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KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS

VOLUME XLIII

No. 6

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CONTENTS

Chemistry. — *On colloidally bound water*. By H. R. KRUYT and H. DE BRUYN Mzn.

(Communicated at the meeting of May 25, 1940.)

I. *Introduction.*

The determination of colloidally bound water has been tried along various ways and the results of these investigations have led to contradictory conclusions.

Leaving the viscosimetric investigations apart (on account of the difficulty they present of correct interpretation), there remain four types of investigation:

1. Determination of water which does not freeze into ice at low temperatures. As a criterion we can take the measurement of the melting heat at a rise of temperature (RUBNER 1), THOENES 2)), or the volume change (JONES and GORTNER 3)).

2. Determination of the actual concentration of a substance added to the sol; the difference between the total quantity of water and that which proves available as solvent, is the colloidally bound water. The concentration may be determined by cryoscopic measurement (NEWTON and GORTNER 4)), or polarimetrically or volumetrically (KOETS 5) in those cases when the liquid can be simply poured away from the colloid.

3. Separation of colloid and medium may also be done by means of ultrafiltration (GREENBERG and GREENBERG 6)). The criterion is, of course, the difference between the total quantity of water and that which serves as a colvent for a molecular dispersion substance (concentration of the ultrafiltrate).

4. Determination of the difference in concentration of a dissolved substance on either side of a dialysis membrane, when on the one side there is colloid, water and dissolved substance, on the other only water and dissolved substance (OAKLEY 7)); the interpretation is again as in 2 and 3.

The remarkable fact is, that methods 1, 2 and 4 have led to similar results, at least as regards the order of magnitude (water binding in the order of 1 gr of water per gram of colloid), whereas method 3 gives an absolutely negative result. This is especially striking in methods 3 (GREENBERG) and 4 (OAKLEY), which differ only in that the one is by ultrafiltration, the other by dialysis.

The calculation of the results of all these methods easily causes errors; the concentrations should be calculated per water unit, not per solution unit. Some communications in the literature are so concise, that it is impossible to check the method of calculation. We have, therefore, thought it necessary to investigate the same objects by different methods and by calculations which are in every respect comparable. We have, however, abstained from checking method 1, because we deem its results impossible of interpretation without suppositions which cannot be verified themselves; method 2 we have left unchecked, because two widely different investigations in this laboratory (KOETS 5), KRUYT and WINKLER 8)) have already confirmed the correctness of the conclusions drawn from this method.

We have, therefore, occupied ourselves with OAKLEY's and GREENBERG's methods, which, for all their similarity, have led to such opposite conclusions. As objects of both methods we took: *sodiumarabinate* for colloid and urea for the moleculary dispersed substance, whose concentration was to be determined. We have applied OAKLEY's and GREENBERG's methods, we have modified them, made supplementary investigations where it seemed necessary, yet *we have had to fully confirm the apparently paradoxical result*: "in OAKLEY's method there is bound water, in GREENBERG's there seems to be none".

In the following experimental part a complete description is given of some of the measurements taken, so that others may check our results and get an idea of the limits of accuracy. In the treatises published on this subject this is often impossible, it has been our wish to prevent this uncertainty in the reader.

II. *Experimental part.*

A. OAKLEY's dialysis method.

1. *Material.*

The *Na arabinate* was prepared by the method of BUNGENBERG DE JONG and THEUNISSEN 9); starting from gum Arabic "Senegal, petit boule blanche I", it was precipitated four times with acetate of sodium and alcohol.

The urea was a *purissimum* preparation, the purity of it was checked by analysis, as described below.

2. *Analysis.*

The urea concentration was determined by the micro-KJELDAHL method namely, following HENWOOD and CAREY 10). This method was first tested by numerous blank determinations, with known quantities of urea.

3. *Investigation*

We made little collodion bags by fabricating four successive collodion films, one on top of the other in ERLEMEYER flasks of 25 cc. These bags were pushed over a length of vacuum tube, in which a glass tube had been introduced. With this glass tube the dialysis membrane was secured and placed in a covered beaker containing the sol-free solution, which was renewed after 1, 2, 4, 6 and 10 days respectively; after a fortnight 10 cc of the inside and 10 cc of the outside liquid were analyzed; moreover, 5—10 cc of the inside liquid were dried in a vacuum drier over P_2O_5, in order to determine the quantities of gum arabic and urea.

Only in experiment 5 did we analyze after a month, after the outside liquid had been renewed 10 times. From the figures obtained in this way we calculated "the quantity of bound water per gram of colloid, h", according to the formula given by GREENBERG:

$$h = \frac{1}{c}\left(1 - \frac{x}{y}\right),$$

in which x is the urea concentration per gram of the total quantity of water in the colloid solution inside the dialysator, y the same concentration but in the beaker, so outside the dialysator, c the colloid concentration in grams per gram of water.

Table I shows that the quantity of bound water is about 1 gram per gram of *sodiumarabinate* and that in an *arabinate* solution of ca 5 %. In 3—8 % concentrations OAKLEY found 1.1 gram.

In principle, therefore, our results confirm those of OAKLEY.

Meanwhile the objection may be raised that possibly in OAKLEY's method the equilibrium has not yet been established after a fortnight and 5 renewals. Although experiment 5, in which, as has been remarked, 10 renewals were made in 30 days, does not point in that direction, we thought it would be well to approach the equilibrium from the other side, namely in two experiments (I and II) we placed a greater urea concentration inside the dialysator than outside, while afterwards in experiment III the concentrations were chosen so that acocrding to experiments I and II there was equilibrium in concentration from the first. In order to promote the establishment of equilibrium this series of experiments was made with somewhat thinner collodion membranes, but owing to this a slight quantity of gum arabic permeated, so that this substance had also to be determined in the outside liquid (Table II).

In this series intermediary determinations were also made. We shall not describe the results extensively, but restrict ourselves to publishing the figures (Table III).

The results of these experiments do not leave any doubt that there is bound water, namely, one to one and a half grams per gram of arabinate.

B. GREENBERG's ultrafiltration method.

1. The material used and the method of analysis were identical to those applied with OAKLEY's method.

TABLE I.

	1		2		3		4		5	
	outside	inside	outside	inside	outside	inside	outside	inside	outside	inside
g. liquid	10.0800	9.9922	10.0082	10.0480	9.9984	10.0677	10.0604	10.0437	10.0863	10.0265
g. urea	0.1210	0.1089	0.1202	0.1071	0.1196	0.1060	0.1202	0.1076	0.1209	0.1073
idem/g. liquid	0.01201	0.01090	0.01201	0.01066	0.01196	0.01053	0.01195	0.01071	0.01199	0.01070
g. liquid		5.1171		10.1044		9.9864		10.0888		8.6494
g. dry substance		0.2996		0.6585		0.6561		0.6516		0.5467
g. gum arabic		0.2438		0.5508		0.5510		0.5435		0.4541
idem/ liquid		0.04764		0.05451		0.05517		0.05387		0.05250
water/liquid	0.9880	0.9415	0.9880	0.9348	0.9880	0.9343	0.9880	0.9354	0.9880	0.9368
h.	0.93		1.06		1.17		0.92		1.05	

TABLE II.

