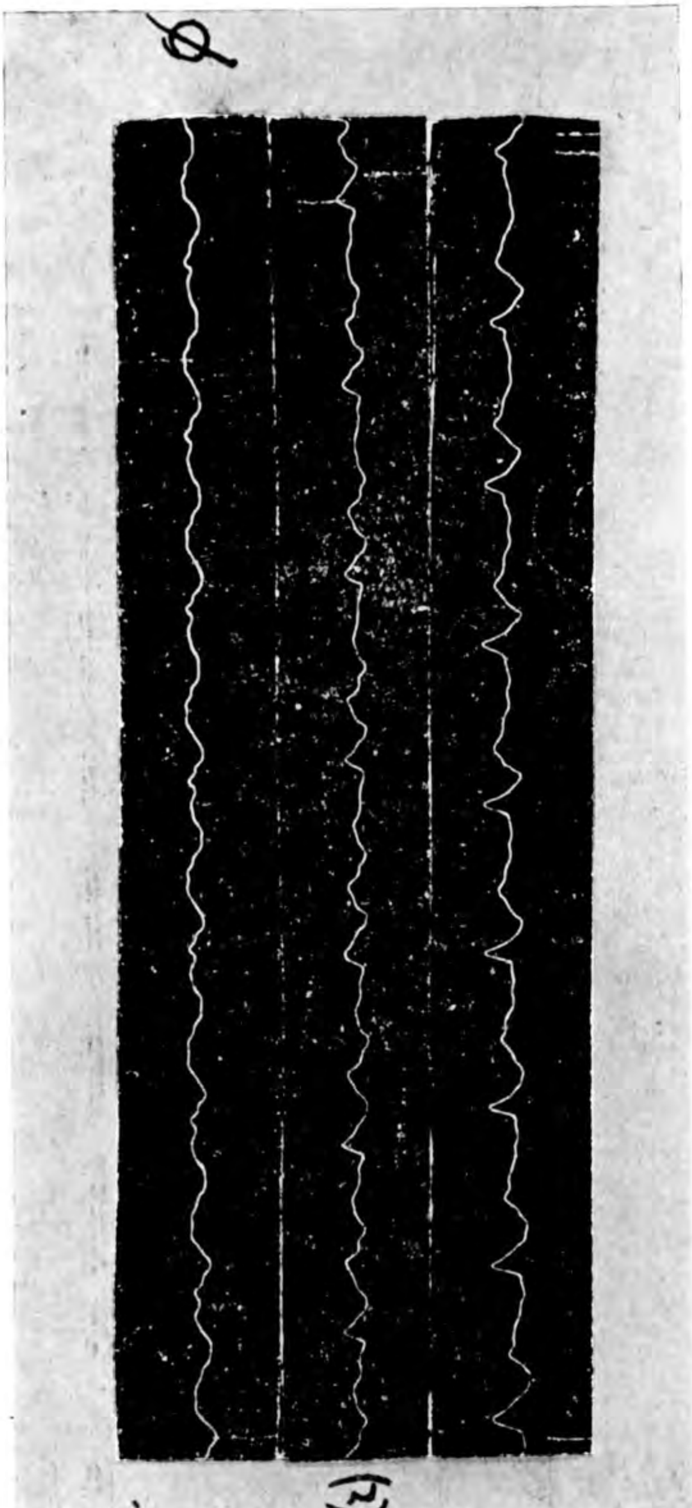


Fig. 3b.

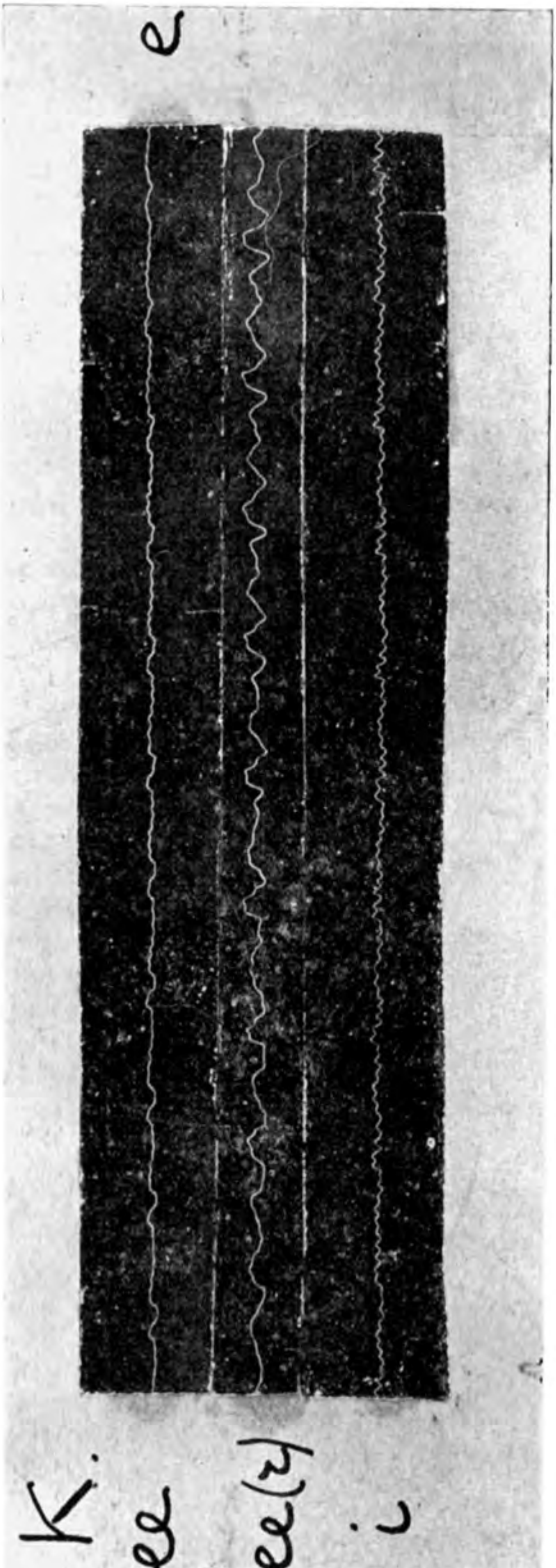


Fig. 3c.

However these results agree fairly well with those stated above, as a whole it did not appear quite satisfactory to me. Then BOUMAN [1]) developed a method to determine the overtones of vowels by means of electric resonance. The results of his investigations made a new explanation of facts acceptable.

Synopsis of results.

The acustical phenomena described above are corresponding in *eu* and *ee*, the reverse in *oo*. The cause of this can only be looked for in the place of articulation, which is the same for *eu* and *ee*, but different for *oo*. The consequence of this may be that in *oo* the narrowing of the opening of the mouth only reduces the space before the place of articulation, and in *eu* and *ee* especially the space behind it. This would imply that in *oo* the formant rises, in *eu* and *ee* the underformant rises, which is the case indeed.

The front resonance chamber in which the formant of *eu* and *ee*, is formed, becomes necessarily smaller with lessening of the distance between the jaws. The narrowing of the mouth opening attending it especially in ee, has a contrary effect and can be the stronger of the two influences at work, i.e. it can cause lowering of the formant : *eu* and *ee*. Widening of the mouth opening has in *oo* the same influence as reducement of the space : rising of the formant.

I cannot explain in details, at any rate not for *eu* and *ee*, how the moving point of the tongue and a small piece of wood or wax can have a similar effect. If BOUMAN's views are right, there is an explanation in them also for these phenomena.

Speaking generally we may say that *oo*, *eu* and *ee* are only pure under special circumstance and that everything which is anyhow detrimental to the pureness of these vowels, brings them in the class of the more or less imperfect vowels (BRÜCKE), where also the so-called short vowels have their place.

As *oo*, *eu* and *ee* are pre-eminently tense vowels and the pronunciation of *r* requires considerable laxity, it is comprehensible that they cannot be joined as they are.

In the Dutch language the joining of *r* to any other vowel is possible. To me this seems to prove that these latter vowels are not tense or at least far less so than *oo*, *eu* and *ee*.

[1]) Archives Néerlandaises de phonétique expérimentale, 2, 1928.

Pathological Anatomy. — *Localisation of the tuberculous lesions in Javanese and Chinese inhabitants of Java.* By A. J. F. OUDENDAL.

(Communicated at the meeting of June 25, 1927)

In about 600 post-mortems of tuberculous patients of the Pathologic Anatomical Department of the S.T.O.V.I.A. at Weltevreden, we exactly registered where and which tuberculous manifestations were found. The collected data comprise :

295 Javanese men

85 Javanese women

208 Chinese men, who all had been suffering from some form of tuberculosis. Only part of the material was submitted to a systematic microscopical examination. I planned to test everything microscopically, but owing to retrenchments in the matter of assistants and subsidy, it became impossible to do so. Therefore only the results of the naked-eye research at the dissecting-table are mentioned.

Before giving a summary of the pathological appearances, I'll first relate something about the length, the weight and the age of the dissected corpses.

The average weight of the Javanese men was 34.1 kilograms $\left(\frac{9831.5}{288}\text{ kgs}\right)$, of the Javanese women 28 kgs $\left(\frac{2331.5}{83}\text{ kgs}\right)$ and of the Chinese men it was 33.7 kgs $\left(\frac{6914.52}{205}\text{ kgs}\right)$.

Concerning the average length, we found for the Javanese men 1.59 m. $\left(\frac{470.86}{295}\text{ m.}\right)$, the Javanese women 1.485 m. $\left(\frac{124.87}{84}\text{ m.}\right)$ and the Chinese men 1.61 m. $\left(\frac{334.50}{207}\text{ m.}\right)$.

Reliable informations about the ages is impossible to get here, so we only can speak about ages at an estimation. The average ages, thus estimated, are:

for the Javanese men 31 years

for the Javanese women 33 years

for the Chinese men 36 years

Scheme of the averages.

	Javanese men	Javanese women	Chinese men
Weight	34.1 kilograms	28 kilograms	33.7 kilograms
Length	1.59 meters	1.48⁵ meters	1.61 meters
Approximate age	± 31 years	± 33 years	± 36 years

The oldest of the Javanese men was 75 years old, of the Javanese women 65 years, the Chinese men 70 years old. The youngest ones were respectively 15, 18 and 16 years old. There was one Chinese child of 9 months old.

We obtained our information from the notes on 2418 post-mortems, collected in about 5½ years. Year by year the proportion of the tuberculous cases in regard to all the post-mortems, showed us for the :

	T.B.C. +	Total numbers of post-mortems	Percentage
1st year (1921—'22)	102	597	16.7 %
2nd year (1922—'23)	141	608	23.2 „
3rd year (1923—'24)	136	397	34.3 „
4th year (1924—'25)	107	364	29.4 „
5th year (1925—'26)	96	331	29 „
6th year (1926—'27 half)	42	121	34.7 „

In the table above all the patients, who died of tuberculosis are counted, so also the Chinese women, european and arabian men and women.

Taking 2418 post-mortems as a whole we found among this number 624 post-mortems of tuberculous patients, or a quarter of the total amount.

As a rule we seldom dissect any child, so about the localisation of the tuberculous lesions in babies and very young people we cannot make any statements. Among the Chinese male patients there was one baby of 9 month old, suffering from bilateral closed tuberculosis of the lungs, with a tuberculosis of the bronchial glands. How the tuberculous inflammations are localized in the body, of the different groups (Javanese men, Javanese women and Chinese men) may be found in the adjoining lists.

Javanese men.

We made 295 post-mortems of native men, and found about the lungs that :

177 × the right lung showed one or more cavities

173 × the left lung did so and

118 × these cavities were found in both lungs at the same time, this makes 68 % of the greatest possible amount (173).

On one side the cavities were found 59 × in the right lung and 56 × in the left lung.

Tuberculosis without cavities was found in 90 cases in the right as well as in the left lung. Bilateral "closed" tuberculosis was to be found in 30 cases (33 %), so 60 times we found a tuberculosis of one lung only without any cavities.

Cavities of the right lung were combined with a closed tuberculosis of the left lung in 48 cases, in 11 cases cavities of the right lung were combined with left lung free from tuberculosis.

Cavities of the left lung were combined with a closed right tuberculosis 52 times, and only 4 times there were left lung-cavities with a "free" right lung. The cavities were mostly found in the cranial part of the lungs, also in Javanese women and Chinese men.

Of the 295 post-mortems :

28 × the right lung was free from tuberculous lesions,

31 × the left lung remained free,

12 × both lungs were free from tuberculosis. Those 12 pairs of "free" lungs were found in :

4 men with caries of the lumbal spine,

1 man with caries of the thoracal spine,

3 men only showed tuberculosis of the bronchial glands,

2 men with a coxitis tuberculosa sinistra,

1 woman with a gonitis tuberculosa dextra,

1 man only showed a tuberculous inflammation of the mesenterial glands.

About the tuberculous lymphadenitis we found :

In 170 out of 295 Javanese men tuberculosis of the bronchial glands (in 57½ %) of the total amount, 88 times (in 29 % of the amount) tuberculosis of the mesenterial glands, and 64 patients with tuberculosis of both glands (22 % of the total amount).

In Javanese men tuberculous bronchial glands were found in :

49 % with bilateral lung cavities,

17 % with cavities of the right lung,

17 % with cavities of the left lung,

9 % with bilateral closed tuberculosis.

Tuberculous mesenterial glands we found in :

52 % with ulcers all along the intestines,

18 % with ulcers in the colon only,

12 % with ulcers in the small intestine only,

17 % without ulcers visible to the naked eye in any part of the intestine.

In 55 out of the same 295 men, we found tuberculous ulcers on the vocal chords, the epiglottis etc (18 %).

These pharynxulcers we found :

29 × with bilateral lung cavities	53 %
15 × with cavities of the left lung	27 %
9 × with cavities of the right lung	16 %
1 × with bilateral closed tuberculosis	2 %
1 × with lungs, free from tuberculosis	2 %

Phneumothorax on one side was found :

4 × at the left lung,

3 × at the right lung.

Menigitis tuberculosa was found in 19 cases (5½ %).

Looking for lesions of the intestines, it struck me how many tuberculous ulcers were found, and very often accompanied by an appendicitis tuberculosa chronica too. Therefore I should like to have an answer to the following questions: Is there any connection between these tuberculous ulcers of the intestines and the great quantity of parasites (hookworms, amoebes), we find here? And isn 't it necessary with patients with a tuberculosis of the lungs or the bones combined with dysenterical troubles, to think more and more about a tuberculous enteritis or colitis? Also for practical purposes with a view to the therapeutics this must be an interesting point.

Of the 295 Javanese men there were:

131 with tuberculous ulcers of the colon 45 %

100 with tuberculous ulcers of the small intestine 34 %

142 didn't show any ulcers of the intestines 47 %

In 79 cases the whole of the intestines showed tuberculous ulcera, so there were:

21 men with ulcers in the small intestine only,

52 men with ulcers in the colon only.

Again there is a visible connection between the open tuberculosis of the lungs and the frequency of the tuberculosis of the intestines. So in 79 cases with tuberculous ulcers all along the intestines, more than half (43 cases) are combined with cavities in both lungs, 14 with cavities in the left lung, and 17 with cavities in the right lung, so from these 79 patients with tuberculosis of the tractus intestinalis, 74 were suffering from an infectious form of "open" tuberculosis of the lung. Only in 5 cases we couldn't find any cavities on examination with the naked eye.

On the other hand it is interesting to note that:

118 corpses with bilateral cavities in the lungs, showed:

43 × tuberculosis of the whole intestines,

20 × tuberculosis of the colon only,

3 × tuberculosis of the small intestine only, so 52 of these cases were free from visible ulcers of the intestines.

We found 12 cases of tuberculosis of the bones:

6 × Caries of the lumbar spine,

5 × Coxitis tuberculosa,

2 × Gonitis tuberculosa,

2 × Caries of the ribs,

1 × Caries of the thoracal spine.

Javanese women.

In 85 Javanese women, we found 36 × one or more cavities of the lungs, as well on the right as on the left hand side. In 24 corpses it was found to be bilateral, follows that cavities on *one* side only were found in 12 cases (right and left).

In 42 right lungs and 38 left lungs we found tuberculous manifestations without cavities, but in 7 corpses the right lung, and 11 times the left lung remained absolutely free from visible tuberculous lesions. Only 3 times *both* lungs seemed to be absolutely free from tuberculosis.

These 3 pairs of free lungs were combined :

1 × with Caries caput femoris dextra,

1 × with Gonitis tuberculosa dextra,

1 × Caries ossis pubis.

We never found a tuberculous inflammation of the bronchial glands or the mesenterical glands only.

Of the lung cavities on one-side 12 × cavities of the left lung were combined with a tuberculosis of the right lung (without cavities) and of the 12 cases with cavities of the right lung 10 were combined with tuberculosis of the left lung without cavities only, and 2 with a "free" left lung.

A bilateral tuberculosis of the lungs without cavities was found in 24 corpses.

We regularly found tuberculous glands ; out of 85 women 40 female showed a tuberculosis of the bronchial glands (47 %), and in 36 cases (41 %) there was a tuberculous inflammation of the mesenterial glands. In 20 cases the bronchial as well as the mesenterial glands were altered.

Of the 40 tuberculous bronchial glands we found :

32½ % with cavities on both sides,

10 % with cavities on the right hand side,

10 % with cavities on the left hand side,

35 % with bilateral tuberculosis of the lungs (without cavities).

The 36 cases of tuberculosis of the mesenterial glands were distributed as follows :

55½ % with tuberculous ulcers all along the intestines,

19 % with tuberculous ulcers in the small intestine only,

5½ % with tuberculous ulcers in the colon only,

19 % without any sign of intestinal tuberculosis.

In 10 women we found a tuberculosis of the larynx, epiglottis, etc. (11 %), always with an open tuberculosis of the lungs with cavities.

5 out of ten with cavities on both sides,

4 with cavities of the right lung,

1 with cavities of the left lung.

No phneumothorax was found in those 85 Javanese women.

About the tuberculosis of the intestines in 31 cases we found tuberculous ulcers all along the intestines, 10 times in the small intestine only, and 12 times in the colon only, respectively in 36½ %, 14 % and 12 %. And 37½ % of the intestines were free from tuberculosis.

These 31 cases of tuberculosis all along the intestines were in 10 cases combined with cavities in both lungs, 8 times with cavities in one lung, and 10 times with a tuberculosis of the lungs without cavities.

Menigitis was found in 5 corpses, tuberculosis of the bones in 6, so:
2 × Gonitis tuberculosa dextra,
1 × Tuberculosis of the right elbow,
2 × Caries of the thoracal spine,
1 × Caries caput femoris,
1 × Caries ossis pubis,
1 × Caries of the ribs.

Chinese men.

In 208 postmortems of Chinese men, we found:

129 × cavities in the right lung, 117 × cavities in the left lung, and in 89 cases we found cavities in both lungs. There were 68 right- and 76 left lungs which showed tuberculous lesions without cavities. In 37 cases there was an inflammation of both lungs, and only 4 Chinese were in possession of both lungs free from tuberculosis, although in 11 cases the right lung and in 15 cases the left lung was free from tuberculosis.

The 4 pairs of "free" lungs were combined with:
2 cases of caries of the spine,
1 case of tuberculosis of the bronchial glands,
1 case of tuberculosis of the bronchial glands with ulcers in the small intestine.

In 40 cases of cavities in the right lung only, 35 were combined with a tuberculosis of the left lung without cavities, so 5 were found in combination with a left lung free from tuberculous lesions.

Out of 28 cases with cavities in the left lung, we found 25 right lungs without cavities, and 3 "free" right lungs.

Tuberculous glands were very often found, in 208 postmortems we found 97 × tuberculous bronchial glands (41 %) and 66 × tuberculous mesenterial glands (31 %).

46 patients possessed tuberculous bronchial as well as mesenterial glands, so in 22 % of the total amount of Chinese corpses.

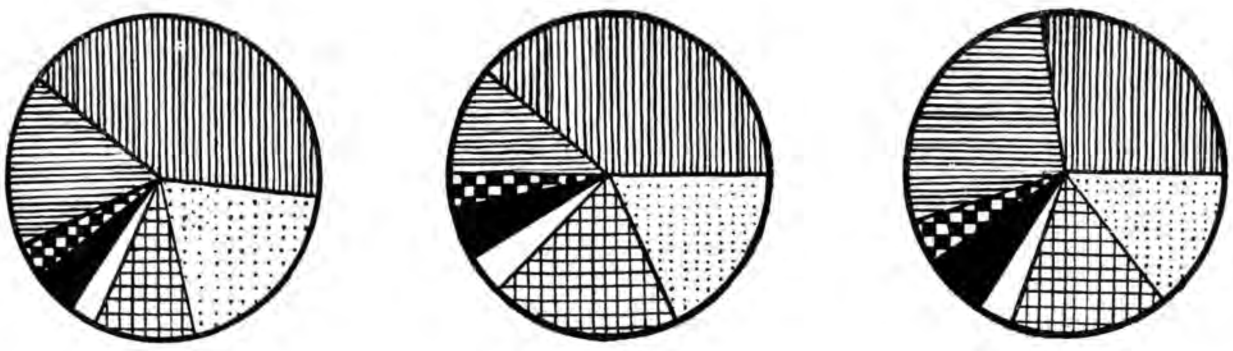
Tuberculous bronchial glands were found in:
54½ % with bilateral cavities of the lungs,
19½ % with cavities of the right lung,
10 % with cavities of the left lung,
12½ % with a bilateral tuberculosis of the lungs without cavities.

Tuberculous mesenterial glands were found in:
51 % with ulcers all along the intestines,
6 % with ulcers in the small intestine only,
22½ % with ulcers in the colon only,
20 % without any visible lesions of the intestines.

We found 39 cases of tuberculous ulcers of larynx and epiglottis:

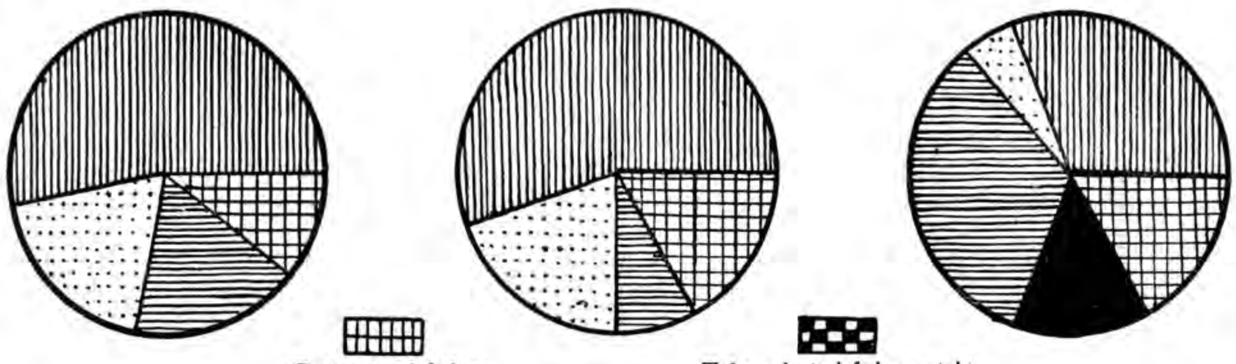
A. J. F. OUDENDAL: PROPORTIONS OF TUBERCULOUS MANIFESTATIONS IN 600 JAVANESE AND CHINESE INHABITANTS OF JAVA.

Chinese men | Javanese men | Javanese women



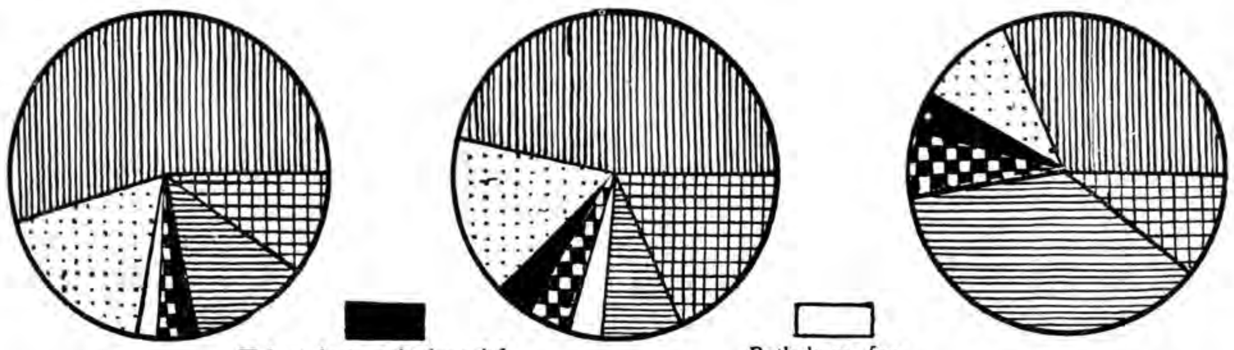
I Proportions of the different manifestations of tuberculosis of the lung.

Cavities in right lung. Cavities in both lungs. Tuberculosis without cavities in both lungs.



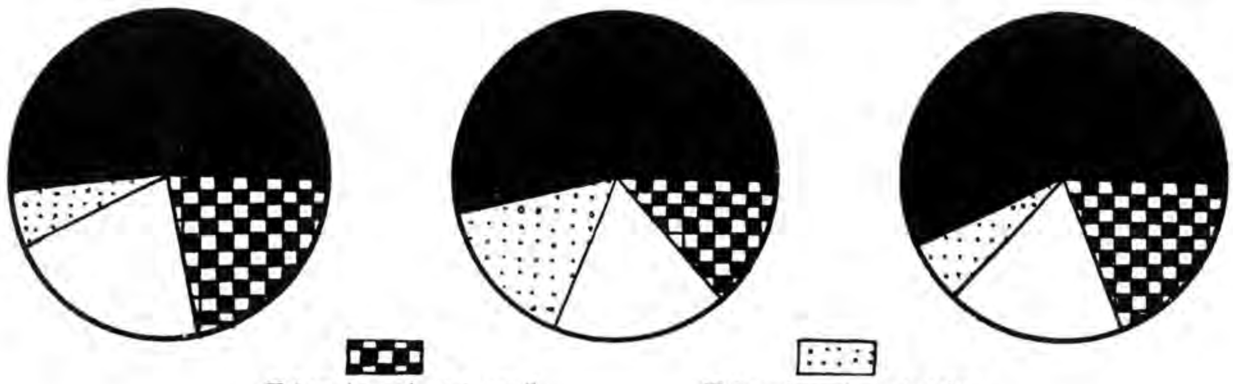
II Tuberculous ulcers all along the intestins compared to the different forms of tuberculosis of the lung.

Cavities in left lung. Tuberculosis left lung, right lung free.



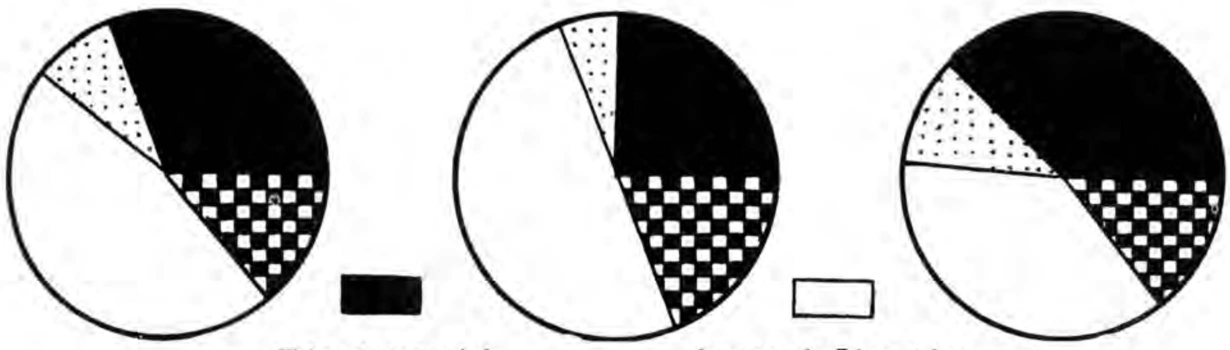
III Tuberculosis of bronchial glands compared to the different forms of tuberculosis of the lung.

Tuberculosis right lung, left lung free. Both lungs free.



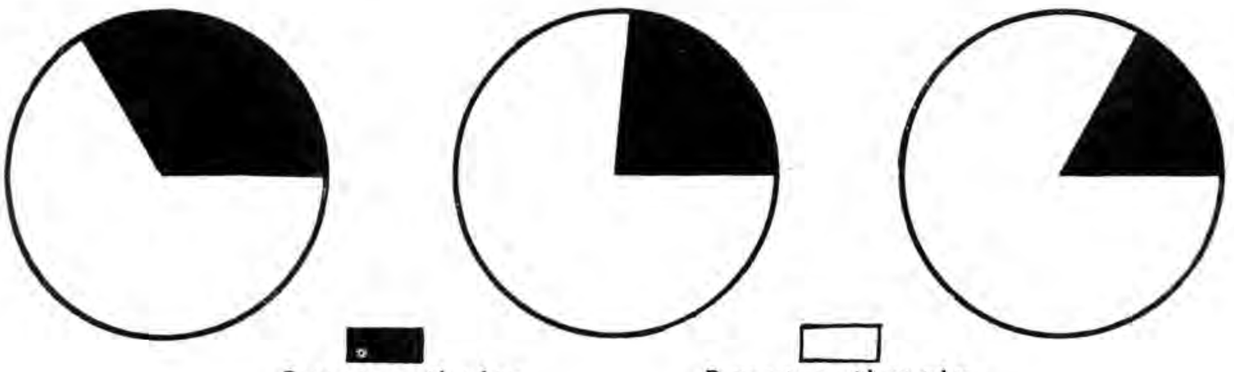
IV Cases with tuberculous mesenteric glands without tuberculosis intestinalis in proportion to those with tuberculous ulcers of the whole intestins and of the small intestine only and colon only.

Tuberculous ulcers in small intestine only. Tuberculous ulcers in colon only.



V Tuberculous ulcers all along the intestins, the colon only and the small intestine only compared to the cases without any intestinal tuberculosis.

Tuberculosis in whole intestines. Intestines free from tuberculosis.



VI Proportions of postmortens with tuberculous affections and postmortens free from tuberculosis.

Postmortens with tuberculosis. Postmortens without tuberculosis.

21 × with bilateral cavities (54 %),
5 × with cavities in the left lung (13 %),
10 × with cavities in the right lung (25 %).

A pneumothorax was found **5** × on the left and 2 × on the right hand side.

Menigitis tuberculosa was found in 12 corpses (6 %).

Tuberculous ulcers all along the intestines was seen **66** ×, in the small intestine only **28** ×, and in the colon only **20** ×, so out of 208 patients only 94 (45 %) were free from tuberculosis of the intestines.

Out of the 66 cases with tuberculosis of the tractus intestinalis 56 were combined with lung cavities (85 %), but only $^1/_3$ of the cases with bilateral lungcavities were free from tuberculosis of the intestines, so out of 89 patients with bilateral lung cavities there were :

35 suffering from tuberculosis all along the intestines,
11 suffering from tuberculous ulcers in the small intestine only,
16 suffering from tuberculous ulcers in the colon only.

Tuberculosis of the bones we found 9 × in 7 Chinese men.

4 × Caries of the lumber spine,
2 × Tuberculosis of the kneejoint,
1 × Tuberculosis of the foot,
1 × Tuberculosis of the sternum,
1 × Coxitis tuberculosa.

We very seldom come across a miliair tuberculosis here. Perhaps according to the fact that we only get very few postmortems of young people. We only found it :

6 × in Javanese men 2 %,
3 × in Javanese women 4 %,
3 × in Chinese men 1½ %.

Pleuritis chronica adhaesiva is seen in about every corps. The amount of corpses without this pleuritis I estimate to be less than 1 %.

As a matter of fact this short summary cannot give you an idea about the dissemination of tuberculous diseases amongst the *living* population.

The distribution of the tuberculous lesions, as we saw them at the post-mortems, cannot give us any certainty about the distribution of the tuberculous lesions as they were during life.

So this summary only has a limited value, and is only meant as a beginning of large statistics, more so as the small groups and the impossibility of having everything microscopically tested, is a definite reason to refrain from drawing general conclusions.

Short summary of the notes on 295 Javanese men.

I.	Right lung with cavities	: 177 ×	; left lung	: 173 ×
	Right lung with tuberculosis without cavities	: 90 ×	left lung	: 90 ×
	Right lung without tuberculous lesions	: 28 ×	left lung	: 31 ×
	Bilateral cavities	: 118 ×		
	Bilateral tuberculosis of the lung without cavities		: 30 ×	
	Both lungs free from tuberculosis	: 12 ×		
II.	Tuberculous bronchial glands		: 170 ×	
	Tuberculous mesenterial glands		: 88 ×	
	Tuberculous ulcers in the small intestine		: 100 ×	
	Tuberculous ulcers in the colon		: 131 ×	
	Tuberculous ulcers on the larynx		: 55 ×	
III.	Tuberculosis of the peritoneum		: 20 ×	
	Tuberculosis of the liver		: 35 ×	
	Tuberculosis of the spleen		: 47 ×	
	Tuberculosis of the kidneys		: 44 ×	
	Menigitis tuberculosa		: 19 ×	
	Phneumothorax left lung		: 4 ×	
	Phneumothorax right lung		: 3 ×	

IV. With 12 pairs of "free" lungs, we found :
4 × Caries of the lumbar spine,
1 × Caries of the thoracal spine,
2 × Coxitis tuberculosa sinistra,
1 × Gonitis tuberculosa dextra,
1 × Tuberculosis of the mesenterial glands only,
3 × Tuberculosis of the bronchial glands only.

V.	Tuberculous ulcers all along the intestines	: 79 ×
	Tuberculous ulcers in the small intestine *only*	: 21 ×
	Tuberculous ulcers in the colon *only*	: 52 ×

VI. In 118 patients with bilateral lung cavities we found :
43 × tuberculous ulcers in the whole bowels,
20 × tuberculous ulcers in the colon *only*,
3 × tuberculous ulcers in the small intestine *only*,
52 × no ulcers.

VII. Of 30 patients with bilateral tuberculosis of the lungs without cavities we found :
5 with tuberculous ulcers all along the intestines,
1 with tuberculous ulcers in the small intestine *only*,
6 with tuberculous ulcers in the colon *only*,
18 without any intestinal ulcers.

VIII. In 59 cases of right lung-cavities only, we found :
17 × tuberculous ulcers all along the intestines,
6 × tuberculous ulcers in the small intestine *only*,
8 × tuberculous ulcers in the colon *only*,
28 × no ulcers.

IX. In 56 cases of left lung-cavities only, we found :
14 × tuberculous ulcers all along the intestines,
5 × tuberculous ulcers in the small intestine *only*,
11 × tuberculous ulcers in the colon *only*,
26 × no ulcers.

X. In 12 cases of bilateral "free" lungs, we found :
5 × tuberculous ulcers in the colon *only*,
2 × tuberculous ulcers in the small intestine *only*,
5 × no ulcers.

XI. Summary of the lungs.

Bilateral cavities	: 118 ×		
Bilateral tuberculosis without cavities	: 30 ×		
Bilateral "free" lungs	: 12 ×		
Cavities of the right lung tuberculosis of the left „	: 41 ×	Cavities of the left lung tubercul. of the right „	: 52 ×
Cavities of the right lung free left lung	: 11 ×	Cavities of the left lung free right lung	: 4 ×
Free right lung Tubercul. of the left lung	: 12 ×	Free left lung Tuberc. of the right lung	: 8 ×

XII. Out of 55 cases of tuberculosis of the larynx there were :
29 with bilateral lung cavities,
9 with cavities of the right lung,
15 with cavities of the left lung,
1 with bilateral tuberculosis of the lungs without cavities,
1 with absolutely free lungs.

XIII. We also found :
3 × tuberculosis of the testes,
2 × tuberculosis of the bladder,
11 × tuberculosis of the prostatus,
3 × tuberculosis of the epididymis,
6 × tuberculosis of the suprarenal body,
1 × tuberculosis of the pancreas,
1 × tuberculosis of the thyroid gland,
6 × tuberculosis of the lumbar spine,
1 × tuberculosis of the thoracal spine,
2 × tuberculosis of the ribs,
5 × coxitis tuberculosa,
2 × tuberculosis of the kneejoint

Summary of 85 Javanese women.

I. Cavities of the right lung : 36 ×; of the left lung : 36 ×
Tuberculosis of the right lung without cavities : 42 ×; left lung : 38 ×
"Free" right lungs : 7 ×; left lung : 11 ×
Bilateral lung-cavities : 24 ×
Bilateral tuberculosis of the lungs without cavities : 24 ×
Bilateral "free" lungs : 3 ×

II. Tuberculosis of the bronchial glands : 40 ×
Tuberculosis of the mesenterial glands : 36 ×
Tuberculous ulcers in the small intestine : 41 ×
Tuberculous ulcers in the colon : 43 ×
Tuberculous ulcers of the larynx etc. : 10 ×

III. Tuberculosis of the peritoneum : 7 ×
Tuberculosis of the liver : 11 ×
Tuberculosis of the spleen : 17 ×
Tuberculosis of the kidneys : 18 ×
Menigitis tuberculosa : 5 ×

IV. With 3 pairs of "free" lungs we found :
1 × gonitis tuberculosa dextra,
1 × caries caput femoris dextra,
1 × caries ossis pubis.

V. Tuberculous ulcers were found :
all along the intestines : 31 ×
In the small intestine *only* : 10 ×
In the colon *only* : 12 ×

VI. In 24 women with bilateral lung cavities we found :
10 × tuberculous ulcers all along the intestines,
4 × tuberculous ulcers in the colon *only*,
4 × tuberculous ulcers in the small intestines only,
6 × no ulcers.

VII. In 24 cases with bilateral tuberculosis of the lungs without cavities, we found :
10 × tuberculous ulcers all along the intestines,
1 × tuberculous ulcers in the colon *only*,
2 × tuberculous ulcers in the small intestine *only*,
11 × no ulcers.

VIII. In 10 patients with cavities in the right lung combined with tuberculosis of the left lung (without cavities) we found:

7 × no ulcers in the intestines,
1 × ulcers all along the intestines,
2 × ulcers in the small intestine *only*.

IX. In 12 patients with cavities of the left lung combined with tuberculosis of the right lung (without cavities) we found:

6 × ulcers all along the intestines,
2 × ulcers in the colon *only*,
4 × no ulcers.

X. Summary of the lungs:

Cavities on the right + left	: 24 ×
Bilateral tuberculosis (without cavities)	: 24 ×
Cavities on the right + tuberculosis on the left	: 10 ×
Cavities on the left + tuberculosis on the right	: 12 ×
Cavities on the right + "free" left lungs	: 2 ×
Cavities on the left + "free" right lungs	: 0 ×
Both lungs "free"	: 3 ×
Left "free", tuberculosis on the right	: 6 × (without cavities)
Right "free", tuberculosis on the left	: 4 × (without cavities)

XI. Tuberculous ulcers of the larynx etc. we found 10 ×

5 × with bilateral lung-cavities,
4 × with lung-cavities of the right,
1 × with lung-cavities of the left.

XII. We also found:

Tuberculosis of the uterus	: 3 ×
Tuberculosis of the tubae	: 3 ×
Tuberculosis of the ovaria	: 5 ×
Tuberculosis of the vagina	: 1 ×
Tuberculosis of the suprarenal body	: 2 ×
Tuberculosis of the knee	: 2 ×
Tuberculosis of the right elbow	: 1 ×
Tuberculosis of the thoracal spine	: 2 ×
Coxitis tuberculosa	: 1 ×
Tuberculosis ossis pubis	: 1 ×
Tuberculosis of the ribs	: 1 ×

Summary of 208 Chinese men.

I.	Right lung with cavities	: 129 ×; left lung	: 117 ×
	Right lung with tuberculosis without cavities	: 68 ×; left lung	: 76 ×
	Right lung "free" from tuberculous lesions	: 11 ×; left lung	: 15 ×
	Bilateral cavities	: 89 ×	
	Bilateral tuberculosis of the lungs without cavities	: 37 ×	
	Bilateral "free" lungs	: 4 ×	

II.	Tuberculosis of the bronchial glands	: 97 ×
	Tuberculosis of the mesenterial glands	: 66 ×
	Tuberculous ulcers in the small intestine	: 94 ×
	Tuberculous ulcers in the colon	: 86 ×
	Tuberculous ulcers of the larynx etc.	: 39 ×

III.	Tuberculosis of the peritoneum	: 19 ×
	Tuberculosis of the liver	: 20 ×
	Tuberculosis of the spleen	: 29 ×
	Tuberculosis of the kidneys	: 23 ×
	Meningitis tuberculosa	: 12 ×
	Phneumothorax left lung	: 5 ×
	Phneumothorax right lung	: 2 ×

IV. With 4 pairs of "free" lungs we found:
2 with tuberculosis of the lumbar spine,
1 with tuberculosis of the bronchial glands *only*,
1 with tuberculosis of the bronchial glands with tuberculous ulcers of the small intestine.

V.	Tuberculous ulcers all along the intestines	: 66 ×
	Tuberculous ulcers in the small intestine *only*	: 28 ×
	Tuberculous ulcers in the colon *only*	: 20 ×

VI. In 89 cases of bilateral lung cavities we found:
35 × tuberculous ulcers all along the intestines,
16 × tuberculous ulcers in the colon *only*,
11 × tuberculous ulcers in the small intestine *only*,
27 × no ulcers.

VII. In 37 patients with bilateral tuberculosis of the lungs we found:
10 × tuberculous ulcers all along the intestines,
1 × tuberculous ulcers in the colon *only*,
10 × tuberculous ulcers in the small intestine *only*,
16 × no ulcers.

VIII. In 40 cases with right lung cavities only we found :
14 × tuberculous ulcers all along the intestines,
1 × tuberculous ulcers in the colon *only*,
1 × tuberculous ulcers in the small intestine *only*,
24 × no ulcers.

IX. In 28 cases with cavities of the left lung only, we found :
7 × tuberculous ulcers all along the intestines,
2 × tuberculous ulcers in the colon *only* ,
3 × tuberculous ulcers in the small intestine *only*,
16 × no ulcers.

X. Together with the 4 pairs of "free" lungs, no ulcers of the intestines were found.

XI. Summary of the lungs:
Bilateral cavities : 89 ×
Bilateral tuberculosis without cavities : 37 ×
Bilateral "free" lungs : 4 ×

Cavities of the right lung with tubercul. on the left :	35 ×	Cavities of the left lung with "free" right lung :	25 ×
Cavities of the right lung with "free" left lung :	5 ×	Cavities of the left lung with "free" right lung :	2 ×
Tuberculosis of the right lung with "free" left lung :	6 ×	Tubercul. of the left lung with "free" right lung :	4 ×

XII. The 39 cases of tuberculosis of the larynx were combined with :
21 × bilateral lung cavities,
10 × cavities of the right lung,
5 × cavities of the left lung,
3 × bilateral tuberculosis without cavities.

XIII. We also found :

Tuberculosis of the bladder	: 2 ×
Tuberculosis of the urethra	: 1 ×
Tuberculosis of the prostatus	: 7 ×
Tuberculosis of the epididymis	: 1 ×
Tuberculosis of the suprarenal body	: 3 ×
Tuberculosis of the pancreas	: 2 ×
Tuberculosis of the sternum	: 1 ×
Coxitis tuberculosa on one side	: 1 ×
Tuberculosis of *one* knee	: 2 ×
Tuberculosis of *one* foot	: 1 ×
Tuberculosis of the lumbar spine	: 4 ×

TABLE I.

Lungs	Chinese (208)	Javanese men (295)	Javanese women (85)	Index I
Bilateral lung cavities	42.5 %	40 %	28 %	% of the total amount of lungs from patients suffering from tuberculosis.
Bilateral tuberculosis without cavities	16 „	10 „	28 „	
Cavities in right lung, left lung without cavities	17 „	16½ „	12 „	
Cavities in left lung, right lung without cavities	12 „	17½ „	14 „	
Cavities in right lung, left lung „free"	2½ „	4 „	2½ „	
Cavities in left lung, right lung „free"	1½ „	1.4 „	0 „	
Tuberculosis of left lung, right lung „free"	2 „	4 „	4½ „	
Tuberculosis of right lung, left lung „free"	6 „	3 „	7 „	
Both lungs free from tuberculosis	2 „	4 „	3½ „	

TABLE II

Glands, Larynx, Peritoneum Meningitis, Liver, Spleen, Kidneys etc.	Chinese	Javanese men	Javanese women	Index II
Tuberculosis of the bronchial glands	46½ %	57½ %	47 %	% of the total amount of patients suffering from tuberculosis
Tuberculosis of the mesenterial glands	31 „	29½ „	41 „	
Tuberculosis of bronchial + mesenterial glands	22 „	22 „	23 „	
Tuberculosis of the larynx	18½ „	18½ „	11½ „	
Tuberculosis of the peritoneum	7 „	9 „	8 „	
Tuberculosis of the liver	12 „	9½ „	13 „	
„ „ „ spleen	16 „	14 „	20 „	
„ „ „ kidneys	15 „	11 „	20 „	
Menigitis tuberculosa	6½ „	5½ „	6 „	
Pneumothorax (left)	1½ „	2 „	0 „	
Pneumothorax (right)	1 „	1 „	0 „	
Miliair tuberculosis	1½ „	2 „	3½ „	
Tuberculosis of the bones	3 „	4 „	7 „	

TABLE III.

Intestinal ulcers	Chinese	Javanese men	Javanese women	Index III
Tuberculous ulcers all along the intestines	31½ %	27 %	36½ %	% of the total amount of intestines from tuberculous corpses
Tuberculous ulcers in the colon *only*	9 „	7 „	14 „	
Tuberculous ulcers in the small intestine *only*	13½ „	18 „	12 „	
No ulcers	46 „	47½ „	37½ „	

TABLE IV

Ulcers in the small and the big intestine at the same time are found with:	Chinese	Javanese men	Javanese women	Index IV
Bilateral lung cavities	53½ %	55 %	32½ %	% of the total number of ulcers of the small and big intestine at the same time
Bilateral tuberculosis of the lungs without cavities	15 „	6 „	32½ „	
Right lung cavities, left no cavities	21 „	20 „	3 „	
Left lung cavities, right no cavities	10½ „	16 „	18½ „	
Both lungs free from tuberculosis		2 „		
Right cavities, left lung free			3 „	
Left cavities, right lung free				
Right tuberculosis without cavities, left lung free			3 „	
Left tuberculosis without cavities, right lung free		½ „	6½ „	

TABLE V.

Tuberculous bronchial glands were combined with	Chinese	Javanese men	Javanese women	Index V
Bilateral cavities of the lungs	51 × or 54 %	84 × or 49 %	13 × or 32½ %	% of the total amount of corpses with tuberculous bronchial glands
Bilateral tuberculosis of the lungs without cavities	13 × „ 12½ „	15 × „ 9 „	14 × „ 35 „	
Right lung cavities, left no cavities	18 × „ 12½ „	29 × „ 17 „	4 × „ 10 „	
Left lung cavities, right no cavities	9 × „ 9 „	29 × „ 17 „	4 × „ 10 „	
Right lung cavities, left lung "free"	1 × „ 1 „	1 × „ ½ „		
Left lung cavities, right lung "free"	1 × „ 1 „	1 × „ ½ „		
Bilateral "free" lungs	2 × „ 2 „	4 × „ 2½ „		
Right tuberculosis without cavities, left lung "free"	1 × „ 1 „	4 × „ 2½ „	2 × „ 5 „	
Left tuberculosis without cavities, right lung "free"	1 × „ 1 „	3 × „ 1½ „	3 × „ 7½ „	

TABLE VI

Tuberculosis of the mesenterial glands are found with	Chinese men			Javanese men			Javanese women		
Tuberculous ulcers all along the intestine	51 %	34 ×	51 %	57 %	46 ×	52 %	64 %	20 ×	55½ %
Ulcers in the colon *only*	25 „	4 ×	8 „	31 „	16 ×	18 „	16 „	2 ×	5½ „
Ulcers in the small intestine *only*	53½ „	15 ×	22½ „	52 „	11 ×	12½ „	58 „	7 ×	19 „
No intestinal ulcers		13 ×	20 „		15 ×	17 „		7 ×	19 „
	of the different kinds of ulcers to be found in the intestines		of the glands	of the different kinds of ulcers to be found in the intestines		of the glands	of the different kinds of ulcers to be found in the intestines		of the glands

Palaeontology. — *A contribution to the knowledge of the fossil fauna and flora of Neede.* By C. H. OOSTINGH and F. FLORSCHÜTZ. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of January 28, 1928).

About two kilometres to the north-west of Neede (in the province of Gelderland) at approximately 52° 9′ N. 6° 35′ E. clay is dug for the manufacture of bricks and tiles on the western slope of the hill by the firm of the Wed. S. TEN BOKKEL HUININK.

This clay occurs between 20 and 30 metres above sea-level and is covered by about 7 metres of typically southern sand and gravel on the surface of which northern erratics are strewn. Borings proved that sand of a fluviatile character is underlying the clay.

The result of an examination of the remains of molluscs, mosses, fruits and seeds (the very abundant wood and rests of leaves are not considered in this article) follows.

A. Molluscs from the clay of Neede.

P. HUFFNAGEL has in 1911 mentioned the occurrence of *Paludina diluviana* [1]) while P. G. KRAUSE has also met at Neede with *Pal. diluviana* associated with a *Valvata*, closely allied to *V. naticina* Menke, and with opercula of *Bithynia tentaculata* (L.) [2]).

Examination of a larger number of Needian molluscs generally led to similar results as I found the following species :

1. *Gyraulus sp.*
2. *Ancylus lacustris* (Linné) ; in limited numbers.
3. *Bithynia tentaculata* (Linné) ; opercula.
4. *Viviparus viviparus (Linné)* (= *V. fasciatus* (Müller)) [3]).

The last is the most conspicuous species and very abundant. Most specimens are crushed in various directions ; a few only are undamaged. The bulk has more or less thick shells and in this respect agrees with the form described as *Paludina diluviana* Kunth [4]).

An examination of a large quantity of specimens had already convinced me that this so called fossil form cannot be sharply separated from the recent *Viviparus viviparus* (= *fasciatus)*. This opinion I found confirmed

[1]) Tijdschr. Nederl. Aardrijksk. Gen., 2nd series, XXVIII, pp. 66—72.

[2]) Z. d. deutschen geol. Ges., 1914, Monatsber., 2.

[3]) *Viviparus* Montfort, 1810, is the oldest available name for this genus, usually named *Paludina* or *Vivipara*. Vivipare Lamarck, 1809, is a vernacular name.

[4]) Z. d. deutschen geol. Ges., 1865, pl. 7, fig. 8 *a—d*.

by a recent paper of K. HUCKE [1]) in which this author proves that *Pal. diluviana (Viviparus diluvianus)* is a habitat-modification of *V. fasciatus* and identical with the still living form from rapidly flowing rivers.

5. *Valvata (Cincinna) piscinalis* (Müller).

Some of the specimens approach the lacustrine and fluviatile forma *antiqua* Sowerby.

6. *Valvata (Cincinna) naticina* Menke ; numerous specimens.

A comparison with recent specimens proved the Needian form undoubtedly to belong to this species. *V. naticina* nowadays lives in larger rivers east of the Oder over a large area between the Baltic and the north-western part of the Pontic seas [2]).

7. *Helicidae spec.* ; a single damaged shell.

8. *Pisidium spec. spec.*

Besides these shells of *Cypris spec. (Ostracoda)*, a small fish-vertebra and a small molar tooth of a mammal were found.

It is a striking fact that the true riverdwellers e.g. *V. viviparus*, forma *diluviana*, and *Valvata naticina* dominate among the molluscs found. Consequently there is no ground for the opinion that the clay in which they occur is a lacustrine deposit. Evidently we must consider it as a fluviatile deposit, probably from a slowly flowing branch of a larger river.

Valvata naticina only is of importance for the determination of the geological age of the Needian clay.

In western Europe this species no longer lives, but as a fossil it occurs in the old-pleistocene for instance at Mosbach near Wiesbaden, at Mauer near Heidelberg and at Hangenbieten near Strassbourg ; besides in that of the hill of Oermten near Geldern. Consequently the Needian clay is probably of old-pleistocene age ; possibly it is old-interglacial or still older.

I do not think that the occurrence of *V. naticina* allows us to form an opinion about the climate of the time at which the Needian clay was deposited, as since pleistocene times the area of this species has been reduced both in an eastern and in a western direction.

Medan (Sumatra), Febr. 1927. C. H. O.

B. The fossil flora of the clay of Neede.

On the plant-remains buried in the clay of Neede I found the following publications :

a. HUFFNAGEL said in "Opmerkingen naar aanleiding van J. VAN BAREN's Morfologische bouw van het diluvium ten oosten van den IJsel" (Tijdschrift van het Koninklijk Nederlandsch Aardrijkskundig Genootschap, 2nd series, volume XXVIII, 1911) that the Needian clay also

[1]) Zeitschr. f. Geschiebeforschung, I, pp. 145—150, 1925.

[2]) LINDHOLM in Archiv f. Molluskenkunde, LIX, pp. 20—33, 1927.

contained a flora which was still being examinated and in which probably a Betula-species was represented.

b. JONGMANS stated in "Jaarverslag der Rijksopsporing van Delfstoffen over 1911": "From younger, pleistocene, layers in the neighbourhood of Neede the districtsgeologist HUFFNAGEL and I gathered clay with plant-remains. However the material obtained was not abundant and scarcely sufficient to prove the sediments to be not older than pleistocene".

c. VAN BAREN mentioned in de "De bodem van Nederland", p. 626, black clay with remains of the fir, broken mammoth-bones, shark-teeth and the freshwatermollusc *Paludina diluviana*.

Among the plant-remains could be recognized:

I. *Archegoniatae:*

Drepanocladus C. Müll. spec.

Homalia trichomanoides (Schreb.) Bryol. eur.

Neckera complanata (L.) Hüben.

II. *Anthophyta:* seeds or fruits from:

Acer campestre L.
Ajuga reptans L.
Alnus spec.
Carex spec.
Chenopodium spec.
Cornus sanguinea L.
Euryale spec.
Nuphar luteum Sm.
Oenanthe spec.
Pirus spec.
Potamogeton spec.
Prunus spinosa L.
Ranunculus sceleratus L.
Ranunculus spec.
Rhamnus cf. *Frangula* L.
Rumex maritima L.
Sambucus nigra L.
Solanum cf. *Dulcamara* L.
Sparganium spec. spec.
Urtica spec.
Viburnum Opulus L.
Vitis vinifera L.

The three recorded mosses still are common in our country: *Drepanocladus* as a bog-dweller whereas *Neckera complanata* and *Homalia trichomanoides* occur in forests on the soil and on trunks of trees.

With the exception of *Euryale*, *Vitis* and *Pirus* the *Anthophyta* found are not foreign to our recent flora. Conspicuous are the large masses of *Alnus*-cones and -seeds.

Untill 1907 *Euryale* was represented in the litterature only by the living species *Euryale ferox* Salisb. which grows in tropical and subtropical Asia from Bengal to Japan and even in the upper Ussuri-region at 45° 56′ N. with a mean January-temperature of —18° C. and a mean year-temperature of scarcely +4° C. (C. A. WEBER: *Euryale europaea* nov. sp. foss. Berichte der deutschen botanischen Gesellschaft, Band XXV, 1907).

In 1907 WEBER (loc. cit.) described a fossil seed from Russia as *Euryale europaea* whilst CL. and E. M. REID gave the same specific name to seeds from the clay of Tegelen (The fossil flora of Tegelen-sur-Meuse, near

Venloo, in the province of Limburg. Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam, Tweede Sectie, Deel XIII, N⁰. 6, 1907) but afterwards substituted it by *Euryale limburgensis*.

Eight years later these British authors described as new species: *Euryale nodulosa* and *Euryale lissa*, respectively from the clays of Reuver and of Brunssum (The pliocene floras of the Dutch-Prussian border. Mededeelingen van de Rijksopsporing van Delfstoffen, N⁰. 6, 1915).

Although damaged, the Needian seed shows some characteristics of an *Euryale*-seed but differs, as may be seen from a comparison of the figures with those in the cited papers of WEBER and of the REIDS, so much from the five recorded species that for the present I feel inclined to consider it as a new species for which I propose the name *Euryale Reidiorum*.

Vitis vinifera nowadays does not occur wild farther north than Mannheim; moreover it is questionable whether at its farthest northern habitats the plant is really wild or escaped from cultivation.

The seed brought to the genus *Pirus* agrees precisely with the cultivated *Pirus communis* L. but it is considerably smaller.

Velp (G.), Holland, F. F.
January 1928.

An anatomical examination of *Staphylea*-seeds, after the dutch text of this article was already printed, threw doubt on the real nature of the fossil figured as N⁰. 2 and referred to the genus *Euryale*. In some characters it resembles *Staphylea*.

We will have to await the finding of less damaged specimens before it will be possible to express a definite opinion.

F. F.

EXPLANATION OF THE PLATES.

Plate I.

Fig. 1 *Vitis vinifera* L. fossil, Neede. 3/1.[1]).
Fig. 2 *Euryale Reidiorum* 3/1[1]).
Fig. 3 „ „ 24/1[1]).
Fig. 4 *Euryale Reidiorum*, testa-surface. 240/1.
Fig. 5 *Euryale ferox* Salisb., testa-surface. 240/1.
Fig. 6 *Euryale europaea* C. A. WEBER, testa-surface. 160/1[2]).
Fig. 7 *Euryale limburgensis* CL. et E. M. REID, testa-surface. 240/1.
Fig. 8 *Euryale lissa* CL. et E. M. REID, testa-surface. 240/1.
Fig. 9 *Euryale nodulosa* CL. et E. M. REID, testa-surface. 385/1.

Plate II.

Fig. 10 *Euryale Reidiorum* cross-selection of testa. 80/1.
Fig. 11 *Euryale ferox* Salisb., cross-selection of testa. 80/1[1]).
Fig. 12 *Euryale europaea* C. A. WEBER, cross-section of testa. 95/1[2]).
Fig. 13 *Euryale limburgensis* CL. et E. M. REID, cross-section of testa. 70/1[1]).
Fig. 14 *Euryale lissa* CL. et E. M. REID, cross-section of testa. 80/1.
Fig. 15 *Euryale nodulosa* CL. et E. M. REID, cross-section of testa. 80/1.

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[1]) Photo C. COOLHAAS.

[2]) Photo C. COOLHAAS after C. A. WEBER.

PLATE I.

Fig. 1.

Fig. 2.

Fig. 3.

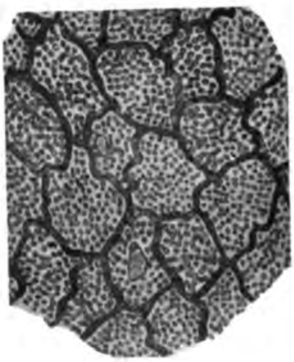

Fig. 4.

Fig. 5.

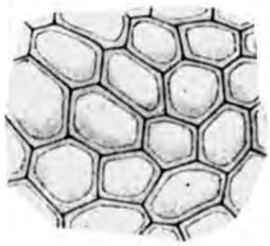

Fig. 6.

Fig. 7.

Fig. 8.

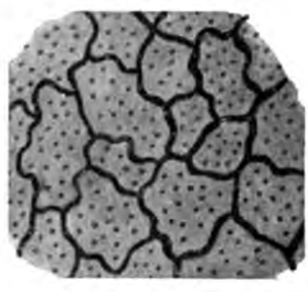

Fig. 9.

PLATE II.

Fig. 10.

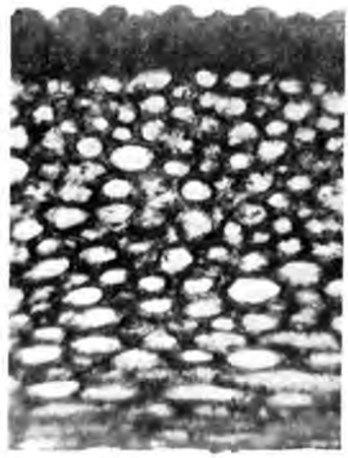

Fig. 11.

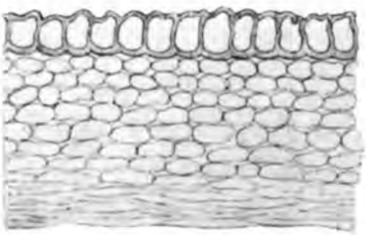

Fig. 12.

Fig. 13.

Fig. 14.

Fig. 15.

Mathematics. — *On non-holonomic connexions.* By Prof. J. A. SCHOUTEN (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of February 25, 1928).

Introduction.

Non-holonomic parameters are well known in mechanics. In geometry they were first used by HESSENBERG [1]), by the author [2]), by CARTAN [3]) and by HLAVATY [4]). Using non-holonomic parameters, the equations of the general linear connexion get another form, given by HORAK [5]).

VRANCEANU [6]) has brought something essentially new. He has shown that in a V_n containing a non-V_m-building field of m-directions there exists a connexion for quantities belonging to the local R_m, and that this connexion can be deduced from the connexion of the V_n by the use of the coefficients of rotation of RICCI. HORAK [7]) has independently found this same connexion and in a paper that is to be published in a short time he will give especially mechanical applications.

Now we get a more general point of view starting from an A_n (X_n with a symmetrical linear connexion) containing a non-X_m-building field of m-directions. Then we can show that in the case of a general $(n\text{-}m)$-direction being given in every point (the case of „Einspannung" of WEYL) there is induced a connexion for all quantities belonging to the local E_m. So we get an A_n^m some of whose properties will be studied more in detail. Especially we will consider the properties of curvature and the generalised equations of GAUSS, which have the same form as in the case of an X_m in A_n, and also something will be said on the geodesics in A_n^m and A_n. Finally the "affine geometry" of an X_n^{n-1} in A_n will be treated. The first paragraph contains a short review on non-holonomic parameters in an A_n.

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1) Vektorielle Begründung der Differentialgeometrie, Math. Ann. **78** (18) 187—217.

2) Die direkte Analysis zur neueren Relativitätstheorie. Verh. Kon. Akad. v. Wet. Amsterdam **12** (18) 6.

3) Sur les variétés à connexion affine, Ann. de l'école normale (3) **40** (23) 325—412.

4) Sur le déplacement linéaire du point, Věstn. České Akademie (24) XIII 1—8.

5) Die Formeln für allgemeine lineare Uebertragung bei Benutzung von nichtholonomen Parametern, Nieuw Archief v. Wisk. **15** (27) 193—201.

6) Sur les espaces non holonomes, Comptes Rendus **183** (26) 825—854, Sur le calcul différentiel absolu pour les variétés non holonomes, Comptes Rendus **183** (26) 1083—1085.

7) (Czechisch) Sur une généralisation de la notion de variété, Publications de la Faculté des sciences de l'université Hasaryk, Brno.

§ 1. *Non-holonomic parameters in A_n.*

In every point of the A_n we introduce besides the measuring vectors $\overset{\nu}{e}$, $\underset{\lambda}{e}$, $\lambda, \mu, \nu = a_1, \ldots, a_n$, belonging to the variables x^ν, an arbitrary system $\overset{k}{e_\lambda}$, $\underset{i}{e^\nu}$, $i, j, k = 1, \ldots, n$. Indicating the components with respect to the latter system with latin indices we have

$$v^k = v^\mu \overset{k}{e}_\mu \quad ; \quad w_i = w_\mu \underset{i}{e}^\mu \quad ; \quad \underset{i}{e}^k = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases} \quad ; \quad \overset{k}{e}_i = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases} \quad . \quad (1)$$

$$A_\lambda^\nu = \begin{cases} 1, \lambda = \nu \\ 0, \lambda \neq \nu \end{cases} \quad ; \quad A_\lambda^k = A_\lambda^\mu \overset{k}{e}_\mu \quad ; \quad A_i^\nu = A_\mu^\nu \underset{i}{e}^\mu \quad ; \quad A_i^k = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases} \quad (2)$$

and

$$v^k = A_\mu^k v^\mu \quad ; \quad w_i = A_i^\mu w_\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the expression

$$(dx)^k = A_\mu^k dx^\mu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the x^k have a signification by themselves if and only if

$$\partial_{[\omega} A_{\mu]}^k = 0 \quad ; \quad \left(\partial_\omega = \frac{\partial}{\partial x^\omega} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In the other case the x^k play the same part as the non-holonomic parameters in mechanics, only their differentials having a signification. We consider just this latter case. A symmetrical linear connexion being given by the parameters $\Gamma_{\lambda\mu}^\nu$ with respect to the system of measuring vectors $\left(\begin{smallmatrix} \nu \\ \lambda \end{smallmatrix}\right)$ we can fix this connexion also by parameters Λ_{ij}^k with respect to the system $\left(\begin{smallmatrix} k \\ i \end{smallmatrix}\right)$, so that

$$\begin{aligned} \nabla_j v^k &= A_{j\gamma}^{\beta k} \nabla_\beta v^\gamma = \partial_j v^k + \Lambda_{ij}^k v^i \\ \nabla_j w_i &= A_{ji}^{\beta\alpha} \nabla_\beta w_\alpha = \partial_j w_i - \Lambda_{ij}^k w_k \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

writing ∂_i for $A_i^\mu \partial_\mu$. It is easily found that

$$\Lambda_{ij}^k = \Gamma_{ij}^k + A_\alpha^k \partial_j A_i^\alpha = \Gamma_{ij}^k - A_i^\alpha \partial_j A_\alpha^k \quad . \quad . \quad . \quad . \quad . \quad (7)$$

hence

$$\Lambda_{[ij]}^k = - A_{ij}^{\alpha\beta} \partial_{[\beta} A_{\alpha]}^k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

So $\Lambda_{[ij]}^k = 0$ is necessary and sufficient for the holonomity of the x^k, hence, notwithstanding the symmetry in λ and μ of the $\Gamma_{\lambda\mu}^\nu$, the parameters Λ_{ij}^k with respect to non-holonomic parameters are not symmetrical in i and j. Of course $\Lambda_{[ij]}^k$ is no affinor. Applicating (6) on $\underset{i}{e}^k$ and $\overset{k}{e}_i$ we get

$$\Lambda_{ij}^k = \nabla_j \underset{i}{e}^k = - \nabla_j \overset{k}{e}_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If the A_n passes into a V_n, and if we choose the orthogonal system $\underset{i}{i^\nu} = \overset{i}{i^\nu}, \overset{k}{i_\lambda}$, as measuring vectors $\underset{i}{e^\nu}, \overset{k}{e_\lambda}$, then

$$\Lambda_{ij}^k = -\nabla_j \overset{k}{i}_i = +\nabla_j \overset{i}{i}_k = \gamma_{kij} = -\gamma_{ikj} \quad . \quad . \quad . \quad . \quad (10)$$

where the γ_{kij} are the coefficients of rotation of RICCI. We will write $\gamma_{ij}^k = -\gamma_{kj}^i$ instead of $\gamma_{ikj} = -\gamma_{kij}$ to render more conspicuous the signification of these coefficients as parameters of a connexion. Hence in orthogonal components the connexion is given by

$$\nabla_j v^k = \nabla_j v_k = \partial_j v^k + \gamma_{ij}^k \, v^i = \partial_j v_k - \gamma_{kj}^i \, v_i \, (= \partial_j v_k + \sum_i \gamma_{ikj} \, v_i). \quad (11)$$

By differentiating and alternating (6) we get

$$\nabla_{[l} \nabla_{j]} v^k = \{\partial_{[l} \Lambda_{|h|j]}^k + \Lambda_{p[l}^k \Lambda_{|h|j]}^p - \Lambda_{[jl]}^p \Lambda_{hp}^k\} v^h \quad . \quad . \quad . \quad (12)$$

hence for the quantity of curvature follows

$$R_{ljh}{}^{k} = -2\, \partial_{[l} \Lambda_{|h|j]}^k - 2\, \Lambda_{p[l}^k \Lambda_{|h|j]}^p + 2\, \Lambda_{[jl]}^p \Lambda_{hp}^k \quad . \quad . \quad . \quad (13)$$

an equation passing into the ordinary form for $\Lambda_{[ij]}^k = 0$, viz. for the case of holonomic parameters.

§ 2. X_n^m *in* X_n.

In every point of an X_n an m-direction be given. In the simplest case these m-directions are X_m-building, viz. it is possible to link them together in such a way that they build a system of ∞^{n-m} X_m. This simplest case will not be considered here. In general there will be a system of $\infty^{n-M} X_M$, $m < M \leqq n$, in such a manner that the given m-direction lies in every point in the local M-direction. We suppose that $M = n$ and give up all simplifications possible for $M < n$. An X_n equipped in this way with local m-directions will be called an X_n^m. An affinor of the local E_m, defined over X_n, will be called an "affinor of the X_n^m". Hence a field of contravariant vectors of the X_n^m is also a field of contravariant vectors of the X_n, but not vice versa, and a field of covariant vectors of the X_n (represented geometrically by two parallel E_{n-1} in every point) forms a field of covariant vectors of the X_n^m (represented geometrically by two parallel E_{m-1}) by intersection with the local E_m. All this is just the same as for an X_m in X_n.

We now introduce in every point of the X_n an $(n-m)$-direction, having no direction in common with the local m-direction. An X_n equipped in this way will be called *rigged* [1]). The essential difference between covariant vectors of the X_n and of the X_n^m disappears in a rigged X_n^m, because here every covariant vector of the X_n^m is in one to one correspondence with the covariant vector

[1]) WEYLS expression "eingespannt" being untranslatable and the $(n-m)$-direction reminding of a hoisted sail on a ship (the local m-direction), the word "rigged" was suggested.

of the X_n, whose E_{n-1} are determined by the E_{m-1} of the former vector and the $(n-m)$-direction of the rigging. Analytically we express this as follows. Besides the system $\binom{\nu}{\lambda}$ belonging to the x^ν we introduce in every point a system $\binom{k}{i}$, in such a manner that the first m contravariant measuring vectors $\underset{a}{e^\nu}$, $a, b, c, d = 1, \ldots, m$, lie arbitrarily in the local m-direction, and that the other ones $\underset{e}{e^\nu}$, $e, f, g, h = m+1, \ldots, n$, lie also arbitrarily in the local $(n-m)$-direction. With respect to this system the quantities of the X_n^m only have components with indices from 1 to m. Writing

$$B_\lambda^\nu = \overset{a}{e}_\lambda \underset{a}{e^\nu} \quad ; \quad C_\lambda^\nu = A_\lambda^\nu - B_\lambda^\nu = \overset{e}{e}_\lambda \underset{e}{e^\nu} \quad \ldots \ldots \quad (14)$$

we have

$$B_a^c = \begin{cases} 1, & a = c \\ 0, & a \neq c \end{cases} \quad , \quad B_e^f = 0 \quad , \quad B_a^f = 0, \quad B_e^c = 0 \quad . \quad . \quad (15)$$

and the X_n^m-component of a vector of X_n is given by the equation

$$v'^\nu = B_\mu^\nu v^\mu \quad ; \quad w'_\lambda = B_\lambda^\mu w_\mu \quad \ldots \ldots \quad (16)$$

or, with regard to the system $\binom{k}{i}$:

$$v'^k = B_\mu^k v^\mu \quad ; \quad w'_i = B_i^\mu w_\mu \quad \ldots \ldots \quad (17)$$

If we write $(dy)^c$ for the $\binom{k}{i}$-components of a translation dx^ν lying in X_n^m:

$$dx^\nu = B_a^\nu \, (dy)^a \quad \ldots \ldots \quad (18)$$

then the y^c are non-holonomic parameters and for B_a^ν follows

$$\boxed{B_a^\nu = \frac{\partial x^\nu}{(\partial y)^a}} \quad \ldots \ldots \quad (19)$$

§ 3. *The connexion induced in a rigged X_n^m in A_n.*

By introducing the $\Gamma_{\lambda\mu}^\nu$ the X_n becomes an A_n. We are going to prove that the connexion of A_n induces a connexion in an X_n^m in A^n provided that this X_n^m is rigged. This connexion is defined as follows:

The covariant differential quotient of a quantity in X_n^m is the X_n^m-component of the covariant differential quotient in A_n.

Thus, indicating the covariant differential quotient in X_n^m by ∇' we have for vectors:

$$\left.\begin{aligned}\nabla'_\mu v^\nu &= B_{\mu\gamma}^{\beta\nu}\,\partial_\beta v^\gamma + B_{\mu\gamma}^{\beta\nu}\,\Gamma_{\lambda\beta}^{\gamma}\,v^\lambda \\ \nabla'_\mu w_\lambda &= B_{\mu\lambda}^{\beta\alpha}\,\partial_\beta w_\alpha - B_{\mu\lambda}^{\beta\alpha}\,\Gamma_{\alpha\beta}^{\nu}\,w_\nu\end{aligned}\right. \quad \ldots\ldots \quad (20)$$

or, with regard to the system $\binom{k}{i}$:

$$\left.\begin{aligned}\nabla'_b\, v^c &= B_{b\nu}^{\mu c}\,\nabla_\mu\, v^\nu = \partial_b\, v^c - B_a^\nu\, v^a\,\partial_b\, B_\nu^c + v^a\, B_{b\nu a}^{\mu c\lambda}\,\Gamma_{\lambda\mu}^{\nu} \\ \nabla'_b\, w_a &= B_{ba}^{\mu\lambda}\,\nabla_\mu\, w_\lambda = \partial_b\, w_a - w_c\, B_\lambda^c\,\partial_b\, B_a^\lambda - w_c\, B_{ba\nu}^{\mu\lambda c}\,\Gamma_{\lambda\mu}^{\nu}\end{aligned}\right\} . \quad (21)$$

From this equation follows for the parameters Λ'^{c}_{ab} of the induced connexion

$$\boxed{\Lambda'^{c}_{ab} = B_{ab\nu}^{\lambda\mu c}\,\Gamma_{\lambda\mu}^{\nu} + B_\nu^c\,\partial_b\, B_a^\nu} \quad \ldots\ldots \quad (22)$$

Quite as in an A_n the alternating part

$$\Lambda'^{c}_{[ab]} = B_\nu^c\,\partial_{[b}\, B_{a]}^\nu \quad \ldots\ldots \quad (23)$$

is no affinor and depends on the choice of the systems $\binom{c}{a}$.

A rigged X_n^m thus equipped with a connexion will be called an A_n^m. Applying (21) to $\underset{a}{e}^c$ and $\overset{c}{e}_a$ we get

$$\Lambda'^{c}_{ab} = \nabla'_b\,\underset{a}{e}^c = -\nabla'_b\,\overset{c}{e}_a. \quad \ldots\ldots \quad (24)$$

but also, in consequence of the choice of the measuring vectors

$$\Lambda'^{c}_{ab} = \nabla_b\,\underset{a}{e}^c = -\nabla_b\,\overset{c}{e}_a = \Lambda^{c}_{ab} \quad \ldots\ldots \quad (25)$$

Starting from a V_n instead of from an A_n and choosing for $\binom{k}{i}$ an orthogonal system, we get easily

$$\Lambda'^{c}_{ab} = \gamma^{c}_{ab}\,(= -\gamma_{cab}) \quad \ldots\ldots \quad (26)$$

Hence the connexion induced in a V_n^m ($= X_n^m$ in V_n) is obtained in a very simple manner by using the coefficients of rotation of RICCI with respect to a suitable chosen system of m congruences of curves [1]).

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[1]) VRANCEANU has found the connexion, induced in V_n^m, just in this way.

§ 4. *Properties of curvature of an* A_n^m *in* A_n.

We define the first and the second *affinor of curvature* in the same way as in an A_m in A_n:

$$\left.\begin{aligned} H_{ba}^{\cdot\cdot\nu} &= B_{ba}^{\mu\lambda}\,\nabla_\mu B_\lambda^{\nu'} = -(\nabla'_b \overset{e}{e}_a)\underset{e}{e}^\nu = \Lambda_{ab}^{e}\,\underset{e}{e}^\nu \\ L_b^{\cdot c}{}_{\cdot\lambda} &= B_{b\nu}^{\mu c}\,\nabla_\mu B_\lambda^\nu = -(\nabla_b \underset{e}{e}^c)\,\overset{e}{e}_\lambda = -\Lambda_{eb}^{c}\,\overset{e}{e}_\lambda \end{aligned}\right\} \quad . \quad . \quad . \quad (27)$$

It strikes that, just as in an A_m in A_n, $H_{ba}^{\cdot\cdot\nu}$ lies with the index ν in the local $(n-m)$-direction and $L_b^{\cdot c}{}_{\cdot\lambda}$ contains with the index λ the local m-direction:

$$B_\alpha^\nu H_{ba}^{\cdot\cdot\alpha} = 0 \quad ; \quad B_\lambda^\alpha L_b^{\cdot c}{}_{\cdot\alpha} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

But here $H_{ba}^{\cdot\cdot\nu}$ is no longer symmetrical in a and b, because

$$H_{[ba]}^{\cdot\cdot\nu} = -B_{ba}^{\mu\lambda}(\nabla_{[\mu}\overset{e}{e}_{\lambda]})\underset{e}{e}^\nu = \Lambda_{[ab]}^{e}\,\underset{e}{e}^\nu \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and this expression vanishes if and only if all vectors $\overset{c}{e}_\lambda$ are X_{n-1}-building viz. if the field of m-directions is X_m-building, the case which we have excluded expressely.

It follows from

$$\left.\begin{aligned} H_{[ba]}^{\cdot\cdot\nu} &= B_{[ba]}^{\mu\lambda}(\partial_\mu B_\lambda^\nu + \Gamma_{\alpha\mu}^{\nu} B_\lambda^\alpha - \Gamma_{\lambda\mu}^{\alpha} B_\alpha^\nu) = B_{ba}^{\mu\lambda}\,\partial_{[\mu} B_{\lambda]}^\nu = \\ &= \partial_{[b}\, B_{a]}^\nu - B_\alpha^\nu\,\partial_{[b}\, B_{a]}^\alpha = C_\alpha^\nu\;\partial_{[b}\, B_{a]}^\alpha \end{aligned}\right\} \quad (30)$$

that the field of m-directions is X_m-building if and only if $C_\alpha^\nu\,\partial_{[b}\,B_{a]}^\alpha$ vanishes.

The ordinary method of obtaining the quantity of curvature is here useless because in an A_n^m *it is generally impossible to construct a parallelogram*, this impossibility being exactly characteristic for a non X_m-building field of m-directions. In fact, if on the one side a translation $\underset{1}{d}y^c$ is followed by a translation $\underset{2}{d}y^c$, and on the other side $\underset{2}{d}y^c$ by $\underset{1}{d}y^c$, then by using (23) and (30) we find for the closing vector the equation

$$2\,\underset{1}{d}\,y^b\,\underset{2}{d}\,y^a\,\partial_{[b}\,B_{a]}^\nu = 2\,\underset{1}{d}\,y^a\,\underset{2}{d}\,y^b\,(H_{[ba]}^{\cdot\cdot\nu} + \Lambda_{[ab]}^{c}\,B_c^\nu) \quad . \quad . \quad . \quad (31)$$

giving the decomposition into one component in the A_n^m and one in the local $(n-m)$-direction. The latter one only vanishes when the field of m-directions is X_m-building, the other one depends on the choice of the systems $\binom{c}{a}$.

So we choose another way and start with $\nabla'_{[l}\,\nabla'_{j]}\,v^k$ which certainly is an affinor. We get

$$\left.\begin{aligned} \nabla'_{[d}\,\nabla'_{b]}\;v^c &= H_{[db]}^{\cdot\cdot\alpha}\,B_\beta^c\,\nabla_\alpha v^\beta + \{H_{[db]}^{\cdot\cdot\alpha}\,(B_a^\beta\,\partial_\alpha\,B_\beta^c - B_{\gamma a}^{c\beta}\,\Gamma_{\beta\alpha}^{\gamma}) + \\ &+ \partial_{[d}\,\Lambda_{|a|\,b]}^{c} + \Lambda_{p\,[d}^{c}\,\Lambda_{|a|\,b]}^{p} - \Lambda_{[b\,d]}^{p}\,\Lambda_{ap}^{c}\}\,v^a \quad ; \quad p = 1, \ldots, m. \end{aligned}\right\} \quad . \quad (32)$$

The expression corresponding with the right hand side of (13)[1]) is here no longer an affinor, but the expression

$$\left.\begin{array}{l} R'^{\,\cdot\cdot\cdot c}_{dba} = -2\,H_{[db]}^{\cdot\cdot\cdot\alpha}\,(B^{\beta}_{a}\,\partial_{\alpha}\,B^{c}_{\beta} - B^{c\beta}_{\gamma a}\,\Gamma^{\gamma}_{\beta\alpha}) - 2\partial_{[d}\,\Lambda^{c}_{|a|\,b]} - 2\,\Lambda^{c}_{p[d}\,\Lambda^{p}_{|a|\,b]} + \\ \qquad + 2\,\Lambda^{p}_{[bd]}\,\Lambda^{c}_{ap} \ \ ; \ \ p = 1, \ldots, m, \end{array}\right\} . \quad (33)$$

which we call the *quantity of curvature* of the A^m_n, is. Using the parameters belonging to the systems $\binom{k}{i}$ and the equations (25) and (29), we may write for the first term of $R'^{\,\cdot\cdot\cdot c}_{dba}$

$$-2\,H_{[db]}^{\cdot\cdot\cdot\alpha}\,B^{\beta}_{a}\,\nabla_{\alpha}\overset{c}{e}_{\beta} = -2\,H_{[bd]}^{\cdot\cdot\cdot e}\,\Lambda^{c}_{ae} = 2\,\Lambda^{e}_{[bd]}\,\Lambda^{c}_{ae} \;. \quad . \quad (34)$$

and this expression can be added to the last term of (33) so that finally

$$\left.\begin{array}{r} R'^{\,\cdot\cdot\cdot c}_{dba} = -2\,\partial_{[d}\,\Lambda^{c}_{|a|\,b]} - 2\,\Lambda^{c}_{p[d}\,\Lambda^{p}_{|a|\,b]} + 2\,\Lambda^{j}_{[bd]}\,\Lambda^{c}_{aj} \\ p = 1, \ldots, m, \\ j = 1, \ldots, n. \end{array}\right\} \quad . \quad . \quad (35)$$

From (32) and (33) follows

$$\nabla'_{[d}\,\nabla'_{b]}\,v^{c} = H_{[db]}^{\cdot\cdot\cdot\beta}\,B^{c}_{\gamma}\,\nabla_{\beta}\,v^{\gamma} - {}^{1}/_{2}\,R'^{\,\cdot\cdot\cdot c}_{dba}\,v^{a} \quad . \quad . \quad . \quad . \quad (36)$$

If the field of m-directions is X_m-building, then $H_{[ba]}^{\cdot\cdot\cdot\gamma}$ vanishes, (36) takes the ordinary form and (33) regains the same form as (13).

If the A_n passes into a V_n, the quantity of curvature passes into

$$\left.\begin{array}{r} K'^{\,\cdot\cdot\cdot c}_{dba} = -2\,\partial_{[d}\,\gamma^{c}_{|a|b]} - 2\,\gamma^{c}_{p[d}\;\;\gamma^{p}_{|a|b]} + 2\,\gamma^{j}_{[bd]}\,\gamma^{c}_{aj} \\ p = 1, \ldots, m, \\ j = 1, \ldots, n. \end{array}\right\} . \quad (37)$$

§ 5. *The generalised equation of* GAUSS.

From the definition of the induced connexion it is easily deduced for a field v^c of the A^m_n:

$$\nabla'_{[d}\,\nabla'_{b]}\,v^{c} = H_{[db]}^{\cdot\cdot\cdot\beta}\,B^{c}_{\gamma}\,\nabla_{\beta}\,v^{\gamma} + L_{[d}^{\cdot\,c}{}_{|\alpha|}\,H_{b]a}^{\cdot\cdot\alpha}\,v^{a} - {}^{1}/_{2}\,B^{\delta\beta c}_{db\gamma}\,R_{\delta\beta\alpha}^{\cdot\cdot\cdot\gamma}\,v^{\alpha} \, . \quad (38)$$

from which follows

$$\boxed{B^{\delta\beta\alpha c}_{dba\gamma}\,R_{\delta\beta\alpha}^{\cdot\cdot\cdot\gamma} = R'^{\,\cdot\cdot\cdot c}_{dba} + 2\,H_{[b|a|}^{\cdot\cdot\cdot\alpha}\,L_{d]}^{\cdot\,c}{}_{\alpha}} \quad . \quad . \quad . \quad : \quad (39)$$

This is the generalised equation of GAUSS for an A^m_n in A_n and we see that it has the same form as the equation for an A_m in A^*_n.

§ 6. *Geodesics in* A^m_n *and in* A_n.

A geodesic in A^m_n is a curve, generated by the pseudoparallel

[1]) This expression has been found by VRANCEANU but the affinor $K'^{\,\cdot\cdot\cdot c}_{dba}$ from (37) does not occur in his papers.

displacement of a contravariant vector in its own direction. t being a parameter on a geodesic, $\frac{dy^b}{dt} \nabla'_b \frac{dy^c}{dt}$ must have the direction of dy^c :

$$\frac{d^2y^c}{dt^2} + \Lambda_{ab}^{\ \ c} \frac{dy^a dy^b}{dt\ dt} = \alpha \frac{dy^c}{dt} \quad \ldots \ldots \ldots \quad (40)$$

Hence a geodesic in A_n^m is also a geodesic in A_n if and only if the vector

$$\begin{aligned} \frac{dx^\mu}{dt} \nabla_\mu \frac{dx^\nu}{dt} - \frac{dx^\mu}{dt} \nabla'_\mu \frac{dx^\nu}{dt} = \frac{dx^\mu}{dt} \left(\nabla_\mu \frac{dx^\alpha}{dt} \right) C_\alpha^\nu = \\ = \frac{dx^\mu}{dt} \frac{dx^\lambda}{dt} \nabla_\mu B_\lambda^\nu = \frac{dy^a}{dt} \frac{dy^b}{dt} H_{ba}^{\cdot\cdot\nu} \end{aligned} \quad \ldots \quad (41)$$

has the direction of dx^ν . In consequence the geodesics in A_n^m are always geodesics in A_n if and only if $H_{(ba)}^{\cdot\cdot\nu}$ vanishes. Thus the alternating part $H_{[ba]}^{\cdot\cdot\nu}$ which is of such a fundamental importance for the non-holonomity of A_n^m, has nothing to do with this question concerning the geodesics. To the case of a geodesic A_m in A_n corresponds the case of an A_n^m with $H_{(ba)}^{\cdot\cdot\nu} = 0$, all geodesics being also geodesics of A_n. If the A_n passes into a V_n, there exist also shortest curves in V_n^m. But it is immediately clear that shortest curves and geodesics are not identical here. In fact, through a point of V_n^m only ∞^{m-1} geodesics pass but generally ∞^{n-1} shortest curves, because every point of the V_n can be connected with every other point by a curve lying wholly in V_n^m. As an example we take the linear complex in R_3 belonging to a system of forces. The field of the 2-directions belonging to every point is not V_2-building and may be given by the equation

$$p_\lambda = a_\lambda + r^\alpha f_{\alpha\lambda} \quad \ldots \ldots \ldots \quad (42)$$

a_λ being a constant vector, $f_{\lambda\mu}$ a constant bivector and r^ν the radius-vector. Writing p for the length of p_λ and i_λ for the unit vector belonging to p_λ we have

$$B_{\mu\lambda}^{\beta\alpha} \nabla_\beta i_\alpha = \frac{1}{p} B_{\mu\lambda}^{\beta\alpha} \nabla_\beta p_\alpha = \frac{1}{p} B_{\mu\lambda}^{\beta\alpha} f_{\beta\alpha} = \frac{1}{p} f'_{\mu\lambda} \quad \ldots \quad (43)$$

and

$$H_{\mu\lambda}^{\cdot\cdot\nu} = -\frac{1}{p} f'_{\mu\lambda} i^\nu \quad \ldots \ldots \ldots \quad (44)$$

$f'_{\mu\lambda}$ being the V_3^2-component of $f_{\mu\lambda}$. The straight lines of the complex are geodesics as well in R_3 as in V_3^2. Obviously two arbitrary points in R_3 can not be connected by a geodesic of V_3^2 but always by a curve lying wholly in V_3^2. The quantity of curvature of V_3^2 is

$$K'_{db\,ac} = -2 H_{[b|a|}^{\cdot\cdot e} H_{d]ce} = -\frac{2}{p^2} f'_{[b|a|} f'_{d]c} \quad \ldots \quad (45)$$

§ 7. *Affine geometry of an X_n^{n-1} in A_n.*

We will prove that an X_n^{n-1} in A_n determines an affine-normal direction in the same way as an X_{n-1} in A_n does, if the following two conditions are fulfilled.