	I		II		III	
	outside	inside	outside	inside	outside	inside
g. liquid		20.5969		20.5167		20.5559
g. urea		0.2424		0.2555		0.2158
id/g liquid	0.01120	0.01177	0.01120	0.01245	0.01197	0.01050
g. gum/g liquid		0.09882		0.09876		0.09895
water/g liquid	0.9888	0.8894	0.9888	0.8888	0.9880	0.8905
after days	12		13		10	
g. liquid	10.0295	9.7520	10.0430	10.0613	10.0390	9.5945
g. urea	0.1137	0.1064	0.1163	0.1114	0.1198	0.1081
id/g. liquid	0.01134	0 01091	0.01158	0.01107	0.01193	0.01127
g. liquid	6.0074	5.5306	6.5000	2.9325	5.3543	3.7023
g. dry subst.	0.1289	0.1937	0.1043	0.0957	0.0722	0.1307
g. gum	0.0608	0.1334	0.0290	0.0632	0.0083	0.0809
id/g. liquid	0.01012	0.02412	0.00446	0.02155	0.00155	0.02404
water/g. liquid	0.9785	0.9650	0.9840	0.9674	0.9865	0.9647
h	1.69		1.56		1.46	

TABLE III.

	After days	h
I	6	—
	12	1.69
II	11	1.03
	13	1.56
III	3	0.78
	8	1.02
	10	1.46

2. *Investigation.*

The ultrafiltration took place at diminished pressure; seeing that in such a case there is a danger of loss of water of the filtrate through evaporation, two calcium chloride tubes were placed behind the suction tube. We found that the weight of the second tube did not increase, while the water received in the first was taken into account.

The first experiments were made with "Ultrafein-Schnell" membrane filters and yielded the results stated in Table IV.

TABLE IV.

	I		II	
	Filtrate	Liquid	Filtrate	Liquid
g. liquid	7.9816		10.0017	
g. urea	0.0964		0.1317	
idem/5. liquid	0.01208	0.01088	0.01317	0.01210
g. gum/g liquid		0.09091		0.09880
water/g liquid	0.9879	0.8982	0.9868	0.8891
h	0.09		— 0.18	

These results, in which *h* is calculated in the same way as in the investigations according to OAKLEY's dialysis method, show that indeed, the bound water thus calculated is practically nil in GREENBERG's ultrafiltration method.

In order to make the two methods as much alike — and hence as comparable — as possible, we applied in the ultrafiltration the same collodion membrane as in the OAKLEY experiments. It was simply placed in a porous beaker (ultrafilter of BECHHOLD–KÖNIG 11)). This gave a double advantage: in the first place the material was the same in both series of experiments, secondly the collodion membrane is much more permeable, so that it is possible to filter at less diminished pressure. For the dialysis method may be conceived as ultrafiltration in which the difference in pressure is nil. In order to achieve different diminished pressures with the same water-jet pump, an adjustable regulator was placed behind the suctionflask and the working pressure was read from an open mercury manometer. It has been said that the thin collodion membranes allow a slight quantity of arabinate to pass, this was determined in the filtrates. The level of filter beaker was kept constant by adding sol and the homogeneousness of the liquid was ascertained by stirring. The results are given in Table V.

These experiments too confirm, therefore, GREENBERG's result that ultra-

TABLE V.

	I				II			
g. urea/g. liquid	original liquid			0.01192				0.01209
g. gum/g. liquid				0.02351				0.02287
g. water/g liquid				0.9646				0.9650
	filtrate	liquid	filtrate	liquid	filtrate	liquid	filtrate	liquid
g. liquid	8.3615		8.6531	8.2633	8.3034		7.5265	8.2450
g. urea	0.1032		0.1062	0.0988	0.1032		0.0949	0.1017
g. urea/g. liquid	0.01234	0.01216	0.01228	0.01196	0.01243	0.01230	0.01261	0.01233
g. liquid	7.3431		7.0880	7.4985	3.6588		3.8708	5.7796
g dry substance	0.0905		0.0915	0.4080	0.0556		0.0559	0.2662
g. gum	—		0.0045	0.3183	0·0101		0.0071	0.1949
idem/g. liquid	—	0.0333	0.00063	0.04245	0.00276	0.0303	0.00183	0.03372
water/g. liquid	0.9877	0.9545	0.9871	0.9456	0.9848	0.9574	0.9856	0.9540
pressure in cm Hg	60		60		20		31	
h	— 0.56		— 0·38		— 0.62		— 0.30	

filtration experiments do not lead to the conclusion that bound water is real, rather do they point to a negative effect.

III. *Discussion.*

Although we can fully confirm the experimental results of the authors mentioned, we yet think we must contradict the conclusion that GREENBERG's method indicates the absence of colloidally bound water.

As well in dialysis as in the ultrafiltration method (at least with infinitely slow ultrafiltration and uninterrupted contact of the two liquids) the potential of the water on either side of the membrane should be equal. The question whether or not the water is bound may be formulated as follows: is the potential of the water modified by the colloid or not? If we give an affirmative answer to this question on the ground of GORTNER's and OAKLEY's experiments, we may express the potential of the water inside the ultrafilter, the dialysis membrane respectively, as

$$[\mu_{H_2O}]_{in} = f(T) + p_{in}\, v_0 + f(c_t)_{in} + f(c_c)_{in} \quad . \quad . \quad . \quad . \quad (1)$$

in which c_t is the concentration of the substance in true solution, c_c that of the colloid, and v_0 the molecular volume of H_2O. Outside the ultrafilter, the dialysis membrane respectively, applies:

$$[\mu_{H_2O}]_{out} = f(T) + p_{out}\, v_0 + f(c_t)_{out} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To hold for equation:

$$[\mu_{H_2O}]_{in} = [\mu_{H_2O}]_{out} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

term $f(c_c)_{in}$, in equation (1) will have to be compensated either by a difference between p_{in} and p_{out}, or by one between $(c_t)_{in}$ and $(c_t)_{out}$. In OAKLEY's method, where the pressure is the same inside and outside the membrane, there is indeed a concentration difference as equilibrium phenomenon. In GREENBERG's method there is a difference in pressure between sol and ultrafiltrate; hence it is by no means certain that a concentration difference is necessary, as long as $p_{in} - p_{out}$ is sufficiently great.

In calculating what difference in pressure agrees with the concentration difference found, we find a value of ca. $\frac{1}{4}$ atmosphere; in our experiments we were always higher than that. Moreover, it should be borne in mind that in the ordinary ultrafiltration the condition of infinitely slow ultrafiltration with permanent maintenance of the equilibrium is by no means fulfilled, owing to which an inversion of the phenomenon for higher pressure values becomes doubtful. The homogeneous liquid phase simply permeates, and as it becomes separated from the membrane, equation (3) is not held for.

GREENBERG's *method, therefore, by no means contradicts the presence of bound water,* OAKLEY's *method confirms the existence of it unequivocally.*

Utrecht, April 1940.

VAN 'T HOFF *Laboratorium of the Rijks Universiteit.*

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Physics. — *Decrease in the intensity of cosmic rays in different directions and the decay of the mesons.* II. By J. CLAY.

(Communicated at the meeting of May 25, 1940.)

In a previous communication [1], we have given a comparison of the decrease of cosmic rays in different directions by the atmosphere and the decrease by a layer of water in a vertical direction. As we had in the vertical direction no measurements from 0—10 m water, it was impossible to compare the influence of smaller layers with the decrease in the atmosphere for small angles with the vertical.

It was possible to make a comparison of the decrease for layers of lead in the vertical direction and the decrease in the atmosphere for small angles. And only for small angles it will be possible to distinguish between the two possibilities, if either the decay of the mesons or the abnormal absorption of the gaseous state of matter is the reason of the abnormal behaviour of the rays in the atmosphere, as was supposed by FERMI [2].

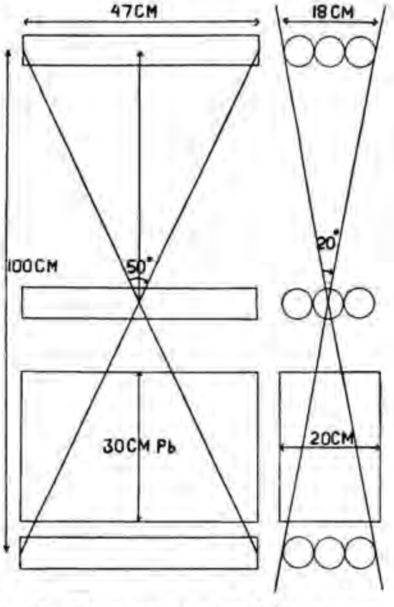

Fig. 1. Countersystem used for measurements in vertical and oblique directions. 30 cm Pb between the counters.

We counted threefold coincidences with counterboxes of 846 cm^2 active surface. The distance between the upper and the lower counter was 100 cm (fig. 1). The boxes with counters and lead were fixed on an iron plate of 2 m length and we could change the inclination of the plate.

Between the lower and the middle counter we had a layer of 30 cm Pb, so that we could be sure that only particles of high penetration power were counted. The number of coincidences in vertical direction was 25.8 per min. The relation of the intensity with inclination can be presented very well by $\cos^2 \varphi$ as can be controled by comparing column 3 and 5 and in fig. 2. In column 7 (Table I) the layer of water is given aequivalent to the layer of lead in 6. 1 cm Pb is aequivalent to 6,9 cm H_2O [3].

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[1] Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 436 (1940).

[2] E. FERMI, Phys. Rev. **56**, 1242 (1939).

[3] J. CLAY and A. V. GEMERT, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 672 (1939).

1	2	3	4	5	6	7	8
φ	sec φ	$\cos^2 \varphi$	a m water-aeq.	I_φ	b' cm Pb for $I = I_\varphi$	b M water for $I = I_\varphi$	$\frac{b}{a-10}$
0	1.00	1.00	0	1.000	0	0	
10	1.015	0.970	10.15	0.968	6	0.41	2.7
15	1.034	0.932	10.34	0.950	15	1.03	3.05
20	1.060	0.881	10.60	0.875	24	1.67	2.80
30	1.155	0.751	11.55	0.755	53	3.66	2.36
40	1.306	0.586	13.06	0.604	90	6.20	2.02
45	1.414	0.500	14.1	0.504	122	8.41	2.05
45	1.414	0.500	14.1	0.484		8.3	2.0
60	0.500	0.250	20.0	0.221		20	2.0
75	3.89	0.067	38.9	0.072		47	1.7
82	7.20	0.020	72	0.019		110	1.8

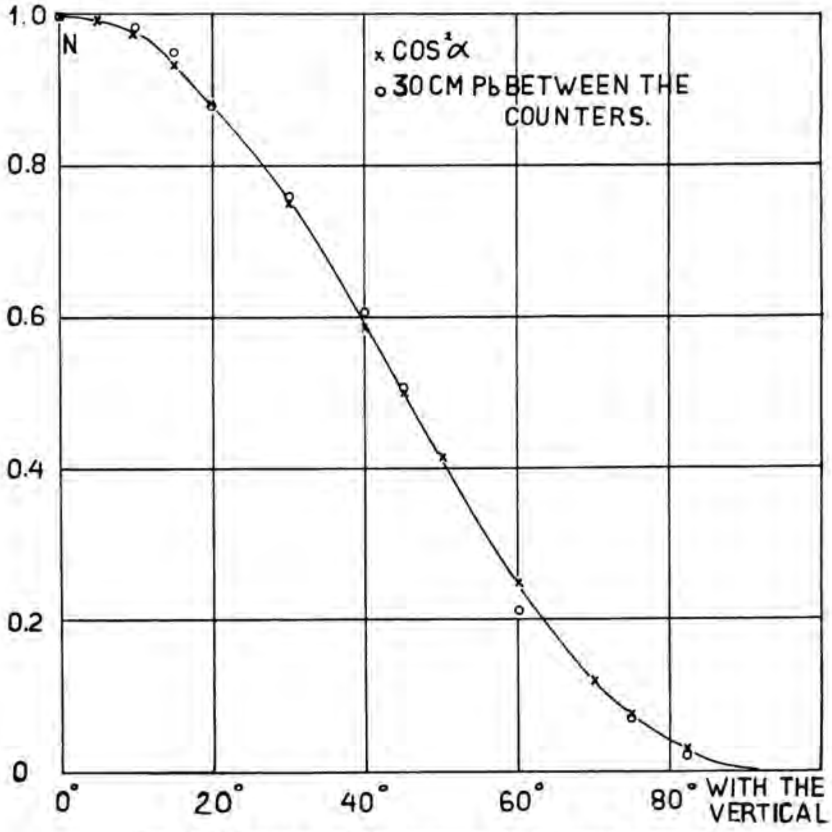

Fig. 2. Variation of the intensity in different directions.

For the series of angles between 45° and 82° only the layer of water could be given, found from the absorption curve in water between 10 and 200 m of depth [1]).

In column 8 the relation is given between the layer of water, transversed in vertical direction and the wateraequivalent of the atmospheric layer, which both give the same decrease of intensity. The relation of these values is also given in fig. 3 for the whole series of inclinations and in fig. 4

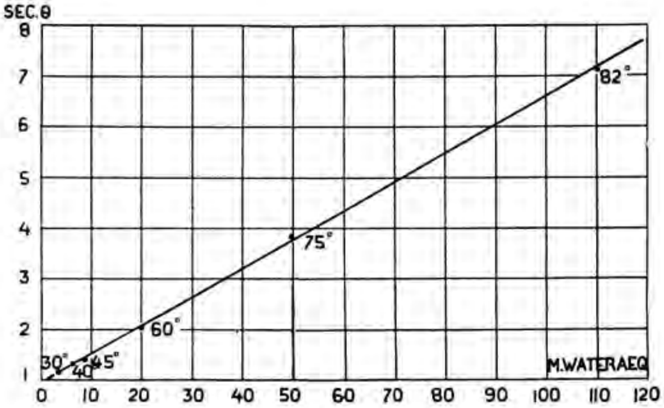

Fig. 3. Relation between sec θ and thickness of waterlayer (b) for aequivalent decrease 20°—82°.