1. The connexion in A_n leaves invariant each volume. (In E_n this condition is always fulfilled).

2. t_λ being a covariant vector having in every point the $(n-1)$-direction of the X_n^{n-1}, the affinor $h_{ba} = B_{ba}^{\mu\lambda} \nabla_\mu t_\lambda$ has the rank $n-1$.

If the connexion A_n leaves invariant every volume, there exists a constant n-vectorfield $P_{\lambda_1 \ldots \lambda_n}$. Every other constant n-vectorfield can be obtained by multiplying $P_{\lambda_1 \ldots \lambda_n}$ with a *constant* scalar. Now if h_{ba} has the rank $n-1$, t_λ can be chosen in a unique way, so that

$$t_{[\mu_1} t_{[\lambda_1} k_{\mu_2 \lambda_2} \ldots k_{\mu_n] \lambda_n]} = P_{\mu_1 \ldots \mu_n} P_{\lambda_1 \ldots \lambda_n} \quad . \quad . \quad . \quad . \quad (46)$$

If the constant n-vectorfield be changed, t_λ only takes a *constant* scalar factor. The affine-normal vector can now be defined by means of the equations

$$\left.\begin{aligned} t_\mu n^\mu &= 1 \\ B_a^\mu (\nabla_\mu t_\lambda) n^\lambda &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

h_{ba} having the rank $n-1$, n^ν is determined but for a *constant* scalar factor. Thus the affine-normal direction is found.

By use of the direction of n^ν just found, the X_n^{n-1} can be rigged, and an affine geometry can be obtained, as indicated in the former paragraphs.

Instead of h_{ba} also $k_{ba} = h_{(ba)}$ or $f_{ba} = h_{[ba]}$ can be used to construct the affine-normal direction.

Geology. — *Concerning the Tertiary of Atcheen.* By Prof. K. MARTIN.

(Communicated at the meeting of March 31, 1928).

In the north of Atcheen young tertiary sediments occur, that have been surveyed by the Survey-office of the Department of mines. More than 6000 mollusca from these deposits were examined to the following effect :

The fossils belong to 347 different species, which sometimes enable us to recognize distinctly a mutation of the species. So, for instance, a marked mutation occurs of the miocene *Melongena gigas*. To designate aberrant forms of species, already known from younger strata, the addition *prior* is adopted. Mutations and priores are of great significance in establishing the age of sediments.

Of the named species from Atcheen only 117 from the tertiary of Java are known, and most of them (77) are found in this island in the pliocene formation, while 66 % are still living in the present time. Moreover, a number of fossils are lacking that characterize the miocene of Java. So the strata of Atcheen must be considered to belong to the pliocene. As yet a perfectly similar formation from the East-Indian Islands is not known, but it may be that in the pliocene of Atcheen sediments occur that correspond to the Sondé-strata of Java.

The Palaeontologic data are insufficient for a grouping of the tertiary of Atcheen into zones of different ages, for the faunae of the consecutive zones, separated by the geologists, reveal a very close affinity. This appears not only from a direct comparison of all fossils, but also from the procentic amount of recent species, known in the several strata. The fossils examined have been gathered in a complex of 6 zones, whose thickness can be estimated at $\pm$ 3000 m. The pliocene geosyncline must, therefore, have subsided very quickly in this region, while the sedimentation was very intense.

The general character of the fauna is purely indopacific. European species are absolutely absent, and FROST's record of the presence of European fish-otoliths is erroneous. It is a mistake due to otoliths being unfit for the determination of the species. The connection between the Indian and the European sea was entirely broken during the pliocene time in the Indian Archipelago, as well as in British-India, where strata occur corresponding to the younger tertiary of Java.

An extensive treatise on the subject will appear in "Wetenschappelijke Mededeelingen", issued by the Mining Department in the Dutch East-Indies.

Chemistry. — *Optically active α-arsoncarboxylic acids.* By H. J. BACKER and C. H. K. MULDER. (Communicated by Prof. F. M. JAEGER.)

(Communicated at the meeting of March 31, 1928).

The optically active sulfocarboxylic acids show remarkable changes in rotatory power on neutralisation.

For the purpose of comparison we have now prepared some corresponding α-arsoncarboxylic acids. These acids are obtained in excellent yield by the reaction of the potassium salts of α-bromocarboxylic acids with potassium arsenite :

$$Br . CHR . CO_2K + K_3AsO_3 = AsO_3K_2 . CHR . CO_2K + KBr.$$

Only the first two terms of the series, arsonacetic acid [1]) and α-arsonpropionic acid [2]), are known. We have prepared α-arsonbutyric and α-arsonvaleric acids :

$$AsO_3H_2 . CH(C_2H_5) . CO_2H \quad AsO_3H_2 . CH(C_3H_7) . CO_2H.$$

α-Arsonbutyric acid crystallises from water in anhydrous prisms and sometimes in plates melting at 127°. α-Arsonvaleric acid, recrystallised from the same solvent, separates in prisms or plates of melting point 114°. These acids may be titrated as dibasic acids, when using phenolphthalein as indicator.

The α-arsonic acids, derived from propionic, butyric and valeric acids, have an asymmetric carbon atom and should thus be resolvable into optically active enantiomorphs.

In fact the resolution has been successful by means of the secondary quinine salts.

α-Arsonpropionic acid gives a diquinine salt with 6 molecules of water ; the corresponding salt of α-arsonbutyric acid crystallises with 5 molecules, that of α-arsonvaleric acid with 4 molecules of water. The solubility of these diquinine salts in water is slight, namely about 0.8, 0.7 and 0.6 % respectively at room temperature.

The quinine salts may be recrystallised from dilute alcohol. They separate in concentric conglomerates of long needles.

Careful decomposition by baryta gave the barium salts, from which hydrochloric acid liberated the active acids.

About six recrystallisations were necessary to obtain the quinine salts in the pure active form.

The rotation of the acids and their salts was measured for different

[1]) PALMER, J. Amer. Chem. Soc. **45**, 3023 (1923). RAMBERG, Svensk Kem. Tidskr. **36**, 119 (1924).

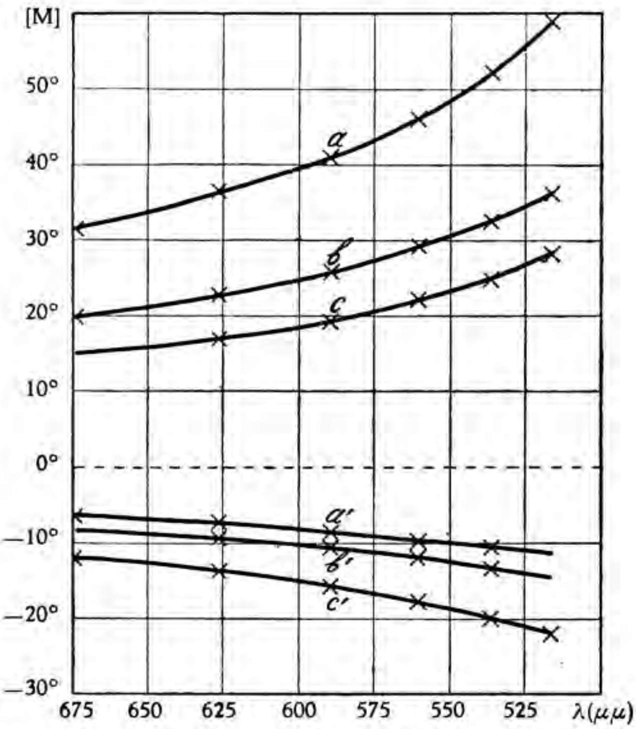
[2]) WIGREN, Svensk Kem. Tidskr. **36**, 127 (1924).

wave-lengths. The curves of rotatory dispersion have been plotted by means of the values, which are given in the following tables.

free acids	λ(μμ)	674.0	626.5	589.3	560.5	536.5	516.0
d-α-arsonpropionic	[M]	+31°.5	+36°.1	+41°.0	+46°.0	+52°.2	+58°.9
d-α-arsonbutyric	„	+19.7	+22.7	+25.7	+29.1	+32.5	+36.3
d-α-arsonvaleric	„	—	+16.7	+19.3	+22.1	+24.8	+28.3

secondary barium salts	λ(μμ)	674.0	626.5	589.3	560.5	536.5	516.0
d-α-arsonpropionic	[M]	— 6°.5	— 7°.3	— 8°.5	— 9°.7	—10°.5	—
d-α-arsonbutyric	„	—	— 9.4	—10.5	—11.9	—13.4	—
d-α-arsonvaleric	„	—10.2	—13.7	—15.6	—17.7	—20.0	—22.0

The tables and the figure show that neutralisation of two acid functions changes the sign of rotation.



Rotatory dispersion of α-arsoncarboxylic acids and their secondary barium salts.

a. α-Arsonpropionic acid *a'*. Salt.
b. α-Arsonbutyric acid. *b'*. Salt.
c. α-Arsonvaleric acid. *c'*. Salt.

The relation between the molecular rotation of the acid and the salt is —4.8 for α-arsonpropionic acid, —2.4 for α-arsonbutyric acid and —1.2 for α-arsonvaleric acid. Thus this relation changes gradually.

The study of the permanence of rotatory power has giv n remarkable results.

Whilst optically active sulfocarboxylic acids are less stable in alkaline than in acid solution, the arsoncarboxylic acids show just the opposite behaviour.

The barium salts of the active arsoncarboxylic acids are perfectly stable in aqueous solution, even at 100°.

The activity of the free acids, however, is labile, especially in the presence of strong inorganic acids.

The racemisation was examined kinetically; it proved to be a monomolecular reaction.

The constant increases almost proportionally to the quantity of hydrochloric acid added. The racemisation constant of the pure acids increases with their molecular weight.

A detailed account of the α-arsoncarboxylic acids and their derivatives will be published elsewhere.

Organic Chemical Laboratory of the State University.

Groningen.

Palæontology. — *The Law of the Necessary Phylogenetic Perfection of the Psychoencephalon.* By Prof. EUG. DUBOIS.

(Communicated at the meeting of March 31, 1928).

Clearly palæontology bears evidence of the growth of life on the Earth.

Complex living beings, highly organized animals and plants have appeared after less complex living beings, simply organized animals and plants, taken on the whole in accordance with the natural systems. Of the Vertebrates — to mention only this group of animals — we find first Fishes, then also Amphibia and Reptiles, and finally Birds and Mammals, and in successive Ages the dominant types are Fishes, Reptiles, Mammals. In the vegetable kingdom appear and become dominant in successive Ages the Cryptogams, the Gymnosperms, the Angiosperms. We see life extend in the water, on the land, in the air. The diversity of the forms of the living beings and of their functions is becoming greater and greater and the efficiency more and more manifold as adaptation to the great diversity of surroundings.

Besides, in successive geological levels we find sequences of animal forms. Following one after another, each form is supplanted by a new one resembling the preceding one closely, but distinguished from it by intensification, transformation, or weakening of some characters.

All these and similar facts can hardly be explained in another way than by descent of the living beings from each other.

The absence of proof of continuous development should, however, also be considered as an other important result of the palæontological research of the earth's crust. The minutest transitions between the members of the many parts of lines of descent with which we became acquainted, were so regularly absent, that this absence can no longer be attributed to "the imperfection of the geological record". Every member of a sequence is stepwise distinguished from the preceding and the following one. Again and again we find pillars of the expected bridges, never arches. Thus, for instance, in the series of the *Equidae*, starting with *Eohippus*, of the *Proboscidea*, starting with *Moeritherium*.

The members of these sequences are intermediate forms, no "transition forms", no "links". Indeed, repeatedly palæontology has removed from the hypothetical stock line its at first much made of transition forms, referring them to side branches of the genealogical tree.

Strictly speaking gradual transition is a priori impossible, because every species living as independent creature, being adapted to particular circumstances of life, must be specialized in its peculiar way.

As it must, however, be assumed that every animal and every plant, in the past as well as at present, owe their existence to another animal and another plant, the present vegetable and animal world must necessarily be connected to those of all the earlier epochs by ties of consanguinity, i.c. by descent or phylogenesis.

That we again and again meet only with pieces of these ties, and that gradual transition between creatures living independently as different species is a priori impossible, leads to the conclusion that the real transitions, the missing pieces of the ties of relationship, the arches of the bridges connecting the species, took place **in the embryonal period of the individual life,** before the independent existence of the individual.

Of this palæontology could not furnish any documents, because they never existed. *Pithecanthropus, Mesohippus, Procamelus, Moeritherium* are pillars of these bridges, the arches of which — we must now assume — will never be found.

But the animals possess a complex of organs, the brain, which by the palæontological way, c a n teach us something about this transition from species to species accomplished in the embryonic state. This applies in particular to the cerebrum or prosencephalon of the Vertebrata and first of all of the Mammals, that part of the encephalon of the latter which is the direct and most highly specialized organ of the psychic functions of the animal, the definite p s y c h o e n c e p h a l o n.

In contrast with almost all the rest of the body, the cells of which multiply by division throughout life, the process of cell division is here entirely restricted to the embryonal period of life. The condition reached in the embryonal period is, therefore, preserved all through life, and the adult psychoencephalon still shows the embryonal cell division term in its volume, which latter, although still increasing subsequently to birth, no longer increases by cell division. This is the reason why the relative volume of this organ (which can be examined by means of natural and artificial endocranial casts also of extinct species of animals) increases in its phylogenesis in the regular proportion of the terms of a geometrical series with the ratio 2, for the cell division is a repeated division into two.

Here appears a direct relation between ontogenesis and phylogenesis.

Phylogenetically the senses of sight, smell, and hearing, placed at the head end of the animal body, "distance receptors", progress apparently concomitant with the cephalisation of the nervous system. With their larger psychoencephalon than that of the earlier forms, the phylogenetically later mammalian forms have also larger eyes, and larger olfactory receptors (we find the cribriform plate of the ethmoid bone extended, indicative of an increased number of olfactory nerve fibres). Compare the skulls of *Lama* with *Procamelus* (Fig. 1), *Martes* with *Mustela* (Fig. 2).

The concomitant perfection of the senses, accompanied with particular or general perfection of the rest of the sensorium, and the motorium, is, however, combined with d i v e r s e m o d e s of specialization of the whole animal, ensuing from adaption through habit with negative selection ("adaptiogenesis"). Evidently the d e g r e e of every kind of this diverse

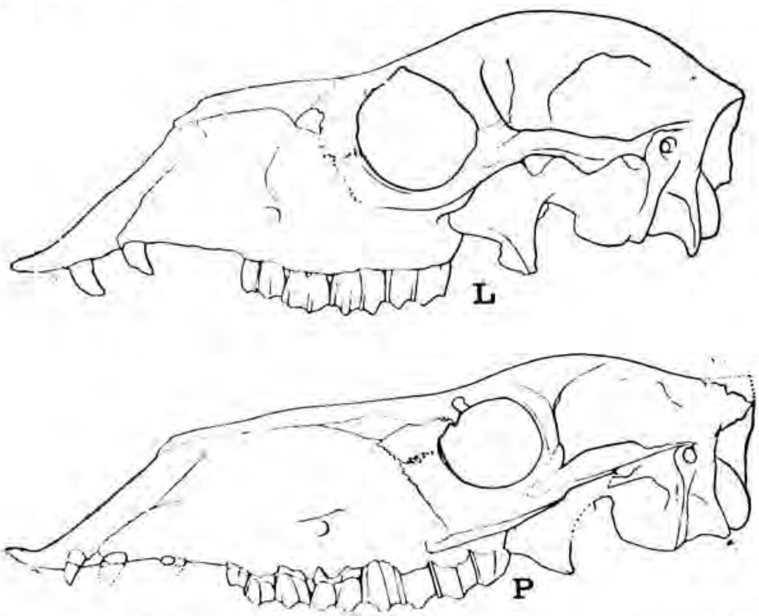


Fig. 1. *L.* Side view of skull of the present day *Lama huanachus* Molina. Accurate photographic outlines. 1/3 Nat. size.
P. Side view of skull of the Upper Miocene *Procamelus gracilis* Leidy. After COPE. 1/3 Nat. size. Bodily both Camelidae, the ancestral form and the descendant, are of the same size and general build.

specialization is determined by the degree of cephalization as given, in the Vertebrates, by the size of the encephalon and this in particular of the most central organ of the psychic functions, that part of the encephalon which in morphology is named the prosencephalon, and may according to its functions be rightly termed the psychoencephalon.

The increase of the size of the prosencephalon in the series of the Vertebrates, from the Fishes to the Mammals, has long been known. It is very striking that such a progress also exists between mammalian species of Early Tertiary and of recent times, belonging to the same group.

With regard to the volume of the whole encephalon in proportion to the volume of the body, this progress was first discovered by ED. LARTET, in 1868, by comparing some mammals (ungulates and carnivores) from the Lower Tertiary of France with recent species. CH. DARWIN made mention of this "remarkable conclusion", in which he saw evidence in support of gradual increase of the brain, in his "Descent of Man" (1871). LARTET thought that by way of explanation, he had to suppose a "tendance de la nature animée vers un perfectionnement dont la cause resterait toujours agissante et la limite indéfinie".

The fact of the small size of the brains of Lower Tertiary mammals was established by O. C. MARSH for a number of species in North-America

(1874—1897). Increase of brain volume was also shown by him for Birds (1880) and Reptiles (summarized 1896) of the Cretaceous period compared with the corresponding forms of recent times.

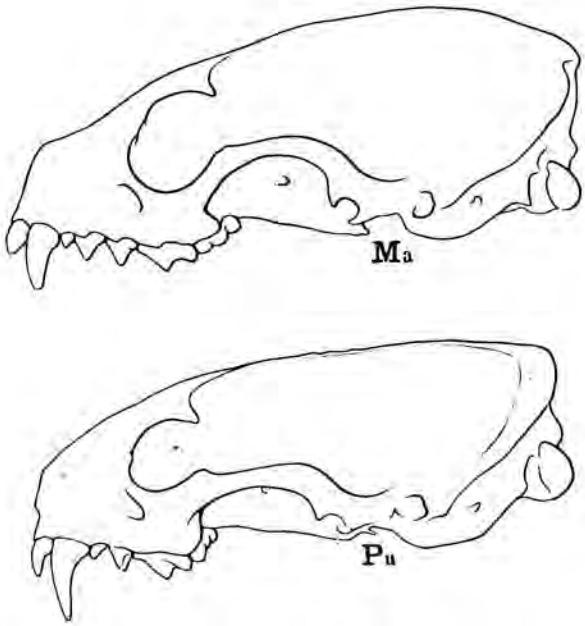


Fig. 2. *Ma.* Side view of skull of a full-grown *Martes foina* Erxl. ♀. Body weight 1160 g (somewhat below average of species).
Pu. Side view of skull of a full-grown *Mustela putorius* L. ♂. Body weight 1220 g (somewhat above average of species). Both accurate photographic outlines. Nat. size.

Concerning the extinct mammals throughout Tertiary time he drew up the following "general law of brain-growth":

1. "All Tertiary mammals had small brains."
2. "There was a general increase in the size of the brain during this period."
3. "This increase was confined mainly to the cerebral hemispheres, or higher portion of the brain."

To this he added:

1. "The brain of a mammal belonging to a vigourous race, fitted for a long survival, is larger than the average brain, of that period, in the same group."
2. "The brain of a mammal of a declining race is smaller than the average of its contemporaries of the same group."

Accordingly, he quite adopted a causal nexus with the Spencerian "survival of the fittest", the Darwinian "natural selection".

Though it was known to him that "in any comparison of the size of the brain in different animals, whether in the same group or in others widely different, it is important to bear in mind that the brain of small animals is proportionally larger in bulk than that of large animals", he was of opinion that this fact "can have only a limited effect, which would not change, essentially, the general results". Thus, as we know now, underrating the effect of the body size, MARSH was led to figure the brains enclosed in the skulls of small and large mammalian species at equal cranial length, through which disproportionally the former became too large, the latter too small.

It is mainly due to this underrating of the effect of the body size that the first two articles of the "general law of brain-growth", drawn up by MARSH, are not valid. Not all the Tertiary Mammals had small brains. Many from the Upper Tertiary are not inferior, as regards the size of the encephalon, to the recent species. Nor has the increase of the size of the brain during the Tertiary period been general. In the lowest cephalized Mammalia of the present day, many belonging to the Insectivora and the Chiroptera, such an increase has not taken place. Further we know now that it was not selection that has increased the volume of the encephalon.

Only the third article of the law of MARSH appears to be valid, but without the word "mainly", for the increase in size was exclusively confined to the cerebral hemispheres, the prosencephalon, in many later mammal species in comparison with earlier species.

Also in the present day animal world this phylogenetic brain-growth, in particular the growth of the prosencephalon, is to be recognized by comparison with allied forms, because some families and genera are more specialized, more progressive, particularly in this organ, than others.

Especially in the Vertebrata it was found in every class, but in the Mammalia in many cases, that the more specialized forms of a group, hence the progressive ones, regularly possess much greater relative brain volume than the conservative species, that are still more closely allied to the original forms.

The "comparative-anatomic" method of inquiry into the phylogenesis thus possible has very particular advantages here, as the size of the body can mostly only be determined accurately in living species.

For in every animal the volume of the brain is not only dependent on cephalization, but evidently also on the size of the body. The volume (and with this the weight) of the brain is really to be considered as the product of two factors. One corresponds to the degree of organization of the organ, the cephalization, the other depends on the bulk of the body. If the factor or coefficient of cephalization is called k, the brain weight E, the body weight P, E appears in general to be proportional to kP , when full-grown animals of two species are compared. Hence the coefficient of cephalization k is proportional to $\frac{E}{P^{5/9}}$. Between two full-grown individuals of one and the same species living in the natural state, the somatic factor is $P^{5/18}$. These are laws valid for all the classes of Vertebrates. For domesticated species (among others Man), the exponent is somewhat smaller.

With the phylogenetic exponent $^5/_9$ we now get not only equal values of

k for animal species which are considered as similar in their whole organization, for instance the members of the very homomorphous family of the *Felidae*, which differ so greatly in body size and weight (the latter varying in the ratio 1 to 100), but also a sequence of the Mammals which — leaving aside certain exceptions to the rule — is nowhere in contradiction to the natural system, if the k is calculated for the psychoencephalon.

If the value of the k of Man is put as unit, the coefficient in the Anthropoid Apes, *Hylobates* included, is $\frac{1}{4}$, in the *Canidae*, *Felidae*, and also the *Bovidae* and *Cervidae* differing little from $^1/_8$, in the *Leporidae* $^1/_{16}$, in the *Muridae* $^1/_{32}$, and in the *Soricidae*, *Centetidae*, and most *Microchiroptera* $^1/_{64}$. Small deviations in some groups are due, as has appeared, to unimportant secondary variations in the cerebellum or some parts of the prosencephalon.

In all these calculations the encephalon with its different, functionally inequivalent parts, was at first considered as a whole. According to what had been observed in the encephalon of Lower Tertiary mammals, strictly the psychoencephalon alone should have been taken into consideration. But between the living mammalian species, leaving the families of the *Insectivora* and *Microchiroptera*, which are on the lowest level of cephalization, out of account, the ratio of volume and weight of the psychoencephalon to the other parts of the brain, which moreover in most cases have a subordinate volume and weight, differ little. Hence the relations of the quantity found for the encephalon to those of the body can in general be considered as also valid for the psychoencephalon. As was already said before, the psychoencephalon has, however, a much smaller relative share of the total volume of the encephalon in the Lower Tertiary Mammals and also in many present day *Insectivora* and *Microchiroptera*. Besides, there are among the living mammalian species some in which the volume of the cerebellum has considerably increased secondarily and independently, namely doubled. In all these cases the cephalization of the psychoencephalon must be taken into consideration in itself, which has been done in the above sequence.

There is no need at present to point out that all the relations of quantity of the brain and the body are based on quantity relations of the cells, the neurones, to the body [1]). We may only remind here, that between the volume of ganglion cells and the body the same relation exists as between the weights of the brain and the body in the individuals of the same species.

This fact and direct counting prove that the number of the brain cells (neurones) is characteristic of the species, and remains the same for all individuals. Only the volume of the brain neurones increases, according to the above relation, with the body weight.

In homoneuric, but larger species, also the number of the brain neurones increases in the same proportion to the body weight.

With higher cephalization the number of the brain neurones increases,

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[1]) I may refer for this to my earlier communications.

independently of the body size, proportionally to the cephalization coefficient. If, therefore, two differently cephalized species of the same body size are compared, the number of nerve elements (neurones) in the brain varies as the value of the cephalization coefficient k, between two species of different body size as $kP^{5/9}$.

The existence of the relations of quantity of the central nervous system, and in particular of the psychoencephalon to the whole body proves that the integrative action of this organ also extends to the ontogenetic growth or growth outside the individual and the phylogenetic growth or growth outside the species.

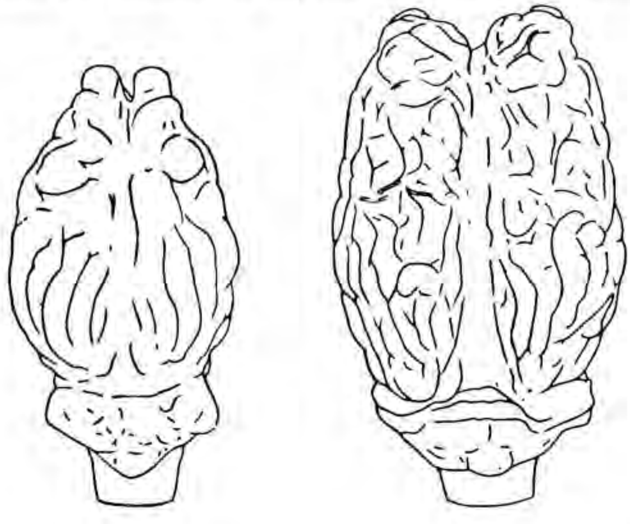
Now not a few species are seen to fall outside the sequences obtained by the method described, which taken on the whole, correspond to the natural zoological system, often outside their order or family, in some cases outside their genus, and this with a value of the cephalization coefficient k which is double or fourfold that of their nearest relations. A single recent form is seen falling outside the series with half the cephalization coefficient of its nearest relations, thus *Tragulus*. Striking is, for instance, the doubling of the coefficient in the genus *Martes* compared with the genus *Mustela* and the fourfold value in the *Megachiroptera* in comparison with most *Microchiroptera* [1]), also in *Tupaia* in comparison with *Sorex*. With regard to some extinct forms there are well-founded suspicions of phylogenetic relation with living species. When also the body size was known to some extent, I have included them in my comparison, thus *Pithecanthropus*, *Mesohippus*, *Procamelus*, *Moeritherium*, (?) *Proviverra (Cynohyaenodon)*. The affinity between *Palaeosyops* and *Tapirus* is only a less close one, but presumably an intermediate form connected these two Perissodactyles. Certainly the selenodont *Oreodon (Merycoidodon)* is still more distantly related with *Tayassus*, with which, however, it possesses some primitive characters in common.

It was really striking how regularly and accurately the determinations of the coefficients of cephalization led to equal results. The coefficient, calculated from the volume of the psychoencephalon, measured as exactly as possible, or from its weight, ascertained as exactly as possible, and the body weight of the species ascertained or estimated as accurately as possible [2]) appeared to increase exactly or very nearly by duplication,

[1]) In this case triplication was formerly found by comparison with the whole encephalon, in consequence of the fact that the cerebellum is very much larger relative to the prosencephalon in the *Microchiroptera* than in the *Megachiroptera*. For the prosencephalon quadruplication holds good.

[2]) In some cases, as in that of *Procamelus gracilis* Leidy and *Lama huanachus* Molina, to a certain extent also that of *Merycoidodon culbertsoni* Leidy and *Tayassus tajacu* L., the animals are of the same size and general build, so that the somatic factor may be eliminated in the equation. In most of the cases of fossil forms the body weight had to be estimated from comparison with similar living forms, however, an estimation of the body weight, even a roughly approximative one, may be serviceable, because it enters into the calculation with the low exponent $^5/_9$.

quadruplication, octuplication. A case of duplication between animals of the same size and general build is represented in Fig. 3. Further than the fourth term *(Moeritherium* to *Elephas)* the geometrical series is not continued in the following table, but undoubtedly the cephalization of some Lower Eocene



P L

Fig. 3. Superior view of endocranial cast of *Procamelus gracilis* Leidy (*P*) and *Lama huanachus* Molina (*L*).
Both accurate photographic outlines. 1/2 Nat. Size.
The casts belong individually to the skulls of Fig. 1.

Ungulata was still less than 1/8 of that of their far relatives of recent times. In so far as their body weight can be estimated by comparison, the Lower Eocene *Phenacodus* and *Coryphodon* possessed only 1/16 of the psychocephalization of the present day *Pecora*, but the Middle Eocene *Uintatherium* already 1/8 of this group, which is at the average level of the recent Mammals. Also *Moeritherium*, of about the same time, possessed 1/8 of the psychocephalization coefficient of *Elephas*, but this latter genus belongs to the most highly cephalized present day Mammals. In the same way the stock forms of *Rhinoceros* and of *Equus*, genera which, as regards cephalization, are above the present average level of the Ungulates, were, in their time, distinguished by a comparatively high cephalization. These facts give further support to the results recorded in the subjoined table. (See following page).

In each of the 17 sequences of this table, mammal forms on a lower level of cephalization precede related forms of a higher level, according to the geometrical progression of the cephalization coefficients. It is a matter of course, that in most of the sequences the mentioned species, genera, and higher groups are not to be considered as direct ancestors of one another. In consequence the lower forms are seldom of smaller size than those on a higher level of cephalization, and this easily may be conceived, since e.g. in such a homomorphous genus as *Felis*, the weights of the species diverge

Geometrical progression (ratio 2) of the psychocephalization

1. *Simiidae* — *Pithecanthropus* (Upper Pliocene) — *Homo*
2. *Callithrix* - *Saimiri* — { *Ateles* / *Cebus* }
3. *Lemur* — *Daubentonia*
4. *Mesohippus* (Middle Oligocene) — *Equus*
5. *Palaeosyops* (Middle Eocene) — ? — *Tapirus*
6. *Tragulus* — *Pecora*
7. *Procamelus* (Upper Miocene) — *Lama*
8. *Merycoidodon* (Middle Oligocene) — *Tayassus*
9. *Hippopotamus* — *Choeropsis*
10. *Moeritherium* (Upper Eocene) — [*Palaeomastodon* (Oligocene)] — [*Mastodon* (Miocene)] — *Elephas* [1])
11. *Mustela* — *Martes*
12. *Ursus* — *Helarctos*
13. *Proviverra* (Middle Eocene) — *Paradoxurus*
14. *Muridae* — { *Dipodinae* / *Leporidae* }
15. *Manis javanica* — { *Manis gigantea* / *Manis tetradactyla* }
16. *Microchiroptera* — *Vampirus* — *Megachiroptera*
17. { *Sorex* / *Centetes* } — { *Talpa* / *Myogale* / *Erinaceus* / *Potamogale* } — *Tupaia*

in the ratio 1 : 100, and this divergence is still greater between Mammals belonging to different genera and of more distant kinship. In this way the lower form *Hippopotamus* (of which, however, several small fossil species are known) may be much larger (indeed 10 times heavier) than the higher form *Choeropsis*. Thus, in each stage of cephalization, we find the forms grow to very different sizes. Moreover they diverge greatly in form and function by specialization, strikingly so in the second term of our series 17. As very important in this respect may be mentioned here the fact, that the psychocephalization of the much specialized and gigantic *Arsinoitherium* from the Lower Oligocene is equal to that of the small present day *Procavia* [2]) and most probably the Lower Oligocene *Megalohyrax* and other *Saghatheriidae*, from which fact we may infer that they all descended from a common stock of the next lower level.

Seemingly between the forms of equal cephalization, there may exist

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[1]) Although the cephalization of *Palaeomastodon* and that of *Mastodon* are not yet ascertained, they have been placed in this series of *Proboscidea* as intermediate (if not transitional) forms, well-known in other respects.

[2]) The estimated weight of *Arsinoitherium* is about 700 times the weight of *Procavia*

gradual transition of specialization, as there is gradual transition in the sizes of the animals. On a nearer view we find this to be impossible, as specialization, depending on adaptation, is only divergence of the mode of organization, not progress of its degree. On the other hand, progress of the degree of organization, indeed of animality, i.e. real evolution of life exists from term to term of psychocephalization.

The discontinuance in the phylogenetic ties thus undeniably exists in respect of the psychocephalization. But this discontinuance is a regular one, resulting from the property of living bodies of growing by cell division, the multiplication of the units that compose them.

Here clearly the development is autonomous, determined by internal factors resulting from a fundamental property of the living beings. Also phylogenesis appears here as a mode of growth, like ontogenesis.

Furthermore, ontogenetic and phylogenetic growth of the animal evidently depending on the integrative action of the psychoencephalon, we may infer that no gradual transition can exist between species of different cephalization, only abrupt change in the form and structure, real metamorphosis. The nevertheless necessary connection, as we saw, must have taken place in the embryonal period of animal life. A smaller phylogenetic distance between species than with doubled psychocephalization and corresponding progress of the whole organization is, therefore, not possible. Quite inconsistent with this conclusion appears to be the phylogenetic gradual transition required by darwinism.

Between animals of equal cephalization gradual transition appears as a matter of fact, manifesting itself in the seemingly unlimited diversity in the specializations of the organisms, of the present day and the past. But this apparent gradual transition, evidently resulting from habit and negative selection, is adaptiogenesis, irregular as depending on the multifarious fluctuation of the environments, to which the mode, not the degree of organization of the living being responds. This modification of the organs, apparently rapid at progress of cephalization, which compellingly excited it, is no phylogenetic growth, which, being autonomous, has a definite direction.

Besides, in the case of the psychoencephalon, phylogenetic growth means at the same time perfection of the organ, continual phylogenetic progress of its functions, for the higher organization is here immediately attained by the increase of the number of cells, which at once leads to more multiple combination, greater functional complexity, direct enlargement of the animal's outer world.

Here is a law of evolution come forth out of the nature of the living being itself, not imposed by the surroundings. CHARLES DARWIN might still think : "There is no evidence of any law of necessary development", and HERBERT SPENCER : "We find progression to result, not from a special, inherent tendency of living bodies, but from a general average effect of their relations to surrounding agencies". From what we now know of the evolution of the psychoencephalon, i.e. the evolution of animality, it appears that there actually does exist such a law of phylogenesis, and that with progression, with perfecting.

It is self-evident that this perfecting, this steady progression cannot have been caused by factors outside the animal, to which darwinism ascribed phylogenesis. External factors can only have effected d i v e r s i t y of the animal forms and functions, by adaptation. It is inconceivable that they should have acted continually in one definite direction, that of perfection.

Physics — *A formula expressing the deflection of the plumb-line in the gravity anomalies and some formulae for the gravity-field and the gravity-potential outside the geoid.* By F. A. VENING MEINESZ.

(Communicated at the meeting of January 28, 1928).

§ 1. In the middle of the 19th century STOKES succeeded in deducing a formula, which expresses the distance between the geoid and some chosen spheroid in the gravity anomalies of the whole earth. These gravity anomalies have to be computed by taking the difference of the observed gravity, reduced to the geoid by free air reduction (reduction of FAYE), and the normal gravity value, corresponding to the spheroid, which has been chosen; that is to say this normal value represents the gravity-field, which would exist on an outside potential surface of some theoretical earth of the same mass as the actual earth, for which this potential surface coincides with the spheroid. This condition determines the normal gravity-field completely, as a well-known potential theorem indicates. To the formula of the normal gravity may however be added any constant or any spherical harmonic of the first degree without changing the result given by the formula of STOKES. This is an advantage, which we will secure also in §§ 2 and 3 for some other formulae, as it renders harmless any error in the constant term of the formula for the normal gravity, which is not impossible so long as gravity is unknown for the greater part of the earth's surface.

The chosen spheroid has to fulfill the following conditions: The volume as well as the centre of gravity must coincide with the volume and the centre of gravity of the actual earth. The radius of curvature has not to deviate more from the earth's radius than in the ratio of the first order of the flattening and we assume further for the spheroid as well as for the geoid that the angle between the normal and the radius towards the earth's centre is of that same order. We suppose lastly that the distance between both surfaces is of the second order of the flattening in proportion to the earth's radius.

With these restrictions the spheroid is wholly arbitrary; it may be an ellipsoid, but this is not necessary and it is not even necessary that it is a body of revolution. It may further be emphasized that the formula of STOKES, corresponding to the fixed relation between gravity-field and shape of the geoid, is independent of the distribution of the masses of the earth, which are the causes of both; it is therefore independent of any assumption about the mass distribution.

It is clear that the existence of this fixed relation between the gravity-field and the shape of the geoid implies also a fixed relation between the gravity and the position of the normal on the geoid, that is to say between the gravity and the deflection of the plumb-line [1]). It is the object of this paper to deduce the corresponding equation, expressing the deflection of the plumb-line in the gravity-anomalies.

The easiest way to deduce such an equation is the differentiation of the distance between the geoid and the spheroid, given by the formula of STOKES, according to a direction tangent to the geoid; we find in this way the angle between the normal on the geoid and the normal on the spheroid, which may be considered as the deflection of the plumb-line with regard to the chosen spheroid. It is however necessary first to clear up two points concerning the equation of STOKES.

The first point is how far this formula gives also the local deviations of the geoid. This is questionable because of the fact, that in deducing it, a constant value R is substituted at a certain moment for the earth's radius. HELMERT for instance expresses doubt about it in "Die Theorien der höheren Geodäsie", and considers the equation as only valid for giving the general shape of the geoid. Would this be true, then the result for the deflection of the plumb-line got in this way, would be valueless; it is therefore necessary to prove, as we think it is possible to do, that this doubt is not founded. In this regard we may draw attention to a remarkable paper of POINCARÉ: "Les mesures de gravité et la géodésie", which is not generally known to geodesists and which appeared in the "Bulletin Astronomique" of 1901. POINCARÉ makes a study of the whole problem and, without knowing apparently about the work of STOKES, deduces the same formula and enlarges specially on the possibility of determining with this formula the local shape of the geoid with this equation. We will look into this question more closely in § 4.

The second point, which has to be examined, is the question concerning the effect of the masses outside the geoid. In deducing the formula of STOKES or one of the formulae of this paper, it is supposed that there are no masses outside the geoid, so that their validity is questionable so long as there are such masses. We will eliminate this difficulty by considering a regulated earth, for which these masses are removed. As the masses outside the geoid are fairly well known, the changing over from the actual earth to the regulated one presents no difficulty: we can compute the effect of these masses and find in this way the changes of the gravity anomaly and of the geoid, caused by the removal of these masses. The change of the gravity anomalies is about the same as the ordinary BOUGUER reduction. After having determined the shape of the regulated geoid by introducing these reduced anomalies in the formula of STOKES, the actual geoid can be derived by applying the difference between both geoids. It may be remarked — LAMBERT drew first attention to this point — that the change of the

[1]) See also of M. MARCEL BRILLOUIN: „Champ de grav. extérieur et densités internes", Comptes Rendus de l'Acad. d. Sciences, t. **180**, and t. **184**.

geoid will also bring along a shifting of the centre of gravity, so that the centre of gravity of the spheroid, which has to coincide with that of the regulated earth, will not quite coincide with that of the actual earth.

In order to lessen the difference between the geoids of the actual earth and of the regulated earth, of which we determine the shape with the formula of STOKES, we may partly compensate the taking away of the masses by adding at the same time corresponding masses inside the geoid. We may do this by applying the condensation method of HELMERT, who adds in the same vertical at a depth of 21 km the same quantity of mass as has been removed. Or we may do it according to the inversion method of RUDZKI, who compensates the removal of an outside mass at a distance l from the centre of the earth, by adding a mass, which is R/l times the former one, at a distance R^2/l from the earth's centre, R being the mean earth's radius. RUDZKI obtains in this way, that the geoid remains unchanged, so that the formula of STOKES gives at once the actual geoid. Or lastly we may follow the isostatic method, adding the same quantity of mass, as has been taken away, and distributing it inside the geoid according to one of the accepted methods. The reduction of the gravity anomalies, corresponding to this last method, is the ordinary isostatic reduction, if we extend this mass-regulation also to the oceanic part of the earth's crust, filling up the oceans till they have normal density with masses removed from the crust below. For the purpose, which we have in view here, this extension to the oceans would not of course be necessary.

Of the three compensation methods the isostatic one gives the greatest shift of the geoid, because the distance between the removed masses and the added masses is greatest. Still it seems to me that this method is preferable to the other ones. The greater shift of the geoid is no serious drawback, as it can be computed, and the method has the advantage that the field of gravity anomalies becomes more regular by the isostatic reduction than by any other: it removes as well as possible the effect of the local mass-irregularities in the crust. This advantage is worth mentioning because it makes each anomaly value representative for a greater area of the earth's surface, so that a certain limited number of anomalies will give a better image of the geoid.

When we adopt one of these methods for removing the outside masses in order to be able to apply the formulae on the regulated earth, we must take care of the following question. As has been remarked in the beginning, we have to reduce the observed gravity by free air reduction to the geoid, that is to say that we have now to reduce to the regulated geoid. We have therefore to apply an extra free air reduction for the distance between both geoids. If we take for instance the isostatic method of regulation we have the following series of reductions to the observed gravity: First the free air reduction to the actual geoid and then the reductions belonging to the regulation of the earth, including first the ordinary isostatic reduction, representing the change of gravity caused by the transport of masses and then the free air reduction from the original geoid to the regulated geoid. This last reduction is the same as the reduction of BOWIE.

Objection might be made that the masses outside the actual geoid have been removed and not the masses outside the regulated geoid. As a matter of fact we ought to remove the masses outside this last geoid. The difference may however be neglected if one of the compensation methods has been used, as in this case the distance between both geoids is small; it will in fact not exceed some twenty meters.

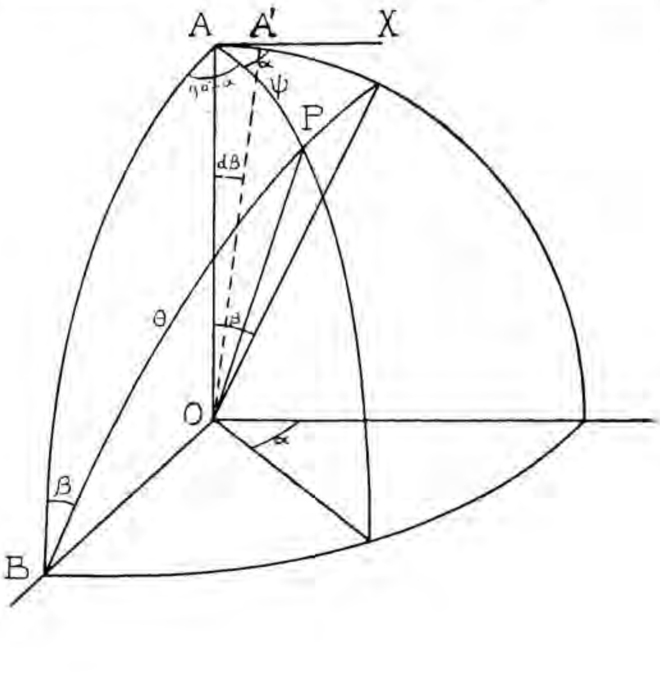
We will now proceed to the deduction of the formula for the deflection of the plumb-line, which we will do by differentiating the formula of STOKES according to a direction tangent to the geoid. This formula may be written :

$$N = \frac{R}{4\pi\gamma}\int^{\sigma} S_T \, \Delta_0 \, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1^A)$$

in which : N = distance between geoid and spheroid,
R = mean earth's radius,
$d\sigma$ = surface-element of a sphere with radius 1,
Δ_0 = gravity anomaly corresponding to the element $d\sigma$,
S_T = function of the angle ψ between the radius of the point A, where N is computed, and the radius of the point P coinciding with $d\sigma$:

$$S_T = \operatorname{cosec} \tfrac{1}{2}\psi + 1 - 6 \sin \tfrac{1}{2}\psi - 5 \cos\psi - 3 \cos\psi \, lg \left[\sin \tfrac{1}{2}\psi \left(1 + \sin \tfrac{1}{2}\psi\right)\right] \quad (1^B)$$

We choose in A an X direction tangent to the geoid and introduce an azimuthal coordinate α, representing the angle between OAX and OAP



Besides the system of coordinates ψ and α we introduce a second system θ and β, θ being the angle between the radii of P and B (OB perpendicular to OAX), and β the angle between AOB and POB.

Let ϑ be the deflection of the plumb-line in A with regard to the spheroid, and let ϑ_x be its component in the X direction. Neglecting quantities of higher order, we have then :

$$\vartheta_x = \frac{\partial N}{\partial x} = \frac{1}{R}\frac{\partial N}{\partial \beta}$$

in which last formula N is expressed in the second system of coordinates θ and β :

$$N = \frac{R}{4\pi\gamma}\int^{\sigma} S_T(\theta, \beta) \times \triangle_0(\theta, \beta)\, d\sigma.$$

For differentiating N according to β we have to determine the value of $N + \frac{\partial N}{\partial \beta}\, d\beta$ in a point A' at a distance $Rd\beta$ from A. The field $\triangle_0$ remains at the same place, but the field S_T shifts with the centre A towards A', that is to say it rotates an angle $d\beta$. If the system of coordinates is kept unmoved, we find :

$$N + \frac{\partial N}{\partial \beta}\, d\beta = \frac{R}{4\pi\gamma}\int^{\sigma} S_T[\theta, (\beta - d\beta)] \times \triangle_0(\theta, \beta)\, d\sigma$$

but we may also shift the system of coordinates in the same way as the field S_T by rotating it an angle $d\beta$, and we find then :

$$N + \frac{\partial N}{\partial \beta}\, d\beta = \frac{R}{4\pi\gamma}\int^{\rho} S_T(\theta, \beta) \times \triangle_0[\theta, (\beta + d\beta)]\, d\sigma$$

The two expressions give the formulae :

$$\vartheta_x = \frac{1}{R}\frac{\partial N}{\partial \beta} = -\frac{1}{4\pi\gamma}\int^{\sigma} \frac{\partial S_T}{\partial \beta}\triangle_0\, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (2^A)$$

$$\vartheta_x = \frac{1}{R}\frac{\partial N}{\partial \beta} = +\frac{1}{4\pi\gamma}\int^{\sigma} S_T\, \frac{\partial \triangle_0}{\partial \beta} d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad (3^A)$$

which are both worth while examining : The first expresses ϑ_x in the gravity anomaly itself and the second expresses ϑ_x in the horizontal gradient of the anomaly.

We will go back to the original system of coordinates, and express both formulae in ψ and α. The spheric triangle *APB* gives :

$$\left.\begin{aligned}\frac{\partial\psi}{\partial\beta} &= \cos\alpha \\ \frac{\partial\alpha}{\partial\beta} &= -\sin\alpha\,\mathrm{cotg}\,\psi\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (4^A)$$

and therefore :

$$\frac{\partial}{\partial\beta} = \cos\alpha\,\frac{\partial}{\partial\psi} - \sin\alpha\,\mathrm{cotg}\,\psi\,\frac{\partial}{\partial\alpha} \quad . \; . \; . \; . \; . \; . \quad (4^B)$$

so that we get :

$$\vartheta_x = -\frac{1}{4\pi\gamma}\int^{\sigma}\cos\alpha\,\frac{\partial S_T}{\partial\psi}\,\triangle_g\,d\sigma \quad . \; . \; . \; . \; . \; . \quad (2^B)$$

$$\vartheta_x = +\frac{1}{4\pi\gamma}\int^{\sigma} S_T\left[\cos\alpha\,\frac{\partial\triangle_0}{\partial\psi} - \sin\alpha\,\mathrm{cotg}\,\psi\,\frac{\partial\triangle_0}{\partial\alpha}\right]d\sigma \quad . \; . \quad (3^B)$$

We may introduce

$$d\sigma = \sin\psi\,d\alpha\,d\psi$$

and multiply with $\varrho'' = \mathrm{cosec}\,1''$, in order to express ϑ_x in seconds. The second formula gives then :

$$\vartheta''_x = \frac{\varrho''}{4\pi\gamma}\int_0^{2\pi}\cos\alpha\,d\alpha\int_0^{\pi}\sin\psi\,S_T\,\frac{\partial\triangle_0}{\partial\psi}\,d\psi - \frac{\varrho''}{4\pi\gamma}\int_0^{2\pi}\sin\alpha\,d\alpha\int_0^{\pi}\cos\psi\,S_T\,\frac{\partial\triangle_0}{\partial\alpha}\,d\psi \quad (3^C)$$

which we will not further examine in this paper.

The first formula gives :

$$\vartheta''_x = \frac{1}{2\pi}\int_0^{2\pi}\cos\alpha\,d\alpha\int_0^{\pi}Q\,\triangle_0\,d\psi \quad . \; . \; . \; . \; . \; . \quad (2^C)$$

with

$$Q = -\frac{\varrho''}{2\gamma}\sin\psi\,\frac{\partial S_T}{\partial\psi}$$

that is to say in introducing the expression for S_T :

$$\left.\begin{aligned}Q = \frac{\varrho''}{2\gamma}\cos^2\tfrac{1}{2}\psi\Big[&\mathrm{cosec}\,\tfrac{1}{2}\psi + 12\sin\tfrac{1}{2}\psi - 32\sin^2\tfrac{1}{2}\psi + \\ &+ \frac{3}{1+\sin\tfrac{1}{2}\psi} - 12\sin^2\tfrac{1}{2}\psi\,lg\,\{\sin\tfrac{1}{2}\psi\,(1+\sin\tfrac{1}{2}\psi)\}\Big]\end{aligned}\right\} \quad (2^D)$$

The following table gives the values of Q for intervals of 10° for ψ ($\triangle_0$ and γ are supposed to be expressed in 0.001 cm) :

ψ	Q	ψ	Q	ψ	Q
1°	+12.35	70°	+0.03	140°	−0.25
10°	+ 1.59	80°	−0.15	150°	−0.16
20°	+ 1.02	90°	−0.29	160°	−0.08
30°	+ 0.79	100°	−0.38	170°	−0.02
40°	+ 0.61	110°	−0.41	180°	−0.00
50°	+ 0.43	120°	−0.40		
60°	+ 0.22	130°	−0.34		

For increasing ψ, Q first decreases rather quickly, afterwards more slowly, is zero for ψ more than 70°, gets to a minimum —0.41 for ψ about 110°, and increases then towards zero, which value is reached for $\psi = 180°$.

For small ψ we can neglect in the formula for Q all but the first term, and we get by introducing in stead of ψ the linear distance $AP = r$, so that $r = 2R \sin \frac{1}{2} \psi$:

$$Q = \frac{1338}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2^E)$$

in which r is expressed in km.

By substituting in (2C) the surface element df in stead of the expression in $d\psi$ and da, we find for the effect of the anomalies in the neighbouring region :

$$\vartheta''_x = \frac{\varrho''}{2\pi\gamma} \int \cos a \frac{\triangle_0}{r^2} df \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We may express this formula in the following way : If we apply to every surface element df of the geoid a mass equal to $\frac{\varrho''}{2\pi\gamma} \triangle_0 df$, the reflection of the plumb-line is the resultant of the attractions, exerted by these masses in A if these attractions act along the radii towards df and are equal to the mass divided by the square of the distance.

The decrease of the influence of a certain anomaly with the distance is therefore greater than for the formula of STOKES : This formula reduces for the neighbouring region to :

$$N = \frac{1}{2\pi\gamma} \int \frac{\triangle_0}{r} df \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

so that the influence on N decreases only in inverse ratio to the first power of r.

Formula (6) is in harmony with formula (5) as this last formula indeed represents the gradient of the former.

The property that the influence of a certain anomaly decreases more quickly with the distance than for the formula of STOKES, gives the advantage that the formula for the deflection of the plumb-line is more independent of the anomalies in other parts of the earth's surface, so that the fact, that as yet the gravity is only very imperfectly known over the globe, need not necessarily prevent its application.

The formula for the deflection of the plumb-line may be useful for connecting the results of both types of geodetic operations : the astronomic observations and the determinations of the gravity. Besides it may be of use, as well as the formula of STOKES itself, to study the local deviations of the geoid for the central area of a region where the gravity has been determined. The anomalies in the other parts of the globe will generally have an equable effect for the whole central area : for the deflection of the plumb-line for instance a small and nearly constant amount. So by neglecting the influence of the anomalies in those distant region we will get a result for the shape of the geoid, which will perhaps be slightly wrong in position, but which will not be much deformed. We may even put the question, if the determination of the geoid along this line will not better answer the purpose, than that, which is founded on the direct observations of the plumb-line deflection. The disadvantage of the uncertainty of the gravity in other parts of the globe, may perhaps be more than compensated by the uncertainty, resulting from the fact that in using this last method, N is determined by integrating a quantity, which is only known in isolated spots.

A provisional computation for the Netherlands has shown, that the differences of the plumb-line deflections for the three stations in the central part : Urk, Wolberg, and Utrecht, are in good harmony with the values which have been deduced from the gravity anomalies : the deviations do not exceed 0″.5. For this computation only the gravity-anomalies in the Netherlands have been used.

It appears doubtful if the accuracy of the result can ever attain such a perfection, that the formula could be used to controll the results of the triangulations by giving for each astronomical station an equation for the latitude and longitude components of the plumb-line deflection. Still it may give a useful control for traverse surveys, of which the errors are so much greater, as is well known : We may considerably improve in this way the control of these surveys by astronomical observations. For this purpose it is of course necessary to have a sufficient number of gravity stations in an area of somewhat greater extension than the area of the survey.

Besides the formulae for the deflection of the plumb-line, it is a simple matter to deduce other formulae from the formula of STOKES by further differentiation. In this way we may derive formulae, expressing the second differential coefficients of N in the gravity anomalies or in the gradient of these anomalies. These second differential coefficients may be considered as the difference of the reciprocal values of the radii of curvature of the geoid and the spheroid, so that the curvature of the geoid may be determined in this way. We will not take up this question in this paper and we will only remark that in this way a formula may be found, connecting the two series of quantities, given by the torsion balance: the horizontal gradients of the gravity and the curvature data. These data are therefore not independent, although the curvature cannot be brought in connection with a single value of the gradient but with the gradient field over the whole earth.

The next paragraphs will treat of the neglected terms of the formula of STOKES and will prove the validity of this formula and of the formulae found by differentiating it, as well for the determination of the general shape of the geoid as for that of the local irregularities. Besides they will treat of some formulae about the gravity field outside the geoid, which we will need for our purpose.

§ 2. *Gravity anomaly in an arbitrary point outside a sphere, which encloses all the masses and on which the anomaly is known.*

Before looking into the question of the validity of the formula of STOKES, we will derive the solution of the above problem, which will be wanted for that research. Practically this will also give the formula for expressing the gravity anomaly outside the geoid in the anomalies on the geoid: the error committed by replacing the geoid by the sphere with a radius equal to the mean earth-radius is of the same order as the error in the STOKES formula i.e. the percentual error will not exceed the order of the flattening.

Let T be the difference of the potentials caused in the same point by the actual earth and by the theoretical earth; by the last we mean one of the infinite number of possible mass-distributions inside the geoid with the same total mass as the earth and with an outside potential surface coinciding with the spheroid. Only values of T in points outside both mass-distributions will be considered, that is to say in points outside the geoid or on the geoid.

Let further:

a = aequatorial radius of the spheroid,

f = flattening of the spheroid,

g = gravity in an arbitrary point, caused by the actual earth,

γ = gravity in an arbitrary point, caused by the theoretical earth,

ϱ = radius of an arbitrary point towards the centre of gravity of geoid and spheroid,

g_0, γ_0, ϱ_0 = values of g, γ and ϱ on the geoid,

g_s, γ_s, ϱ_s = values of g, γ and ϱ on the spheroid,

N_0 = distance between geoid and spheroid,

R = radius of the sphere, which is considered,

$\Delta_0 = g_0 - \gamma_s$ = gravity anomaly on the geoid.

As we suppose N_0 to be of the order of $(a f^2)$, Δ_0 of the order of $(g f^2)$ and the angle between ϱ and g or γ of the order of (f), we have:

$$N_0 = \frac{T_0}{\gamma_0} + (af^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^A)$$

$$g_0 - \gamma_0 = -\frac{\partial T_0}{\partial \varrho} + (gf^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^B)$$

$$\gamma_0 - \gamma_s = N_0 \frac{\partial \gamma_0}{\partial \varrho} + (gf^4) = \frac{\partial \gamma_0}{\partial \varrho} \frac{T_0}{\gamma_0} + (gf^4) \quad . \quad . \quad . \quad . \quad (7^C)$$

Therefore:

$$\Delta_0 = -\frac{\partial T_0}{\partial \varrho} + \frac{\partial \gamma_0}{\partial \varrho} \frac{T_0}{\gamma_0} + (gf^4)\,^{1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^D)$$

In the same way as the anomaly Δ_0 has been defined for the geoid, we may define the anomaly Δ for a point outside the geoid as the difference of the actual gravity in this point and the gravity of the theoretical earth in a point in the same vertical, where the potential is equal to the potential of the actual earth in the first point. By rough approximation we may say that the second point is at the same distance outside the spheroid as the first point is outside the geoid. We have again:

$$\Delta = -\frac{\partial T}{\partial \varrho} + \frac{\partial \gamma}{\partial \varrho} \frac{T}{\gamma} + (gf^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^E)$$

If the spheroid were a sphere, the differential-quotient $\frac{\partial \gamma}{\partial \varrho}$ would be:

$$\frac{\partial \gamma}{\partial \varrho} = -2\frac{\gamma}{\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For a spheroid it has to be multiplied by a factor $1 + (f)$ (see e.g. HELMERT, Theor. d. h. Geod. II, page 94) and so we get:

$$\Delta = -\frac{\partial T}{\partial \varrho} - 2\frac{T}{\varrho} + (gf^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

with a corresponding equation for Δ_0. As Δ is of the order of (gf^2) the last term in (9) is small with regard to the others. It will in fact not surpass 0.0001 or 0.0002 cm in Δ. We will neglect it as well as all the last terms of the formulae (7).

The problem which we want to solve is to express Δ in Δ_R, Δ_R representing the value of Δ on the sphere. It is well known that for a point outside the attracting masses, T can be written in the shape:

$$T = \frac{k^2}{\varrho}\left[\frac{K_2}{\varrho^2} + \ldots + \frac{K_n}{\varrho^n} + \ldots\right]. \quad . \quad . \quad . \quad . \quad (10^A)$$

[1]) In the original paper (Kon. Akad. v. Wet. Amsterdam, Dl. **37** N⁰. 1) the second term of the right member of (7^D) has been neglected. The reasoning remains the same when taking it into account.

in which k^2 is the gravitational constant and $K_2 \ldots K_n \ldots$ are spherical harmonics of the $2^{nd} \ldots n^{th}$-degree. There are no spherical harmonics of zero and first degree, because the total mass of the actual and of the theoretical earths are the same, so that the differential masses, which cause the potential T, have a total mass zero, and because the centres of gravity of the masses of the actual and the theoretical earths coincide with the origin of the radius ϱ.

We have now:

$$g-\gamma=-\frac{\partial T}{\partial \varrho}=\frac{k^2}{\varrho}\left[3\frac{K_2}{\varrho^2}+\ldots+(n+1)\frac{K_n}{\varrho^n}+\ldots\right] \quad (10^B)$$

$$\Delta=-\frac{\partial T}{\partial \varrho}-2\frac{T}{\varrho}=\frac{k^2}{\varrho^2}\left[\frac{K_2}{\varrho^2}+\ldots+(n-1)\frac{K_n}{\varrho^n}+\ldots\right] \quad (10^C)$$

and for $\varrho = R$:

$$\Delta_R=\frac{k^2}{R^2}\left[\frac{K_2}{R^2}+\ldots+(n-1)\frac{K_n}{R^n}+\ldots\right] \quad (10^D)$$

$$T_R=\frac{k^2}{R}\left[\frac{K_2}{R^2}+\ldots \qquad +\frac{K_n}{R^n}+\ldots\right] \quad (10^E)$$

When Δ_R is known we can develop Δ_R into a series of spherical harmonics, which gives us $K_2 \ldots K_n \ldots$ and by substituting these values in the formula for Δ, the problem is solved because formula (10^C) is convergent when $\varrho \geqq R$. We can however deduce an equation which allows a simpler computation, by expressing Δ directly in Δ_R. This gives besides a more useful formula for the case Δ_R is not completely known over the whole globe, which is of course the actual state of things. The deduction is quite analogous to the method of STOKES, who solved the problem of computing T_R (and thereby $N_R = T_R/\gamma$) from Δ_R, by expressing T_R directly in Δ_R, so that the elaborate way of first computing all $K_2 \ldots K_n \ldots$ can be avoided.

The spherical harmonic of the n^{th} degree of Δ_R is:

$$\frac{2n+1}{4\pi}\int_0^\sigma P_n\, \Delta_R\, d\sigma \quad (11)$$

in which: $d\sigma$ = surface-element corresponding to Δ_R of a sphere with radius 1.

P_n = LEGENDRE's spherical harmonic of the angle ψ between the radius ϱ of the point A where we want Δ and the radius R of the point P where we have Δ_R.

This gives:

$$\Delta=\sum_2^\infty\left(\frac{R}{\varrho}\right)^{n+2}\times\frac{(2n+1)}{4\pi}\int_0^\sigma P_n\, \Delta_R\, d\sigma$$

which is convergent because $\varrho \geqq R$. Therefore:

$$\triangle = \frac{1}{4\pi}\int_0^{\pi}\left[\sum_2^{\infty}(2\,n+1)\,P_n\left(\frac{R}{\varrho}\right)^{n+2}\right]\triangle_R\,d\sigma$$

or

$$\triangle = \frac{1}{4\pi}\int_0^{\pi} S_{\triangle}\,\triangle_R\,d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12^A)$$

with

$$S_{\triangle} = \sum_2^{\infty}(2\,n+1\,P_n\left(\frac{R}{\varrho}\right)^{n+2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12^B)$$

Now $S_{\triangle}$ can be expressed in R, ϱ and ψ by making use of the well-known formula:

$$\sum_2^{\infty} P_n\left(\frac{R}{\varrho}\right)^n = \frac{\varrho}{r} - 1 - \frac{R}{\varrho}\cos\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

in which r is the distance AP, given by:

$$r = (\varrho^2 - 2\,\varrho\,R\cos\psi + R^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Multiplying (13) by $\left(\frac{R}{\varrho}\right)^{\frac{1}{2}}$, differentiating according to $\frac{R}{\varrho}$ and multiplying by $2\left(\frac{R}{\varrho}\right)^{2\frac{1}{2}}$ makes the first member equal to $S_{\triangle}$, and we find:

$$S_{\triangle} = \frac{R^2(\varrho^2 - R^2)}{\varrho r^3} - \frac{R^2}{\varrho^2} - 3\,\frac{R^3}{\varrho^3}\cos\psi \quad . \quad . \quad . \quad . \quad . \quad (15)$$

(12^A) combined with (15) gives $\triangle$ expressed in $\triangle_R$.

Still we have to be careful if we want to use these formulae for determining the difference between $\triangle$ and $\triangle_R$: If $\triangle_R$ does not contain spherical harmonics of zero and first degree as has been assumed, the difference can indeed be found simply by subtracting $\triangle_R$ from formula (12^A), but if in the formula for $\triangle_R$ those spherical harmonics are not zero, we must first subtract those terms from $\triangle_R$ before taking the difference with $\triangle$. $\triangle$ is certainly free from those terms even if $\triangle_R$ is substituted in (12^A) without correction, because $S_{\triangle}$ does not contain spherical harmonics of zero and first degree. The terms of zero and first degree in $\triangle_R$ are easily found by applying formula (11) and in this way we find for the difference $\triangle - \triangle_R = \delta$:

$$\delta = \frac{1}{4\pi}\int_0^{\pi} S_{\delta}\,\triangle_R\,d\sigma - \triangle_R \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16^A)$$

with:

$$S_{\delta} = \frac{R^2(\varrho^2 - R^2)}{\varrho r^3} + \frac{(\varrho^2 - R^2)}{\varrho^2} + 3\,\frac{(\varrho^3 - R^3)}{\varrho^3}\cos\psi \quad . \quad . \quad . \quad . \quad (16^B)$$

in which formulae $\triangle_R$ may be substituted as it is, without taking away its terms of zero and first degree. δ will not contain those terms.

Formula (16^B) converges towards zero when $\varrho - R$ is converging towards zero. If the difference $\varrho - R$ which we will represent by h, is small, we can write, neglecting second and higher powers of h in S_3:

$$S_3 = \frac{h}{R}\left[2\frac{R^3}{r^3} + 2 + 9\cos\psi\right] \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16^C)$$

Applying these formulae to the geoid in stead of to a sphere with radius R, we have to substitute for R the mean earth's radius and for h the elevation above the geoid.

For giving an idea of the magnitude of the variation of $\triangle$ outside the geoid, we will apply the formulae to a special case.

Supposing a circular patch of anomalies, defined by the formula:

$$\triangle_R = \triangle_C\left(1 - \frac{u^2}{l^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17^A)$$

in which $\triangle_C$ is the value in the centre, u the horizontal radius from $\triangle_R$ to this centre and l the outer limiting radius. Suppose l small with regard to the earth's radius. In this case we can neglect the second and third term of (16^C). We find for a point at an elevation h above the centre:

$$\delta_h = -2\frac{h}{l}\left[\sqrt{1 + \frac{h^2}{l^2}} - \frac{h}{l}\right]\triangle_C \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17^B)$$

For $h = 0.36\ l$ we get $\delta_h = 0.5\ \triangle_C$, that is to say that the anomaly has diminished to half its value for an elevation about one sixth of the diameter of the patch.

This allows the conclusion, that, generally speaking, the effect of the diminution of $\triangle$ with the elevation is too small to warrant a corresponding reduction in bringing back the result of a gravity determination to sea level: it will seldomly exceed 0.001 or 0.002 cm $\sec^{-2}$ per 1000 m elevation.

The formulae (12), (15), (16) and (17) may be applied to all other quantities which can be represented by a series like:

$$q = \frac{C}{\varrho^2}\left[a_2\frac{K_2}{\varrho^2} + \ldots . + a_n\frac{K_n}{\varrho^n} + \ldots .\right] \quad . \quad . \quad . \quad . \qquad (18)$$

in which c, $a_2 \ldots . a_n \ldots .$ are constants and therefore independent of the radius ϱ or the angle ψ. We have simply to substitute q and q_R for $\triangle$ and $\triangle_R$. In this way they may for instance be used to express the variation of $g - \gamma$ outside the geoid in $g_0 - \gamma_0$ on the geoid, $g - \gamma$ representing the difference of the actual and the theoretical gravity in the same point.

§ 3. *Potential outside the geoid.*

To express the variation of T outside the geoid in T_0 on the geoid, we have analogous formulae deduced in the same way:

$$T = \frac{1}{4\pi}\int_0^\tau S_T\, T_0\, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19^A)$$

$$S_T = \frac{R(\varrho^2 - R^2)}{r^3} - \frac{R}{\varrho} - 3\frac{R^2}{\varrho^2}\cos\psi \quad . \quad . \quad . \quad . \quad . \quad (19^B)$$

For the difference $t = T - T_0$ we find:

$$t = \frac{1}{4\pi}\int_0^\sigma S_t\, T_0\, d\sigma - T_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20^A)$$

$$S_t = \frac{R(\varrho^2 - R^2)}{r_3} + \frac{(\varrho - R)}{\varrho} + 3\frac{(\varrho^2 - R^2)}{\varrho^2}\cos\psi \quad . \quad . \quad . \quad (20^B)$$

and for a small elevation h:

$$S_t = \frac{h}{R}\left[2\left(\frac{R}{r}\right)^3 + 1 + 6\cos\psi\right] \quad . \quad . \quad . \quad . \quad . \quad (20^C)$$

For expressing the distance N in some outside point between corresponding potential surfaces of the actual and theoretical earths, we have to substitute for T: γN and for T_0: $\gamma_0 N_0$.

For local irregularities in T or N of an extension, which is small with regard to the earth's radius, we can neglect the second and third terms of (20^C) and we find the same formula for the diminution with h as for Δ. This is also true for the general case, if Δ_R and T_0 are free of spherical harmonics of zero and first degree, because the difference between S_J and S_t contains only spherical harmonics of zero and first degree.

Lastly we will ask to express T outside the geoid in the anomaly Δ_0 on the geoid, or if we replace again the geoid by the sphere with radius R, to express T in Δ_R. This is an enlargement of the problem of STOKES; by making the radius ϱ of the point, where we want T, equal to R, it is brought back to the identical problem.

Following the same way of deduction as for the formula for Δ we get:

$$T = \frac{R}{4\pi}\int_0^\sigma S_{\Delta T}\, \Delta_R\, d\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21^A)$$

with

$$S_{\Delta T} = \sum_2^\infty \frac{(2n+1)}{(n-1)} P_n \left(\frac{R}{\varrho}\right)^{n+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21^B)$$

and we find $S_{\triangle T}$ by multiplying (13) by $\left(\frac{R}{\varrho}\right)^{\frac{1}{2}}$, differentiating according to $\frac{R}{\varrho}$, multiplying by $2\left(\frac{R}{\varrho}\right)^{-1\frac{1}{2}}$, integrating according to $\frac{R}{\varrho}$ and multiplying by $\left(\frac{R}{\varrho}\right)^2$:

$$S_{\triangle T} = 2\frac{R}{r} + \frac{R}{\varrho} - 5\frac{R^2}{\varrho^2}\cos\psi - 3\frac{Rr}{\varrho^2} - 3\frac{R^2}{\varrho^2}\cos\psi \log nat \frac{[\varrho - R\cos\psi + r]}{2\varrho} \quad (21^C)$$

By putting $\varrho = R$, which gives $r = 2R\sin\frac{1}{2}\psi$, we get back to the formula of STOKES. The distance N is of course found by dividing T by γ. In this way the formula gives the outside potential surfaces of the earth, when the gravity anomaly on the geoid is known and provided the theoretical outside potential surfaces have been computed. N may of course be differentiated in the same way as has been done in the first paragraph; we find then the deflection of the plumb-line in a point outside the geoid, expressed in the gravity anomalies on the geoid, or in their horizontal gradient. Executing the same thing with the formula for N deduced from formula (19), we can get this plumb-line deflection expressed in the value of N_0 on the geoid or in the plumb-line deflections on the geoid.

§ 4. *The validity of the formula of* STOKES.

To find the order of magnitude of the neglected terms of the formula of STOKES, we suppose a fictitious earth with a mass equal to the total mass of the real earth, and of which the outside potential surface is a sphere with a radius R. The difference of R and the earth's radius is of the order of the flattening. The gravity on this sphere is, according to a potential theorem, constant over the whole surface.

To this fictitious earth is added a mass-distribution of positive and negative masses with a zero total mass, in such a way, that the combination of these masses with the fictitious earth gives a geoid on which the gravity anomalies (i.e. the gravity minus the above mentioned constant value) are the same as the gravity anomalies $\triangle_0$ of the real earth in corresponding points of the real geoid; for corresponding points we may for instance take points with the same geographical coordinates. We will henceforth indicate this added mass-distribution with the letter M.

If $\triangle_0$ is supposed to be known, the formulae of §§ 2 and 3 allow the complete determination of the outside gravity-field of this mass-distribution M without any further neglections than those, given by the last terms of the formulae (7); in the same way the formula of STOKES may give the distance N_R between the sphere and the geoid of the above combination without neglecting more than these terms of (7).

We will now combine the mass-distribution M with the theoretical earth, which has been defined in the beginning of § 2, and of which the outside potential surface is the spheroid. This addition to the theoretical earth causes the potential surface to shift from the spheroid to a geoid, which will nearly coincide with the real geoid. If we suppose $\triangle_0$ to be known, the distances N'_0 between this geoid and the spheroid, and the anomaly $\triangle'_0$ on this geoid, may be computed with the formulae of §§ 2 and 3; we have to introduce in these formulae for h the distance between the speroid and the sphere.

We will now prove that the difference $\delta' = \triangle'_0 - \triangle_0$ is of the order of gf^3, i. e. of the order of the flattening with regard to $\triangle_0$ itself. If this is true we may neglect δ' as we have already neglected quantities of the same order in formula (9); these neglections are insignificant considering the accuracy of the determination of the gravity anomalies. And secondly; if we neglect the difference between the anomaly $\triangle'_0$ and the real anomaly $\triangle_0$ we may confound the geoid, which we have got by combining M with the theoretical earth, with the real geoid, so that the distance N'_0 can be considered to be also valid for this last geoid.

We need not doubt that δ' is of the order of gf^3 as far as δ' is given by the second and third terms of (16^C) in combination with (16^A), because $\frac{h}{R}$ is of the order of the flattening, while we assumed that $\triangle_0$ is of the order of gf^2. The only doubt, which might arise, concerns the effect of the first term of (16^C) for small r. We have seen in § 2, page 13, that because of this term, the difference δ, caused by local anomalies, may get a value of the order of $\frac{2h}{l}\triangle_0$, in which l is the horizontal extension of the anomaly. So we see that, locally, values of δ' may occur, exceeding the order of magnitude of gf^3. It is clear that these local values of δ' may be neglected for the determination of N'_0 in some point A of the spheroid which is far away; a more thorough investigation which we will not repeat here, confirms this opinion. The question is however, if they have no effect if they occur near to A.

In order to prove that this is not the case, we will suppose that our sphere with radius R is tangent to the spheroid in A, while the radius is supposed to coincide with the smallest radius of curvature of the spheroid in that point (see the supposition about this radius on page 1), so that the whole sphere is inside the spheroid. This is necessary if we want to apply the formulae of the previous paragraphs, because we assumed there: $\varrho \geqq R$. This supposition makes it of course impossible that the centre of the sphere should coincide with the centre of gravity of the earth, but it may easily be seen, that this only affects the deduction in this way, that ϱ in formula (8) for $\frac{\partial\gamma}{\partial\varrho}$ is not measured from the centre of gravity of the earth, but from the centre of the sphere at some distance of the order of fR. This means a deviation of the formula (9) of the order of gf^3, which does not exceed the term, which has already been neglected.

If we introduce in A an azimuthal coordinate a, representing the angle between the vertical plane through A in which the sphere and the spheroid osculate, and the vertical plane through A and through the point P, where we suppose that the value of δ' occurs, we find in P a distance h between the sphere and the spheroid, which may approximately be given by:

$$h = (f)\,\frac{r^2}{R} \sin^2 a$$

in which (f) means a constant of the order of the flattening, while r represents again the distance AP.

We find thus that δ' is of the order of:

$$(f)\,\frac{r^2}{Rl} \times \triangle_P$$

in which $\triangle_P$ is the value of $\triangle_0$ in P. We see therefore that, even for small r, we need not fear that δ' would exceed the order of gf^3.

We may conclude that, if we neglect in $\triangle_0$ quantities of the order of gf^3, our problem is brought back to the determination in A of the distance N'_0, corresponding to the mass-distribution M, which has been defined in the beginning of this paragraph. As A is also a point of the sphere, N'_0 equals N_R, so that it can be computed by applying the formula of STOKES: we have to substitute in this formula the anomalies $\triangle_0$ and the radius R. We may notice however, that it will doubtless be somewhat better to substitute in this formula for R the mean earth's radius; we only chose a slightly different value for R in order to be able to apply the formulae of the previous paragraphs.

The conclusion at which we arrive, is that we are justified in using the formula of STOKES: the neglections in N will not exceed the order of Rf^3, i. e. one metre. We may apply the formula of STOKES as well for the determination of the general shape of the geoid as for the deduction of its local shape and obviously we may follow the same reasoning and arrive at the same conclusion for the formulae, derived from the formula of STOKES by differentiating it once or twice according to a direction tangent to the geoid.

Physiology. — *Annotations on the physiology and the anatomy of a dog, living 38 days, without both hemispheres of the cerebrum and without cerebellum.* By Dr. G. G. J. RADEMAKER and Dr. C. WINKLER.

(Communicated at the meeting of April 28, 1928).

Until now, different workers have not succeeded in keeping long enough alive a dog, after the removal of both hemispheres and cerebellum, to do physiological and anatomical researches on it.

Such researches may have a great interest for:

1⁰. a physiology of the higher parts of the central system, wanting to be freed from the psychological nomenclature in which it is, necessarily at present, bound;

2⁰. an anatomy, wanting to know the systematic architecture of those higher parts, which till now, is only known very roughly.

Dr. RADEMAKER has tried to produce such animals for research, forced by considerations, from which he gives the following report.

Total or partial removal of the cerebellum in normal young animals has not answered in a sufficient way the question: what function may be practised by the cerebellum?

It was his endeavour to solve this question, by removing the cerebellum in animals without hemispheres.

It could be expected, that the loss of all conditioned reflexes after extirpation of the hemispheres might simplify the symptoms of the removal of the cerebellum.

Firstly, the comparison of the behaviour of animals without hemispheres before and after the loss of the cerebellum should probably produce more distinct facts to enable us to understand the cerebellar function.

Secondly: the study of the compensation of the symptoms produced by cerebellar removal in normal animals, compared with those after such removal in animals without hemispheres, should open the possibility of determining the part, played by the hemispheres in compensating disorders caused by cerebellar removal.

For most workers believe, that the transient cerebellar disorders, which gradually pass to a certain extent, after cerebellar removal in normal animals, are compensated for by the function of the hemispheres, though nobody has made special experiments upon this question.

And lastly. Only by removal of the cerebellum after that of the hemispheres, can it be for certainty fixed, which function the brainstem may perform without being influenced by cerebrum and cerebellum.

The dog, now presented to you, is an animal, living 38 days without hemispheres and without cerebellum. At the first operation, on December 9th 1926, 97 days before death, the left hemisphere has been taken away; 20 days afterwards, on December 29th 1926, 77 days before death, the right hemisphere was extirpated, and the 7th of February 1927, 38 days before death, the cerebellum was removed.

All operations were very well supported. Difficulties in the regulation of the warmth of the animal, of its feeding, of its uncleanliness were overcome by the well-trained nursery-knowledge and the devotion of Mrs. RADEMAKER. But a fortnight before death, an infectious distemper of dogs made its entrance into the kennels. Fever, purulent rhinitis and conjunctivitis, frequent respiration, coughing and sneezing occurred. The animal died at the 17th of March 1927. At the autopsia were found, purulent bronchitis and pneumonia and an empyema of the right frontal sinus.

In the short time in which research was possible, different interesting facts were noticed.

After the removal of the left hemisphere, the animal walked mostly in a circle to the left, counter-clockwise.

The preference for walking to the left, persisted after the removal of the right hemisphere, contrarely to what is ordinarily seen. This had been the only asymetrical disorder seen in the animal after the removal of both hemispheres.

It is difficult to prove, that this symptom was caused by the removal of the striatum at the left side and the nearly absolute sparing of the striatum at the right side, as the anatomy taught us (fig. 43—96 of the series).

After the extirpation of both hemispheres the animal had the ordinary symptoms, described by many investigators.

1°. the animal is blind, deaf and cannot smell, if one regards the extensive reactions, which the normal animal produces on optic, acustic and olfactive stimulations, as a measure for seeing, hearing and smelling.

2°. the animal walks practically as a normal dog walks, setting his paws on the earth in the proper position and making no errors in keeping his equilibrium (fig. 1).

3°. the animal has lost all the reactions of position of the legs, and all the reactions of correction made by the extremities, originated by the surface of its body (fig. 2).

4°. the supporting-tonus of the hind-legs has much diminished. They bend promptly at a slight pression upon the pelvis or when a sandbag of 2 kg is put on the back of the animal. But the tonus of support has not altered in the forelegs.

Now, as in this animal without hemispheres, the removal of the cerebellum was executed, the same disorders were seen as in a normal animal after the decerebrate section of SHERRINGTON. The animal had stiffened. Without trying to stand up, the animal placed on its back, retained this position with its neck stretched backward, with its stiffened back bent convexely, and with the four extremities totally stretched and stiffened, resisting violently every passive flexion. The stretched tonus was the so-called plastic tonus : it presented the clasp-knife phenomenon and did not disappear after passive flexion of the distal joints of the extremity. After a week, this rigidity could still be seen, in the second-third week it disappeared (fig. 3).

After varying the position of the head against the trunk and still more after varying the position of the head in space, the extension-rigidity could be seen diminishing or augmenting (tonic cervical- and labyrinthic-reflexes of MAGNUS and DE KLEYN).

However, at the end of the first week, the extension-tonus of the hind-legs still being augmented, only a very slight supporting-tonus could be shown in them. A pressure of 2 kg exerced on the sole was sufficient to bend them (fig. 4).

On the contrary a pression of 10 kg, exerced on the sole of the foreleg was supported, without bending, and if a sandbag of 5 kg was placed upon the shoulders, the forelegs did not bend (the dog's weight was 6.5 kg) (fig. 4).

In the course of the second to the fourth week after the cerebellar removal, the rigidity disappeared and the animal made attempts to right himself.

The labyrinth-righting-reflexes returned, the supporting-tonus of the hind-legs augmented, positive and negative reactions of support and different other reactions of the extremities were seen again.

If the animal was put on his hind-legs and the trunk moved in a backward direction, the animal made an alternative movement backward with the hind-legs, and the moving of the trunk to the left or to the right, was followed by a running movement to left or to the right, of the hind-legs.

Also, if the animal only was supported by one hind-leg, this leg followed the movement of the trunk forward, backward, to the right or to the left, by hopping in the same direction, as the movement of the trunk was made.

If the animal was held in the air by the shoulders, and then was permitted to descend on an oblique plane (with the back-part of its body), the hind-legs adjusted to the inclination of the plane and the animal walked exactly up or down the incline as was desired (fig. 5).

The described hop-movements of the legs play, in normal animals, an extensive part in keeping and in restoring equilibrium. The dog *Robbie* however, if placed, at that time free on his legs, always fell on the ground, because the hopping movements of the hind-legs came much too late and those of the fore-legs were not yet restored.

If, at this stage, the animal was placed on the left side, in its cage, it soon lifted the head and the forepart of the body and remained in a rolled-up position in the manner of a normal dog.

Sometimes the animal tried to take an upright position, but, as the hind-part of the body remained in a reclining position, it turned round in clockwise-movements to the left. Sometimes, the animal succeeded in getting, by a leap, on its four legs, but always then it fell to the left side. It never lifted the head and fore-part of the body so well from the left side, as it did from the right side.

The animal did not govern well the movements of the head. If it was free in the air, the head fell backward or lateral-ward. If the head was damaged by this movement at the bars of its cage, the animal could whine vehemently.

The restitution of different movements of the extremities in dog *Robbie* as f.i. the hopping movements after trunk-motion, the running backward or foreward of the hind-legs according to trunk-motion — all movements also missing in the first days after cerebellum-extirpation in normal dogs — proves, that they may be restituted without intervention of the hemispheres.

Still other symptoms of the animal are worth mentioning. If chopped-meat was placed in the mouth of the animal, it was well masticated and swallowed. But crusts of bread were rejected from the mouth. And if meat and crusts of bread were given together, the meat was swallowed, the bread-crusts rejected.

If the animal was irrigated with water, it shook itself like a normal dog. If its nose was moistened, the dog licked it off. Tickling of the mucous membrane of the nose was answered by sneezing.

After turning the animal along a dorso-ventral axis, vivid eye-nystagmus is seen.

Interesting was the intense reaction made by the animal on accustic stimuli. The answer to the sharp sound, made by sucking in the air with closed lips, was: adjusting of the auricles, lifting of the head and a movement with the legs. The animal without hemispheres answered this stimulus more intensely, after the removal of the cerebellum than before.

Many tactile stimuli gave vivid reflexes; so the cornea-reflexes were found, as also the movements of the eyelids after touching of the lashes, and movements of the auricles after touching of its hairs.

The animal discharged spontaneously urine and faeces. In the urine neither albumines, nor glucose was found, even in the first days after the cerebellar-removal.

The animal lived in two alternatingly varying periods. In one it was restless, it tried to make walking movements and to lift itself. In the other period it seemed to sleep. If shaken or pinched, it opened its eyes, yawned and moved its hairs as a dog that is awaking.

And lastly it must be mentioned, that the animal, after being set upon its legs, never had "uncontrolled" (LUCIANI's astasia) movements. Those movements do not appear immediately but only after some time, if the cerebellum-removal in normal animals with intact hemispheres has taken place.

Reckoning with the short space of time lived by the animal after the last operation, and with the appearance of an infectious distemper in the third week after it, the conclusion is nearly certain, that the animal has not at all attained the maximum of compensation, and that the possible restoration of the post-operative phenomena might have been much more evident.

The anatomical annotations upon the dog *Robbie*, are the following.

1⁰. If comparing the rest of *Robbie's* brain with normal brains of a dog, with the naked eye on a photo (fig. 6), it is not only seen that the removed nervous mass is very important, but also that the border lines of the removed hemispheres demarcate the brainstem at the left and at the right side in a very unequal way.

2⁰. It seems that the left hemisphere has been removed more completely than the right one. The remaining part of the right hemisphere surpasses widely that of the left in a frontal direction. The most frontal piece of the right hemisphere is transformed in a spherical mass of nervous tissue, connected with the dorsal wall of the brainstem, which at the right side is nearly twice as broad as on the left side (fig. 6). The cerebellum is taken away on both sides, except a few lateral lamellae only slightly connected by fibres with the brainstem.

3⁰. Research by uninterrupted series of sections gives a solution of the difference between the left and right hemisphere.

At the left side, the striatum has been completely removed with the extirpation of the pallium. On the contrary on the right side, the striatum has been spared. Only at its most frontal end it possibly may be a little damaged.

A more detailed research makes it obvious, that the sphaerical mass of nervous system (fig. 7, N⁰. 27 of the series) contains the dorsally opened frontal end of the lateral ventricle, with its medial and lateral walls. The latter is connected with the striatum (fig. 7, Nos 27 and 69 of the series) in which all the constituant parts, nucleus caudatus, nucleus lentiformis with its putamen and globus pallidus and the nucleus accumbens septi are found. The operating knife has gone lateral to the external capsule towards the olfactory convolutions (fig. 7, Nos 69, 83, 96 of the series).

The striatum is very rich in fibres (fig. 7, Nos 83—111). This may be compared with the status marmoratus, which OSCAR and CÉCILE VOGT have described in pathological cases of the striatum in overfunction.

It is sending out a very strong system of fibres along the "Kammsystem of EDINGER" and the ansa lenticularis (fig. 7, Nos 111 and 126), crossing the capsula interna, wherein the cortical fibres are strongly degenerated but not all lost.

At the left side neither a "Kammsystem" nor an ansa lenticularis is found. Together with the totally removed striatum, they have disappeared at that side.

At the right side a part of the pyriform gyrus with the nucleus amygdalae has remained and from there a certain number of fibres enter well-myelinated in the anterior commissure, that has been totally lost at the left side.

At the left side, the small remainder of the striatum which has been spared (fig. 7, N⁰. 83 of the series), appears as a small field of circular form between brainstem and the left corpus callosum. In this field some fibres appear, being fibres of the capsula interna.

In this way the experimental border-line between removed cortex + striatum and the remaining diencephalon is found (fig. 7, N⁰. 96 of the series) as composed of three parts. In the middle layer is seen the internal capsule, ventrally the cornu AMMONIS with fimbria and columna fornicis, at the dorsal end the corpus callosum, with the sulcus medialis of the hemispheres (fig. 7, Nos 96 and 111 of the series). They are united in a mass of fibres, forming the caudal border of the cicatricial line (fig. 7, N⁰. 111 of the series), where it ends.

The first section, in which no longer operation-lines are passing through the hemispheres is in fig. 7, N⁰. 126 of the series. There it is seen, that all fibres going from the striatum into the "Kammsystem" or ansa lenticularis have fallen out; neither are they found in more caudal sections. Whereas at the right side those fibre-systems have developed very intensely and are hypertrophied. Dr. MORRISON of Boston is working out the sequences of the loss of those systems at the left side.

However, it may already be mentioned here, that, influenced by the well-preserved striatum of the right side, there are found cell-territories in the middle of the right diencephalon, with well preserved middle-sized and small nerve-cells, missing completely in the left, where the striatum has been removed (fig. 8).

Both diencephala are severely damaged by the removal of both hemispheres and have lost the greater part of their cells. But in the middle of the left diencephalon, in the nucleus medialis and in a medial part of the nucleus ventralis, are seen only holes, where formerly cells have been, while, in the same region of the right diencephalon are found well preserved cells (fig. 8).

To this region the frontal radiation of the red nucleus can be followed (to the diencephalon) and there are reasons for accepting, that at the right side, going from the red nucleus to the thalamus, from there *to the nucleus caudatus* and farther to the nucleus pallidus, the Kammsystem and ansa lenticularis, a system has remained intact and functioning.

At the left side cells in the substantia nigra have disappeared (fig. 9). At the right side (fig. 9), those depending from the striatum, and they are the majority, have all remained, only those depending from the cortex, have disappeared. In the corpus subthalamicum of the left side many of the larger cells in the lateral part have disappeared, while nearly none in the medial parvocellular part of this nucleus are missing.

It would have been of great interest, if the degenerated and fallen out systems between the left striatum and the tegmentum of the pons VAROLI, medulla oblongata and spinalis, as well as the strongly developed paths from the right striatum and the corresponding parts of the nervous system, had not been complicated by the presence of the cortical and cerebellar pathways. But the animal has not lived long enough to bring a total loss of the cortical and cerebellar fibres.

This may be illustrated by the study of the restiform body and the brachium pontis of the dog *Robbie*, compared to those parts in animals living 136 days or 360 days without a cerebellum.

In all those animals the cell-preparations of the nervous system do not differ very much. In the spinal cord, large cells in the posterior horns and most in the columns of CLARKE are all degenerated after 38 days, and after 136 days or 360 days have all disappeared. In the medulla oblongata in all cases the cells are degenerated and most of them have disappeared in the nuclei olivares inferiores, the nuclei funiculi laterales, in the nuclei proprii corporis restiformis, together with some of the smaller cells in the nuclei of GOLL and BURDACH. In the ventral formation of nuclei in the pons VAROLI all cells have disappeared after 38, 136 and 360 days.

But fibre-preparations differ intensily of cell-preparations after 38 days or longer, because the degeneration of fibres follows the degeneration of cells, but the loss of fibres follows the loss of cells in a much slower tempo.

In dog *Robbie*, WEIGERT-preparations give the impression that the corpus restiform and the brachium pontis are unaltered.

After 136 and 360 days all fibres in the restiform body have disappeared except the long dorsal spino-cerebellar pathway, which never disappears totally.

And the brachium pontis in *Robbie*, seems to have no fibres missing, it contains all fibrae transversae and all fibrae rectae pontis, whereas after 136 or 360 days, all fibres in the brachium pontis, together with the fibrae transversae et rectae, have disappeared with the cells.

We have not been able to have in *Robbie*, the fibre-systems of the right striatum, well developed but un-complicated, because not all cortical and cerebellar fibres had been destroyed. This would have been possible, if the animal had lived longer.

G. G. J. RADEMAKER AND C. WINKLER: ANNOTATIONS ON THE PHYSIOLOGY AND THE ANATOMY OF A DOG, LIVING 38 DAYS, WITHOUT BOTH HEMISPHERES OF THE CEREBRUM AND WITHOUT CEREBELLUM.

Fig. 1. The dog *Robbie*, after removal of both hemispheres of the brain.
Exposition on the 5th of February 1927.

Fig. 2. Absence of reflexes of posture after removal of both hemispheres of the brain. If the head or the ventral surface of the body were touching a table, the extremities were not placed upon the table.

Fig. 3. The dog *Robbie*, one week after the removal of the cerebellum. The animal has extension-tonus of the extremities on both sides.
Exposition on the 13th of February 1927.

Fig. 4. 1. The hind-legs are bending at a pressure upon the soles of $1^1/_2$ kilogram.
2. The fore-legs resist bending at a pressure upon the soles of 11 kilogram.
3. If a sandbag of $5^1/_2$ kilogram is hung on the shoulders of the animal, the fore-legs do not bend. Weight of the dog $6^1/_2$ kilogram.
Exposition on the 13th of February 1927.

Fig. 5.

1. The animal is lowered with the hind-part of its body upon an inclined plane.
2. The animal does not sit, but, by moving its legs backward, it tries to remain standing and walks down the inclined plane.
3 and 4. Here also the animal is lowered with the hind-legs upon an inclined plane. It again tries by moving its legs backward to keep from sitting and so walks backward up the inclined plane.

Exposition on the 23th February 1927.

G. G. J. RADEMAKER AND C. WINKLER: ANNOTATIONS ON THE PHYSIOLOGY AND THE ANATOMY OF A DOG, LIVING 38 DAYS, WITHOUT BOTH HEMISPHERES OF THE CEREBRUM AND WITHOUT CEREBELLUM.

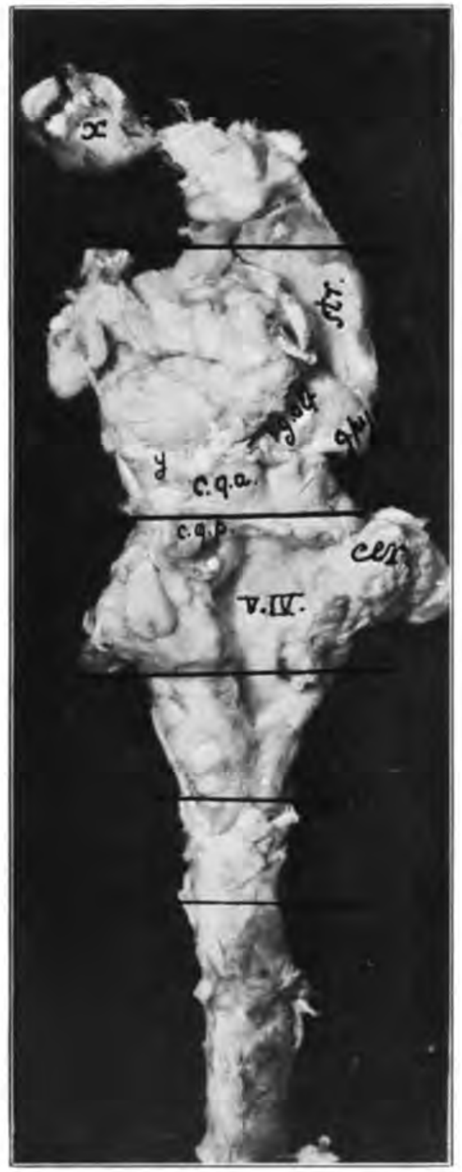


Fig. 6.
The rest of the brain of the dog *Robbie* seen from a dorsal view.

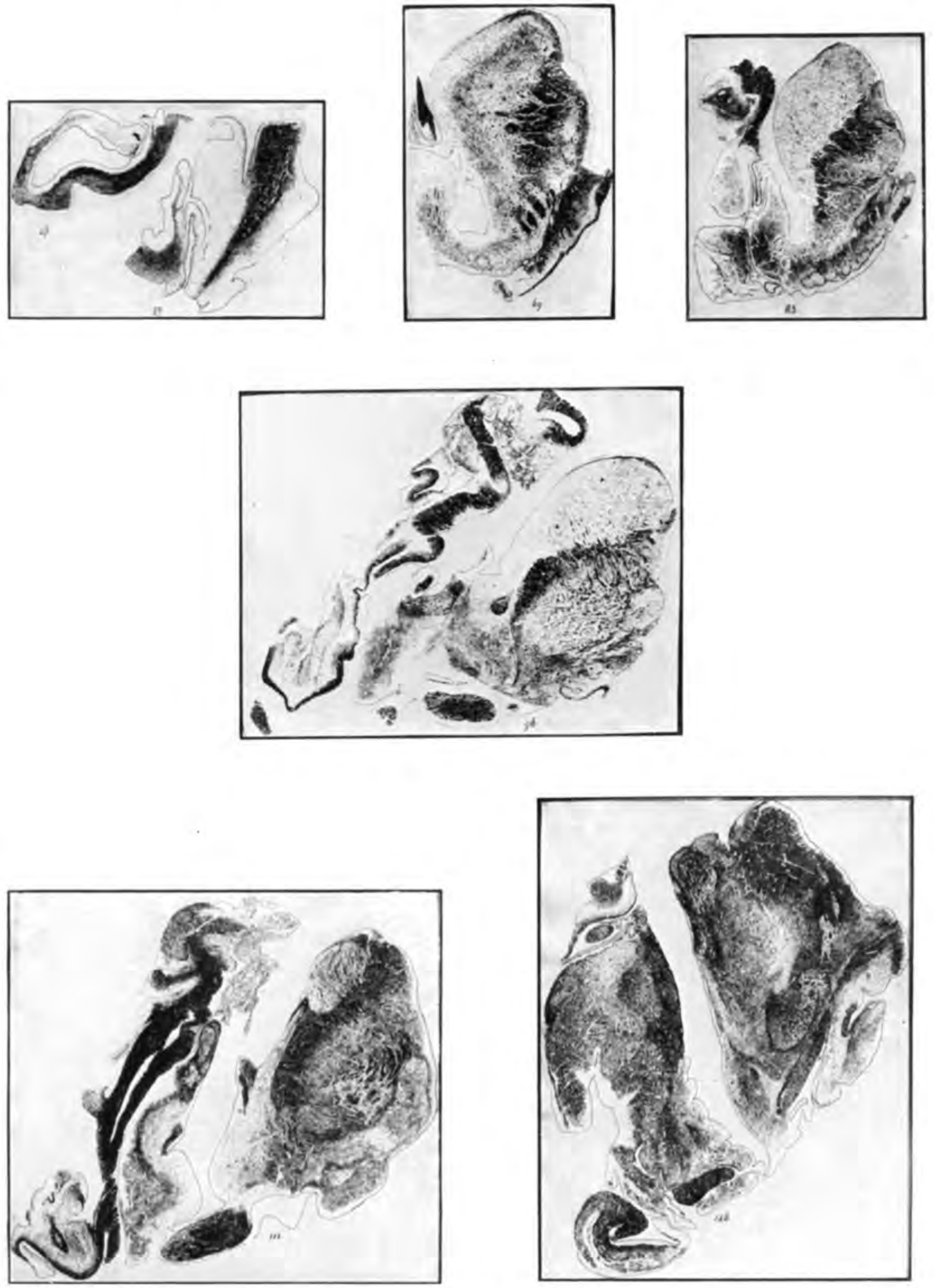


Fig. 7.
6 drawings of sections through the frontal end of the brain of dog *Robbie*.
At the left side the striatum is removed.
At the right side the striatum has been nearly totally spared.

Fig. 8.
Transverse section through the commissura media thalami.
At the left: in *a*. total degeneration of cells in the medial part of the thalamus.
At the right: in *b*. several well preserved cells are found in that medial part.

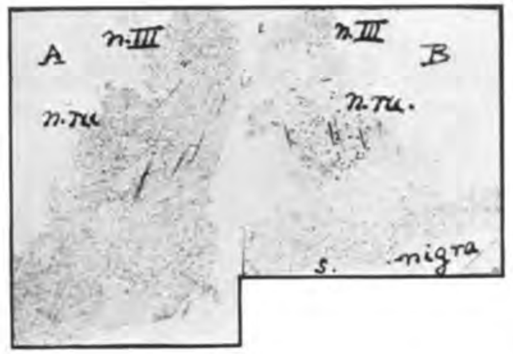


Fig. 9.
Two sections through the pedunculus cerebri.
A. At the left: loss of nearly all cells in the substantia nigra.
B. At the right: nearly all cells in the substantia nigra are well preserved.

Anatomy. — *On the Developmental History of the müllerian duct in the Sterlet (Acipenser ruthenus).* By W. MOOY. (Communicated by Prof. J. W. VAN WIJHE.)

(Communicated at the meeting of February 25, 1928).

As an assistant and under the supervision of Prof. J. W. VAN WIJHE I have studied the early development of the Müller's duct in Acipenser ruthenus.

I was prompted to make this inquiry after a publication of A. OSTROUMOFF. However this worker wanted the first stages in the development of Müller's duct, Prof. VAN WIJHE succeeded in providing me with the missing stages.

OSTROUMOFF (1908) examined fishes to the length of 18 mm in which he could not detect anything of the oviduct. The following stages up to that of 35 mm were missing in his material. In that of 35 mm he recognized the müllerian duct, in which he distinguished 3 parts :

1^0. the front-end, in the form of a thickened stripe of peritoneal epithelium, in his opinion a rudiment indicative of a phylogenetic relation to the posterior margin of the pronephros.

2^0. a median part, the ostium abdominale passing into

3^0. a channel developing posteriorly as a solid foundation between the peritoneal epithelium and the wolffian duct.

Furthermore I found in the literature on this subject the following particulars :

FELIX (1906) writes : "Unbekannt ist die Eileiterentwicklung der Ganoiden und Dipnoer". OSCAR HERTWIG (1910) does not express himself about the development of Müller's duct in Ganoids.

The most interesting and at the same time the most recent study on this subject is that of MASCHKOWZEFF (1924). He writes :

"Der grosze Eileitertrichter bildet sich aus einigen zusammengeschmolzenen sekundären Nephrostomaltrichtern des Mesonephros. In kaudaler Richting von der Wurzel des Eileitertrichters an wachsen die aus Auswüchsen der sekundären Baumanschen [1]) Kapseln entstandenen sekund. Nephrostomalkanälchen (N_2) ebenso unter dem Harnleiter ; sie bilden aber keine offene Nephrostomaltrichter, sondern laufen unter das Peritoneum der Rumpfhöhle blind aus.

Durch Verschmelzung dieser blinden sekundären Nephrostomaltrichter

[1]) MASCHKOWZEFF writes mistakenly „Bauman" instead of Bowman.

(N_2) bildet sich ein Zellenband, das vorn in den Eileitertrichter übergeht. In der Nähe des Eileitertrichters bildet sich eine Höhlung innerhalb des kompakten Zellenbandes, das sich somit zu einem Kanal umgestaltet (Abb. 6, El.).

Das Wachstum des Eileiters in kaudaler Richtung geht ziemlich langsam vor sich und im steten Verhältnis zu der Abbildung der sekundären Baumannschen Kapseln (M_2) und der sich aus diesen entwickelnden sekundären Nephrostomalkanälchen und seiner blindgeschlossenen Nephrostomaltrichtern, die schlieszlich miteinander verschmelzen.

Der Eileiter der Knorpelganoiden entwickelt sich also aus den Trichtern der sekundären Nephrostomalkanälchen.

Ein Zusammenhang des Eileiters mit den Baumannschen Kapseln (N_2) läszt sich nur für die Anfangsstadien feststellen, später aber fallen die sekundären Nephrostomalkanälchen auseinander.

Unsere Untersuchungen haben uns zu folgenden Ergebnissen geführt · es sind zwei Typen von Eileitern zu unterscheiden : pronephrische Eileiter einerseits, mesonephrische andererseits. Der mesonephrische Typus zerfällt seinerseits in zwei Gruppen : Eileiter die sich aus sekundären Nephrostomaltrichtern entwickeln, und solche die aus primären Nephrostomaltrichtern gebildet werden.

I. *Selachia, Amphibia, Dipnoi und Amniota.*

Der Eileitertrichter entwickelt sich aus den Pronephrostrichtern, der Eileiterkanal aber aus einer Spaltung des primären Harnleiters in zwei Kanäle.

Die Samenleiter bilden sich aus dem primären Harnleiter, der Hodenzentralkanal — aus den primären Nephrostomaltrichtern.

II. *Knorpelganoiden.*

Der Eileiter entwickelt sich aus den sekundären Nephrostomaltrichtern des Mesonephros und mündet in das Ende des primären Harnleiters.

Der primäre Harnleiter dient zur Ausführung der Samen, der Hodenzentralkanal bildet sich aus den primären Nephrostomaltrichtern.

III. *Teleostei und Crossopterigii.*

Eileiter und Samenleiter stammen von den primären Nephrostomaltrichtern, und beide sind dem Hodenzentralkanal der beiden ersten Gruppen homolog."

As it will appear lower down, my results concerning the development of the oviduct in Acipenser ruthenus are absolutely different from those of MASCHKOWZEFF.

Concise exposition of the investigation.

For this inquiry I have made, after embedding in paraffin, series of transverse sections of $7\frac{1}{2}$—$10\,\mu$ thickness, which I stained with hematoxylin and eosin.

The microphotos have been taken by Mr. P. J. DE VRIES, instrument-maker 1st class at the laboratory.

Sterlet, 21 mm in length.

I did not succeed anymore than OSTROUMOFF in finding an oviduct in fishes up to 18 mm in length. However, my next-following specimen, 21 mm long, exhibits on the left side the ostium abdominale, just come forth, with the foundation of the duct, while these parts are still absent on the right side.

On photo 1 the ostium is observed under the middle of the section of the wolffian duct and laterad open to the body cavity. The lateral (dorsal) lip of the ostium leans right against that duct. The medial lip is formed by a laterally inclined fold of the somatopleura. This fold can still be followed as far as 17 sections towards the rostrum. It gradually becomes sagittal and smaller, while shifting medially to disappear under the medial margin of Wolff's duct.

Now let us follow the section of photo 1 candad; we shall then see on the next-following section (photo 2) that through the fusion of its two lips, the ostium is changed into a compact mass of cells, which can still be followed two sections further. The last of these two sections has been represented on photo 3. Here the conglomeration of cells is, as it were, hemmed in between the somatopleura and Wolff's duct. It is striking that the wall of the duct is two cells thick at the spot where it is in contact with the conglomeration, while for the rest it consists of a single layer of cells.

Sterlet 28 mm long.

In this stage of development the oviduct has appeared on either side. Also in this specimen the ostium abdominale is situated under the wolffian duct and laterad open to the body cavity. The lateral lip of the ostium appears as a slight swelling of the somatopleura towards the coelon. The medial lip is a laterally inclined fold of the somatopleura. This fold is present at about 25 sections rostrally from the closure of the ostium abdominale, which is engendered by fusion of the two lips. This fold reveals the same changes as the corresponding one of the sterlet of 21 mm. When tracing the sections caudad from the closure of the ostium abdominale the müllerian duct appears to contain a lumen at 75 sections.

Here follows a description of the successive sections of the end of the müllerian duct on the right side of the animal.

Photo 4. We see that the müllerian duct lies closer to the wolffian duct than to the somatopleura. We also see that the wall of the müllerian duct, turned towards the wolffian duct, is richer in cells than the wall turned towards the somatopleura. Thirdly the wolffian duct is involute at the place of the müllerian duct.

Finally the wall of this involute portion of the wolffian duct is obviously thickened, it consists of several cell-layers, whereas the rest of its wall has only one cell-layer. The anomaly in its wall in the right top corner of the photo is due to the opening of a tubule of the mesonephros in the wolffian duct.

On the next section the lumen of the müllerian duct appears to have disappeared.

Photo 5. The boundary line between the blind end of the müllerian duct and the wolffian duct becomes vaguer. In the middle we see between them a cell, of which it is difficult to say whether it belongs to the one duct or the other.

Photo 6. This section presents a striking similarity to the schemata 477 and 478 on p. 506 of O. HERTWIG's Textbook.

In connection with the preceding photo it appears distinctly that the blind end of the müllerian duct goes on growing at the expense of the wolffian duct.

Photo 7. The involution of the wolffian duct has flattened. Of the cells in situ it cannot be said whether they belong to the caudal end of the müllerian duct, or to the wolffian duct.

On the next section the involution has flattened still more. For the rest it is like the preceding one.

Photo 8. The wall of the wolffian duct turned towards the coelom is now entirely parallel to the somatopleura. The thickening of the wolffian duct has disappeared.

We also examined fishes of 25 and 33 mm. Our experience with them agreed with that of the discussed sterlet of 28 mm. For the sake of brevity I omit a detailed discussion of these fishes.

Discussion and Conclusion.

MASCHKOWZEFF writes that the oviduct of the cartilaginous ganoids develops from the funnels of the secondary Nephrostomal-tubules. His argument is expressed in the above. He adds a reconstruction-scheme of a mesonephros-segment near the beginning of the müllerian duct in Acipenser, which I subjoin.

Beforehand I will observe that I borrowed the *literal* explanation of the figures from MASCHKOWZEFF.

Nowhere did I find in the larvae, cut by myself, the mesonephros derivates N_2. Neither in the sterlet of 21 mm, which assuredly presents an incipient stage of the development of Müller's duct.

Truly MASCHKOWZEFF adds :

"Ein Zusammenhang des Eileiters mit den Baumanschen Kapseln (N_2) läszt sich nur für die Anfangsstadien feststellen, später aber fallen die sekundären Nephrostomalkanälchen auseinander", but then at any rate segmental thickenings in the extremity of the müllerian duct should be observable and I never detected them in my preparations.

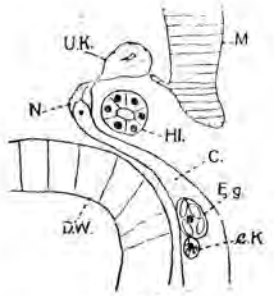


Abb. 5.

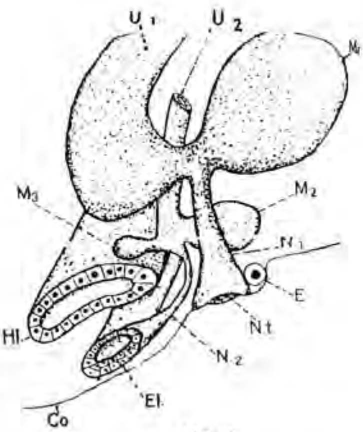


Abb. 6

Abb. 5. Querschnitte durch den Embryo vom Accipenser Stellatus im Alter von 73 Stunden. *M*. Miotom. *Hl*. Harnleiter. *US*. Ursegmentstiel. *Dw*. Darmwand. *Eg*. große primäre Geschlechtszelle. *ek*. Kleine primäre Geschlechtszellen. *N*. Primäre Nephrostomaltrichter. *Ur*. Primäre Urnierenkanälchen. *C*. Colomhöhle.

Abb. 6. Plastische Rekonstruktion eines Mesonephrossegmentes im Bereich der Eileiteranlage beim Accipenser Ruthenus.
El. Eileiter. N_2 Sekundäre Nephrostomalkanälchen. M_2 Sekundäre Malpighische Körperchen. U_2 Sekundäre Urnierenkanälchen. M_3 Anlage der tertiären Malpighischen Körperchen. *Nt*. primäre Nephrostomaltrichter. *E* primäre Geschlechtszelle. N_1 primäre Nephrostomalkanälchen. M_1 primäre Malpighische Körperchen. U_1 primäre Urnierenkanälchen. *Co*. Colomepitellium.

I never found in any of the stages I examined the thickened stripe of peritoneal epithelium, which is a continuation of the medial fold at the beginning of the müllerian duct anteriorly, which OSTROUMOFF found in fishes of 35 mm and which he considers as a rudiment pointing to a phylogenetic relation to the posterior margin of the pronephros.

Contrary to OSTROUMOFF who, when speaking of the müllerian duct, says: "Welcher Kanal nach hinten in Gestalt einer soliden Anlage zwischen dem Peritonealepithel und dem Wolffschen Kanale weiterwächst", and who consequently assumes an independent growth of the solid extremity of the müllerian duct of Acipenser ruthenus, my investigations convinced me that the müllerian duct of Acipenser ruthenus grows under a supply of material from the wolffian duct. That OSTROUMOFF did not observe this mode of growth is quite comprehensible, because in his youngest specimen (35 mm long) the duct already extended over the same segments as in the full-grown animal that he examined ($1\frac{1}{2}$ m in length). In both cases he found the posterior margin of the ostium in the 22^d^ and the termination of the duct in the 27^th^ segment. It is quite possible that the müllerian duct in a sterlet of 35 mm or more receives little or no material from the wolffian duct, as it has already reached its terminal. Indeed, during the growth of the animal the duct gets longer just as the myotomes (length measured parallel to the body-axis), but this may be due to its own (already acquired) material, just as is the case with the myotomes.

In accordance with OSTROUMOFF I found in a sterlet of 78 mm that the müllerian duct extends over five segments. Its blind caudal end is, indeed, in direct contact with the wolffian duct, but this is no longer thickened and is not involute, so that it does not seem any longer to give off any more cells.

As known, the end of the müllerian duct in Acipenser lies at a considerable distance in front of the urogenital orifice. Referring to this OSTROUMOFF says of his full-grown specimen: "zwischen seinem hinteren Ende und dem unpaaren Genitalsinus liegen noch 10 Segmente". In the common terminal of the wolffian- and the müllerian duct still kidney-tubules are emptying, and this terminal to which also the "Genitalsinus" is to be counted, must of necessity be more than double the length of the müllerian duct.

It will not do, therefore, to say with MASCHKOWZEFF (see supra) about the Cartilaginous ganoids: "Der Eileiter mündet in *das Ende* des primären Harnleiters".

According to what has been said above, the pronephros duct in the sterlet has been split into two tubes only along 5 segments, and is, therefore, more primitive than in the Selachians, in which it is divided into two tubes over its whole length.

Also the front end of the pronephros duct (as far as the 22^{d} segment according to OSTROUMOFF) remains unsplit in the sterlet, and in this respect it seems to agree rather with Amphibians and Amniotes than with the Selachians.

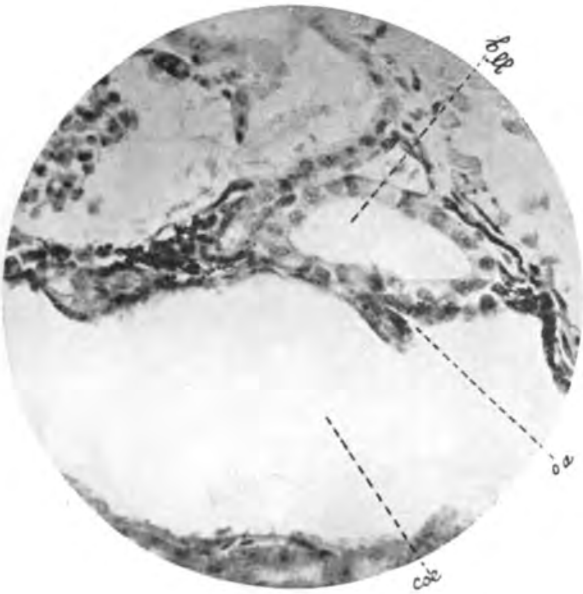
Summary.

1. Already at its first appearance the müllerian duct is in direct contact with the wolffian duct. This applies to both cases, as well when assuming as the beginning (front-end) of the müllerian duct the hind-end of the ostium abdominale (photo 2), as when considering as such the thickening in the wolffian duct, two sections further caudally. (Photo 3.)

2. The blind end of the müllerian duct, which goes on growing, keeps this contact also later on and in this place the thickening of the wolffian duct in young larvae is strong, while it consists everywhere else only of a single cell-layer.

3. The blind end of the müllerian duct is involved into this thickening and in younger specimens than the youngest (35 mm) examined by OSTROUMOFF the boundaries between the two ducts have entirely or partly disappeared. Here cells can be observed passing from the thickening to the müllerian duct (cf. photos 5, 6 and 7).

W. MOOY: "ON THE DEVELOPMENTAL HISTORY OF THE MÜLLERIAN DUCT IN THE STERLET (ACIPENSER RUTHENUS)".



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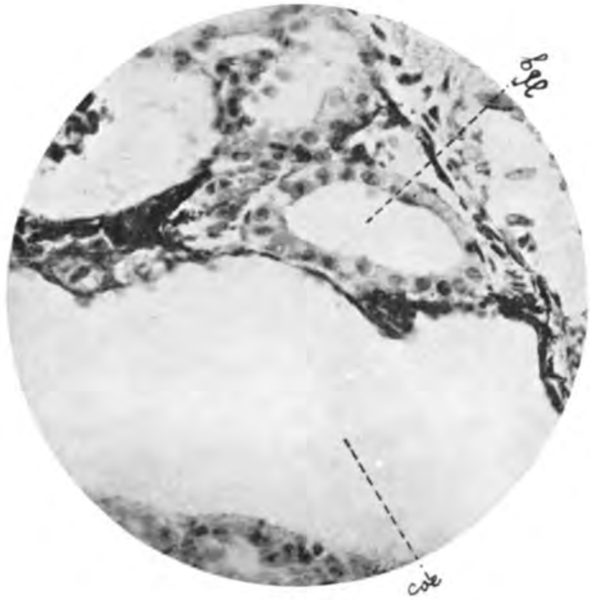

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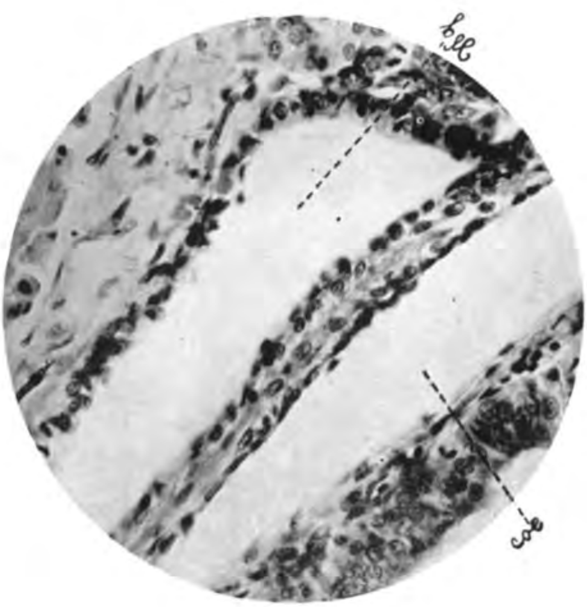

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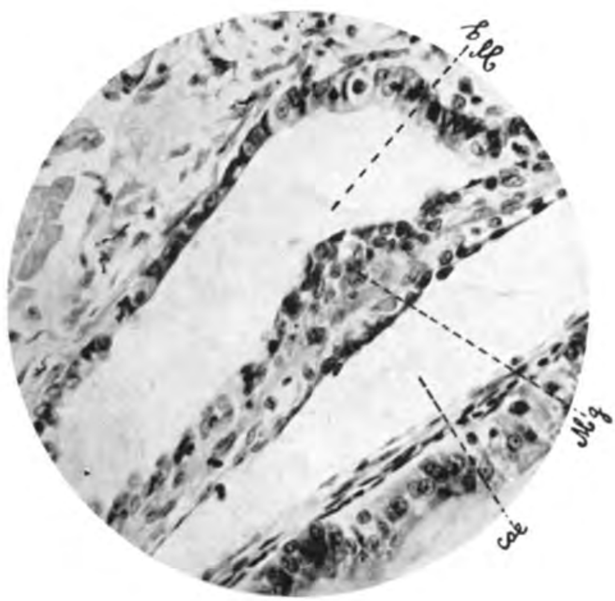

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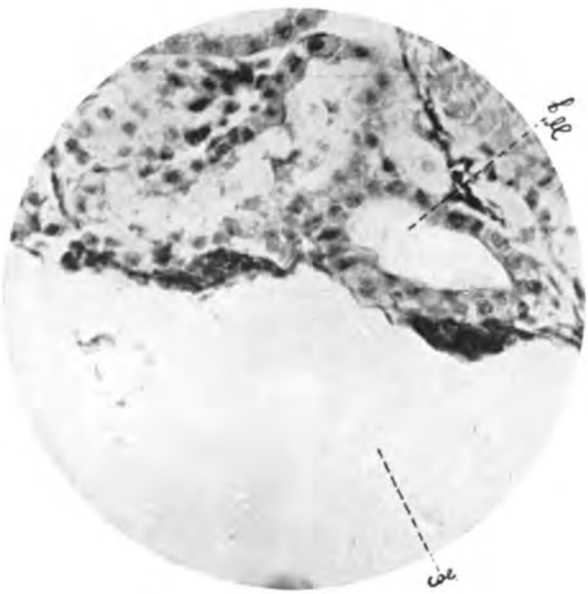

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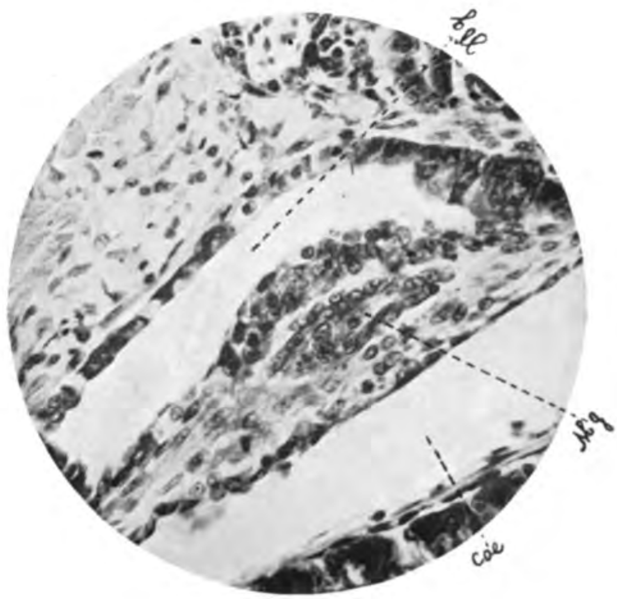

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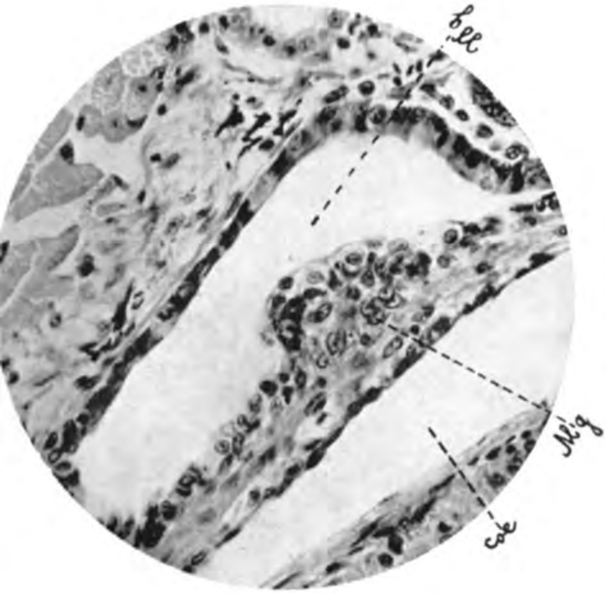

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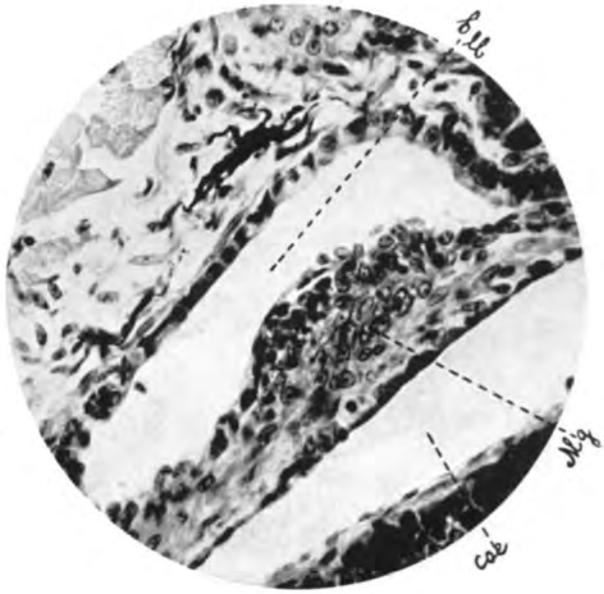

Foto 5.

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MASCHKOWZEFF, A. Entwicklung des Urogenitalsystems bei den Knorpelganoiden und einigen anderen Wirbeltieren. Anatomischer Anzeiger. Centralblatt für die gesamte wissenschaftliche Anatomie. Bd. **59**, 1924—1925.

OSTROUMOFF, A. (Kazan) Zur Entwicklungsgeschichte des Sterlets (Acipenser ruthenus). VI Der Müllersche Gang. Zool. Anzeiger, Bd. **33**, 1908.

EXPLANATION OF THE PLATE.

The photos 1, 2, 3 belong to a sterlet 21 mm long;
Fig. 2 photo first section after that of fig. 1;
Fig. 3 „ third „ „ „ „ „ 1;
The photos 4, 5, 6, 7, 8 belong to a sterlet 28 mm long;
Fig. 5 photo second section after that of photo 4;
„ 6 „ third „ „ „ „ „ 4;
„ 7 „ fourth „ „ „ „ „ 4;
8 „ sixth „ „ „ „ „ 4;

Wg = wolffian duct; Mg = müllerian duct;
o a = ostium abdominale; coe = coelom.

Physics. — *The principal susceptibilities of Manganese Ammoniumsulphate crystals at low temperatures.* By L. C. JACKSON and W. J. DE HAAS. (Comm. Number 187*c* from the Physical Laboratory at Leiden.

(Communicated at the meeting of November 26, 1927).

Introduction. Crystalpowder of manganese ammoniumsulphate is known to follow the law of CURIE down to the lowest temperatures at which it has been examined [1]). The data available at the time of the research, seemed to indicate that the $1/\chi$, T-lines for the principal susceptibilities of paramagnetic crystals (if these lines are straight) are parallel to each other. It therefore seemed important to examine the principal susceptibilities of a crystal of manganese ammonium-sulphate. We might namely expect either the three principal susceptibilities to be equal and consequently the crystal to be magnetically isotropic, though belonging to the monocline system, or, what was more probable both positive and negative values of the constant $\triangle$ to occur in the formula $\chi(T+\triangle) = \text{constant}$.

It seemed however, just as well possible that each of the three principal susceptibilities follows the law of CURIE.

§ 1. *Experimental research.* We grew large well-formed crystals of manganese ammoniumsulphate and made sections from these in the shape of small cylinders with the aid of a crystalgrinding-goniometer, so that the axes of these cylinders form a definite known angle with the crystallografic axis. In order to determine the three principal susceptibilities and the angle between the susceptibility χ_1 and the *c*-axis three differently orientated disks from the crystal are necessary, while the fourth equation is given by the known value of the susceptibility $^1/_3\,(\chi_1+\chi_2+\chi_3)$.

The susceptibilities in the directions of the axes of the disks were then determined by measuring the force exercised on the small cylinders in a non-homogeneous field, that is with the method of CURIE. The value of $H\dfrac{dH}{dX}$ being not yet known at the place where the disk was put, the apparatus was calibrated with the aid of a thin celluloid capsule exactly of the same form as the little disks. It was fixed in the magnetic field in exactly the same position as the disks and filled with the finely divided material. The force on the empty capsule was proved to be neglectable.

1) L. C. JACKSON and H. KAMERLINGH ONNES, Proc. Roy. Soc. London Oct. 1923, Comm. N⁰. 168*a*.

Thus the susceptibilities of the crystaldisks were determined in the temperatureregion of liquid hydrogen, boiling under different pressures. The apparatus used (somewhat changed for our purpose) has been described Comm. N⁰. 139*b*.

§ 2. *Method of calculation.* The axes of the three crystal disks were parallel to the „*b*"-axis, being the axis of symmetry of the crystal, perpendicular to the „*c*" (0 . 0 . 1) plane and perpendicular to the „*p*" (1 . 1 . 0) plane of the crystal respectively. When the susceptibilities in these directions are called *b*, *c* and *p*, respectively while *m* is the mean susceptibility, the main susceptibilities χ_1, χ_2, χ_3 (χ_3 coincides with the „*b*"-axis, χ_1 and χ_2 lie in the plane of symmetry) can be determined from the following equations:

$$\chi_3 = b \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$\chi_1 \cos^2 \{\psi + (\beta - 90)\} + \chi_2 \sin^2 \{\psi + (\beta - 90)\} = c \quad \ldots \quad (2)$$

$$\chi_3 \cos^2 a + (\chi_1 \sin^2 \psi + \chi_2 \cos^2 \psi) \sin^2 a = p \quad \ldots \quad (3)$$

$$\chi_1 + \chi_2 + \chi_3 = 3m \quad \ldots \ldots \ldots \quad (4)$$

where ψ is the angle between χ_1 and the „*c*" axis of the crystal and a the angle between the normal on the „*p*" plane ((1 . 1 . 0) plane) and the „*b*" axis, while β represents the angle between the „*a*" and „*c*" axes.

From these equations χ_1, χ_2, χ_3 and ψ can be solved.

§ 3. *Experimental results.* The following tables give the values of the susceptibilities as they have been observed.

Crystal disk "*b*"		Crystal disk "*c*"		Crystal disk "*p*"	
T	$\chi \cdot 10^6$	T	$\chi \cdot 10^6$	T	$\chi \cdot 10^6$
$20^\circ.3_7$ K.	547	$20^\circ.3_5$ K.	534	$20^\circ.3_7$ K.	577
18.9_4	592	19.1_4	583	19.7_7	611
16.9_4	660	17.8_2	613	17.6_0	669
15.0_7	740	16.4_0	662	15.2_2	766
		15.0_0	738	14.8_5	791

In order to make the further calculation a set of values is necessary, which have been reduced to the same temperatures. Consequently graphs have been drawn of the values mentioned above of the χ's for the „*b*", „*c*" and „*p*" disks as functions of the temperature. The values of the „*b*", „*c*", „*p*" are then graphically reduced to $T = 20^\circ.35$, $19^\circ.0$, $17^\circ.0$ and $15^\circ.0$ K.

These values follow here:

	Crystal disk			mean susceptibility of the powder „m"
	„b"	„c"	„p"	
T.K.	$\chi \cdot 10^6$			
20°.35 K.	548	534	578	542
19.0	589	572	620	580
17.0	662	639	690	649
15.0	748	723	780	735

Moreover the following data [1]) are still required:

$$a : b : c = 0.7360 : 1 : 0.4972$$

$$\beta = 107^\circ 2'$$

$$p\,(1\,.\,1\,.\,0) : p\,(1\,.\,\bar{1}\,.\,0) = 70^\circ 16'$$

The final results of the calculation are:

T	$\cos^2 \psi$		
20°.35 K.	0.8518	of	0.1471
19.0	0.7813	„	0.2176
17.0	0.7936	„	0.2063
15.0	0.8366	„	0.1663

T	$\chi_1 \cdot 10^6$	$\chi_2 \cdot 10^6$	$\chi_3 \cdot 10^6$
20°.35 K.	754	323	548
19.0	819	333	589
17.0	926	359	662
15.0	1038	420	748

The values in the third column (0.1471 etc.) have been accepted provisionally as the exact ones, as they come nearer to the values, which are known for the other members of the family of the monoclinic double

[1]) See GROTH, Chemische Kristallographie.

sulphates, to which manganese ammoniumsulphate belongs. No great accuracy may be expected however in the determination of ψ. The values of the principal susceptibilities will be accurate to about two percent. From these values we calculated the molecular susceptibilities, taking into account the diamagnetic property of the anion, the crystallization water and the ammoniumsulphate. The results are given in the following table. The values for χ'_{3m} are those, which have been observed with the aid of the crystaldisk „*b*".

T	χ'_{1m}	χ'_{2m}	T	χ'_{3m}
$20^{\circ}.3_5$ K.	0.295	0.127	$20^{\circ}.3_7$ K.	0.214
19.0	0.321	0.130	18.9_4	0.232
17.0	0.362	0.140	16.9_4	0.258
15.0	0.406	0.165	15.0_7	0.290

§ 4. *Results.* The inverse values of the corrected molecules susceptibilities are plotted against the absolute temperature. In this way the variation of the different susceptibilities with T has been represented

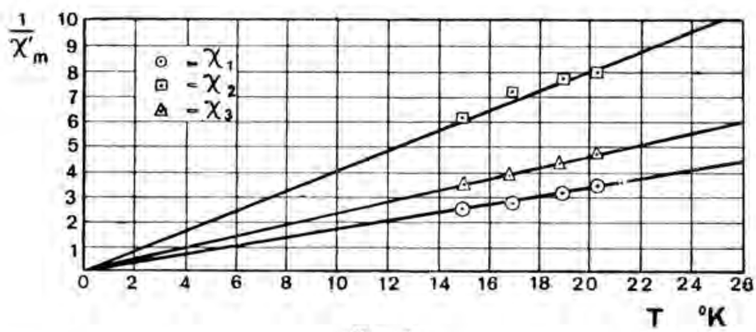

Fig. 1.

graphically. We see from Fig. 1 that the $\frac{1}{\chi} . T$ lines are neither coïnciding, nor parallel. The values are lying as well as possible on three straight lines, which all pass through the origin.

This means that the principal susceptibilities follow the law of CURIE $\psi . T = C$, but with different values for the C.

The mean susceptibility of the crystalpowder of manganese ammoniumsulphate obeys the law of CURIE, because each of the principal susceptibilities follows this law.

Physics. — *Research about the question whether grey tin becomes supraconductive or not.* By W. J. DE HAAS, G. J. SIZOO and J. VOOGD. (Communication N⁰. 187*d* from the Physical Laboratory at Leiden.)

(Communicated at the meeting of December 17, 1927).

Introduction. The physical condition of a metal is known to have a great influence up on the conductivity. Also the form of the cristal-lattice has a predominant influence on different physical properties of the metals. This induced us to investigate, whether grey tin, in the region of the temperatures obtainable with liquid helium behaved in a different way as white tin which, as we know, becomes supraconductive at about 3.6° K.

A difficulty for the research of grey tin is that it can only be obtained as a rough powder. The size of the grains of this powder ranges from 0.5 to 0.1 mm.

It was kindly made for us at the Van 't Hoff Laboratory at Utrecht by Professor COHEN, later during the latter's stay in America by Professor KRUYT. We heartily thank both gentlemen.

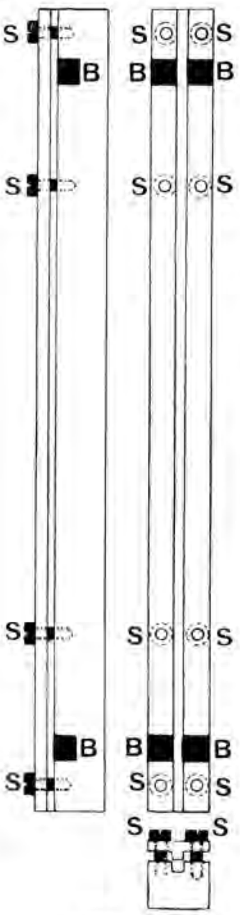

Fig. 1.

§ 1. The first experiments were made with a quantity of tin-powder pressed into a barshape. For that purpose a small piece of ivory was made with a groove in the length-direction of 1 mm width and depth. At some distance from the extremities four amalgamated brass pieces B (see fig.) were fixed to which the wires for the resistance-measurements were soldered. The groove was filled with grey tin and a lid, provided with a ledge exactly fitting into the groove, was screwed to the ivory by means of the screws. As it has no sense to measure accurately the resistance of such a badly defined resistance body and as moreover this was not our intention, the measurements were made with a plain Kohlrausch bridge. The resistances of the feeding wires were measured separately, so that a correction could be made for them.

The results of this research are given in Table I.

TABLE I.

Temperature	
300° K.	112.8
4.2	38.3
3.8	38.3
3.6	34.3
1.2	34.3

We can readily criticize this experiment. The current density becomes very great at the points of contact of the different metal grains and consequently a number of real resistances might arise either by heat-development or, what is more probable, by a too large magnetic field of the current itself, which, as is known, can disturb the supraconductive state. Also the grains might change their positions during the cooling, so that it would be preferable to press the ivory ledge into the groove with strong springs.

Anyhow, it would be desirable to repeat the experiment in a different way.

§ 2. With the second method we followed a suggestion of Prof. P. EHRENFEST and made use of the persistent currents in analogy with a method for gallium used by TUYN and KAMERLINGH ONNES.

A bar of grey tin was made inside a glass tube long 12 cm and with a diameter of 7 mm. An external coaxial field of 400 to 500 Gauss was excited and made to disappear. At its maximum strength such a field most probably was strong enough to destroy the supraconductive state of the metals; the supraconductive property of tin disappears namely between the temperatures 3.6° K. and 1.5° K. with fields ranging from 0 to 230 Gauss, for indium these values are from 3°.4 K. to 2°.3 K.; the fields from 0 to 130 Gauss, for mercury from 4°.2 K. to 1°.87 K., the fields from 0 to 340 Gauss, for lead from 7°.2 K. to 3°.7 K. and the fields from 0 to 600 Gauss.

If now we let the maximal field disappear we cross the eventual maximal magnet temperature threshold value of the metal and rouse the persistent currents in every grain. At the extremities of the bar of grey tin outside the cryostate an easily revolving magnet has been placed to discover the existence of the persistent currents. The experiments were made at the boiling point-pressures mentioned below and at the temperatures of liquid helium.

TABLE II.

Boiling point of Helium.	Temperature
774 mm.	4°.21 K.
350	3.50
165	3.00
70	2.55
45	2.35
40	2.30
36	2.26
33	2.23
27	2.15
20	2.04
10	1.80

§ 3. *Conclusions.* No declinations of the magnetic needle were stated. We may conclude from this that grey tin does become supraconductive even at the lowest temperatures used. We might still object that the grey tin might have been in the supraconductive state continually, also at the highest temperatures and in the highest fields used. A rise from 0 to 500 Gauss and a vanishing again of the fields should have had no influence in that case. However, this supposition is hardly probable with a view to the experiments under § 1. And yet, in order to get absolute certainty we have repeated the experiment with the bar of grey tin in such a way that we excited the coaxial magnetfield, before the cryostate with the bar of grey tin had been cooled. Only after the gradual cooling of the Helium-cryostate-glass and the reducing of the boiling-point of the Helium to the low temperature of 1.8° K., the magnetic field was made to vanish. This time too the bar of grey tin did not show any persistent currents.

While KAMERLINGH ONNES [1]) came to the conclusion that a loose atom connection and a great distance between the atoms were to advantageous for the creation of the supraconductive state the above experiment shows that the phenomenon is more complicated.

Grey tin has a smaller density than white tin.

[1]) Leiden. Comm. Suppl. N⁰. 50.

Physics. — *On the change of colour of crystals at low temperatures.* By I. OBREIMOW and W. J. DE HAAS. (Comm. N⁰. 191a from the Physical Laboratory Leiden.)

(Communicated at the meeting of January 28, 1928).

Introduction. In contrast with the spectra of gases, the absorption spectrum of solid bodies and liquids consists as a rule of very large, sometimes very diffuse bands. But in the last few years more and more experiments have shown that, by sufficiently lowering the temperature, the crystalspectra are transformed from large bands into sharp spectral-lines. 1906 already JEAN BECQUEREL observed, that the crystals of the rare earths, when immersed in liquid air, gave absorption spectra, which consisted of a large number of very fine bands [1]).

It must be mentioned that already in some cases of the absorption-spectra, the spectral terms have been found and that the classification of the lines has succeeded [2]).

The well-known change of colour [3]), which appears at the cooling of crystals, suggests the formation of absorption-bands and a change in the structure of the bands.

The purpose of this research is to examine, whether

1⁰. the change of colour, which has been observed down to the temperature of liquid air continues in the temperature-region of liquid hydrogen and

2⁰. whether in this case the absorption-bands are perhaps transformed into spectral lines.

§ 1. Three different crystals were subjected to a preliminary research viz. the crystals of azobenzol, potassium bichromate and iodine. All three crystals change their colours at low temperatures. Azobenzol and potassium bichromate become citrine-yellow and iodine dark red.

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1) J. BECQUEREL, Le Radium, **4**, 328, 1907. See for further literature among others.
V. HERI, Proceedings of the Optical Institute I, 2.
M. DE SÉLINCOURT, Proc. R.S. **107**, p. 247, 1925.
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RUBENS und G. HERTZ, Stzb. d. Preuss. Akad. I, p. 256, 1912.
A. KRONENBERGER und P. PRINGSHEIM, Z. f. Phys. **40**, p. 75, 1926.
B. GUDDEN und R. POHL, Ph. Zs. p. 481, 1925.

2) A. G. S. VAN HEEL, Dissertation 1925. See for the theory P. EHRENFEST, Memorial Volume H. KAMERLINGH ONNES, Leiden 1922.
See for literature also P. PRINGSHEIM, Fluorescenz und Phosphorescenz im Lichte der Neueren Atomtheorie.

3) See among others KOURBATOFF, Journ. Russ. Chem. Ges. **39**, II, p. 134, 1907.

The method of investigation was exceedingly simple and hardly wants a detailed explanation.

The light of an arc-lamp is cast through the crystal plates. After having first been carefully cooled by the vapour of liquid hydrogen, these are immersed in the liquid hydrogen boiling under atmospheric pressure.

The vacuum-vessel in which the hydrogen boiled, was made of quartz. Small windows of plane quartz glass had been fused in the walls to let the light pass, perpendicularly to the axis of the vessel.

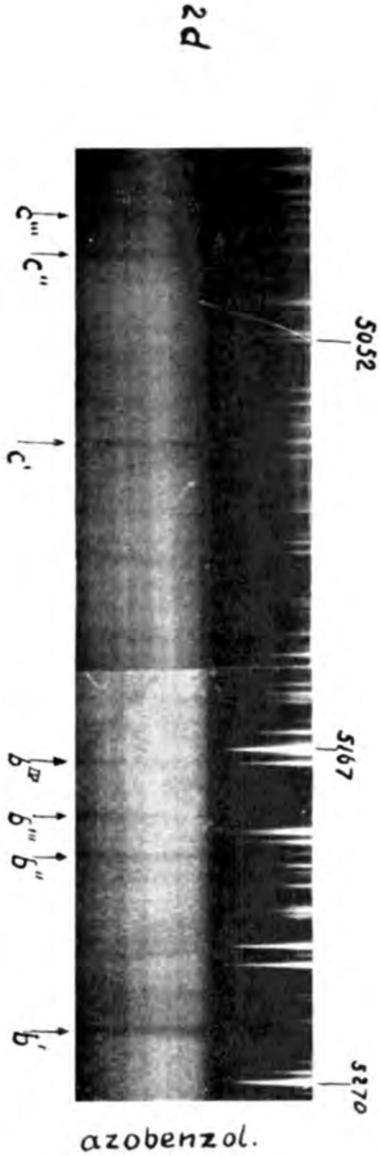
The light through the first two quartz windows, falls on the crystal which, with the aid of cork is fixed by mean of small springs in a holder, passing through the cryostatecap. The light after passing through the crystals leaves the vessel by two quartz windows and falls on the slit of a Steinheil spectrograph. The dispersion of this instrument is such that the *D*-lines are separated over a distance of about $^1/_3$ mm. For the determination of the wave-lenghts of the absorption-lines the spectrum of an iron arch is cast at the same time on the photographical plate. On account of their large absorption the crystals have sometimes been thinly cut (for azobenzol about 0.1 mm, for potassium bichromate about 0.2 to 0.3 mm and for iodine also about 0.2 to 0.3 mm). Their dimensions were prescribed by these of the opening in the cork holders 2 to 3 mm).

§ 2. *Potassium-bichromate and iodine.* At room temperature potassium bichromate has a continuous absorption-band, which begins at nearly 5500 Å and extends into the extreme ultraviolet (fig. 1*a*). At 20° T.K. the red side of the absorption-band is split up into some sharp lines (see fig. 1*b*). The potassium bichromate shows a distinct pleochroism; when we place a nichol in the beam of rays of the crystal, the spectra change with the azimuth of the nichol. Fig. 1*b* shows the enlarged spectrum of a potassium bichromate crystal (0.2 mm thick) taken with an incident polarized beam of rays. The orientation of the crystal had not been determined. We intend to repeat these experiments and give here only this preliminary communication.

Iodine-sheets (0.1 mm thick) are not transparent at ordinary and liquid air temperatures. These same sheets at 20° T.K. (liquid hydrogen) however, transmit dark red light ($>$ 6700 Å). In the transmitted light we see very weak, periodical bands. Pleochroism may be very distinctly observed. The experiments with iodine have not been continued for the present, as the sheets have to be made thinner and have to be studied at still lower temperatures.

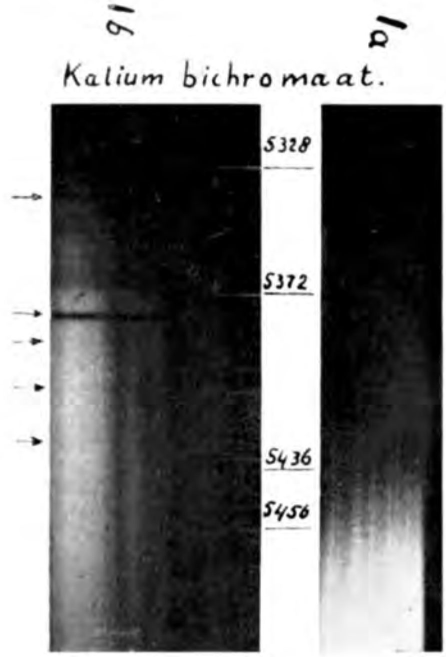
§ 3. Azobenzol, $C_6H_5 - N = N - C_6H_5$ crystallizes in the monocline system ($a : b : c = 2.11 : 1 : 1.33$; angle $(a, c) = 114°26'$). From an alcoholic solution we can precipitate the crystals in the shape of thin sheets by evaporation. The base of these sheets is *c* (001). Plane of the axes $\perp$ to the plane of symmetry (010). When we illuminate these sheets by white

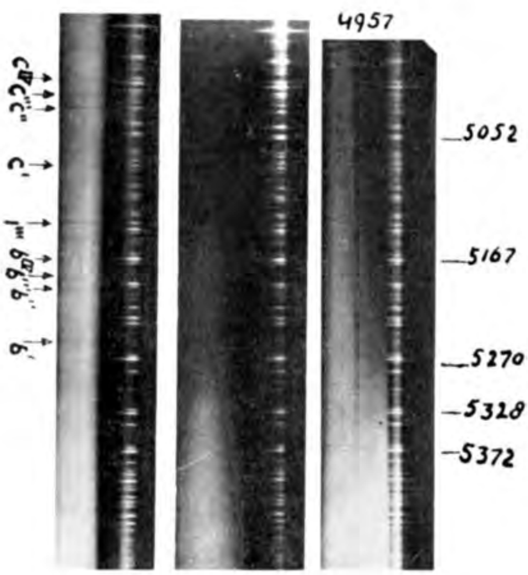
I. OBREIMOW AND W. J. DE HAAS: ON THE CHANGE OF COLOUR OF CRYSTALS AT LOW TEMPERATURES.



azobenzol.

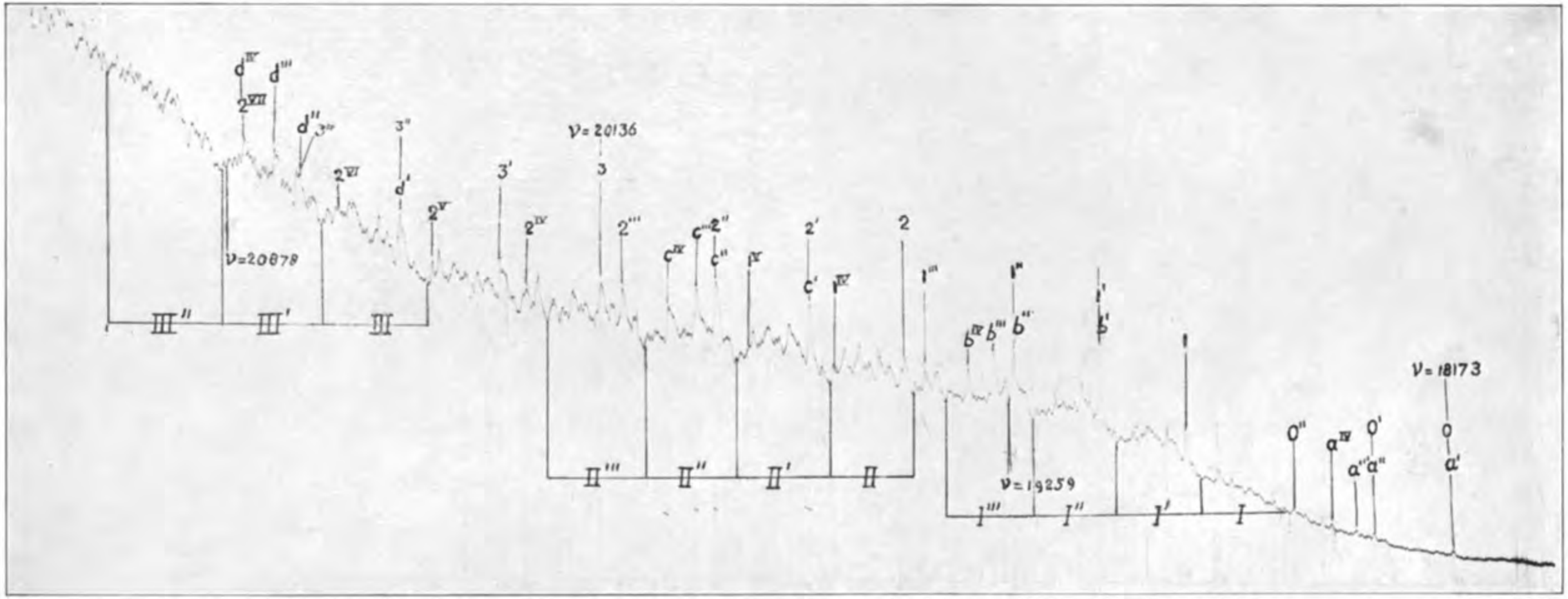
Kalium bichromaat.





azobenzol.

2c 2b 2a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light they are very pleochroistic. They show an orange-colour, if the electric lightvector is parallel to the *b*-axis and a citrine-colour, if the lightvector is perpendicular to the direction mentioned above. At a temperature of —180° C one component becomes of a citrine-yellow colour and the other of an almost white very light green colour. The spectroscopical research shows that in the case of the second component, the plate is transparent for the whole visible spectrum.

We examined the „red" component only. Fig. 2*a* shows the „red" absorption-spectrum at room temperature. It doesn 't show any structure. Fig. 2*b* shows the same absorption spectrum at —196 C (liquid nitrogen). The edge of the very large absorption-band breaks up into a series of narrower but still large and diffuse bands. About 13 bands could be counted; in the fig they are not very well visible; in the blue the absorption becomes too strong to be able to observe still anything. At a temperature of 20° T.K. (liquid hydrogen) azobenzol shows a splendid absorption-line-spectrum (fig. 2*c*). In the case of azobenzol, as well as in the other cases described above, the change of colour continues because of the narrowing of the absorption-bands. In fig. 2 a small part of a diagram of azobenzol at usual temperature and one at 20° K.T. have been reproduced above each other. We see from this fig. 2*b*, 2*c* that the absorption-bands are not perceptibly shifted, but that the bands are split up into lines. In the reproduction the pictures lose considerably in contrast. Moreover the same part of the absorption-spectrum has been represented 10 × enlarged in fig. 2*d*.

§ 4. We cannot yet communicate the classification of the lines and we have the intention to examine the symptoms partly at still lower temperatures. Yet some regularities and periodicities are so striking that we will mention them here.

The periodicity is very striking for azobenzol. We have taken a photogram from the spectrum of the azobenzol represented in fig. 3. We can state distinct periodical variations of the intensities indicated by the figures I, I′, I″, I‴, II, II′, II″, II‴, III, III′, III″, etc. The medium width of the periodicity I is 17 mm, of II 18 mm of III 21 mm.

We want to draw the attention to the group of lines marked *a′*, *a″*, *a‴*, *a⁗*; *b′*, *b″*, *b‴*, *b⁗*; *c* etc., *d* etc. This group of 4 lines repeats itself four times. The distances between the lines of a group are given below:

	a	*A*	*b*	*B*	c	*C*	*d*	*D*	mean
I	0	0	0	0	0	0	0	0	0
II	15.8	1	17.2	1	18.7	1	20.9	1	1
III	19.3	1.22	21.1	1.23	22.4	1.20	25.2	1.20	1.21
IV	24.1	1.52	26.2	1.53	28.1	1.55	31.3	1.50	1.51

In this table the distances *a′*—*a″*, *b′*—*b″* etc. have been taken as a unit for the vertical columns a, b, etc. and from these the numbers A, B, etc. have been calculated.

The microphotogram (fig. 3) shows that we have succeeded in splitting up the band into isolated lines as is for instance evident for the lines a', a'', a''', a'''', b', b'', b''', b'''', c', c''. But towards the violet end of the spectrum the absorption lines are lying so closely together that it is difficult to find the real place of their centre.

For, if

$$y_1 = f_1(\nu) \quad y_2 = f_2(\nu)$$

represents the distribution of the absorbed light in two isolated lines and if both coïncide, the light absorption caused by both the lines, will be represented by

$$y = y_1 + y_2 = f_1(\nu) + f_2(\nu).$$

The maximum absorption will lie at

$$y'_1 + y'_2 = 0.$$

viz there where the slopes of both absorption-curves are equal and opposite, but not at the values of ν where

$$y'_1 = 0 \text{ or } y'_2 = 0.$$

As however these latter values of ν give the centres of the lines it is desirable to fix only the place of the good undisturbed lines. The fixing of weak lines may lead to a mistaking of the place of the maximum absorption for that of the centre of a single line.

Consequently we have not fixed the place of all maxima, but it seems that the total spectrum is a periodical repetition of the same line-groups. It seems that the lines marked as 1, 1′, 1″, 1‴, possess the same distance in frequency.

This is also the case with the lines 2, 2′, 2″, 2‴, 2⁗, 2′′′′′, 2′′′′′′, and with the lines 3, 3′, 3″, 3‴, and with the lines 0, 0′ and 0″.

The experiments will be continued with some larger dispersion of the spectrograph and if necessary at lower temperature.

Finally we kind thank Mr. P. M. VAN ALPHEN, who made the microphotograms for us and Mr. G. J. FLIM for his valuable help with the construction of the apparatus.

Ophthalmology. — *On the Analysis of Ocular movements.* By C. D. VERRIJP. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of February 25, 1928).

As the spatial position of an object is known from the location of three fixed points on the object, the analysis of the eye-movements may be imagined to be effected through the localization in a number of successive positions of three points indicated or supposed on the eye. If the assumption be admitted that the eye can perform such movements that all points of it describe paths parallel to some level plane (e.g. the horizontal), it will suffice for the analysis of *these* movements to localize two fixed points. We are justified in assuming that, at least for the horizontal and the vertical plane, such movements are possible. I will confine myself to this kind and first of all consider in particular those movements in which all points describe paths parallel to a horizontal plane.

Now, if of two fixed points the place is known for a number of successive positions, we can determine how the eye has shifted, respectively has rotated, and that with a precision that depends besides on the accuracy of the observations also on the minuteness of the displacements about which observations are made. As, in the experiment it is not possible to accurately establish the place of the fixed points indicated or supposed on the eye, I have relinquished this method. The object in view may also be attained if with every position a line fixed with respect to the eye, and one fixed point are localized. The visual line [1]) I take for fixed line, for

[1]) The current nomenclature distinguishes (HELMHOLTZ's Handb. der physiol. Optik:) 1⁰ the "Blicklinie", i.e. the straight line, drawn from the point of fixation to the centre of rotation of the eye: 2⁰ the "Gesichtslinie" (visual line) consisting of two parts, the front-portion of which is a straight line, joining the fixation-point to the first nodal point, the posterior portion being the straight line running from the second nodal point to the fovea centralis. Besides these lines also "Visierlinie" was spoken of, i.e. a line drawn through two points covering each other, which means that the point is situated in the middle of the dispersed image of the other; since the cone of rays, which forms the dispersed image, is limited by the opening of the pupil, these "Visierlinien" (lines of sight) should intersect each other in the centre of the pupil. In the third edition of HELMHOLTZ's manual GULLSTRAND observes that, since for "visieren" (aiming, directing) central vision is required, only one line of sight can be concerned, which after the refraction touches the central point of the fovea centralis. In connection with the fact that it is not likely that we may speak of one single centre of rotation, I call the visual line ("Blicklinie") the straight line which in the normal, not squinting eye joins the point of fixation with the middle of the fovea centralis, the place of the highest visual acuity.

fixed point the intersection of this line and the anterior surface of the cornea.

I purpose to describe in a following paper the experiments by which this is realized.

The problem is now reduced to the analysis of the displacement of a portion of a straight line in a level plane.

This level plane is the horizontal plane in which the visual line is moving. The experimental data are a number of successive positions of this line and the spots, (expressed in co-ordinates) which the intersection of the anterior surface of the cornea with the visual line, occupies every time.

When a portion of a line moves in a flat plane from a position AB to a position $A'B'$, a point P can *always* be indicated that is equidistant from all corresponding points of AB and $A'B'$ (the intersection of two lines which divide vertically in two the connecting lines of two pairs of corresponding points, as may easily be proved). The portion AB may then have come in $A'B'$ through a rotation in which all the points have described arcs around the centre P. It is clear, however, that *in general* this rotation may be assumed only when the described angle of rotation is infinitely small. If this should not be so, the displacement from AB to $A'B'$ may also be conceived as originating from one or more shifts of AB parallel to itself, combined with one or more rotations, whereby the place of the centres of rotation and the extent of the shiftings *remain in that case absolutely undetermined*. However, the movement may always be considered as the succession of infinitely small rotations, every following one round another centre of rotation, now *altogether determined*; the object of the analysis is to find the line described by this ever varying centre of rotation.

The centres of rotation thus found are properly speaking the intersections of the *axes*, round which the eye rotates, with the flat plane, in which the visual line is moving.

But apart from this it is also evident that, in connection with the foregoing, we should not a priori speak of one single centre of rotation, without knowing the movement. Every experimental method that *starts* from the hypothesis that there is only one centre of rotation, is objectionable. This is the more cogent as i.a. on anatomic grounds we are induced to doubt a simple rotation round one single point.

It should be required of the experimental method that it shows us the place occupied by the eye in a number of successive positions. The analysis of these data furnishes the curve of the instantaneous centres of rotation.

When looking up the literature to see how the various investigators have conceived the problem, it seems to me that they have, generally speaking, not formed a correct kinematic idea about it. When we determine the lines of sight for the different positions of the eye, and we find that they intersect in one and the same point, it is clear that this point need not be the centre of rotation of the eye. Already BERLIN pointed to this fact. Conversely, when we find that these lines do not intersect in one point, it is not admissible, to conclude from it that there has not been one centre of rotation. But by the methods in which "*the*" centre of rotation is not assumed beforehand on a definite line, we do not attain our purpose either. At best they serve to prove that *the* centre of rotation does not exist. If I am not mistaken, only one method has been described that meets the requirements. It is that of FICK, which J. J. MÜLLER [1]) used in an investigation. He gives the place of the line of sight and that of the intersection of this line with the anterior surface of the cornea for a number of successive positions of the eye, each of them forming a rather small angle (about 5°—7°) with the following one. From the fact that the named intersections lie approximately on a circle, he concludes that there is only one centre of rotation, viz the centre of this circle.

This conclusion, however, would be justified only if the lines of sight were tangents of a circle with the same centre (or if they intersected in the same centre). But since the author states that this is most often not the case, it seems to me that the results of this inquiry go rather against the hypothesis of one single centre of rotation.

I shall not dwell further on the literature [2]), but will only enter into a speculation of the method described by KOSTER [3]).

True, this method is founded originally on the assumption of one centre of rotation. However, the author has made complementary measurements, with which after all an image of the eye-movements can be projected.

In connection also with the fact that it is difficult to judge of the accuracy of the data already obtained, I thought it in every respect expedient to undertake new experiments, in which I should start directly from the above speculation.

As stated before, in every new position of the eye the direction of the visual line and the co-ordinates of a fixed point on that line are measured. The position is then established completely. The following considerations now lead to the construction of the curve of the centre of rotation (see fig. 1).

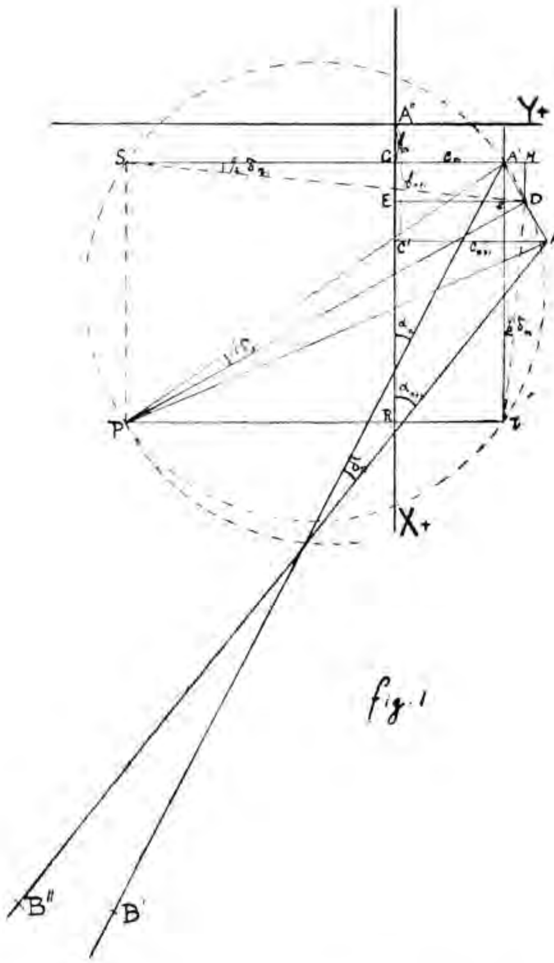
In the figure let the X-axis be positive downward and the Y-axis positive to the right. Let the zero-position of the visual line coincide with the X-axis; let the fixed point A^0 on it be in that zero-position in the intersection of X-, and Y-axis; now the eye rotates and the visual line takes up e.g. a position $A'B'$. The angle described by the rotating eye we call a_n. The co-ordinates of the point A' we call in the direction of X-, and Y-axis respectively b_n and c_n. Now let us watch the movement, when the eye removes from the position $A'B'$ to the position $A''B''$ and let us thereby assume for a moment a rotation round one single point P. The co-ordinates of A'' are b_{n+1} and c_{n+1}; the angle over

[1]) J. J. MÜLLER, Arch. f. Ophth. XIV Jahrg., Abt. 3, p. 183; 1868.

[2]) A bibliography is given by BRENNECKE (Klin. Monatsbl. f. Augenh., Bd. **68** p. 227, 1922) who himself seems to have determined only the intersections of lines of sight; also by KOSTER, and by C. SCHAAP, Thesis. Leiden 1927.

[3]) W. KOSTER, Arch. Néerl. des sciences exactes et nat. T. XXX, p. 370. 1897.

which the eye has moved from the zero-position we call α_{n+1}. From $A'B'$ to $A''B''$ the angle of rotation was $A'PA'' = \alpha_{n+1} - \alpha_n$. This angle we name δ_n. Of course the angle between the visual lines is the same,



just as the angle between $A'B'$ and the X-axis is α_n. When drawing from A' the line $A'r$ parallel to the X-axis, and from P the line Pr normal to it, then the line PA' is a diameter of a circle passing through the points P, A' and r, because $\angle PrA' = 90°$; this circle cuts the line $A'A''$ in D and it will easily be seen that, as now also $\angle A'DP = 90°$, and consequently $\triangle PA'D \cong \triangle PA''D$,

$$A'D = A''D \text{ and } \angle A'PD = \angle A''PD = \tfrac{1}{2}\delta_n.$$

When producing the perpendicular $A'C$ as far as S and erecting the perpendiculars DE on the X-axis and DH on $A'C$, then it will be seen that also the angles $A'SD$ and $A'rD = \frac{1}{2}\delta_n$. Let the co-ordinates of the centre of rotation P be called ξ and η then

$$\xi = A^oR = A^oE + ER = A^oE + er = A^oE + eD \cot \angle erD \text{ and}$$
$$\eta = CS = CH - SH = ED - HD \cot \angle HSD,$$

from which are derived the general formulae

$$\xi = \tfrac{1}{2}(b_{n+1} + b_n) + \tfrac{1}{2}(c_{n+1} - c_n) \cot \tfrac{1}{2}\delta_n \quad . \; . \; . \; . \; . \quad (1)$$

and

$$\eta = \tfrac{1}{2}(c_{n+1} + c_n) - \tfrac{1}{2}(b_{n+1} - b_n) \cot \tfrac{1}{2}\delta_n \quad . \; . \; . \; . \; . \quad (2)$$

So the location of the centre of rotation can be readily computed from the coordinates of A' and A'' and the angle of rotation, which is likewise the angle between the two visual lines.

This speculation was founded on the assumption that the displacement from $A'B'$ to $A''B''$ occurred through rotation round the point P; which assumption is admissible only if the angle between $A'B'$ and $A''B''$ is infinitely small. In this respect the above formulae may still be simplified. For first of all $\frac{1}{2}(b_{n+1} + b_n)$ draws near to b_n and $\frac{1}{2}(c_{n+1} + c_n)$ to c_n. The co-ordinates of A' are both functions of a. We write $b = F(a)$ and $c = f(a)$. It can now easily be seen that the last terms of the second sides of the above equations are nothing but the differential coefficients of these functions, for e.g.

$$\tfrac{1}{2}(c_{n+1} - c_n) \cot \tfrac{1}{2}\delta_n = \frac{c_{n+1} - c_n}{2 \times t_g \frac{1}{2}\delta_n} = \frac{c_{n+1} - c_n}{\delta_n} \rightarrow f'(a),$$

when δ_n and a are expressed in radials. If a is expressed in degrees this expression becomes $\frac{180}{\pi} \times f'(a) = = 57.2958 \times f'(a)$. The equations (1) and (2) change into the following:

$$\xi = F(a) + 57{,}2958\, f'(a) \quad . \quad (3)$$

and

$$\eta = f(a) - 57{,}2958\, F'(a) \quad . \quad (4)$$

These equations can also be derived directly (Fig. 2).

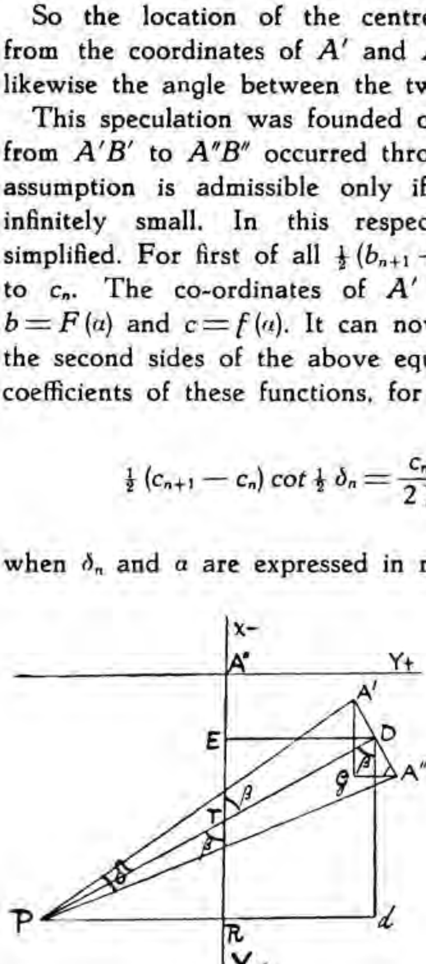


fig. 2.

Here the distance $A'A''$ represents a very small displacement, the rotation taking place over an angle δ, round the centre of rotation P; PD we call ϱ, the angle this line makes with the X-axis β. Then is

$\beta = \angle DT A_0 = \angle PTR = \angle PDd = \angle A' A'' G$. Further we have: $\xi = A^0R = A^0 E + ER = A^0E + Dd$.

Now $A^0 E = b + \frac{1}{2} \triangle b$, when expressing the very small increase of b : $A' G$ by $\triangle b$. Furthermore

$$Dd = \varrho \cos \beta = \varrho \frac{GA''}{A'A''} = \varrho \times \frac{\triangle c}{\varrho \delta}$$

(δ is expressed in radials and $A'A''$ may be looked upon as an arc): now

$$\frac{\triangle c}{\delta} = \frac{\triangle c}{\frac{\pi}{180} \triangle a} = 57{,}3 \times \frac{\triangle c}{\triangle a}$$

when $\triangle a$ represents, expressed in degrees, the increment of the angle which the visual line forms with its zero-position (direction of the X-axis); naturally this increment is of the same magnitude as $\angle A'PA''$. So we get:

$$\xi = b + \tfrac{1}{2} \triangle b + 57{,}3 \times \frac{\triangle c}{\triangle a}.$$

This expression has for limit, if $\triangle b$, $\triangle c$ and $\triangle a$ are infinitely small,

$$\xi = F(a) + 57{,}3 \times f'(a).$$

In the same way we can derive:

$$\eta = RP = Rd + dP = ED + dP = c + \tfrac{1}{2} \triangle c - \varrho \sin \beta = c + \tfrac{1}{2} \triangle c -$$

$$- \varrho \times \frac{\triangle b}{\varrho \delta} = c + \tfrac{1}{2} \triangle c - \frac{\triangle b}{\frac{\pi}{180} \times \triangle a},$$

the limit of which expression is again:

$$\eta = f(a) - 57{,}3 \times F'(a).$$

It is evident that only the formulae (3) and (4) are theoretically quite correct. If for a definite point of the curve, described by A, we wish to establish the corresponding centre of rotation (corresponding with the small piece of the curve of which A (or D) is the middle) we should try to construct $F(a)$ and $f(a)$ from the experimental data about b and c, after which ξ and η can be computed for the said point A. But then it is necessary, we should have very accurate data concerning the curve, described by A: every slight deviation in this curve may bring on great changes in the place of the centre of rotation.

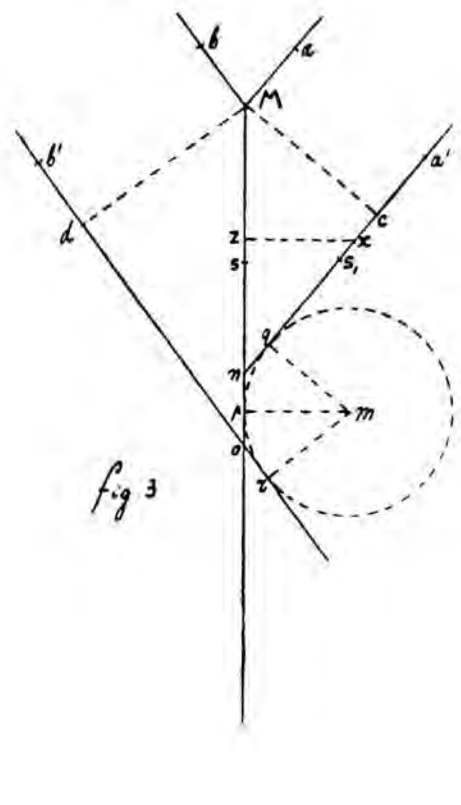
Meanwhile with the aid of the formulae (1) and (2) an image may be projected that, especially when making an adequate number of observations, need not differ much from reality.

The difference with the second method, which uses (3) and (4) is, that the first method considers only two points of the A curve, not the points between them nor those round about them, and two visual lines, whereas the secound method gives a speculation *in connection with the surrounding points*; this is quite rational owing to the fact that the curve must show some gradation, that is to say that the location of all the points affects in any case that of the neighbouring points. But errors in the observations will count all the more now. With the second method, not with the first, the exact condition might be derived from correct observations. Yet with observations, not perfectly correct, there is every chance to project an image with the first method, which is more like the real image than the one obtained by the second method. I shall have an opportunity to demonstrate this when discussing the data furnished by KOSTER's observations.

Discussion of KOSTER's *method.*

This method was published as early as 1897. Strange to say, the notion of one centre of rotation has continued in the literature, in spite of KOSTER's results, which give evidence to the contrary.