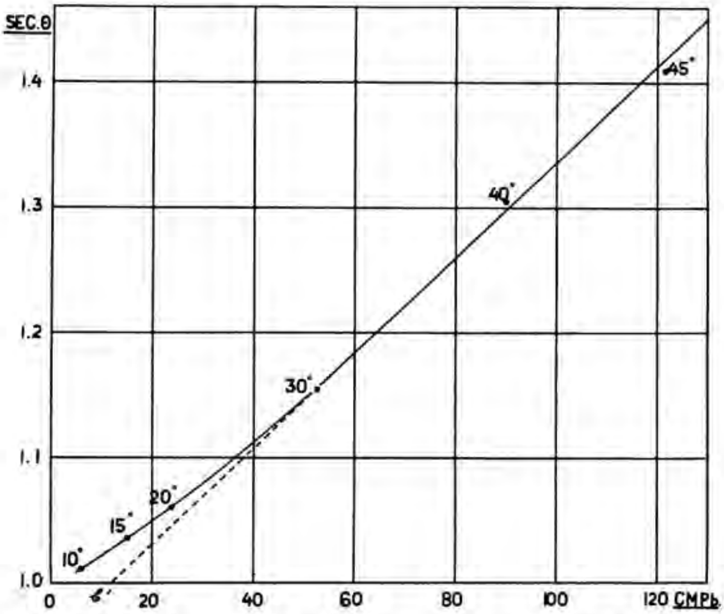

Fig. 4. Relation between sec θ and leadthickness for small angles with the vertical.

for the inclinations to 45° and the accurate values of lead. The decrease in vertical direction can be found in fig. 5.

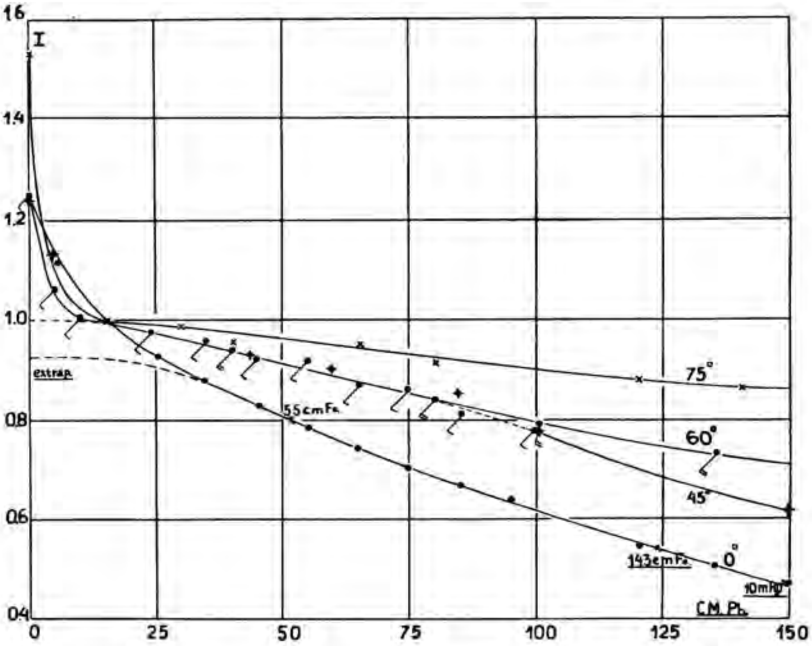

Fig. 5. Decrease of intensity in lead in vertical direction.

If the reason of the abnormal decrease of the atmosphere was the distance of the atoms in the gaseous state, as was supposed by FERMI, there had to be a b (and b′) proportional to sec θ down to $\theta = 0$.

We see from the graph that this is not according to experiment. If the decay of mesons is the origin of the abnormal high decrease in the atmosphere the relation sec θ to b will be linear for higher values of θ as well, but will be changed for the lower values as is expected from the calculations of BRUINS [4]).

We see that the experimental results are in agreement with this second supposition.

The value of the inclination of the line $\frac{d(\sec \theta)}{db'} \times 146 = 0{,}61$, means that the lifetime of the particles is $2{,}6 \times 10^{-6}$ sec. vid BRUINS [4]) graph. 2. As db' is given in cm Pb we have to multiply by 146 as 10 m water is aequivalent to 146 cm Pb.

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[4]) E. M. BRUINS, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, p. 700 (1940).

Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4.* (Dritte Mitteilung). Von R. WEITZENBÖCK.

(Communicated at the meeting of May 25, 1940.)

Ich setze im Anschlusse an die vorige Mitteilung die Ermittlung der zu einer allgemeinen Erzeugenden der Fläche F gehörigen linearen Räume fort, wobei die Voraussetzung gemacht wird, dass die (einfachste) Differentialinvariante Q nicht verschwindet. Als besondere Tatsache sei hervorgehoben, dass die Heftebene von der zur Erzeugenden „assoziierten" oder Bei-Geraden getroffen wird.

§ 9.

Die durch (32) gegebene Heftebene m_{ijk} der Erzeugenden 0_{ik} kann nach Gleichung (35) auch in der Gestalt

$$m\,(0_{ik}) = (1^2 H \pi^2) = 0$$

dargestellt werden. Wir ersetzen nun 0_{ik} nach (60) durch

$$\varphi_{ik} = \tfrac{3}{2}\,(HH_1)_{ik} + \lambda\,.\,0_{ik},$$

d.h. wir berechnen

$$m_{ijk}\,(\varphi_{ik}) = m\,(\varphi) = (\varphi_1{}^2\, H\,(\varphi)\,\pi^2).$$

Nach (62) und (70) wird

$$m\,(\varphi) = 3\,(HH_2H\,(\varphi)\,\pi^2) + \tfrac{3}{2}\,\lambda\,.\,(1^2\,H\,(\varphi)\,\pi^2) + \tfrac{3}{2}\,\lambda'\,(0^2\,H\,(\varphi)\,\pi^2) =$$
$$= \tfrac{3}{2}\,\lambda^2\,(\lambda - 8\,Q)^2\,.\,(1^2\,H\pi^2),$$

also

$$m_{ijk}\,(\varphi) = \tfrac{3}{2}\,\lambda^2\,(\lambda - 8\,Q)^2\,.\,m_{ijk}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

wovon als besonderer Fall für $\lambda = 4\,Q$ folgt:

$$m_{ijk}\,(a) = 3.2^7\,.\,Q^4\,.\,m_{ijk}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

weiters haben wir nach (59)

$$m_{ik} = a_{ik} + 4\,Q\,.\,0_{ik}.$$

Hieraus wird

$$m_{ik}\,(\varphi) = a_{ik}\,(\varphi) + 4\,Q\,(\varphi)\,.\,\varphi_{ik}.$$

also nach (76) und (74)

$$m_{ik}(\varphi) = \lambda^3 (\lambda - 8\,Q)^3 \,.\, m_{ik}. \quad \dots \quad (80)$$

Hieraus wieder für $\lambda = 4\,Q$:

$$m_{ik}(\alpha) = -(4\,Q)^6 \,.\, m_{ik}. \quad \dots \quad (81)$$

Aus (80) folgt noch, dass der nach (42) zu berechnende Punkt M für die Gerade φ_{ik}, also der Punkt $M(\varphi)$, wieder der Geraden m_{ik} angehört.