The method starts from the assumption of one centre of rotation. Three positions of the eye are considered, a central position, in which the visual line Mo (fig. 3) runs straight forward, and to the left and to the right a



lateral-position, of which the visual lines *do* and *cn* are at the same angle a with the central position. This is realized in the experiment in the following way : on a graduated arc of which the centre is in M, a movable sector is mounted, so that with every movement it rotates round M. To this sector is applied a small aiming-telescope for fixation that can be shifted parallel to itself, as well as in the direction of the axis, which shiftings are measurable. Now it is first arranged so as to make the visual line fall along Mo. Subsequently the telescope is moved over an angle a, measured on the protractor, first to the one side and then to the other, every time determining how far it has to be shifted parallel to itself (Mc, respectively Md) to render fixation possible again. The direction of the visual lines is then indicated qc, respectively by rd. If now it might be assumed that the eye had rotated continually round the one point m, the place of point m could be derived from the data obtained, for a simple computation shows that

$$Mp = \frac{Mc + Md}{2 \sin a} \quad \text{and} \quad pm = \frac{Md - Mc}{2 \sin a \, tg \, \frac{1}{2} a}.\ ^{1)}$$

So, if there is one centre of rotation, its place can, indeed, be determined by help of this method. Additionally by also measuring Ms: the distance from M to the intersection of the anterior surface of the cornea with the visual line in the middle-position, the place of the centre of rotation in respect to that intersection can be indicated. If, however, different values are found for Mp and pm for different values of a, it follows that we cannot speak any longer of one centre of rotation. On the other hand, we are not sure that, if the same values are always found for Mp and mp, there is really one centre of rotation. For the only thing we have established, is the position of the lines along which three visual lines are running, of which the middle one makes equal angles a with the other two : this does not at all determine the piece of the line the eye cuts off in the three different positions. Suppose *all* visual lines should touch the circle with mp for radius, so that the same point m would always be found, it would be possible that, without one's noticing it, the distance from one and the same point, e.g. s, of the eye to this point m always varies. If such be the case, there is of course no question about one single *centre of rotation*. KOSTER found different points m ; from this it follows that there is not one centre of rotation. But besides this *the eye has never rotated round one of those points m*. This could be the case only if three positions were con-

[1]) If considering one direction of vision straight forward and two side-positions on *the same* side the one forming an angle α, the other an angle 2α with the first, we get:

$$Mp = \frac{2\,Mc\,(1 + \cos a) - Md}{2 \sin a} \quad \text{and} \quad pm = \frac{2\,Mc \cos a - Md}{2\,(1 - \cos a)}$$

sidered, of which the middle one would make a very (infinitely) small angle with the outer ones. Even then it would still depend on the magnitude of the shifting along the visual line how far we should approximate the situation of the centre of rotation with the situation of the point m.

When confining ourselves to determinations of a, Mc, and Md there is no reason to make these determinations for more than two values of a, for the values computed for the point m have no meaning in themselves. Indeed, KOSTER has recognized the error of this method, when he says that "it is not impossible that the eye moves either posteriorly or anteriorly, in the sense of the visual line" (p. 377). He, therefore, gives in the same table, in which he imparts the values for ps and pm (designated respectively X and x) a column with the values for the displacement V in the lateral position. He computes this V as follows ; he assumes that in the right lateral position the front point of the cornea is found, once e.g. in x (fig. 3). He knows this place, as he continually measures the distance cx. Now it is clear in connection with the foregoing, that it is just these measurings of cx, which after all enable us to analyse the movement to some degree (the number of observations and the accuracy are not very great). For the direction of the visual line and the situation of a fixed point of it, are known for a number of positions. The values of cx are not given as such, but KOSTER says he has calculated V as follows :

$$V = s_1x = nx - ns_1 \; ; \; nx = nc - xc = \frac{Mc}{tg\ a} - xc \text{ and } ns_1 = ps = X.$$

It might be fancied, that KOSTER made a mistake here, as it would be expected that not $ps = ns_1$, but $ps = qs_1$, considering that s_1 is to represent the anterior point of the cornea, without displacement. However from what follows, I must conclude that ns_1 was KOSTER's real intention but that, without expressing it distinctly, he relinquishes the method just described, and considers the point m no longer as a centre of rotation. The movement of the eye is then described as follows : (p. 382) in order to move from the central position, say, into the right lateral position, the eye first shifts over a distance np anteriorly, then rotates round n, until the visual line occupies the position ns_1 (and then $ps = ns_1$) and then again shifts in the direction of the visual line over a distance s_1n. The point p is then considered as a fixed point of rotation *in the orbit*, and we might conceive all eye-movements in the way just described, as consisting of two displacements combined with a rotation round the fixed point. It may be contended first of all that this view does not tally with KOSTER's own observations ; for then all visual lines must go through the centre of rotation and in KOSTER's observations this was certainly not the case. True, it might be assumed that the eye first rotates (round p), then shifts twice, but this does not square with the description just given. KOSTER's exposition, which is not quite clear, gives us an impression that a compromise is aimed at between his own results and those of VOLKMANN and of WOINOW. It would follow from the experiments of these researchers which KOSTER refers to, that the visual lines intersect in one point. Such a view, which agrees with that of HERING, who assumes a centre of rotation that is supposed to be fixed in the orbit and not fixed with respect to the eye, cannot be recommended, as it is *always* possible to bring about a change of position in a plane surface by means of two shifts and one rotation, in which process it does not matter where the centre of rotation is situated.

Meanwhile there are, as I observed, sufficient data at hand to compute the position of the eye for the different angles of rotation. I proceeded as follows:

The intersection of the axes of the co-ordinates is laid in s (fig. 3); the one axis, which I do not call the X-axis, but the ξ-axis to prevent confusion with KOSTER's nomenclature, falls in the direction so (this direction positive), the other axis I call η-axis (positive to the right). Now let us consider the point x, of which the co-ordinates sz and zx are designated b and c in accordance with the foregoing (p. 359). We then have

$$b = sz = sn - zn = (sp - np) - zn = X - x\,tg\,\tfrac{1}{2}\,a - (X + V)\cos a,$$

from which we deduce

$$b = X(1 - \cos a) - x\,tg\,\tfrac{1}{2}\,a - V\cos a.$$

Furthermore $c = zx = (V + X)\sin a$.

So we have the general formulae:

$$b_n = X_n(1 - \cos a_n) - x_n\,tg\,\tfrac{1}{2}\,a_n - V_n \cos a_n \quad . \quad . \quad . \quad . \quad (5)$$

$$c_n = (V_n + X_n)\sin a_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

with the aid of which the values of b and c can be computed for every case from KOSTER's data. Now if we assume that in the displacement from every position observed, to the following, rotation takes place round one point every time, so that the expressions for b_n and c_n can be substituted in the equations (1) and (2), we find for the co-ordinates ξ and η of these centres of rotation after reduction:

$$\xi = \frac{X_{n+1}\sin\frac{1}{2}a_{n+1}\cos\frac{1}{2}a_n - X_n\sin\frac{1}{2}a_n\cos\frac{1}{2}a_{n+1}}{\sin\frac{1}{2}(a_{n+1} - a_n)} -$$

$$-\tfrac{1}{2}(x_{n+1}\,tg\,\tfrac{1}{2}a_{n+1} + x_n\,tg\,\tfrac{1}{2}a_n) + \frac{(V_{n+1} - V_n)\sin\frac{1}{2}(a_{n+1} + a_n)}{2\sin\frac{1}{2}(a_{n+1} - a_n)}$$

$$\eta = -\frac{(X_{n+1} - X_n)\sin\frac{1}{2}a_{n+1}\sin\frac{1}{2}a_n}{\sin\frac{1}{2}(a_{n+1} - a_n)} + \tfrac{1}{2}(x_{n+1}\,tg\,\tfrac{1}{2}a_{n+1} - x_n\,tg\,\tfrac{1}{2}a_n) \times$$

$$\times \cot\tfrac{1}{2}(a_{n+1} - a_n) + \frac{(V_{n+1} - V_n)\cos\frac{1}{2}(a_{n+1} + a_n)}{2\sin\frac{1}{2}(a_{n+1} - a_n)}.$$

In this way the data of the following table have been obtained (to the right the angles are positive, to the left negative).

Left eye: horizontal movement; opened normally; lengths in mm.

α	$\delta_{(\alpha_{n+1}-\alpha_n)}$	X	x	V	b	c	ξ	η
− 45°		13.2	1.5	0.1	4.4	− 9.4		
	15°						13.5	− 0.4
− 30°		13.8	1.1	− 0.3	2.4	− 6.8		
	10°						15.4	− 3.2
− 20°		14.6	− 2.3	− 1.6	2.0	− 4.4		
	10°						16.7	− 5.4
− 10°		14.4	− 13.7	− 3.4	2.4	− 1.9		
	10°						12.1	12.6
0°				0	0	0		
	10°						16.5	10.9
10°		14.4	− 13.7	3.1	− 1.6	3.0		
	10°						14.4	0.1
20°		14.6	− 2.3	2.3	− 0.9	5.8		
	10°						8.7	− 3.4
30°		13.8	1.1	0.8	0.9	7.3		
	15°						10.9	0.8
45°		13.2	1.5	0.5	2.9	9.7		

The difference between the points now found as centres of rotation and the points *m* is distinct. It is striking that, whereas the centres of rotation in a rotation from the zero-position to +10° and −10°, are both situated more than 10 mm to the right, the corresponding point *m* lies nearly 14 mm to the left. It would be wrong to ascribe these differences to the inaccuracy of the observations: the fact that the points have another meaning is responsible for them.

KOSTER's results for the wide-opened eye (in which, as pointed out already by J. J. MÜLLER, the bulbus protrudes) have been obtained in the same way. The results are corresponding; they have been tabulated in the second table. According to KOSTER the point s now lies 0.8 mm to the front ; I have assumed that the data of his table relate to this new point s, so that my results are also numerical values concerning the new central-position of the eye.

Finally I have made a graph of most data of the first table. (The line $y_4\ y_3 \ldots . x_3\ x_4$ illustrates the path of the cornea, for the positions from $-45°$ to $+45°$; $Q_4, Q_3 \ldots P_3, P_4$ represent the corresponding centres of rotation, while m_1, m_2, m_3 and m_4 are the centres of KOSTER for rotations over angles of respectively $\mp 10°, \mp 20°, \mp 30°$, en $\mp 45°$.)

Whether the flexion in the curve at y_1 points to a discontinuity in the movement, or whether it is caused by the inaccuracy or the small number of observations, I will not venture to decide. It is obvious what difficulties one would meet, when working out these data in the equations (3) and (4) ; I, therefore, abstained from that operation.

Left eye: horizontal movement; wide-opened; lengths in mm.

α	$\delta_{(\alpha_{n+1}-\alpha_n)}$	X	x	V	b	c	ξ	η
− 45°		13.9	2	0.3	4.7	−10.0		
	15°						15.8	0.4
− 30°		13.7	1.5	− 0.1	2.3	− 6.8		
	10°						15.5	− 2.5
− 20°		14.3	− 1.7	− 1.3	1.8	− 4.4		
	10°						12.6	11.7
− 10°		13.8	− 16.6	− 0.4	− 0.8	− 2.3		
	10°						12.9	− 6.0
0°				0	0	0		
	10°						16.3	12.3
10°		13.8	− 16.6	3.6	− 1.9	3.0		
	10°						12.3	− 6.2
20°		14.3	− 1.7	1.3	− 0.1	5.3		
	10°						10.3	− 0.4
30°		13.7	1.5	0.4	1.1	7.1		
	15°						11.8	− 1
45°		13.9	2	− 0.4	3.5	9.5		

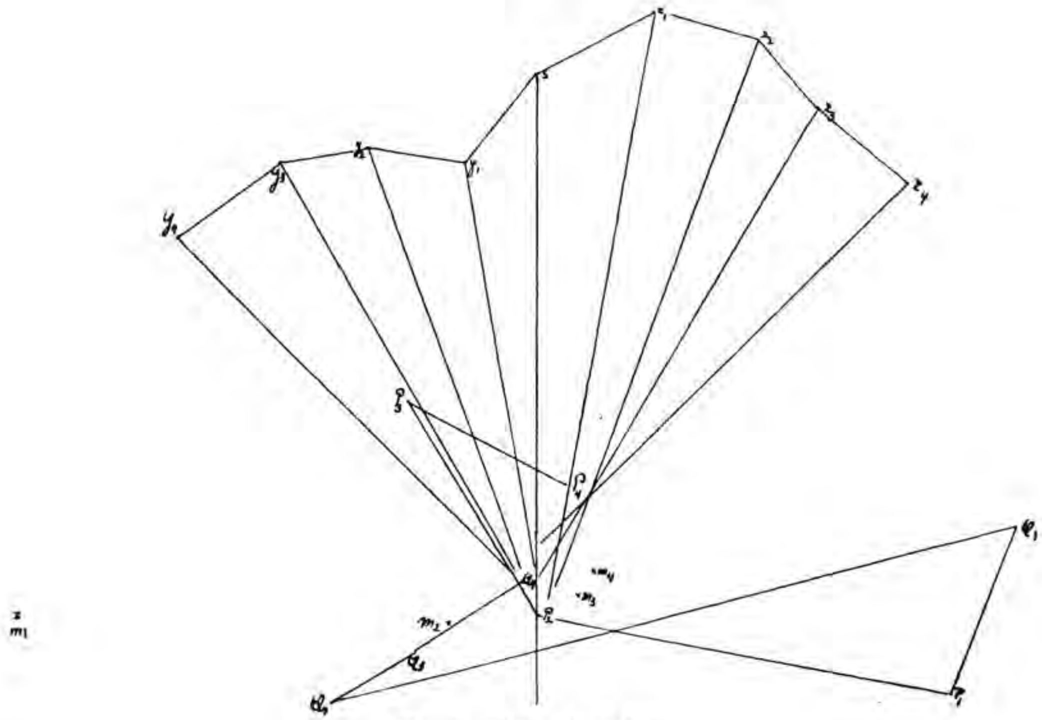


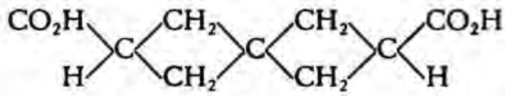
Fig. 4. (Scale about 5/1).

Chemistry. — *Optical resolution of a spirocyclic compound of the allene type.* By H. J. BACKER and H. B. J. SCHURINK. (Communicated by Prof. F. M. JAEGER.)

(Communicated at the meeting of April 28, 1928).

Recent publications [1]) on spirocyclic compounds of the asymmetric allene type xyC : C : Cxy induce us to publish some results of our investigations in this field.

In order to test the stereochemical theory, that such spiranes should exist in enantiomorphs, we have studied *spiroheptanedicarboxylic acid :*



FECHT [2]) has obtained this acid in small yield, starting from tetrabromopentaerythritol and malonic ester.

After having improved the preparation until the yield was about 80 %, we attempted the resolution by means of the dibrucine salt, which crystallises with six molecules of water in short prisms and decomposes at about 135°.

This brucine salt, recrystallised carefully from water and decomposed by ammonia, gives an optically active ammonium salt. The rotation is feeble and reaches its maximum value after about 5 crystallisations.

The rotatory power is given in the following table for different wave lengths.

Rotatory dispersion of ammonium spiroheptanedicarboxylate.

$\lambda(\mu\mu)$	656.3 (C)	589.5 (D)	546.3 (Hg)	486.1 (F)
$[\alpha]$	$+0°.11$	$0°.13$	$0°.15$	$0°.19$
[M]	$+1°.9$	$2°.3$	$2°.6$	$3°.4$

The free acid being only little soluble in water cannot be studied accurately in aqueous solution. Its ethereal solution shows a positive rotation : $[M]_D = +1°.9$.

We believe that this is the first observation of an optically active spirane of this type, whose resolution is completely reproducible.

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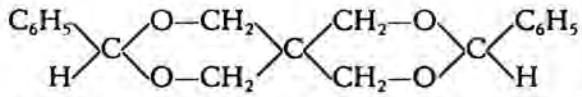
[1]) PFEIFFER and BACKES, Ber. **61**, 434 (1928); BÖESEKEN and FELIX, Ber. **61**, 787 (1928).

[2]) Ber. **40**, 3888 (1907).

Further we studied a compound of analogous structure, the *dibenzalpentaerythritol* :



READ [1]) has sought already in 1912 for optical activity in the case of this and related compounds.

BÖESEKEN and FELIX [2]) publish to have observed for this compound on one occasion an optical activity, which could not be reproduced.

Our results, already obtained in 1927, may be shortly summarized.

The dibenzal compound, prepared under different conditions, had always the same melting point (162°) and the same chemical properties. Thus the occurrence of a cis-trans isomerism, which might be possible in the case of a pyramidal distribution of the valences of the central carbon atom, is improbable.

The substance was obtained in large hexagonal crystals with edges up to half an inch in length.

The Röntgen analysis, made in the inorganic chemical laboratory by Mr. VAN MELLE, showed that the crystal has a three-fold screw axis, perpendicular to the basal plane. Thus dextro- and laevorotatory crystals should be possible.

Indeed both kinds of crystals were obtained. The rotation could not be measured accurately with our polarisation apparatus ; the value is about $\alpha_D = \pm 2°/mm$.

The Röntgen analysis of dibenzalpentaerythritol, the results of which will be published elsewhere, has proved, that the molecules in the crystal possess three mutually perpendicular two-fold axes, according to the symmetry of an orthorhombic bisphenoïde. Thus the central carbon atom has a tetrahedral (not a pyramidal) distribution of its valences and the phenyl groups must be placed in the long axis of the molecule, for instance :

The optical activity of the crystals is caused by the asymmetric distribution of the molecules.

None of the crystals, when dissolved in alcohol, ethyl acetate or chloroform, has shown a trace of optical activity.

Groningen, April 1928.

Organic chemical laboratory of the State University.

[1]) J. chem. Soc. **101**, 2090 (1912).

[2]) Loc. cit.

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Mathematics. — *Intuitionistische Betrachtungen über den Formalismus* [1]).
By Prof. L. E. J. BROUWER.

(Communicated at the meeting of December 17, 1927).

LITERATUR.

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Intuitionistische Zerlegung mathematischer Grundbegriffe, Jahresber. d. D. M. V. 33 (1925) (hier zitiert als I 8).

Ueber Definitionsbereiche von Funktionen, Math. Annalen 97 (1927) (hier zitiert als I 9).

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[1]) In ungefähr gleichlautender Form am 16. 2. 1928 der Berliner Akademie der Wissenschaften vorgelegt.

§ 1.

Die Richtigkeitsdifferenzen zwischen der formalistischen Neubegründung und dem intuitionistischen Neubau der Mathematik werden beseitigt sein, und die Wahl zwischen beiden Beschäftigungen sich auf eine Geschmacksangelegenheit reduzieren, sobald die folgenden in erster Linie auf den Formalismus bezüglichen, aber in der intuitionistischen Literatur zuerst formulierten Einsichten allgemein durchgedrungen sein werden. Dieses Durchdringen ist deshalb nur eine Zeitfrage, weil es sich um reine Besinnungsergebnisse handelt, die kein diskutables Element enthalten und zu denen jederman der sie einmal verstanden hat, sich bekennen muss. Von den vier Einsichten ist bisher für zwei dieses Verständnis und dieses Bekenntnis in der formalistischen Literatur erreicht. Das Eintreten der gleichen Sachlage für die beiden übrigen wird das Ende des Grundlagenstreites in der Mathematik bedeuten.

Erste Einsicht. *Die Einteilung der formalistischen Bemühungen in einen Aufbau des „mathematischen Formelbestandes" (formalistischen Bildes der Mathematik) und eine intuitive (inhaltliche) Theorie der Gesetze dieses Aufbaues, sowie die Erkenntnis, dass für die letztere Theorie die intuitionistische Mathematik der Menge der natürlichen Zahlen unentbehrlich ist.*

Zweite Einsicht. *Die Verwerfung der gedankenlosen Anwendung des logischen Satzes vom ausgeschlossenen Dritten, sowie die Erkenntnis, erstens dass die Erforschung des Berechtigungsgrundes und des Gültigkeitsbereichs des genannten Satzes einen wesentlichen Gegenstand der mathematischen Grundlagenforschung ausmacht, zweitens dass dieser Gültigkeitsbereich in der intuitiven (inhaltlichen) Mathematik nur die endlichen Systeme umfasst.*

Dritte Einsicht. *Die Identifizierung des Satzes vom ausgeschlossenen Dritten mit dem Prinzip von der Lösbarkeit jedes mathematischen Problems.*

Vierte Einsicht. *Die Erkenntnis dass die (inhaltliche) Rechtfertigung der formalistischen Mathematik durch den Beweis ihrer Widerspruchslosigkeit einen circulus vitiosus enthält, weil diese Rechtfertigung auf der (inhaltlichen) Richtigkeit der Aussage, dass aus der Widerspruchslosigkeit eines Satzes die Richtigkeit dieses Satzes folge, d. h. auf der (inhaltlichen) Richtigkeit des Satzes vom ausgeschlossenen Dritten beruht.*

1. Die erste Einsicht fehlt noch in F 2 (vgl. insbesondere den mit derselben in Widerspruch stehenden Absatz V auf S. 184—185). Nachdem sie durch POINCARÉ stark vorbereitet worden war, tritt sie zum ersten Mal in der Literatur auf in I 1, wo S. 173—174 die genannten Teile der formalistischen Mathematik als *mathematische Sprache* und *Mathematik 2. Ordnung* unterschieden und der intuitive Charakter des letzteren Teiles betont wird [2]). Mit der Bezeichnung der Mathematik

[2]) Eine mündliche Erörterung der ersten Einsicht Herrn HILBERT gegenüber hat im Herbst 1909 in mehreren Unterhaltungen stattgefunden.

2. Ordnung als *Metamathematik* ist sie in F 4 (vgl. insbesondere S. 165 u. 174) in der formalistischen Literatur durchgebrochen. Der Anspruch der formalistischen Schule, mit dieser dem Intuitionismus entnommenen Einsicht den Intuitionismus ad absurdum zu führen (vgl. Math. Zeitschr. 26, S. 3), ist wohl nicht ernst zu nehmen.

2. Die gedankenlose Anwendung des logischen Satzes vom ausgeschlossenen Dritten findet sich noch in F 2 und F 3 (vgl. z. B. F 3, S. 413, Z. 11—4 v. u., und insbesondere F 2, S. 182, Z. 16—19 v. o., S. 182, Z. 2 v. u. —S. 183, Z. 2 v. o., S. 184, Z. 21—13 v. u., wo jedesmal der Satz vom ausgeschlossenen Dritten als mit dem Satz vom Widerspruch im wesentlichen gleichbedeutend angesehen wird). Zum ersten Male findet sich die zweite Einsicht in der Literatur in I 2, und sodann mehr oder weniger ausführlich in jeder der Veröffentlichungen I 3—8. Abgesehen von der mit ihr aufs engste verbundenen Erkenntnis der intuitionistischen Widerspruchslosigkeit des Satzes vom ausgeschlossenen Dritten, bricht sie in der formalistischen Literatur durch in F 5 [3]), wo einerseits die beschränkte inhaltliche Gültigkeit des Satzes vom ausgeschlossenen Dritten anerkannt (vgl. insbesondere S. 155—156), andererseits die widerspruchslose Kombination einer logischen Formulierung des Satzes vom ausgeschlossenen Dritten mit anderen Axiomen im Rahmen der formalistischen Mathematik als Aufgabe gestellt wird. Besonders eloquent wird dann auf die beschränkte inhaltliche Gültigkeit des Satzes vom ausgeschlossenen Dritten hingewiesen in F 6, S. 173—174, wo aber die Erweiterung seiner Anzweiflung auf die übrigen Aristotelischen Gesetze über das Ziel hinausschiesst.

3. Während der Zeit der gedankenlosen Anwendung des Satzes vom ausgeschlossenen Dritten in der formalistischen Literatur wird das Prinzip von der Lösbarkeit jedes mathematischen Problems zunächst in F 1, S. 52 als Axiom bzw. Ueberzeugung, sodann in F 3, S. 412—413 in zwei verschiedenen Formen (in welchen statt von „Lösbarkeit" der Reihe nach von „prinzipieller Lösbarkeit" und von „Entscheidbarkeit durch eine endliche Anzahl von Operationen" gesprochen wird) als Gegenstand noch zu erledigender Probleme hingestellt. Aber auch nach der Erörterung der dritten Einsicht in I 2, S. 156, I 4, S. *80*, I 6, S. 203—204, und nach dem Durchbruch der zweiten Einsicht in der formalistischen Literatur wird in F 6, S. 180, wo das Problem der Widerspruchsfreiheit des Axioms von der Lösbarkeit eines beliebigen mathematischen Problems als Beispiel einer „in den mathematischen Denkbereich fallenden Frage grundsätzlicher Art, an die man sich früher nicht heranmachen konnte" hingestellt wird, diese Frage als unabhängig von der Sicherung der (die Widerspruchsfreiheit des Satzes vom ausgeschlossenen Dritten mit umfassenden) Grundlagen der mathematischen Wissenschaft noch offenstehend vorgeführt.

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[3]) Nachdem schon in F 4, S. 160 Aufmerksamkeit auf den Satz vom ausgeschlossenen Dritten bekundet wird.

4. Die vierte Einsicht wird zum Ausdruck gebracht in I 9, S. 64. In der formalistischen Literatur findet sich von ihr bisher keine Spur, wohl aber manche ihr widersprechende Aeusserung, z.B. in F 1, S. 55—56 und vor allem in F 6, wo S. 162—163 noch ausgerufen wird: „Nein, wenn über den Nachweis der Widerspruchsfreiheit hinaus noch die Frage der Berechtigung zu einer Massnahme einen Sinn haben soll, so ist **es doch** nur die, ob die Massnahme von einem entsprechenden Erfolge begleitet wird" [4]).

Nach dem Vorstehenden hat der Formalismus vom Intuitionismus nur Wohltaten empfangen und weitere Wohltaten zu erwarten. Dementsprechend sollte die formalistische Schule dem Intuitionismus einige Anerkennung zollen, statt gegen denselben in höhnischem Ton zu polemisieren und dabei nicht einmal die richtige Erwähnung der Autorschaft einzuhalten. Ueberdies sollte die formalistische Schule bedenken, dass im Rahmen des Formalismus von der eigentlichen Mathematik bisher noch immer *nichts* gesichert ist (weil ja der metamathematische Widerspruchsfreiheitsbeweis des Axiomensystems nach wie vor aussteht), wogegen der Intuitionismus auf der Grundlage seiner konstruktiven Mengendefinition und seiner Haupteigenschaft der finiten Mengen [5]) schon einige Lehrgebäude der eigentlichen Mathematik in unerschütterlicher Sicherheit neu errichtet hat. Wenn also die formalistische Schule nach ihrer Aeusserung in F 6, S. 180 beim Intuitionismus Bescheidenheit bemerkt hat, so sollte sie darin Anlass finden, in bezug auf diese Tugend dem Intuitionismus nicht nachzustehen.

§ 2.

In I 7, S. 3 wurde bemerkt, dass bei den Bestrebungen zur Durchführung des Widerspruchsfreiheitsbeweises der formalistischen Metamathematik die intuitionistische Widerspruchslosigkeit des Satzes vom ausgeschlossenen Dritten als ermutigender Umstand gelten kann.

Wenn die kombinierte Aussage des Satzes vom ausgeschlossenen Dritten für endlichviele mathematische Eigenschaften bzw. für eine beliebige Spezies von mathematischen Eigenschaften [6]) als *mehrfacher Satz vom ausgeschlossenen Dritten erster* bzw. *zweiter Art* bezeichnet wird, dann ist einerseits klar, dass die Widerspruchslosigkeit des einfachen Satzes vom ausgeschlossenen Dritten keineswegs unmittelbar diejenige des mehrfachen Satzes vom ausgeschlossenen Dritten nach sich zieht, andererseits,

[4]) Uebrigens liegt bei derartigen Aeusserungen genau genommen doch wieder eine gedankenlose Anwendung des Satzes vom ausgeschlossenen Dritten, mithin eine Verdunkelung der zweiten Einsicht vor.

[5]) Vgl. F 9, S. 66 (Theorem 2).

[6]) D.h. die Existenzaussage eines Simultangesetzes, das für sie alle die Richtigkeit oder Absurdität entscheidet.

dass die im vorigen Absatze in Erinnerung gebrachte Bemerkung erst dann ihre volle Tragweite erlangt, wenn nicht nur für den einfachen Satz vom ausgeschlossenen Dritten, sondern auch für den mehrfachen Satz vom ausgeschlossenen Dritten erster Art, die Widerspruchslosigkeit feststeht. In der Tat wird im Folgenden der Beweis der letzteren Widerspruchslosigkeit erbracht. Des weiteren wird sich ergeben, dass der (für die formalistischen Hoffnungen belanglose) mehrfache Satz vom ausgeschlossenen Dritten zweiter Art keine Widerspruchslosigkeit mehr besitzt.

Die Widerspruchsfreiheit des mehrfachen Satzes vom ausgeschlossenen Dritten erster Art beweisen wir mittels vollständiger Induktion. Es sei n eine natürliche Zahl, es sei die Widerspruchsfreiheit der kombinierten Aussage des Satzes vom ausgeschlossenen Dritten für n beliebige mathematische Eigenschaften bewiesen, es seien $n + 1$ mathematische Eigenschaften $a_1, a_2, \ldots, a_{n+1}$ vorgegeben, und es sei einen Augenblick angenommen, dass die kombinierte Aussage des Satzes vom ausgeschlossenen Dritten für $a_1, a_2, \ldots, a_{n+1}$ zu einem Widerspruch führte. Das würde heissen, dass jede der 2^{n+1} Kombinationen von $a_1, a_2, \ldots, a_{n+1}$ je mit dem Richtigkeits- oder mit dem Absurditätsprädikat versehen, zu einem Widerspruch führte.

Wir behaupten, dass unter dieser Annahme der einfache Satz vom ausgeschlossenen Dritten für a_{n+1} notwendig absurd sein muss. Denn wäre a_{n+1} richtig bzw. absurd, so wäre auf Grund der Widerspruchsfreiheit der kombinierten Aussage des Satzes vom ausgeschlossenen Dritten für n beliebige mathematische Eigenschaften die Kombination der Richtigkeit bzw. Absurdität von a_{n+1} mit der kombinierten Aussage des Satzes vom ausgeschlossenen Dritten für $a_1, a_2, \ldots, a_n$ widerspruchsfrei, entgegen der Annahme des vorigen Absatzes. Die Annahme des vorigen Absatzes hat sich also als unstatthaft erwiesen, und die kombinierte Aussage des Satzes vom ausgeschlossenen Dritten für $n + 1$ beliebige mathematische Eigenschaften hat sich als widerspruchsfrei herausgestellt.

Gegenbeispiele der Widerspruchsfreiheit des mehrfachen Satzes vom ausgeschlossenen Dritten zweiter Art liefert der aus der Haupteigenschaft der finiten Mengen folgende Satz, dass bei Zerlegung des Einheitskontinuums in zwei Teilspezies eine dieser Teilspezies mit dem Einheitskontinuum identisch und die andere leer ist. Aus diesem Satze folgt nämlich, dass die kombinierte Aussage des Satzes vom ausgeschlossenen Dritten für eine beliebige Eigenschaft in bezug auf *alle* Punktkerne des Einheitskontinuums dann und nur dann widerspruchsfrei ist, wenn die Eigenschaft entweder für alle Punktkerne des Einheitskontinuums richtig oder für alle Punktkerne des Einheitskontinuums absurd ist. Insbesondere sind die beiden folgenden Aussagen kontradiktorisch:

1. *Alle Punktkerne des Einheitskontinuums sind entweder rational oder negativ-irrational.*

2. *Für alle Punktkerne des Einheitskontinuums ist die Rationalitätsfrage entweder entscheidbar oder unentscheidbar.*

Formulieren wir in Analogie mit dem Obigen folgende drei Fassungen des Prinzips der Reziprozität der Komplementärspezies:

1. *Jedes Element der Komplementärspezies der Komplementärspezies von R gehört zu R* (einfaches Prinzip der Reziprozität der Komplementärspezies).

2. *Jede endliche Spezies von Elementen der Komplementärspezies der Komplementärspezies von R gehört zu R* (mehrfaches Prinzip der Reziprozität der Komplementärspezies erster Art).

3. *Jede Spezies von Elementen der Komplementärspezies der Komplementärspezies von R gehört zu R* (mehrfaches Prinzip der Reziprozität der Komplementärspezies zweiter Art).

Alsdann besteht auch hier die Widerspruchsfreiheit nur für 1. und 2., und nicht für 3. Ein Gegenbeispiel liefert die Spezies *G* derjenigen Punktkerne des Einheitskontinuums *C*, für welche die Rationalitätsfrage entscheidbar ist. Denn die Komplementärspezies der Komplementärspezies in *C* von *G* ist mit *C* identisch (weil nämlich die Komplementärspezies in *C* von *G* leer ist), während wir oben gesehen haben, dass *G* unmöglich mit *C* identisch sein kann.

Mathematics. — *Beweis dass jede Menge in einer individualisierten Menge enthalten ist* [1]). By Prof. L. E. J. BROUWER.

(Communicated at the meeting of December 17, 1927).

Sei M eine beliebig vorgegebene Menge. Wir zählen zunächst die Menge der ersten Wahlen von M durch eine Fundamentalreihe ab, zählen sodann die Menge der zweiten Wahlen von M als Produkt zweier Fundamentalreihen (mittels des Diagonalverfahrens) wiederum durch eine Fundamentalreihe ab, zählen darauf die Menge der dritten Wahlen von M als Produkt der letzteren Fundamentalreihe mit einer neuen Fundamentalreihe (mittels des Diagonalverfahrens) gleichfalls durch eine Fundamentalreihe ab, usw. Alsdann bekommt jede Wahl eine neue Nummer, und wenn wir alle hierbei nach bestimmter erster, zweiter, ... bis einschliesslich n-ter Wahl für die $(n+1)$-te Wahl nicht vorkommenden Nummern als gehemmte Nummern betrachten, bekommen wir eine mit M identische „monotone" Menge N, d.h. eine mit M identische Menge N, in welcher *nach* einer ungehemmten n-ten Nummer a nur Nummern $\geqslant a$ als $(n+1)$-te Nummer ungehemmt sein können, und in welcher für beliebiges n keine zwei ungehemmte n-te Wahlen mit gleichen Nummern vorkommen.

Nun nehmen wir *in der Menge* N ein Fundamentalreihe von Aenderungen a_1, a_2, ... vor, wobei für jedes beliebige n die Erzeugnisse der 1-sten, 2-ten, ... bis einschliesslich $(n—1)$-ten Wahl nicht von a_n beeinflusst werden. Und zwar ändern wir für a_1 zunächst in der Menge N die Reihe der ersten Wahlen derweise, dass jede gehemmte Wahl gehemmt bleibt, während eine ungehemmte Wahl dann und nur dann ungehemmt bleibt, wenn ihr in der Fundamentalreihe keine andere Wahl vorangeht, welche das gleiche Zeichen erzeugt; sodann erklären wir *nach* einer beliebigen ungehemmt *gebliebenen* ersten Wahl σ diejenigen und nur diejenigen zweiten Wahlen ungehemmt, welche zuvor nach einer das gleiche Zeichen wie σ erzeugenden ersten Wahl ungehemmt waren, und die ganze Mengenfortsetzung dieser ungehemmten zweiten Wahlen wird bei diesem „Transport" ungeändert gelassen. Hierdurch bekommen wir eine mit N identische monotone Menge N_1, in welcher gleiche Elemente immer nur aus gleichen ersten Wahlen hervorgehen.

Die Aenderung a_2 wirkt in solcher Weise auf N_1, dass für eine beliebige ungehemmte erste Wahl σ von N_1 zunächst für die auf σ folgenden zweiten

[1]) Für die Definition der Menge, der individualisierten Menge und der finiten Menge vgl. Math. Annalen **93**, S. 244—245. Im Folgenden werden wir sowohl eine beliebige Zeichenreihe wie *nichts*, kurz als „Zeichen" bezeichnen.

Wahlen die gleiche Aenderung ausgeführt wird, welche oben bei a_1 mit der Reihe der ersten Wahlen vorgenommen wurde, sodann *nach* einer beliebigen auf σ folgenden, ungehemmt *gebliebenen* zweiten Wahl τ diejenigen und nur diejenigen dritten Wahlen ungehemmt erklärt werden, welche in N_1 nach einer das gleiche Zeichen wie τ erzeugenden, auf σ folgenden zweiten Wahl ungehemmt waren, und die ganze Mengenfortsetzung dieser ungehemmten dritten Wahlen bei diesem „Transport" ungeändert gelassen wird. Hierdurch bekommen wir eine mit N und N_1 identische monotone Menge N_2, in welcher gleiche Elemente immer nur aus gleichen ersten und zweiten Wahlen hervorgehen.

Die Aenderung a_3 wirkt in solcher Weise auf N_2, dass für beliebiges σ und τ mit den auf σ und τ folgenden dritten Wahlen und ihren Mengenfortsetzungen die gleiche Aenderung ausgeführt wird, welche oben zunächst bei a_1 mit der Reihe der ersten Wahlen und ihren Mengenfortsetzungen und sodann bei a_2 mit den auf σ folgenden zweiten Wahlen und deren Mengenfortsetzungen vorgenommen wurde.

In dieser Weise bestimmen wir der Reihe nach $N_1, N_2, N_3, \ldots$, wobei jedesmal N_ν aus $N_{\nu-1}$ hervorgeht mittels Hemmung eines Teiles der vorher ungehemmten Folgen von ν Wahlen und dementsprechender Umordnung der (alle ungehemmt bleibenden) ungehemmten Folgen von $\nu + 1$ Wahlen. Hierbei bemerken wir, dass einer ungehemmten Folge von $\nu + 1$ Wahlen mit Indizen $\leqslant m$ in $N_{\nu+1}$ eindeutig eine die gleichen Zeichen erzeugende und gleiche Indizes besitzende ungehemmte Folge von $\nu + 1$ Wahlen in N_ν und der letzteren der Reihe nach in $N_{\nu-1}, N_{\nu-2}, \ldots, N_1, N$ eindeutig je eine die gleichen Zeichen erzeugende ungehemmte Folge von $\nu + 1$ Wahlen mit Indizen $\leqslant m$ entspricht. Mithin brauchen wir, um von Indizes $\leqslant m$ besitzenden Folgen von $\nu + 1$ Wahlen in $N_{\nu+1}$ die Gehemmtheit bzw. die erzeugten Zeichen festzustellen, der Reihe nach in $N, N_1, N_2, \ldots, N_\nu$ ebenfalls nur Folgen mit Indizen $\leqslant m$ in Betracht zu ziehen, so dass die betreffende Feststellung einen endlichen d.h. ausführbaren Prozess darstellt.

Wenn wir nun die Menge P dadurch definieren, dass sie für jedes ν in der Wirkung der Folgen von $1, 2, 3, \ldots, \nu$ Wahlen mit N_ν übereinstimmt, dann ist die Menge P individualisiert and enthält N, also M.

Wenn wir sagen, dass zwei Mengenelemente *differieren*, wenn für passendes n die Erzeugnisse ihrer ersten n Wahlen verschieden sind, und dass zwei Mengen *übereinstimmen*, wenn keine von beiden ein von allen Elementen der anderen differierendes Element enthalten kann, so folgert man im Falle einer *finiten* Menge M mittels der Haupteigenschaft der finiten Mengen [2]) leicht, dass die zugehörige Menge P mit M übereinstimmt. *Mithin ist jede finite Menge in einer mit ihr übereinstimmenden individualisierten Menge enthalten.*

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[2]) Vgl. Math. Annalen **97**, S. 66 (Theorem 2).

Mathematics. — *A Representation of a quadrifold set of Twisted Cubics on the Points of a Linear Four-dimensional Space.* By J. W. A. VAN KOL. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of February 25, 1928).

§ 1. The twisted cubics k^3 that pass through two given points H_1 and H_2 and cut two given lines a_1 and a_2 twice, may be represented on the points of a linear four-dimensional space R_4 in the following way. In R_4 we choose two quadratic spaces $\Omega^2{}_1$ and $\Omega^2{}_2$ that have a double line l_1 resp. l_2. We suppose a projective correspondence to be established between the points of a_1 and the planes in $\Omega^2{}_1$ and another one between the points of a_2 and the planes in $\Omega^2{}_2$. Let a curve k^3 cut a_1 in A_1 and A'_1 and a_2 in A_2 and A'_2 and let R_1, R'_1, R_2 and R'_2 be the spaces that touch $\Omega^2{}_1$ resp. $\Omega^2{}_2$ along the planes associated to the said points. To k^3 we shall associate as image point the point where the plane of intersection of R_1 and R'_1 and that of R_2 and R'_2 cut each other. Inversely an arbitrary point in R_4 is the image of one curve k^3.

§ 2. Through an arbitrary point of l_1 resp. l_2 there pass two tangent spaces of $\Omega^2{}_1$ resp. $\Omega^2{}_2$. In this way in $\Omega^2{}_1$ and $\Omega^2{}_2$ there are defined quadratic involutions of planes to which quadratic involutions of points I_1 and I_2 on a_1 resp. a_2, are associated. Each of the ∞^3 curves k^3 that cut a_1 resp. a_2 in a pair of points of I_1 resp. I_2, has its image point on l_2 resp. l_1.

l_1 and l_2 are cardinal lines; an arbitrary point P of l_1 e.g. is the image of each of the ∞^2 curves k^3 that pass through the points of a_2 which are associated to the planes where $\Omega^2{}_2$ is touched by its spaces of contact through P.

The transversal t_1 resp. t_2 of a_1 and a_2 through H_1 resp. H_2 is completed by the conics through H_2 resp. H_1 that cut a_1, a_2 and t_1 resp. t_2, to ∞^3 curves k^3 that are represented in the points of the plane of intersection σ_1 resp. σ_2 of the spaces which touch $\Omega^2{}_1$ and $\Omega^2{}_2$ in the planes associated to the points of intersection of a_1 and a_2 with t_1 resp. t_2.

There are two singular planes σ_1 and σ_2 both of which cut l_1 and l_2; an arbitrary point P of σ_1 is the image of the ∞^1 curves k^3 formed by t_1 and the conics that pass through H_2, cut t_1 and cut a_1 and a_2 in the points corresponding to the planes where $\Omega^2{}_1$ an $\Omega^2{}_2$ are touched by its spaces of contact through P which are different from the spaces of contact $l_1\,\sigma_1$ and $l_2\,\sigma_1$.

$\sigma_1 \sigma_2$ is a cardinal point that represents the ∞^2 curves k^3 formed by t_1, t_2 and the transversals of a_1 and a_2.

§ 3. Our set contains ∞^2 curves k^3 that are singular for the representation, viz. the curves k^3 that cut a_1 in a pair of points of I_1 and a_2 in a pair of points of I_2. Each of these curves k^3 has ∞^1 image points, viz. all the points of a transversal of l_1 and l_2.

§ 4. $\Omega^2{}_1$ and $\Omega^2{}_2$ are the loci of the image points of the curves k^3 that touch a_1 resp. a_2.

The surface of intersection O^4 of $\Omega^2{}_1$ and $\Omega^2{}_2$ is the locus of the image points of the curves k^3 that touch a_1 as well as a_2.

§ 5. Let us investigate the representation of the system Σ_1 of the curves k^3 that have a given chord b. The curves of Σ_1 cut a_1 as well as a_2 in pairs of points of a quadratic involution. To these quadratic point involutions on a_1 and a_2 there correspond quadratic plane involutions in $\Omega^2{}_1$ resp. $\Omega^2{}_2$. These involutions have the property that two spaces which touch $\Omega^2{}_1$ resp. $\Omega^2{}_2$ in planes that correspond to each other through this involution, have a plane of intersection lying in a fixed space through l_1 resp. l_2.

The plane of intersection α_b of these spaces is apparently the image plane of Σ_1.

Two planes α_{b_1} and α_{b_2} cut each other in one point. Hence:

There is one twisted cubic that passes through two given points and has four given chords.

O^4 and α_b cut each other in four points.

There are four twisted cubics that pass through two given points, have a given chord and touch two given lines.

§ 6. Let us call the image surface of the system Σ_2 of the curves k^3 that pass through a given point P, O_P. We determine the degree of O_P by examining the intersection of it and a plane α that touches $\Omega^2{}_1$ as well as $\Omega^2{}_2$. As there is one curve k^3 of Σ_2 that passes through a given point of a_1 as well as through a given point of a_2, α cuts O_P besides in the points $\alpha\, l_1$ and $\alpha\, l_2$ in one more point. l_1 and l_2 are single lines of O_P as through two given points of l_1 and l_2 there passes one curve k^3 of Σ_2. As, accordingly, α cuts O_P in all in three points, O_P is a cubic surface. We can show that O_P has *one conic that passes through the points $\sigma_1\, l_1$, $\sigma_1\, l_2$* and *$\sigma_1\, \sigma_2$ in common with σ_1 and one conic that passes through $\sigma_2\, l_1$, $\sigma_2\, l_2$ and $\sigma_1\, \sigma_2$ with σ_2.*

O_P and α_b have one point in common besides the points $\alpha_b\, l_1$ and $\alpha_b\, l_2$. Hence:

There is one twisted cubic that passes through three given points and has three given chords.

By applying the method indicated in § 8 we find that O_P and O_Q cut each other outside l_1 and l_2 in singular points only, whence:

There is no twisted cubic that passes through four given points and has two given chords.

The intersection of O^4 and O_P gives:

There are four twisted cubics that pass through three given points and touch two given lines.

§ 7. Let Ω_l be the image space of the system Σ_3 of the curves k^3 that cut a given line l. We determine the degree of Ω_l by means of the intersection with a line p that touches $\Omega_1{}^2$ as well as $\Omega_2{}^2$. p is the locus of the image points of the curves k^3 that pass through a definite point A_1 of a_1, through a definite point A_2 of a_2, and cut a_1 and a_2 outside A_1 resp. A_2 in points that correspond to each other through a certain projective correspondence between the points of a_1 and those of a_2. The number of points of intersection of p and Ω_l is, therefore, equal to twice the number of curves of Σ_3 that pass through two given points of a_1 as well as through a given point of a_2. This number is equal to two as the twisted cubics that pass through five given points and cut a given line, form a surface of the fifth degree that has triple points in the given points. Ω_l is, accordingly, of the fourth degree. We can show that l_1 and l_2 are double lines and that σ_1 and σ_2 are single planes of Ω_l.

§ 8. The intersection of Ω_l and Ω_m consists of σ_1, σ_2 and a surface O_{lm} of the degree 14, which is evidently the image surface of the system Σ_4 of the curves k^3 that cut two given lines l and m. l_1 and l_2 are quadruple lines of O_{lm} and σ_i has a curve of the sixth order that has triple points in the points $\sigma_i l_1$ and $\sigma_i l_2$ and a double point in the point $\sigma_1 \sigma_2$ in common with O_{lm}.

The intersection of O_{lm} successively with α_b and O^4 gives:

There are six twisted cubics that pass through two given points, have three given chords and cut two given lines.

There are 24 twisted cubics that pass through two given points, touch two given lines and cut two other given lines.

According to a theorem of PIERI [1]) the number of points of intersection of O_{lm} and O_P outside l_1 and l_2 is found by subtracting from the product of the degrees of O_{lm} and O_P the product of the multiplicities of l_1 on O_{lm} and O_P, the product of the multiplicities of l_2 on O_{lm} and O_P and the classes of the envelopes of the spaces through l_1 or l_2 that touch O_{lm} and O_P at the same point of one of these lines. The class of the envelope of the spaces through l_1 that touch O_{lm} and O_P at the same point of l_1, is equal to the number of spaces that pass

[1]) Rend. del Circolo Mat. di Palermo, t. V, 1891.

through an arbitrary point S and through l_1 and touch O_{lm} and O_P at the same point of l_1. It is easily proved that an arbitrary space through l_1 cuts O_P along l_1 and a conic that cuts l_1 once; accordingly this space touches O_P once, viz. in the point of intersection of l_1 and the said conic. An arbitrary space through l_1 cuts O_{lm} along the line l_1, which must be counted four times, and a curve of the tenth order that cuts l_1 in six points; consequently this space touches O_{lm} six times, viz. in the points of intersection of l_1 and the said curve of the tenth order. To an arbitrary point L_1 of l_1 we shall now associate the six points L'_1 of l_1 where O_{lm} is touched by the space that is defined by S and the plane touching O_P at L_1. Inversely through this correspondence there are associated to an arbitrary point L'_1 the four points L_1 where O_P is touched by the four spaces that are defined by S and the four planes touching O_{lm} at L'_1. The (4, 6)-correspondence between the points L_1 and L'_1 arising in this way, has 10 coincidences, hence the class in question is ten. Consequently the number of points where O_P and O_{lm} cut each other outside l_1 and l_2, is equal to $3 \times 14 - 2 \,.\, 1 \,.\, 4 - 2 \,.\, 10 = 14$. This number contains 4 points where the intersections of O_P and O_{lm} with σ_1 cut each other outside the points $l_1\sigma_1$, $l_2\sigma_1$ and $\sigma_1\sigma_2$, 4 points where the intersections of O_P and O_{lm} cut each other outside the points $l_1\sigma_2$, $l_2\sigma_2$ and $\sigma_1\sigma_2$ and the point $\sigma_1\sigma_2$ itself, which must be counted twice. There remain, accordingly, 4 points that are neither singular nor cardinal points. Thus we have found the following number, which, however, may be derived more simply in a direct way:

There are four twisted cubics that pass through three given points, have two given chords and cut two given lines.

If we apply the method indicated above to two surfaces O_{lm} and O_{no}, we find:

There are 36 twisted cubics that pass through two given points, have two given chords and cut four given lines.

§ 9. The intersection of O_{lm} and Ω_n consists of the lines l_1 and l_2, which must be counted eight times, two curves of the sixth order lying resp. in σ_1 and σ_2 and a curve k_{lmn} of the order 28 that is the image of the system Σ_5 of the curves k^3 that cut three given lines l, m and n. k_{lmn} cuts l_1 and l_2 in 14 points, as the number of points of intersection of k_{lmn} and l_1 as well as the number of points of intersection outside l_1 of k_{lmn} and a tangent space of $\Omega^2{}_1$ is equal to the number of curves of Σ_5 that pass through a given point of a_1. The number of points of intersection of k_{lmn} and σ_1 is equal to the number of conics that pass through H_2 and cut the six lines a_1, a_2, t_1, l, m and n (in different points). The conics that pass through H_2 and cut a_1, a_2, l, m and n form a surface of the degree 18 [1]) that is cut by t_1 outside the points of inter-

[1]) Cf. SCHUBERT, Kalkül der abzählenden Geometrie, p. 96, where the numbers of conics $P\nu^6 = 18$ and $P^2\nu^4 = 4$ are derived.

section of t_1 with a_1 and a_2, which are quadruple lines of the surface, in ten points. Accordingly σ_1 and σ_2 are cut by k_{lmn} in ten points.

The intersection of $\Omega^2{}_1$ and k_{lmn} gives:

There are 28 twisted cubics that pass through two given points, have a given chord, cut three given lines and touch another given line.

§ 10. We can further investigate the representations of several other systems, as the systems of the curves k^3 that touch one, two or three given planes, that touch a given plane and at the same time cut one or two given lines, that touch a given plane and at the same time pass through a given point and others.

The numbers that may be deduced in this way and those already found above are the following ones:

$P^4B^2 = 0$	$P^3B^2\nu^2 = 4$	$P^2B^3\nu^2 = 6$	$P^2B^2\nu^4 = 36$
$P^3B^3 = 1$	$P^3B^2\nu\varrho = 8$	$P^2B^3\nu\varrho = 12$	$P^2B^2\nu^3\varrho = 72$
$P^2B^4 = 1$	$P^3B^2\varrho^2 = 16$	$P^2B^3\varrho^2 = 24$	$P^2B^2\nu^2\varrho^2 = 144$
$P^3T^2 = 4$			$P^2B^2\nu\varrho^3 = 288$
$P^2BT^2 = 4$			$P^2B^2\varrho^4 = 576$
$P^3BT\nu = 4$	$P^2B^2T\nu = 4$	$P^2T^2\nu^2 = 24$	$P^2BT\nu^3 = 28$
$P^3BT = 8$	$P^2B^2T\varrho = 8$	$P^2T^2\nu\varrho = 48$	$P^2BT\nu^2\varrho = 56$
		$P^2T^2\varrho^2 = 96$	$P^2BT\nu\varrho^2 = 112$
			$P^2BT\varrho^3 = 224$

Here P indicates the condition that a twisted cubic pass through a given point, B that it have a given chord, ν that it cut a given line, T that it touch a given line and ϱ that it touch a given plane.

§ 11. From the above we can derive properties of different surfaces formed by systems of ∞^1 curves k^3 [1]).

The curves k^3 that touch a_1 and a_2 and cut a given line l, form a surface of the degree 24 that has 12-fold points in H_1 and H_2; a_1 and a_2 are eightfold lines and l is a quadruple line of this surface.

The curves k^3 that touch a_1 and cut two given lines l and m, form a surface of the degree 28 that has 14-fold points in H_1 and H_2; a_1 is an eightfold line, a_2 is a twelvefold line and l and m are quadruple lines of this surface. Etc.

[1]) Cf. also these Proceedings **30**, p. 1016 (1927).

Bacteriology. — *On the Transmuted Tubercle bacilli type BTTx., and their significance for the Diagnosis and the Therapy of Tuberculosis* By C. H. H. SPRONCK and W. HAMBURGER.

(Communicated at the meeting of October 29, 1927).

The classical law of the invariableness of bacteria had been advanced under the influence of the great discoveries of ROBERT KOCH. But gradually facts became known that negatived this theory. Morphological and biological properties may be lost, and new ones may be acquired. At present the variability of bacteria is wellnigh an ascertained fact; it is considerable with some species, small with others. But bacteria are by no means kaleidoscopical beings, their invariability being subject to laws still unknown.

According to NEUFELD [1]) the tubercle bacillus belongs to the non-fluctuating bacteria. But it has long been known, that the tubercle bacillus possesses a great adaptive power, and that it is possible to cultivate it on all sorts of media. Now, we detected that on media that are gradually altered and become poorer and poorer, the properties of the parasite alter considerably at a given moment. When we saw this for the first time, we thought of course of contamination, and months passed by before we got the conviction, that we had to do with a variety of the parasite. Small though the variability of the tubercle bacillus may seem, in reality also this micro-organism is undoubtedly capable of change, and of an intense change.

The stability of the transmuted parasite seems even to be still greater than that of the typical tubercle bacilli. Anyhow, in spite of all our efforts we have not yet succeeded in inducing the transmuted parasite to resume the state of a typical tubercle bacillus or of another variety. Moreover, the transmuted parasite has not only lost certain properties, it has also acquired new ones, which is the reason why we have not spoken of modified but of transmuted tubercle bacilli, and have styled the transmuted tubercle bacillus bacillus tuberculosis transmutatus x (BTTx) [2]).

Compared with typically human and bovine strains the transmutant grows with amazing rapidity, but only at 37—38° C. and in the presence of oxygen. It does not grow at room-temperature.

Already in bacilli, only 24—48 hours old that do not show granulation,

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1) Deutsche med. Wochenschr.: 1924, N⁰. 1.

2) JOLLOS (Zentralbl. f. Bakteriol. Abt. I, Orig. Bd. **93**), indeed, has cogently maintained that in the case of bacteria it is not right to speak of mutation, because with asexual propagation it is hardly possible to judge of heredity, and as the bacteria do not possess a nucleus, the cytological analysis is out of the question. We might use LEHMAN's term "Klonumwandlung", but most bacteriologists use the word mutation, which is understood in all countries.

an oval, protruding spore may appear, which always lies near one of the extremities of the rod. In one and the same cell we never find more than one spore. Before long the spore is mature, the rod degenerates and the spore gets free. The elements of the transmutant, (threads, bacilli, and spores) are non-acidfast, neither do they stain after GRAM. Only the younger still immature spores stain occasionally after GRAM. The spores do not possess great resistance, still they seem to be instrumental in preventing the destruction of the culture by heating to 56—59° C. during one hour, even when before the heating 0.5 % carbolic acid has been added. The transmuted bacilli, therefore are at least as resistant as the typical bacilli. If the latter are killed in a solution of 0.01 % formol within 24 hours (CALMETTE) the transmuted bacilli are undoubtedly much more resistant. Cultures of typical tubercle bacilli protected from exsiccation and light, survive at the most for 1½ years when kept in a cool place; but most cultures die off much sooner. Under the said conditions cultures of the transmuted tubercle bacillus appeared without exception capable of reproduction after 1½ years.

The transmutant has completely lost virulence for caviae. It does not produce tuberculin, nor is the substance contained in its elements. Whereas typical tubercle-bacilli are rich in fat and waxy substances, the transmuted ones are very poor in this respect.

Thus far we have transmuted 7 strains of tubercle bacilli, viz. 6 human ones and 1 bovine. The transmutation never having failed, no matter whether the strain had been cultivated in the laboratory already for some time, or was completely fresh, we consider the transmutation of every strain possible, and that of the third type of virulent tubercle bacilli, viz. bacillus gallinaceus we deem probable. For the transmutation, which in our laboratory is sometimes called emaciation-curve, at least 2 or 3 months are required.

It is remarkable that always the same transmutant came forth from 7 strains. True, there are differences among the transmuted strains inter se, but they are of a quantitative character. One strain grows quicker than the other, also the agglutinability varies.

It has long been known, that in cultures of typical tubercle bacilli not only acid-fast but also non-acid-fast rods are observed. They were suspected to be degenerated or imperfect rods of KOCH, or young, still undeveloped bacilli and they were even considered as a variety. In BALDWIN, PETROFF and GARDNER's [1]) recently published work "Tuberculosis" we read: "All attempts to separate non-acid-fast from acid-fast forms have failed, so that they must be considered to represent immature or imperfect forms rather than a separate variety".

We consider the non-acid-fast rods of the transmutant to be similar to those of the cultures of the typical tubercle-bacillus, but we assume the latter to be part of the parasite, which is not a simple

1) The Trudeau Foundation Studies, 1927, p. 33.

schizomycete, but a higher micro-organism, which belongs to the genus actinomyces. In its growth outside the infected organism it forms two sorts of rods, acid-fast and non-acid-fast, both originating from non-acid-fast threads and multiplying through fission. On the best media the parasite forms few non-acid-fast rods, but their number increases when the medium grows poorer, and now at one moment mutation takes place which under the modified circumstances largely benefits the existence and the propagation. It reminds us of a self-defence in distress.

We have decidedly observed the same mutation, but till now only once, after inoculating tissue of a tuberculous cavia-spleen on a poor medium, absolutely unfit to grow tubercle bacilli. After the medium, which had been guarded against exsiccation, had remained at 38° C. for two months without evincing any sign of growth, a suspicious speck became visible that gradually got bigger. Of course we suspected contamination, the more so since the culture exhibited a number of red specks, which had never been observed in the transmutant. On closer inspection we learned that it really was the transmutation, which did not exhibit the red spots any more in the next following culture.

Long before the appearance of the transmutant a typical culture had been obtained from the same spleen tissue, and the transmutation was commenced in the ordinary way. The transmutant thus obtained, and the one cultivated directly from the tuberculous spleen-tissue, could initially be distinguished by the latter's slower growth and smaller agglutinability. However, these differences stayed away in further cultivation.

In the tuberculous spleen-tissue that had been spread on the surface of the poor medium processes had presumably been going on that were analogous to those on the gradually altered media that are used for the transmutation. We assume that in the spleen-tissue a small typical culture has arisen that was soon short of foodstuffs; that the parasite adjusting itself by little and little to this shortage, had passed into the transmutant, which thrives better on the poor medium than the typical parasite on the best media hitherto known. The direct, or rather quasi direct culture of the transmutant is now tried again and again, but as yet without success.

Of late years the properties of the transmuted tubercle bacillus have been examined in different directions. It appeared thereby that they are significant for the diagnosis as well as for the therapy of tuberculosis.

After our communication concerning the transmutant as a tuberculosis-diagnosticum, the agglutination-titre of the blood-serum has been examined of another 100 subjects, suffering from, or suspected to suffer from tuberculosis, of patients suffering from other diseases, or apparently healthy persons. The titre was determined as in the first investigation. Only in some cases has the titre been established more precisely.

The cases of indubitable tuberculosis have now been divided into the following groups: 1⁰. sufferers from tuberculosis of the skin and the mucous membranes of mouth, throat, and nose (Table I), 2⁰. sufferers from tuberculosis of lymphatic glands, bones, and joints (Table II), 3⁰. sufferers from pulmonary tuberculosis (Table III) and 4⁰. sufferers from abdominal-tuberculosis. In Table V the cases are reported, in which we suspected the existence of tuberculosis: in Table VI sufferers from other diseases or apparently healthy people. In all the tables the cases have again been arranged according to the titre the better to determine the relation between the number of positive titres (1 : 100 and higher) and that of the negative ones.

The new observations justify us as yet in considering 1 : 100 as the lowest titre-limit of the positive reaction.

TABLE I. Tuberculosis of skin and mucous membranes.

N⁰.	Diagnosis	Titre	Notes
1	Scrophuloderma colli. (Prof. V. D. VALK, Groningen.)	1 : 10	About 2 years ago begun with a swollen gland. Patient looks healthy and now leaves clinic quite cured.
2	Tuberculosis cutis verrucosa. (Prof. V. D. VALK, Groningen.)	1 : 10	Verrucae at left middlefinger and fore-arm.
3	Lupus vulgaris nasi. (Prof. V. D. VALK, Groningen.)	1 : 50	Boy of 12 years, pale, small, slender; no fever. For three years lupus of nose and cheek. Röntgenization: large calcified hilusglands and a calcified focus in the right inferior lobe.
4	Lupus faciei, mucosae oris et nasi. (Prof. V. D. VALK, Groningen.)	1 : 50	Boy 17 years old, lupus for three years. Pirquet very strong +.
5	Lupus nasi. (Prof. BENJAMINS, Groningen.)	1 : 50	Woman 38 years old.
6	Lupus vulgaris faciei. (Prof. V. D. VALK, Groningen.)	1 : 100	
7	Lupus vulgaris disseminatus. (Prof. V. D. VALK, Groningen.)	1 : 100	Face and arms.
8	Lupus vulgaris in scrophuloderma-scars. (Prof. V. D. VALK, Groningen.)	1 : 100	
9	Lupus. (Prof. BENJAMINS, Groningen.)	1 : 100	Boy 15 years old.
10	Lupus faciei et mucosae nasi. (Dr. LA CHAPELLE, Assen.)	1 : 100	Vocal chords also affected. Lupus of 6 years standing, still small foci of it are persisting.
11	Lupus. (Prof. BENJAMINS, Groningen.)	1 : 200	Girl of 16 years.
12	Lupus mucosae nasi. (Prof. V. D. VALK, Groningen.)	1 : 200	v. Pirquet +.
13	Lupus. (Prof. BENJAMINS, Groningen.)	1 : 300	Woman 49 years old.

TABLE II. Tuberculosis of lymphatic glands, bones, joints

Nº.	Diagnosis	Titre	Notes
1	Tuberculosis of the knee-joint. (Dr. LA CHAPELLE, Assen.)	1 : 10	Röntgenization; focus in the epiphysis of the femur. The aspirated fluid gave a positive cavia test.
2	Tuberculous jugular glands. (J. HOOGKAMER, The Hague.)	1 : 10	
3	Tuberculosis of the ileo sacral joint. Formation of abscess. (Dr. LA CHAPELLE, Assen.)	1 : 50	25 years, general condition excellent. Formerly glandular abscess at the neck.
4	Tuberculosis of the left foot, perforation of the outer ankle. (Dr. LA CHAPELLE, Assen.)	1 : 100	21 years; formerly pleuritis.
5	Tuberculosis of jugular glands.	1 : 100	22 years, also laryngitis tuberculosa sputum: cavia-test not yet ended.
6	Tub. of right elbow and malum Pottii. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Girl of 16 years. Processes are coming to rest.
7	Coxitis tub. many fistulae. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Process seems to be active.
8	Tuberculosis jugular glands with abscesses. (Dr. LA CHAPELLE, Assen.)	1 : 100	Besides old tub. of the lungs.
9	Tub. of metacarpus of the left hand; fistulae. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Girl of 14 years. No fever.
10	Tub. of the tarso metatarsal joint I; fistulae. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Girl of 22 years. Resection 8 Aug. '27. Remains small fistula and pressure-pain. Titre determined 29 Sept. '27.
11	Tub. of the left large trochanter. (M. C. A. KLINKENBERGH and A. VAN BEEK, Utrecht.)	1 : 100	Man 49 years has been recently operated.
12	Most probably tuberculous coxitis. (Dr. LA CHAPELLE, Assen.)	1 : 200	Boy of 5 years, photo atrophy, no focus, v. Pirquet first time —, second time strong +.
13	Coxitis tuberculosa, cured. (Dr. LA CHAPELLE, Assen.)	1 : 200	26 years, photo only morbid growth of bone.
14	Spondylitis tuberculosa. (J. HOOGKAMER, The Hague.)	1 : 200	29 years. Formerly symptoms of tub. of the lungs.
15	Spondylitis tuberculosa abscess descended along carious ribs. (Dr. LA CHAPELLE, Assen.)	1 : 200	29 years. Formerly tub. of the lungs.
16	Osteomyelitis tuberculosa. (J. HOOGKAMER, The Hague.)	1 : 200	Woman 40 years, also tub. of the lungs, lingering for years.
17	Gonitis tuberculosa sin. (M. C. A. KLINKENBERGH and A. VAN BEEK, Utrecht).	1 : 200	Man 33 years, is feverish.
18	Tub. of the hand. (Dr. LA CHAPELLE, Assen.)	1 : 300	37 years, recovering v. Pirquet very weak positive.
19	Tuberculous jugular glands, fistulae. (Dr. LA CHAPELLE, Assen.)	1 : 300	40 years; fistulae from his 19th year. For the rest healthy.

With a titre of 1 : 100, we may think of the resorption of millions of tubercle bacilli, so that there is no question about an insignificant tuberculous focus, but of a progressive tuberculous process, that exists or has existed for some time, for the production of antibodies proceeds still some time after the resorption of the antigens has stopped. On the other hand the titre 1 : 100 does not indicate anything about the nature and the extent of the tuberculous process. Not infrequently a small focus contains far more tubercle bacilli than a large one and the resorption is larger at one place than at the other.

From a practical point of view it is significant that already in incipient

TABLE III. Tuberculosis of the Lungs.

N⁰.	Diagnosis	Titre	Notes
1	Chronic pulmonary tub. (W. v. HASELEN, IJmuiden and S. v. SLOOTEN, Haarlem).	1 : 10	Severe, hopeless case.
2	Tub. of the Lungs. (L. WEIJL, Middelburg).	1 : 10	Far advanced.
3	Chronic extensive tub. of the lungs. (M. LE HEUX, Doorn).	1 : 50	Boy of 18 years. Far advanced.
4	Chronic tub. of lungs and larynx. (J. HOOGKAMER, Den Haag.)	1 : 100	26 years, far advanced, pleuritis and spina ventosa.
5	Chronic tub. of the lungs. (DR. KERSBERGEN and C. J. BOUWER, Haarlem).	1 : 100	
6	Chronic tub. of the lungs. (DR. KERSBERGEN and C. J. BOUWER, Haarlem).	1 : 100	
7	Chronic tub. of the lungs. (DR. KERSBERGEN and C. J. BOUWER, Haarlem).	1 : 100	
8	Chronic tub. of the lungs. (J. BURCK, 's Graveland).	1 : 100	29 years.
9	Chronic tub. of the lungs. (F. A. v. D. BREGGEN, Haarlem).	1 : 100	
10	Chronic tub. of the lungs. (J. HOOGKAMER, The Hague.)	1 : 100	Tubercle bacilli in sputum.
11	Chronic tub. of the lungs. (J. WACHTERS, Culemborg).	1 : 100	Also coxitis tuberculosa and abdominal tub.
12	Tub. of the lungs (formerly). (DR. STENVERS, Utrecht).	1 : 100	Now influenza and acute encephalitis.
13	Chronic tub. of the lungs. (Dr. LA CHAPELLE, Assen.)	1 : 200	Besides tub. cutis at neck and left shoulderblade. Now hydrops genu after trauma; titre of the aspirated fluid 1 : 50.

tuberculosis of the lungs the titre is 1 : 100 or higher. Because recent lung-foci generally contain many tubercle bacilli and there is in the lungs a favourable opportunity for resorption, it is easily understood that the titre is high already when the symptoms are still vague.

In rapidly progressing exsudative forms of pulmonary tuberculosis the titre has not yet been examined. Probably it will appear to be low, because here the blood is easily overcharged with antigens, which the organism cannot destroy. Table III again shows, that in lingering forms the titre is regularly high and remains so for a long time; it falls only when the disease is far advanced, so that a serological diagnosis is superfluous.

Also when the lung-process has come to rest or is cured, a fall of the titre is sure to reveal itself after some time. That in the cases mentioned the titre is lower than 1 : 100 does not lessen the practical significance of the agglutination-test for the recognition of the pulmonary tuberculosis If we compare the low titre with the clinical observation, its significance can easily be interpreted.

Probably a titre of 1 : 100 is also attained in a short time in the case of tuberculosis of the peritoneum (Table IV). The large surface of the peritoneum and the initially unimpeded resorption are favourable factors.

TABLE IV. Tuberculosis of the peritoneum.

N⁰.	Diagnosis	Titre	Notes
1	Peritonitis tuberculosa. (Dr. LA CHAPELLE, Assen.)	1 : 50	Much fluid in the belly. Also both lungs diffusely affected. High temperatures, v. Pirquet —.
2	Tubercles on the bowels. (Dr. LA CHAPELLE, Assen.)	1 : 90	Found at an operation of the abdomen, probably originated a short time ago in the lactation-period.
3	Tub. peritonei. (Dr. LA CHAPELLE, Assen.)	1 : 100	Previously pleuritis, fever, v. Pirquet +.
4	Tub. of the abdomen. (Dr. LA CHAPELLE, Assen.)	1 : 200	Recovering, v. Pirquet weak +.
5	Peritonitis t.b.c. (DR. LA CHAPELLE, Assen.)	1 : 300	Regressive process.

During an operation of the abdomen Dr. LA CHAPELLE detected tubercles in the serosa of the ileum, that had probably originated in the lactation-period and had not yet induced any sympton. Nonetheless the titre rose as high as 1 : 90 (Table IV n⁰ 2). The severe far advanced case n⁰. 1 (tuberculosis of abdomen and lungs) shows a fall of the titre (1 : 50) accompanied by negative tuberculin-reaction (negative anergy of v. HAYEK).

With lupus the tubercle bacilli are extremely rare. In the skin temperature and light are detrimental to their reproduction. It cannot be expected, therefore, that a small lupus-spot, a dissection-tubercle, a tuberculosis verrucosa cutis would induce a titre of 1 : 100. In 13 cases of lupus and other forms of tuberculosis cutis (Table I) the titre was lower than 1 : 100 in four cases (30 %). In the other cases it varied from 1 : 100 to 1 : 300.

Also in tuberculosis of lymphatic glands, bones and joints (Table II) the titre can be lower than 1 : 100. The resorption is decidedly tardier here, and the number of tubercle bacilli is as a rule smaller than in the lungs, so that the whole organism cannot join so soon in the strife against the local process. In 3 (15 %) of the cases examined the titre was lower than 1 : 100, in the others it varied from 1 : 100 to 1 : 300.

In case N⁰. 1, a patient of Dr. LA CHAPELLE (Assen) suffering from a tuberculous inflammation of the knee-joint, the titre appeared to be smaller than 1 : 10 on the first examination, and more than a month later it had risen only little (rather more than 1 : 10). Röntgenization distinctly revealed a tuberculous focus in the epiphysis of the femur and by puncture he obtained a little viscous fluid, in which tubercle bacilli could be demonstrated with the aid of the cavia-test. But the liquid injected subcutaneously on August the 10th contained only few bacilli, as in August and in September no symptom of tuberculous infection was noticeable. Not before October did a small infiltrate appear on the place of the infection and some time later swelling of the regionary glands occurred.

From Table V it appears again that in cases that remind us of tuberculosis the agglutination-test often turns out positive, which speaks for its validity. Out of the 36 sera of this group 21 (58 %) had a titre of 1 : 100 to 1 : 300. It goes without saying that we tried to verify the serological diagnosis. In one case there were afterwards distinct, clinical manifestations of tuberculosis (n⁰. 17). In another (n⁰. 22) tubercle bacilli were found in the urine, and the extirpated kidney appeared to be tuberculous. In a third (n⁰. 32) the cavia-test demonstrated tubercle bacilli in the sputum.

Altogether we examined up to now $36 + 33 = 69$ doubtful cases with $21 + 17 = 38$ (55 %) positive results.

The group of 20 sufferers from divers other diseases or apparently healthy persons. (Table VI) gives in as many as 7 cases (35 %) a positive result. It is as if this high percentage renders the practical significance of the agglutination-test as illusive as that of the positive tuberculin-reaction. On closer inspection of the positive cases, however, the surmise rises that the experiment might contribute to the recognition of the tuberculous character of diseases, whose cause is still unknown, or the predisposing influence which a tuberculous focus exerts on the origin of other diseases.

Since much has been written about the connection between erythema induratum of BAZIN and tuberculosis and JADASSOHN considers this disease even as a form of tuberculosis cutis, we looked for cases of this disease and thanks to the assistance of Dr. TER MATEN we had an opportunity to examine the blood-serum in two cases. In the one case (n⁰. 16) the titre was 1 : 200, in the other 1 : 50 (n⁰. 12). Truth to tell, we had expected a

TABLE V. Tuberculosis?

N⁰.	Diagnosis	Titre	Notes
1	Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 10	After an operation irregular fever.
2	Vague complaints. (DR. ENKLAAR, Amsterdam).	1 : 10	
3	Throat complaints, headache struma. (J. HOOGKAMER, The Hague.)	1 : 10	Tuberculous relations.
4	Divers complaints, not specific laryngitis. (J. HOOGKAMER, The Hague.)	1 : 10	
5	Suspicious cutaneous inflammation. (J. HOOGKAMER, The Hague.)	1 : 10	
6	Presumably no tub. (J. HOOGKAMER, The Hague.)	1 : 10	
7	Suppuration of the ear after radical operation. Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 10	Radical operation years ago. No reason to think of tuberculosis.
8	Headache. (W. HAMBURGER, Utrecht).	1 : 50	
9	Chronic rheumatic pedarthritis of Poncet. (Dr. LA CHAPELLE, Assen.)	1 : 50	X-photo does not give a decision.
10	Tuberculosis(?). (Dr. LA CHAPELLE, Assen.)	1 : 50	Boy of 6 years. Subcutaneous, granulating cavity in the middle under the chin.
11	Process in the thoracic cavity. Tuberculosis? (Dr. LA CHAPELLE, Assen.)	1 : 50	Patient has been attended in a sanatorium, has had a pleurisy, that was soon cured. Now increasing oppressiveness, cause unknown.
12	Knee irritated through paving, meniscus-luxatio. (Dr. LA CHAPELLE, Assen.)	1 : 50	Röntgenization negative.
13	Tuberculosis of the lungs? (J. WACHTERS, Culemborg).	1 : 50	
14	Tuberculosis? (Dr. LA CHAPELLE, Assen.)	1 : 50	Woman 29 years, abdominal complaints. Scars in the neck of glands in youth.
15	Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 70	The one physician thinks of tub. the other does not.
16	Tuberculosis? (DR. KERSBERGEN and C. J. BOUWER, Haarlem).	1 : 100	
17	Tuberculosis? (DR. ENKLAAR, Amsterdam).	1 : 100	Serological diagnosis, afterwards confirmed.
18	Presumably tub. of the lungs. (J. HOOGKAMER, The Hague.)	1 : 100	
19	Tabes mesaraica. (DR. E. H. B. VAN LIER, Utrecht).	1 : 200	20 years old.

TABLE V. Tuberculosis? *(continued)*.

N⁰.	Diagnosis	Titre	Notes
20	Rise of temperature, Cause unknown. (J. HOOGKAMER, The Hague.)	1 : 100	
21	Bronchial asthma. Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 100	
22	Tub. of the kidney? (J. WESTENBURG, Bloemendaal).	1 : 100	Urine appeared to contain tubercle bacilli (cavia-test). The extirpated kidney was found to be tuberculous.
23	Rheumatic pains, reminding of Poncet. (Dr. LA CHAPELLE, Assen.)	1 : 100	
24	Tuberculous suppuration of the ear? (J. HOOGKAMER, The Hague.)	1 : 100	
25	General weakness.	1 : 100	Girl 18 years. Feels ever tired and weak in the muscles.
26	Chronic bronchitis. Tuberculosis. (DR. FREERICKS, 's Hertogenbosch).	1 : 100	No tubercle bacilli in the sputum (cavia-test).
27	Incipient tub. of the lung(?). (CHR. EGGINK, Amersfoort).	1 : 100	
28	Tuberculosis? (Dr. LA CHAPELLE, Assen.)	1 : 100	Girl, 11 years, weak, colipyelitis. v. Pirquet strong +.
29	Incipient tub. of the lungs. (J. WACHTERS, Culemborg).	1 : 100	
30	Painful knee, slight swelling of the capsule. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Girl of 15 years.
31	Tubercle under the retina? Hoarseness, inflammation of epiglottis and larynx-tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 100	
32	Asthma bronchiale. Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 200	The cavia-test confirms the serol. diagn.
33	Incipient tub. of the lung? (Our personal observation).	1 : 200	Young woman, debility, fatigue, small rise of temperature. Does not track or cough. Lungs carefully examined by internists. Doubt about affection of the apex. Röntgenization negative.
34	Suspicious pulmonary disease. (J. BAST, Oudehaske).	1 : 200	First cavia-test (sputum) negative; second one not ended yet.
35	Otitis media perforativa suppurativa for years. (J. HOOGKAMER, The Hague.)	1 : 200	This patient is the mother of No. 25. Previously she was glandulous.
36	Bilateral suppuration of the ear. Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 300	Previously patient was glandulous.

TABLE VI. Suffers from other diseases, or apparently healthing.

N⁰.	Diagnosis	Titre	Notes
1	Lupus erythematosus. (E. SWAAB, Breda).	1 : 10	Diagnosis made by several dermatologists.
2	Sepsis. (L. ALKEMADE, St. Oedenrode).	1 : 10	Purulent infiltration of the lower leg appeared to be rich in micrococcus catarrhalis.
3	Psoriasis. (J. HOOGKAMER, The Hague.)	1 : 10	No trace of tuberculosis.
4	Apparently healthy. (Our personal observation).	1 : 10	Robust youth.
5	Chronic intestinal phenomena. (W. HAMBURGER, Utrecht).	1 : 10	Nervous, weak man.
6	Sterile abscess between hepar and pancreas. High temperature. (Dr. LA CHAPELLE, Assen.)	1 : 10	Afterwards pleurisy, clear sterile liquid.
7	Hoarseness. (J. HOOGKAMER, The Hague.)	1 : 10	Vocal chords do not close up well. (pression on nervus recurrens?) No symptoms of larynx- or lung-tuberculosis.
8	Morbus Basedowi. (J. HOOGKAMER, The Hague.)	1 : 10	
9	Ozena. (J. HOOGKAMER, The Hague.)	1 : 50	
10	Lues. (J. HOOGKAMER, The Hague.)	1 : 50	
11	Throat complaints. (J. HOOGKAMER, The Hague.)	1 : 50	
12	Erythema induratum Bazin. (Dr. TER MATEN, Amsterdam.)	1 : 50	
13	Chronic Bronchitis. (J. HOOGKAMER, The Hague.)	1 : 50	74 years old.
14	Spondylitis type Bechterew; light psoriasis. (Dr. LA CHAPELLE, Assen.)	1 : 100	
15	Psoriasis and rheumatic complaints. (W. HAMBURGER, Utrecht.)	1 : 100	51 years; no tub. in the family.
16	Erythema induratum Bazin. (Dr. TER MATEN, Amsterdam.)	1 : 200	The father of the 32-year-old patient died of haemoptoe.
17	Ozena. (P. BUTTER, Kampen.)	1 : 200	Girl, 22 years. For the rest quite healthy, the 24-years-old sister (also ozena) has been suffering from a light tub. of the lung. Parents and 7 brothers and sisters healthy.
18	Apparently healthy. (J. HOOGKAMER, The Hague.)	1 : 200	Miss B. no reason to think of tub. Tub. not in the family.
19	Apparently healthy. (J. HOOGKAMER, The Hague.)	1 : 200	Afterwards a kidney-process was detected. Examination of the urine: cavia-test 5 Sept. up to now no manifestations of tub. The kidney already extirpated had large purulent cavities. Result of anatomic examination not yet known.
20	Rheumatism. (J. HOOGKAMER, The Hague.)	1 : 300	Suffering for a long time. Treated with all sorts of remedies without a positive result, also with strepto-staphylococci vaccin.

positive result in both cases. The high titre in case N⁰. 16 cannot be at all surprising.

Concerning the 22-year-old ozena-sufferer N⁰. 17, titre 1 : 200, for the rest in rude health, we heard that her sister, older by two years, who also suffers from ozena, is less healthy and has passed through a light tuberculosis of the lungs, so that the high titre is probably the consequence of a favourably regressing or an arrested tuberculous process.

As for case N⁰. 19, titre 1 : 200, a diseased kidney was detected afterwards. The examination of the urine is not yet terminated (cavia-test), but in the mean time the kidney has been removed. It showed large suppurating hollows. The result of the anatomic investigation is not yet known.

There remain still 4 cases (20 %) whose high titres cannot be accounted for as yet. But it is remarkable that of these persons one is suffering from chronic rheumatism, one from rheumatism and psoriasis, and one from BECHTEREW's type of spondylosis and light psoriasis.

Also in our first investigation in the group of 35 subjects suffering from divers diseases or apparently healthy people, there were three inexplicable cases two of which were rheumatism.

Thus far 9 cases of rheumatism have been examined, in six of them a titre was found of 1 : 100 or higher (in one case a titre of 1 : 10 and in two cases of 1 : 50). We may think that rheumatism and tuberculosis may occur side by side, or perhaps of tuberculous rheumatism of PONCET. We may also think of para-agglutinins.

In our previous agglutination-tests with acidfast tubercle bacilli para-agglutinins were very cumbersome. When using the transmutant nothing was noticed as yet of the very high titres, previously revealed in other diseases and also in tuberculosis. In tuberculosis we, therefore, expected some times titres of 1 : 500 and higher; a higher titre than 1 : 400 (one case) has however not been found as yet.

So far as we know the literature does not contain any indication that formerly difficulties have been met with in chronic rheumatism, so that it seems expedient to determine the titre for a larger number of cases. It is not out of the bounds of probabilities, that the transmutant points to tuberculous cases in the chaos of chronic rheumatism. Should para-agglutinins appear to come into play, this will detract little from the value of the diagnostic agglutination-test, since pulmonary tuberculosis and chronic rheumatism seldom go together.

In virtue of our personal investigation the rather weak and transient tuberculosis-immunity can be analyzed into two components: an allergic (hyper-sensitiveness to tuberculin) and an antitoxic-anti-infectious one, which induce, the one as well as the other, a small degree of immunity in the cavia. This immunity is so weak, that, if the infectious dosis is only a little larger than the dosis minima, little or nothing can be noticed of a greater power of resistance in the previously treated animals.

In the transmuted tubercle bacilli the antigen of the tuberculin-allergy is altogether absent. They evoke in the cavia only a bacillar protein-allergy, which probably has nothing to do with the tuberculosis-immunity. On the other hand they do contain antigens of the antitoxic-anti-infectious component.

To our mind there is often in the tuberculous patient a shortage of either component, while most colleagues will be inclined to assume only a shortage of the antitoxic-anti-infectious component. The reason of these different views is that some consider great sensitiveness to tuberculin others the absence of this allergy as a favourable sign. We believe that the allergy is evoked by an antibody, which is eagerly absorbed by the cells, so that there is often a paucity of this substance in the blood. So the tuberculin-reaction occurs in cells, and the positive reaction is in our opinion a sign that the blood contains little antibody. If on the contrary the blood is rich in antibody, the tuberculin-adsorption takes place not within but outside the cells, and we speak of negative reaction. So, in our opinion, the negative reaction is preferable to the positive, except in case of negative anergy (v. HAYEK).

A vaccine, consisting of killed typical tubercle-bacilli would be preferable, because all the antigens are contained in it. But vaccinetherapy cannot employ typical tubercle bacilli, because the blood is most often poor in the antibody of the allergic component, so that dangerous adsorption processes are to be feared in the tuberculous cells that are richest in this antibody, and the subcutaneous injection also evokes infiltrations and abscesses due to the very difficult resorption of bacilli rich in fat.

To our mind, therefore, the use of transmuted tubercle-bacilli is making a virtue of necessity, for they contain only antigens of the antitoxic-anti-infectious component. The same drawback is observed in the tuberculin-therapy, which applies only the antigen or rather the haptene of the allergic component, and consequently can reinforce only the allergic component.

A vaccine consisting of transmuted tubercle bacilli and tuberculin would, therefore, be preferable, but we thought it expedient in the first place to apply the transmuted tubercle bacilli as such, in order to ascertain whether they are, indeed, active, and if so, to what extent.

We used exclusively bacilli 2×24 hours old that had been killed with formol (0.2 %). Very likely the use of living bacilli would be better, but seeing that the resistance is not inconsiderable and the parasite of the tuberculosis possesses a great adaptive power, we did not think that the injection of living transmuted bacilli into man was allowable, harmless though it may seem.

Of the killed bacilli a vaccine is made in the ordinary way, which was called transmutan to avoid the ominous word tuberculosis, and now consists of the 7 strains at our disposal. In order to obtain a stock-transmutan, rightly called polyvalent, more and more transmuted strains will be taken up in the vaccine. In many lingering cases, it is quite possible to employ auto-transmutan, which has so far been tried in only two cases.

Two years ago the first trial was made with transmutan, to which SPRONCK invited some of his ex-pupils, who acquitted themselves so well of this task, that their results of the first year published by Dr. ENKLAAR [1]), were confirmed further in the second year, in a still higher degree. While in the first year 9 colleagues made experiments, in the second year 128 partook of the experiments out of their own impulse.

Also in the second year we chiefly tried transmutan in frequently occurring pulmonary tuberculosis, although it would be difficult to name a second disease, in which the trial of a new remedy is more difficult. In all countries KOCH's tuberculin has been tried again and again for years, with the result that some consider the tuberculin therapy useful, and others call it useless. The difficulty in judging of a new remedy for pulmonary tuberculosis arises from the following circumstances. The disease offers a great variety. The anatomical lesions may be of small extent, yet of a serious character; they may also be less dangerous in spite of large extent. Moreover tuberculosis of the lungs is rather apt to spontaneous cure, and its course is so capricious that the ablest specialist is absolutely at a loss to make a prognosis.

It is evident that the activity of transmutan would have to be very great to cause an agreement concerning its utility in this disease in a few years.

New experience confirms the harmlessness of the vaccine. If the blood should contain too many antigens, as occurs in acute and far-advanced, lingering cases or with high fever, no good can be expected from transmutan, which even may be pronouncedly inimical.

The experiences of the second year also show that in not a few cases the sufferers manifest an improvement during the treatment, which with the application of other means had been expected for a long time in vain. This improvement consisted in subjective and objective recuperation and in clinical cure with or without a return of the power to work. As a rule recuperation or recovery was obtained only after long-continued treatment, but sometimes also surprisingly quick.

The patients that have improved during the vaccine-treatment or have been cured clinically have no advantage over those that are cured spontaneously. As might be expected a priori, a relapse may also occur in the treated patient.

In the case of recidivation it has struck us that when the treatment was resumed, improvement again was noticeable. In a more severe case of pulmonary tuberculosis the general condition improved considerably during the vaccine-treatment and the symptoms of the disease disappeared almost completely. But in the morning still some sputum was ejected occasionally that appeared to contain tubercle bacilli. Contrary to the advice of the physician the treatment was discontinued, while the patient got to his work again. A few months later again symptoms appeared, which induced him to

[1]) Geneeskundige Bladen, XXV, n⁰. 4, 1926.

stop his occupation. Immediately the vaccine-treatment was resumed and again improvement set in just as the first time.

The same observation was made in abdominal tuberculosis. During the treatment the subjective and the objective symptoms disappeared, and the treatment was discontinued because clinically the patient seemed to be restored. Some time later a relapse followed. After another vaccine-treatment the patient got better again. In these cases the treatment concerned outpatients, so that additional favourable circumstances such as bed-rest, better nutrition or attendance did not come into play.

Most often the cases of pulmonary tuberculosis did not belong to the light forms but to the severer ones. In the latter, however, transmutan has been tried, in the hope to save the sufferer, but sometimes it was too late. Anyhow, the favourable impression of a large number of physicians was not at all attained in the treatment of light, incipient cases, in which it is a pity that transmutan has hardly been tried.

In cases of abdominal tuberculosis very favourable results have been obtained and the rapid improvement was striking. Little is to be said about surgical cases, for lack of experience. But the number of surgeons, who try transmutan, increases so that after another twelvemonth they will have in one way or other a settled opinion about it.

In order to judge of the use of transmutan the most suitable experiments can be made in tuberculosis of the skin and of the mucous membranes of nose, mouth and pharynx, because the result can be verified so well. We are, therefore, highly pleased with Prof. VAN DER VALK's interest in the subject, who since November of last year has applied transmutan in his clinic and polyclinic at Groningen in a number of cases of lupus and other forms of tuberculosis cutis, and it was gratifying to learn that he has held a demonstration of patients, who had improved so much and so quickly by the administration of transmutan, that many colleagues were induced to ask us for it. The results of his examination, which is still going on, will no doubt be published in due time bij Prof. VAN DER VALK. We are also happy to state that Prof. BENJAMINS of Groningen has put this inquiry in hand in his clinic, and we are not less pleased to say that he obtained already some positive results, which encouraged him to further experimentation.

Several colleagues observed that during the transmutan-treatment a tuberculous process can not only be checked, but that also existing anatomic, tuberculous lesions can recover. Regression of the anatomic changes has also been observed in tuberculous processes in lungs, abdomen, etc., but cases of tuberculosis of the skin and the mucous membranes are to be considered in the first place as suitable to establish regression with certainty, and to make out whether complete cure can be attained.

An indicium would be very convenient to prevent deceptions in the transmutan-treatment. In cases of pulmonary tuberculosis a division into light, middle severe, and vere severe (fatal) cases is no doubt of importance, and a priori the best results of transmutan can be expected from light cases. But experiences like the following, show that it is difficult to foretell anything. In an establishment transmutan would be applied in four cases of tuberculosis of the lungs, three of which were rather severe cases, and one so severe that a good result was not expected. But in view of the harmlessness of the vaccine, the experiment was made also in this case to prevent the discouraging influence of exclusion. The result was that the four patients improved and contrary to our expectation the severest improved most.

It is obvious to suppose that the best results of transmutan are yielded by patients, possessing the property of being good antibody-producers. Now some observations make us suspect that the agglutination-titre of the blood-serum of the sufferer might give us an indication in this respect. In this connection we do not think of an influence of the agglutinins, but of the probability that the agglutination-titre is an index of the quantity of other antibodies attacking the parasite, that are present in the blood of the sufferer, for experience taught us that with bacterial infections and also with artificial immunization, the various antibodies increase at about the same rate.

If a tuberculous process is on the way to spontaneous recovery or was cured spontaneously a short time ago, the agglutination-titre seems to be high. Examples of this are the cases of abdominal tuberculosis mentioned in Table IV, nr. 4 (titre 1 : 200) and nr. 5 (titre 1 : 300); again in Table II the case of tuberculous coxitis cured spontaneously nr. 13 (titre 1 : 200) and the spontaneously recovering case of tuberculosis of the hand, nr. 18 (titre 1 : 300).

That in the stage of recovery we find a higher agglutination-titre, that is at a moment when the circulation of a large quantity of active antibodies in the blood can be assumed, favours the supposition that also in tuberculosis the production of the different antibodies goes on at the same rate, so that the agglutination-titre in connection with the clinical observation, is an indicium whether the patient is to be regarded as a good or a less good producer of antibodies.

Therefore, we are now ascertaining whether the transmutan-treatment of patients, whose blood-serum has a higher amount of specific agglutinins already before the treatment, yields a better result than the treatment of patients with a lower titre.

Clinical observations enable us already now to interpret low titres as follows: If the disease is running an acute course, we may assume too many antigens in the blood, which the organism cannot absorb. If the lingering disease is already far advanced, reduced production as well as large consumption of antibodies (adsorption by antigens) may be the cause of the low titre. In both cases the low titre is considered as a warning that the administration of transmutan is hazardous. In all other cases a low titre is regarded as an index that the whole organism does not or little partake of the strife against the local process, or that the sufferer is less capable of producing antibodies. In the first case the transmutan-treatment may be deemed rational. In the second case it is still being tried at haphazard.

That the results became gradually better, may perhaps be attributed to gradual change of the vaccine, which originally was prepared from one transmuted strain and now from seven strains. Doubts are also entertained as to the duration of the activity of the vaccine. Since a few months back it has been recommended not to order the whole series of increasing doses at once but only 6 at a time, that the vaccine might be applied as fresh as possible. In order to have fresh transmutan at our disposal the preparation takes place at intervals of a few weeks and the amount left of the preceding preparations is destroyed.

SUMMARY.

1. From 7 strains of typical, virulent tubercle bacilli 7 strains of non-acidfast, avirulent tubercle bacilli have been cultivated, which may be called indentic.

2. As explanation it has not been assumed, that we have at length succeeded in isolating the long known non-acidfast rods, present in every culture of the typical tubercle bacillus, from the acid-fast rods, but that the parasite of tuberculosis, adapting itself to a poor existence, saves itself by transmutation.

3. The transmutant thrives well on the poor medium, on which the typical parasite does not propagate itself. Its resistance and stability are greater, so that its survival is more safeguarded than that of the typical parasite.

4. In the blood-serum of sufferers from tuberculosis specific agglutinins are demonstrable by means of the transmutant.

5. The significance of this simple agglutination-test for the diagnosis of tuberculosis is getting more and more obvious.

6. The agglutination-test begins to be useful also in estimating the specific resisting power of the tuberculous patient.

7. The favourable impression of the vaccinotherapeutic application of the transmutant, obtained in the first year, was confirmed and increased in the second.

8. The utility of this therapy is attributed to strengthening of the antitoxic-anti-infectious component of the immunity against tuberculosis.

Physics. — *Isotherms of monatomic substances and their binary mixtures.* XXVI. *Isotherms of helium at —183.0 and —201.5° C. and pressures of 3 to 8 atmospheres.* By G. P. NIJHOFF and W. H. KEESOM. (Comm. N⁰. 188*b* from the Physical Laboratory at Leiden.)

(Communicated at the meeting of October 29, 1927).

§ 1. *Introduction.*

In a former communication [1]) we pointed to the importance of measurements of gases in slightly compressed states in order to obtain values as exact as possible of the first virial coefficients. Especially for helium this is of great importance with a view to thermometry at low temperatures. Moreover, as we then also remarked already, the now examined pressure region for helium with its low critical pressure, is of particular importance, because comparison, by means of the principle of the corresponding states, becomes possible with other gases, for which the equation of state in the corresponding pressure region has already been exactly examined.

We found it a great difficulty that the divided manometer constructed by us for pressures of 1 to 4 atmospheres, which, as has also been described in the former paper, must be read with a cathetometer, is not very fit for a fast use, so that the measurements with it want comparatively much time. With a view to this we have constructed two closed manometers, which are being shortly described here.

§ 2. *The closed manometers M 6 and M 20.*

These have been arranged in the same way as those described by KAMERLINGH ONNES and HYNDMAN [2]).

The first, destined for pressures between 2 and 6 atmospheres, consists of a glass stem 150 cm long, with a section of about 0.08 cm^2 and a capacity of about 12 cm^3. To this was blown at the upper end a small reservoir of 6 cm^3 and at the lower end a reservoir of 18 cm^3. So if we read the top of the mercury meniscus to $^1/_{10}$ mm, when the mercury during the measurements is standing quite at the upper end in the stem we have still an accuracy of $^1/_{7500}$, whereas at a lower place of the mercury the relative accuracy increases proportionally to the volume.

[1]) G. P. NIJHOFF and W. H. KEESOM, These Proc. **28**, 963, 1925, Comm. Leiden N⁰. 179*b*.

[2]) H. KAMERLINGH ONNES and H. H. F. HYNDMAN, These Proc. **4**, 776, 1902, Comm. Leiden N⁰. 78*c*.

The second manometer in principle quite identical to the first has the following dimensions: large reservoir 51.3 cm^3, stem 7 cm^3 with a length of 143 cm, small reservoir 2.9 cm^3.

It must still be observed that it did not seem desirable to us to join the manometer for the reading of the lowest pressures to the piezometer by means of compressed air as is usually done for the other manometers at Leiden. Piezometer and manometer were being filled out of the same mercury reservoir and formed a couple of communicating vessels. In order to calculate the pressure in the piezometer, we must of course make corrections for the difference in height of the two mercury columns and for a possible temperature difference of the two water jackets.

§ 3. *The measurements and results.*

The measurements are made in the same way as indicated in our former communication [1]).

We were very pleased that, for a great part of these, Prof. BORIS ILIIN from Moscow was willing to work with us. We have collected the results of these measurements in a separate communication [2]). The remaining ones we give in table I. In table II the values of B_A are given then, which we have calculated from our measurements and from those of ours and of Professor ILIIN.

For our calculations we took the value $\alpha_A = 0.0036618$ used until now

TABLE I.

θ °C.	p int. atm.	pv_A	d_A	$O-C\,(pv_A)$
— 183.07	3.2279	0.33112	9.748[5]	+0.00012
	4.0064	.33144	12.088	+ 7
	5.1853	.33197	15.620	+ 4
	7.3067	.33277	21.957	— 17
	8.1456	.33327	24.441	— 6
— 201.52	2.9347	.26323	11.149	—0.00004
	3.5627	.26351	13.520	— 5
	6.0023	.29471	22.669	+ 5
	6.5786	.26478	24.845	— 10

[1]) G. P. NIJHOFF and W. H. KEESOM, These Proc. **28**, 963, 1925, Comm. Leiden N⁰. 179*b*.

[2]) See the following communication N⁰. 188*c*. These Proceedings **31**, 408, 1928.

TABLE II.

θ 0° C.	$B_A \cdot 10^3$	θ 0° C.	$B_A \cdot 10^3$
Measurements of NIJHOFF, KEESOM and ILIIN			
− 103.29	+ 0.366	− 235.77	+ 0.0295
− 146.50	+ 0.256	− 249.80	− 0.0085
− 183.07	+ 0.158	− 252.57	− 0.0100
− 201.52	+ 0.120	− 255.84[5]	− 0.0237
− 224.94	+ 0.059	− 258.99	− 0.0340
Measurements of BOKS and KAMERLINGH ONNES.			
+ 20	+ 0.550	− 103.64	+ 0.364
0	+ 0.523	− 142.02	+ 0.270
− 37.40	+ 0.480	− 183.34	+ 0.185
− 70.30	+ 0.428		
Measurements of HOLBORN and OTTO.			
0	+ 0.5282	− 183.0	+ 0.1562
− 50	+ 0.4344	− 208.0	+ 0.0998
− 100	+ 0.3366	− 252.8	− 0.0093
− 150	+ 0.2295	− 258.0	− 0.0337
− 183.0	+ 0.1537		

at Leiden. We have given up the idea of a recalculation according to the determination of the fundamental pressure coefficient of helium by KEESOM and Miss VAN DER HORST [1]) since the values of B_A will only slightly change. But the values of $B = \frac{B_A}{A_A}$ will be changed somewhat. It is also for this reason that we have preferred to give the values of B_A in this communication, instead of those of B.

At the same time we herewith give the values of B_A, which were calculated by us from the measurements of BOKS and KAMERLINGH ONNES [2]) and for comparison the values of B_A, reduced to Leiden units, which have been found in Berlin [3]).

[1]) W. H. KEESOM and Miss H. VAN DER HORST, These Proc. **30**, 970, 1927, Comm. Leiden N⁰. 188*a*.

[2]) J. D. A. BOKS and H. KAMERLINGH ONNES, Comm. Leiden N⁰. 170*a*.

[3]) L. HOLBORN and J. OTTO, Zs. f. Phys. **30**, 320, 1924 and **37**, 359, 1926.

These different values of B_A are represented in fig. 1.

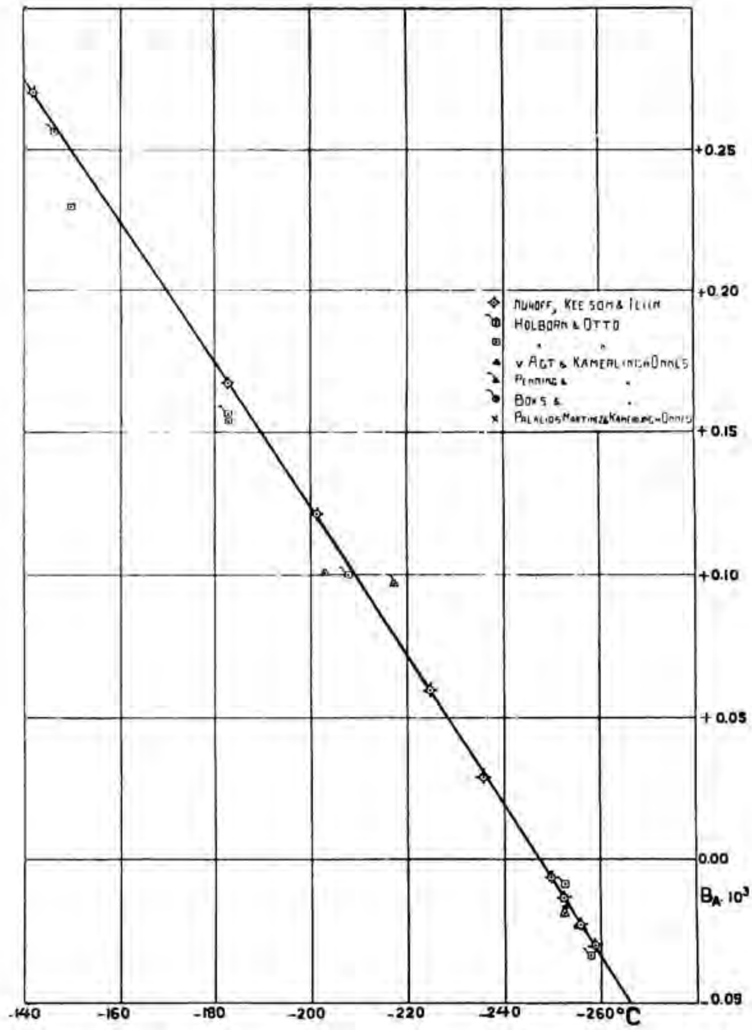

Fig. 1.

Herewith we observe that the German results, as far as the temperature region, which still can be reached with liquid oxygen, are lying lower than ours, while at the temperatures of liquid hydrogen, though the dispersion of the Berlin values is larger, the agreement can yet be called very satisfactory.

Besides it is very remarkable that at lower temperatures the B_A-values tend to depend linearly from the temperature and that after extrapolation this line passes through values found in the liquid helium region.

Finally we thank Miss A. SOLLEWIJN GELPKE for her help as well at the measurements, as especially at the calibration of the manometers.

Physics. — *Isotherms of monatomic substances and their binary mixtures. XXVII. Isotherms of helium between —103.6° C. and —259.0° C. and at pressures of 1.5 to 14 atmospheres.* By G. P. NIJHOFF, W. H. KEESOM and B. ILIIN. (Comm. N⁰. 188c from the Physical Laboratory at Leiden.)

(Communicated at the meeting of October 29, 1927).

We have measured isotherms of helium in the same way as has been described in the communication about oxygen [1]) with the same arrangement of the piezometer and the manometers.

The temperatures were obtained with the aid of liquid ethylene, liquid hydrogen and with the hydrogen vapour cryostat.

The piezometer with which the volumes were measured consisted of the same stem of 108 cm^3, with which also the oxygen isotherms have been measured; however, with a view to the so much lower temperatures at which we wished to measure, the gas reservoir had been replaced by a larger one of 400 cm^3 so that the normal volume with which we worked, amounted to about 500 cm^3. The small reservoir in the cryostat had a capacity of 20 cm^3, so that our greatest density amounts to about 25 Amagat-units.

The following table gives the values found by us. The B_A's, which can be calculated from these, have been published in the preceding communication [2]). The last column gives the differences between the observed values of pv_A, and those which have been calculated with the just mentioned B_A's.

As could also be expected from an estimation of that term using the reduced virial coefficients VII 1 [3]), it appears from the column $O—C$ that the term with C_A in the development of pv_A to ascending powers of v_A^{-1} does not come into account in the region of temperatures and pressures treated here. It is evident that this benefits the exactness of the determination of B_A [4]).

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[1]) G. P. NIJHOFF and W. H. KEESOM, These Proc. **28**, 963, 1925, Comm. Leiden N⁰. 179*b*.

[2]) G. P. NIJHOFF and W. H. KEESOM, These Proc., page 404; Comm. Leiden N⁰. 188*b*.

[3]) H. KAMERLINGH ONNES and W. H. KEESOM, Comm. Leiden Suppl. N⁰. 23, § 36.

[4]) For the interest which exact measurements of the isotherms of gases and in connection with this, the determination of the attraction quantity of VAN DER WAALS have for the theory of absorption, compare B. ILIIN, Phil. Mag. (6) **48**, 193. 1924. Zs. f. Phys. **33**, 435, 1925.

TABLE I.

θ °C.	p int. atm.	pv_A	d_A	$O-C\,(pv_A)$
−103.30	14.242	0.62988	22.610	+0.0002
	9.910	.62743	15.792	+ 2
− 146.62	9.6830	.46792	20.694	
− 224.94	4.4156	.17769	24.848	0.00000
	4.1425	.17759^5	23.326	0
	2.7909	.17721	15.747	+ 6
− 235.77	3.4229^5	.13737	24.966	+0.00004
	2.9017	.13720	21.150	− 1
	2.4784	.13713	18.074	+ 1
	2.1585	.13709	15.745	+ 4
− 249.80	2.1315^5	.085008	25.075	+0.00002
	2.0766	.085017	24.426	+ 1
	1.9183	.085027	22.561	+ 2
	1.7267	.085035	20.308	− 3^5
− 252.57	1.8842	.074865	25.168^5	+0.00002
	1.8816	.074865	25.133	+ 2
	1.8373	.074865	24.542	+ 1
	1.4710	.074882^5	19.644^5	− 2
− 255.84^5	1.5793	.062504^5	25.249	−0.00001
	1.5430	.062533	24.661	0
	1.4611	.062565	23.326	+ 0^5
	1.3109^5	.062673	20.299	+ 4
− 258.99	1.28305	.050756	25.279	+0.00001
	1.12572	.050850^5	22.075	0

Physics. — *Isotherms of di-atomic substances and their binary mixtures.* XXXIV. *Isotherms of hydrogen at temperatures of 0° C. and +100°* C. By G. P. NIJHOFF and W. H. KEESOM. (Comm. N⁰. 188*d* from the Physical Laboratory at Leiden.)

(Communicated at the meeting of December 17, 1927).

The isotherms of +100° C. and 0° C. have been measured with a piezometer with a capacity of 1500 cm³, the same with which VAN URK and KAMERLINGH ONNES [1]) have measured part of their nitrogen isotherms. To this we connected a reservoir with a capacity of 33.6 cm³. The pressures were measured with the aid of the closed manometer M 60. The temperature of 0° C. was obtained with the aid of finely planed ice made from water of the main, whereas for that of + 100° C. the steam apparatus described by KAMERLINGH ONNES in Comm. N⁰. 27 [2]) was used, still in the old shape [3]). The temperature of the vapour in this apparatus was measured with a BECKMANN-thermometer of which the steam point had been determined separately.

Beforehand, for the sake of control, we have first measured three points at +20° C., which agreed well with SCHALKWIJK's isotherm. In the last column of table I we give the differences between the observed pv_A's and the values of pv_A calculated with the aid of values of B_A and C_A, which we communicate in table II and which, in order to make them correspond as well as possible, we chose somewhat differing from the values of SCHALKWIJK [4]) and from the later ones of KAMERLINGH ONNES, CROMMELIN and Miss SMID [5]).

Concerning the isotherms of +100° C. and 0° C., the region of pressures in which we have measured does not seem to be too favourable for the determination of the values of B; for the C plays a not to be neglected part here.

For 100° C. we determined B_A and C_A as follows. HOLBORN and OTTO [6]), who have measured to about 100 atmospheres, give for +100° C. in their development according to ascending powers of the pressure, only a second term, whereas they don't want a quadratic term. If we compare

[1]) A. TH. VAN URK and H. KAMERLINGH ONNES, Comm. Leiden N⁰. 169*d*. For the calibration see also A. TH. VAN URK, Thesis Leiden.

[2]) H. KAMERLINGH ONNES, Verslagen Kon. Ak. v. Wet. Amsterdam **5**, 79, 1896, Comm. Leiden N⁰. 27.

[3]) Compare W. H. KEESOM and Miss H. V. D. HORST, These Proc. **30**, 970, 1927; Comm. Leiden N⁰. 188*a*.

[4]) J. C. SCHALKWIJK, These Proc. **4**, 107, 1902; Comm. Leiden N⁰. 70.

[5]) H. KAMERLINGH ONNES, C. A. CROMMELIN and Miss E. I. SMID, These Proc. **18**, 465, 1915, Comm. Leiden N⁰. 146*b*.

[6]) L. HOLBORN and J. OTTO, Zs. f. Physik. **33**, 1, 1925.

their development according to p with a development in series according to d_A, then from the coefficients of the first development in series, we can calculate the corresponding coefficients of the last mentioned series. In this way we calculated the value of B_A and C_A communicated in table II. In the last column of table I we give the differences between the observed values of pv_A and the values, which we calculated with the values just mentioned of B_A and C_A. We conclude from these that the values of B_A and C_A derived from HOLBORN and OTTO give sufficient correspondence also for our measurements.

As the value $C_A = 0.606 \times 10^{-6}$ calculated from the measurements of AMAGAT for the same temperature, is only little different, we can also put some trust into the value used by us $C_A = 0.635 \times 10^{-6}$.

For the isotherm of 0° C. we find the best correspondence with $B_A = 0.605 \times 10^{-6}$ and $C_A = 0.565 \times 10^{-6}$, whereas from the measurements of HOLBORN and OTTO follows $B_A = 0.620 \times 10^{-3}$ and $C_A = 0.760 \times 10^{-6}$, and from AMAGAT has been calculated for $C_A = 0.670 \times 10^{-6}$.

The measured quantities are following here:

TABLE I.

θ °C.	p int. atm.	pv_A	d_A	$O-C\,(pv_A)$
+20	32.006⁵	1.0947	31.063⁵	−0.0003
	40.098	1.0990	36.485	0
	45.771⁵	1.1021	42.527	− 6
+100	39.964	1.3929	28.691	− 2
	43.852	1.3951	31.431⁵	+ 3
	48.746	1.3987	34.850	− 0⁵
	54.603	1.4028	38.924	0
	59.391	1.4061	42.237	+ 1
0	32.312⁵	1.0188	31.715	−0.0003⁵
	32.323	1.0190	31.721	− 1⁵
	33.524	1.0194	32.885	− 5
	34.875	1.0206	34.171	− 1
	36.306⁵	1.0217	35.536	+ 1
	37.883	1.0226	37.047	0
	39.545⁵	1.0231	38.652	− 5
	42.905⁵	1.0260	41.817	+ 3
	44.085	1.0264⁵	42.949	+ 0⁵
	44.119	1.0266	43.284	0

In the following table we collect the values of B_A and C_A found by the different observers.

TABLE II.

	$B_A \cdot 10^3$	$C_A \cdot 10^6$
0° C.		
AMAGAT	0.669	0.670
KAMERLINGH ONNES and BRAAK	0.580	0.670
WITKOWSKI	0.619	
CHAPPUIS	0.605	
HOLBORN and OTTO	0.620	0.760
VERSCHOYLE	0.626	0.560
NIJHOFF and KEESOM	0.605	0.565
20° C.		
SCHALKWIJK	0.667	0.993
KAMERLINGH ONNES, CROMMELIN and Miss SMID	0.657	1.119
VERSCHOYLE	0.698	0.533
NIJHOFF and KEESOM	0.677	0.797
100° C.		
AMAGAT	1.057	0.606
WITKOWSKI	0.920	
HOLBORN and OTTO	0.937	0.635
KAMERLINGH ONNES and BRAAK	0.863	0.606
NIJHOFF and KEESOM	0.937	0.635

In fig. 1 the most important values of B_A for this temperature region have been indicated.

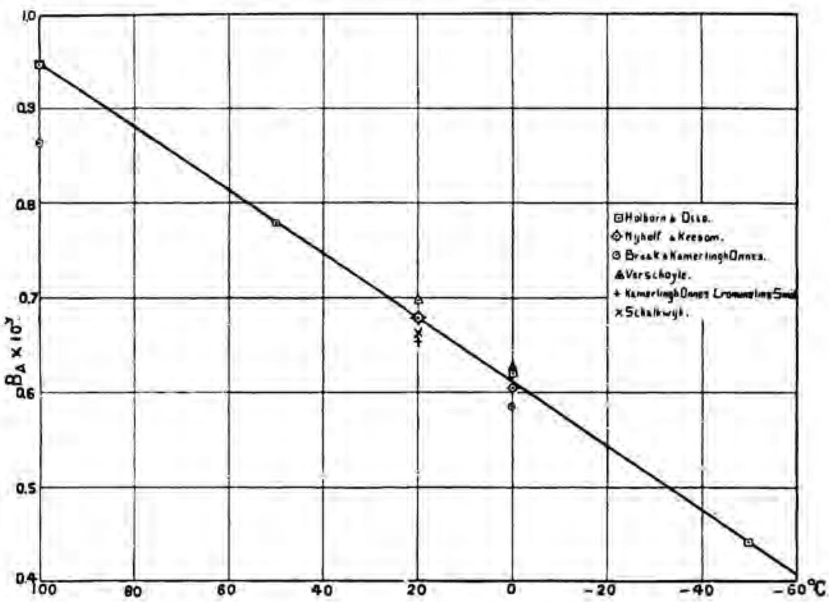

Fig. 1.

Physics. — *Isotherms of di-atomic substances and their binary mixtures.* XXXV. *Isotherms of hydrogen at temperatures of —225.5° C. to —248.3° C. and pressures of 1.6 to 4.2 atmospheres.* By G. P. NIJHOFF and W. H. KEESOM. (Comm. N⁰. 188e from the Physical Laboratory at Leiden.)

(Communicated at the meeting of January 28, 1928).

In the same way as has been indicated in a preceding communication [1]) we have measured isotherms of hydrogen at temperatures of —225.5° C. to —248.3° C. The piezometer used for this purpose was the same as the one used in the previous case, while the pressures were measured with the closed manometer as was also described in Communication N⁰. 188*b*. The temperatures were obtained by the hydrogen-vapour-cryostat.

From the measurements which are given below, we have found the following values for B_A:

θ °C.	$B_A \cdot 10^3$
—225.54	—0.268
—231.52	.309
—236.56	.340
—241.84	.390
—248.32	.439

The deviations of the observed values of pv_A, calculated with the aid of the given B_A-s are added in the last column of the following table.

The values of B_A mentioned here, agree well with those which VAN AGT and KAMERLINGH ONNES [2]) have measured with the aid of the thermometer of constant volume and variable density at temperatures of liquid hydrogen.

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[1]) G. P. NIJHOFF and W. H. KEESOM, These Proceedings p. 404, 1928, Comm. Leiden N⁰. 188*b*.

[2]) F. P. G. A. J. VAN AGT and H. KAMERLINGH ONNES. These Proceedings **28**, 614, 1925, Comm. Leiden N⁰. 176*b*.

θ °C.	p int. atm.	pv_A	d_A	$O-C\ (pv_A)$
—225.54	2.7313	0.16982	16.086	+ 0.00012
	3.4419	.16855	20.417	+ 1
	3.9391	.16772	23.481	0
	4.1727	.16727	24.950	— 5
—231.52	2.4830	0.14676	16.915	— 0.00010
	2.8937	.14593	19.830	— 5
	3.4443	.14477	23.786	+ 1
	3.6201	.14440	25.063	+ 3
—236.56	2.0808	0.12808^5	16.246	— 0.00008
	2.4734	.12695	19.483	— 11
	2.7297	.12628	21.616	— 5
	3.0178	.12565	24.001	+ 13
	3.1453	.12523	25.101	+ 8
—241.84	2.5531	0.10487	24.346	0.00000
	1.7959	.10773	16.665	— 13
	2.0727	.10682	19.399	+ 3
	2.3408	.10577	22.121	+ 4
—248.32	1.6290	0.08185	19.925	— 0.00005
	1.7349	.08128	21.303	— 2
	1.9614	.07984	24.581	— 1
	2.0078	.07968	25.209	+ 10

Geology.— *The potential energy of the gas in the oil bearing formations.* By J. VERSLUYS.

(Communicated at the meeting of May 26, 1928).

Mineral oil accumulated in pools in the earth is as a rule saturated with gas at the prevailing pressure and in many cases some free gas is still found in the highest parts of the oil bearing anticlines and domes. If in some cases the oil was not saturated with gas at the initial pressure, the pressure would decline in the vicinity of the borehole as soon as the oil bearing stratum was struck and some oil and gas had escaped. On account of this some gas would be set free in the vicinity of the borehole and even the pressure of the edgewater would not re-establish the original pressure near the borehole, until all liberated gas had been dissolved again. So for some time the pressure observed would be approximately equal to that at which the oil would be saturated with the absorbed gas and this would be the pressure measured in the borehole.

If an oil has absorbed gas and afterwards the pressure is reduced a part of this gas can be set free and expand after it has been set free. Owing to this the gas is able to exert energy and this energy is supposed to yield the principal force expelling the oil from the porous rocks into the wells.

The object of this paper will be to find a mathematical expression for the extent of the potential energy present in this form. The work performed by the gas if it is set free and expands is supplied by the molecular energy and cooling would be the result. Assuming that the heat needed to restablish the initial temperature is supplied immediately we may accept that the process is isotherm and further we will disregard all deviations from the laws of BOYLE and HENRY.

Should a volume of oil q be under a pressure of p atmospheres and should it be saturated with gas at that pressure, the coefficient of absorption being a, that volume of oil would contain a quantity of gas, which would occupy at atmospheric pressure a volume

$$a\,q\,p. \qquad (1)$$

If the pressure declines by dp a certain quantity of gas would be set free, occupying at the atmospheric pressure a volume

$$aq\,dp \qquad (2)$$

and at the pressure p under which the oil and gas are:

$$aq\frac{dp}{p} \qquad (3)$$

The volume of the oil and the gas associated with it, being q at the beginning, is increased by the volume expressed under (3). The work performed by the gas is then:

$$dA_1 = a\, \alpha q\, dp, \quad \ldots \ldots \quad (4)$$

if the pressure of one atmosphere equals a units of power per unit of area.

If the pressure declines from P_2 at which the oil is saturated with the gas it contains, to a pressure P_1, in this manner, i.e. at being set free, the gas will perform an amount of work:

$$A_1 = a\, \alpha q\, (P_2 - P_1) \quad \ldots \ldots \quad (5)$$

As the pressure decreases between those limits P_2 and P_1 the gas liberated while the pressure declined from p to $p—dp$ ($P_2 > p > p—dp > P_1$), will still expand owing to the pressure declining from $p—dp$ to P_1. The work performed by the quantity of gas, which would occupy the unit of volume at atmospheric pressure, should the pressure decline from p_2 to p_1 is:

$$a \log \frac{P_2}{P_1} \quad \ldots \ldots \quad (6)$$

according to a familiar formula.

Hence the quantity of gas set free between the limits of pressure p and $p—dp$ would, by expansion owing to the decline of pressure from $p—dp$ to P_2, perform work:

$$dA_2 = a\, \alpha q \log \frac{p}{P_1}\, dp \quad \ldots \ldots \quad (7)$$

The work performed by the expansion of the gas set free if the pressure declines from P_2 to P_1 is:

$$A_2 = a\, \alpha q \left\{ \int_{P_1}^{P_2} \log p\, dp - \log P_1 \int_{P_1}^{P_2} dp \right\} = a\, \alpha q \left\{ P_2 \log \frac{P_2}{P_1} - (P_2 - P_1) \right\}. \quad (8)$$

Hence, the total energy exerted by the gas, if the pressure declines from P_2 at which the oil is saturated to a smaller pressure P_1, is:

$$A = A_1 + A_2 = a\, \alpha q\, P_2 \log \frac{P_2}{P_1} \quad \ldots \ldots \quad (9)$$

The product $\alpha q\, P_2$ in this equation is the volume which would be occupied by all the gas originally absorbed in the oil at atmospheric pressure. If this volume be put at

$$\alpha q\, P_2 = Q_2 \quad \ldots \ldots \quad (10)$$

(9) becomes converted into:

$$A = a\, Q_2 \log \frac{P_2}{P_1}. \quad \ldots \ldots \quad (11)$$

A represents the energy which the gas being set free from the oil between the limits of pressure P_2 and P_1 exerts in two manners; viz. owing to the liberation and owing to the expansion. We must keep in mind that the gas still remaining absorbed in the oil would at atmospheric pressure occupy a volume

$$Q_1 = aq\,P_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and the gas set free between the limits of the pressure P_2 and P_1, would at atmospheric pressure occupy a volume:

$$aq\,(P_2 - P_1) = Q_2 - Q_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence the energy expressed by (11) is exerted by a quantity of gas $Q_2 - Q_1$ in the two manners exposed above.

But the work expressed by (11) also equals the work which would be performed by a quantity of free gas Q_2 in expanding between the same limits of pressure P_2 and P_1 according to (6).

So we have deduced: if a volume of oil q at the pressure P_2 and P_1 ($P_2 > P_1$) would be saturated by quantities of gas occupying volumes respectively Q_2 and Q_1, at atmospheric pressure, the volume of gas set free if the pressure after saturation declined from P_2 to P_1, would occupy a volume $Q_2 - Q_1$ at atmospheric pressure and this quantity of gas would during this process perform work equal to that performed by all the gas (Q_2) associated with the oil if it were free and expanded between the same limits of the pressure P_2 and P_1, this work being expressed by (11).

If the oil is — which often is the case in the highest parts of the structure — at the pressure P_1 not only associated with the quantity of gas Q_2, saturating it at this pressure, but also with a quantity of free gas, occupying a volume Q_f at the atmospheric pressure, this free gas would if the pressure declined from P_2 to P_1 exert an energy

$$a\,Q_f \log \frac{P_2}{P_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and the total work performed by the quantities of gas Q_2 and Q_1 would be

$$a\,(Q_2 + Q_b) \log \frac{P_2}{P_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

So in general for the work performed by all the gas (absorbed and free) associated with the oil in a pool if the *reservoir pressure ("rock-pressure")* P_r be smaller than or equal to the pressure at which the oil would be saturated with the gas present, may be written

$$aQ \log \frac{P_r}{P_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

if P_o represents the pressure under which the oil and the gas leave the formation and Q the volume, all the gas, free and absorbed in the oil would occupy at atmospheric pressure.

Geology. — *Determination of the pressure in gas containing strata.* By J. VERSLUYS.

(Communicated at the meeting of May 26, 1928).

The pressure in the oil bearing strata is a factor of importance in oil recovery, because from this pressure the quantity of gas dissolved in the oil can be calculated. In a porous rock containing liquid without gas the pressure can be computed from the specific gravity of the liquid and the level to which it rises in a well. The pressure in strata holding only gas may be deduced from the pressure of the gas in a closed well near the surface.

Generally it is accepted that the pressure exerted by the weight of the gas in the well may be neglected which would mean that the pressure at the bottom would be the same as the pressure at the head of the well. This however is not by any means the case, in deep wells great pressures occur. The specific gravity of the gas under a great pressure may not always be neglected. The initial pressure of the gases and fluids in the pores of the rocks as a rule does not differ much from the hydrostatic pressure of a column of water which would fill the borehole up to the surface. This was stated i.a. by J. H. GARDNER (III) in 1916, the phenomenon has since been observed in many districts and is mentioned by several authors.

According to this observation, in a sand at a depth of 1000 metres the pressure would be about 100 atmospheres. With this pressure a gas, the specific gravity of which in grams/cub. centimeter is 0.001 at atmospheric pressure, would have a specific gravity of $^1/_{10}$ of that of water. If 100 atmospheres pressure are measured at the head of a closed well the weight of the column of gas would equal that of a column of water of 100 metres, so there must be added about 10 atm. to the pressure at the head of the well and the gas pressure in the sand would be about 110 atmospheres. These figures are arbitrarily chosen, but it is clear that the pressure in the well-head can considerably differ from the pressure in the earth layer.

The correction to be applied to the measured gas pressure at the head of a well can be simply deduced. If the pressure at a height y above a certain level in the well is p atmospheres and the specific gravity of the gas γ, then the pressure exerted by the column of the gas enclosed by the levels y and $y + dy$ in units of power per unit of area is

$$p\gamma \, dy \qquad (1)$$

If the pressure of one atmosphere is equal to a units of power per unit of area, we may write:

$$p\gamma \, dy = -a \, dp \quad \ldots \ldots \ldots \quad (2)$$

or

$$dy = \frac{a}{\gamma} \frac{dp}{p} \quad \ldots \ldots \ldots \quad (3)$$

Integrating this equation between the limits y_2 and y_1, respectively p_2 and p_1, we get:

$$y_2 - y_1 = -\frac{a}{\gamma} \log \frac{p_2}{p_1} \quad \ldots \ldots \ldots \quad (4)$$

and if y_1 and y_2 be taken for the heights of the gas-bearing stratum and the head of the well $y_2 - y_1 = D$ is the depth of the well. So

$$D = \frac{a}{\gamma} \log \frac{p_1}{p_2} \quad \ldots \ldots \ldots \quad (5)$$

or

$$p_1 = p_2 \, e^{\frac{D\gamma}{a}} \quad \ldots \ldots \ldots \quad (6)$$

Hitherto p_2 and p_1 have been expressed in atmospheres; this formula (6), however, holds if any other unit of pressure be chosen.

If D be expressed in meters, then $a = 10.33$ and then the pressure p_1 in the gas-bearing sand would be computed from the formula:

$$p_1 = p_2 \, e^{\frac{D\gamma}{10.33}} \quad \ldots \ldots \ldots \quad (7)$$

in which p_2 is the pressure in the closed well, measured at the surface, γ the specific gravity of the gas in grams per cubic centimeter and D the depth of the well in meters. If the depth be expressed in feet and the specific gravity in pounds per cubic foot this formula becomes

$$p_1 = p_2 \, e^{\frac{D\gamma}{2093.6}} \quad \ldots \ldots \ldots \quad (8)$$

This pressure p_1 is generally called "*rockpressure*" which term however in recent times is being replaced by the more correct expression "*reservoir pressure*" (see I page 363 and II).

If we multiply the volume of a gas pool by its average porosity and the reservoir pressure p_1, we find the amount of gas contained in the pool disregarding phenomena of adsorption.

Should gas and liquid be associated in a porous bed, the reservoir pressure cannot be calculated from the pressure at the head of the closed well, this as a rule partly being filled with liquid. W. B. HEROY (I, p. 371) states that in literature many very low figures for the reservoir pressure are encountered, these figures practically giving the pressure read at the head of the closed well (the so-called closed pressure).

In W. B. HEROY's paper two methods are described of determining the reservoir pressure in oil-bearing strata. One of these methods (see I, p. 366) is to pump gas into the well, forcing the liquid from the well into the porous rock. The gas pressure required to produce this result is the pressure p_2 in (7) and (8) and the reservoir pressure is p_1 computed with the aid of one of these formulae.

Multiplying the volume of oil in the subterranean reservoir by the absorption coefficient and the reservoir pressure, according to the HENRY law, we get the amount of gas absorbed in the oil.

LITERATURE.

I. W. B. HEROY: "Rockpressure". *Bull. Am. Ass. Petr. Geol.*, XII, pp. 355—384, 1928.

II. Communication from the "SUBCOMMITTEE TO CALIFORNIA OPERATORS GENERAL COMMITTEE ON GAS CONSERVATION", entitled: „Increase oil recovery, conserve gas", in *The Oil and Gas Journal*, May 10th 1928, pp. 38, 142 and 143.

III. J. H. GARDNER: "The Mid-Continent oil fields", *Bull. Geol. Soc. Am.*, XXVIII, pp. 700—702, 1916.

Bacteriology. — *Von dem Stoffwechsel und der Verbreitung der Gärungssarcinen. (Sarc. ventriculi* GOODSIR *und Sarc. maxima* LINDNER.) By JAN SMIT. (Communicated by Prof. W. SCHÜFFNER.)

(Communicated at the meeting of March 31, 1928)

Durch die vortreffliche Untersuchung der Gärungssarcinen, die Herr Prof. BEIJERINCK zusammen mit Herrn Dr. GOSLINGS in den Jahren 1905 und 1911 [1]) verrichtete, wurde die damals ebenso neue als überraschende Tatsache festgestellt, dass man Kulturen der Magensarcinen (*Sarc. ventriculi* GOODSIR), deren Anwesenheit bei einigen Magenstörungen man schon seit 1842 kannte, die aber immer für unkultivierbar gegolten hatten, sehr leicht gewinnen kann, indem man von Gartenerde ausgeht, wobei die zuckerhaltige Nahrungsflüssigkeit (Malzwürze oder Bouillon mit Zucker) so stark angesäuert wird (mit 8 cc N. Phosphorsäure oder 6—7 cc N. Salzsäure pro 100 cc), dass sie nur noch für die in der Erde befindlichen Sarcinen geniessbar ist. Nach 24 Stunden bei 37° verrät sich ihr Wachtstum durch eine sehr starke Gasbildung, und das mikroskopische Präparat des Erdsatzes weist eine grosse Menge sehr grosser Sarcinenpakete auf, deren Bild vollkommen übereinstimmt mit dem der im Magen aufgefundenen Sarcinen.

Anderseits wurde von LINDNER [2]) und HENNEBERG [3]) auf die Tatsache hingewiesen, dass sich in spontan gegorenen Mehlbreien und in „buttersauren Maischen" bisweilen ebenfalls eine grosszellige Sarcine findet, die äusserlich stark der Magensarcine gleicht, von genannten Forschern aber nicht näher untersucht wurde, wahrscheinlich da sie sich nicht rein züchten liess.

In den genannten Veröffentlichungen BEIJERINCKS wird diese Sarcinenart mit denen aus Boden und Magen identifiziert, zwar ohne näheren Beweis. Es kam mir daher wichtig vor, diese Sarcinen dreifacher Herkunft einer eingehenderen, vergleichenden Untersuchung zu unterwerfen.

Es war dazu an erster Stelle notwendig, obengenannte Säuregrade der von BEIJERINCK benutzten Kulturflüssigkeiten auf eine besser zu vergleichende Basis zu stellen, durch die Bestimmung der Wasserstoffionenkonzentration, wobei der Anreicherungsversuch mit Erde am besten gelingt und auch durch Beobachtung der Widerstandsfähigkeit der Reinkultur der 3 Mikroben verschiedenen Säuren gegenüber.

Als Resultat dieser anderweitig [4]) veröffentlichten ausführlichen Unter-

[1]) Siehe diese Proceedings, **7**, 580 (1905) und **13**, 1237 (1911).

[2]) LINDNER, Mikroskop. Betriebskontrolle in den Gärungsgewerben 3[0] Ed. S. 342.

[3]) HENNEBERG, Gärungsbakt. Praktikum S. 100.

[4]) Siehe Ned. Tijdschr. v. Hygiene I, 201 (1927) und II, 210 (1927).

suchung erwähne ich hier nur, dass bei Anwendung von Malzwürze, angesäuert mit HCl bis zu einem p_H Wert von 1.3—1.5 (8—10 cc Norm. HCl pro 100 cc Würze) gewöhnlich die besten Erfolge erzielt werden, dass aber auch dann zahlreiche Erdmuster kein Wachstum der Sarcine veranlassen, welche Misserfolge bei Anwendung von Phosphorsäure (bis p_H = ca. 2—2.2 oder 10—15 cc N. Säure per 100 cc) noch bedeutend häufiger waren. Ausser den genannten Säuren waren auch Salpetersäure, Schwefelsäure und Milchsäure mehr oder weniger geeignet, während Essigsäure und Oxalsäure Misserfolge herbeiführten. Von einer so weitgehenden Widerstandsfähigkeit freier Salpetersäure gegenüber (p_H = 1.5 war ungefähr die Grenze) wird dies wohl das erste Beispiel sein.

Mit den Reinkulturen der Sarcinen liessen sich die Grenzen noch genauer bestimmen. Es zeigte sich, dass Ansäuerung mit Salzsäure ertragen wurde bis zu einem p_H von 0.8 (20 cc Norm. pro 100 cc), Salpetersäure bis zu 1.5, Milchsäure bis 2.9. Die Widerstandsfähigkeit gegen Phosphorsäure ist sehr gross: 70 cc N. per 100 cc wird noch ausgehalten, wobei aber das p_H nur bis auf 1.45 gefallen ist.

Auf der alkalischen Seite liegt die Grenze bei p_H = 9.8. Das Wachstumsgebiet liegt also innerhalb sehr weiten Grenzen.

Zwei Kennzeichen fanden sich, wodurch *S. maxima* sich von den beiden andern unterschied, die indertat in all ihren Eigenschaften sich als identisch herausstellten und daher unter dem Namen *S. ventriculi* zusammengefasst bleiben dürfen. Es waren: die Zellulose-reaktion mit Chlorzinkjodium, die für *S. ventriculi* positiv, für *S. maxima* negativ ausfällt, und der Stoffwechsel, der sehr grosse Unterschiede aufweist.

Schon BEIJERINCK hat *S. ventriculi* sehr richtig als obligate Zuckermikrobe beschrieben, welche die Eiweisse von Pepton, Bouillon, Hefenwasser, Malzextrakt u.ä. braucht, um wachsen zu können, daneben aber noch Zucker (Glukose, Laevulose, Saccharose, Laktose [1]), Maltose) erfordert. Sie ist aber nur zur Verarbeitung weniger Prozente Zucker, 1½ bis 2 %, imstande, sodass die meistens stürmisch einsetzende Gärung nach wenigen Tagen aufhört, während noch genügend Stickstoff vorrätig ist sowie aller Zucker, der über genannten Betrag hinaus hinzugefügt sein dürfte. Es zeigte sich, dass *S. maxima* noch weniger Zucker verarbeiten konnte: nur ½ bis 1 %. Ueber die Stoffwechselprodukte war aber so gut wie nichts bekannt. Ausser den beiden Gasen Kohlensäure und Wasserstoff (Verhältnis ihres Volumens ungefähr 3 : 1) findet man bei BEIJERINCK Milchsäure als Produkt erwähnt. Es zeigte sich mir aber, dass davon entweder nichts oder nur wenig entsteht, während dagegen ungefähr 10 % vom verschwundenen Zucker an Essigsäure festgestellt wurde. Die Hauptprodukte der Zuckerumwandlung von *S. ventriculi* sind aber Kohlensäure und Alkohol in Mengen, die sich der der Alkoholhefe nähern. Zufügung von Kreide hat

[1]) Dass BEIJERINCK diesen Zucker zu den nicht brauchbaren rechnet, muss auf einem Irrtum beruhen. Sowohl in Roh- als Reinkultur war die Vergärung der Laktose vollkommen normal.

auf die Mengenverhältnisse nur wenig Einfluss. Nur ist *S. maxima* imstande, 2 % Zucker zu verarbeiten.

Stoffwechsel der Sarcinen.

	S. ventriculi			S. maxima
Produkte (in Proz. des verschwundenen Zuckers)	Hefenwasser 2 % Glukose (verschwunden 1.52 %)	Hefenwasser 2 % Laevulose (verschwunden 1.94 %)	Peptonwasser 2 % Glukose (verschwunden 1.82 %)	Hefenwasser 2 % Glukose (verschwunden 0.6 %)
Kohlensäure	41.7 %	45.2 %	42.2 %	36.3 %
Wasserstoff	0.58	0.7	0.78	2.55
Aethylalkohol	40.3	50.6	40.2	Spore
Ameisensäure	1.08	—	2.4	1.05
Essigsäure	9.0	3.7	13.0	10.0
Buttersäure	—	—	—	37.1
Bernsteinsäure	—	—	—	3.44
Milchsäure	3.05	—	—	10.7
Acetylmethylkarbinol	Spore	0.06	1.2	—
Total	95.72 %	100.26 %	99.8 % [1]	101.14 %

In obiger Tabelle findet man die Zahlen vereinigt, auch für die Produkte von *S. maxima*, unter denen der Alkohol gänzlich fehlt, doch von einer grossen Menge Buttersäure ersetzt wird. Weiter findet sich auch Milchsäure unter den Produkten, während sich das Verhältnis der Kohlensäure zum Wasserstoff merklich verringert hatte (an Volumen 1 : 1.4), wodurch es sich zeigt, dass auch in dieser Hinsicht diese Sarcine den echten Buttersäurebakterien nahe steht (für *Gran. saccharobutyricum* findet DONKER [2]) Werte von ungefähr 1 : 1.2). Dagegen findet sich das Volumenverhältnis der Gase bei *S. ventriculi* gerade bei der fakult. anaeroben *Cl. polymyxa*, von der einige Stämme (u.a. DONKER's Stämme 11 und 4) auch weiter in ihrem Stoffwechsel einigermassen der Sarcine gleichen.

Oben erwähnte ziemlich zahlreiche Misserfolge der Anreicherungsversuche mit Erde führten mich zu einer nähern Untersuchung einer grossen Zahl von Erd-, Schlamm-, und Sandmustern aus verschiedenen Gegenden Hollands, nebst einigen Mustern getrockneter Erde, die mir das Kolonial-

[1]) Dank der Freundlichkeit von Herrn Prof. KLUYVER in Delft ist diese Analyse im dortigen Laboratorium mit dem grossen Gärungsapparat (60 L. Inhalt) geschehen, wofür ich ihm und seinen Assistenten, den Herrren Ing. VAN NIEL und LEEFLANG, zu Dank verpflichtet bin.

[2]) H. J. DONKER, Dissertation, Delft 1926.

institut in Amsterdam freundlichst überliess. Aus dieser Untersuchung liess sich folgern, *dass sich in allen oberflächlichen Erdschichten hierzulande und in Indien Sarcinen vorfinden*, dass aber verschiedene Bodenarten dafür oft ganz verschiedene Säuregrade der Anreicherungsflüssigkeit brauchen. Während manche Bodenarten angesäuerte Malzwürze verlangen und in ungesäuerter Flüssigkeit nur Bakterien wachsen lassen, ist für andere Arten jede Ansäuerung vom Uebel. Es stellte sich als möglich heraus, für jedes Bodenmuster der Erdoberfläche denjenigen Säuregrad auszusuchen, wobei Sarcinenentwicklung möglich war und also zu dem obigen Ausspruch von der Allgegenwart zu gelangen. Hier und in Indien in grösserer Tiefe genommene Muster stellen sich als sarcinenfrei heraus.

Die von diesen Tatsachen eingegebene Erklärung, als sollte ihre Anwesenheit an der Erdoberfläche mit der Düngung zusammenhängen, konnte nicht richtig sein, da sich einerseits in normalen menschlichen und tierischen Fäkalien nie Sarcinen zeigten, anderseits in gänzlich unkultiviertem Sandboden: Dünensand, sowohl am Strande als in den innern Dünen, diese Mikroben sich immer fanden.

Diese Wahrnehmungen waren die Veranlassung zur Untersuchung von Sandmustern verschiedener Herkunft: Sand aus einer öffentlichen Anlage in Amsterdam, für Strassenbauzwecke herbeigeführter Sand, Heidesand, und Sand aus einem Sandbett der Wasserleitung in Helmond. Negativ waren nur das unter den Plaggen gesammelte Muster Heidesand und einige Muster Flusssand, die sich durch Reinheit auszeichneten, und wobei man also die Abwesenheit einer Infektion mit dem Schmutz der Erdoberfläche voraussetzen muss, ebenso wie dies der Fall ist mit dem Muster Dünensand, an einer Stelle genommen, wo vor kurzem ein Sandrutsch stattgefunden hatte, wodurch auf einer Tiefe von ungefähr 2 M. eine Schicht blossgelegt worden war. Dagegen waren, wie bereits bemerkt wurde, die Muster des bewohnten und unbewohnten Meeresstrandes immer positiv.

Ferner konstatierte ich ihre Anwesenheit bis in tiefen Schichten des Wasserfilters in Helmond, der die letzten 4 Jahre ohne Sanderneuerung im Betrieb gewesen war und bei dem Infektion mit Erde nur möglich gewesen ist durch das Wasser selbst, das vor der Filtrierung ein langes, offenes Dekantationsbassin durchläuft. Es sieht danach aus, dass Wind und Staub bei der Verbreitung wohl einen bedeutenden Anteil haben. Dies müsste also ihre allgemeine Anwesenheit in der Luft mit sich bringen, was bisher nicht konstatiert worden war. Von vornherein stand das Vorkommen der schweren, grosszelligen Sarcinen in der Atmosphäre auch nicht zu erwarten. Dennoch liess sich ihre Anwesenheit darin aufzeigen, wozu ich eine Schüssel mit sterilem Sand auf einen offenen, vom Laboratorium durch eine Glastür getrennten Balkon stellte. Nachdem dieser dort ungefähr 3 Wochen dem Wind und Wetter ausgesetzt gewesen war, zeigte es sich, dass der erst negativ reagierende Sand positiv geworden war.

Zweifellos spielen Wind und Staub gleichfalls eine grosse Rolle bei der Verbreitung von *S. maxima* auf Getreidearten, wo sie sich so häufig zeigen. Um sich davon zu überzeugen braucht man nur verschiedene Kleie-arten, wie früher beschrieben wurde [1]), mit angesäuerter Saccharoselösung bei 37° zu kultivieren. Besonders günstig erwies sich Roggenkleie.

Die Tatsache, dass dies stets bis auf 1 Mal, *S. maxima* war, während aus allen Mustern Erde (2 ausgenommen), Schlamm, Abwasser, Sand und auch aus dem menschlichen Magen *S. ventriculi* isoliert wurde, ist zu auffallend, um nicht einen Augenblick dabei stillzustehen. Man könnte meinen, dass die verschiedenen Kulturflüssigkeiten (angesäuerte Malzwürze für Erde u.s.w., angesäuerte Saccharoselösung für Kleie) für das verschiedene Resultat verantwortlich sein könnten. Die Möglichkeit lässt sich indertat nicht leugnen, und folgende Wahrnehmung spricht auch für diese Auffassung.

Während ja aus einem bestimmten Kleiemuster, bei Kultivierung in der beschriebenen Weise in angesäuerter Saccharoselösung, nur einmal aus vielen *S. ventriculi* isoliert wurde, gewann ich diese Art dreimal hinter einander aus der gleichen Kleie, wenn sie in der Weise wie Sand, Erde u.s.w. in Malzwürze ($p_H = 2.2$) gezüchtet wurde. Die Kleie beherbergt also, wie zu erwarten war, beide Arten und die verschiedenen Nahrungsstoffe sind offenbar die Ursache der verschiedenen Ergebnisse. Dass sie dies nicht unter allen Umständen sind, beweisen obengenannte Ausnahmefälle mit Erde, wobei *S. maxima* auch in Malzwürze gewonnen wurde, wobei man noch die Wahrnehmung hinzufügen kann, dass auch aus dem halbverdauten Mageninhalt eines Kaninchens in dieser Weise *S. maxima* gezüchtet werden konnte. Dagegen ist es uns nie gelungen, diese Art aus Erde, Sand u.s.w. mit Saccharoselösung zu gewinnen, auch nicht wenn sterilisierte Kleie in der gebräuchlichen Menge hinzugefügt wurde. *S. maxima* scheint daher ausser auf Getreide-arten nicht stark verbreitet zu sein.

Ueber die Weise, wie *S. ventriculi* in den Magen kommt, kann wenig Zweifel bestehen. Dass sie kein ständiger Bewohner ist, steht fest: in gesunden Magen findet sie sich nicht und sie muss daher als ein Gelegenheitsgast betrachtet werden, mit der Nahrung mitgekommen, der seine Gegenwart nur kundtut, wenn er die Gelegenheit sich zu entwickeln bekommt, wie dies der Fall ist, wenn die Nahrung länger als normal im Magen verharrt und genügend freie Salzsäure anwesend ist. Infolge der oben angezeigten Anwesenheit in der Luft des Dunstkreises lässt sich das Eindringen in den Magen mit der Nahrung sehr gut erklären. Dass man mikroskopisch und kulturell immer *S. ventriculi* findet und nie *S. maxima*, bleibt inzwischen unaufgeklärt.

Die weite Verbreitung über die Erdoberfläche bringt mit sich, dass auch aus Amsterdamer Grabenwasser einige Male *Sarc. ventriculi* gezüchtet werden konnte; mit Abwasser gelang dies stets, ebenso wie mit Abwasserschlamm, „activated sludge", Meerschlamm aus der Südersee und ähnlichem Material. Ich halte es denn auch wohl für wahrscheinlich, dass die nie

[1]) l.c.

gezüchtete und nur einige Male abgebildete *S. paludosa* (SCHRÖTER) [1]) mit der Gärungssarcine identisch ist. Ich selbst nahm sie einige Male in Abwasser wahr und fand ihr Aeusseres und ihren Umfang indertat vollkommen gleich.

Von noch einer andern Sarcinenart ist die Anwesenheit in Schlamm und Abwasser beschrieben worden und zwar von der Methansarcine, die einige fettsaure Salze in Methan, Kohlensäure und Kohlensäuresalze umwandelt [2]). Ihr Aussehen ist einigermassen dem der Gärungssarcine ähnlich und es ist auffallend, dass man letztere auch stets antrifft in den mehr oder weniger gereinigten Rohkulturen der Methangärung. Diese Sarcinen hat man aber nie in Reinkultur erhalten, was mit der *S. ventriculi* wohl der Fall ist und letztere sind auch nicht zur Methangärung in dazu geeigneten Acetatflüssigkeiten imstande, wie es sich herausstellte. Ebensowenig konnte ich diese Gärung erhalten mit Sandmustern, mit denen man wohl Entwicklung der *S. ventriculi* bewirkt. Die Folgerung ist daher berechtigt, dass die letzte Art in den Methankulturen nur als Verunreinigung auftrete und dass beide Arten ganz gewiss verschieden seien.

Einen ganz andern Blick auf die Anwesenheit und die Verbreitung der Gärungssarcinen in der Natur gewann ich aber, als ich es versuchte, die Anwesenheit auch auf mikroskopischem Wege in den Materialen darzutun, in denen sie den Kulturexperimenten zufolge so häufig vorkommen. Erde und Kleie waren dafür, aus früher (l.c.) genannten Gründen ungeeignet.

Als ich nun die leichte Kultivierbarkeit aus allerlei Sandmustern gefunden hatte, wurden damit ähnliche Versuche in der Hoffnung auf bessern Erfolg wiederholt, besonders als es sich mir zeigte, dass man solche Muster unter einem tüchtigen Wasserstrahl reinwaschen und weiter einige Monate unter Wasser aufbewahren kann, ohne dass dadurch das Vermögen der Sarcinengärung eingebüsst wird. Ich hielt es daher für wahrscheinlich, dass in dem so behandelten, von allem feinen Schmutz befreiten Sande, in dem alle Organismen genügend Gelegenheit bekommen hatten, sich mit Feuchtigkeit zu sättigen, durch genaues Mikroskopieren Sarcinenpakete oder etwa Kokken von einer in diesen Paketen vorkommenden Grösse ($2\frac{1}{2}$ bis 3 μ), aufgezeigt werden müssten.

Diese Erwartung erfüllte sich aber keineswegs. Wurde der rohe Sand mit Wasser geschüttelt und nachdem er sich gesetzt, die obenstehende Flüssigkeit centrifugiert, so war zwar in den Anreicherungsversuchen das Centrifugat stets positiv, aber das mikroskopische Präparat trug ebenso wenig etwas ein als die langwierigen Versuche, in dem reinen, noch positiv reagierenden Sande die gesuchten Organismen zu sehen.

In der Hoffnung, in dieser schwierigen Sache etwas weiter zu schreiten,

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1) SCHRÖTER, Kryptogamenflora Schlesiens.
WILHELMI, Komp. d. Biol. Beurt. d. Wasser.
WEYL's Handbuch d. Hygiene, II. Band, 3. Atb.

2) N. L. SÖHNGEN, Dissertation, Delft 1906.

versuchte ich der Entwicklung der Sarcinen auf dem Fusse zu folgen, um auf diese Weise etwaige junge Entwicklungsstadien beobachten zu können. Während nun ungefähr 16 Stunden nach dem Anfang des Experiments die Gärung bemerkbar wurde und eine grosse Anzahl Sarcinenpakete sich gebildet hatte, konnten schon nach der 11. Stunde die ersten Pakete gefunden werden. Sie haben dann schon die Grösse von 8 oder mehr Kokken, während daneben nie Pakete von geringerem Umfang, und ganz bestimmt kein allgemeines Kokkenstadium, wahrzunehmen waren. Unsre Kenntniss über die Form, in der sie auf Naturmaterialen vorkommen, wird durch diese Versuche nicht erweitert. Man kann bloss mit Bestimmtheit sagen, dass diese Form eine andre und viel widerstandsfähigere sein muss als wir sie in unsern Reinkulturen kennen, denn da haben sie nur eine Lebensdauer von 2 Tagen, während Naturmateriale in unbeschränkter Zeitdauer für den Sarcinenversuch dienlich sind.

Man könnte dabei zunächst an Sporen denken. Die Zahl der sporenbildenden Sarcinenarten ist gering: BEIJERINCK erwähnt die Bildung bei *S. ureae*, während auch *S. pulmonum* (HAUSER) sporentragend zu sein scheint. Bei *S. ventriculi* hat man sie nie wahrgenommen, auch nicht SURINGAR bei seinen äusserst sorgfältigen Untersuchungen. Ich selbst habe sie auch nie finden können, weder in lebendigen noch in gefärbten Präparaten. In unsern Kulturen fehlt den Sarcinen gewiss die Zeit sie zu bilden, denn sie sind schon lange vorher abgestorben. Dass die Gelegenheit in der Natur besser wäre, lässt sich nicht dartun, solange ihr Lebenszyklus noch so ungenügend bekannt ist.

Man darf ruhig annehmen, dass bei Sarcinenpaketen, die in die Aussenwelt geraten, bald ein Absterben stattfindet, ohne Sporenbildung. Die baldige Sterilität der Sarcinenkulturen mit sterilisierter Erde oder sterilisiertem Sand, beweist das. Etwaige Sporen müssten also von der resistenten Form hervorgebracht werden.

Dass die resistente Naturform nicht die Sarcinenform sein kann wird noch durch nachfolgenden merkwürdigen Versuch bestätigt. Lässt man das Wachsen der Sarcinen in Malzwürze in Gegenwart von Sand, Kleie, Kreide oder Erde, die steril sind, stattfinden und befreit man nachher diese Stoffe durch wiederholte Waschungen mit sterilem Wasser von der Kulturflüssigkeit, so verrät das mikroskopische Bild dieser ausgewaschenen Stoffe die Anwesenheit von einer grossen Zahl Sarcinenpaketen. Impft man diese Stoffe in neue Nahrungsflüssigkeit, so lässt sich auch indertat die Sarcine wieder zum Wachstum bringen, und das gleiche ist bisweilen auch am folgenden Tage noch der Fall. Nach 48 Stunden sind aber die Pakete zu einer weitern Entwicklung nicht mehr imstande, während das mikroskopische Bild noch unverändert ist. Trocknet man den Sand nach dem Auswaschen der Nahrungsflüssigkeit, entweder an der Luft oder in einer Kohlensäureatmosphäre und bei niedriger Temperatur, so bleiben die Sarcinenpakete gut sichtbar, aber die Lebensdauer ist nicht länger als in nassem Zustande. Es gelingt nicht, durch Reiben in einem Mörser die Pakete im reingewaschenen Sande zu Kokken zu reduzieren. Wird der trockne Sand stark gerieben so werden sie wohl in Fetzen zerrissen, fallen aber nicht in Kokken aus einander. Verlängerung der Lebensdauer wird durch alles dies nicht erzielt. Dagegen ergiebt das Trocknen eines Musters ausgewaschenen Fluss- oder Meersandes ein Präparat, in dem sich kein einziges Paket entdecken lässt, das aber trotzdem, und sogar Jahre nachher, in angesäuerter Malzwürze eine reiche Sarcinenkultur veranlasst.

Unternimmt man den gleichen Versuch mit nicht-sterilem, sarcinenhaltigem Sand, so ist das Ergebnis das gleiche : sind einmal die Sarcinen zum Wachsen gebracht, so ist ihre Lebensdauer nicht länger als 2 Tage.

Infolge dieser starken Empfindlichkeit der Sarcinenkulturen gegen das Aufbewahren, die es nötig macht, die Reinkulturen alle 2 Tage zu übertragen, darf man, glaube ich, die Folgerung ziehen, *dass die Sarcinenpakete eine sehr empfindliche Form dieses Bakteriums darstellen, deren Abwesenheit man in denjenigen Naturmaterialen ruhig annehmen darf, die auch nach langem Aufbewahren noch positiv auf die Sarcinenprobe reagieren. Darin kommt sie offenbar in einer latenten und sehr widerstandsfähigen Form vor, deren mikroskopische Gestalt noch nicht festgestellt werden konnte. Lässt man sie sich zu Sarcinenpaketen auswachsen, so ist damit die Haltbarkeit auch ganz verschwunden.* Es gelingt auch nicht, die einmal gebildeten Pakete wieder zur haltbaren Form zurückzubringen : fügt man den lebenden Kulturen Erde, Sand,Kreide oder Kleie, die steril sein müssen, bei, so verlängert man die Lebensdauer nur unbedeutend. Nur wenn man die Röhren auf dem Höhepunkt der Gärung mit Kohlensäuregas anfüllt und sie darauf zuschmilzt, kann man die Lebensdauer auf wenige Wochen ausdehnen. Aber auch dieses Mittel ist nicht unfehlbar.

Es kam mir nun wichtig vor zu beobachten, ob sich auch in andrer Hinsicht ein Resistenzunterschied zwischen den Sarcinenpaketen und der latenten Form zeigte. Dazu wurde das Verhalten bei Erhitzung in beiden Fällen beobachtet. Die Reinkulturen der Paketform, in Malzwürze mit Kreide gewachsen, stellten sich als sehr empfindlich heraus : schon 30 Min. bei 50° und 10 Min. bei 55° C. war tödtlich. Der rohe Sand dagegen musste während 10 Min. auf 75° C. erhitzt werden, bevor er das Vermögen verlor, in saurer Malzwürze Sarcinenwachstum zu ergeben. Die Versuche wurden so ausgeführt, dass in Reagenzgläsern ungefähr 3 cc Sand mit 10 cc Wasser zusammengefügt wurden. Diese Reagenzgläser wurden dann in einem Wasserbad von der bestimmten Temperatur getaucht. Kontrollbeobachtungen ergaben, dass unter diesen Umständen innerhalb 3 Min., bei Schütteln in noch kürzerer Zeit, Wasser und Sand die Temperatur des Wasserbades angenommen hatten. Die angegebene Zeitdauer ist die des jeweiligen Untertauchens.

Um aber noch etwaige Unterschiede in der Resistenz durch die Anwesenheit des Sandes zu beseitigen, wurden die Versuche in etwas andrer Weise wiederholt. Einerseits wurde roher Sand mit Wasser geschüttelt und, nachdem er sich gesetzt, das Oberste abgegossen. Die sehr trübe Flüssigkeit wurde in Reagenzgläser gefüllt und ergab darin einen Niederschlag, dessen Volumen und Feinheit denen der Kreide in den Röhren der Reinkulturen zu vergleichen waren. Eine andre Reihe von Röhren wurde folgendermassen behandelt. Nachdem sich die Sarcinen in Röhren mit Malzwürze, Kreide und sterilem Sand entwickelt hatten, wurde die Flüssigkeit abgegossen und der Sand durch wiederholtes Waschen mit sterilem Wasser gereinigt. Er enthielt dann noch eine grosse Anzahl von Paketen. Es wurden nun wieder 10 cc Wasser in die Röhren geschenkt und die Behandlung fand weiter in der gleichen Weise statt wie bei den Versuchen mit rohem Sand. Ebenso wie dort, wurde das Wasser nach Erhitzung und geschwinder Abkühlung abgegossen und durch Malzwürze ersetzt.

Untenstehende Tabelle giebt eine Uebersicht von den Ergebnissen dieser 4 Versuchsreihen.

Resistenz von S. ventriculi gegen Erhitzung.

Zeitdauer	Temper.	Anzahl Röhren positiv			
		Reinkultur in Malzwürze und Kreide (Sarcinenform)	Satz von rohem Sand (latente Form)	Reinkultur in sterilem Sand (Sarcinenform)	Roher Sand (latente Form)
15 min.	50°	5 von 6	—	—	—
15 „	55°	keine	—	2 von 5	—
30 „	55°	keine	—	1 von 6	—
10 „	60°	keine	6 von 6	2 von 5	—
20 „	60°	keine	6 von 6	1 von 6	6 von 6
10 „	65°	keine	6 von 6	keine	6 von 6
10 „	70°	keine	6 von 6	keine	2 von 6
10 „	75°	keine	4 von 6	keine	keine
10 „	80°	keine	keine	keine	keine

Es erhellt daraus, dass die latente Form, sowohl im rohen Sande als im sandfreien Besatz, offensichtlich widerstandsfähiger ist als die Sarcinenform unter den gleichen Umständen.

Ferner musste noch die Frage beantwortet werden, ob vielleicht im Sande Organismen vorkommen, die, wiewohl sie selbst keine Sarcinenform besitzen, diese dennoch bei ihrer Entwicklung in saurer Malzwürze hervorbringen können. Dabei muss man den etwa aufsteigenden Gedanken, als hätte man hier mit Kunstprodukten zu schaffen, die in den stark sauren Flüssigkeiten entstünden, fahren lassen, wenn man bedenkt, dass auch aus Kleie in neutralem Zuckerwasser gute Kulturen von *S. maxima* zu erhalten sind, während auch die Reinkulturen in neutralen Flüssigkeiten vollkommen normal wachsen und gären. Wohl wäre es möglich, dass die Pakete die anaerobe Form aerober Organismen andrer Gestalt darstellten. Dass dafür nur die Kokkenform oder die Sarcinenform in Betracht kommt, liegt auf der Hand. Die Frage tat tich also auf: lassen sich in Sand oder sonstwo Kokken oder Sarcinen finden, die unter anaeroben Verhältnissen in Malzwürze das Bild und die Gasbilding der Gärungssarcinen zeigen?

Ich säte nun eine Menge gewaschenen Sand, bei der sich die Möglichkeit der Sarcinengärung herausgestellt hatte, in Platten und in hohen Säulen Malzagar aus. Unter der grossen Zahl von Kolonien, die ich erhielt, kamen viele Kokken vor, die isoliert und dann auf ihr Gärungsvermögen hin untersucht wurden. Daneben wurden alle in diesem Laboratorium anwesenden Arten von Mikrokokken und Sarcinen und noch einige Arten dieser letztern, die erhalten worden waren auf Agarplatten, welche der Luftinfektion ausgesetzt gewesen waren, in die Untersuchung hineinbezogen. Das Ergebnis war vollständig negativ. Mit einigem Staunen stellte ich die Tatsache fest, dass sich in keiner der Kulturplatten und in keiner der Agarröhren Kolonien der Gärungssarcinen fanden, wiewohl doch die Möglichkeit der Koloniebildung unter beiden Umständen feststeht, wofern man von gut gärenden Flüssigkeitskulturen ausgeht. Koloniebildung aus der latenten Form findet also offenbar viel schwieriger statt, obgleich man bei der Beurteilung dieser Frage zu bedenken hat, dass auch bei der Aussaat von gut gärendem Material ein sehr grosser Prozentsatz unverändert in den Platten liegen bleibt und nur ein sehr geringer Teil sich zu Kolonien auswächst.

Hat vielleicht die unsichtbare, latente Form der Sarcine die Abmessungen eines filtrierbaren Organismus? Wie unwahrscheinlich dies auch erscheinen mag, gerade bei diesem grosszelligen Organismus, es war dennoch notwendig, auch darüber Gewissheit zu erlangen.

Nun stellte es sich heraus, dass sich die Organismen von Material wie Sand nicht leicht loslösen: sogar wenn man den Sand mit Wasser in einem Mörser reibt, das obere schmutzige Wasser abgiesst und sich setzen lässt, so sind der ausgewaschene Sand und auch der Satz positiv, aber mit der klaren Flüssigkeit ist keine Sarcinengärung zu erzielen. Filtriert man dann auch das genannte schmutzige Wasser durch eine sterile Kerze, so bleibt mit diesem Filtrat die Gärung ebenfalls aus. So ist daher obenstehende Frage zu verneinen.

Zum Schlusse: ist es möglich, die Bedingungen zu finden, wobei sich in Stoffen wie Sand Zunahme der „latenten" Sarcinen zeigt? Diese Frage wurde von der Erwägung eingegeben, dass ein sporenfreier Organismus, er mag so widerstandsfähig sein wie er will, doch nicht so allgemein in der Natur vorkommen kann und dort unter so stark wechselnden Umständen sein Dasein zu jeder Zeit behaupten kann, ohne dass Vermehrung stattfindet, wodurch sich der Vorrat wieder anfüllt. Zweifellos wirkt der Wind mit bei der Verbreitung dieser Bakterie, doch auch dies erklärt ihre unbedingte Allgegenwart ungenügend, wenn man nicht auch auf ihre Vermehrung rechnen darf.

Sich auswachsen zu Sarcinen, wofür Zucker und „Pepton" nötig ist, kann hierbei keine Rolle spielen, da dies die Resistenz praktisch auf Null herabsetzt. Eine Vermehrung der Bakterie kann also nur dann eine bleibende erhöhte Anzahl bewirken, wenn auch die Nachkommenschaft die resistente Form besitzt. Vermehrung derselben wird sich zeigen, indem man zum Sarcinenversuch geeigneten Sand unter verschiedenen mehr oder weniger naturähnlichen Umständen aufbewahrt und jedesmal die Menge Sand bestimmt, womit ein positives Ergebnis dieses Versuches zu erhalten ist. Nähme diese Menge merklich ab, so wäre damit eine Vermehrung der latenten Keime bewiesen.

Versuche diese Vermehrung aufzuzeigen, wurden folgendermassen gemacht. Eine grosse Menge Sand wurde durch Waschen vom schlimmsten Schmutz befreit, nachher bei niedriger Temperatur getrocknet und gut gemischt. Von diesem Sand wurden Mengen von 50 gr. in sterilen Kolben mit 50 cc von einer der Lösungen, die man in untenstehender Tabelle findet, übergossen. Nach einer Woche, bei Zimmertemperatur, wurde dann die Flüssigkeit abgesogen und der Sand mit wenig Wasser nachgewaschen und wieder bei niedriger Temperatur getrocknet. Von den so erhaltenen Sandmustern wurden dann je 6 Portionen von 3 gr. und 6 Portionen von 2 gr. in sterilen Stöpselflaschen von 50 cc abgewogen und diese wurden weiter mit Malzwürze angefüllt, die mit Salzsäure angesäuert war zu $p_H = 1.75$. Die Tabelle giebt die Ergebnisse dieser Versuche, wobei noch zu

Art der Flüssigkeit	6 × 3 gr. Sand Anzahl posit.	6 × 2 gr. Sand Anzahl posit.	Mittl. Sarc. titer	Mikr. Bild nach Behandlung	Bemerkungen
1. Ursprünglicher Sand, gewaschen und getrocknet	3	1	7.5 gr.	keine Sarc.	
2. Leitungswasser	4	3	4.3 „	idem	Wasser blieb klar
3. idem + 0.01 % Pepton	3	0	10 „	„	sehr geringe Trübung
4. idem + 0.1 % Pepton	5	5	3 „	„	schwache Trübung
5. idem + 0.5 % Pepton	4	3	4.3 „	„	starkes Anwachsen der Bakterien (Faulung)
6. idem + 1 % Saccharose	3	1	7.5 „	„	Buttersäure gärung
7. idem + 0.1 % NH_4Cl	3	4	4.3 „	„	Flüssigkeit klar
8. idem + 0.1 % Asparagine	4	3	4.3 „	„	Flüssigkeit klar

erwähnen ist, dass der rohe, nicht ausgewaschene Sand schon mit 300—500 mg die Sarcine zum Wachsen brachte.

Aus diesen Zahlen lässt sich, dünkt mich, folgern, dass unter den gegebenen Verhältnissen von einer starken Vermehrung der latenten Sarcinen keine Rede ist. Es lässt sich nicht leugnen, dass das Aufbewahren unter einer 0.1 Proz. Peptonlösung das Ergebnis der Experimente verbessert (0.5 Proz. Pepton veranlasst ein zu starkes Wachstum der Bakterien, das eine Zunahme der latenten Sarcinen verhindert), aber auch durch Leitungswasser allein geht das Titer schon von 7.5 gr. auf 4.3 gr. rückwärts. Dass durch diese Behandlung keine Sarcinenpakete noch auch irgend welches Vorstadium derselben im Sand sichtbar werden, kann nicht wundernehmen, wenn man an die verhältnismässig grossen Mengen Sand denkt, die zur Sarcinenentwicklung notwendig sind.

Ueber die Lebensweise der Sarcinen in der Natur bringen diese Versuche nur wenig Aufklärung und die vielen Rätsel, welche die geheimnisvolle latente Form umgeben, bleiben gleichfalls ungelöst.

Laboratorium der Hygiene der Universität Amsterdam.

Amsterdam, im März 1928.

Physics. — *On* OSEEN's *theory for the approximate determination of the flow of a fluid with very small friction along a body.* By J. M. BURGERS. (Mededeeling N⁰. 9 uit het Laboratorium voor Aerodynamica en Hydrodynamica der Technische Hoogeschool te Delft.) (Communicated by Prof. P. EHRENFEST.)

(Communicated at the meeting of December 17, 1927).

§ 1. The determination of the flow of a viscous fluid along a body of given contour is one of the most important problems of hydrodynamics. When we consider the stationary flow along a body at rest and start from the equations of motion for an incompressible viscous fluid, the solution is required of a system of four partial differential equations with the velocity components and the pressure as variables. This solution has to satisfy the boundary conditions, which for the case are that the velocity of the fluid along the surface of the body is zero, and that at infinity the flow asymptotically approaches to a parallel motion with given constant velocity. In the following we will suppose that this latter velocity has the value V and is directed along the negative x-axis. Until now the rigorous solution of these equations presents unsurpassable difficulties. However, many applications of hydrodynamics do not require a solution of general validity, but a solution for the case in which the internal friction of the fluid is very small and may be put nearly equal to zero. We shall confine ourselves to this case.

The so called irrotational flow of DIRICHLET may be considered as the first attempt to find a solution of the problem before us. The deduction of this particular solution is based upon the principle that, when the internal friction of the fluid becomes infinitely small, no objections arise against the supposition that the fluid layers will glide over each other in some places. In consequence, even when the condition that the fluid in contact with the body must be at rest, is observed rigorously, we may accept a tangential velocity differing appreciably from zero at a small distance from the surface. The region in which this velocity appears is separated from the surface by a thin layer in which gliding occurs. Calculating the magnitude of the rotation of the flow in this layer by means of the well known formulae, we shall get a very high value; in other words a strong vortex motion is present in this layer. The thickness of this layer may be supposed the smaller as the friction is less.

The normal velocity component, in contrary to the tangential one, will not acquire an appreciable value at a very small distance from the surface, as this would be in contradiction with the equation of continuity. Therefore the boundary conditions may be simplified into the single condition that only the normal velocity component has to be zero.

We know that in a fluid without friction vortex motion once present cannot be destroyed, and that at the other hand in the interior of the fluid no vortex motion can arise. The flow being free from rotation at some distance in front of the body, we are led to suppose that it nowhere possesses vortex motion. In this case the flow can be characterized by a potential function. This potential, which we will call Φ_0, is deduced from LAPLACE's equation (which in itself is a consequence of the equation of continuity):

$$\triangle \Phi_0 = 0,$$

and from the boundary conditions:

$$\frac{\partial \Phi_0}{\partial n} = 0 \text{ along the surface of the body;}$$

$$\frac{\partial \Phi_0}{\partial x} = -V, \quad \frac{\partial \Phi_0}{\partial y} = \frac{\partial \Phi_0}{\partial z} = 0 \text{ at infinity.}$$

As is known, these equations admit a solution. Moreover it can be proved easily that a pressure field may be calculated from the velocity distribution in such a way that the general equations of motion are satisfied. The flow determined in this way will be called the DIRICHLET flow.

The experiments proved, however, that this DIRICHLET flow differs appreciably from the real state of motion unless we consider a very thin body; in particular the region of vortex motion occurring in the "wake" down stream is lacking in the theoretical solution, while the pressure distribution calculated from it gives zero resultant, which is in contradiction with the observed facts.

We thus have to seek other approximate solutions in which it is not supposed a priori that the entire flow is free from vortex motion. An example of such a solution is the well known discontinuous motion investigated by HELMHOLTZ, KIRCHHOFF, RAYLEIGH and others. We will not enter into a discussion of this solution, which is exposed in numerous text books and communications, and may be regarded as generally known [1]). On the contrary I would treat in the following lines the theory developed by OSEEN. OSEEN originally started from researches concerning the flow with great friction. In the last years, however, several communications

[1]) Compare f.i. H. LAMB, Hydrodynamics (Cambridge), art. 76, 77, 78; PH. FRANK u. R. VON MISES, Die Differential- und Integralgleichungen der Mechanik und Physik (Braunschweig 1927) II, p. 775 and seq.

both from OSEEN himself and from ZEILON have appeared, treating the case of a very small viscosity [2]).

Although OSEEN has given his theory for three dimensional as well as for two dimensional flow, and ZEILON has developed examples of both cases, I will confine myself in the following lines entirely to the two dimensional case, thereby aiming at the most simple formulation in order to exhibit the striking points as clear as possible.

§ 2. In order to understand the leading principles of the theory we must pay attention to the properties of vortex motion. From the hydrodynamical equations it follows that the vortices are transported by the fluid elements; the directions of the rotation vectors follow the changes in the orientation of the fluid particles, and the intensity of the rotation is modified in such a way that its product with the cross section of a fluid element retains the same value. From this follows that the strength of a vortex tube is not affected by the flow of the fluid. Moreover the action of the viscosity has to be taken into account, producing a diffusion of the vortex motion, which is gradually spread out over the entire field. The equations of motion show that no vortex motion can arise in the interior of the field as long as the motion remains regular; the only source of vortex motion has to be looked for in the forces which the surface of a solid body exerts upon the adjacent fluid particles.

We now confine ourselves to the two dimensional motion. Then all vortex vectors are perpendicular to the direction of the flow and we only have to discriminate between positive and negative vorticity; hence the property just mentioned is simplified into the following one: the vorticity is dragged along with the flow, does not alter its strength, but diffuses at the same time from the fluid particles to which it was bound to the surrounding ones. From this we deduce that in stationary two dimensional flow of a fluid with infinitely small friction (in which case the diffusion of the vorticity becomes infinitely slow), the vorticity will be distributed in such a way that the vortex strength has a constant value along every stream line.

In applying this conclusion to the flow in the vicinity of a body, we can add that the vorticity may be found along those stream lines or parts of stream lines only, which extend from the body down stream: for up stream of the body the flow does not supply any vorticity and the diffusion is so weak that it cannot produce an appreciable extension of the vorticity

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[2]) The most important of these papers are: C. W. OSEEN, Zur Theorie des Flüssigkeitswiderstandes, Nova Acta Reg. Soc. Scient. Upsaliensis (IV. 4) 1914; Beiträge zur Hydrodynamik I, Ann. d. Physik **46**, p. 231, 1915; and in particular: Hydrodynamik (Bd. I der Sammlung Mathematik in Monographien und Lehrbüchern, Leipzig 1927), p. 211 and seq.; N. ZEILON, On potential problems in the theory of fluid resistance, Kungl. Svenska Vetenskapsakademiens Handlingar, III Ser. I: 1, 1924; Beiträge zur Theorie des asymptotischen Flüssigkeitswiderstandes, Nova Acta Reg. Soc. Scient. Upsaliensis 1927.

against the direction of the flow. Stream lines passing the body at some distance will not carry any vorticity.

This idea seems to render little use when we do not know the course of the stream lines, which are found only when the problem is solved.

Now OSEEN has put the question whether we may get a satisfactory approximation by determining the distribution of the vorticity in the field, starting from a simple flow pattern, which needs not correspond to the real flow. Accepting such an imaginary "transport flow for the vorticity", the lines along which the vorticity is constant could be determined. Then we can try to choose the vorticity in such a way that the field belonging to this vortex distribution, superposed upon a suitable irrotational flow, leads to a flow satisfying the boundary conditions along the surface of the body and the condition :

$$u = -V, \quad v = 0 \text{ at infinity.}$$

We will not investigate this idea in its most general aspect, but will consider a simple case, the first one treated by OSEEN, to which, until now, most attention has been paid.

In this case for the "transport flow" the most simple one imaginable is accepted ; viz. a flow with a constant velocity V parallel to the x-axis. This flow enters the body at one side and leaves it at the other. We will consider this as unimportant, as this flow does not represent the solution we look for, but is only a means for arriving at the distribution of the vorticity.

The vortex distribution now becomes very simple. At the upstream side of the body we can only have a thin vortex layer, as the vorticity will not diffuse upstream ; in the wake of the body, however, in a region of equal breadth, extending down stream to infinity, vorticity will be present, with a strength independent of x, and therefore being a function of y only (compare fig. 1). We shall write for this function : $\zeta(y)$.

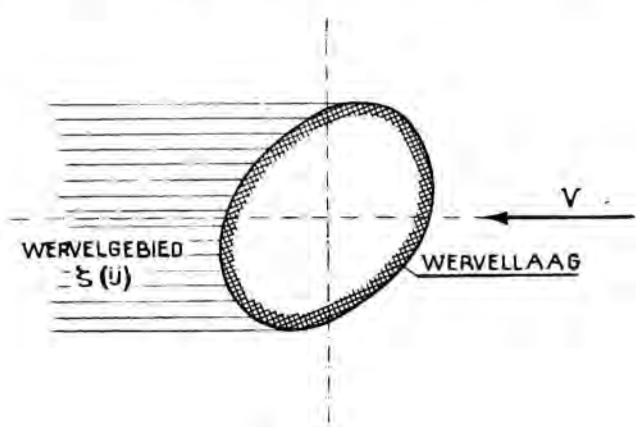

Fig. 1.