Dem in (42) stehenden Ausdruck

$$R_0 = 2_{05,03}$$

lässt sich, wenn man die Ableitung Q' von $Q = 0_{13,23}$ benutzt, eine andere Gestalt geben. Wir haben

$$Q' = 1_{13,23} + 0_{23,23} + 0_{14,23} + 0_{13,33} + 0_{13,24}.$$

Hier verschwindet rechts der zweite Term, der vierte ist R (Vgl. (41)). Auch der dritte Term ist nach (1) Null. Beim ersten Term haben wir auf Grund der Gleichungen (1) und der zyklischen Symmetrie:

$$1_{13,23} = -2_{13,13} - 3_{13,12} = +\tfrac{1}{2}\,1_{33,12} = -\tfrac{1}{6}\,1_{33,03} = +\tfrac{1}{6}\,0_{33,13} = -\tfrac{1}{6}\,R:$$

und beim letzten Term wird analog

$$\begin{aligned} 0_{13,24} &= -\tfrac{2}{3}\,0_{13,33} - \tfrac{2}{5}\,0_{13,15} = -\tfrac{2}{3}\,R + \tfrac{2}{5}\,1_{03,15} = -\tfrac{2}{3}\,R - \tfrac{1}{5}\,5_{03,11} = \\ &= -\tfrac{2}{3}\,R + \tfrac{1}{5}\,5_{03,02} \quad = -\tfrac{2}{3}\,R + \tfrac{1}{5}\,2_{05,03} = -\tfrac{2}{3}\,R + \tfrac{1}{5}\,R_0. \end{aligned}$$

Also kommt schliesslich

$$R_0 = 2_{05,03} = 5\,Q' - \tfrac{5}{6}\,R \quad , \quad Q' = \tfrac{1}{6}\,R + \tfrac{1}{5}\,2_{05,03}. \quad \dots \quad (82)$$

Diese Grösse R_0 tritt auch in der Gleichung der Schmiegebene h_{ijk}

$$(h'\pi)^2 = (\pi^2\,HH_1H_2) = 0 \quad , \quad h_{ijk} = (HH_1H_2)_{ijk}. \quad \dots \quad (83)$$

im Punkte H an die Kurve C_H auf. Nach (21) haben wir

$$\begin{aligned} (\pi^2\,HH_1\,H_2) = (0H_1\,H_2\,\pi^2)\,0_{22} = \tfrac{1}{2}\,(0^2\,H_2\,\pi^2)\,(H_1)_{22} - \\ - \tfrac{1}{2}\,(0^2\,H_1\,\pi^2)\,(H_2)_{22} - (0^2\,H_1\,H_2\,\pi)\,\pi_{22}. \end{aligned}$$

Dies kann mit Hilfe von (54) und (55) umgeformt werden auf

$$(h'\pi)^2 = (\pi^2\,HH_1\,H_2) = \tfrac{4}{3}\,Q\,(\pi_{02,04} + \pi_{02,22}) + \tfrac{1}{3}\,\pi_{02,03}\,(H_2)_{22}. \quad . \quad (84)$$

Hier enthält der letzte Term die Heftebene m_{ijk} (Vgl. (32)) und für den Koeffizienten $(H_2)_{22}$ ergibt sich nach (55):

$$(H_2)_{22} = 4.1_{22,23} + 2.0_{22,33} + 2.0_{22,24}.$$

Nun ist wie oben:

$$\begin{aligned} 1_{22,23} &= -3_{22,12} = +\tfrac{1}{3}\,3_{22,03} = -\tfrac{1}{6}\,0_{22,33} = \tfrac{2}{9}\,R \\ 0_{22,34} &= -\tfrac{4}{3}\,R \\ 0_{22,24} &= -\tfrac{2}{3}\,0_{22,33} - \tfrac{2}{5}\,0_{22,15} = \tfrac{8}{9}\,R + \tfrac{2}{5}\,1_{22,05} = \tfrac{8}{9}\,R - \tfrac{4}{5}\,2_{12,05} = \\ &= \tfrac{8}{9}\,R + \tfrac{4}{15}\,2_{03,05} = \tfrac{8}{9}\,R - \tfrac{4}{15}\,R_0. \end{aligned}$$

Also wird

$$(H_2)_{22} = -\tfrac{8}{15}\,R_0 = \tfrac{4}{9}\,R - \tfrac{8}{3}\,Q'. \quad . \; . \; . \; . \; . \; . \qquad (85)$$

Der Schmiegraum H' im Punkte H an die Kurve C_H wird gegeben durch die Kovariante

$$(H'x) = (HH_1\,H_2\,H_3\,x) = 0. \quad . \; . \; . \; . \; . \; . \; . \qquad (86)$$

Dies kann man am einfachsten nach (84) entwickeln indem man dort $\pi_{ik} = (H_3\,x)_{ik}$ einsetzt. Wir erhalten

$$\begin{aligned} (H'x) = \tfrac{4}{3}\,Q\,[x_{04}\,(H_3)_{02} - x_{02}\,(H_3)_{04} + x_{22}\,(H_3)_{02} - x_{02}\,(H_3)_{22}] &+ \\ + \tfrac{1}{3}\,(H_2)_{22}\,[x_{03}\,(H_3)_{02} - x_{02}\,(H_3)_{03}]& \end{aligned}$$

$$\left.\begin{aligned} (H'x) = -x_{02}\,[\tfrac{4}{3}\,Q\,(H_3)_{04} + \tfrac{4}{3}\,Q\,(H_3)_{22} + \tfrac{1}{3}\,(H_2)_{22}\,(H_3)_{03}] &+ \\ + x_{03}\,.\,\tfrac{8}{9}\,Q\,(H_2)_{22} + \tfrac{32}{9}\,Q^2\,[x_{04} + x_{22}]& \end{aligned}\right\} \quad . \qquad (87)$$

Hieraus ist schliesslich für $x = H_4$ ein Ausdruck für die Invariante

$$\triangle_H = (HH_1\,H_2\,H_3\,H_4) \quad . \; . \; . \; . \; . \; . \; . \; . \qquad (88)$$

erhältlich, der $\triangle_H$ durch die Grössen $(H_i)_{rs}$ ausdrückt.

§ 10.

Wenn eine Regelfläche F abwickelbar ist, so verschwindet $(M'_{02}\,x)$ und daher auch (Hu') identisch. Ich behaupte, dass für diesen Fall die Differentialkomitante erster Ordnung

$$E_0 = 0_{1\pi}\,(0u') = (01^2\,\pi^2)\,(0u') = -(10^2\,\pi^2)\,(1u') \quad . \; . \; . \; . \qquad (89)$$

den zugehörigen Kurvenpunkt (bei einem Kegel die Spitze) und die zugehörige Schmiegebene darstellt.

Beweis. Bei einer Torse haben wir $0_{ik} = (yy')_{ik}$, also

$$E_0 = (1\,yy'\,\pi^2)\,(1u') = -\tfrac{1}{2}\,(yy'\,y''\,\pi^2)\,.\,(yu');$$

bei einem Kegel mit Spitze s ist $0_{ik} = (sy)_{ik}$, also

$$E_0 = (1sy\,\pi^2)\,(1u') = -\tfrac{1}{2}\,(syy'\,\pi^2)\,.\,(su').$$

Wir berechnen E_0 für die Gerade φ_{ik} von (60). Nach (62) wird:

$$E_0(\varphi) = 3\,(\varphi\, H H_2 \pi^2)\,(\varphi\, u') + \lambda\,.\,(\varphi\, 1^2 \pi^2)\,(\varphi\, u') + \lambda'\,.\,(\varphi\, 0^2 \pi^2)\,(\varphi\, u').$$

Bringt man hier die beiden Reihen φ in den Klammerfaktor und setzt

$$\varphi_{ik} = \tfrac{3}{2}\,(H H_1)_{ik} + \lambda\,.\,0_{ik},$$

so kommt

$$\left.\begin{aligned} E_0(\varphi) = (H u')\,.\,\{\tfrac{9}{2}\,(H H_1 H_2 \pi^2) - \tfrac{9}{4}\,\lambda\, \pi_{02,04} - \tfrac{3}{4}\,\lambda\, \pi_{02,22} - \lambda'\, \pi_{02,03}\} + \\ + \lambda\,(H_1 u')\, \pi_{02,03} + \lambda^2 E_0 \end{aligned}\right\} \quad . \qquad (90)$$

Hier kann man noch $(H H_1 H_2 \pi^2)$ nach (84) ausdrücken.

Setzen wir in (90) $\lambda = 0$, so geht φ_{ik} über in die Tangente $(H H_1)_{ik}$ an C_H im Heftpunkte H; E_0 zerfällt demgemäss in $(H u')$ und die dazugehörige Schmiegebene h_{ijk}, Gleichung (84).

Bei $\lambda = 8\,Q$ wird φ_{ik} mit m_{ik} identisch. Da auch die Geraden m_{ik} eine abwickelbare Regelfläche bilden, muss auch für diesen Fall $E_0(m)$ in zwei Faktoren zerfallen, von denen der eine den durch (42) dargestellten Punkt M, der andere die zugehörige Schmiegebene geben muss. Also muss nach (32) der zweite Faktor die Form $\pi_{02,03}$ sein.

Das dies in der Tat der Fall ist, kann man verifizieren, wenn man von einer Identität Gebrauch macht, die wir jetzt ableiten wollen. Setzt man in (90) $\lambda = 8\,Q$, so entsteht im letzten Term $64\,Q^2 E_0$ und es lässt sich $Q E_0$ durch andere Komitanten ausdrücken.

Es sei $\dot{0}_{ik}$ mit 0_{ik} äquivalent. Dann können wir in dem Produkt

$$Z = 0_{ik,\alpha\beta}\, \dot{0}_{\gamma\delta,\varepsilon\eta} = (0\, i^2 k^2)\, \dot{0}_{\gamma\delta}\, 0_{\alpha\beta}\, \dot{0}_{\varepsilon\eta}$$

die Reihe $\dot{0}$ des zweiten Faktors in den Klammerfaktor bringen und erhalten

$$Z = 0_{ik,\varepsilon\eta}\, 0_{\gamma\delta,\alpha\beta} + 2\,(\dot{0}0\, i k^2)\, \dot{0}_{\varepsilon\eta}\, i_{\gamma\delta}\, 0_{\alpha\beta} + 2\,(\dot{0}0\, i^2 k)\, \dot{0}_{\varepsilon\eta}\, k_{\gamma\delta}\, 0_{\alpha\beta}.$$

Hier werden die beiden letzten Terme, wenn wir auch die zweite Reihe $\dot{0}$ in den Klammerfaktor schaffen, wegen $(\dot{0}^2 0) = 0$ gleich

$$+\, i_{0k,\gamma\delta}\, 0_{\varepsilon\eta,\alpha\beta} + k_{0i,\gamma\delta}\, 0_{\varepsilon\eta,\alpha\beta},$$

und dies gibt wegen der zyklischen Symmetrie von $i_{0k,\gamma\delta}$ bezgl. $i, 0, k$:

$$-\, 0_{ik,\gamma\delta}\,.\,0_{\varepsilon\eta,\alpha\beta}.$$

Also wird

$$Z = 0_{ik,\varepsilon\eta}\, 0_{\gamma\delta,\alpha\beta} + 0_{ik,\gamma\delta}\, 0_{\varepsilon\eta,\alpha\beta}$$

und damit erhält man die Identität:

$$0_{ik,\alpha\beta}\, 0_{\gamma\delta,\varepsilon\eta} + 0_{ik,\gamma\delta}\, 0_{\varepsilon\eta,\alpha\beta} + 0_{ik,\varepsilon\eta}\, 0_{\alpha\beta,\gamma\delta} = 0, \quad . \ . \ . \ . \qquad (91)$$

d.h. wir haben im Produkte Z zyklische Symmetrie in den drei Paaren $\alpha\beta$, $\gamma\delta$ und $\varepsilon\eta$.

Bei der Ableitung von (91) können statt dieser Paare auch Reihen u', v', w' stehen. Steht z.B. u' statt $\varepsilon\eta$, so haben wir

$$0_{ik,\alpha\beta} \,.\, 0_{\gamma\delta}(0u') = -0_{ik,\gamma\delta} \,.\, (0u')\, 0_{\alpha\beta} - 0_{ik}(0u') \,.\, 0_{\alpha\beta,\gamma\delta}$$

Dies kann man anwenden für den Fall

$$0_{ik,\alpha\beta} \,.\, 0_{\gamma\delta}(0u') = 0_{13,23} \,.\, 0_{1\pi}(0u') = Q \,.\, E_0$$

und erhält

$$QE_0 = -0_{13,1\pi} \,.\, (0u')\, 0_{23} - 0_{13}(0u') \,.\, 0_{23,1\pi}$$

oder schliesslich

$$QE_0 = -\tfrac{1}{2}\,\pi_{02,03} \,.\, (0u')\, 0_{23} + \tfrac{3}{16}(Hu') \,.\, \pi_{02,04} \quad . \quad . \quad . \quad . \qquad (92)$$

§ 11.