When it is preferred to leave aside the idea of a special transport flow for the vorticity, one might as well start from the supposition that in the wake of the body, in a region of equal breadth, the strength of vorticity may be represented by a function $\zeta(y)$, and that a concentrated vortex layer is present at the front side of the body, while outside of these two regions no vorticity occurs in the flow.

Outside of the vortex region we must now get irrotational motion characterized by a potential function $\Phi(x, y)$. We shall suppose that this potential may be continued analytically without singularities into the vortex region in the wake of the body. Then in this region a flow has to be superposed on it determined by the distribution of the vorticity which is zero outside of this region. Writing now in the whole field for the components u and v of the velocity of the resultant flow :

$$u = \frac{\partial \Phi}{\partial x} + u', \quad v = \frac{\partial \Phi}{\partial y} + v' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the quantities u' and v' will be zero everywhere outside of the vortex region. Within this region they satisfy the equations

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0, \quad \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now it is of importance to remark that we can find the function Φ without knowing in advance the distribution of the vorticity. The boundary conditions tell that the velocity must be zero on the surface of the body. Now an infinitely thin vortex layer is present on the anterior side. Just outside of this layer the condition for the tangential component of the velocity may be left aside, as it can allways be satisfied later on by giving a suitable strength to the vortex layer. Hence only the condition regarding the normal component has to be observed. In this way for the potential, which satisfies the equation of LAPLACE

$$\triangle \Phi = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we get :

$$\frac{\partial \Phi}{\partial n} = 0 \text{ at the front side of the body} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

just as in the case of the DIRICHLET flow.

In the wake of the body we have to determine a velocity distribution u', v', that satisfies (2) and is zero everywhere outside of the vortex region. Such a velocity field can be found by supposing that v' is zero everywhere, and that u' is a function of y only, determined by the equation

$$\frac{\partial u'}{\partial y} = -\zeta(y) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

When we suppose that over the breadth of the vortex region $\int \zeta \, dy$ has the value zero, equation (5) possesses a solution for u', which is zero at both sides of this region. The fact that ζ does not depend on x of course is of predominant importance; if this was not the case, also u' would depend on x and the equation of continuity would show that v' cannot be put equal to zero.

Both velocity components must vanish at the back of the body. Substituting $v' = 0$ in form. (1), we get at the back:

$$\frac{\partial \Phi}{\partial x} + u' = 0, \quad \frac{\partial \Phi}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The condition

$$\frac{\partial \Phi}{\partial y} = 0 \quad \text{at the back of the body,} \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

together with the condition (4) just mentioned and the conditions:

$$\frac{\partial \Phi}{\partial x} = -V, \quad \frac{\partial \Phi}{\partial y} = 0 \quad \text{at infinity} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

are sufficient to determine the potential function Φ: once Φ being found, the relation

$$u' = -\left(\frac{\partial \Phi}{\partial x}\right) \quad \text{at the back of the body} \quad . \quad . \quad . \quad . \quad (6b)$$

gives the velocity component u', from which the vortex field ζ can be deduced.

In this way we come to a new type of potential fields, differing from those of the DIRICHLET flow.

§ 3. As a first example we shall consider the flow around a circular cylinder with radius a. In order to obtain a solution by means of the methods of theory of functions, the complex stream function χ is introduced, which is composed of the potential function Φ and the stream function Ψ:

$$\chi = \Phi + i\,\Psi.$$

Writing $z = x + iy$, we shall get:

$$w = \frac{d\chi}{dz} = \frac{\partial \Phi}{\partial x} + i\,\frac{\partial \Psi}{\partial x} = u - i\,v = c\,e^{-i\theta},$$

where c represents the absolute value of the velocity, and θ the angle between the direction of flow and the positive x-axis. Putting

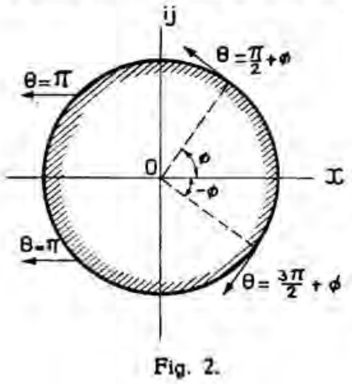

Fig. 2.

$$\omega = lg\, w = lg\, c - i\,\theta. \quad . \quad . \quad (8)$$

ω will be a function for which the solution of the boundary condition problem is much easier than it is for Φ[3]), as the value of the imaginary part of ω, i.e. $-i\,\theta$, is known in all points of the circumference of the circle. In fact the boundary conditions (4) and (6a) lead to the expressions (compare fig. 2):

$$\left.\begin{array}{ll} \text{for} \quad 0 < \Phi < \frac{\pi}{2}: & \theta = \frac{\pi}{2} + \Phi \\ \frac{\pi}{2} < \Phi < \frac{3\pi}{2}: & \theta = \pi \\ -\frac{\pi}{2} < \Phi < 0: & \theta = \frac{3\pi}{2} + \Phi \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Hence the function ω can be found from the integral

$$\omega = -\frac{1}{\pi}\int d\Phi \frac{d\theta}{d\Phi}\, lg\,(z - a\, e^{i\Phi}) + \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

taken over the circumference. In our case the particularity occurs that in the point $\Phi = 0$ the angle θ changes discontinuously; therefore a term taking into account the amount of the change must be added to the integral.

On account of the relation:

$$\frac{d\theta}{d\Phi} = 1 \quad \text{for} \quad -\frac{\pi}{2} < \Phi < \frac{\pi}{2},$$

which follows from the data mentioned above, we get

$$\omega = -\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Phi\, lg\,(z - a\, e^{i\Phi}) + lg\,(z - a) + C_1 + i\,C_2 \quad . \quad . \quad (11)$$

3) In treating this example, and also in the following one (the oblique plate) ZEILON uses a theorem given by HILBERT in order to solve the boundary problem for Φ. This leads to a rather tedious calculation.

In a paper by the present author published in these Proceedings, Vol. **23**, p. 1082, 1921, the flow around a cylinder has been calculated by means of a FOURIER expansion for the potential. This method does not converge quickly. Limiting the expansion to 4 terms, for the coefficient in the expression (15) for Q was found: 2,36, with 11 terms: 2,30, the rigorous value being $2(\pi - 2) = 2{,}283$. In the paper mentioned a diagram of the streamlines etc. has been given.

In order to check this formula and at the same time to determine the value of the constant C_2, we make the point z approach to a point $a\,e^{i\Phi_0}$ $(0 < \Phi_0 < \pi)$ of the circle (see fig. 3). We have to bear in mind that the argument of $z - a\,e^{i\Phi}$ takes the value $\frac{1}{2}(\Phi + \Phi_0 + \pi)$ at the points $a\,e^{i\Phi}$ for which $\Phi_0 - \pi < \Phi < \Phi_0$, while it takes the value $\frac{1}{2}(\Phi + \Phi_0 - \pi)$ at the points $a\,e^{i\Phi}$ for which $\Phi_0 + \pi > \Phi > \Phi_0$. The argument of $a\,e^{i\Phi_0} - a$ being equal to $\frac{1}{2}(\Phi_0 + \pi)$, we find for the imaginary part of ω:

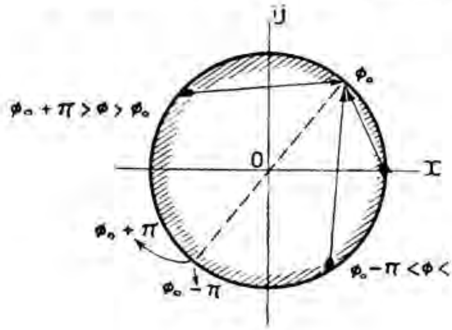

Fig. 3.

$$I(\omega) = -\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\Phi_0} d\Phi\,(\Phi + \Phi_0 + \pi) - \frac{1}{2\pi}\int_{\Phi_0}^{\frac{\pi}{2}} d\Phi\,(\Phi + \Phi_0 - \pi) +$$

$$+ \frac{1}{2}(\Phi_0 + \pi) + C_2 = -\Phi_0 + \frac{\pi}{2} + C_2,$$

when $0 < \Phi_0 < \frac{\pi}{2}$; and:

$$I(\omega) = -\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Phi\,(\Phi + \Phi_0 + \pi) + \frac{1}{2}(\Phi_0 + \pi) + C_2 = C_2,$$

when $\frac{\pi}{2} < \Phi_0 < \pi$.

Therefore we have to take $C_2 = -\pi$.

At a point on the lower half of the circle, for which $-\pi < \Phi_0 < 0$, the same expression can be used when $\frac{1}{2}(\Phi_0 - \pi)$ is written for the argument of $a\,e^{i\Phi_0} - a$ in stead of $\frac{1}{2}(\Phi_0 + \pi_0)$.

When we calculate the value of ω at a point of the field outside of the circle, the determination of the arguments does not give any difficulty: as soon as the argument of $z - a\,e^{i\Phi}$ has been fixed for any value of Φ, its magnitude for other values, and also the argument of $z - a$ will follow in an unequivocal way.

Considering now the behaviour of the flow at great distances of the cylinder, we may expand the logarithms and shall get:

$$lg\,(z - a\,e^{i\Phi}) = lg\,z - \frac{a\,e^{i\Phi}}{z} - \frac{a^2 e^{2i\Phi}}{2z^2} - \dots$$

$$lg\,(z - a) = lg\,z - \frac{a}{z} - \frac{a^2}{2z^2} - \dots$$

which leads to:

$$\omega = -\frac{a}{z}\left(1 - \frac{2}{\pi}\right) - \frac{a^2}{2z^2} \dots + C_1 - i\pi.$$

As the velocity at infinity is V, C_1 must have the value $lg\ V$; hence:

$$\omega = lg\ V - i\pi - \frac{\pi - 2}{\pi}\frac{a}{z} - \frac{a^2}{2z^2} - \dots \qquad (12)$$

From this we find for the value of w:

$$w = u - iv = e^{\omega} = -V\left\{1 - \frac{\pi - 2}{\pi}\frac{a}{z} - \dots\right\} \qquad (13)$$

and finally we get for the complex stream function:

$$\chi = -V\left\{z - \frac{\pi - 2}{\pi} a\, lg\, z - \dots\right\} \qquad (14)$$

In the last expression after the term representing a parallel flow with the constant velocity $-V$, a logarithm appears, representing a flow radially outwards, of the total strength

$$Q = 2(\pi - 2)\, a\, V = 2{,}283\, a\, V \qquad (15)$$

We can deduce this also from the stream function Ψ, the expression of which is:

$$\Psi = -V\left\{y - \frac{\pi - 2}{\pi} a\, arctg\, \frac{y}{x} - \dots\right\}$$

showing that the stream line $\Psi = 0$, which upstream of the cylinder is directed along the $+x$-axis, is determined down stream by:

$$y = \frac{\pi - 2}{\pi} a\, arctg\, \frac{y}{x} - \dots\dots$$

For $x \rightarrow -\infty$ this equation approaches asymptotically to:

$$y = \pm(\pi - 2)\, a = \pm 1{,}14\, a.$$

Both branches lie outside of the vortex region (which is included within the lines $y = +a$ and $y = -a$); therefore they also appear in the complete field.

In order to deduce the value of u', we must calculate the value of ω

for the points of the semi-circle between $\Phi = \frac{\pi}{2}$ and $\Phi = \frac{3\pi}{2}$. As $\frac{\partial \Phi}{\partial y}$ is zero here, we find u' from the equation:

$$u' = -\frac{\partial \Phi}{\partial x} = -\frac{d\chi}{dz} = -e^{\omega} = e^{R(\omega)}.$$

We can confine ourselves to consider the real part of ω only, as the imaginary part has the constant value $-\pi i$.

Now for $z = a e^{i\Phi_0}$ we have:

$$|z - a e^{i\Phi}| = 2 a \sin \frac{\Phi_0 - \Phi}{2},$$

therefore we find:

$$R(\omega) = -\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Phi \, lg \sin \frac{\Phi_0 - \Phi}{2} + lg \sin \frac{\Phi_0}{2} + lg\, V =$$

$$= lg\, V + lg \sin \frac{\Phi_0}{2} + \frac{1}{\pi}\left(\Phi_0 - \frac{\pi}{2}\right) lg \sin\left(\frac{\Phi_0}{2} - \frac{\pi}{4}\right) -$$

$$-\frac{1}{\pi}\left(\Phi_0 + \frac{\pi}{2}\right) lg \sin\left(\frac{\Phi_0}{2} + \frac{\pi}{4}\right) + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Phi\, (\Phi - \Phi_0) \cot \frac{\Phi - \Phi_0}{2}.$$

The last term occurring in this expression can be reduced to a function of the form:

$$\varphi(\xi) = \frac{2}{\pi}\int_0^{\xi} d\xi\, \xi \cot \xi,$$

which has been calculated by ZEILON for values of ξ ranging from 0 till $\frac{3\pi}{4}$ [1]).

Then:

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Phi\, (\Phi - \Phi_0) \cot \frac{\Phi - \Phi_0}{2} = \varphi\left(\frac{\Phi_0}{2} + \frac{\pi}{4}\right) - \varphi\left(\frac{\Phi_0}{2} - \frac{\pi}{4}\right).$$

Let us consider f.i. the points determined by $\Phi_0 = \frac{\pi}{2}$ and $\Phi_0 = \pi$.

[1]) Compare N. ZEILON, On potential problems, etc., p. 34, table I.

According to the table given by ZEILON we have:

$$\varphi(0)=0;\quad \varphi\left(\frac{\pi}{4}\right)=0{,}465;\quad \varphi\left(\frac{\pi}{2}\right)=lg\,2=0{,}693\ ^{5)};\quad \varphi\left(\frac{3\pi}{4}\right)=0{,}229.$$

Therefore the point $\Phi_0=\frac{\pi}{2}$ gives:

$$R(\omega)=lg\,V+\frac{1}{2}\,lg\,2,$$

from which

$$u'_{90^\circ}=V\sqrt{2}=1{,}414\ V.$$

At $\Phi_0=\pi$ we get:

$$R(\omega)=lg\,V+0{,}111,$$

and

$$u'_{180^\circ}=1{,}117\,V.$$

From the fact that u' is not equal to zero for $y=a$, while for values of y just higher than a, u' must be 0, it follows that the line $y=a$ (and also the line $y=-a$) represents a vortex layer, along which ζ becomes infinite in such a way that

$$\lim_{\varepsilon=0}\int_{a-\varepsilon}^{a+\varepsilon}\zeta\,dy=u'_{(y=a)}=V\sqrt{2}.$$

We note the relation:

$$Q=\int_{-a}^{+a}u'\,dy\quad\ldots\ldots\ldots\ldots\quad(16)$$

where Q is the quantity determined by (15). This is immediately deduced from the consideration that Q denotes the amount which the flow represented by Φ carries from the cylinder outward to infinity; this

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5) That the integral

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\xi\,\xi\cot\xi=-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\xi\,lg\sin\xi$$

has the value $lg\,2$, is easily proved from the relations:

$$\int_0^{\frac{\pi}{2}}d\xi\,lg\sin\xi=\int_0^{\frac{\pi}{2}}d\xi\,lg\cos\xi=\frac{1}{2}\int_0^{\frac{\pi}{2}}d\xi\,lg\,\frac{\sin 2\xi}{2}=\frac{1}{4}\int_0^{\pi}d\xi\,lg\,\frac{\sin\xi}{2}=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}d\xi\,lg\,\frac{\sin\xi}{2}=\frac{1}{2}\int_0^{\frac{\pi}{2}}d\xi\,lg\sin\xi-\frac{\pi}{4}\,lg\,2.$$

amount is replenished by the flow u' directed towards the cylinder. Mathematically expressed, when ds denotes an element of the circle:

$$Q = \int ds \frac{\partial \Phi}{\partial n} = -\int_{-a}^{+a} dy \left(\frac{\partial \Phi}{\partial x}\right) \text{ (at the back of the cylinder)} = +\int_{-a}^{+a} u' \, dy.$$

§ 4. As a second example we can take the flow along a plate of breadth $2a$, making an angle α with the x-axis (see fig. 4 where AB represents the plate). In connection with the treatment given by ZEILON we shall represent the flow along AB in the z-plane conformally upon

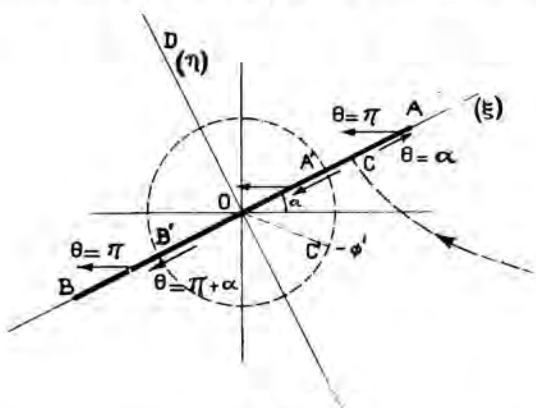

Fig. 4.

the field outside of a circle with radius $\frac{a}{2}$ in a τ-plane, related to the z-plane by means of the formula:

$$z e^{-i\alpha} = \tau + \frac{a^2}{4\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In this expression the complex variable $\tau = \xi + i\eta = \frac{a}{2} e^{i\varphi}$ is referred to the axes OA and OD. When now we suppose the values which ω takes in the points of both sides of the line AB to be transported to the corresponding points of the circle, we shall get for θ:

$$\left.\begin{array}{l} \text{along the segment } 0 < \Phi < \pi, \text{ corresponding to the back of the} \\ \quad \text{plate} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \theta = \pi \\ \text{along the segment corresponding to the front side of the plate:} \\ \quad \text{for } \pi < \Phi < 2\pi - \Phi' \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \theta = \pi + \alpha \\ \quad \text{for } -\Phi' < \Phi < 0 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \theta = \alpha \end{array}\right\} \quad (18)$$

where $-\Phi'$ is the argument of the point C' corresponding to the stagnation point at the front side of the plate. The position of this point is unknown for the present.

As θ has a constant value in every one of the three intervals defined above, and only shows abrupt changes at the points A', B, C' respectively of the amounts $\pi - \alpha, \alpha, -\pi$, (10) gives us the three terms only:

$$\omega = -\frac{\pi-\alpha}{\pi} lg\left(\tau - \frac{a}{2}\right) - \frac{\alpha}{\pi} lg\left(\tau + \frac{a}{2}\right) + lg\left(\tau - \frac{a}{2} e^{-i\Phi'}\right) + C_1 + iC_2 . \quad (19)$$

Applying this formula to the points of the circle, we find: $iC_2 = -i\pi - \frac{1}{2} i(\alpha - \Phi')$. When z becomes infinite, τ approaches asymptotically to $z e^{-i\alpha}$, and the expansion of ω becomes:

$$\omega = C_1 - i\left(\pi + \frac{\alpha - \Phi'}{2}\right) + \frac{a}{2z} e^{i\alpha}\left(\frac{\pi - 2\alpha}{\pi} - e^{-i\Phi'}\right) + \dots$$

The velocity must approach here to: $u = -V$, $v = 0$, hence ω has to take the value $lg\, V - i\pi$, from which follows $C_1 = lg\, V$, $\Phi' = \alpha$. Therefore:

$$\omega = lg\, V - i\pi - \frac{a}{2z}\left(1 - \frac{\pi - 2\alpha}{\pi} e^{i\alpha}\right) + \dots \quad . \quad . \quad . \quad (20)$$

which leads to:

$$w = e^{\omega} = -V \left\{ 1 - \frac{a}{2z}\left(1 - \frac{\pi - 2\alpha}{\pi} e^{i\alpha}\right) + \dots \right\} . \quad . \quad . \quad . \quad (21)$$

and finally:

$$\chi = -V \left\{ z - \frac{a}{2}\left(1 - \frac{\pi - 2\alpha}{\pi} e^{i\alpha}\right) lg\, z + \dots \right\} \quad . \quad . \quad . \quad (22)$$

In this case too a logarithm occurs in χ, now, however, having a complex coefficient. The flow shows a divergence amounting to:

$$Q = \pi a V \left(1 - \frac{\pi - 2\alpha}{\pi} \cos \alpha\right) \quad . \quad . \quad . \quad . \quad . \quad (23a)$$

and a circulation of the magnitude:

$$C = a\, V(\pi - 2\alpha) \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23b)$$

Q and C both become zero in the trivial case $\alpha = 0$; moreover C is zero, as might be expected, when $\alpha = \frac{\pi}{2}$ [6]). In this case Q becomes $\pi a V$.

We can deduce the vortex strength of the plate by calculating the value of $R(\omega)$ for $\tau = \frac{a}{2} e^{i\Phi}$, where $0 < \Phi < \pi$.

§ 5. Besides the distribution of the velocity a knowledge of the pressure is of importance. In calculating the pressure it becomes evident

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6) At small values of α the circulation C becomes equal to $\pi a V \alpha$. As ZEILON remarks, this is half the value given by the theory of KUTTA and JOUKOWSKY.

that the flow does not satisfy the exact equations of motion. In ZEILON's first paper, and also in OSEEN's Hydrodynamik the pressure distribution therefore is deduced from the approximate equations on which the calculations of the flow are based.

If the exact equations of motion for the stationary flow are written in the form:

$$\left.\begin{aligned}\frac{\varrho}{2}\frac{\partial}{\partial x}(u^2+v^2)-\varrho v\zeta&=-\frac{\partial p}{\partial x}+\mu\triangle u\\ \frac{\varrho}{2}\frac{\partial}{\partial y}(u^2+v^2)+\varrho u\zeta&=-\frac{\partial p}{\partial y}+\mu\triangle v,\end{aligned}\right\}.$$

where, as before,

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},$$

the approximate equations, which were the starting point of OSEEN's calculations, will be:

$$\left.\begin{aligned}\frac{\varrho}{2}\frac{\partial}{\partial x}(u^2+v^2)\qquad\qquad&=-\frac{\partial p}{\partial x}+\mu\triangle u\\ \frac{\varrho}{2}\frac{\partial}{\partial y}(u^2+v^2)-\varrho V\zeta&=-\frac{\partial p}{\partial y}+\mu\triangle v\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad (25)$$

The difference between the latter equations and the exact ones can be expressed by saying that in the terms in which the vortex intensity ζ occurs multiplied by the velocity, the components of the actual velocity u, v are replaced by the components of the "transport flow of the vorticity" as introduced into § 2, i.e. by $-V$ and 0 respectively.

If μ goes to zero we can integrate the system (25); using (5) we find:

$$p=const.-\frac{1}{2}\varrho(u^2+v^2)-\varrho Vu' \quad . \quad . \quad . \quad . \quad . \quad (26)$$

For the constant term we take $\frac{1}{2}\varrho V^2$, in order to make p zero at an infinite distance from the vortex region. Along the front side of the body, where $u'=0$, (26) reduces to the ordinary BERNOULLI formula. At the back side we have $u=v=0$ and therefore:

$$p_b=\frac{1}{2}\varrho V^2-\varrho Vu' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26a)$$

In the case of the cylinder u' lies between $V\sqrt{2}$ at $\Phi=90^\circ$ and 1,117 V at $\Phi=180^\circ$; hence p_b is comprised between $-0{,}914\,\varrho V^2$ and $-0{,}617\,\varrho V^2$.

In order to determine the resistance experienced by the body, we put in formula (26):

$$u=\frac{\partial\Phi}{\partial x}+u', \quad v=\frac{\partial\Phi}{\partial y}.$$

which gives:

$$p = \frac{1}{2}\varrho V^2 - \frac{1}{2}\varrho\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right\} - \varrho\frac{\partial\Phi}{\partial x}u' - \frac{1}{2}\varrho u'^2 - \varrho V u'.$$

Bearing in mind that $u' = 0$ at the front of the body and that $\partial\Phi/\partial x = -u'$ at the back side, we can write for the value of p at the points of the surface of the body:

$$p = -\frac{1}{2}\varrho\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right\} + \frac{1}{2}\varrho(u' - V)^2 \quad . \quad . \quad . \quad (27)$$

We shall consider first the term

$$p_I = -\frac{1}{2}\varrho\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right\}.$$

By means of some reductions the resultant of this part of the pressure is found to be [7]):

$$\int p_I \cos(n,x)\, ds = -\varrho Q V + \varrho\int_{-a}^{+a} u'^2\, dy.$$

(directed along the negative x-axis). The second part of the expression (27) gives a pressure:

$$p_{IIf} = \frac{1}{2}\varrho V^2,$$

at the front side of the body, and a pressure:

$$p_{IIb} = \frac{1}{2}\varrho(u' - V)^2;$$

at the back side. The resultant of these pressures has the magnitude (again taken along the negative x-axis):

$$\int_{-a}^{+a} dy\,(p_{IIf} - p_{IIb}) = \varrho Q V - \frac{1}{2}\varrho\int_{-a}^{+a} u'^2\, dy.$$

The total resistance now becomes:

$$W = +\frac{1}{2}\varrho\int_{-a}^{+a} u'^2\, dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

[7]) In order to prove this formula, it is necessary to add to $\int p_I \cos(n,x)\, ds$ the quantity

$$\varrho\int\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2 dy - \frac{\partial\Phi}{dx}\frac{\partial\Phi}{dy}\, dx\right\},$$

which is again substracted afterwards. Compare ZEILON, On potential problems, p. 17 and OSEEN, Hydrodynamik, p. 295; the calculation is performed here in a slightly different way, as these authors allways consider a moving body within a fluid at rest.

By numerical integration ZEILON deduced for the case of the cylinder:

$$W = 0{,}657\,\varrho\,V^2 d \quad \text{(28a)}$$

where $d = 2a =$ diameter of the cylinder.

In the case of the oblique plate we get a resistance, and besides a force in the direction of y.

We note that on account of the discontinuity of u and of the presence of the term $-\varrho V u'$ formula (25) for the pressure is discontinuous along the lines $y = +a$ and $y = -a$, which confine the vortex region. Proceeding from $y = a + \varepsilon$ to $y = a - \varepsilon$ (where ε is a small positive quantity) in the case of the cylinder u increases from a value $-u_1$ (depending on x) to $-u_1 + V\sqrt{2}$; u' increases from 0 to $V\sqrt{2}$. Therefore the pressure decreases suddenly by the amount $\varrho V^2 (1 + \sqrt{2} - u_1 \sqrt{2}/V)$. From a physical point of view this result is impossible. Of course this is due to the fact that with the supposed transport flow we have to apply a system of special forces to the field of motion, which at the same time influence the distribution of the pressure.

In his second paper ZEILON tried to escape from this difficulty. For the determination of the field of flow it was essential that the vorticity ζ was a function of y only; the magnitude of the "transport velocity" does not affect this function. Therefore the calculations of §§ 2—4 are not modified when we take any function $-U(y)$ for this velocity in stead of $-V$, provided that it is again directed along the y-axis. Then, however, the second equation of (25) is changed into:

$$\frac{\varrho}{2}\frac{\partial}{\partial y}(u^2 + v^2) - \varrho U \zeta = -\frac{\partial p}{\partial y} + \mu \triangle v \quad \text{(29)}$$

and for the pressure distribution an other solution is obtained. ZEILON requires that in the immediate vicinity of the body the hydrodynamical equations have to be satisfied as well as possible; hence immediately behind the body the pressure may not change discontinuously when we enter into the vortex region. As the velocity is zero everywhere along the back of the body, and the terms $\mu\triangle u$, $\mu\triangle v$, occurring in the equations of motion, vanish here at the same time with μ, p has to be constant along the back side. It takes the value which is given by the formula (valid at the front side):

$$p_f = \frac{1}{2}\varrho V^2 - \frac{1}{2}\varrho (u^2 + v^2)$$

when applied to the limiting points, where the front transfers into the back.

In the case of the cylinder this value is (with $u_{90°} = V\sqrt{2}$, $v_{90°} = 0$):

$$p_a = -\frac{1}{2}\varrho V^2 \quad \text{(30)}$$

This value is higher than that which is found from (26); therefore ZEILON finds a smaller resistance:

$$W = 0{,}523\,\varrho V^2 d \quad \text{(31)}$$

In the case of the plate this consideration, however, is not valid. The flow described by (19) gives an infinitely high velocity at the edges of the plate, and therefore an infinite negative pressure, which cannot be continued as in the case of the cylinder.

ZEILON tried finally to obtain still other pressure distributions by supposing a modified vortex distribution, in which the vortex layer, occurring at the front side of the body extends over some distance on both sides along the back [8]). Along these parts of the back side we no more have to satisfy the boundary conditions: $u=0$, $v=0$, and therefore we cannot decide whether $\partial\Phi/\partial y=0$. ZEILON now introduces a certain hypothesis about the angle θ considered in § 3; then it is still possible to solve the boundary problem in a simple way. By means of an appropriate assumption about the course of θ, ZEILON in the case of the cylinder succeeds in obtaining a pressure distribution which fits the experimental results rather well. However, this modification of the original hypothesis takes away its former simplicity, and contains an element of arbitrariness.

§ 6. We might ask if the resistance could be found by applying the theorem of momentum. As the flow does not satisfy the hydrodynamical equations of motion, and therefore neither the fundamental equations of mechanics, this seems to be impossible.

However, there is a characteristic particularity in the solutions given by OSEEN's theory, which makes the application of this theorem to be not entirely without prospects. Therefore we direct our attention to the infinite part of the field.

According to OSEEN's theory the vorticity is zero everywhere in the field, with the exception of the region extending down stream of the body. Outside of this latter region the flow is characterized by a potential and therefore it also possesses a complex stream function. The latter can be expanded in the following progression:

$$\chi = -V z + (A_1 - i A_2)\, lg\, z + \frac{B_1 + i B_2}{z} + \quad . \quad . \quad . \qquad (32)$$

which leads to the potential function (with $r=\sqrt{x^2+y^2}$):

$$\Phi = -V x + A_1\, lg\, r + A_2\, arctg\, \frac{y}{x} + \frac{B_1 x + B_2 y}{r^2} + \quad . \quad . \qquad (33)$$

The term $A_1\, lg\, r$, representing a flow directed radially outward of the total strength $Q=2\pi A_1$, is characteristic for the theory. This term does not occur in the case of the DIRICHLET flow. On the contrary the term $A_2\, arctg\frac{y}{x}$, representing a circulation, is well known from the aerofoil theory of KUTTA and JOUKOWSKY.

[8]) ZEILON, Beiträge zur Theorie, ... p. 35.

As in reality no resultant radial flow can occur, the flow has to be replenished by a flow directed inward, which evidently can occur only in the wake of the body. This is confirmed by the experiments. They show, however, that the breadth of the vortex region does not remain finite, but gradually increases down stream. The compensational flow is therefore spread over a greater extent, and its velocity decreases in inverse proportion.

Let us now start from the supposition that the flow outside of the vortex region may be characterized by a potential function Φ of the type (33) and that within this region it is allowed to put:

$$u=\frac{\partial \Phi}{\partial x}+u', \qquad v=\frac{\partial \Phi}{\partial y}+v' \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where u' and v' satisfy the equation of continuity:

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

It is not a priori certain that the potential can allways be expressed by a series of the type (33) and then can be continued analytically throughout the entire vortex region. In the case of the discontinuous flow described by HELMHOLTZ and KIRCHHOFF for the case of a plate, transverse to the x-axis, we should have found an expansion for χ, beginning with the terms:

$$\chi=-Vz+\beta\sqrt{z}+\ldots=-Vr\,e^{i\varphi}+\beta\sqrt{r}\,e^{\frac{i\Phi}{2}}+\ldots$$

where the argument of $\sqrt{z}$ is determined by the condition $-\pi<\Phi<+\pi$.[9])

However, we will adhere, be it by the way of a hypothesis, to formula (33).

Now it is necessary to gather some data concerning the flow in the vortex region. Although the limits of this region will not be defined clearly in general, we may suppose that it is wholly included between two curves

$$y=f_1(x), \qquad y=f_2(x).$$

With great probability we may expect that at a great distance down stream of the body the functions f_1 and f_2 satisfy the conditions:

$$f_1 \ll |x|, \qquad f_2 \ll |x|,$$

so that also the breadth $b=f_1-f_2$ of the vortex region only slowly increases with x. We arrive at this supposition when bearing in mind that at great distances of the body, where eventual irregular motions are damped out, the spreading out of the vortex region is determined only by the frictional forces. At a great distance from the body the velocity components u and v will approach asymptotically to $-V$, 0,

[9]) In the following terms of this series also a logarithm occurs.

so that here OSEEN's equation for the vortex motion:

$$-\varrho V \frac{\partial \zeta}{\partial x} = \mu \triangle \zeta$$

will hold with continually increasing approximation.

The greater part of the following considerations, however, are valid also when the breadth b does not increase indefinitely but reaches a finite limit.

The velocity u' which occurs in the vortex region, is subjected to the relation:

$$\int_{f_1}^{f_2} u' \, dx = Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

When b increases without limit, u' must decrease indefinitely, as stated above.

If we bear in mind that $\partial u'/\partial x$ is of the order of u'/r, and that according to the equation of continuity (35) the same must hold for $\partial v'/\partial y$, we deduce that in the vortex region the quantity v' is at most of the order of bu'/r. Therefore $\partial v'/\partial x$ will be of the order of bu'/r^2. We now put the equation for $\partial p/dy$ into the following form:

$$\frac{\partial}{\partial y} \left\{ p + \frac{\varrho}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 + \frac{\varrho}{2}\left(\frac{\partial \Phi}{\partial y}\right)^2 \right\} =$$
$$= -\varrho \left[\frac{\partial \Phi}{\partial x}\frac{\partial v'}{\partial x} + u'\left(\frac{\partial^2 \Phi}{\partial x \, \partial y} + \frac{\partial v'}{\partial x}\right) + \frac{\partial \Phi}{\partial y}\frac{\partial v'}{\partial y} + v'\left(\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial v'}{\partial y}\right)\right] + \mu \triangle v'.$$

Investigating the order of magnitude of the terms occurring at the right hand side of this expression, the highest will be found to be bu'^2/r^2. The variations of the expression between $\{ \}$ on the left hand side, along a line drawn parallel to the y-axis through the vortex region, therefore are at most of the order: $b^2 u'^2/r^2$. As $\int u'^2 \, dy$ over the breadth of the vortex region cannot get an infinite value, not only these variations themselves but also the integral of the variations of the expression $\{ \}$ over the breadth of the vortex region will become zero when r increases indefinitely. We may deduce from these considerations that the pressure in the vortex region at great distances from the body will be given with sufficient accuracy by the approximate formula:

$$p = const. - \frac{1}{2}\varrho \left\{ \left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 \right\} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

After inserting the value of Φ from (33), we get:

$$p = const. + \varrho V \frac{A_1 x - A_2 y}{r^2} + \text{terms of the order } r^{-2} \ldots \quad (37a)$$

We may now apply the theorem of momentum to the region bounded by a circle with very great radius R around the origin. Then for the x-component of the resultant of the pressure forces on this circle we find:

$$K = \int_0^{2\pi} p\, R \cos \Phi \, d\Phi = \pi \varrho\, V A_1 = \frac{1}{2} \varrho\, V Q$$

(taken in the negative direction of x). The transport of momentum along the $-x$-axis into the region within this circle is:

$$I = \int_0^{2\pi} \varrho\, u\, (u \cos \Phi + v \sin \Phi)\, R\, d\Phi.$$

Inserting from (34), I becomes:

$$I = \varrho \int_0^{2\pi} \left\{ \left(\frac{\partial \Phi}{\partial x} \right)^2 \cos \Phi + \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial y} \sin \Phi \right\} R\, d\Phi +$$

$$+ \varrho \int_0^{2\pi} \left(2u' \frac{\partial \Phi}{dx} + u'^2 \right) R \cos \Phi \, d\Phi + \varrho \int_0^{2\pi} \left(v' \frac{\partial \Phi}{\partial x} + u' \frac{\partial \Phi}{\partial y} + u' v' \right) R \sin \Phi \, d\Phi.$$

The first integral has the value $-3\pi \varrho V A_1 = -\frac{3}{2} \varrho V Q$. In the second integral we put $R \cos \Phi \, d\Phi = -dy$; as $\int u'\, dy$ is finite, we may take $\partial \Phi / \partial x = -V$ in this formula and get:

$$\varrho \int_{f_1}^{f_2} (2u'V - u'^2)\, dy = 2 \varrho V Q - \varrho \int_{f_1}^{f_2} u'^2\, dy.$$

The greatest one of the terms, occurring in the expression between ( ) in the third integral, is of the order of bu'^2/R; as also $\sin \Phi$ becomes infinitely small for the points of the vortex region, the integral may be neglected entirely.

The resistance experienced by the body is equal to the sum of K and I. Hence finally we find:

$$W = \varrho V Q - \varrho \int_{f_1}^{f_2} u'^2\, dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

If we adhere to the supposition that the breadth of the vortex region increases indefinitely down stream, the second term expires; therefore:

$$W = \varrho V Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38a)$$ [10]

[10] When the compensational flow u', v' is not introduced, we find: $K + I = -\varrho VQ$. This result has been obtained by LAGALLY; compare M. LAGALLY, Zeitschr. f. angew. Mathematik u. Mechanik, 2, p. 409, 1922.

The appearance of the resistance therefore is essentially connected with the presence of a logarithmic term in the potential. And this term is obtained only then, when $\partial\Phi/\partial n$ is not zero at all points of the surface of the body.

In accordance with this result is the experimental observation that the streamlines which follow the contour of the body at the front side, leave this contour at certain points, usually situated in the neighbourhood of the section of maximum breadth or of a sharp edge. The points where this occurs limit the region in which $\partial\Phi/\partial n = 0$; beyond these points we evidently have $\partial\Phi/\partial n > 0$.

In the case of a cylinder the potential determined from the boundary conditions (4) and (6*a*) of § 2, according to OSEEN's theory, gives $Q = 2{,}283\, aV$, which leads to a much too high value of the resistance. Hence we must look for other boundary conditions for the potential; and the modification of DIRICHLET's condition is to be varied in such a way, that Q decreases. The researches of ZEILON mentioned at the end of § 5, possibly constitute a beginning in this direction.

In deducing form. (38*a*) it is supposed of course that in the infinite region of the field the flow determined by the functions Φ and u' approaches to the flow that exactly satisfies the true equations of motion. However, it is not required that this flow is entirely stationary; the deductions also hold when the flow in the immediate vicinity of the body fluctuates about a certain mean value, if only the fluctuations decrease without limit when we go farther and farther away from the body. The formula then gives the mean value of the resistance.

Mathematics. — *The Congruence of the Twisted Cubics that Cut Five given Lines Twice.* By Prof. JAN DE VRIES.

(Communicated at the meeting of February 25, 1928).

§ 1. The twisted cubics k^3 that have the lines b_1, b_2, b_3, b_4 and b_5 as bisecants, form a congruence Γ. In order to represent this on a field of points we establish a projective correspondence between the point range (C) on b_1 and the tangents (c) of a conic c^2. As the image of the k^3 that cuts b_1 in C_1 and C_2, we shall consider the point of intersection F of the tangents c_1 and c_2 corresponding to C_1 and C_2.

The k^3 that cut a line l, form a system Λ of which we shall determine the image curve.

The k^3 through C_1 that have b_2, b_3, b_4, b_5 as bisecants, form a congruence, which has been investigated by GODEAUX [1]). The k^3 of this congruence that rest on l, form a surface of the 9th degree with a triple point C_1. There are, accordingly, *six* k^3 that cut b_1 in one more point C_2. Consequently the *system* Λ is *represented* in a *curve* λ^6 and the k^3 through C_1 form a surface $(C)^6$.

§ 2. Any conic k^2 that cuts a line b_k twice and rests on the four other lines b, is completed by each of the two transversals t'_k, t''_k of these four b to a k^3 of the congruence Γ.

The curves k^2 in planes through b_1 that cut b_2, b_3, b_4, b_5 and m, form a surface O^8. The line t'_1 cuts O^8 in four points that do not lie on a line b; hence the k^2 that cut b_1 twice and rest on b_2, b_3, b_4, b_5 and t'_1, form a surface O'^4_1 with *double line* b_1.

On b_1 these k^2 define a correspondence (2, 2) between the points C_1 and C_2; the system Σ'_1 of the figures (k^2, t'_1) is, therefore, represented on a conic ω'^2_1. Its points of intersection with c^2 are the images of the figures that touch b_1.

There is no k^2 with bisecant b_1 that rests on the other four b and at the same time on the transversals t'_1 and t''_1. Accordingly any point of intersection of the conics ω'^2_1 and ω''^2_1 is the image of more than one k^3 and is, therefore, a *singular point.*

§ 3. This is confirmed by the following consideration. The *hyperboloid* H_{234} with directrices b_2, b_3, b_4 contains the transversals t'_5 and t''_5 of

[1]) L. GODEAUX, *Sur une congruence linéo-linéaire de cubiques gauches* (Bull. Acad. de Belgique 1908, N⁰. 4, p. 531). This congruence can also be investigated by the aid of the above mentioned representation; however this has no singular points.

b_1, b_2, b_3, b_4; their points of intersection with b_1 are indicated by C'_{1234} and C''_{1234}. If B'_{2345} and B''_{2345} are the points of intersection of H_{234} and b_5, H_{234}, contains a system of ∞^1 curves k^3 that pass through C'_{1234}, C''_{1234}, B'_{2345}, and B''_{2345}, cut b_2, b_3, b_4 twice, and belong, therefore, to the congruence Γ [1]). All these k^3 are represented in the *singular point* $S_5 \equiv c'_5\, c''_5$ defined by C'_{1234} and C''_{1234}.

Analogously we find the *singular points* S_2, S_3 and S_4 as images of the systems on the *hyperboloids* H_{345}, H_{245} and H_{235}.

The four points S_k are evidently the points of intersection of the conics ω'^2_1 and ω''^2_1.

§ 4. As l cuts each of the four hyperboloids twice, the *image curve* has *double points* in S_k. As λ^6 is a rational curve it has 6 more double points; they are due to curves k^3 that cut l twice. Hence a line chosen at random is a *bisecant* of *six* curves of the congruence.

Two curves λ^6 have 20 points F in common; hence on two lines there rest 20 k^3 and the curves resting on l form a surface Λ^{20}. On this the lines b_k are *sextuple* and the ten transversals t'_k, t''_k are *quadruple* (l has four points in common with O'_4).

Two surfaces Λ have the lines b_k, t'_k, t''_k and 20 curves k^3 in common.

§ 5. The system Σ'_2 of the k^3 consisting of a k^2 and the transversal t'_2 of b_1, b_3, b_4, b_5, is represented on the points of the tangent c'_2 that corresponds to the point of intersection of b_1 and t'_2.

The systems Σ'_2 and Σ_3 have the degenerate k^3 in common consisting of t'_2, t'_3 and the line of intersection of the planes $b_2\, t'_3$ and $b_3\, t'_2$. This k_3 has the point $c'_2\, c'_3$ as image.

Each point $c'_k\, c'_l$, $c'_k\, c''_l$, $c''_k\, c''_l$ $(k, l = 2, 3, 4, 5)$ is the image of a k^3 consisting of three parts. Also the non singular points of intersection of $c'_k\, (c''_k)$ with ω'^2_1 and ω''^2_1 are images of such composite k^3.

Hence Γ contains *fourty* k^3 consisting of *three straight lines*.

§ 6. Besides the hyperboloids considered in § 3 there are six more hyperboloids each of which is defined by three lines b, that contain ∞^1 k^3 of Γ; they are not represented in singular points.

Two hyperboloids have a k^3 in common only when they have a common directrix b. For H_{123} and H_{345} cut each other along b_3 and a k^3 that has b_3, hence all b_k, as bisecants.

As H_{345} is represented in S_2, H_{245} in S_3 and H_{123} has a k^3 in common with each of these two hyperboloids, the image of H_{123} passes through S_2 and S_3. Analogously the image of H_{145} contains the points S_4 and S_5.

[1]) If the points of H_{234} are projected out of a point of this surface on a plane, the images of the k^3 form a pencil of rational curves c^3 that have their double point in one of the cardinal points of the representation. To them belong the images of four composite k^3.

As H_{123} has only one k^3 in common with H_{145}, their *images* are *straight lines*, viz. the lines $S_2 S_3$ and $S_4 S_5$; we shall indicate them by h_{123} and h_{145}.

Hence the *ten hyperboloids* are represented as carriers of curves of Γ in the points S_k and the sides of the *complete quadrilateral* defined by them.

§ 7. On the conic Ψ^2 which H_{123} has in common with a plane Ψ, the k^3 of H_{123} define an involution I^3. Hence H_{123} contains *four* k^3 that touch Ψ. Consequently the image curve of the system Ψ of the k^3 touching Ψ has *quadruple points* in S_k and cuts each line h in four more points; it is, therefore, a Ψ^{12} (S^4).

With $\omega'_1{}^2(S)$ it has eight points F in common that are images of curves k^3 on $O'_1{}^4$; in six of these composite k^3 the k^2 touches Ψ and the point of intersection of Ψ and t'_1 defines a figure which must be counted twice.

According to $\Psi^{12}(S^4)$ and $\lambda^6(S^2)$ Ψ is a surface of the degree 40. As a k^3 of Γ chosen at random has only points of b_k in common with Ψ, the lines b are *twelvefold* lines; the transversals t', t'' are evidently *eightfold* on Ψ.

§ 8. The *surface* $(C)^6$ of the k^3 that cut b_1 in a point C, is *represented* on the points of the line c (§ 1) corresponding to C. A line k^3 that does not lie on $(C)^6$ cuts b_1 in two points; the other 16 points of intersection lie on the other four b; these are, therefore, *double lines* of $(C)^6$.

The plane Cb_2 contains two k^2 that form two k^3 on $(C)^6$ together with t'_2 and t''_2; the tangents at C to these k^2 are not coplanar with b_1, hence C is a *double point* on $(C)^6$; the transversal through C of b_2 and b_3 cuts $(C)^6$ twice in C and twice on b_2 and on b_3.

As the image line c has two points in common with each of the conics $\omega'_1{}^2$ and $\omega''_1{}^2$, t'_1 and t''_1 are *double lines*; the other eight lines t', t'' are *single* on $(C)^6$.

§ 9. The surface $(B_2)^6$ of the k^3 that cut b_2 in the point B, has in common with $(C)^6$ the lines $b_1, b_2, t'_1, t''_1, t'_2, t''_2$, which must be counted double, the lines b_3, b_4, b_5, which must be counted four times, and the six lines t'_k, t''_k $(k = 3, 4, 5)$; besides these lines they have two curves k^3 in common. Hence through *two points* chosen at random on b_k and b_l, there pass *two* curves of Γ.

Consequently the image curve of the system on $(B_k)^6$ is a *conic* $\beta_k{}^2$. The conics $\beta_k{}^2$ form a system with index 2 for the k^3 represented in a point F belongs to 2 points B_k.

As $(B_k)^6$ contains one k^3 of each of the systems Σ'_l, Σ''_l $(l \neq k, 1)$ (§ 5), $\beta_k{}^2$ passes through the three singular points S_l. The image curves $\beta^2_2(S_3 S_4 S_5)$ and $\beta^2_3(S_2 S_4 S_5)$ have two points F in common: this shows again that two surfaces $(B_k)^6$ and $(B_l)^6$ have two k^3 in common.

§ 10. A line d of the image plane is the image of a surface $\triangle^6$ that passes through b_1 and has the lines b_2, b_3, b_4, b_5 as *double lines*. For d cuts each line c in one point and each β_k^2 in two points.

Also t'_1, t''_1 are *double lines*, because each of the conics ω'^2 has two points in common with d.

The *field of rays* $[d]$ is, therefore, the image of a *net* of *surfaces* $\triangle^6$ with a basis consisting of *six* double lines and *nine* single lines (b_1 and eight lines t', t''). This figure together with each k^3 of Γ forms the basis of a pencil belonging to this net.

There are evidently *four* more similar nets.

§ 11. As a surface Λ^{20} has a sextuple line in b_1 (§ 4), the k^3 that cut a line m resting on b_1 in C outside C, form a surface M''; on this t'_1, t''^4 are *double lines*, the other 8 lines t are *triple lines*. (To the systems Σ there correspond surfaces O^4).

Any hyperboloid H that does not contain b_1, has two points in common with m; accordingly the image curve μ of M^{14} has double points in S_k. μ has three points F and the double point S_k in common with the image line c'_k or c''_k of a system Σ_k; it is, therefore, a $\mu^5(S^2)$.

It has 14 points F in common with $\lambda^6(S^2)$; this shows again that the degree of the surface M is 14.

A line c contains 5 points F of μ^5; hence b_1 is a *quintuple* line of M^{14}. The other lines b_k are *quadruple*; this appears when we consider the intersection of this surface and an arbitrary k^3. Combination of $\mu^5(S^2)$ with $\omega^2(S)$ and with c'_k (c''_k) gives as result that M^{14} has t'_1 and t''_1 as *double lines* and that the other 8 transversals are *triple* lines. When the line l rests on b_1, $\Lambda^{20}(b_k^6, t_k^4)$ decomposes into $(C)^6(b_1, b_k^2)$ and M^{14} $(b^1{}_5, b_k^4)$.

If l cuts two lines b, Λ^{20} is replaced by two surfaces $(B)^6$ and a surface $M^8(b_k^3\, b_l^3\, b_p^2\, b_q^2\, b_r^2)$.

§ 12. A ray n of the *plane pencil* (N, ν) is a *bisecant* of *six* ϱ^3 (§ 4). On the intersection of ν and a hyperboloid H the k^3 define an I^3; two of the lines containing the pairs of this I^3 pass through N. Hence H contains two k^3 of the system of the k^3 that cut a ray n twice.

The line h_{123} contains accordingly two points F of the image curve and the double points S_2, S_3; it is, therefore, a $\nu^6(S^2)$.

As $\nu^6(S^2)$ and $\lambda^6(S^2)$ have 20 points F in common, the k^3 of the system form a *surface* N^{20} with *sextuple* b_k and *quadruple* t_k; for ν^6 cuts a c 6 times, a c'_k 4 times.

§ 13. Let E be the system of the k^3 of which a *tangent* passes through the point N.

The tangents of the k^3 on H form a congruence [3, 4]. For through a point P of H there passes a line that is a bisecant of $\infty^1 k^3$, hence a tangent of two k^3, and the line that touches the k^3 through P at P. And as the aforesaid I^3 has four double points, a plane contains four tangents.

Consequently the image curve of E has *triple* points in S_k and as h_{123} also contains three points F this curve is an $\varepsilon^9\,(S^3)$.

It has 30 points F in common with $\lambda^6\,(S^2)$; accordingly the k^3 of the system E form a *surface* E^{30}. On this surface the lines b are *ninefold*, the lines t *sixfold*. As a check the fact may serve that E^{30} has 30 curves k^3 in common with Λ^{20} besides the multiple lines b and t.

§ 14. A conic ω^2 through the four points S is the image of a surface O^4, for $\omega^2\,(S)$ has four points F in common with $\lambda^6\,(S^2)$. As ω^2 has two points in common with any line c and one point F with any β^2, b_1 is double line and O^4 contains the four lines b_k $(k \neq 1)$. Also the eight lines t_k lie on O^4.

The *pencil* (ω^2) is, therefore, the image of a *pencil* (O^4) of which the basis consists of the double line b_1 and the 12 lines b_k and t_k.

Chemistry. — *Osmosis of ternary liquids. General considerations* V.
By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of April 28, 1928).

The diffusing mixture and the real membrane.

In the preceding communications (Gen. III and IV) we have seen in what way the composition of the diffused mixture can be found and how from this the directions in which the different substances pass through the membrane may be deduced, the composition of the diffused liquid, etc. We have seen that the composition of the diffused mixture, which we shall call L_0, is represented by the point of intersection S_0 of two conjugated chords. This is only the case, however, when the membrane itself does not contain the diffusing substances; as, however, the membrane does contain these substances, we shall call the first a "theoretical" and the second a "real" membrane.

In fig. 1 two points of the one branch of an osmosis-path are represented by 1 and 2 and two points of the other branch by 1′ and 2′. In point 1 (and 1′) of the path we imagine an osmotic system with a real membrane; we represent this by:

$$l_1 \times L_1 \mid m_1 \times M_1 \mid r_1 \times L'_1 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

On the left side of the membrane there are l_1 quantities of a liquid L_1 and on the right side r_1 quantities of a liquid L'_1. We represent the composition of L_1 by:

$$x_1 \text{ quant. of } X + y_1 \text{ quant. of } Y + (1 - x_1 - y_1) \text{ quant. of } W . \quad (2)$$

and that of L'_1 by:

$$x'_1 \text{ quant. of } X + y'_1 \text{ quant. of } Y + (1 - x'_1 - y'_1) \text{ quant. of } W . \quad (3)$$

Although the substances will not be equally spread in the membrane, yet we may say that it contains a definite quantity of liquid of a definite composition. We shall say that the membrane contains m_1 quantities of a liquid M_1, the composition of which we shall represent by:

$$\alpha_1 \text{ quant. of } X + \beta_1 \text{ quant. of } Y + (1 - \alpha_1 - \beta_1) \text{ quant. of } W . \quad (4)$$

We imagine this liquid M_1 to be represented in fig. 1 by the point 1″. In the point 2 (and 2′) of the path we then have an osmotic system:

$$l_2 \times L_2 \mid m_2 \times M_2 \mid r_2 \times L'_2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In order to represent the compositions of these liquids we imagine in (2), (3) and (4) the index 1 to be substituted by 2. The liquid M_2 of the membrane has been represented in fig. 1 by point 2″.

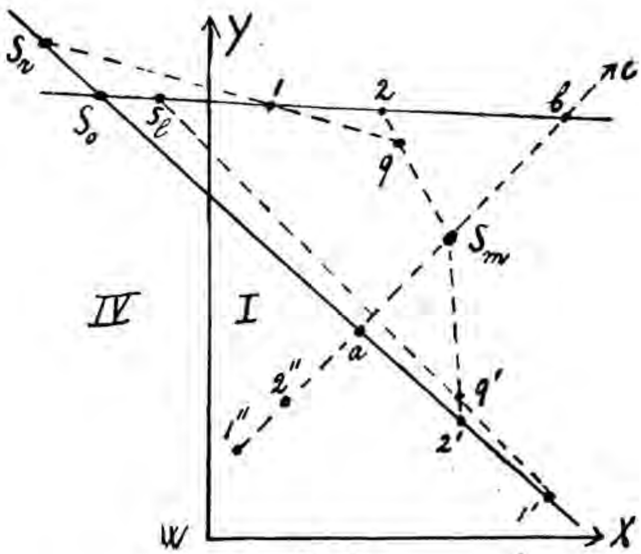

Fig. 1.

Consequently the osmosis-path of the system does not consist of 2, but of three branches; if namely the left side liquid travels along the part 1 . 2, and, therefore, the right side liquid along the part 1′ . 2′, then the membrane travels along the part 1″ . 2″ of the path.

If the left side liquid gives off a certain quantity u of a substance, then the right side liquid will take in a quantity, smaller or larger than u, according as the membrane will take in or give off a little of this substance during that time; it may even be supposed that as well the left side liquid as the right side liquid give e.g. the substance X to the membrane or take it from the membrane. For this reason we cannot speak any more of a single diffused mixture, but we shall distinguish three mixtures; we call them L_l, L_m and L_r and represent them in fig. 1 by the points s_l, s_m and s_r.

L_l is the mixture which is taken in or given off by the left side liquid. As L_1 has passed into L_2 by taking in or giving off a certain quantity of L_l, s_l, therefore, must be situated somewhere on the line 1 . 2. If s_l is situated in the way drawn in fig. 1, then this mixture has been given off by the left side liquid. We represent the composition of this mixture by:

$$x \text{ quant. of } X + y \text{ quant. of } Y + (1 - x - y) \text{ quant. of } W. \quad (6)$$

L_m is the mixture, which is taken in or given off by the membrane; consequently the point s_m is situated somewhere on the line 1″ . 2″. In fig. 1 it has been assumed that this mixture has been taken in by the membrane. We represent its composition by:

α quant. of $X + \beta$ quant. of $Y + (1 - \alpha - \beta)$ quant. of W . . (7)

L_r is the mixture which is taken in or given off by the right side liquid; consequently the point s_r is situated somewhere on the line $1' . 2'$. In fig. 1 we have assumed that this mixture has been taken in by the right side liquid. We represent its composition by:

x' quant. of $X + y'$ quant. of $Y + (1 - x' - y')$ quant. of W . (8)

Now we shall assume that the left side liquid has given off l quantities of L_l and the membrane has taken in m quantities of L_m and the right side liquid r quantities of L_r.

It appears from the liquids L_1 and L_2 on the left side of the membrane that during the osmosis $l_1 - l_2$ quantities have disappeared here; they contain $l_1 x_1 - l_2 x_2$ quantities of X and $l_1 y_1 - l_2 y_2$ quantities of Y; of course the remainder is the substance W. Consequently we find:

$$l = l_1 - l_2 \quad x = \frac{l_1 x_1 - l_2 x_2}{l} \quad y = \frac{l_1 y_1 - l_2 y_2}{l} \quad . \ . \ . \ (9)$$

From the liquids in the membrane and on the right side of it, follows:

$$m = m_2 - m_1 \quad \alpha = \frac{m_2 \alpha_2 - m_1 \alpha_1}{m} \quad \beta = \frac{m_2 \beta_2 - m_1 \beta_1}{m} \quad . \ . \ (10)$$

$$r = r_2 - r_1 \quad x' = \frac{r_2 x'_2 - r_1 x'_1}{r} \quad y' = \frac{r_2 y'_2 - r_1 y'_1}{r} \quad . \ . \ (11)$$

As the quantities of the substances do not change during the osmosis, we have:

$$\left.\begin{array}{l} l_1 + m_1 + r_1 = l_2 + m_2 + r_2 \\ l_1 x_1 + m_1 \alpha_1 + r_1 x'_1 = l_2 x_2 + m_2 \alpha_2 + r_2 x'_2 \\ l_1 y_1 + m_1 \beta_1 + r_1 y'_1 = l_2 y_2 + m_2 \beta_2 + r_2 y'_2 \end{array}\right\} \quad . \ . \ . \ . \ (12)$$

With the aid of (9), (10) and (11) we find from this:

$$l = m + r \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ (13)$$

$$l x = m \alpha + r x' \qquad l y = m \beta + r y'$$

If we consider the compositions of the mixtures L_l, L_m and L_r then it follows from (13):

$$l \times L_l = m \times L_m + r \times L_r \quad . \ . \ . \ . \ . \ . \ . \ (14)$$

This expresses that the entire mixture, which has disappeared from the left side liquid, has been taken in by the membrane and the right side liquid. As this speaks for itself, we are able to write down (14) without any further deduction. It now follows from (14):

the point s_l is situated between s_m and s_r and divides the line into two parts, which are determined by:

$$s_l s_r : s_l s_m = m : r \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (15)$$

We shall now assume that, besides the compositions of the left side and right side liquids, their quantities l_1, l_2, r_1 and r_2 are known also. Then we are able to determine l, x and y from (9) and r, x' and y' from (11). Next we find m, α and β from (13). Consequently we know the quantities and the compositions of the three diffused mixtures.

Matters are otherwise, however, if we only know the compositions of the left side and right side liquids. Then the point S_0 is indeed known but not the points S_l and S_r, which are essential in order to determine the directions in which the substances now pass through the membrane. Yet in many cases it is possible to find these directions with the aid of point S_0 only, as will be shown later on.

When system (1) has passed into (5), then the membrane has taken in m quantities of L_m. We now put these m quantities in the left side liquid L_2; this now changes its composition and passes again into a liquid L_q which has been represented in fig. 1 by point q. This liquid has been formed from l_2 quantities of L_2 and m quantities of L_m; consequently point q is situated on the line $2 . S_m$ and it divides this line into two parts, which are determined by:

$$q . 2 : q . s_m = m : l_2 \quad \ldots \ldots \quad (16)$$

So, instead of system (5) we now get the system:

$$(l_2 + m) \times L_q \mid m_1 \times M_1 \mid r_2 \times L'_2 \quad \ldots \ldots \quad (17)$$

so that the membrane has not changed its composition in passing from (1) to (17); consequently the diffused mixture is represented by the point of intersection of the lines $1 . q$ and $1' . 2'$. As the state of things on the right side of the membrane in (17) is now the same as in (5), the mixture L_r must have diffused; consequently point s_r is the point of intersection of the lines $1 . q$ and $1' . 2'$.

If we put the m quantities of L_m in the right side liquid, then we get the system:

$$l_2 \times L_2 \mid m_1 \times M_1 \mid (r_2 + m) \times L'_q \quad \ldots \ldots \quad (18)$$

The liquid L'_2 of (5) has now been replaced by a liquid L'_q which is situated in fig. 1 on the line $2' . s_m$; for this obtains:

$$q' . 2' : q' s_m = m : r_2 \quad \ldots \ldots \quad (19)$$

As the state of things on the left side of the membrane now is the same in (18) as in (5), the point of intersection of the lines $1 . 2$ and $1' . q'$ falls in the point s_l. Consequently we find:

s_l is the point of intersection of the chord $1 . 2$ with the line $1' . q'$
s_r is the point of intersection of the line $1 . q$ with the chord $1' . 2'$.

In fig. 1 s_r is situated on the left side and s_l on the right side of s_0; this is the case when s_m is situated between a and b; if, however, s_m

is situated between $2''$ and a, so that point q' comes on the other side of the chord $1'.2'$, then we see that both points come on the left side of s_0. If, however, s_m is situated between b and c (we imagine c at infinite distance) then both points fall on the right side of s_0.

In the transition-cases, viz. when s_m is situated in a or in b, then s_l or s_r coincide with s_0.

We are able to express the length of the lines $s_0 s_l$ and $s_0 s_r$ in different ways. A simple way of expressing it is among others:

$$s_0 s_l = \frac{m}{l} \cdot \beta \qquad s_0 s_r = -\frac{m}{r} \cdot \alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

in which, however, α and β have another meaning than above, though they are connected with them. We are able to deduce these equations in different ways, one of which we shall briefly indicate.

It is namely possible to prove that the equations (13) obtain for each system of coordinates; we now choose the point s_0 as origin, the line $s_0 1'$ as X-axis and the line $s_0 1$ as Y-axis. Then α is the distance of s_m to the line $s_0 1$, measured along a line parallel to the line $s_0 1'$; β is the distance of s_m to the line $s_0 1'$, measured along a line parallel to $s_0 1$.

For the point s_l now obtains $x = 0$ and $y = s_0 s_l$ and for point s_r we find $x' = s_0 s_r$ and $y' = 0$. Substituting these values in (13) we find (20).

In fig. 1 α and β are both positive; now (20) gives a positive value for $s_0 s_l$ and a negative value for $s_0 s_r$; we see that this is in accordance with fig. 1.

From what precedes it appears that the points s_l and s_r can be situated in different ways with respect to s_0; it follows from (20) that the smaller m is with respect to l and r, the nearer they will be to s_0. For the present we shall leave it so; to the other case we are going to refer later on.

Above we have noticed already that we have to know the points s_l and s_r in order to determine the composition of the diffusing mixtures and the directions in which they and the substances pass through the membrane. In order to consider what mistakes may arise, if we use the point s_0 instead of these points, we shall discuss a few cases.

First we shall suppose the point s_0 to be in field IV not too close to one of the sides (or their prolongations) of the triangle, so that the points s_l and s_r will be situated in this field as well.

If we were to use the point s_0 now, then we should make a mistake in the determination of the mixtures L_l and L_r which really have diffused and of course this will always be the case, when the points do not coincide.

The directions, in which the three substances and their mixtures pass through the membrane, are, however, also indicated by the point s_0.

The same obtains also for other fields, except, as we shall see later on, towards the end of the osmosis.

Now we assume that during the osmosis point s_0 passes from field IV to field I; as in this case no divergences of the substances Y and W occur, we need only consider the substance X.

As long as s_0 is situated in field IV, the mixture contains a negative quantity of X; as liquid 1 gives off this mixture, a negative quantity of X will go towards the right; so the substance X will really go towards the left.

If s_0 comes on the side WY, so that the mixture does not contain X, then no X will consequently pass through the membrane.

If s_0 comes in field I then the mixture contains a positive quantity of X; now the substance X passes through the membrane towards the right. Point s_0 now yields the simple scheme:

$$\underleftarrow{X} \qquad \underline{X} \qquad \underrightarrow{X} \quad \ldots\ldots \quad (21)$$

The first symbol obtains when s_0 is situated in field IV, the second when s_0 is situated in field I; the dash indicates that at that moment no X passes through the membrane.

Each of the points s_l and s_r indicates a similar scheme as (21); yet there is a difference viz. the transition-symbol occurs in these three schemes at lifferent moments of the osmosis.

We now shall assume that the points are situated with respect to one another as shown in fig. 1; then point s_l will be the first to come on the side WY, next point s_0 and at last s_r. If the three systems are combined to form one, we get scheme (22).

	1	2	3	4	5	6	7	
S_l	←	—	→	→	→	→	→	S_l
S_r	←	←	←	←	←	—	→	S_r
S_0	←	←	←	—	→	→	→	S_0

. . (22)

The top row (at the beginning and the end of which s_l has been placed) represents the s_l-scheme; we call these arrows and directions the s_l-arrows and s_l-directions. Consequently these arrows indicate what has really been happening to the left side liquid. Therefore, an arrow pointing to the left indicates that the left side liquid takes in X, an arrow pointing to the right that this left side liquid gives off X.

The middle row is the s_r-scheme. The s_r-arrows indicate, therefore, what really happens to the right side liquid. Consequently an arrow pointing to the left indicates that the right side liquid gives off X, an arrow pointing to the right that this right side liquid takes in X.

The lowest row is the s_0-scheme; consequently the s_0-arrows only indicate what can be deduced from point s_0; now we must consider in how far they are in accordance with reality.

For this purpose we divide (18) into seven groups.

Group 1 obtains as long as the three points are still situated in field IV.

Group 2 obtains when s_l comes on the side WY.

Group 3 obtains when s_l is in field I but the other points still in IV.

Group 4 obtains when s_0 comes on the side WY.

Group 5 obtains when s_r is still in IV, but the other points already in I.

Group 6 obtains when s_r comes on the side WY.

Group 7 obtains when the three points are situated in field I.

From this now appears:

in group 1 and 7, so when the three points are situated at the same time either in field IV or in field I, the s_0-arrow indicates the real direction, in which the substance X passes through the membrane; in the transition-groups 2—6 this is no more the case.

Let us e.g. take group 2. It appears from the s_l-dash that the left side liquid does not absorb or give off X at that moment; it follows from the s_r-arrow that the right side liquid gives off X; consequently this has been absorbed by the membrane.

The s_0-arrow, therefore, does not completely concur with reality; for it indicates that X from the right side liquid has gone towards the left side liquid; in reality, however, X has gone from the right side liquid towards the membrane.

It is also clear that the s_0-arrows cannot fully represent the state of things in the other transition-groups either; in each of these groups the s_l- and s_r-arrows namely have contrary direction; consequently they cannot be replaced by a single arrow.

Further consideration of what is really happening in the groups 3—6 is left to the reader.

So we find the following:

the s_0-scheme corresponds with reality as long as point s_0 does not approach one of the sides of the triangle too much; in the vicinity of side WY, however, it obtains no more for the substance X and we have to replace it by the s_l- and s_r-schemes. In general the quantities of X duffusing during these transitions, are small.

Of course the same things obtain for the substances Y or W when point s_0 arrives in the vicinity of the sides WX or XY.

Previously (Gen. III fig. 3) we have already seen that point s_0 can also pass from field IV towards VII during the osmosis; this is the case when the chords 1 . 2 and 1'. 2' (fig. 1) become parallel.

As long as s_0 is situated in field IV we have the scheme:

$$\underset{\longleftarrow}{X} \quad \underset{\longrightarrow}{Y} \quad \underset{\longrightarrow}{W} \quad \ldots \ldots \ldots \quad (23)$$

Then namely the left side liquid gives off a mixture s_0, which contains a negative quantity of X but a positive quantity of Y and W.

When s_0 comes in field VII, then the left side liquid absorbs this mixture; this, however, now contains a positive quantity of X and a negative quantity of Y and W; consequently we again get scheme (23).

At the moment that s_0 passes from field IV towards VII, scheme (23) obtains as well as we have previously (Gen. III) seen. Remarkable is only that at this moment as much of X diffuses towards the left as Y and W towards the right.

Consequently we get the scheme (23) no matter whether s_0 is situated in field IV or VII; as this also obtains for the points s_r and s_l we find, therefore:

when point s_0 passes from field IV towards VII; then the s_0-scheme indicates the directions in which the substances really pass through the membrane.

It is easy to see, however, that this is no more the case for the directions, in which the mixtures L_l and L_r pass through the membrane. For this another scheme with five transition-groups may be deduced; in these cases, however, the quantities of the diffusing mixtures are generally small. The deduction of these schemes I leave to the reader.

In a system with a theoretical membrane the composition of the mixture which really passes through the membrane, is represented by s_0; this point, therefore, can never be situated as shown in figs. 2 or 3. In fig. 2 namely s_0 is situated between the points 1 and 2 and it is clear that L_1 can never pass into L_2 by absorbing or giving off this mixture s_0. Consequently the point s_0 cannot be situated between 1′ and 2′ either. Of course s_0 cannot be situated as drawn in fig. 3, for then L_1 as well as L_2 should have to absorb this mixture.

Fig. 2.

Fig. 3.

In a system with a real membrane the mixtures, really absorbed or given off on the left and on the right side, are represented by s_l and s_r. It is clear now that s_l, which must be situated on the chord 1 . 2, can never be situated between the points 1 and 2; the point s_r which must be situated on the chord 1′ . 2′ can never be situated between the points 1′ and 2′. Although in a system with a theoretical membrane the point s_0 cannot be situated as drawn in figs. 2 and 3, this is actually the case in a system with a real membrane, especially when the points 2 and 2′ draw nearer to each other.

We imagine that system (1) approaches its final condition without changing the composition of its membrane; then the liquids become equal on both sides of the membrane. We represent this system by:

$$l_e \times L_e \mid m_1 \times M_1 \mid r_e \times L_e \quad \ldots \ldots \ldots \quad (24)$$

The final liquid l is now represented in fig. 4 by a point e on the

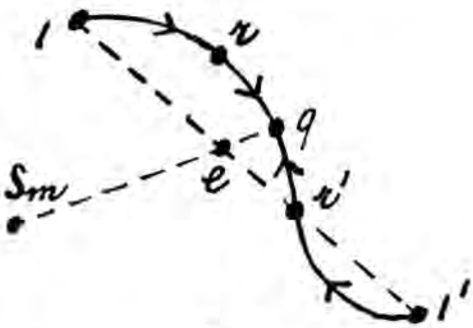

Fig. 4.

line 1 . 1′. As, however, the membrane has absorbed m quantities of a mixture L_m, the system (24) will not come into existence in reality, but a system:

$$l_q \times L_q \mid m_q \times M_q \mid r_q \times L_q \quad \ldots \ldots \ldots \quad (25)$$

This liquid L_q has another composition than L_e and has been represented in fig. 4 by point q. If we represent the composition of L_m by the point s_m, then q must be situated on the prolongation of the line $s_m e$. We now find:

$$e\,q = \frac{m}{r_1 + l_1 - m} \times e\,S_m \quad \ldots \ldots \ldots \quad (26)$$

Consequently the position of the point q depends on the quantity of m and the composition of the mixture L_m which has been absorbed by the membrane. From this appears among other things that even the position of point q depends on the dimensions of the membrane.

Instead of a path with the final point e on the line 1 . 1′ we consequently get a path, which has been shifted a little and has its point in q.

Branch 1′. q of this path now intersects the line 1 . 1′ in a point r'; consequently a point r conjugated with r' must be situated on branch 1 . q.

If the liquid 2 is situated between r and q consequently 2′ between r' and q then we see that the point of intersection s_0 of the chords 1 . 2 and 1′. 2′ is situated as drawn in fig. 2.

If branch 1′. q did not intersect the line 1 . 1′ then s_0 would be situated as drawn in fig. 3.

Consequently we see that towards the end of the osmosis point s_0

can be situated as drawn in the figs. 2 and 3; it is clear that in this case no s_0-scheme can be formed any more.

So the following things appear among others from our considerations.

If we only know the compositions of the liquids on the left and the right side of the membrane, then we can only find the point s_0; this gives only approximated values for s_l en s_r.

In general the s_0-scheme indicates the exact directions, in which the different substances and the mixtures L_l and L_r go through the membrane; it is, however, no longer absolutely valid for any of the substances, when the point s_0 comes in the vicinity of one of the sides of the triangle (e.g. for the substance X in the vicinity of the side WY).

neither does it obtain absolutely any more for the direction of the diffusing mixture when s_0 is situated at infinite distance (e.g. with the transition from field IV towards field VII).

It is of no use towards the end of the osmosis.

If, however, not only the compositions, but also the quantities of the liquids on the left and the right side of the membrane are known, then we can find L_l and L_r and they enable us to determine accurately what has been happening during the osmosis.

(To be continued.)

Leiden, Lab. of Inorganic. Chemistry.

Mathematics. — *Invarianten der Integranden vielfacher Integrale in der Variationsrechnung.* II.*) By Prof. L. KOSCHMIEDER. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of November 26, 1927).

II. *Invarianten der Grundfunktion F.*

Alles folgende bezieht sich auf F (14) als Grundfunktion; in § 3 wurden deren Besonderheiten gegenüber (3) erörtert und dem weiteren dienliche Bezeichnungen eingeführt.

1. *Die erste Variation.*

§ 6. *Die einfachsten Invarianten.*

Wir beginnen mit der sogleich zu benutzenden Bemerkung, dass die Determinante

$$\xi = \begin{vmatrix} \xi_1 & \xi_2 & \dots & \xi_{n+1} \\ x_{1,1} & x_{2,1} & \dots & x_{n+1,1} \\ \dots & \dots & \dots & \dots \\ x_{1,n} & x_{2,n} & \dots & x_{n+1,n} \end{vmatrix} = \theta_i \xi_i \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

in der die ξ_i den in § 3, (18) erklärten Sinn haben, nach (18), (20) eine Punktinvariante vom Gewichte -1 ist,

$$\xi' = D^{-1}\,\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Betrachten wir den Schnitt $x_i = \grave{x}_i$ der Überfläche $\mathfrak{f}$ (2) und einer andern $\grave{\mathfrak{f}}$ mit der Parameterdarstellung $\grave{x}_i = \grave{\varphi}_i(\grave{u}_\alpha)$! Wir bilden bei dieser die Ableitungen $\grave{x}_{i,\alpha} = \partial\grave{\varphi}_i/\partial\grave{u}_\alpha$ und bezeichnen die den θ_k entsprechenden Determinanten der Matrix $\|\grave{x}_{i,\alpha}\|$ mit $\grave{\theta}_k$; dann ist nach (41) die Grösse $\grave{\theta}_i\xi_i$ punktinvariant vom Gewichte -1. Schreiben wir gemäss (16) kurz $\Gamma(x_i, \theta_i) = F$, $\Gamma(\grave{x}_i, \grave{\theta}_i) = \grave{F}$, so finden wir, indem wir nach dem beim Beweise des Satzes 1 gesagten die ξ_i durch die $\partial F/\partial\theta_i$ (21) ersetzen, den Ausdruck

$$\mathfrak{T}_{\mathfrak{f},\grave{\mathfrak{f}}} = \frac{\partial F}{\partial \theta_i}\,\grave{\theta}_i$$

als eine absolute Punktinvariante [34]). Nun ist, wenn $\mathfrak{f}$ eine Extremale des

*) I. in diesen Proceedings, Vol. 31, N⁰. 2, (1928), S. 140.

[34]) Man kann dies sofort aus (20), (21) entnehmen; uns lag an dem Zusammenhange mit (40).

Variationsproblems $\delta I = 0$ ist, $\mathfrak{T}_{\mathfrak{k}, \dot{\mathfrak{k}}}$ die Grösse, deren Verschwinden die *Transversalität* [35]) von $\mathfrak{k}$ zu $\dot{\mathfrak{k}}$ aussagt; daher erweist sich diese Beziehung als invariant. Da sich ferner bei derselben Aufgabe die *E-Funktion* in der Form darstellen lässt [36])

$$E(x_i, \theta_i, \dot{\theta}_i) = \dot{F} - \mathfrak{T}_{\mathfrak{k}, \dot{\mathfrak{k}}},$$

so ist wegen der Invarianz von F und $\mathfrak{T}$ auch E absolut invariant.

Mit Rücksicht darauf, dass die Variationen δx_i (25) Veränderliche der Art ξ_i (18) sind, entnimmt man aus (40), (41) weiter, dass die Grösse

$$v = \theta_i \, \delta x_i, \quad . \quad . \quad . \quad (42) \qquad\qquad v' = D^{-1} v \quad . \quad . \quad . \quad (43)$$

eine Punktinvariante vom Gewichte -1 ist.

Wir wenden uns jetzt zur *Variation der Grundfunktion*; es ist

$$\delta F = \frac{\partial F}{\partial x_i} \delta x_i + \frac{\partial F}{\partial x_{i,\alpha}} \delta x_{i,\alpha} = \frac{\partial}{\partial u_\alpha}\left(\frac{\partial F}{\partial x_{i,\alpha}} \delta x_i\right) + W\theta_i \, \delta x_i, \quad . \quad (44)$$

wo die Ausdrücke $W\theta_i = W_i$ die Variationsableitungen [1]), (7), (8) bedeuten. Nach (19) und Satz 3 ist δF absolut invariant,

$$\delta F' = \delta F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Da dies ferner nach der in (9) enthaltenen Formel

$$\frac{\partial F}{\partial x_{i,\alpha}} = \frac{\partial F'}{\partial x'_{k,\alpha}} \frac{\partial x'_k}{\partial x_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

und nach (25. 1) für die Klammer im ersten Gliede

$$t = \frac{\partial}{\partial u_\alpha}\left(\frac{\partial F}{\partial x_{i,\alpha}} \delta x_i\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

auf der rechten Seite von (44) und daher nach Satz 2 für dieses selbst gilt, so ist dort auch das zweite Glied vW absolut invariant,

$$v' W' = v W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Nach (43) ist mithin W eine Punktinvariante vom Gewichte 1,

$$W' = DW \text{ [37])} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Auf demselben Wege [38]) kann man auch die Parameterinvarianz der Funktion W dartun. Zunächst ist nach (37) und Satz 4

$$\delta \bar{F} = \mathfrak{D} \, \delta F \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

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[35]) Nach RADON [20]), S. 58.

[36]) Ebd. S. 60.

[37]) Dieser kurze Beweis ist dem von BOLZA [9]), S. 349 bei einfachen Integralen gegebenen nachgebildet.

[38]) In anderer Weise leitet RADON [20]), S. 57 die Invarianz (57) her. — A. a. O. [1]), S. 189 habe ich (49), (57) aus der ausdrücklichen Darstellung der W_i entwickelt.

In der Formel (44)

$$\delta F = t + vW \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

nimmt die Grösse t (47) bei dem Wechsel (5) gleichfalls den Faktor $\mathfrak{D}$ an. Denn es ist nach [1]), (31)

$$\left.\begin{aligned} \frac{\partial \bar{F}}{\partial x_{i,\bar{\alpha}}} &= \frac{\partial F}{\partial x_{i,\lambda}} \mathfrak{D} \frac{\partial \bar{u}_\alpha}{\partial u_\lambda}, \\ \bar{t} &= \mathfrak{D} \frac{\partial}{\partial u_\lambda}\left(\frac{\partial F}{\partial x_{i,\lambda}} \delta x_i\right) + \frac{\partial F}{\partial x_{i,\lambda}} \delta x_i \frac{\partial}{\partial \bar{u}_\alpha}\left(\mathfrak{D} \frac{\partial \bar{u}_\alpha}{\partial u_\lambda}\right) \end{aligned}\right\}; \quad . \quad . \quad . \quad (52)$$

infolge der bekannten Beziehung

$$\frac{\partial}{\partial \bar{u}_\alpha}\left(\mathfrak{D} \frac{\partial \bar{u}_\alpha}{\partial u_\lambda}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

bleibt

$$\bar{t} = \mathfrak{D} t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Aus (50), (54) schliesst man mit Hilfe von (51), dass auch

$$\bar{v}\,\overline{W} = \mathfrak{D} v W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

ist; da sich nach (42), (38), (36, 1)

$$\bar{v} = \mathfrak{D} v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

ergibt, erkennt man W als absolute Parameterinvariante,

$$\overline{W} = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

§ 7. *Die Funktion F_1 von* De Donder.

Es sei auf f (2)

$$\theta^2 \equiv \theta_i \theta_i > 0 \text{ [39])} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Wir setzen

$$\frac{1}{\theta^2} \frac{\partial^2 F}{\partial x_{i,\alpha} \partial x_{i,\beta}} = \Phi_{\alpha,\beta} \text{ [40])}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

$$\text{det. } \Phi_{\alpha\beta} = \Phi \ , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Zunächst bringen wir die Determinante Φ zu der Funktion F_1 von De Donder[5]) in Beziehung. Aus der Formel

$$\frac{\partial F}{\partial x_{i,\alpha}} = \frac{\partial F}{\partial \theta_l} \theta_{l,i\alpha} \,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

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39) Vgl. [1]), S. 181; $\theta > 0$.

40) $\Phi_{\alpha,\beta}$ ist in [1]), (5) mit $F_{\alpha,\beta}/\theta^2$, in [6]), (17) mit $F^*_{\alpha,\beta}$ bezeichnet. Φ (60) hat in der Schreibweise [6]), S. 141 den Wert $\Phi \mathfrak{G}^{-2n}$.

in der $\theta_{l,i\alpha} = -\theta_{i,l\alpha}$ die algebraische Ergänzung des Elementes $x_{i,\alpha}$ bezüglich der Determinante θ_i bedeutet ($\theta_{i,i\alpha}$ [41]) $= 0$), erhalten wir

$$\frac{\partial^2 F}{\partial x_{i,\alpha}\,\partial x_{p,\beta}} = \frac{\partial^2 F}{\partial \theta_l \partial \theta_m}\,\theta_{l,i\alpha}\,\theta_{m,p\beta} - \frac{\partial F}{\partial \theta_l}\,\frac{\partial \theta_{i,l\alpha}}{\partial x_{p,\beta}}.$$

Ist $p = i$, so fällt der Subtrahend rechts fort, weil $x_{i,\alpha}$ in θ_i nicht auftritt (§ 3); multipliziert man diese einfachere Gleichung mit $x_{h,\alpha}\,x_{k,\beta}$, wendet die elementaren Determinantenformeln

$$\theta_{p,q\gamma}\,x_{r,\gamma} = j_{qr}\,\theta_p - j_{rp}\,\theta_q, \quad j_{lm} = \begin{cases} 0 \text{ für } l \neq m, \\ 1 \text{ für } l = m \end{cases}$$

an und beachtet die aus (17) durch Differentiation folgende Beziehung

$$\theta_l\,\frac{\partial^2 F}{\partial \theta_l \partial \theta_m} = 0, \text{ [42]} \quad \ldots \ldots \ldots \quad (62)$$

so findet man nach Summation über i

$$\Phi_{\alpha\beta}\,x_{h,\alpha}\,x_{k,\beta} = \frac{\partial^2 F}{\partial \theta_h\,\partial \theta_k}. \quad \ldots \ldots \ldots \quad (63)$$

In der $(n+1)$-reihigen Determinante der Grössen rechts hat die algebraische Ergänzung f_{hk} des Elementes $\partial^2 F / \partial \theta_h\,\partial \theta_k$ nach (63) den Wert

$$f_{hk} = (-1)^{h+k}\,\text{det.}\,(\Phi_{\alpha\beta}\,x_{H,\alpha}\,x_{K,\beta}),$$

wo H und K die Zahlen von 1 bis $n+1$ mit Ausnahme von h und k vorstellen; da hier die rechte Seite sich in das Produkt $\Phi\,\theta_h\,\theta_k$ umformen lässt, stimmt die Determinante Φ in der Tat mit der von DE DONDER durch die Formel

$$f_{hk} \equiv \text{adj.}\,\frac{\partial^2 F}{\partial \theta_h\,\partial \theta_k} = F_1\,\theta_h\,\theta_k \quad \ldots \ldots \ldots \quad (64)$$

erklärten Funktion F_1 überein, *es ist*

$$\Phi = F_1. \quad \ldots \ldots \ldots \ldots \quad (65)$$

Was ferner die Transformation der Ausdrücke $\Phi_{\alpha\beta}$, Φ betrifft, so ergibt sich aus [1]), (20), (23) und der nach u. (38), (58) geltenden Gleichung

$$\bar{\theta} = \mathfrak{D}\theta, \quad \ldots \ldots \ldots \ldots \quad (66)$$

dass

$$\Phi'_{\alpha\beta} = D^2\,\Phi_{\alpha\beta}, \quad \ldots \quad (67) \qquad \overline{\Phi}_{\alpha\beta} = \mathfrak{D}^{-1}\,\Phi_{\lambda\mu}\,\frac{\partial \bar{u}_\alpha\,\partial \bar{u}_\beta}{\partial u_\lambda\,\partial u_\mu}, \quad \ldots \quad (68)$$

$$\Phi' = D^{2n}\,\Phi \text{ [43]}), \quad \ldots \quad (69) \qquad \overline{\Phi} = \mathfrak{D}^{-n-2}\,\Phi \quad \ldots \ldots \quad (70)$$

---

[41]) Hier wird über i nicht summiert.

[42]) Vgl. RADON [20]), S. 55. Die dortigen Grössen $\Phi_{\alpha,\beta}$ und unsere $\Phi_{\alpha,\beta}$ (59) erweisen sich auf Grund von (63) und [20]), (7) als gleich.

[43]) Nach (65) ist also $F_1' = D^{2n} F_1$; nicht $F_1' = D^2 F_1$, wie a. a. O. [5]) irrtümlich angegeben.

ist. Wir setzen Φ wie auch F in dem betrachteten Gebiete [44]) etwa als positiv voraus; dieses Vorzeichen bleibt bei den Verwandlungen (69), (70) und (19), (37) erhalten.

Mit Hilfe der vorstehenden Formeln gewinnt man die Transformation mehrerer später wichtiger Grössen. Nach (67), (69) ist

$$\Phi'_{\alpha 3}\, \Phi'^{-\frac{1}{n}} = \Phi_{\alpha 3}\, \Phi^{-\frac{1}{n}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

hier wie im folgenden verstehen wir unter der mehrdeutigen Potenz deren reellen positiven Wert. Setzt man weiter

$$F^{n+2}\, \Phi = f, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

so findet man aus (19), (37), (69), (70)

$$f' = D^{2n} f, \quad . \quad . \quad (73) \qquad\qquad \bar{f} = f \quad . \quad . \quad . \quad . \quad (74)$$

Die Ausdrücke

$$\vartheta_i = f^{\frac{1}{2n}}\, \theta_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

verwandeln sich auf Grund von (20), (38), (73), (74) wie folgt:

$$\vartheta'_k = \frac{\partial x_i}{\partial x'_k}\, \vartheta_i, \quad . \quad . \quad . \quad (76) \qquad\qquad \bar{\vartheta}_i = \mathfrak{D}\, \vartheta_i \quad . \quad . \quad . \quad (77)$$

Vermöge (69) leitet man aus W (49) die absolute Punktinvariante

$$M = W \Phi^{-\frac{1}{2n}}, \quad . \quad . \quad (78) \qquad\qquad M' = M \text{ [45])} \quad . \quad . \quad (79)$$

her; darüber hinaus zeigen (73), (74), dass die Grösse

$$S = W f^{-\frac{1}{2n}} = W\, (F^{n+2}\, \Phi)^{-\frac{1}{2n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

eine absolute Punkt- und Parameterinvariante ist [46]),

$$S' = S, \quad . \quad . \quad . \quad (81) \qquad\qquad \bar{S} = S \quad . \quad . \quad . \quad . \quad (82)$$

§ 8. *Das Grundintegral als Anlass zu einer Massbestimmung.*

P. FINSLER [47]) und L. BERWALD [48]) haben die Differentialgeometrie eines N-stufigen Raumes aufgebaut, in der auf jeder Kurve die Bogenlänge

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[44]) Dieses ist hinsichtlich der x_i der Bereich X_{n+1} [s. (1)]; die $x_{i,\alpha}$ haben beliebige endliche Werte von der Art, dass (58) gilt.

[45]) Diese Punktinvariante hat DE DONDER a. a. O. [5]) gefunden; die dort mitgeteilte Gestalt $WF_1^{-\frac{1}{2}}$ ist nach [43]) durch $WF_1^{-\frac{1}{2n}}$ zu ersetzen.

[46]) Wir nannten $-S$ a. a. O. [1]), (14) die "mittlere extremale (Ueberflächen-) Krümmung".

[47]) Ueber Kurven und Flächen in allgemeinen Räumen: Dissertation, Göttingen 1918.

[48]) *a*) Jahresber. d. Deutsch. Math.-Ver. **34** (1925), S. 213—220; *b*) Math. Zeitschr. **25** (1926), S. 40—73; *c*) Lotos (Prag) **74** (1926), S. 43—51; *d*) Journ. f. d. reine u. angew. Math. 156 (1927), S. 191—222.

durch das einfache Grundintegral eines Variationsproblems erster Ordnung [49]) erklärt ist. Sieht man in ähnlicher Weise bei jeder im $n+1$-stufigen Raume X_{n+1} gelegenen Überfläche X_n (2) das Integral (15) als Mass ihres Inhaltes an, so ermöglicht dieses n-fache Grundintegral, wie ich im folgenden andeuten will, eine *Massbestimmung in* X_{n+1}. BERWALD bildet seine Entwickelungen in der Form dem Sonderfalle nach, in dem das Integral die Bogenlänge in einem RIEMANNschen Raume $\mathfrak{X}_N$ angibt [50]); entsprechend vergleichen wir hier das allgemeine Integral (15) mit demjenigen besonderen, welches den Inhalt $\mathfrak{J}$ eines Teiles der in einen RIEMANNschen Raum $\mathfrak{X}_{n+1}$ gebetteten Überfläche $\mathfrak{X}_n$ (2) darstellt. Das Linienelement in $\mathfrak{X}_{n+1}$ sei $d\mathfrak{s}^2 = \mathfrak{a}_{ik}\, dx_i\, dx_k$; die $\mathfrak{a}_{ik}$ sind Funktionen der x_l allein. Mit Hilfe ihrer algebraischen Ergänzungen $\mathfrak{A}_{ik}$ bezüglich der Determinante $\mathfrak{a}$ der $\mathfrak{a}_{ik}$ drückt $\mathfrak{J}$ sich in der Gestalt aus

$$\mathfrak{J} = \int^{(n)} \mathfrak{F}\, du, \quad . \quad . \quad (83) \qquad\qquad \mathfrak{F}^2 = \mathfrak{A}_{ik}\, \theta_i \theta_k \text{ [51])}; \quad . \quad . \quad (84)$$

ersichtlich besteht die Beziehung

$$\mathfrak{A}_{ik} = \tfrac{1}{2} \frac{\partial^2 \mathfrak{F}^2}{\partial\theta_i\, \partial\theta_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Demgemäss führt man bei dem Integrale (15) zur Erklärung eines Grundtensors in dem *allgemeinen Raume* X_{n+1} zunächst die Grössen

$$A_{ik} = \tfrac{1}{2} \frac{\partial^2 F^2}{\partial\theta_i\, \partial\theta_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

ein; entsprechend (84) gilt dann die Formel

$$F^2 = A_{ik}\, \theta_i\, \theta_k\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

da F (17) in den θ_i homogen von erster Stufe ist. Mit Ausnahme des Falles (84), in dem die $\mathfrak{A}_{ik}$ reine Ortsfunktionen sind, hängen die A_{ik} ausser von den x_l auch von den θ_l ab.

Die Determinante

$$A = \text{det.}\, A_{ik} = \text{det.} \left(\frac{\partial F}{\partial\theta_i} \frac{\partial F}{\partial\theta_k} + F \frac{\partial^2 F}{\partial\theta_i\, \partial\theta_k} \right)$$

lässt sich leicht berechnen. Schreibt man sie nämlich als Summe von 2^{n+1} Determinanten, deren Elemente teils von der Gestalt $\sigma_{ik} = \frac{\partial F}{\partial\theta_i} \frac{\partial F}{\partial\theta_k}$ sind, teils die Form $\tau_{ik} = F \frac{\partial^2 F}{\partial\theta_i\, \partial\theta_k}$ besitzen, so verschwinden von diesen Determinanten alle, bei denen mindestens zwei Spalten Elemente σ_{ik} enthalten; ferner diejenige, welche aus lauter Elementen τ_{ik} besteht, weil dies nach (64) von ihrer Reziproken $F^{n(n+1)}$ det. f_{ik}

49) $n = 1$, $q = 1$ in der Bezeichnung (3).

50) Vgl. [47]) a), S. 216.

51) Vgl. z. B. [2]), S. 253.

gilt. Es bleiben die Determinanten zu summieren, bei denen die σ_{ik} genau in einer Spalte auftreten; da dann die algebraischen Ergänzungen der σ_{ik} die Werte $F^n f_{ik}$ haben, ergibt sich nach (64), (65), (17), (72)

$$A = \frac{\partial F}{\partial \theta_i} \frac{\partial F}{\partial \theta_k} F^n \Phi \theta_i \theta_k = F^{n+2} \Phi = f \quad . \quad . \quad . \quad . \quad (88)$$

Demnach ist $A \neq 0$. Unter Heranziehung der Grössen (75) kann man also die Invariante F^2 (87) in der Form $A^{-\frac{1}{n}} A_{ik} \vartheta_i \vartheta_k$ darstellen; dabei bilden nach (76) die ϑ_i einen bei dem Wechsel (1) kovarianten Vektor. Folglich sind die Ausdrücke $a^{ik} = A^{-\frac{1}{n}} A_{ik}$ die kontravarianten Komponenten eines Tensors zweiter Stufe; die diesen Sachverhalt ausdrückenden Formeln

$$a'^{lm} = a^{ik} \frac{\partial x'_l}{\partial x_i} \frac{\partial x'_m}{\partial x_k}$$

sind an Hand von (86), (88), (20), (73) leicht auch unmittelbar zu bestätigen. Wir sehen den — übrigens parameterinvarianten — Tensor der a^{ik} als den massbestimmenden *Grundtensor* in X_{n+1} an; seine kovarianten Komponenten sind die mit $A^{\frac{1}{n}-1}$ multiplizierten algebraischen Ergänzungen der Elemente A_{ik} in Bezug auf A. Die a_{ik} sind Funktionen der x_l und — abgesehen von dem Sonderfalle (84) — der θ_l; es ist det. $a_{ik} = A^{\frac{1}{n}}$. Wir erklären jetzt das *Linienelement* in X_{n+1} durch die Formel

$$ds^2 = a_{ik}\, dx_i\, dx_k.$$

Weitere differentialgeometrische Ausführungen in der durch das Vorstehende bezeichneten Richtung werde ich an anderer Stelle folgen lassen.

II. *Die zweite Variation.*

§ 9. *Die Verallgemeinerung der Transformation von* UNDERHILL.

Die zweite Variation der Grundfunktion F ist nach (51)

$$\delta^2 F = \delta t + W \delta v + v \delta W, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

Da aus (42) durch Variation und Multiplikation mit W die Formel

$$W \delta v = W \delta \theta_i\, \delta x_i + W \theta_i\, \delta^2 x_i \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

hervorgeht und ferner gemäss [1]), (8)

$$\delta W_i\, \delta x_i = v \delta W + W \delta \theta_i\, \delta x_i$$

ist, ergibt sich

$$W \delta v + v \delta W = \delta W_i\, \delta x_i + W_i\, \delta^2 x_i, \quad . \quad . \quad . \quad . \quad . \quad (91)$$

$$\delta^2 F = \delta t + \delta W_i\, \delta x_i + W_i\, \delta^2 x_i. \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Der zweite Summand rechts hat, wie aus der a.a.O. [6]), § 1 durchgeführten Rechnung [vgl. dort die Formeln (12), . . . , (15)] ohne Aenderung übernommen werden kann, den Wert

$$\delta W_i\, \delta x_i = \frac{\partial W_i}{\partial x_k}\, \delta x_i\, \delta x_k - F_{ik,\alpha\beta}\, \delta x_i\, \delta x_{k,\alpha\beta}$$
$$+ \left(\frac{\partial^2 F}{\partial x_i\, \partial x_{k,\alpha}} - \frac{\partial^2 F}{\partial x_{i,\alpha}\, \partial x_k} - \frac{\partial}{\partial u_\beta} \frac{\partial^2 F}{\partial x_{i,\beta}\, \partial x_{k,\alpha}} \right) \delta x_i\, \delta x_{k,\alpha}\, ;$$

dem Zeichen $F_{ik,\alpha\beta}$ kommt dabei die a.a.O. [6]), S. 133 bzw. [1]), S. 188 angegebene Bedeutung zu. Schreibt man

$$- F_{ik,\alpha\beta}\, \delta x_i\, \delta x_{k,\alpha\beta} = \frac{\partial F_{ik,\alpha\beta}}{\partial u_\beta}\, \delta x_i\, \delta x_{k,\alpha} - \delta x_i\, \frac{\partial}{\partial u_\beta}\, (F_{ik,\alpha\beta}\, \delta x_{k,\alpha})$$

und zieht den Ausdruck [6]), (16) bzw. [1]), (30) der Vierzeigergrössen heran [40]), so wird weiter

$$\left.\begin{aligned} \delta W_i\, \delta x_i &= \frac{\partial W_i}{\partial x_k}\, \delta x_i\, \delta x_k - \delta x_i\, \frac{\partial}{\partial u_\beta}\, (\Phi_{\alpha\beta}\, \theta_i\, \theta_k\, \delta x_{k,\alpha}) \\ &+ \left[\frac{\partial^2 F}{\partial x_i\, \partial x_{k,\alpha}} - \frac{\partial^2 F}{\partial x_{i,\alpha}\, \partial x_k} + \tfrac{1}{2} \frac{\partial}{\partial u_\beta} \left(\frac{\partial^2 F}{\partial x_{i,\alpha}\, \partial x_{k,\beta}} - \frac{\partial^2 F}{\partial x_{i,\beta}\, \partial x_{k,\alpha}} \right) \right] \delta x_i\, \delta x_{k,\alpha}. \end{aligned}\right\} \quad (93)$$

Hier greift nun die *Hilfsformel*

$$\left.\begin{aligned} &\frac{\partial^2 F}{\partial x_i\, \partial x_{k,\alpha}} - \frac{\partial^2 F}{\partial x_{i,\alpha}\, \partial x_k} + \tfrac{1}{2} \frac{\partial}{\partial u_\beta} \left(\frac{\partial^2 F}{\partial x_{i,\alpha}\, \partial x_{k,\beta}} - \frac{\partial^2 F}{\partial x_{i,\beta}\, \partial x_{k,\alpha}} \right) \\ &= \theta_{i,k\alpha}\, W + \Phi_{\alpha\beta} \left(\frac{\partial \theta_i}{\partial u_\beta}\, \theta_k - \frac{\partial \theta_k}{\partial u_\beta}\, \theta_i \right) \end{aligned}\right\} . \quad (94)$$

ein. Sie ist a.a.O. [6]), § 2, (28) im Falle einer Extremale hergeleitet, d.h. unter der Annahme $W = 0$, von der wir uns hier freimachen. Die Beziehungen [6]), (20), . . . , (26) sind von letzterer unabhängig. Auf der rechten Seite von [6]), (26) ersetzt man das erste Glied nach [1]), (7) durch den Ausdruck $\theta_{k,i\alpha} \left(\dfrac{\partial F}{\partial x_k} - W_k \right)$ [52]) ; die weitere Rechnung unterscheidet sich also von der a.a.O. [6]) angestellten lediglich dadurch, dass man dort rechts das Glied $\theta_{i,k\alpha}\ W_k$ [52]) hinzufügt. Statt [6]), (28) erhält man mithin (94).

Trägt man (94) in (93) ein, beachtet, dass dann rechts $\theta_{i,k\alpha}\, \delta x_{k,\alpha} = \delta \theta_i$ ist, und addiert beiderseits $W \theta_i\, \delta^2 x_i$, so findet man laut (91), (90) mit Rücksicht auf [1]), (8), auf (42) und die Symmetrie der Grössen (59)

$$v\, \delta W + W \delta v = \frac{\partial W_i}{\partial x_k}\, \delta x_i\, \delta x_k - v\, \frac{\partial}{\partial u_\beta}\, (\Phi_{\alpha\beta}\, \theta_k\, \delta x_{k,\alpha})$$
$$- v \Phi_{\alpha\beta}\, \frac{\partial \theta_k}{\partial u_\beta}\, \delta x_{k,\alpha} + W \delta v,$$

$$v\, \delta W = \frac{\partial W_i}{\partial x_k}\, \delta x_i\, \delta x_k - v\, \frac{\partial}{\partial u_\alpha} \left(\Phi_{\alpha\beta} \frac{\partial v}{\partial u_\beta} \right) + v\, \delta x_k\, \frac{\partial}{\partial u_\alpha} \left(\Phi_{\alpha\beta}\, \frac{\partial \theta_k}{\partial u_\beta} \right).$$

[52]) Hier ist nach k nicht zu summieren.

Wenn man daher

$$\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\theta_i}{\partial u_\beta}\right)=P_i \text{ [53]} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (95)$$

$$\tfrac{1}{2}\left(\frac{\partial W_i}{\partial x_k}+\frac{\partial W_k}{\partial x_i}+P_k\,\theta_i+P_i\,\theta_k\right)=L_{ik}=L_{ki} \text{ [54]} \quad . \; . \quad (96)$$

setzt, so wird

$$v\,\delta W=L_{ik}\,\delta x_i\,\delta x_k-v\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v}{\partial u_\beta}\right) \quad . \; . \; . \; . \quad (97)$$

Da bei Einführung der Bezeichnung

$$\frac{\partial W}{\partial x_i}+P_i=Q_i \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (98)$$

nach [1]), (8) die Formeln gelten

$$L_{ik}=\tfrac{1}{2}(Q_i\,\theta_k+Q_k\,\theta_i),\quad L_{ik}\,\delta x_i\,\delta x_k=vQ_i\,\delta x_i, \quad . \; . \; . \quad (99)$$

entnimmt man aus (97) als Wert der Variation von W

$$\delta W=Q_i\,\delta x_i-\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v}{\partial u_\beta}\right) \quad . \; . \; . \; . \; . \; . \quad (100)$$

Durch Einsetzung der Werte (97) bzw. (100) in (89) gewinnt man diejenigen beiden *Darstellungen der Grösse* δ^2F, *welche* UNDERHILLS auf den Fall $n=1$ bezügliche *Transformation der zweiten Variation* [8]) *auf unser Grundintegral* (15) *verallgemeinern.*

Wir zerlegen in § 10 den als Bestandteil von δ^2F auftretenden Ausdruck

$$\delta(vW)=W\delta v-v\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v}{\partial u_\beta}\right)+vQ_i\,\delta x_i, \quad . \; . \; . \; . \quad (101)$$

der nach (48) und Satz 3 eine absolute Punktinvariante ist, in weitere invariante Summanden.

§ 10. *Die Punktinvariante* Ψ.

Es werde in (101) rechterhand statt v (42), (43) die laut (43), (69) absolut punktinvariante Grösse

$$\Omega=\Phi^{\frac{1}{2n}}v, \quad . \; . \; . \quad (102) \qquad\qquad \Omega'=\Omega \quad . \; . \; . \quad (103)$$

eingeführt. Aus $v=\Phi^{-\frac{1}{2n}}\Omega$ folgt

$$\delta v=\Phi^{-\frac{1}{2n}}\delta\Omega-\frac{1}{2n}\,\Omega\Phi^{-\frac{1}{2n}}\frac{\delta\Phi}{\Phi}. \quad . \; . \; . \; . \; . \; . \quad (104)$$

---

[53]) P_i sind die a. a. O. [6]), (33) mit A_i bezeichneten Ausdrücke.

[54]) Ist $n=1$, so stimmen die Grössen L_{11}, $L_{12}=L_{21}$, L_{22} mit den von UNDERHILL [3]), S. 327 eingeführten L_1, M_1, N_1 überein.

und es ist

$$\frac{\delta\Phi}{\Phi}=\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_i}\delta x_i+\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\delta x_i\right)-\delta x_i\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\right)\quad.\quad(105)$$

Dabei kann man die Ausdrücke $\partial\Phi/\partial x_{i,\alpha}$ auch nach Art der Grössen (61) im Sinne von (65) geschrieben denken. Das zweite Glied auf der rechten Seite von (101) bringt man leicht auf die Form

$$v\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v}{\partial u_\beta}\right)=\Omega\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\,\Phi^{-\frac{1}{n}}\frac{\partial\Omega}{\partial u_\beta}\right)+\Omega^2\,\Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right);(106)$$

man erhält daher statt (101)

$$\left.\begin{aligned}\delta(vW)=W\Phi^{-\frac{1}{2n}}\delta\Omega-\frac{1}{2n}\Omega W\Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\delta x_i\right)&\\-\Omega\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\Phi^{-\frac{1}{n}}\frac{\partial\Omega}{\partial u_\beta}\right)-\frac{1}{2n}\Omega W\Phi^{-\frac{1}{2n}}\left[\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_i}-\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\right)\right]\delta x_i&\\+\Omega\Phi^{-\frac{1}{2n}}Q_i\,\delta x_i-\Omega^2\Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right).&\end{aligned}\right\}(107)$$

Wir werden jetzt zeigen, dass in dieser, wie erwähnt absolut invarianten Verbindung jedes der drei ersten Glieder $\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3$ rechts die Invarianzeigenschaft besitzt; ist dies nachgewiesen, so ist eine neue Invariante ermittelt, nämlich die Summe $\mathfrak{Z}$ der drei letzten Glieder auf der rechten Seite von (107).

Die Invarianz $\mathfrak{Z}_1'=\mathfrak{Z}_1$ folgt aus (79), (103) und Satz 3. Was $\mathfrak{Z}_2$ betrifft, so ist nach (69)

$$\frac{\partial\Phi'}{\partial x'_{i,\alpha}}=D^{2n}\frac{\partial\Phi}{\partial x_{k,\alpha}}\frac{\partial x_k}{\partial x'_i},$$

sodass durch Division mit (69) und Zusammensetzung mit den $\delta x'_i=\delta x_l\,.\,\partial x'_i/\partial x_l$ sich ergibt

$$\frac{1}{\Phi'}\frac{\partial\Phi'}{\partial x'_{i,\alpha}}\delta x'_i=\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\delta x_i;\quad.\quad.\quad.\quad.\quad.\quad.\quad.\quad(108)$$

mit Hilfe des Satzes 2 und kraft (79), (103) bestätigt man, dass in der Tat $\mathfrak{Z}_2'=\mathfrak{Z}_2$ ist. In $\mathfrak{Z}_3$ ist nach (71) und Satz 2 zunächst der Inhalt der Klammer invariant, daher auch $\mathfrak{Z}_3=\mathfrak{Z}_3'$.

Nach der an (107) angeschlossenen Bemerkung ist jetzt die Invarianz $\mathfrak{Z}'=\mathfrak{Z}$ dargetan. Mit $\mathfrak{Z}$ ist auch die Grösse $\mathfrak{Z}/\Omega=\Psi$, also *der Ausdruck*

$$\left.\begin{aligned}\Psi(x_i,x_{i,\alpha},x_{i,\alpha\beta},x_{i,\alpha\beta\gamma},\delta x_i)=\frac{1}{2n}W\Phi^{-\frac{1}{2n}}\left[-\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_i}+\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\right)\right]\delta x_i&\\+\Phi^{-\frac{1}{2n}}Q_i\,\delta x_i-\Omega\,\Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right)&\end{aligned}\right\}(109)$$

absolut punktinvariant.

§ 11. *Die Punktinvariante U.*

Da man die Variationen δx_i (25) als Veränderliche ξ_i (18) auffassen kann, treffen auf die Invariante (109) die Voraussetzungen des Satzes 1 zu. Ersetzt man daher in Ψ die δx_i durch die Werte $\partial F/\partial\theta_i$ und somit Ω (102) gemäss (42), (17) durch $\Phi^{\frac{1}{2n}} F$, so entsteht eine Invariante der Art g vom Gewichte 1, aus der nach Multiplikation mit $\Phi^{-\frac{1}{2n}}$ zufolge (69) eine absolute Invariante hervorgeht. Indem wir diese mit $-FU$ bezeichnen, *gelangen wir* bei irgendeiner Überfläche $\mathfrak{f}$ (2) *zu der absoluten Punktinvariante*

$$\left.\begin{aligned} U(x_i, x_{i.\alpha}, x_{i.\alpha\beta}, x_{i.\alpha\beta\gamma}) = &- F^{-1}\,\Phi^{-\frac{1}{n}}\,Q_i\frac{\partial F}{\partial\theta_i} + \Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right)\\ &+\frac{1}{2n}F^{-1}\,W\Phi^{-\frac{1}{n}}\frac{\partial F}{\partial\theta_i}\left[\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_i} - \frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i.\alpha}}\right)\right], \end{aligned}\right\} \quad (110)$$

der Verallgemeinerung der Invariante K von UNDERHILL [55]), die sich aus U für $n=1$ ergibt. Das erste Glied $\mathfrak{Q}$ auf der rechten Seite von (110) kann man nach (17) und (99) auch schreiben

$$\mathfrak{Q} = - F^{-2}\Phi^{-\frac{1}{n}}\,L_{ik}\frac{\partial F}{\partial\theta_i}\frac{\partial F}{\partial\theta_k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

Ist $\mathfrak{f}$ im besonderen eine *Extremale*, also $W=0$, so lassen sich die Grössen L_{ik} durch eine einzige L darstellen in der Form [56])

$$L^*_{ik} = L\theta_i\theta_k \;{}^{57)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

daher vereinfacht sich die Invariante U zu

$$U^* = - F^{-1}\,\Phi^{-\frac{1}{n}}Q^*_i\,\frac{\partial F}{\partial\theta_i} + \Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right). \quad . \quad . \quad (113)$$

bzw.
$$U^* = - \Phi^{-\frac{1}{n}}L + \Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right) \quad . \quad . \quad . \quad . \quad (114)$$

Hiernach ist die Funktion L [6]), (41) der Gestalt fähig

$$L = F^{-1}\,Q^*_i\,\frac{\partial F}{\partial\theta_i}.$$

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[55]) A. a. O. [3]), S. 330.

[56]) Das Zeichen * deutet an, dass die mit ihm versehenen Grössen für eine Extremale zu berechnen sind

[57]) A. a. O. [6]), S. 138. — Die Grösse L verallgemeinert den von K. WEIERSTRASS für $n=1$ mit F_2 [vgl. [9]), S. 226] bezeichneten Ausdruck.

§ 12. *Die Punkt- und Parameterinvariante* Ψ_0.

Um über die Punktinvariante Ψ (109) hinaus zu einem Ausdrucke fortzuschreiten, der ausserdem parameterinvariant ist, variieren wir die Grösse S (80). Diese ist absolut invariant in beiderlei Sinne (81), (82); dieselbe Eigenschaft hat nach den Sätzen 3 und 4 der Ausdruck

$$\delta S = S\left[\frac{\delta W}{W} - \frac{1}{2n}\frac{\delta\Phi}{\Phi} - \left(\frac{1}{2}+\frac{1}{n}\right)\frac{\delta F}{F}\right].$$

Da das laut (19), (45), (37), (50) auch von der Grösse $\delta F/F$ gilt, ist

$$\zeta = \frac{\delta W}{W} - \frac{1}{2n}\frac{\delta\Phi}{\Phi}$$

eine absolute Punkt- und Parameterinvariante, wie man mit Hilfe von (49), (57), (69), (70) auch unmittelbar feststellt.

Zu geeigneter Darstellung von ζ wendet man auf den Minuenden die Formel (100) an; man bedient sich an Stelle von v (42), (56) des Ausdrucks

$$V = F^{-\frac{1}{2}+\frac{1}{n}}\Phi^{\frac{1}{2n}}v = F^{-\frac{1}{2}+\frac{1}{n}}\Omega, \quad . \quad . \quad . \quad . \quad . \quad (115)$$

der sich vermöge (103), (70) als absolut invariant in beiderlei Sinne erweist,

$$V' = V, \quad . \quad . \quad . \quad (116) \qquad\qquad \bar{V} = V. \quad . \quad . \quad . \quad (117)$$

Indem man bei der Umrechnung von (100) die den Faktor V enthaltenden Glieder zusammenfasst und den Subtrahenden in ζ ähnlich wie in (105) umformt, findet man

$$\left.\begin{aligned}\zeta = W^{-1}Q_i\,\delta x_i - F^{-\frac{1}{2}-\frac{1}{n}}\Phi^{-\frac{1}{2n}}S^{-1}V\frac{\partial}{\partial u_\alpha}\left[\Phi_{\alpha\beta}\frac{\partial}{\partial u_\beta}\left(F^{\frac{1}{2}-\frac{1}{n}}\Phi^{-\frac{1}{2n}}\right)\right]\\ -\frac{1}{2n}\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_i}\delta x_i + \frac{1}{2n}\delta x_i\frac{1}{F}\frac{\partial}{\partial u_\alpha}\left(\frac{F}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\right)\\ -\frac{1}{FS}\frac{\partial}{\partial u_\alpha}\left(F^{1-\frac{2}{n}}\Phi^{-\frac{1}{n}}\Phi_{\alpha\beta}\frac{\partial V}{\partial u_\beta}\right) - \frac{1}{2n}\frac{1}{F}\frac{\partial}{\partial u_\alpha}\left(\frac{F}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\delta x_i\right).\end{aligned}\right\} \quad (118)$$

Nun ist hier rechts sowohl das vorletzte Glied $-\mathfrak{z}_4$ als auch das letzte $-\mathfrak{z}_5/2n$ invariant in beiderlei Sinne. Zunächst ist nämlich $\mathfrak{z}_4' = \mathfrak{z}_4$ infolge der Beziehungen (71), (116) und des Satzes 2. Ferner ergibt sich im Hinblick auf (37), (70), (68), (117), (82)

$$\bar{\mathfrak{z}}_4 = \mathfrak{D}^{-1}F^{-1}S^{-1}\frac{\partial}{\partial \bar{u}_\alpha}\left(\mathfrak{D}F^{1-\frac{2}{n}}\Phi^{-\frac{1}{n}}\Phi_{\lambda\mu}\frac{\partial \bar{u}_\alpha}{\partial u_\lambda}\frac{\partial \bar{u}_\beta}{\partial u_\mu}\frac{\partial V}{\partial u_\nu}\frac{\partial u_\nu}{\partial \bar{u}_\beta}\right);$$

benutzt man rechts die (in $u, \bar{u}$ statt in x, x' geschriebenen) Formeln [1]), (27) und differenziert die Klammer als ein Produkt, dessen einer Faktor

$\mathfrak{D} \,.\, \partial u_\alpha / \partial u_\lambda$ ist, so erhält man kraft (53) $\bar{\mathfrak{z}}_4 = \mathfrak{z}_4$. Aus (19), (108) und Satz 2 folgt $\mathfrak{z}_5' = \mathfrak{z}_5$; da nach (70)

$$\frac{1}{\Phi}\frac{\partial \Phi}{\partial x_{i,\alpha}} = \frac{1}{\Phi}\frac{\partial \bar{\Phi}}{\partial x_{i,\lambda}}\frac{\partial \bar{u}_\alpha}{\partial u_\lambda}$$

ist, stellt sich auf Grund von (36) und (53) auch

$$\bar{\mathfrak{z}}_5 = \frac{1}{\mathfrak{D}F}\frac{\partial}{\partial u_\alpha}\left(\mathfrak{D}\frac{\partial u_\alpha}{\partial u_\lambda}\frac{F}{\Phi}\frac{\partial \Phi}{\partial x_{i,\lambda}}\delta x_i\right) = \mathfrak{z}_5$$

heraus. Die Invarianten $-\mathfrak{z}_4$ und $-\mathfrak{z}_5/2n$ trennt man von der Invariante ζ (118) ab; indem man deren verbleibenden Bestandteil mit S multipliziert und

$$\frac{S}{W}Q_i - \frac{1}{2n}\frac{S}{\Phi}\frac{\partial \Phi}{\partial x_i} + \frac{1}{2n}\frac{S}{F}\frac{\partial}{\partial u_\alpha}\left(\frac{F}{\Phi}\frac{\partial \Phi}{\partial x_{i,\alpha}}\right) = R_i$$

setzt, gewinnt man die gewünschte *absolute Invariante in beiderlei Sinne*

$$\Psi_0'(x_i, x_{i,\alpha}, x_{i,\alpha\beta}, x_{i,\alpha\beta\gamma}, \delta x_i) = R_i\,\delta x_i$$

$$- VF^{-\frac{2}{n}}\Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial \Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right) - VF^{-\frac{1}{2}-\frac{1}{n}}\frac{\partial}{\partial u_\alpha}\left(\Phi^{-\frac{1}{n}}\Phi_{\alpha\beta}\frac{\partial F^{\frac{1}{2}-\frac{1}{n}}}{\partial u_\beta}\right).$$

Durch die vorgenommene Auflösung des zweiten Gliedes auf der rechten Seite von (118) in die beiden mit dem Faktor V behafteten Glieder von Ψ_0' wird die Beziehung dieses Ausdrucks zu Ψ' (109) ersichtlich:

$$\left.\begin{aligned}\Psi_0' = F^{-\frac{1}{2}-\frac{1}{n}}\Psi' - VF^{-\frac{1}{2}-\frac{1}{n}}\frac{\partial}{\partial u_\alpha}\left(\Phi^{-\frac{1}{n}}\Phi_{\alpha\beta}\frac{\partial F^{\frac{1}{2}-\frac{1}{n}}}{\partial u_\beta}\right)\\ + \frac{1}{2n}WF^{-\frac{3}{2}-\frac{1}{n}}\Phi^{-1-\frac{1}{2n}}\frac{\partial F}{\partial u_\alpha}\frac{\partial \Phi}{\partial x_{i,\alpha}}\delta x_i\,.\end{aligned}\right\} \quad (119)$$

§ 13. *Die Punkt- und Parameterinvariante U_0.*

Auf die Invariante Ψ_0', in der von den Veränderlichen der zweiten Reihe nur die absolut parameterinvarianten $\delta x_i = \xi_i$ auftreten, und zwar linear homogen, lassen sich die Sätze 1,6 anwenden. Wenn man demgemäss in Ψ_0' die δx_i durch die Grössen $\partial F/\partial \theta_i$ und daher V (115) durch $f^{\frac{1}{2n}}$ (72) ersetzt, erhält man aus Ψ_0' einen Ausdruck T, der punktinvariant vom Gewichte 1 und absolut parameterinvariant ist. Infolge von (73), (74) wird der Quotient $-T/f^{\frac{1}{2n}}$, also *die Grösse*

$$\left.\begin{aligned}U_0(x_i, x_{i,\alpha}, x_{i,\alpha\beta}, x_{i,\alpha\beta\gamma}) = U_0 = F^{-\frac{2}{n}}U\\ + F^{-\frac{1}{2}-\frac{1}{n}}\frac{\partial}{\partial u_\alpha}\left(\Phi^{-\frac{1}{n}}\Phi_{\alpha\beta}\frac{\partial F^{\frac{1}{2}-\frac{1}{n}}}{\partial u_\beta}\right) - \frac{1}{2n}WF^{-2-\frac{2}{n}}\Phi^{-1-\frac{1}{n}}\frac{\partial F}{\partial u_\alpha}\frac{\partial F}{\partial \theta_i}\frac{\partial \Phi}{\partial x_{i,\alpha}}\end{aligned}\right\} (120)$$

eine absolute Punkt- und Parameterinvariante. Sie verallgemeinert die von UNDERHILL *bei einfachen Integralen angegebene Invariante* K_0 [58]), die aus U_0 für $n=1$ hervorgeht.

Für eine *Extremale* hat U_0 den Wert

$$U_0^* = F^{-\frac{2}{n}} U^* + F^{-\frac{1}{2}-\frac{1}{n}} \frac{\partial}{\partial u_\alpha}\left(\Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial F^{\frac{1}{2}-\frac{1}{n}}}{\partial u_\beta}\right) \quad . \quad . \quad (121)$$

oder, wie sich auf Grund von (114) nach kurzer Umrechnung ergibt,

$$U_0^* = -F^{-\frac{2}{n}} \Phi^{-\frac{1}{n}} L + F^{-\frac{1}{2}-\frac{1}{n}} \Phi^{-\frac{1}{2n}} \frac{\partial}{\partial u_\alpha}\left[\Phi_{\alpha\beta} \frac{\partial}{\partial u_\beta}\left(F^{\frac{1}{2}-\frac{1}{n}} \Phi^{-\frac{1}{2n}}\right)\right]. (122)$$

Auf die geometrische Bedeutung von U_0 [59]) bei RIEMANNschen $\mathfrak{X}_{n+1}$ und allgemeinen Räumen X_{n+1} (§ 8) werde ich, wie gesagt, bei anderer Gelegenheit eingehen. Hier sei noch die *übersichtliche Form* vermerkt, die man *der Grösse* $\delta^2 I$ bei einer festberandeten *Extremale* $\mathfrak{f}^*$ mit Hilfe von U^*, U_0^* geben kann, indem man gewissen von UNDERHILL für $n=1$ aufgestellten Formeln [60]) entsprechende beim Integrale (15) nachbildet.

Man bringt unter den genannten Annahmen die zweite Variation

$$\delta^2 I^* = \int^{(n)} \delta^2 F\, du = \int^{(n)} v\, \delta W\, du \quad ^{61)}$$

an Hand von (107), (99), (112) leicht auf die Gestalt

$$\delta^2 I^* = -\int^{(n)} \Omega \left[\frac{\partial}{\partial u_\alpha}\left(\Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial \Omega}{\partial u_\beta}\right) + U^* \Omega\right] du.$$

Die Einführung des Ausdrucks V (115) an Stelle von Ω liefert in Verbindung mit (121)

$$\delta^2 I^* = -\int^{(n)} V \left[\frac{\partial}{\partial u_\alpha}\left(F^{1-\frac{2}{n}} \Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial V}{\partial u_\beta}\right) + U_0^*\, V\, F\right] du;$$

durch teilweise Integration des ersten Gliedes rechts erhält man, indem man das Verschwinden von V längs des Randes berücksichtigt,

$$\delta^2 I^* = \int^{(n)} \left(F^{-\frac{2}{n}} \Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial V}{\partial u_\alpha} \frac{\partial V}{\partial u_\beta} - U_0^*\, V^2\right) F du.$$

Diese Formel gibt Anlass zu einer Bemerkung [62]) über das Vorzeichen

58) A. o. O. [3]), S. 334.

59) Diejenige von K_0 ist von UNDERHILL [3]), S. 338 angegeben. — Die Invarianten K [55]), K_0 von UNDERHILL treten in den neuesten Arbeiten über die Differentialgeometrie des Variationsproblems der Art [3]) auf: BERWALD 48*b*), S. 61; *c*), S. 46, 52; *d*), S. 192.

60) A. a. O. [3]), S. 335 f.

61) Vgl. u. (89).

62) Sie rührt für $n=1$ von UNDERHILL her; [3]), S. 336.

von $\delta^2 I^*$: Ist die quadratische Form $\mathfrak{q} = \Phi_{\alpha\beta}\, w_\alpha\, w_\beta$ etwa definit positiv [63]), und ist $U_0^* < 0$ auf $\mathfrak{f}^*$, so gilt dort $\delta^2 I^* > 0$.

III. *Zerlegung der zweiten Variation in parameterinvariante Summanden.*

Bei der Darstellung (89) der zweiten Variation $\delta^2 F$, die nach (37) und Satz 4 eine Parameterinvariante [64]) vom Gewichte 1 ist, haben gemäss (54), (56), (57) die drei Glieder rechts die gleiche Eigenschaft,

$$\overline{\delta t} = \mathfrak{D}\delta t, \quad \overline{W\delta v} = \mathfrak{D} W\delta v, \quad \overline{v\delta W} = \mathfrak{D} v\delta W \quad . \quad . \quad . \quad (123)$$

Wir werden jetzt das dritte Glied $v\delta W$ weiter in fünf invariante Summanden zerspalten. Dabei nehmen wir in der a.a.O. [6]), S. 141 geschilderten Weise auf die invariante quadratische Differentialform $q_{\alpha\beta}\, du_\alpha\, du_\beta$ [65]) Bezug, deren Grundtensor die kontravarianten Komponenten

$$q^{\alpha\beta} = \theta \Phi_{\alpha\beta}$$

besitzt [40]). Wie dort bedienen wir uns der — bei invarianten χ, ψ gleichfalls invarianten — Differentiatoren

$$\nabla(\chi,\psi) = q^{\alpha\beta} \frac{\partial \chi}{\partial u_\alpha}\frac{\partial \psi}{\partial u_\beta}, \quad \triangle_1(\chi) = \nabla(\chi, \chi), \quad \triangle(\chi) = \frac{1}{\theta}\frac{\partial}{\partial u_\alpha}\left(\theta q^{\alpha\beta}\frac{\partial \chi}{\partial u_\beta}\right).$$

Mit diesen Zeichen kann man unter Verwendung der nach (56), (66) absolut invarianten Grösse

$$\omega = \theta^{-1} v, \quad . \quad . \quad . \quad (124) \qquad\qquad \overline{\omega} = \omega \quad . \quad . \quad . \quad (125)$$

in dem Ausdrucke [s. (101)]

$$v\delta W = -\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\, v \frac{\partial v}{\partial u_\beta}\right) + \Phi_{\alpha\beta}\frac{\partial v}{\partial u_\alpha}\frac{\partial v}{\partial u_\beta} + v Q_i\, \delta x_i$$

das erste und zweite Glied rechts schreiben

$$-\frac{1}{2}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v^2}{\partial u_\beta}\right) = -\frac{1}{2}\theta \triangle(\omega^2) - 2\omega \nabla(\theta, \omega) + \omega^2 [\theta^{-1}\triangle_1(\theta) - \triangle(\theta)],$$

$$\Phi_{\alpha\beta}\frac{\partial v}{\partial u_\alpha}\frac{\partial v}{\partial u_\beta} = \theta^{-1}\triangle_1(\theta\omega) = \theta\triangle_1(\omega) + 2\omega\nabla(\theta, \omega) + \omega^2\theta^{-1}\triangle_1(\theta) \quad {}^{66)}.$$

Setzt man für Q_i den Wert (98) ein, fügt den Summanden $\omega^2 P_i \theta_i$ hinzu und zieht ihn in der Gestalt $\omega\theta^{-1}\,\theta_k P_k \theta_i\, \delta x_i$ wieder ab, so erhält man

$$\left.\begin{aligned} v\delta W = &-\frac{1}{2}\theta\triangle(\omega^2) + \theta\triangle_1(\omega) + \omega^2[2\theta^{-1}\triangle_1(\theta) - \triangle(\theta) + P_i\theta_i] \\ &+ \theta\omega\frac{\partial W}{\partial x_i}\delta x_i + \omega\theta^{-1}\theta_k(P_i\theta_k - P_k\theta_i)\,\delta x_i. \end{aligned}\right\} \quad (126)$$

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63) Der Faktor von $\mathfrak{q}$ ist nach § 7 positiv.

64) Da in III ausschliesslich von Parameterinvarianten die Rede ist, nennen wir diese weiterhin kurz Invarianten.

65) Die a. a. O. mit $g_{\alpha\beta}$ bezeichneten Grössen nennen wir hier $q_{\alpha\beta}$.

66) Vgl. [6]), (51).

Von den Gliedern rechts ist nach (66) und (125) das erste und zweite, nach (57), Satz 5 und (36 . 1) das vierte invariant vom Gewichte 1. Das dritte Glied hat dieselbe Eigenschaft; da nämlich darin der Faktor Θ von ω^2 gemäss [6]), (68) die Umformung gestattet

$$\Theta \equiv 2\,\theta^{-1} \triangle_1(\theta) - \triangle(\theta) + P_i\,\theta_i = \theta^{-1}\,[\triangle_1(\theta) - \sum_i \triangle_1(\theta_i)], \quad (127)$$

so lehrt [6]), (58), dass $\bar{\Theta} = \mathfrak{D}\Theta$. Jetzt schliesst man aus (126) und (123,3), dass auch das fünfte Glied invariant vom Gewichte 1 ist; man überzeugt sich davon auch unmittelbar, indem man in ihm laut (95)

$$P_i\,\theta_k - P_k\,\theta_i = \frac{\partial}{\partial u_\alpha}\left[\Phi_{\alpha\beta}\left(\frac{\partial\theta_i}{\partial u_\beta}\,\theta_k - \frac{\partial\theta_k}{\partial u_\beta}\,\theta_i\right)\right]$$

schreibt und dann (38), (68), (53) heranzieht.

Vermöge (89), (123), (126), (127) erscheint die Grösse $\delta^2 F$ als eine Summe von sieben Invarianten des Gewichtes 1. Demgemäss *zerlegt sich die zweite Variation*

$$\delta^2 I = \int^{(n)} \delta^2 F\, du$$

in sieben parameterinvariante Integrale

$$\left.\begin{aligned} \delta^2 I = &\int^{(n)} \delta t\, du + \int^{(n)} W\delta(\theta\omega)\, du - \frac{1}{2}\int^{(n)} \triangle(\omega^2)\,\theta du \\ &+ \int^{(n)} \triangle_1(\omega)\,\theta du + \int^{(n)} \omega^2\,\theta^{-1}\,[\triangle_1(\theta) - \sum_i \triangle_1(\theta_i)]\, du \\ &+ \int^{(n)} \omega\,\frac{\partial W}{\partial x_i}\,\delta x_i\,\theta\, du + \int^{(n)} \omega\theta^{-1}\theta_k\,(P_i\,\theta_k - P_k\theta_i)\,\delta x_i\, du. \end{aligned}\right\} \quad (128)$$

Bei festgehaltener $(n-1)$-stufiger Begrenzung verschwinden das erste und dritte von ihnen.

Dass unter dieser Annahme die Formel (128) für eine Extremale $W = 0$ in diejenige übergeht, welche ich in diesem Sonderfalle a.a. O.[6]), (69) angegeben habe, bestätigt man so: Weil dann nach [6]), S. 139

$$\frac{\partial W_k}{\partial x_i} + P_i\,\theta_k = \frac{\partial W_i}{\partial x_k} + P_k\,\theta_i$$

ist, vereinfacht sich die Summe des sechsten und siebenten Integranden nach [1]), (8) und nach (42), (124) wie folgt:

$$\omega\theta^{-1}\theta_k\left(\frac{\partial W_k}{\partial x_i} + P_i\,\theta_k - P_k\theta_i\right)\delta x_i = \omega\theta^{-1}\,\theta_k\,\frac{\partial W_i}{\partial x_k}\,\delta x_i = \omega^2\,\frac{\partial W_k}{\partial x_k}.$$

Physics. — *The best method of measurement of a resistance thermometer.* (21st Communication of results obtained by the aid of the "VAN DER WAALS-Fund"). By A. MICHELS and P. GEELS. (Communicated by Prof. J. D. VAN DER WAALS Jr.).

(Communicated at the meeting of December 17, 1927).

The present communication forms a continuation of the 19th communication [1]) in which the replacement of the ice-point of the thermometer scale by another fixed point, reproducible to within $^1/_{4000}{}^\circ$, was proposed. In connection with the desired accuracy, it was found necessary to investigate the factors, which determine the accuracy of a resistance thermometer, and to find how the influence of these factors could be reduced to a minimum.

The following considerations are also partially applicable to other observations. The use of a resistance thermometer depends on the change of the resistance of a measuring wire with temperature, and other influences which result in an alteration to the resistance (for example the pressure effect) are amenable to similar treatment.

Besides the external factors, such as the choice of galvanometer, the accuracy of the resistance boxes used etc., which influence any resistance measurement, the most troublesome factor in the use of a resistance thermometer is the temperature rise of the measuring wire resulting from the measuring current.

In an absolute temperature measurement it is therefore desirable not to work with current which results in a temperature rise of the wire greater than the accuracy with which it is desired to establish the temperature. [2])

The temperature rise is determined by two factors, the amount of heat evolved by the Joule effect and the velocity with which this heat is dissipated to the surroundings. The latter is very greatly influenced by the construction of the thermometer and the best construction will be that with which the heat is dissipated as rapidly as possible, in other words a thermometer with as small a lag as possible, a factor also very desirable for other reasons.

The lower limit of this lag will be largely determined by other conditions such as insulation, stability etc. which will not be considered further.

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[1]) These Proceedings, **30**, p. 1017 (1927).

[2]) N.B. Actually it should be permissible to work with a constant temperature rise, but it would then be necessary to be certain of the constancy.

Even though it is understood that the best arrangement has been chosen so far as these factors are concerned, a large variation may still be made as to the length and diameter of the wire and the measuring current.

The temperature of the wire is given by the expression:

$$\delta t = \beta \frac{i^2}{d^3}$$

where β is a constant, i the measuring current and d the diameter of the wire. This relation between δt and i has already been tested and established [1]). The temperature rise of the wire is therefore proportional to the square of the measuring current.

From the 20[th] communication (to be published in the following number of these Proceedings, vol. **31**) it follows that, if i is the current strength in the measuring wire and dR an arbitrary alteration to the resistance R, then the galvanometer deflection is given by

$$a = C \frac{i \dfrac{dR}{R} \times R}{\sqrt{g + R_0 + R}},$$

when a moving coil galvanometer within its aperiodic limits is used.

If a moving coil galvanometer in a constant field is used, this expression only differs in the numerator, the root being replaced by the first power.

As a moving coil galvanometer is usually used, the derivation will be given for this instrument in its aperiodic limit and only the result given for the other case, which may be obtained in exactly the same way.

It is hardly necessary to mention that the formulae used hold for almost any circuit, whether a potentiometer, a differential or a bridge method is used. In the last circuit it is understood that $n \gg 1$ (loc. cit. for the notation used). These conditions are not sufficient in the case of the THOMSON Bridge, but this bridge is of little importance for the present purpose. The only alteration that can occur is in the value of R_0, which disappears in some cases, and which is always small compared to the galvanometer resistance g (loc. cit.).

In order to simplify the present calculations $g + R_0$ has been replaced by G, so that the above expression becomes

$$a = C \frac{i \dfrac{dR}{R} \times R}{\sqrt{G + R}}.$$

It is at once apparent from this expression that the accuracy of the measurement is directly proportional to i, whilst, as already observed, δt is proportional to i^2. These are therefore two opposed influences.

[1]) 17[th] Communication of the VAN DER WAALS Fund. These Proceedings **30**, p. 47.

The limit of the measurable temperature interval is that interval which is just equal to the temperature rise of the wire itself.

It is thus necessary to know the length and diameter of the wire for a given resistance R, with which a minimum temperature rise is obtained.

As already indicated

$$\delta t = \beta \frac{i^2}{d^3}$$

whilst the resistance R is given by

$$R = \gamma \frac{l}{d^2}$$

where γ is a constant and 1 the length of the wire.

From these expressions it follows that the smallest temperature rise with a given current is obtained when d is as large as possible, and, therefore, for a given R, when the wire is as long as possible.

This is also clearly shown by eliminating d from the above two expressions to give

$$\delta t = A i^2 \left(\frac{R}{l}\right)^{3/2}$$

(A is a constant).

Thus, from either point of view, it is desirable to make the wire as long as possible. Other external factors, such as the winding space, necessary distance for insulation etc., will determine the value of l. If it is assumed that l is made as large as possible in relation to the method of measurement, l may be considered as a constant and will disappear as a variable from the equations.

The problem may then be solved as follows.

Let $\triangle t$ be the temperature alteration which it is desired to measure and δt the temperature increase which may be tolerated (this may be left undecided, if a relation is afterwards established between $\triangle t$ and δt).

As δt has been chosen, it may be treated as a constant.

The deflection of the galvanometer is given by

$$a = C \frac{i \frac{dR}{R} R}{\sqrt{G+R}}$$

in which $\frac{dR}{R}$ is proportional to $\triangle t$.

Put $C \frac{dR}{R} = D \triangle t$

$$a = D \triangle t \frac{iR}{\sqrt{G+R}}$$

It is now necessary to find the minimum value of $\triangle t$ under the given condition that the temperature rise is not greater than δt. This is a limiting condition capable of mathematical determination

$$\delta t = \beta \frac{i^2}{d^3} \quad \text{whilst} \quad R = \gamma \frac{l}{d^2}.$$

Eliminating d and bringing all the constants (including δt) under one letter, the limiting condition may be expressed as

$$i^4 R^3 = E.$$

There is an experimental value of α, which is the smallest value observable. Let this be μ, then the smallest value of $\triangle t$ is given by

$$\mu = D \triangle t \frac{iR}{\sqrt{G+R}}$$

when $\frac{iR}{\sqrt{G+R}}$, which may be represented by z, is made as large as possible within the limiting conditions.

If z is plotted in a space diagram as a function of i and R (the x- and y-axis respectively), the question is reduced to the determination of the maximum of a surface with a border condition. This condition defines a space curve on the surface. From the expression $z = \frac{iR}{\sqrt{G+R}}$ it appears that the surface is regular and that the boundary lines go through the R-axis and run parallel to the $(z-i)$ surface. The surface therefore possesses no *absolute* maximum, although it reaches a maximum value on the boundary, and the question is therefore reduced to the determination of the maximum of the space curve defined by the two equations

$$z = \frac{iR}{\sqrt{G+R}}$$

$$i^4 R^3 = E.$$

This determination is made as follows:

$$F = K \frac{iR}{\sqrt{G+R}} + \lambda (i^4 R^3 - E) \qquad i^4 R^3 - E = 0$$

$$F'_{(i)} = K \frac{R}{\sqrt{G+R}} + 4 \lambda i^3 R^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$F'_{(R)} = K \frac{i}{\sqrt{G+R}} - \frac{1}{2} K \frac{iR}{\sqrt{G+R}^3} + 3 \lambda i^4 R^2 = 0 \quad . \quad . \quad (2)$$

(1) and (2) give:

$$\frac{K}{\sqrt{G+R}} + 4\lambda i^3 R^2 = 0. \quad \ldots \ldots \quad (3)$$

$$\frac{K}{\sqrt{G+R}} - \frac{1}{2}\frac{KR}{\sqrt{G+R}^3} + 3\lambda i^3 R^2 = 0.$$

$$\frac{1}{2}\frac{KR}{\sqrt{G+R}^3} + \lambda i^3 R^2 = 0$$

substituting in 3

$$\frac{K}{\sqrt{G+R}} - 2\frac{KR}{\sqrt{G+R}^3} = 0$$

$$R = G.$$

The result $R = G$ is independent of the value of E and therefore of δt and hence holds for the case chosen $\delta t = \triangle t$.

In the latter case it is possible to obtain a simpler solution, using the same proof that the maximum lies on the border curve, as the expression $\triangle t = \beta \frac{i^3}{a^3}$ is not then a condition for the maximum, but is an absolute equation.

Eliminating d between this equation and $R = \gamma \frac{l}{d^2}$ gives

$$\triangle t^2 = H i^4 R^3.$$

Solving for i and substituting the value in

$$a = D\triangle t \frac{iR}{\sqrt{G+R}}.$$

$$= L(\triangle t)^{3/2} \frac{R^{1/4}}{\sqrt{G+R}} \qquad (H \text{ and } L \text{ constant})$$

or for minimum $a = \mu$

$$\mu = L(\triangle t)^{3/2} \cdot \frac{1}{\sqrt[4]{\frac{(R+G)^2}{R}}}.$$

The smallest value of $\triangle t$ is obtained when $\frac{(G+R)^2}{R}$ is a minimum. Differentiation gives $G = R$.

It thus appears that the best value is found when R is made equal to G. R is therefore determined, and where l is fixed, d is also established. A simple numerical calculation shows that the maximum is not very pronounced and that a variation of 50 % in d does not make any appreciable alteration to the best conditions.

The value of $\triangle t$ corresponding to the value of R can only be calculated when the necessary experimental data relating to the radiation, galvanometer sensitivity etc. are known.

The above deduction is only practicable when the value $R = G$ lies within the limits in which the variable shunt resistance can be regulated.

A similar calculation for a moving coil galvanometer in a constant field gives $R = {}^1/_3\, G$.

In conclusion a few notes on a circuit with overlapping shunts will be given in connection with the above.

In this circuit it is only the difference between the two currents passing through the galvanometer circuits, that acts as a directing current on the galvanometer. This results in a large current being passed through each of the galvanometer coils and the limitations of the galvanometer current being reached before those of the current in the measuring wire. This inconvenience may be avoided in the following way:

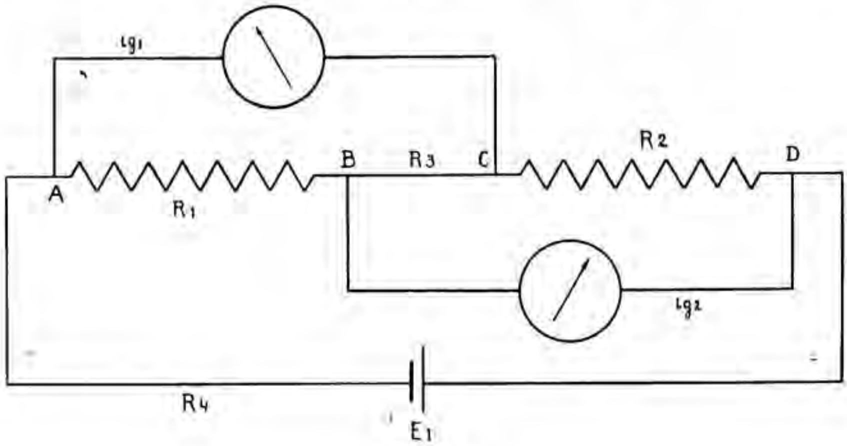

Fig. 1.

Fig. 1 is a schematic diagram of a Kohlrausch circuit, the current commutation being omitted.

Take, as is sufficient for the present derivation, the case when the two resistances R_1 and R_2 are equal. The galvanometer is adjusted to give no deflection. An alteration of the resistance from $R \rightarrow R + dR$ gives a deflection, which is determined by the algebraic sum of i^{g_1} and i_{g_2}.

In the calculation of the deflection dR may be replaced by an *E.M.F.* idR, E left out of consideration and R_4 broken.

In the equilibrium condition an *E.M.F.* in R_3 will also make no alteration to the algebraic sum of i_{g_1} and i_{g_2}.

According to the superposition law an *EMF* may be introduced into R_3 as well as into R_4. The two *EMF*'s together will not influence the algebraic sum $i_{g_1} + i_{g_2}$. The latter, and therefore the galvanometer deflection, will remain exclusively determined by idR. If E_1 and E_2 are chosen opposite in sign i_{g_1} and i_{g_2} may be both reduced to a very small value by the exact choice of E_1 and E_2.

The Steinwehr commutator must be modified to incorporate this addition.

It is desirable to place a shunt across both E_1 or E_2 in order to obtain an exact regulation.

Figure 2 gives the potential fall in the main circuit.

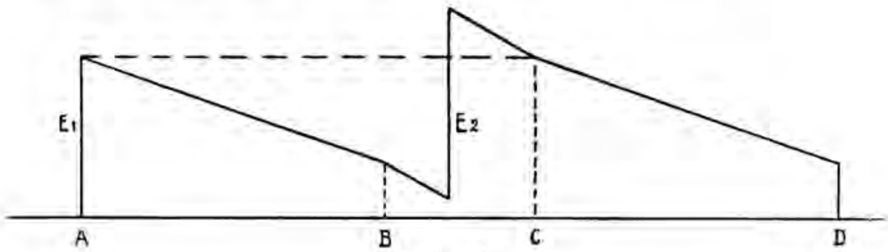

Fig. 2.

Geology. — *Alkaline rocks of the volcano Merapi (Java) and the origin of these rocks.* By H. A. BROUWER.

(Communicated at the meeting of February 25, 1928).

Since HARKER [1]) and BECKE [2]) have separated an atlantic from a pacific facies of eruptive rocks, of wich the atlantic rocks with the same acidity are richer in alkalies, many exceptions to this rule have been found. The opinion that alkaline rocks are connected with radial and sub-alkaline rocks with tangential movements of the earth-crust has been carried very far, but it has been stated, that in different cases there may be but a remote connection with tectonic movements as f.i. by the influence of these movements on the erosion, by which products of differentiation formed at various depths, can be found now at different places at the surface.

Of late different opinions on the origin of alkaline rocks have been put forward, partly emphasizing the absorption of limestone by sub-alkaline magmas (DALY) [3]) partly emphasizing differentiation during crystallization (BOWEN) [4]).

Nearly all the volcanoes of Java have produced pyroxene andesites and basalts, near the north coast of Java (Moeriah, Loeroes, Ringgit, island of Bawean) however, there are volcanoes, from which leucite- and nepheline-bearing rocks have been erupted.

For the study of the differentiation in a volcanic magma, which is not accessible for direct observation, the study of the xenoliths in the volcanic rocks is of much importance. If reaction with limestone really produces alkaline magma the study of exomorphic and endomorphic contactmetamorphism of xenoliths could give a decisive answer. Till now my studies on contactmetamorphism in the East-Indies did not give any indication on the production of alkaline magma in connection with xenoliths of limestone. The minerals formed under the influence of the magma in the limestone are

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1) A. HARKER. The Natural History of Igneous Rocks. Article in Science Progress VI, 1896, p. 12 and in bookform 1909.

2) F. BECKE. Die Eruptivgebiete des böhmischen Mittelgebirges und der amerikanischen Andes. Atlantische und Pazifische Sippe der Eruptivgesteine. Tscherm. Min. Petr. Mitt. XXII, 1903, p. 209—265.

3) R. A. DALY. Origin of alkaline rocks. Bull. Geol. Soc. America Vol. 21. 1910. p. 87. Ibid. Igneous Rocks and their origin. 1914. blz. 410. Ibid. Genesis of alkaline rocks. Journ. of Geol. XXVI. 1918. p. 97.

4) N. L. BOWEN. The later stages of the evolution in igneous rocks. Journ. of Geol. XXIII, suppl. N⁰. 8. 1915. Ibid. Crystallization-Differentiation in igneous magmas. Journ. of Geol. XXVII, 1919, blz. 393. Ibid. The behaviour of inclusions in igneous magmas. Journ. of Geol. XXX, suppl. p. 513.

the same as those in other volcanic regions; they are in the first place lime-bearing minerals, as wollastonite, pyroxene, idocrase, garnet, anortite. By the influence of volatile constituents of the magma several other minerals are also formed as for instance in the metamorphosed xenoliths of limestone in Middle- and South-Italy, of which the Somma is a well-known locality.

Rocks and Xenoliths of the Merapi.

The sediments of most of the volcanoes on the island of Java belong to the pyroxene andesites and basalts and the rocks of the Merapi are no exception on this rule. VERBEEK and FENNEMA [1]) mention different pyroxene andesites mostly with a small olivine content, pyroxene andesite with some amphibole is also found.

The top of the volcano is formed by a lava dome of pyroxene andesite in which amphibole and very little olivine is found [2]). At my request Dr. G. L. L. KEMMERLING collected a number of xenoliths in the volcanic rocks of the Merapi, which were sent to me by the "Dienst van den Mijnbouw" and this collection was completed by myself during an ascension of the volcano in October 1923. Most of these xenoliths have been collected in rocks of the lava dome, some samples of the lava dome were studied under the microscope, they are pyroxene andesites with hyperstene in a much smaller quantity and in smaller crystals than augite. The numerous plagioclase phenocrysts have a zonal structure with frequent alternations of more basic and more acid zones so that the margin is only slightly more acid than the bytownitic central part. Larger crystals of ore are also found. In some slides were found rests of brown amphibole, which are mostly strongly resorbed. Some large crystals of amphibole which are up to several centimeters in length can more likely be considered as xenoliths than as real elements of the volcanic rocks.

The groundmass consists of plagioclase, pyroxene, iron ore and a varying quantity of glass.

Of the xenoliths we only mention those of sedimentary origin. They are principally metamorphic limestones, sandstones and arkoses. Only the metamorphic limestones are of importance for our present subject.

Metamorphic Limestones.

There are different mineral associations belonging to the metamorphic limestones, which appear partly in the same xenoliths but are also found separately as different xenoliths. Calcite is found in several xenoliths,

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[1]) R. D. M. VERBEEK en R. FENNEMA. Description géologique de Java et Madoera I. p. 322.

[2]) G. L. L. KEMMERLING. De hernieuwde werking van den vulkaan G. Merapi (Midden-Java) van begin Augustus 1920 tot en met einde Februari 1921. Vulkanologische mededeelingen. N⁰. 3. 1921. p. 28.

Fig. 1. Trachyte with porous groundmass with much glass and numerous small lath-shaped crystals of orthoclase. Enlarged $\times$ 40.

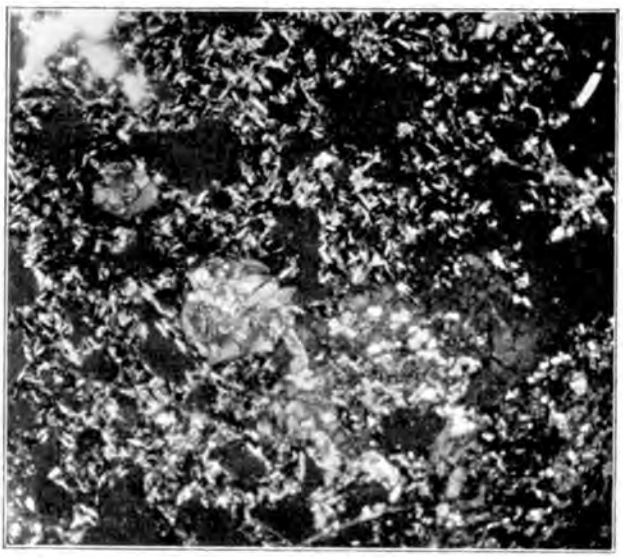

Fig. 2. Leucite phonolite with phenocrysts of leucite in a groundmass with much glass and lath-shaped crystals of orthoclase. In the fig. four leucite phenocrysts with strong optic anomalies are visible. Enlarged $\times$ 40.

sometimes in very small quantities only. The following mineral associations can be distinguished.

1. Wollastonite-diopside. They are partly fine-grained mixtures of these minerals with small quantities of ore, plagioclase and carbonate. More coarsely grained parts in these xenoliths consist of wollastonite and diopside, in which small pyroxene crystals often are enclosed by larger wollastonite crystals. At the contact of the andesite and the xenoliths a narrow zone of a substance rich in iron ore is found.

In other xenoliths parts with much wollastonite alternate with green parts with much pyroxene. Along the contact of andesite and xenoliths is found a zone, which is rich in pyroxene and iron ore with plagioclase and brownish glass; a light-brown glass is also found between and in the crystals of wollastonite.

2. Garnet-wollastonite-epidote. They are xenoliths with much carbonate. Yellowish-green garnet and a mineral of the epidote group with a low double-refraction are found in a fine-grained mixture. Wollastonite is principally found outside this mixture and is also intergrown with carbonate.

3. Garnet-wollastonite-epidote-plagioclase-diopside. Parts of the xenolith consist of a mixture of strongly double-refracting epidote and light-brownish yellow garnet, which sometimes enclose parts which nearly entirely consist of wollastonite. Other parts consist of a plagioclase-pyroxene mixture. The xenolith forms the central part of a homoeogeneous xenolith consisting of a fine-grained mixture of plagioclase and diopside which at the margin of the metamorphic limestone is characterized by a strong increase of the pyroxene content with regard to the plagioclase. The xenolith has a rather considerable carbonate content.

4. Contactmetamorphic limestone with zonal structure. A block of large dimensions ($50 \times 50 \times 30$ cm) collected on the lahar field of the Kali Batang has a zonal structure and has been described by KEMMERLING as a greyish-green schist, probably belonging to the diabase- and chlorite schists, which are found at the surface among the pre-Tertiary rocks of the Djiwo mountains South of Klaten (KEMMERLING loc. cit. p. 29). It is however a contactmetamorphic limestone, in which zones or lenses of different mineralogical composition and colour alternate.

The mineral associations in a certain zone often change from place to place, in different zones certain minerals are predominating, which can however appear also in smaller quantities in other zones. There are for instance light-coloured zones, which consist of a mixture of wollastonite and carbonate in varying quantities, whether or no with plagioclase and augite, which also can predominate together. Leucite is a common mineral which is found in different zones, also with basic plagioclase, biotite and augite. Some zones consist of orthoclase and augite with some calcite. The leucite-bearing zones represent phanerites with preponderant leucite, which are extremely scarce among the igneous rocks.

Of great importance are phonolitic and trachytic zones mostly with phenocrysts of leucite or orthoclase and with much glass in the groundmass. These will be described in some more detail.

Trachytes and Phonolites.

As has been mentioned above these are found as zones of varying composition in the contactmetamorphic limestones with zonal texture. Two main types can be distinguished :

a. phenocrysts of orthoclase, sometimes with the fissures of sanidine, are imbedded in a groundmass with much glass and numerous small lath-shaped crystals of orthoclase or acid plagioclase. Pyroxene microlites also occur, they partly have a small extinction-angle and are optical negative which indicates the presence of the aegirine molecule (fig. 1).

b. The groundmass differs from that mentioned sub *a* by the abundance of leucite in more or less idomorphic or rounded crystals, leucite is also found as small phenocrysts or is restricted to the phenocrysts (fig. 2). The optic anomalies, sometimes with distinct polysynthetic twins, a strong potash reaction and the positive double-refraction confirm the determination as leucite.

In both types (*a* and *b*) calcite, apparently of magmatic origin, is locally found. Both types form transitions into mineral associations, in which the glass-bearing groundmass disappears. These partly leucite-bearing phanerites have been mentioned already above. Besides, orthoclase is not the only mineral, which appears as phenocrysts, for exceptionally a rather basic plagioclase is also found, which shows the exsistence of transitions to more andesitic types. Also augite is locally found as small phenocrysts in the zones mentioned sub *a* and *b*. The great importance however lies in the presence of zones with the composition of trachytes and leucite phonolites, while for the rest the magma of the Merapi has produced pyroxene andesites only.

A connection between the origin of the trachytic and phonolitic zones and a reaction of the pyroxene andestic magma with the metamorphic limestones, in and at the margin of which the zones are found, is abvious. The peculiar texture and the great dimensions of the block of metamorphic limestone might be indications that it is a part of the wall-rock, which had already been metamorphosed, before it was detached and enclosed in the magma. The great importance of the study of the xenoliths lies in the conclusions that can be obtained with regard to the processes, that take place in the deeper parts of a volcano. There are no experimental results available which illustrate the assimilation of limestone by an andesitic or basaltic magma, while the batholitic assimilation can only be judged by its consequences, which are explained in different ways. The phenomena, which have been described above have not exactly the same value as an experiment because the supposition is possible that a concentration of the volatile constituents with the alkalies took place in the conduit and that the association with limestone took place afterwards. If however such a differentiation by crystallization produced alkaline rocks in the Merapi, the expectation would be founded, that from the Merapi and from the numerous other volcanoes of Java, which produced pyroxene andesites and basalts only, at least a single piece of alkaline rock, without connection with limestone, would have become known. These expectations do not agree with the facts.

Distribution of alkaline rocks on Java.

In how far assimilation of limestone or differentiation by crystallization can be considered to be the cause of the origin of the alkaline magmas must be considered in every separate case. And though BOWEN states that assimilation has been but a small factor in the production of the great variety of eruptive rocks, be does not exclude the possibility of limestone-assimilation f.i. for the formation of melilite basalt and some other alkaline rocks.

Although this question cannot be decided for the volcanoes of Java, it is of importance to consider in how far it is possible, that assimilation of limestone has played a part in the production of the alkaline rocks.

Alkaline rocks are only known near the north coast of the eastern part of Java (Moeriah near Semarang, island of Bawean north of Soerabaya, Loeroes west of and Ringgit east of Besoeki). The division between this region, where alkaline rocks are found and by far the greatest part of Java where the volcanoes erupted pyroxene andesites and basalts only, cannot be made too sharply for besides alkaline rocks subalkaline rocks are found among the products of the same volcanoes. Leucite basalt, amphibole andesite, basalt (partly orthoclase-bearing) and trachyandesite are known from the Loeroes, and from the Ringgit we know leucitite, leucite basalt, tephrite and basanite and also trachyandesite without leucite, andesite and basalt. On the other hand the Merapi, a typical representative of the Javanese andesite- and basalt volcanoes has produced trachyte and phonolite although in a small quantity. As a matter of course our knowledge of the substratum of the volcanoes is incomplete, but a facies rich in limestones is characteristic for the Tertiary near the north coast of Rembang and in Madoera, while thick beds of limestone which must have given a great possibility for assimilation, are found near alkaline rocks in South-Celebes f.i. in the vicinity of the Peak of Maros. Perhaps the dip of the limestones of the Gg. Toegoe, south-east of Klaten can be connected with the occurrence of xenoliths of limestone in the products of the Merapi. The various explanations which have been given for the origin of alkaline magmas show certain features in common despite great differences of emphasis, as has been stated already by SMYTH [1]), who considers the origin

[1]) C. H. SMYTH. The chemical composition of the alkaline rocks and its significance as to their origin. Amer. Journ. of Science. XXXVI, 1913, blz. 33.

of alkaline rocks to be principally affected through the agency of mineralizers, the influence of which is also taken into consideration by other authors.

The pneumatolytic phenomena, which take place in the magma during the long periods of dormancy of a volcano could favour the production of an alkaline magma on a larger scale than we described for the volcano Merapi. An important addition of lime will not be prevented by the progressive crystallization of the magma if the magma is very fluid and rich in fugitive constituents. Without reaction with limestone the alkalies could also be concentrated but there is not one of the numerous volcanoes of Java, which produced pyroxene andesites and basalts only, where we can find an indication, that alkaline differentiates have been produced in this way.

Lastly also the mechanical hypothesis to explain the distribution of alkaline rocks cannot be left entirely out of consideration. For Java the difference in stability between north- and southcoast is a striking feature, but there are no data from which could be concluded that the differentiation is influenced by the doubtless strongly changing crustal movements.

A further study of the limestone-xenoliths in the products of the different volcanoes of Java will be of interest. In the first place the attention can be drawn to those volcanoes near the north coast, of which no alkaline rocks are known.

Physics. — *Experiments on the velocity distribution in the boundary layer along a rough surface; determination of the resistance experienced by this surface*. By B. G. VAN DER HEGGE ZIJNEN. (Mededeeling N⁰. 10 uit het Laboratorium voor Aerodynamica en Hydrodynamica der Technische Hoogeschool te Delft.) (Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of February 25, 1928).

§ 1. *Introduction.*

In a discussion of the experimental data concerning the resistance experienced by the flow through pipes and channels with rough walls, HOPF [1]) deduced that this resistance is proportional to the square of the mean velocity, at least if the state of motion is entirely turbulent. According to FROMM [2]) the same law holds for the pressure drop in a rectangular channel, the walls of which are composed of surfaces having a well defined roughness. Evidently in these cases the resistance coefficient is independent of REYNOLDS' number and is determined entirely by the dimensions of the channel and of the projections on its surface.

However, the numerous experiments on the resistance of flow still need completion and extension; researches on the distribution of the velocity in the boundary layer, on the development of the boundary layer along the surface and more detailed experiments concerning the resistance of rough surfaces are of importance for a better understanding of the phenomena observed by several experimenters.

The opportunity for carrying out such researches in the Laboratory for Aerodynamics and Hydrodynamics of the Technical University at Delft presented itself in the beginning of 1925, when by the courtesy of Prof. VON KÁRMÁN a sheat of "waffle-plate", of about the same aspect as used by FROMM in his researches on rectangular channels, was put at our disposal. The experiments were finished in 1927, when the measurements on the velocity distribution could be completed and checked by weighing the resistance experienced by a board, covered on both sides with waffle-plate, directly on a balance.

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[1]) L. HOPF. Abhandl. a. d. Aerodynamischen Institut der Techn. Hochschule Aachen III, 1924, p. 1; Zeitschr. f. angewandte Math. u. Mech. 3, 1923, p. 329.

[2]) K. FROMM. Abhandl. a. d. Aerodynamischen Institut der Techn. Hochschule Aachen III, 1924; Zeitschr. f. angewandte Math. u. Mech., 3, 1923, p. 339.

§ 2. *Summary of the principal theoretical data about the motion in the boundary layer.*

Before describing the experiments performed and discussing their results, some formulae concerning the flow in the boundary layer along a rough surface may be deduced.

The following considerations are based upon the hypothesis of Prof. VON KÁRMÁN [3]) that the resistance of a rough surface is entirely due to the head resistance of the projections. It might be expected that as this head resistance, at least above a certain value of REYNOLDS' number, follows the quadratic law, the same will be the case with the resistance experienced by the entire surface; this supposition is supported by the work of HOPF and FROMM mentioned above.

The velocity distribution in a section of the boundary layer will now be supposed to satisfy a relation of the form:

$$u = V\left(\frac{y}{\delta}\right)^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where u = velocity (parallel to the surface) in an arbitrary point of the section considered; V = velocity of the undisturbed flow just outside of the boundary layer; y = distance of the point in question from the surface; and δ = thickness of this section of the boundary layer.

This relation has to be verified by experiment. If the surface is not too rough, and if REYNOLDS' number for the boundary layer $\left(R_\delta = \frac{V\delta}{\nu}\right)$ is sufficiently high, it may be expected from the results of other researches that form. (1) will hold.

Although about the flow in the vicinity of the projections nothing can be predicted with certainty, following VON KÁRMÁN the velocity at the top of a projection of height h may be expressed by:

$$u_t = aV\left(\frac{h}{\delta}\right)^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a is an unknown numerical constant.

The resistance of a projection will now be proportional to u_t^2, and the resistance experienced by the surface per unit area may be written:

$$\tau_0 = c\varrho V^2\left(\frac{h}{\delta}\right)^{2n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In order to deduce a relation between δ and the distance x of the section

[3]) TH. VON KÁRMÁN. Ueber die Oberflächenreibung von Flüssigkeiten, Vorträge aus dem Gebiete der Hydro- und Aerodynamik, Innsbruck 1922 (Berlin 1924), p. 146.

considered from the leading edge of the surface, we may use the formula given by VON KÁRMÁN [4]) for the loss of momentum. It has to be born in mind, however, that the experiments described below relate to a surface mounted in a wind tunnel, in which case the velocity V outside of the boundary layer is not constant, but increases down stream in consequence of the narrowing of the passage of the flow, caused by the growth of the boundary layers along the tunnel walls and along the surface.

The increase of V is rather small; putting:

$$\frac{dV}{dx} = \beta V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we may neglect terms as $\beta^2 x^2$, $\beta^3 x^3$, ..., in the following formulae, even when x is equal to the length l of the plate.

In this case the integral of (4) is:

$$V = V_0 (1 + \beta x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4^a)$$

The loss of momentum in the boundary layer per unit length will be:

$$\frac{d}{dx}\int_0^\delta u^2\, dy - V\frac{d}{dx}\int_0^\delta u\, dy = -\frac{\tau_0}{\varrho} + \delta V \frac{dV}{dx} \quad . \quad . \quad . \quad . \quad (5)$$

Inserting form. (1), (3) and (4) and dividing by V^2, equation (5) is reduced to [5]):

$$\frac{n}{(2n+1)(n+1)}\frac{d\delta}{dx} + \frac{2n^2+3n}{(2n+1)(n+1)}\beta\delta = c\left(\frac{h}{\delta}\right)^{2n}.$$

the integral of which is, when $\delta = 0$ for $x = 0$:

$$\delta = \left\{\frac{(2n+1)^2(n+1)\,c}{n}\right\}^{\frac{1}{2n+1}} x^{\frac{1}{2n+1}}\, h^{\frac{2n}{2n+1}} \left\{1 - \frac{2n+3}{2}\beta x\right\}. \quad . \quad (6)$$

Hence from form. (3), inserting the value of V from (4a) and that of δ from (6), we get:

$$\tau_0 = c\varrho V_0^2 \left\{\frac{n}{(2n+1)^2(n+1)\,c}\right\}^{\frac{2n}{2n+1}} \left(\frac{h}{x}\right)^{\frac{2n}{2n+1}} \{1 + (2n^2+3n+2)\beta x\} \quad . \quad (7)$$

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[4]) TH. VON KÁRMÁN, Ueber laminare und turbulente Reibung, Zeitschr. f. angewandte Math. u. Mech., 1, 1921, p. 235 form. (5).

[5]) The relations for δ, τ_0 and c_f which are given here for the case of an accelerated flow, have been deduced by Prof. BURGERS.

The integral of (7) with respect to x, multiplied by the breadth b of the plate, gives the total resistance:

$$\left.\begin{aligned} W &= b\int_0^x \tau_0\,dx \\ &= c\varrho V_0^2 b x^{\frac{1}{2n+1}} h^{\frac{2n}{2n+1}} \left(\frac{1}{2n+1}\right)^{\frac{2n-1}{2n+1}} \left(\frac{n}{(n+1)c}\right)^{\frac{2n}{2n+1}} \left\{1+\frac{2n^2+3n+2}{2n+2}\beta x\right\} \end{aligned}\right\} \quad (8)$$

This expression is simplified by introducing:

$$I = \varrho\int_0^\delta u\,(V-u)\,dy = \frac{n}{(2n+1)(n+1)}\,\varrho\,V^2\,\delta \quad . \quad . \quad . \quad . \quad (9)$$

The quantity I will be called the defect of momentum in the boundary layer. It has to be noted that in (9) occurs the local value of the velocity, V, and not V_0.

By means of (4a), (6) and (9) form. (8) leads to:

$$W = bI\left\{1+\frac{(2n+1)^2}{2(n+1)}\beta x\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If n and c are found from experimental results for a known value of β, the behaviour of δ, τ_0 and the value of W in the case of an unlimited flow (where V is constant), are easily deduced by putting everywhere $\beta = 0$ in the formulae.

§ 3. *Experimental arrangements.*

In order to collect data about the distribution of the velocity in the neighbourhood of rough surfaces and to check the formulae deduced for δ and for the resistance, experiments were performed with two sheats of "waffle plate" of dural, as used f.i. for covering treadles. The first plate served only for the measurements in the boundary layer; later on, when a greater sheat of the metal was put at our disposal, a second plate with projections differing but slightly from those of the first, was used for measuring directly on a balance the total resistance experienced by a board covered on both sides with it and for determining the loss of momentum in the wake down stream. In order to compare the results of these experiments with those performed on the first plate, a few measurements of the velocity distribution in the boundary layer along the second plate were performed.

Waffle plate I. The surface may be characterized as follows: The projections had the shape of quadrilateral pyramids, arranged in regular horizontal and vertical rows without change; their mean height was

TABLE I. Velocity in the boundary layer (in cm/sec).

Waffle plate I, $V = 811$ cm/sec.

y cm.	x = 25 t	v	50 t	v	75 t	v	100 t	v	125 t	v	150 t	v	175 t	v
0.025	380	39	354	36	230	35	310	35	324	19	300	18	317	22
0.050	428	90	384	81	325	90	350	74	377	58	361	31	375	66
0.075	462	152	425	140	384	121	371	125	400	106	396	64	396	122
0.100	476	236	439	214	409	227	400	196	425	176	416	116	425	185
0.125	485	331	450	304	434	304	409	270	441	256	441	184	436	262
0.150	520	396	466	361	445	361	425	313	452	341	450	270	445	327
0.175	529	434	485	400	459	389	445	342	461	379	459	334	461	374
0.200	540	450	490	412	485	416	450	374	471	409	471	379	470	404
0.250	573	493	515	450	490	450	476	412	489	432	484	415	485	434
0.300	592	525	529	471	511	459	485	445	501	459	494	441	501	450
0.400	621	573	557	498	529	493	511	484	520	489	525	476	520	490
0.500	666	618	579	539	549	529	539	493	541	510	539	494	532	520
0.600	709	653	621	573	573	557	561	529	557	529	551	515	551	549
0.700	729	689	640	601	592	573	573	557	573	552	568	532	568	568
0.800	759	713	665	621	621	590	592	573	586	568	581	557	573	573
0.900	770	749	681	651	638	620	618	581	600	576	592	573	581	581
1.00	789	762	709	666	658	627	621	610	620	592	621	592	592	592
1.25	811	784	740	715	689	665	658	630	640	620	640	621	621	621
1.50	811	806	762	762	715	709	681	666	666	645	665	651	640	640
1.75	811	811	789	780	749	715	713	702	688	668	688	673	665	666
2.00	811	811	806	806	780	759	729	729	715	709	713	698	689	689
2.50	811	811	811	811	806	784	770	762	761	749	750	722	715	722
3.00	811	811	811	811	811	811	806	784	790	780	770	761	762	762
4.00	—	—	—	—	811	811	811	811	811	811	811	811	811	811
5.00	—	—	—	—	811	811	811	811	811	811	811	811	811	811
X =	200	200	175	175	175	175	175	175	200	200	200	200	200	200
Bar.	768	767	768	766	770	768	773	773	769	769	765	766	766	765
Temp.	18.4	17.3	17.8	17.8	16.1	17.4	18.6	17.5	18.0	19.6	18.0	18.4	18.4	17.1

Waffle plate I ($V = 1600$, 2400, 3200 cm/sec.) and **Waffle plate II** ($V_p = 2400$ cm/sec., 198 cm.)

y cm.	V = 1600: 25 v	50 v	75 v	100 v	125 v	150 v	175 v	V = 2400: 25 v	50 v	75 v	100 v	125 v	150 v	175 v	V = 3200: 25 v	50 v	75 v	100 v	125 v	150 v	175 v	Waffle II t	v	y cm.
0.025	139	117	74	62	56	27	112	261	228	114	139	114	324	217	368	388	210	221	185	335	243	1018	158	0.025
0.050	256	208	176	132	92	109	201	420	370	285	262	188	458	370	594	590	416	385	346	511	392	1142	295	0.050
0.075	369	316	272	236	184	202	282	584	515	429	393	316	615	520	761	761	625	510	538	686	548	1211	445	0.075
0.100	484	405	384	345	295	292	421	745	655	566	532	455	772	676	955	949	820	701	778	773	730	1260	615	0.100
0.125	638	529	549	466	426	412	548	888	790	790	701	616	924	848	1185	1130	1060	911	961	973	956	1296	824	0.125
0.150	734	635	645	579	530	538	615	1056	912	950	851	771	1045	980	1386	1295	1320	1120	1204	1290	1160	1340	1011	0.150
0.175	790	692	721	656	630	625	681	1176	1025	1060	950	911	1130	1050	1586	1415	1460	1225	1384	1475	1340	1384	1116	0.175
0.200	853	751	780	691	691	697	761	1253	1135	1160	1020	1050	1185	1120	1650	1585	1595	1410	1544	1545	1450	1421	1190	0.200
0.250	900	818	800	725	780	773	810	1421	1232	1240	1090	1220	1240	1225	1864	1705	1740	1500	1665	1700	1630	1513	1310	0.250
0.300	999	870	830	809	821	818	841	1513	1310	1310	1175	1285	1332	1320	1975	1805	1850	1620	1798	1780	1730	1568	1384	0.300
0.400	1082	942	900	890	898	906	911	1681	1460	1440	1360	1430	1430	1415	2210	1935	1990	1820	1850	1936	1884	1648	1513	0.400
0.500	1163	1010	961	955	950	949	961	1823	1545	1550	1460	1520	1513	1500	2395	2095	2110	1950	1971	2061	1990	1714	1600	0.500
0.600	1246	1082	1024	989	988	992	1010	1936	1640	1620	1555	1580	1570	1582	2580	2200	2245	2015	2100	2135	2060	1772	1714	0.600
0.700	1332	1150	1056	1024	1015	1015	1040	2061	1722	1660	1620	1640	1632	1620	2721	2315	2315	2100	2172	2190	2125	1815	1772	0.700
0.800	1436	1176	1096	1050	1035	1045	1060	2190	1840	1760	1660	1680	1675	1665	2860	2430	2400	2150	2256	2245	2200	1849	1823	0.800
0.900	1460	1246	1130	1120	1070	1070	1080	2304	1918	1820	1740	1740	1725	1710	2962	2520	2460	2237	2315	2305	2237	1884	1884	0.900
1.00	1513	1325	1183	1175	1100	1090	1095	2333	1955	1900	1760	1770	1765	1770	3062	2600	2525	2375	2375	2375	2294	1936	1936	1.00
1.25	1600	1421	1262	1262	1200	1155	1175	2400	2162	2005	1900	1880	1835	1840	3200	2860	2745	2500	2525	2440	2440	2043	1989	1.25
1.50	1600	1475	1355	1320	1262	1230	1262	2400	2245	2200	1990	1980	1935	1930	3200	3025	2845	2610	2610	2580	2540	2116	2061	1.50
1.75	1600	1521	1450	1380	1320	1310	1320	2400	2315	2240	2120	2050	1995	2010	3200	3115	2950	2765	2740	2725	2663	2190	2116	1.75
2.00	1600	1560	1500	1450	1360	1330	1380	2400	2345	2300	2240	2110	2060	2080	3200	3180	3100	2875	2845	2800	2767	2247	2190	2.00
2.50	1600	1600	1550	1500	1515	1430	1450	2400	2372	2380	2300	2240	2190	2180	3200	3200	3140	3080	2950	2960	2920	2381	2285	2.50
3.00	1600	1600	1585	1570	1560	1515	1520	2400	2400	2390	2380	2360	2275	2290	3200	3200	3160	3140	3114	3105	3030	2430	2333	3.00
4.00	1600	1600	1600	1600	1600	1600	1570	2400	2400	2400	2400	2400	2360	2375	3200	3200	3200	3200	3200	3180	3160	2510	2500	4.00
5.00	1600	1600	1600	1600	1600	1600	1600	2400	2400	2400	2400	2400	2400	2400	3200	3200	3200	3200	3200	3200	3200	2560	2560	5.00
X =	200	200	175	175	200	200	200	200	200	175	175	200	200	200	200	200	175	175	200	200	200	129	129	cm.
Bar.	758	756	764	765	771	761	765	758	757	758	765	770	760	768	755	758	758	766	773	762	763	767	767	mm.
Temp.	19.7	19.1	18.8	15.4	18.6	19.1	17.0	18.8	17.7	18.0	19.2	16.8	18.6	18.6	19.2	19.5	19.9	18.9	19.7	20.1	19.7	13.1	14.3	°C.