Konstruiert man zu vier Erzeugenden von F die assoziierte Gerade G_5 und lässt dann die vier Erzeugenden auf 0_{ik} zusammenrücken, so geht für $Q \neq 0$ G_5 in die zur Erzeugenden 0_{ik} assoziierte Gerade, oder, wie wir kurz sagen wollen, in die „Bei-Gerade" g_{ik} über. Ich habe vor Kurzem bewiesen [1]), dass g_{ik} gegeben wird durch

$$g_{ik} = 0_{ik} \,.\, P - 1_{ik} \,.\, 12\,Q\,(3\,Q\,s + R^2) + 2_{ik} \,.\, 36\,Q^2 R - 3_{ik} \,.\, 72\,Q^3. \qquad (93)$$

Hierbei sind Q, R, S und T durch (41) gegeben und P ist der Ausdruck

$$P = 2R^3 + 12\,QRS + 9\,Q^2 T \quad . \quad . \quad . \quad . \quad . \quad . \qquad (94)$$

Er ist vom φ'-Gewichte $\varrho = 30$ und vom λ-Gewicht $\sigma = 15$.

Setzen wir in der Gleichung $\pi_{02,03} = 0$ der Heftebene $\pi_{ik} = g_{ik}$, so entsteht Null, *die Beigerade schneidet also die Heftebene*. Wir nennen den Schnittpunkt G. Seine Gleichung findet man, wenn man einen nicht in Tangentialraum M'_{02} liegenden Punkt mit der Heftebene zu einem R_3 verbindet und diesem dann mit g_{ik} schneidet. Man erhält bei $Q \neq 0$:

$$(Gu') = R \,.\, (Hu') + 4\,Q \,.\, (3u')\, 3_{02} = 0 \quad . \quad . \quad . \quad . \qquad (95)$$

Wir haben dann ausser dem Heftpunkt H in der Heftebene m_{ijk} zwei invariante Punkte: G und den durch (42) gegebenen Punkt M. Hiermit lässt sich auf der Erzeugenden 0_{ik} selbst ein zweiter invarianter Punkt F konstruieren, nämlich der Schnittpunkt von 0_{ik} mit der Verbindungslinie GM. Als Gleichung von F erhält man:

$$F = M + G = (Hu')\,(2R + \tfrac{3}{5}\, 2_{05,03}) - 16\,Q \,.\, (0u')\, 0_{23} \quad . \quad . \qquad (96)$$

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[1]) Diese Proceedings **43**, 325—333 (1940).

oder auch, nach (82):

$$(u'F) = (Hu') \,.\, (\tfrac{3}{2} R + 3\,Q') - 16\,Q \,.\, (0u')\,0_{23} = 0 \quad . \quad . \qquad (96a)$$

F, G und M sind Differentialkontravarianten mit den Gewichten $\varrho = 14$ und $\sigma = 8$.

Die Gerade

$$\varphi_{ik} = \tfrac{3}{2}\,(HH_1)_{ik} + \lambda \,.\, 0_{ik}$$

im Büschel um H in der Heftebene lässt sich dann durch die beiden Basisgeraden $(HM)_{ik}$ und $(HG)_{ik}$ wie folgt darstellen

$$\varphi_{ik} = \frac{1}{64\,Q^2}\,\{(HM)_{ik} \,.\, (3\lambda - 8\,Q) + (HG)_{ik} \,.\, (3\lambda - 24\,Q)\} \quad . \qquad (97)$$

Hieraus folgt z.B. dass die Gerade MG die Tangente $h_{ik} = (HH_1)_{ik}$ im Punkte $M + 3G$, die Heftgerade a_{ik} aber im Punkte $M - 3G$ schneidet.

Die Beigerade g_{ik} gibt mit der Erzeugenden 0_{ik} verbunden den kovarianten R_3

$$(B'x) = R \,.\, x_{02} - 2\,Q \,.\, x_{03} = 0 \; . \; . \; . \; . \; . \; . \; . \qquad (98)$$

und zwei aufeinander folgende dieser Räume schneiden sich in der Ebene

$$\left.\begin{aligned} &(\tfrac{2}{3} R^2 + 2\,QR' - 2\,Q'R) \,.\, \pi_{02,03} - 2\,QR \,.\, (\pi_{02,13} + \pi_{02,04}) + \\ &\qquad\qquad + 4\,Q^2 \,.\, (\pi_{03,13} + \pi_{03,04}) = 0. \end{aligned}\right\} . \qquad (99)$$

Man rechnet leicht nach, dass diese Ebene auch die Beigerade schneidet.

Mathematics. — *Beiträge zur Theorie der Systeme* PFAFF*scher Gleichungen.* IV. *Beweis des Haupttheorems für den Fall, dass der Rang einen beliebigen Wert hat.* Von J. A. SCHOUTEN und W. VAN DER KULK.

(Communicated at the meeting of April 27. 1940.)

1. *Konstruktion der Anfangswerte* $\overset{1}{p}_0, \dots, \overset{\varrho+1}{p}_0$.

Der Teil (a) des Haupttheorems ist bewiesen für $p = 2\varrho$ und $p = 2\varrho + 1$ 1). Nehmen wir also an es sei $p \geqq 2\varrho + 2$ und ausserdem, dass der Beweis schon geliefert sei für einen um eins geringeren Wert von p und denselben Wert von ϱ. Durch umnumerieren kann man stets erreichen dass (III 7) übergeht in

$$Q_{[23} \dots Q_{2\varrho\, 2\varrho+1]} \neq 0; \quad Q_{ba} \overset{\text{def}}{=} 2\, B^{\omega}_{[b}\, (\partial_{\omega|}\, B^{\lambda}_{a]})\, Q_{\lambda}. \qquad (1)$$

(III 8) ist gleichbedeutend mit

$$P_{a_1 \dots a_\varrho} \overset{\text{def}}{=} B^{\lambda_1}_{a_1} \dots B^{\lambda_\varrho}_{a_\varrho}\, P_{\lambda_1 \dots \lambda_\varrho} \neq 0; \quad a_1, \dots, a_\varrho = 1, \dots, \mathrm{p}. \qquad (2)$$

und (III 9) mit

$$Q_{[ab}\, P_{a_1 \dots a_\varrho]} = 0; \quad a, b, a_1, \dots, a_\varrho = 1, \dots, \mathrm{p}. \qquad (3)$$

Wir beweisen jetzt, dass nicht *alle* Bestimmungszahlen $P_{b_1 \dots b_\varrho}$; $b_1, \dots, b_\varrho = 2, \dots, 2\varrho + 1$ verschwinden können. Dazu betrachten wir in dem Gebiet der Indizes $2, \dots, 2\varrho + 1$ die einfachen $(\varrho - 1)$-Vektoren

$$P_{b_2 \dots b_\varrho (k)} \quad ; \quad k = 1, 2\varrho + 2, \dots, \mathrm{p}. \text{ }^{2)}. \qquad (4)$$

Wäre nun die Behauptung unrichtig, so würde aus (3) für $a_1 = k$, $a_2 = b_2, \dots, a_\varrho = b_\varrho$; $a, b, b_2, \dots, b_\varrho = 2, \dots, 2\varrho + 1$ folgen

$$Q_{[ab}\, P_{b_2 \dots b_\varrho] (k)} = 0 \quad ; \quad k = 1, 2\varrho + 2, \dots, \mathrm{p}. \qquad (5)$$

Nun ist aber Q_{ab} laut (1) im Gebiete der Indizes $2, \dots, 2\varrho + 1$ vom höchsten Range. Infolge des Hilfssatzes (III, Fussnote S. 455) wäre also

$$P_{b_2 \dots b_\varrho k} = 0; \quad b_2, \dots, b_\varrho = 2, \dots, 2\varrho + 1; \quad k = 1, 2\varrho + 2, \dots, \mathrm{p}. \qquad (6)$$

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1) Beiträge zur Theorie der Systeme PFAFFscher Gleichungen, III. Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 453—462 (1940), hier weiter zitiert als III.