1.7 mm; their distance in longitudinal direction 6.5 mm, in transverse direction 6.33 mm. The tops of the projections, as well as the valleys at their root, were slightly rounded off.

Comparing this with the rough surfaces used by FROMM, our "waffle plate I" seems to approach FROMM's "Waffelblech II", although the height of the projections of the latter plate was less (height 0.858 mm, distance of the tops 6.62 mm).

This surface was nailed on a rectangular wooden board of 189.5 $\times$ 50 cm, the leading edge of which was sharpened over 15 cm. The dimensions of the metal sheat allowed to cover one side only; however, the metal was bent around the leading edge of the board and covered the back over 10 cm. Two wooden clamps were provided at the uncovered side to give the board a greater stiffness.

This board was mounted vertically in the wind tunnel at a distance of 175 to 225 cm from the honey comb at the entrance (this distance is called X in table I).

The arrangement for the determination of the velocity distribution in the boundary layer was the same as that used in the experiments on a glass plate, described before [6]). The measurements were carried out with a hot wire anemometer (length of wire 37 mm, diameter 0.05 mm, heated to about 600° C above the surrounding atmosphere), which was mounted in a screw micrometer with displacement perpendicular to the plate. The distance between the hot wire and the surface could be regulated to 0.01 mm. The screw micrometer was mounted at the outside of the tunnel on a strong iron frame, which carried at the same time four round bars, two of which supported the upper edge of the plate and the two others the lower edge.

The displacement of the hot wire in the direction of the flow was performed by shifting the micrometer along the iron frame, without altering the position of the plate.

The mean velocity of the flow in the tunnel, V, was determined with a Pitot-tube, connected to an alcohol micromanometer; the value of V was kept constant by the experimenter. This Pitot-tube was mounted 268 cm behind the honey comb at midheight of the tunnel and 60 cm from the vertical tunnel wall facing the experimenter; the velocity indicated by this Pitot-tube will be called V_p.

Waffle plate II. The surface of the second metal sheat, which is called further on "waffle plate II", consisted of quadrilateral pyramids, 1.5 mm high, at distances of 6.65 mm in the longitudinal direction and of 6.4 mm in the transverse direction; hence it differed slightly from the surface of plate I.

Two sheats of this metal were fixed to the sides of a wooden board, which was sharpened at the leading edge over 15 cm, while the upper- and

[6]) B. G. VAN DER HEGGE ZIJNEN. Measurements of the velocity distribution in the boundary layer along a plane surface (thesis Delft, 1924).

the lower edges were sloped over 5 cm in order to get a sharp edge in the direction of the plate. The sheats were fixed as tightly as possible to the board by means of small nails ; at the leading edge, where the wood was thin, the sheats were riveted. The dimensions of the covered board are : length of the metal sheat : 199.4 cm, height (measured along the curved sheat) : 50 cm ; thickness of the board : 2.4 cm.

The arrangement for the measurement of the resistance is represented in fig. 1. The board C is supported by two thin steel wires D, E (diameter 0.33 mm, length 290 cm) in the vertical plane of symmetry of the tunnel ; they passed the top of the tunnel with sufficient clearance. At the lower edge of the board two pairs of steel wires F and G were provided in order to avoid transverse oscillations ; they were stretched slightly, so that the plate was free to swing longitudinally.

From the leading edge a steel wire H led to a sensitive balance B outside of the tunnel, the slope of which was 30°.30′ ; its end was fixed to the tunnel wall at K and a certain tension was given to this wire. The measurement of the forces acting on the model was performed in such a way that the other arm of the balance B was loaded with a determined weight, after which the wind speed in the tunnel was regulated until B was in balance. The resistance experienced by the plate was afterwards deduced from the weighed forces by resolving them graphically. During the measurements the distance of the leading edge of the board from the honey comb (X) was 158 cm ; the lower edge of the board was 17.1 cm above the bottom of the tunnel, while the distance between board and the front wall of the tunnel was 41.6 cm. The Pitot-tube A was fixed at 143 cm behind the honey comb, 20 cm above the bottom and 52.4 cm from the front wall. The velocity V_p indicated by it will practically be equal to V_0 at the beginning of the board ($X = 158$ cm).

The resistance measured on the balance will be affected by the resistance of the steel wires carrying the board, and on the other hand by the suction at the trailing edge. In order to arrive at the true surface friction, corrections have to be applied for both influences.

The wire resistances are of minor importance ; it is sufficient to deduce them from their length and diameter by means of the diagram for the resistance coefficients of cylinders given by PRANDTL [7]).

The suction at the trailing edge of the model is determined according a method also given by PRANDTL [8]) by mounting a brass tube of 9 mm diameter, provided with 9 holes of 0.8 mm in the space left between both metal sheats at the rear. This tube was closed at the bottom, while the upper

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[7]) L. PRANDTL. Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen II (München 1923), p. 24.

[8]) L. PRANDTL. Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen I (München 1921), p. 120.

end, projecting above the board, was connected with an alcohol micromanometer; the holes were directed down stream. For various values of V the suction in this space was compared to the static pressure on Pitot-tube A, which was supposed to be equal with sufficient accuracy to the pressure at

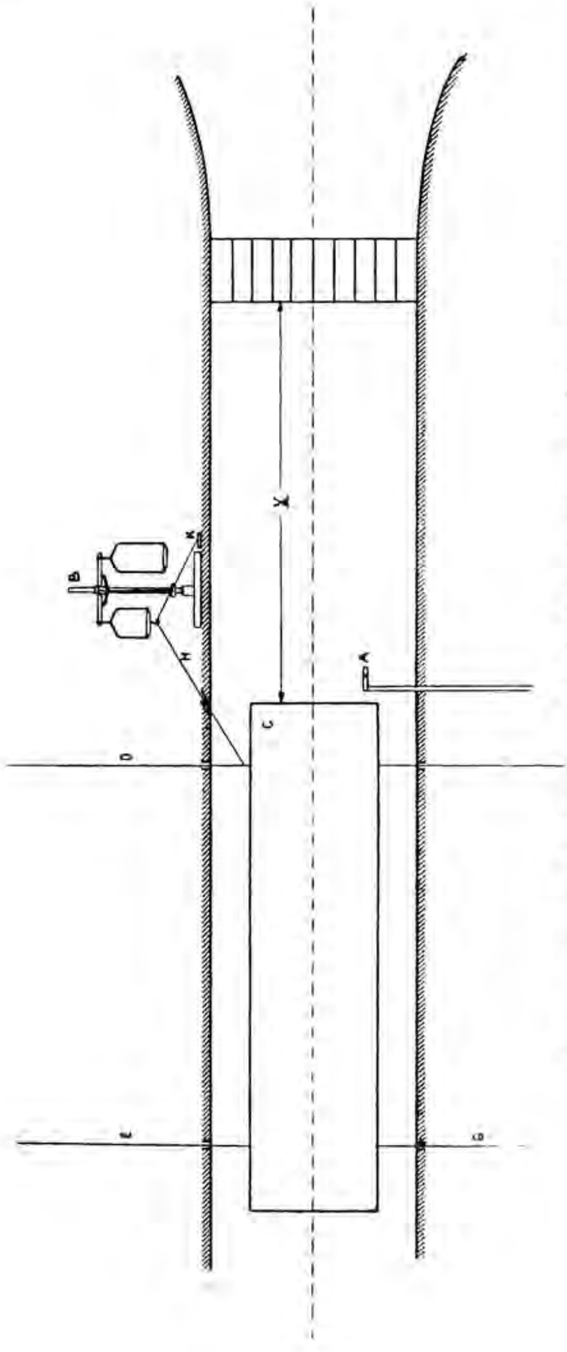

Fig. 1. Scheme of the arrangement for the direct determination of the resistance.

the leading edge of the plate. A second manometer was used for the determination of the air speed.

The pressure differences at the leading and at the trailing edge of the board, multiplied by the area of the trailing edge gives the suction experienced by the model.

This brass tube was absent during the resistance measurements with the balance.

The few measurements of the velocity distribution in the boundary layer along plate II were performed in the same way as those in the case of plate I, with the exception that here the first pair of iron bars, which held the board, was taken away and replaced by steel wires in order to avoid any disturbance of the flow as far as possible. The results of the velocity measurements with plate I had proved that these bars might cause a disturbance of the flow in the boundary layer in down stream sections. The pair of bars in the neighbourhood of the hot wire anemometer, however, could not be dispensed with, so that only the foremost pair, which gave the greater trouble, was removed.

§ 4. *Measurements in the boundary layer along plate I.*

The measurements on the velocity distribution along plate I were performed at a great number of distances from the leading edge and at various values of V.

As stated in § 2, the velocity outside of the boundary layer will increase down stream. However, during the experiments the wind speed was regulated in such a way that V had the same value at every value of x, as this made a check and a comparison of the results more easy.

As it proved to be impossible to determine the velocity gradient in the immediate neighbourhood of the surface in order to evaluate the shearing stresses, the velocity in the boundary layer was measured only at values of $y \geqslant 0.025$ cm. Some of the results have been collected in table I (the last columns of this table give the results of the measurements on plate II). Every velocity mentioned here is obtained as the mean of 6 readings. The indices t and v respectively relate to the measurements in which the distance y is estimated from a "top" or from a "valley" between the pyramids. The series t were observed immediately behind the series v and therefore the distance x had in this case to be increased by about 3 mm.

The experiments have been performed with $V = 811$ cm/sec in the sections $x = 5$, 10, 15, 20, 25, 37.5, 50, 62.5, 75, 87.5, 100, 125, 150 and 175 cm, y being reckoned in this case from a top and from a valley respectively; with $V = 1200$, 1600, 2000, 2400, 2800 and 3200 cm/sec the velocity was observed in the sections: $x = 25$, 50, 75, 100, 125, 150 and 175 cm for the series v only.

As might be expected, the series t and v show appreciable differences at the smaller values of y; they disappear for the greater part, however, when in the series t the value of y is increased by the height h of the projections, $h = 1.7$ mm.

The results were plotted on a logarithmic scale (log u against log y): one of these diagrams, relating to the measurements at $x = 175$ cm, has been reproduced in fig. 2. By means of the diagrams it was inquired

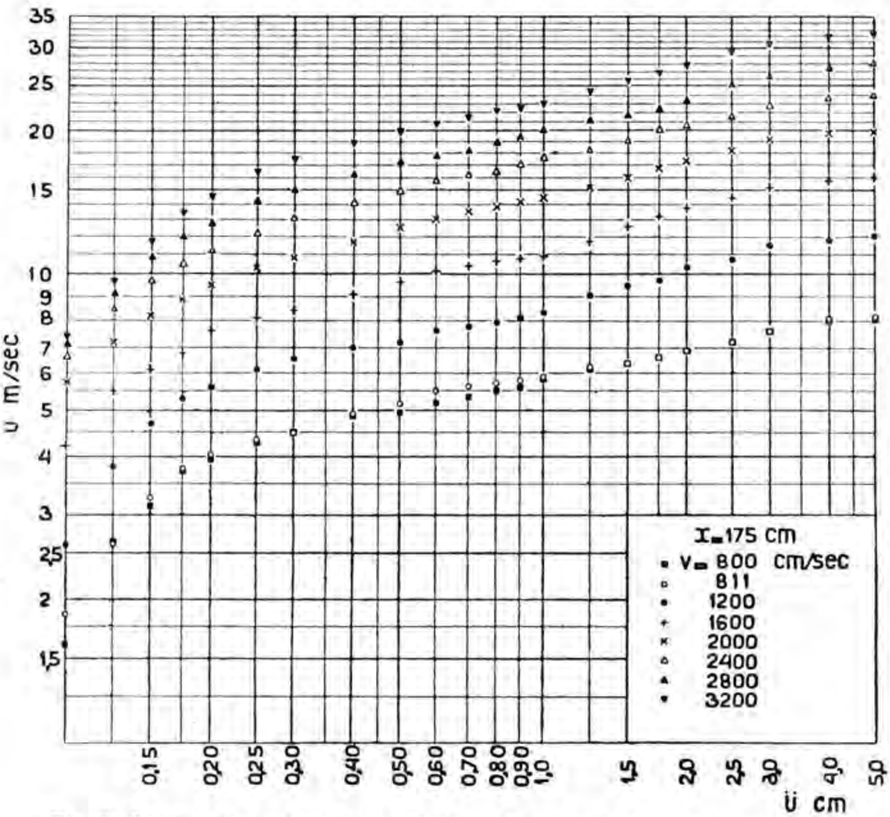

Fig. 2. Logarithmic scale diagram of the velocity in the boundary layer with $x = 175$ cm.

whether the velocity in the boundary layer could be represented by the formula $u = V\left(\frac{y}{\delta}\right)^n$ with a reasonable degree of accuracy. This proved to be the case; however, for all series the observed values of u seem to be arranged rather wavingly along the mean straight line [9]). This circumstance renders it difficult to determine the value of n with sufficient accuracy. The mean values, deduced from these diagrams, have been collected in table II.

They show a clearly marked decrease when x increases; probably this is due to the fact that the turbulent state of motion is not yet fully developed when x is small. On the contrary, n seems to be independent

[9]) The same phenomenon, although less pronounced, has been observed in the researches on a smooth surface, mentioned in note [6]) (p. 28, fig. 8).

TABLE II. Values of n.

x	V cm/sec									Mean
	800(v)	811(v)	811(t)	1200(v)	1600(v)	2000(v)	2400(t)	2800(v)	3200(v)	
15	—	0.33	0.38	—	—	—	—	—	—	0.35
20	—	0.34	0.37	—	—	—	—	—	—	0.35
25	—	0.32	0.36	0.34	0.37	0.36	0.36	0.38	0.37	0.36
37.5	—	0.30	0.32	—	—	—	—	—	—	0.31
50	—	0.30	0.31	0.33	0.34	0.33	0.34	0.33	0.33	0.33
62.5	—	0.285	0.31	—	—	—	—	—	—	0.30
75	0.27	0.275	0.29	0.27	0.27	0.27	0.27	0.27	0.27	0.27
87.5	—	0.275	0.28	—	—	—	—	—	—	0.28
100	—	0.27	0.28	0.29	0.30	0.29	0.28	0.29	0.29	0.29
125	—	0.24	0.25	0.24	0.27	0.24	0.24	0.24	0.26	0.25
150	—	0.25	0.24	0.25	0.26	0.24	0.25	0.245	0.25	0.25
175	0.24	0.24	0.24	0.25	0.24	0.25	0.25	0.25	0.25	0.25

TABLE III. Values of δ (in cm).

x cm	V cm/sec								Mean
	800	811	1200	1600	2000	2400	2800	3200	
5	—	0.4	—	—	—	—	—	—	0.4
10	—	0.63	—	—	—	—	—	—	0.63
15	—	0.90	—	—	—	—	—	—	0.90
20	—	1.00	—	—	—	—	—	—	1.00
25	—	1.20	1.16	1.14	1.10	1.06	1.02	1.08	1.11
37.5	—	1.50	—	—	—	—	—	—	1.50
50	—	2.00	1.90	1.80	1.96	1.80	1.64	1.76	1.84
62.5	—	2.18	—	—	—	—	—	—	2.18
75	2.55	2.55	2.55	3.1	2.7	2.5	2.55	2.45	2.62
87.5	—	3.00	—	—	—	—	—	—	3.00
100	—	3.20	3.20	3.0	2.85	2.9	2.9	3.0	3.01
125	—	3.75	3.80	4.0	4.0	3.5	3.7	3.1	3.69
150	—	3.70	4.0	4.0	4.1	4.0	3.5	3.3	3.80
175	3.80	3.80	4.0	3.8	3.6	3.7	3.65	3.5	3.74

of V. With $x \geqslant 100$ cm, n becomes about 0.25. Our researches are not sufficient to prove whether this is the limiting value, or whether n will decrease still further when x increases to the value $^1/_7$ which holds for a smooth surface [10]).

The logarithmic diagrams give at the same time the thickness δ of the boundary layer; the results have been collected in table III (all numbers mentioned here relate to the series v).

The value of δ decreases, though not much, when V increases, contrary to what has been supposed in § 2. However, δ never can be determined with great accuracy and the sensibility of the hot wire decreases at higher values of V.

From a numerical integration of the value of $u\,(V-u)$ over the thickness of the boundary layer the defect of momentum I has been determined, according to form. (9). Writing:

$$c_i = \frac{I}{^1/_2\,\varrho V^2 x} = \frac{2I}{\varrho V^2 x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the coefficient c_i has the values given in table IV (again for the series v):

TABLE IV. Values of $c_i \times 1000 = \dfrac{2I}{\varrho V^2 x} \times 1000.$

x	V cm/sec								Mean
	800	811	1200	1600	2000	2400	2800	3200	
5	—	21.6	—	—	—	—	—	—	21.6
10	—	15.8	—	—	—	—	—	—	15.8
15	—	15.8	—	—	—	—	—	—	15.8
20	—	14.0	—	—	—	—	—	—	14.0
25	—	12.7	13.2	13.76	13.4	12.56	12.60	13.62	13.12
37.5	—	10.1	—	—	—	—	—	—	10.1
50	—	10.6	11.1	11.48	12.18	11.30	10.48	10.80	11.13
62.5	—	8.4	—	—	—	—	—	—	8.4
75	9.08	9.46	9.30	10.12	9.50	8.64	8.86	8.66	9.22
87.5	—	8.90	—	—	—	—	—	—	8.90
100	—	8.60	8.60	8.48	7.84	7.80	8.40	8.32	8.29
125	—	7.39	7.10	7.40	7.40	6.80	7.94	7.00	7.29
150	—	6.52	6.72	6.84	6.98	6.68	6.52	6.08	6.62
175	5.68	5.68	5.84	5.92	5.48	5.56	6.04	5.66	5.73

These values do not vary systematically with V.

[10]) Compare the papers mentioned in notes [4]), (p. 238) and [6]) (fig. 7 and 8).

§ 5. *Discussion of the results.*

It was of importance to compare the experimental results collected in tables II to IV with the formulae deduced in § 2.

In the first place this has been done with the exponent n. Writing form. (11) by means of (9):

$$c_i = \frac{2n}{(2n+1)(n+1)} \frac{\delta}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and taking the mean values of δ and c_t for every value of x, we arrive at the following results for the expression $\frac{n}{(2n+1)(n+1)}$:

TABLE V.

x	δ	δ/x	c_i	$\frac{(2n+1)(n+1)}{n}$
25	1.11	0.0444	0.01312	7.77
50	1.84	0.0368	0.01140	6.63
75	2.63	0.0351	0.00922	7.61
100	3.01	0.0301	0.00830	7.25
125	3.69	0.0295	0.00748	7.89
150	3.80	0.0253	0.00662	7.64
175	3.72	0.0213	0.00574	7.42
				Mean... 7.44

This gives for n about 0.25 (exactly 0.255), which is in agreement with the value deduced from the measurements on the velocity distribution for the higher values of x.

If this value of n is valid over the whole surface, form. (6) will become:

$$\delta = 5{,}02\, c^{2/5} \left(\frac{x}{h}\right)^{3/5} (1 - 1{,}75\, \beta x)$$

Now it was found in the experiments on waffle plate II that the velocity outside of the boundary layer increased by 160 cm/sec over the full length (198 cm) of the model, when V_0 was equal to 2400 cm/sec. This leads to $\beta = 0.000337$. If we accept that the same value of β holds for all series the factor $(1—1.75\,\beta\, x)$ will become equal to $1—0.00059\, x$.

Now $h = 0.17$ cm; therefore we get for the constant the values mentioned in table VI:

TABLE VI.

x	δ	$c^{2/3}$	δ calculated from form. (13)
25	1.11	0.0473	1.21
50	1.84	0.0493	1.89
75	2.63	0.0555	2.44
100	3.01	0.0531	2.92
125	3.69	0.0571	3.32
150	3.80	0.0527	3.70
175	3.72	0.0459	4.03
		Mean... 0.0516	

With the mean value 0.0516, the expression for δ becomes

$$\delta = 0{,}259 \left(\frac{x}{h}\right)^{2/3} h\,(1 - 0{,}00059\,x) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The values of δ derived from this formula have also been mentioned in table VI. From form. (12) we now find for c_i :

$$c_i = 0{,}0691 \left(\frac{x}{h}\right)^{-1/3} (1 - 0{,}00059\,x) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Table VII gives the comparison between the values of c_i calculated from this expression and those derived from the observed values of I.

TABLE VII.

x	(c_i) calculated	(c_i) experimental
25	0.0130	0.01312
50	0.0111	0.01140
75	0.00871	0.00922
100	0.00779	0.00829
125	0.00709	0.00729
150	0.00660	0.00662
175	0.00617	0.00573

Although the differences between both sets of numbers are important, it may be said that in general the results found for n (as deduced from the logarithmic diagrams for the velocity distribution), for the thickness of the boundary layer and for the defect of momentum in the boundary layer, confirm the relations deduced in § 2.

With the value of the constant mentioned before, form. (3) gives for τ_0:

$$\tau_0 = 0{,}01'7\, \varrho V^2 \sqrt{\frac{h}{\delta}} \quad \ldots \ldots \ldots \quad (15)$$

The resistance experienced by a single wall of length l and breadth b, in a flow with the constant velocity V, now becomes according to (8) (with $\beta = 0$) :

$$W = 0{,}0690 \left(\frac{h}{x}\right)^{1/5} \tfrac{1}{2} \varrho V^2 bx \quad \ldots \ldots \ldots \quad (16)$$

§ 6. *Measurements on the velocity distribution along plate II.*

As stated in § 3, two series of measurements on the velocity distribution in the boundary layer along plate II were performed in order to check the results found with plate I.

The plate was mounted in the same way as mentioned at the end of § 3, with the leading edge at $X = 129$ cm behind the honey comb of the wind tunnel. The measurements were carried out at a distance of $x = 198$ cm from the leading edge, starting from a valley; a second series was performed 3 mm down-stream starting from a top. The hot wire anemometer was put at 40 cm above the bottom of the tunnel. Measurements have been performed only for the value $V_p = 2400$ cm/sec (Pitot-tube 42 cm behind honey comb; 52 cm from the front wall and 20 cm above the tunnel bottom), in which case V_x proved to be 2560 cm/sec according to the readings of the hot wire, which instrument had been calibrated carefully before. This leads to the velocity increase of 160 cm/sec along the surface, as stated before.

The results have been collected in the last columns of table I. They prove, as was the case with plate I at the higher values of x, that the velocity in the boundary layer increases with the 0.25-power of y; here too in a logarithmic diagram the observed values of the velocity are grouped more or less wavingly along the mean straight line. The exact evaluation of n and δ becomes rather difficult on account of this phenomenon.

The value of δ proved to be about 4.8 cm.

§ 7. *Determination of the resistance of plate II.*

The total resistance of the model, as found by means of the balance, has been represented in fig. 3 as a function of V_0 ($+ \ldots + $); in the same diagram the suction at the trailing edge ($\times \ldots \times$) and the resistance of the suspension wires (d) have been given. When the total resistance is diminished by the suction and by the resistance of the

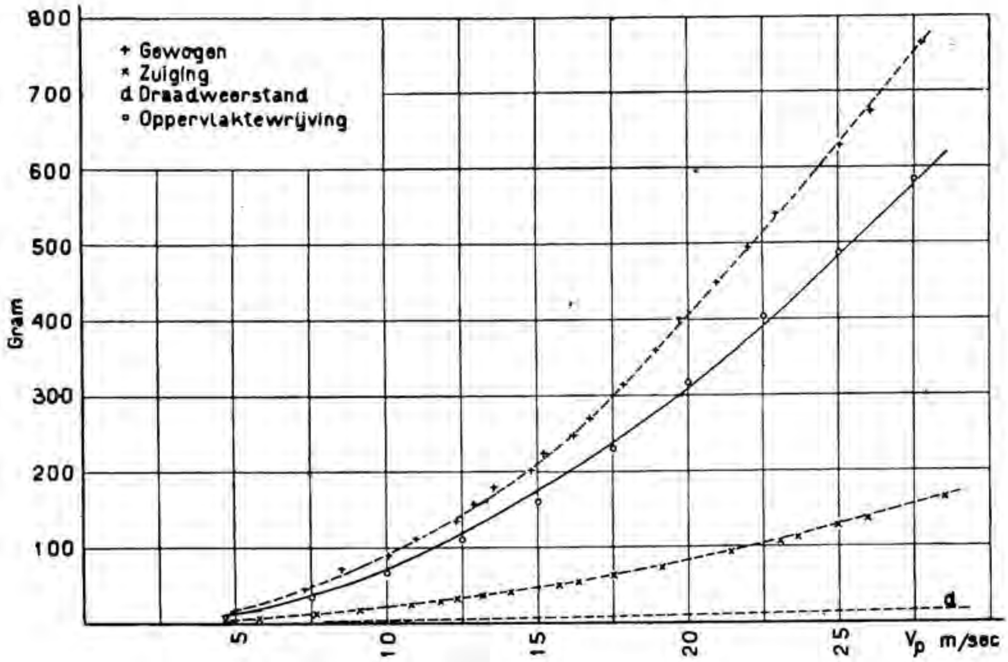

Fig. 3. Total resistance and surface friction as function of the velocity.

suspension wires we get the surface friction. From the diagram the following values have been deduced for this surface friction (∘ . . . . . ∘).

TABLE VIII.

V_0 m/sec	Surface friction (both sides), in grammes	V_0 m/sec	Surface friction (both sides), in grammes
5	12	17.5	230
7.5	34	20	316
10	66	22.5	404
12.5	110	25	487
15	160	27.5	584

They may be represented by the interpolation formula :

$$W = 0{,}77\,V_0^2 \;(V_0 \text{ in m/sec}).$$

which is in agreement with the supposition that the resistance is proportional to the square of the velocity. This formula is represented in fig. 3 by the full drawn line.

The results of the weighing have now been compared with the value of the resistance as deduced from the experiments on the velocity distribution in the boundary layer. To this end the value of

$$I=\varrho\int_0^\delta u\,(V-u)\,dy$$

has been calculated from the measurements mentioned in § 6.

It has to be born in mind that the velocity in the first part of the boundary layer cannot be found with sufficient accuracy. Starting from a valley f.i. we might expect that the hot wire anemometer, which is at first more or less screened by the pyramids, will be influenced at higher values of y by the v- and w-components of the velocity. On the other hand, when we start from a "top", and call the distance y between the hot wire and the surface zero when the wire just touches the pyramids, as has been done before, we still have to take into account the flow between top and valley, which region of course was not accessible with the apparatus used by us. We have allowed ourselves to extrapolate the u-y-curve from top to valley i.e. over the distance h until the value $u=0$ is reached at the base of a pyramid. The region between top and valley accounts for about 4 % of the total defect of momentum, and therefore an estimation of the flow in this part of the boundary layer will not lead to important errors.

In this way we found for the expression $\frac{I}{\varrho}=\int_0^\delta u\,(V-u)\,dy$ when starting from a valley: 3469000 cm^3/sec^2, and when starting from a top, taking the flow between top and valley into account, 3242000 cm^3/sec^2 (as mentioned in § 6, V_p was 2400 cm/sec). The difference between the two values is due to the region of high velocities (i.e. high values of y), where the smaller sensibility of the hot wire anemometer leads to less accurate readings of the velocity than in the region of low air speed near the surface.

The mean value of $\frac{I}{\varrho}=3346000$ cm^3/sec^2 gives for the surface friction on both sides of the plate together at $V_p=2400$ cm/sec, according to form. (10) with $\varrho=\frac{1}{8000}$ and $b=50$ cm:

$$W=\frac{3346000}{8000}(1+0{,}9\,\beta x)=444 \text{ grammes}$$

when we suppose that the velocity distribution is the same for all heights.

The surface friction as determined by weighing is, according to fig. 3, at $V_p=2400$ cm/sec: 454 grammes, which differs little from the value deduced from the measurements on the velocity distribution in the boundary layer.

It is of importance to compare this value of W with that given by the formula deduced from the measurements with plate I, taking x equal to 198 cm and $h = 0.15$ cm. Form. (13) leads in this case to:

$$\delta = 4{,}13 \text{ cm}$$

hence form. (9) gives:

$$\frac{I}{\varrho} = \frac{4{,}13}{7{,}5} (V_{198})^2 = \frac{4{,}13}{7{,}5} (1{,}066)^2 \, V_0^2 = 0{,}626 \, V_0^2.$$

and form. (10):

$$W = 0{,}83 \, V_0^2 \; (V_0 \text{ in m/sec}).$$

which leads to $W = 478$ gram at $V_0 = 24$ m/sec. The check must be considered fair, especially as it is not easy to determine h with sufficient accuracy.

§ 8. *Resistance deduced from total head loss in the wake.*

Moreover the resistance experienced by the model has been evaluated from the total head loss of the flow in the wake of the model. We suppose the flow to be stationary and will neglect the components of the velocity perpendicular to the direction of the undisturbed wind as the model is a long, thin board, mounted in the direction of the flow. Calling the static pressure in a section of the tunnel in front of the model p_A and the static pressure in a section behind the model p_B we get as the resultant of the pressure forces for a section of unit height:

$$W_p = \int (p_A - p_B) \, dy.$$

The change of momentum per unit time of the air passing the sections A and B is found from the total head loss. The mass of air entering A per unit time is $\int \varrho u_A \, dy$; its momentum: $\int \varrho u_A^2 \, dy$; the momentum of the air leaving B is: $\int \varrho u_B^2 \, dy$. Consequently:

$$W = \int (p_A + \varrho u_A^2) \, dy - \int (p_B + \varrho u_B^2) \, dy \quad . \quad . \quad . \quad . \quad (17)$$

By introducing the total head: $H = p + \frac{1}{2} \varrho u^2$, (17) is reduced to:

$$W = \int \triangle H \, dy + \int {}^1/_2 \, \varrho u_A^2 \, dy - \int {}^1/_2 \, \varrho u_B^2 \, dy \quad . \quad . \quad . \quad (18)$$

The second and third members at the right hand side of (18) are nearly equal; taken together they can be treated as a correction applied to the first term. They can be simplified when u_A and u_B are written as the sum of the undisturbed velocity V and a disturbing velocity u_a, u_b: in this way the correction becomes:

$$\left.\begin{aligned}\int {}^1/_2\,\varrho\,(V^2+2u_aV+u_a^2)\,dy-\int {}^1/_2\,\varrho\,(V^2+2u_bV+u_b^2)\,dy=\\ =\varrho V\int (u_a-u_b)\,dy+{}^1/_2\,\varrho\int (u_a^2-u_b^2)\,dy\end{aligned}\right\} \quad (19)$$

The equation of continuity, applied to the region between A and B, proves that the first term at the right hand side of (19) is zero. Putting $u_A = V =$ constant which holds if the section A is taken far enough in front of the model and the flow is measured not too near to the honey comb, u_a will be zero, and (18) is reduced to:

$$W=\int \triangle H\,dy-{}^1/_2\,\varrho\int u_b^2\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where the integral is extended only over the region of turbulent motion in the wake.

In evaluating the experimental data, the correction proved to be of the order of 10 % of the uncorrected value of W. On the other hand it was found that the values of H varied in a rather important and unsystematic manner at various heights above the tunnel bottom. On account of this it was considered superfluous to calculate the correction in a more refined manner.

The section B was chosen 40 cm behind the trailing edge of the model, while the values of u_A and H_A were derived from a Pitot-tube put at 42 cm behind the honey comb, 20 cm above the bottom of the tunnel and 52 cm from the front wall. The values in the section B were determined by means of a Pitot-tube which could be shifted perpendicularly to the plate. In order to observe the variations of $\triangle H$ with the height, H_B was read at the following values of z: 12, 15, 17, 20, 30, 40, 44, 48, 52, 56, 60, 63, 65 and 68 cm (measured from the tunnel bottom). In a direction perpendicular to the plate the Pitot-tube was set at the following values of y: 20, 22, 23, 24, 25, 26, 28, 30, 31, 32, 33, 34, 34½, 35, 35½, 36, 36½, 37, 37½, 38, 38½, 39, 39½, 40, 40½, 41, 41½, 42, 42½, 43, 43½, 44, 44½, 45, 45½, 46, 47, 48, 49, 50, 52, 54, 55, 56, 57, 58, 60 cm (measured from the front wall).

During these experiments the model was fixed by means of thin steel wires, in the same way as during the measurements with the balance. On account of the dimensions of the tunnel and of the position of the section B, the distance X had to be reduced to 129 cm; the distance from the model to the vertical front wall was 41.5 cm, measured from half way the thickness of the board.

While reading H_B the velocity of the air in the tunnel as indicated by Pitot-tube A, was kept constant at $V_p = 2400$ cm/sec. At every position of B 10 readings were taken of the differences of the static pressures and of those of the dynamic pressures $(p + \frac{1}{2}\varrho V^2)$ on A and B, at intervals of 5 seconds. A third manometer connected to A allowed the determination of the velocity V_p. When the pressures in a certain position had been observed, the Pitot-tube B was shifted to the next position and again 10 readings were taken; etc. During the readings at a constant value of z the fan of the wind tunnel was kept in action; it had to be stopped when z was modified.

The values of $\triangle H$ were plotted as function of y for every value of z; the areas of the curves obtained in this way were determined by means of a planimeter. In first approximation the area J of each of them represents the resistance of the model per cm height at the particular value of z at which $\triangle H$ has been determined. Then the correction

$$C = \int \tfrac{1}{2}\,\varrho\, u_b^2\, dy$$

was deduced from $\triangle H$ and $\triangle p$ for the same values of z. Taking this correction into account, so that $J_{corr.} = J - C$, the area of the curve for $J_{corr.}$ as a function of z gives the resistance of the whole model. The results are collected in table IX (J being expressed in mm water $\times$ cm, i.e. in kg/m^2 $\times$ cm).

TABLE IX. Total head loss in the wake of waffle plate II.

z cm	12	15	17	20	30	40	44	48	52	56	60	63	65	68
J mm.cm	18.3	104.7	116.2	111.6	138.9	145.0	132.5	129.8	127.9	107.4	91.4	98.8	97.6	12.3
Correction	?	6.6	8.8	7.9	12.3	14.0	11.2	10.7	10.6	7.8	5.7	6.3	4.8	?
J corr.	18.3	98.1	107.4	103.7	126.6	131.0	121.3	119.1	117.3	99.6	85.7	91.5	92.8	12.3

Integrating the value of $J_{corr.}$ with respect to z, we find:

$$W = \int J_{corr.}\, dz = 561.5 \text{ grammes.}$$

Taking into account the suction at the trailing edge, the value of which is 120 grammes for $V_p = 2400$ cm/sec, the surface friction on both sides of the model will be 441.5 grammes.

Notwithstanding the uncertainty in the determination of the resistance from the total head loss, the agreement between the resistance found in this way and that deduced from other methods seems to be fair.

§ 9. *Summary.*

From measurements of the distribution of the velocity in the boundary layer the value of the exponent n in the relation $u = V\left(\frac{y}{\delta}\right)^n$ has been deduced. This exponent proves to be independent of the velocity V and seems to be determined entirely by geometrical relations. The results provisionally lead to the conclusion that n approaches to the limit of 0.25 for increasing values of δ.

The value of the thickness of the boundary layer δ deduced from the experiments has been compared with the relations found in § 2 for the fully developed turbulent state of motion along rough surfaces. As may be expected from the formulae, δ does not depend on V and is determined entirely by x, by the height h of the pyramids on the surface, and by a numerical factor. This factor, which also occurs in the formula for the resistance, is connected with the shape and the arrangement of the pyramids on the surface. It may be expected that the relations for δ, for the shearing stresses near the surface and for the resistance coefficient deduced from our measurements, will hold for a series of rough surfaces similar to those investigated. The experimental material, however, is not sufficient to decide whether this supposition is of general validity.

The surface friction of both sides of plate II, as weighed on the balance, is in agreement with the resistance as determined from the total head loss in the wake by means of a Pitot-tube; the surface friction deduced from the loss of momentum in the boundary layer (determined with a hot wire anemometer) leads to the same value. The measurements with the balance lead to the formula:

$$W = 0{,}77\, V_0^2 \;\; (V_0 \text{ in m/sec}),$$

while form. (10) deduced from the measurements performed at plate I, inserting $\beta = 0{,}000337$, gives:

$$W \cong 0{,}83\, V_0^2.$$

The results of the determinations of the surface friction of both sides together according to several methods may be summarized as follows (the length of the board being 198 cm, its breadth 50 cm and the velocity $V_0 = 2400$ cm/sec):

Plate II, from measurements with the balance	454 gr.
from total head loss in the wake	441 gr.
from loss of momentum in the boundary layer	444 gr.
from form. (8), $x = 198$, $h = 0.15$ cm, $\beta = 0.000337$	478 gr.

Mathematics. — *On the Motion of a Plane Fixed System with Two Degrees of Freedom.* (Second Communication [1])). By Prof. W. VAN DER WOUDE.

(Communicated at the meeting of March 31, 1928).

§ 1. By the motion of a fixed system we always understand, at least in purely kinematical considerations, the motion of two fixed systems relative to each other: already in the usual indication of the problem we notice the peculiar lack of symmetry that strikes us in the further treatment [2]). In the simplest case, where this motion depends on one parameter, this lack is very little troublesome: it seems to me that already in the next case — the motion depending on two parameters — it is certainly worth while to pass on to a more symmetrical representation.

As in this case we choose a system of axes that is not fixed to either of the systems, the formulas are in the beginning slightly more complicated than in the usual method: this disadvantage disappears, however, as soon as we give this system of axes the movement that is prescribed by the problem so that the symmetry remains intact.

The followed method is briefly this: With the exception of a few special cases (§ 3) there always exists a definite line *d*, that is not fixed to either of the systems and that is the locus of the possible poles of rotation; this line may be chosen as the X-axis of a system of axes. It is then obvious that a definite point must be chosen as origin. In this way the formulas for the motion have become so simple that the known conclusions may be read at once.

For the sake of a more outward consequence the usual expressions "fixed" and "movable system" have been replaced by "the systems Σ_1 and Σ_2".

§ 2. Let OXY be a rectangular system of axes; the coordinates (x, y) of any point always relate to *this* system; whenever there is question of the components of a vector we always mean the projections of this

[1]) The earlier communication (these Proceedings, Vol. **29**, p. 652) gives a list of literature. I received for perusal an article on this subject of Dr. H. J. E. BETH, which will appear in the next number of the Nieuw Archief voor Wiskunde (2e reeks, deel XV, vierde stuk). The method of Dr. BETH, however, is entirely different from mine. Dr. BETH has always treated also the non holonomous cases; I restrict myself in this paper to the holonomous cases, although an extension would not be difficult.

[2]) Except in a few chapters of:

R. BRICARD. Leçons de Cinématique. Tome I. Paris, Gauthier—Villars; 1926.

vector on *this* system. Let further Σ_1 and Σ_2 be two plane fixed systems the motion of which relative to each other depends on two parameters u and v; we shall assume that the motion of both relative to OXY also depends on u and v, that, however, it is possible that the motion of one, e.g. of Σ_1, depends on one parameter, e.g. of u, but that in this case the motion of Σ_2 relative to OXY depends either on both u and v or on v only.

For convenience' sake there follows here first a summary of the formulas that express the displacements occurring in this motion. The elementary displacement relative to OXY of a point (x, y) that is fixed to Σ_i $(i = 1, 2)$ is defined by

$$\left.\begin{aligned}\delta x &= (\xi_1^{(i)} - \omega_1^{(i)}\, y)\, du + (\xi_2^{(i)} - \omega_2^{(i)}\, y)\, dv \\ \delta y &= (\eta_1^{(i)} + \omega_1^{(i)}\, x)\, du + (\eta_2^{(i)} + \omega_2^{(i)}\, x)\, dv\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

Here

$$\xi_1^{(i)},\ \xi_2^{(i)},\ \eta_1^{(i)},\ \eta_2^{(i)},\ \omega_1^{(i)},\ \omega_2^{(i)}$$

have the well known signification; between these quantities exist the relations [1])

$$\frac{\partial \omega_1^{(i)}}{\partial v} = \frac{\partial \omega_2^{(i)}}{\partial u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2^{\alpha})$$

$$\left.\begin{aligned}\frac{\partial \xi_1^{(i)}}{\partial v} - \frac{\partial \xi_2^{(i)}}{\partial u} &= \eta_2^{(i)}\, \omega_1^{(i)} - \eta_1^{(i)}\, \omega_2^{(i)} \\ \frac{\partial \eta_1^{(i)}}{\partial v} - \frac{\partial \eta_2^{(i)}}{\partial u} &= -\, \xi_2^{(i)}\, \omega_1^{(i)} + \xi_1^{(i)} \omega_2^{(i)}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2^{\beta})$$

In the same way the elementary displacement relative to Σ_1 of a point (x, y) that is fixed to Σ_2 is given by

$$\left.\begin{aligned}\delta x &= (\xi_1 - \omega_1\, y)\, du + (\xi_2 - \omega_2\, y)\, dv \\ \delta y &= (\eta_1 + \omega_1\, x)\, du + (\eta_2 + \omega_2\, x)\, dv\end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (3)$$

ω_1 and ω_2 are the rotations of Σ_2 in its motion relative to Σ_1; ξ_1 is the projection on OX of the vector that expresses the velocity relative to Σ_1 of the point (o, o) fixed to Σ_2. If we consider (x, y) as a point that is fixed to Σ_1 and if in (3) we replace ξ_1 by $-\xi_1$ etc. the displacement relative to Σ_2 of a point fixed to Σ_1 is expressed by these equations.

In this case it follows from (1) and (3) that

$$\left.\begin{aligned}\xi_1 = \xi_1^{(2)} - \xi_1^{(1)};\ \xi_2 = \xi_2^{(2)} - \xi_2^{(1)};\ \eta_1 = \eta_1^{(2)} - \eta_1^{1};\ \eta_2 = \eta_2^{2} - \eta_2^{(1)} \\ \omega_1 = \omega_1^{(2)} - \omega_1^{(1)};\ \omega_2 = \omega_2^{(2)} - \omega_2^{(1)}\end{aligned}\right\} \quad . \quad (4)$$

[1]) Cf. e.g. G. DARBOUX: Théorie des Surfaces I, p. 67, 71 (Gauthier—Villars, Paris), or L. P. EISENHART: A Treatise on Differential Geometry, p. 168, 170 (Ginn and Co., Boston, New York, London). Our formulas (2) however are only identical with the cited ones when we replace $\xi_1^{(i)}$ by $-\xi_1^{(i)}$ etc. as l.c. always the inverse motion, the motion of OXY relative to Σ_1, is considered.

The equations (2^β) do not hold good for $\xi_1 \ldots \omega_2$. (2^α) remains valid; we can see this by filling in $i=1$ and $i=2$ in (2^α) and by subtracting the two equations from each other. It appears that

$$\frac{\partial \omega_1}{\partial v} = \frac{\partial \omega_2}{\partial u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

§ 3. We start from (3) and we consider, therefore, the motion of Σ_2 relative to Σ_1. We shall call the possible movements depending on two parameters the system $[\mathfrak{M}^2]$.

The locus of the poles is found from:

$$\delta x = \delta y = 0$$

and has, therefore, as equation

$$\begin{Vmatrix} \xi_1 - \omega_1 y & \xi_2 - \omega_2 y \\ \eta_1 + \omega_1 x & \eta_2 + \omega_2 x \end{Vmatrix} = 0$$

or

$$(\xi_1 \omega_2 - \xi_2 \omega_1) x + (\eta_1 \omega_2 - \eta_2 \omega_1) y + \xi_1 \eta_2 - \xi_2 \eta_1 = 0$$

If for the moment we exclude the cases where

$$\omega_1 = \omega_2 = 0$$

or

$$\begin{Vmatrix} \xi_1 \, \eta_1 \, \omega_1 \\ \xi_2 \, \eta_2 \, \omega_2 \end{Vmatrix} = 0,$$

(6) represents a straight line d.

It is evident that

$$\omega_1 = \omega_2 = 0$$

means that $[\mathfrak{M}^2]$ contains only translations and that

$$\begin{Vmatrix} \xi_1 \, \eta_1 \, \omega_1 \\ \xi_2 \, \eta_2 \, \omega_2 \end{Vmatrix} = 0$$

indicates that the system $[\mathfrak{M}^2]$ depends on one parameter only.

In the future we shall always exclude these cases.

§ 4. We shall now make the condition that d coincide with OX; for this it is necessary and sufficient that

$$\xi_1 = \xi_2 = 0$$

For the sake of a further simplification we first remark that owing to (5) we can introduce a new variable θ through

$$2\, d\theta = \omega_1\, du + \omega_2\, dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^a)$$

We denote an integrating factor of $\eta_1\, du + \eta_2\, dv$ by $\dfrac{1}{2H(u, v)}$ so that we can put

$$2H\, d\tau = \eta_1\, du + \eta_2\, dv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7^b)$$

We shall further introduce in H the variables θ and τ defined through (7^a) and (7^b) but for a constant, in stead of u and v.

The displacement relative to Σ_1 of a point (x, y) of Σ_2 is now expressed by

$$\left.\begin{aligned}\delta x &= \qquad\quad - 2\, y\, d\theta \\ \delta y &= 2\, H\, d\tau + 2\, x\, d\theta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

It is evident that $d\theta = 0$ means a translation and $d\tau = 0$ a rotation round the origin; also that θ is twice the angle between two lines one of which is fixed to Σ_1 and the other to Σ_2 and that τ may be replaced by any function of τ without the form of (I) changing.

We shall now represent the displacement relative to OXY of a point $(x\, y)$ that is fixed to Σ_1 by

$$\left.\begin{aligned}\delta x &= (U_1 - \Omega_1\, y)\, d\tau + [U_2 - (\Omega_2 - 1)\, y]\, d\theta \\ \delta y &= (V_1 - H + \Omega_1 x)\, d\tau + [V_2 + (\Omega_2 - 1)\, x]\, d\theta\end{aligned}\right\}; \quad . \quad . \quad . \quad (8)$$

accordingly for the displacement relative to OXY of a point of Σ_2 we have

$$\left.\begin{aligned}\delta x &= (U_1 - \Omega_1\, y)\, d\tau + [U_2 - (\Omega_2 + 1)\, y]\, d\theta \\ \delta y &= (V_1 + H + \Omega_1 x)\, d\tau + [V_2 + [\Omega_2 + 1)\, x]\, d\theta\end{aligned}\right\}. \quad . \quad . \quad . \quad (9)$$

From (8) as well as from (9) there follow relations (see 2^a and 2^3) between U_1, U_2, V_1, V_2, Ω_1, Ω_2 and H; through addition and subtraction these are simplified to

$$\left.\begin{aligned}&\frac{\partial \Omega_1}{\partial \theta} = \frac{\partial \Omega_2}{\partial \tau} && \frac{\partial U_1}{\partial \theta} - \frac{\partial U_2}{\partial \tau} = V_2\, \Omega_1 - V_1\, \Omega_2 - H \\ &V_1 + H\Omega_2 = 0 && \\ &\frac{\partial H}{\partial \theta} = U_1 && \frac{\partial V_1}{\partial \theta} - \frac{\partial V_2}{\partial \tau} = U_1\, \Omega_2 - U_2\, \Omega_1\end{aligned}\right\} . \quad . \quad (10)$$

§ 5. Let a definite displacement out of $[\mathfrak{M}^2]$ be defined by

$$\frac{d\theta}{d\tau} = \lambda;$$

if $(x, 0)$ is the pole of rotation for the motion of Σ_1 and Σ_2 relative to each other, we have

$$H + \lambda\, x = 0$$

Now in the system Σ_1 d turns about the point $(x', 0)$ for which in (9)

$$\delta\, y = 0$$

hence about the point that is defined by

$$V_1 - H + \Omega_1\, x' + \lambda\, [V_2 + (\Omega_2 - 1)\, x'] = 0;$$

in the system $\Sigma_2\, d$ turns about the point $(x'', 0)$ that is defined by

$$V_1 + H + \Omega_1\, x'' + \lambda\, [V_2 + (\Omega_2 + 1)\, x''] = 0.$$

The former two points coincide when

$$\Omega_1 x^2 + (V_1 - \Omega_2 H) x - V_2 H = 0 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

if this is the case $(x, 0)$ and $(x'', 0)$ also coincide as might be expected. Through (11) two points are defined — at least if $\Omega_1 \pm 0$ —; we can call them the *stationary poles of rotation* (for the given position). For the moment we shall further put

$$\Omega_1 \neq 0.$$

In the future we shall always choose the middle between these stationary poles of rotation as origin of our system of coordinates of which so far we had defined the X-axis, not the origin. Now we have always

$$V_1 - H \Omega_2 = 0.$$

In connection with one of the formulas (10) it follows from this that

$$V_1 = \Omega_2 = 0.$$

It is impossible that H is identically equal to zero as in this case the motion of Σ_1 and Σ_2 relative to each other would only have one degree of freedom.

The formulas (8), (9), and (10) are now greatly simplified. We have already found

$$V_1 = \Omega_2 = 0,$$

further in (10)

$$\frac{\partial \Omega_1}{\partial \theta} = 0$$

hence Ω_1 is a function of τ only. Accordingly we can again denote $\int \Omega_1 \, d\tau$ by a new variable; if this is again called τ we have — cf. (10) —

$$\frac{\partial H}{\partial \theta} = U_1 \, ; \; \frac{\partial V_2}{\partial \tau} = U_2$$

$$\frac{\partial U_1}{\partial \theta} - \frac{\partial U_2}{\partial \tau} = V_2 - H.$$

SUMMARISING. The displacement relative to Σ_1 of a point of Σ_2, hence any displacement out of $[\mathfrak{M}^2]$, is expressed by

$$\left.\begin{aligned} \delta x &= \qquad\quad -2\, y d\theta \\ \delta y &= 2\, H d\tau + 2\, x d\theta \end{aligned}\right\} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

the displacement relative to OXY of a point of Σ_1 by

$$\left.\begin{aligned} \delta x &= \left(\frac{\partial H}{\partial \theta} - y\right) d\tau + \left(\frac{\partial V_2}{\partial \tau} + y\right) d\theta \\ \delta y &= (-H + x)\, d\tau + (V_2 - x)\, d\theta \end{aligned}\right\} ; \quad . \quad . \quad . \quad . \quad . \quad (II)$$

the displacement relative to OXY of a point of Σ_2 by

$$\left.\begin{aligned}\delta x &= \left(\frac{\partial H}{\partial \theta} - y\right) d\tau + \left(\frac{\partial V_2}{\partial \tau} - y\right) d\theta \\ \delta y &= (H + x)\, d\tau + (V_2 + x)\, d\theta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (III)$$

Between the functions $H(\tau, \theta)$ and $V_2(\tau, \theta)$ there only exists the relation

$$\frac{\partial^2 H}{\partial \theta^2} - \frac{\partial^2 V_2}{\partial \tau^2} + H - V_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (IV)$$

§ 6. *Simple Results.* Let any displacement be given by

$$\frac{d\theta}{d\tau} = \lambda.$$

The pole of rotation for the displacement of Σ_1 and Σ_2 relative to each other is the point $P\left(-\frac{H}{\lambda}, 0\right)$; in the plane Σ_1 d turns round the point Q_1 $\left(\frac{H - \lambda V_2}{1 - \lambda}, 0\right)$ for which

$$\delta y = 0;$$

in Σ_2 d turns round Q_2 $\left(\frac{-H - \lambda V_2}{1 + \lambda}, 0\right)$.

In any position there exists a projective correspondence between P and Q_1 and also between P and Q_2 [1]).

Special cases:

1. The three points coincide in the stationary poles of rotation.

2. If O is the pole of rotation V_2 and $-V_2$ are the abscissae of Q_1 and Q_2.

3. If Σ_1 and Σ_2 have a translation relative to each other (P lies at infinity; $\lambda = 0$), d turns round Q_1 (H, o) in Σ_1, round $(-H, o)$ in Σ_2.

4. If a has a translation in Σ_1, $(-H, o)$ is the pole of rotation; if d has a translation in Σ_2, (H, o) is the pole of rotation.

O is always the middle between the found pairs of points.

The motion of d in Σ_1 depends on one parameter only in the case that

$$-H + V_2 = 0$$

for then in (II)

$$\delta y = 0$$

for any point ($x = 0$) if the displacement is defined by

$$d\tau - d\theta = 0;$$

---

[1]) Cf. BETH l.c. who derives a complete classification of the movements with two parameters, including the non-holonomous ones, from the projective relations between P, Q_1 and Q_2.

in this case all points of d move on d and d is a fixed line in Σ_1. For any other displacement d turns in Σ_1 about the point (H, o).

But on the same condition the motion of d in Σ_2 depends on only one parameter: the displacement defined by

$$d\tau + d\theta = 0$$

leaves d at rest in Σ_2; for any other displacement it turns about the point $(-H, o)$. This gives the

THEOREM OF KOENIGS. *If the displacement of d in Σ_1 depends on only one parameter, this is also the case with the motion of d in Σ_2. The displacements that leave d at rest in Σ_1 and those that leave it at rest in Σ_2, are different.*

A second interesting special case is the following one. Suppose

$$V_2 = 0.$$

In this case the two stationary poles of rotation coincide in O; they correspond to the displacement for which

$$d\tau = 0.$$

In order to examine the displacement of O relative to Σ_1 and Σ_2 we have only to calculate δx and δy in (II) and (III) for $O(o, o)$ and to replace them by their opposites. Then evidently

$$\delta x = \delta y = 0.$$

The origin is accordingly at rest in Σ_1 and Σ_2, in other words:

If V_2 is identically equal to zero the system of movements $[\mathfrak{M}^2]$ of the systems Σ_1 and Σ_2 relative to each other contains a finite rotation about a point that is fixed to Σ_1 and to Σ_2.

This leads to the problem: when does the system $[\mathfrak{M}^2]$ contain finite rotations about a point that is fixed to Σ_1 and to Σ_2?

The pole of rotation must be fixed in Σ_1 and Σ_2: d always passes through the pole, hence d turns about the same point in Σ_1 and Σ_2, which point is accordingly one of the stationary poles of rotation. The problem is therefore: is it possible that for the *finite* movement defined by

$$\sqrt{H}\, d\tau + \sqrt{V_2}\, d\theta = 0$$

the pole of rotation $(\pm \sqrt{H V_2}, o)$ is fixed to Σ_1. The vector of the displacement of a point (x, y) relative to Σ_1 is given by the components $dx - \delta x$, $dy - \delta y$ when δx and δy are taken from (II) and if for $\frac{d\theta}{d\tau}$ $(\pm \sqrt{HV_2}, o)$ is substituted.

It is therefore necessary that for one of the points $(\pm \sqrt{H V_2}, o)$

$$dx - \delta x = 0$$

or

$$3\left(H\sqrt{V_2}\,\frac{\partial V_2}{\partial \tau} + V_2\sqrt{H}\,\frac{\partial H}{\partial \theta}\right) + V_2\sqrt{\theta_2}\,\frac{\partial H}{\partial \tau} + H\sqrt{H}\,\frac{\partial V_2}{\partial \theta} = 0.$$

If H and V_2 satisfy this equation besides (IV), $[\mathfrak{M}^2]$ contains a finite rotation about a fixed point.

As might be expected this equation is always satisfied if $V_2 = 0$.

Does the system $[\mathfrak{M}^2]$ contain *finite* rectilinear translations? The translation is always parallel to d (cf. (I)); however we have seen that d turns about a point at finite distance in Σ_1 as well as in Σ_2 and, accordingly, has neither a fixed direction in Σ_1 nor in Σ_2. A finite rectilinear translation can, therefore, only be expected in the case that we have excluded until now; in that case it is in fact contained in $[\mathfrak{M}^2]$ (cf. §§ 5, 7; $\Omega_1 = 0$).

§ 7. We shall now briefly discuss the case that has so far been excluded where (cf. § 5)

$$\Omega_1 = 0$$

If we suppose in the first place

$$V_1 - \Omega_2 H \neq 0 \,. \quad \ldots \ldots \quad (a)$$

it appears from (II) § 5 that:

there is only one stationary pole of rotation (or the other one lies at infinity).

If we choose this stationary pole of rotation (not at infinity) as origin we have:

$$V_2 = 0.$$

It appears further from (10) that Ω_2 is a function of θ only and $\Omega_2 H^2$ of τ only, hence

$$H = \frac{\varphi(\tau)}{\sqrt{\Omega_2}}$$

We shall introduce $\int \varphi(\tau)\, d\tau$ as new variable; if we call this variable again τ the form of (I) does not change and H is a function of θ only. Further

$$\Omega_2 = \frac{1}{H^2}\,,\quad V_1 = -\frac{1}{H}\,,\quad U_1 = \frac{\partial H}{\partial \theta}.$$

Any displacement relative to Σ_1 of a point of Σ_2 is still expressed by

$$\left.\begin{aligned} \delta x &= \qquad\qquad\quad - 2\, y\, d\theta \\ \delta y &= 2\, H(\theta)\, d\tau + 2\, x\, d\theta \end{aligned}\right\} \quad \ldots \ldots \quad (I)$$

the displacement relative to OXY of a point of Σ_1 by

$$\left.\begin{aligned} \delta x &= \quad \frac{\partial H}{\partial \theta}\, d\tau + \left[U_2 - \left(\frac{1}{H^2} - 1\right) y \right] d\theta \\ \delta y &= -\left(\frac{1}{H} + H\right) d\tau + \left(\frac{1}{H^2} - 1\right) x\, d\theta \end{aligned}\right\}; \quad \ldots \quad (II)$$

the displacement relative to OXY of a point of Σ_2 by

$$\left.\begin{aligned}\delta x &= \frac{\partial H}{\partial \theta}\, d\tau + \left[\, U_2 - \left(\frac{1}{H^2}+1\right) y \,\right] d\theta \\ \delta y &= -\left(\frac{1}{H} - H\right) d\tau + \left(\frac{1}{H^2}+1\right) x\, d\theta\end{aligned}\right\} \quad \ldots \quad \text{(III)}$$

Between the functions U_2 and H there exists the relation

$$\frac{\partial^2 H}{\partial \theta^2} - \frac{\partial^2 U^2}{\partial \tau^2} + \frac{1}{H^3} + H = 0.$$

This proves: *if always (i. e. for any pair of values of u and v) $\Omega_1 = 0$, the system $[\mathfrak{M}^2]$ contains a rectilinear finite translation of Σ_1 and Σ_2 relative to each other, corresponding to $d\theta = 0$. If besides $H = \pm 1$ the motion of d in Σ_1 as well as in Σ_2 depends on one parameter only.*

There remain the possibilities

$$\Omega_1 = 0, \quad V_1 - \Omega_2 H = 0, \quad V_2 \neq 0 \quad \ldots\ldots \quad (\beta)$$

d does not contain any stationary pole of rotation.

$$\Omega_1 = 0, \quad V_1 - \Omega_2 H = 0, \quad V_2 = 0. \quad \ldots\ldots \quad (\gamma)$$

Any point of d is a stationary pole of rotation.

If, accordingly, Ω_1, $V_1 - \Omega_2 H$ and V_2 are always equal to zero, about any point of d a finite rotation is possible. I intend to come back to this remarkable case in a later short paper.

In the following discussion of the quantities of the second order this case $\Omega_1 = o$ is again excluded.

§ 8. *The quantities of the second order.*

If we start from a given original position, for a given $\frac{d\theta}{d\tau}$ the tangent to the path in Σ_1 of any point of Σ_2 is defined (and inversely); the quantities of the second order, e.g. the radius of curvature of any point in its path, are not defined before also $\frac{d^2\theta}{d\tau^2}$ is given.

We shall now put

$$\frac{d\theta}{d\tau} = \lambda, \quad \frac{d^2\theta}{d\tau^2} = \lambda'$$

and we shall only consider the system of infinitesimal displacements where λ is kept constant and λ' is variable. In other words, *we choose movements from $[\mathfrak{M}^2]$ with a fixed pole of rotation* where to any point a definite tangent to its path is already assigned.

For the present we assume that the movement of Σ_2 relative to Σ_1 is given by the functions ξ, η, ω that depend on one parameter t (the time) and that the movement of Σ_1 relative to OXY is given by $\xi^{(1)}, \eta^{(1)}, \omega^{(1)}$ that also depend on t only. For the components of the velocity- and acceleration relative to Σ_1 of a point (x, y) of Σ_2 we have

$$v_x = \xi - \omega y, \quad v_y = \eta + \omega x.$$

$$J_x = \frac{dv_x}{dt} + \omega^{(1)} v_y = \frac{d\xi}{dt} + \omega^{(1)} \eta - (\eta^{(1)} + \eta)\,\omega - \frac{d\omega}{dt}\,y - \omega^2 x,$$

$$J_y = \frac{dv_x}{dt} - \omega^{(1)} v_x = \frac{d\eta}{dt} - \omega^{(1)} \xi + (\xi^{(1)} + \xi)\,\omega + \frac{d\omega}{dt}\,x - \omega^2 y$$

The radius of curvature of a point (x, y) of Σ_2 in its path in Σ_1 is given by

$$\frac{1}{R^2} = \frac{(V_x J_y - J_x V_y)^2}{V^6}$$

PROOF. If we use fixed axes we have

$$\frac{1}{R^2} = \frac{(x' y'' - y' x'')^2}{(x'^2 + y'^2)^3}$$

i.e. $\frac{1}{R^2}$ is equal to the square of the vector product of the velocity and the acceleration divided by the sixth power of the velocity. That is exactly what the above mentioned formula expresses.

In the same way it appears that the center of curvature in the path of $M(x, y)$ is the point

$$\mu\left(x - \frac{V_y}{V_x J_y - V_y J_x}\,V^2, \quad y + \frac{V_x}{V_x J_y - V_y J_x}\,V^2\right)$$

We now pass on to the case in question by the following substitutions (cf. the formulas (I) and (II), § 5)

$$\xi = 0, \quad \eta = 2H, \quad \omega = 2\lambda$$

$$\xi^{(1)} = \frac{\partial H}{\partial \theta} + \lambda \frac{\partial V_2}{\partial \tau}, \quad \eta^{(1)} = -H + \lambda V_2, \quad \omega^{(1)} = 1 - \lambda,$$

$$\frac{d}{dt} = \frac{\partial}{\partial \tau} + \lambda \frac{\partial}{\partial \theta}.$$

These entirely determine the elements of the second order.

§ 9. As a first example we shall determine the system of inflexional circles for these displacements.

First we find

$$V_x = -2\lambda y; \quad V_y = 2H + 2\lambda x$$

$$J_x = 2\dot{H} - 2\lambda^2 V_2 - 4\lambda^2 - 2\lambda' y$$

$$J_y = 2\frac{\partial H}{\partial \tau} + 4\lambda \frac{\partial H}{\partial \theta} + 2\lambda^2 \frac{\partial V_2}{\partial \tau} - 4\lambda^2 y + 2\lambda' x.$$

The equation of the inflexional circle, i.e. the locus of the points where the curve has an infinite radius of curvature, runs:

$$V_x J_y - V_y J_x = 0$$

or

$$2\lambda^3(x^2+y^2) - (\lambda H + 2\lambda^2 H - \lambda^3 V_2)\,x - \\ - \left(\lambda\frac{\partial H}{\partial \tau} + 2\lambda^2\frac{\partial H}{\partial \theta} + \lambda^3\frac{\partial V_2}{\partial \tau} - \lambda' H\right) y - H(H - \lambda^2 V_2) = 0.$$

As λ is a constant, λ' a variable parameter, this gives:

For all displacements about the same pole of rotation the inflexional circles form a pencil; the base points lie on d; one of them is the pole of rotation, the other a point $H\left(\frac{H - \lambda^2 V_2}{\lambda^2}, 0\right)$. *In all these displacements H describes a point of inflexion.*

We found above: if $\mu(\xi, \eta)$ is the center of curvature of $M(x, y)$ we have

$$\xi = x - \frac{V_y}{V_x J_y - J_x V_y} V^2, \quad \eta = y + \frac{V_x}{V_x J_y - V_y J_x} V^2.$$

We substitute the values indicated for V_x, V_y, J_x and J_y but at the same time, in order to simplify the formulas, we choose the pole $\left(-\frac{H}{\lambda}, 0\right)$ of the movement as new origin, i.e. we put

$$\bar{x} = x + \frac{H}{\lambda}, \quad y = \bar{y}$$

$$\bar{\xi} = \xi - \frac{H}{\lambda} = \bar{x} - \frac{V_y}{V_x J_y - V_y J_x} V^2, \quad \bar{\eta} = \eta.$$

Thus we find

$$\bar{\xi} = \frac{(2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y})\,\bar{x}}{8\lambda^3(\bar{x}^2 + \bar{y}^2) + 2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y}},$$

$$\bar{\eta} = \frac{(2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y})\,\bar{y}}{8\lambda^3(\bar{x}^2 + \bar{y}^2) + 2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y}}$$

Here B and C are functions of λ, τ and θ, i.e. in all the considered elementary displacements — where the initial position and the value $\lambda = \frac{\partial \theta}{\partial \tau}$ have been chosen — they have the same values; λ' is a parameter that assumes any value.

If for the moment we choose also λ' constant, any point M has a definite center of curvature μ; in this case the aforesaid formulas express a well known quadratic correspondence between M and μ; it is especially interesting that in the inverse movement M is the center of curvature of μ. Analytically this means that $\bar{x}$ and $\bar{y}$ are expressed in $\bar{\xi}$ and $\bar{\eta}$ through similar formulas.

The correspondence is quadratically involutory; to any line l described by M (or μ), there corresponds a conic as locus of μ (or M). The latter is sometimes called the conic of RIVALS, associated to l.

If now again we consider λ' as a parameter, it is at once evident [1]) that:

The conics of RIVALS *that in the different displacements about the same pole of rotation correspond to the same line, form a pencil. The points of d — and these only — always have the same center of curvature.*

[1]) H. J. E. BETH, l.c.

Pathology. — *On Anophelism without malaria in the vicinity of Amsterdam* [1]. (2nd communication.) By N. H. SWELLENGREBEL, A. DE BUCK and E. SCHOUTE. (From the Institute for tropical hygiene, Department of the Royal Colonial Institute at Amsterdam.) (Communicated by Prof. W. SCHÜFFNER.)

(Communicated at the meeting of April 28, 1928).

In the vicinity of Amsterdam, including a "region I" where malaria occurs, and a "region II" where it is absent or rare, we described [2]) two types of *A. maculipennis*: a long one (i.e. a longwinged form, bodylength and length of wing showing a high degree of correlation) and a short one, the last named being characterized moreover by a greater number of maxillary teeth. As a general rule the long type occurs in region II, the short one in region I. The difference between the two cannot be accounted for by external conditions (e.g. the salinity of the breedingplaces), because it persists after cultivation under exactly identical conditions. We therefore considered the two types as populations in which a short race or a long race dominate. We found no difference in their ability to act as a host for the parasite of simple tertian. The long form showed less appetite for human blood under experimental conditions. The difference of behaviour during hibernation proved to be the most important. Among the long forms hibernation begins in September; it is a complete one, i.e. hardly any blood is taken and there occurs a wide-spread and intensive development of the fat body. In the short forms semihibernation is the rule: feeding continues during the greater part of the winter, provided the temperature is not too low, and there is little development of the fat body. This behaviour of the long-winged population prevents its taking part in the transmission of malaria during the winter.

The necessity of proving the real existence of these differences by examining large numbers of mosquitoes and the fact that the behaviour during hibernation had only been established provisionally, made a reexamination desirable to check and extend our experience. The main facts resulting from this investigation are recorded here, for the details we refer to our extensive paper to be published later on.

Our present investigation includes 46 stations where we measured 11122 ♀ and established the percentage of females carrying blood, fat or eggs among 25065 ♀. Out of this material 4739 resp. 8945 ♀ came from the stations 4 (region I) and 29 (region II). We have regularly examined the Anopheline population of these two stations for a period of over two years. If nothing else could, this should convince us of the reality of the

[1]) Part of this investigation was carried out with financial support from the malaria commission of the League of Nations.

[2]) These Proceedings **30**, p. 61.

MAP I

The numbers indicate the stations. The upper dotted line approximately indicates the southern limit of the area of endemic-, the lower one that of the area with sporadic malaria. The last one is the frontier between region I (North) and II (South).

I. The black sectors indicate the $^0/_0$ ($90° = 25\,^0/_0$) of Anophelines with a *wing above 134 units* (1 unit = 41.7 μ). The half circles to the right represent stable-mosquitoes, those to the left mosquitoes from shelters.

morphological difference of their populations. The results for each station separately may be gathered from the maps I and II; our general results are as follows:

	Winter 1926—'27:	Autumn 1927:
Stations in region II with pure longwinged population	wing: 131.4 ± 0.20; $^0/_0$ fat: 58;	
Id. in reg. I with pure short-winged population	wing: 122.2 ± 0.14; „ „ 7;	wing: 119.2 ± 0.30; $^0/_0$ fat: 7.5
	Difference: 9.2 ± 0.25	
Stations, with segregation mainly in reg. II, "shelters"	wing: 129.2 ± 0.28; $^0/_0$ fat: 45;	wing: 129.6 ± 0.32; $^0/_0$ fat 66.5
"stables"	„ : 123.7 ± 0.30; „ „ : 21;	„ : 120.7 ± 0.31; „ „ 21.
	Difference: 5.5 ± 0.41	Difference: 8.9 ± 0.42

Both periods:

Station 4 (reg. I): wing: 121.5 ± 0.27; % fat: 6.5; max. teeth: 17.9 ± 0.05;	correlation length of wing and number of max. teeth	$+0.215 \pm 0.034$
Station 29 (reg. II): wing: 130.8 ± 0.27; „ „ : 79. ; „ „ 17.1 ± 0.06;		$+0.111 \pm 0.038$
Difference: 9.3 ± 0.39 Difference: 0.8 ± 0.08		

N.B. "Wing" = length of wing in units of 41.7 μ
"max. teeth" = number of maxillary teeth.

MAP II

II. Like map I but the sectors ($90° = 50\%$) indicate the *fat Anophelines.*

The measurement of Anophelines bred from larvae which had been cultivated under exactly identical conditions and only differed in the place the eggs came from (either region I or region II) confirmed last year's experience that the difference between the mosquitoes from these regions cannot be explained by external conditions. The result of the measurement of 514 ♀ and 398 ♂ from region II, 476 ♀ and 500 ♂ from region I is as follows:

	wing ♀	wing ♂	max. teeth
region II.	111.8 ± 0.32	102.5 ± 0.41	16.7 ± 0.04
region I.	108.8 ± 0.43	93.4 ± 0.31	17.7 ± 0.05
Difference:	3.0 ± 0.54	9.1 ± 0.51	1.0 ± 0.07

These results seem to confirm last year's. As a matter of fact there exists a notable difference. Our statement that the shortwinged race predominated in region I can be maintained (apart from a few exceptions) but its counterpart: predominance of the longwinged race in region II should not stand unmodified. For in this region a mixture of both races occurs almost everywhere [1]). This is brought to evidence in a very marked way during the winter by the occurrence of a *"segregation"* of the Anopheline population in two distinct groups: One, living in stables, is almost or quite identical with the short-winged type of region I; it continues to ingest blood and does not grow fat: it shows semihibernation. The other occurs in "shelters", i.e. uninhabited sheds, barns etc., also garrets and other appartments in human habitations; it is long-winged, fat [2]) (at least during the first two months of hibernation) and fasting. In most of the stations in region I there is no segregation and, consequently, very little difference between the mosquitoes from stables and shelters, both, as a rule, are short-winged, with little fat and taking blood under favourable conditions. In other words the short-winged population of region I is fairly pure, the long-winged one in region II is very mixed.

This is also born out by the positive correlation between the length of the wing and the number of maxillary teeth, which we failed to establish last year. This has changed by chosing our material as unmixed as possible. The purest material we have: laboratory-bred mosquitoes from stat. 4 in region I shows a correlation of $+ 0.445 \pm 0.040$. But similar mosquitoes from stat. 29 in reg. II do not show a higher figure than $+ 0.150 \pm 0.044$, whereas among Anophelines caught in nature in various stations of region II the correlation practically disappears ($+ 0.05 \pm 0.03$). Experimental or incidental mixing of long and short forms shows this (seeming) absence of correlation to be due to a gradual increase in the length of wing being accompanied at first by a corresponding rise in the number of teeth but afterwards by its decrease, an anomaly which can be eliminated by dividing the components of the mixed material according to their place of origin. This phenomenon is an important support of our view regarding the racial nature of our long and short types because, according to this view, the phenomenon is easily explained as a consequence of the short type having a larger number of teeth than the long one, whereas rejection of this standpoint leaves no room but for a highly artificial explanation.

Owing to the fact that the majority of wintering mosquitoes in region II are to be found in the "shelters" (i.e. long-winged ones) the average length of the wing in many stations is a high one. Stations with "medium-

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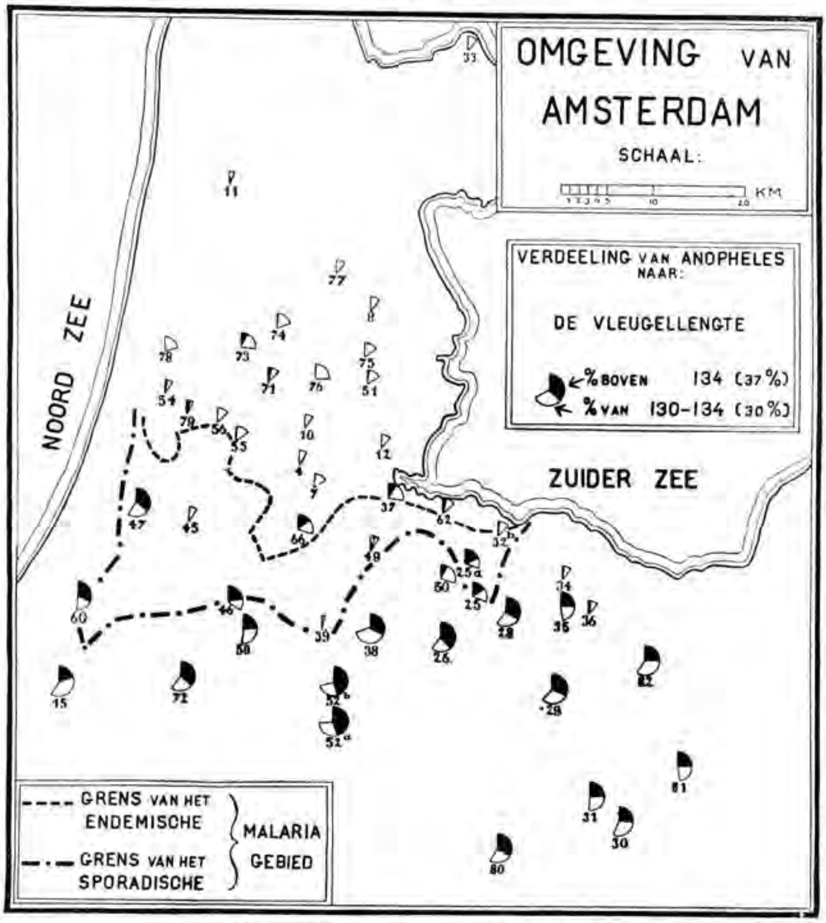
[1]) Last year there were a great number of stations not showing segregation, with long, fat and fasting mosquitoes in the stables. But many of them have turned to segregation in the autumn of 1927. This is why there is a blank under the heading "pure long-winged mosquitoes (autumn of 1927)" in the summary on p. 532. Moreover the difference between the mosquitoes from stables and shelters has increased.

[2]) There exists a fairly high positive correlation (0.42—0.52) between length of wing and fatness, among the mixed Anopheline population of the segregating stations. Among the pure population of region I it is insignificant (0.09—0.13). On the whole we take it to be a false correlation, a consequence of the general fatness during hibernation numbering among the racial characters of the long-winged type, but not of any direct correlative link between length of wing and fatness (as there exists e.g. between length of wing and number of maxillary teeth).

sized" Anophelines (as we called them last year) do no longer figure as harbouring a separate type of Anophelines but as segregating stations with a comparatively high number of stable mosquitoes (i.e. short ones). Consequently the distribution of the long- and short-winged type has changed very little indeed since the autumn of 1925—'26. But it has lost

MAP III



The sectors (90° = 25 %) indicate *long-winged Anophelines*, the white ones: wing of 130—134, black ones: wing above 134 units. The difference with map I is that in this case an average is established taking account of the mosquitoes in stables and shelters according to their numbers in each station. In this way the map can be compared with the one published in our 1st communication.

much of its value as an explanation of Anophelism without malaria, now we have become aware of the existence in region II of the short-winged type, which we must consider as the real vector of malaria, at least in winter.

There are two circumstances going a long way to help us out of this difficulty, relating to *A. maculipennis* visiting human habitations and to local variations of its incidence.

Region II does not belong to those areas with Anopheles but without malaria where the mosquito is limited to the stables and does no longer enter houses. On the contrary, it is quite common there, in spring and autumn even more so than in the houses of region I. In summer it is less numerous in the houses of region II than of region I, but even then its relative incidence (compared to the numbers found in stables) is higher in the first [1]). A preliminary survey of the blood ingested by Anopheles in region I and II, by means of the precipitine-test, did not reveal any marked difference between the two with regard to their appetite for human blood [2]).

But the behaviour of Anophelines of region II visiting houses in winter is different from that in region I. The former belong to the "shelter"-mosquitoes, which carry on a complete hibernation (long-winged, fat, no blood); whereas the latter show semihibernation (short-winged, with little fat, blood).

Consequently the behaviour of the house-visiting mosquitoes in region I favours malarial infection in winter, whereas in region II it does not, whether there is segregation or not.

The preceding explanation does not account for malarial infection in summer. Even if the long-winged race should prove to be of little importance in this respect [3]) the presence of the short-winged race in region I can no longer be neglected. The local incidence of Anopheles, as shown in map IV, materially lessens this difficulty, because it is comparatively low in region II. The exceptions (e.g. stat. 52*a*, 52*b*, 58, 72) which would cause much trouble, if we wished to use the differences in the density of the Anopheline population as the sole and only explanation for the absence of malaria, do not raise any difficulty. For, although the Anophelines are numerous, they mainly belong to the long-winged type. This shows that the racial factor should not be neglected.

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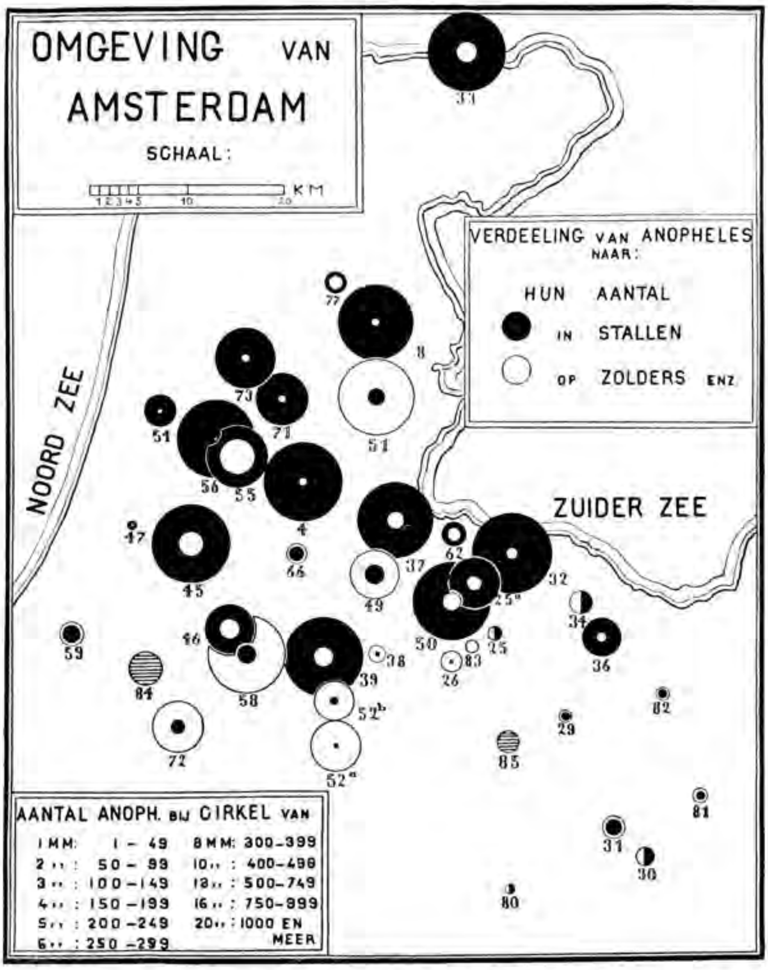
[1]) In spring and autumn 14—48 per house in region II, 12—18 in region I. In summer 1 per house, against 129 per stable in region II, and 5 against 1066 (1 : 213) in region I.

[2]) In the region without malaria 48 % of house-mosquitoes contained human blood against 14 % in the malarious region. But this requires confirmation by a more extensive investigation.

[3]) We are not nearly so sure about this as we are in the case of winter infection. Still the supposition is supported by the result of our feeding experiments with human blood which we have been carrying on for a period of 296 days with an average daily number of 80 ♀ from station 4 and 47 ♀ from stat. 29 in cages of $45 \times 30 \times 50$ cm^3. During all this time the long-winged type showed less appetite and in summer it suffered of a considerably higher mortality than the short-winged type. We do not wish to explain this difference by assuming a special "misanthropy" of the former but simply by its being evidently less able to stand confinement. This is of sufficient importance because the conditions required for *A. maculipennis* to act as an efficient malarial vector (as determined by JAMES *Brit. med. Jrl.*, Aug. 27, 1927), really amount to a close confinement like that in our cages.

There is the less cause for such a neglect as we have reason to believe that the low Anopheline incidence in region I is itself dependent on the

MAP IV



The circles, by the length of their radius, indi ate the largest *number of Anophelines* found in each station per stable (black) and per shelter (white), in Sept.—Oct. 1927. The circles are lying one on the other, the smaller one uppermost. When of equal diameter, two half circles take their place. The two striped circles indicate catches in July.

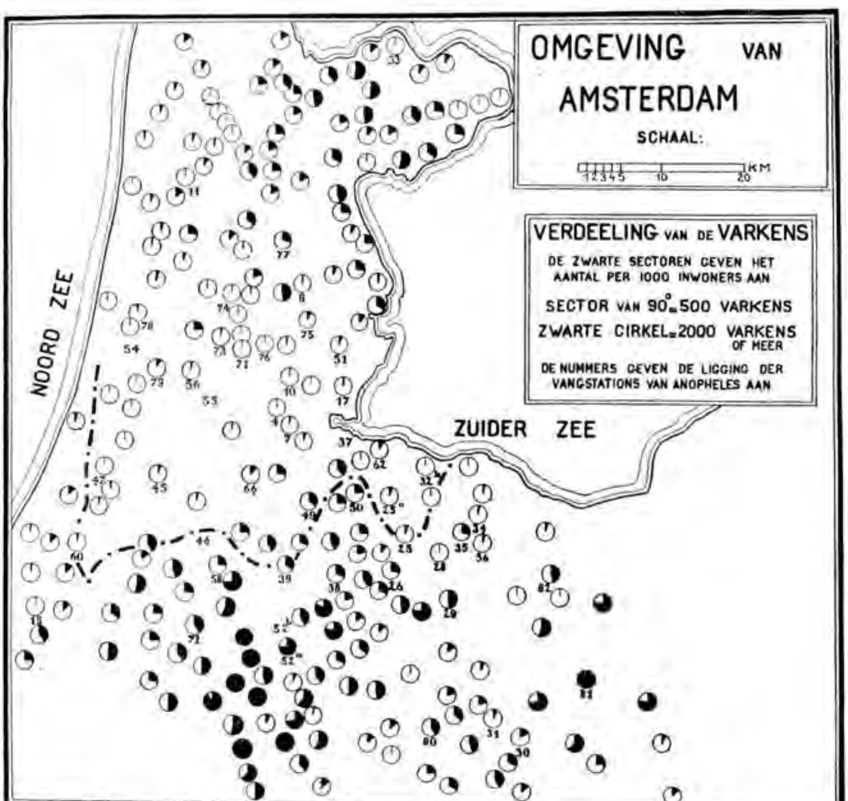
N.B. In region II the white circles indicate long-winged Anophelines, the black circles usually short-winged ones. In region I both circles indicate short-winged mosquitoes (save for a few exceptions).

racial difference, by means of the larval incidence of the breedingplaces, which corresponds closely to the mosquito-incidence in its local variations, being low in region II and high in region I.

The low larval incidence in region II could not be explained by a general inferiority of its breedingplaces. The farthest our observations allowed us to go was to assume an inhibition of the larval development in these breedingplaces affecting the shortwinged race inhabiting region II, as a consequence of the fresh water; our experience distinctly pointing to the favourable influence a certain degree of salinity exercices on these larvae in region I. A comparison of the food of the larvae in regions I and II revealed the former as less selective in the choice of their food; in summer they were not so much restricted to the use of green algae and flagellates [1]) as the latter, a circumstance which should affect their chances of survival and, consequently, the multitude of the race to which they belong.

There remains a notable difference between the regions I and II which we should not neglect although, for the moment, we cannot yet estimate its proper value, viz. the number of pigs, an animal of especial interest because the precipitine tests had shown it to provide more blood than any other. The distribution of pigs seems to open the perspective of a possible explanation of the absence of malaria in the frontier-stations of region II, with a numerous short-winged Anopheline population (e.g. No. 39, 46, 50).

MAP V



The black sectors (90° = 500 pigs) indicate the *numbers of pigs per 1000 inhabitants* in each municipality. The numbers show the situation of the stations mentioned in the maps I—IV.

1) In accordance with VAN THIEL (*Bull. Soc. path. exot.* 1927, XX, 366) whose interpretation is somewhat different from ours.

A numerous porcine population might affect the local incidence of adult[1]) Anophelines by distributing the same number of adults over a greater number of stables. A comparison of the maps IV and V shows that the districts with a numerous porcine population are by no means confined to region II (North eastern portion of the province of N. Holland) but that there is a better agreement in the south.

Our last year's conclusion, explaining the absence of malaria in region II by the biological characteristics of our long-winged race which represents *A. maculipennis* in that region, is confirmed in so far as our present investigation supports the view that this race really exists and that its behaviour diminishes its importance as a malarial vector. But the phenomenon of segregation has shown us that the short-winged race likewise exists in region II. Still the investigation of the racial factor, although unable to explain, unaided, the Anophelism without malaria, has helped us to estimate at their proper value the local incidence of Anophelines (in general and in human habitations) and — provisionally — of pigs, as ancillary explanatory factors.

1) The larval incidence could not, of course, be influenced in this way.