2) Die Abdrosselung (k) deutet an, dass wir nicht einen ϱ-Vektor, sondern $p - 2\varrho$ $(\varrho - 1)$-Vektoren betrachten.

und infolge der gemachten Annahme auch

$$P_{a_1 b_2 \ldots b_\varrho} = 0 \quad ; \quad b_2, \ldots, b_\varrho = 2, \ldots, 2\varrho + 1 \quad ; \quad \mathrm{a}_1 = 1, \ldots, \mathrm{p}. \tag{7}$$

Jetzt betrachten wir die einfachen $(\varrho - 2)$-Vektoren

$$P_{b_3 \ldots b_\varrho (k)(l)} \quad ; \quad b_3, \ldots, b_\varrho = 2, \ldots, 2\varrho + 1 \quad ; \quad k, l = 1, 2\varrho + 2, \ldots, \mathrm{p}. \tag{8}$$

Aus (3) und (7) würde dann für $a = k$, $\mathrm{a}_1 = l$; $\mathrm{a}_2 = b_2, \ldots, \mathrm{a}_\varrho = b_\varrho$ folgen

$$\left.\begin{aligned} Q_{[b b_2} P_{b_3 \ldots b_\varrho](k)(l)} = 0 \; ; \; k, l = 1, 2\varrho + 2, \ldots, \mathrm{p} \; ; \\ b, b_2, \ldots, b_\varrho = 2, \ldots, 2\varrho + 1. \end{aligned}\right\} \quad . \; . \tag{9}$$

und daraus ginge unter Verwendung desselben Hilfssatzes hervor

$$P_{\mathrm{a}_1 \mathrm{a}_2 b_3 \ldots b_\varrho} = 0 \; ; \; b_3, \ldots, b_\varrho = 2, \ldots, 2\varrho + 1 \; ; \; \mathrm{a}_1, \mathrm{a}_2 = 1, \ldots, \mathrm{p}. \tag{10}$$

In dieser Weise fortfahrend liesse sich schliesslich dartun dass $P_{\mathrm{a}_1 \ldots \mathrm{a}_\varrho} = 0$ wäre; $\mathrm{a}_1, \ldots, \mathrm{a}_\varrho = 1, \ldots, \mathrm{p}$ und dies ist im Gegensatz zu (2). Nicht alle Bestimmungszahlen $P_{\mathrm{a}_1 \ldots \mathrm{a}_\varrho}$ mit Indizes $2, \ldots, 2\varrho + 1$ verschwinden also und durch eine Umnummerierung, die auf (1) keinen Einfluss hat, lässt sich demnach stets erreichen, dass

$$P_{2 \ldots \varrho+1} \neq 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \tag{11}$$

ist.

In der X_{n-1} $x^1 = \underset{0}{x}{}^1$, mit den Koordinaten x^α; $\alpha, \beta = 2, \ldots, n$, betrachten wir jetzt die Grössen

$$\left.\begin{aligned} \overset{0}{B}{}_b^\alpha (x^2, \ldots, x^n) &\overset{\text{def}}{=} B_b^\alpha (\underset{0}{x}{}^1, x^2, \ldots, x^n) \; ; \; b = 2, \ldots, \mathrm{p} \; ; \; \alpha, \beta = 2, \ldots, n \\ \overset{0}{C}{}_\beta^\varkappa (x^2, \ldots, x^n) &\overset{\text{def}}{=} C_\beta^\varkappa (\underset{0}{x}{}^1, x^2, \ldots, x^n) \; ; \; \varkappa = \mathrm{p} + 1, \ldots, \mathrm{n}. \end{aligned}\right\} \tag{12}$$

Da infolge (III 10) [3])

$$\overset{0}{B}{}_b^\alpha \, \overset{0}{C}{}_\alpha^\varkappa = 0 \; ; \; b = 2, \ldots, \mathrm{p} \; ; \; \varkappa = \mathrm{p} + 1, \ldots, \mathrm{n} \quad . \; . \; . \tag{13}$$

ist, sind

$$\overset{0}{C}{}_\beta^\varkappa \, dx^\beta = 0 \; ; \; \varkappa = \mathrm{p} + 1, \ldots, \mathrm{n} \quad . \; . \; . \; . \; . \; . \tag{14}$$

und

$$\overset{0}{B}{}_b^\alpha \, \partial_\alpha f = 0 \; ; \; b = 2, \ldots, \mathrm{p} \quad . \; . \; . \; . \; . \; . \; . \tag{15}$$

in dieser X_{n-1} adjungierte Systeme. Wir behaupten nun, dass die Komponenten Q_β; $\beta = 2, \ldots, n$ und $P_{\beta_1 \ldots \beta_\varrho}$; $\beta_1, \ldots, \beta_\varrho = 2, \ldots, n$, die in dieser

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[3]) Wir denken uns die B_b^α wieder in der Form (III 10) geschrieben.

X_{n-1} einen Vektor bzw. ϱ-Vektor in $x^\alpha = \underset{0}{x}^\alpha$ darstellen, i.b. auf das PFAFFsche System (14) Bedingungen genügen, die für dieses System die Bedeutung von (III 6, 7, 8, 9) haben. Aus (III 6) folgt in $x^\alpha = \underset{0}{x}^\alpha$

$$\overset{0}{B}{}^\beta_b \, Q_\beta = B^\lambda_b \, Q_\lambda - B^1_b \, Q_1 = 0 \; ; \; b = 2, \ldots, \mathrm{p}, \quad . \quad . \quad . \quad (16)$$

da $B^1_b = 0$ ist für $b \neq 1$ infolge (III 10), und dies ist die Gleichung (III 6) für das System (14) in X_{n-1}. Aus demselben Grunde ist

$$\left.\begin{aligned} 2\,\overset{0}{B}{}^\beta_{[b} \, \partial_{|\beta|} \, \overset{0}{B}{}^\alpha_{a]} = 2\,(B^\beta_{[b} \, \partial_{|\beta|} \, B^\alpha_{a]})_{x^1 = \underset{0}{x}^1} = 2\,(B^\omega_{[b} \, \partial_{|\omega|} \, B^\alpha_{a]})_{x^1 = \underset{0}{x}^1} \, ; \\ b, \mathrm{a} = 2, \ldots, \mathrm{p} \end{aligned}\right\} . \quad (17)$$

sodass aus (1) folgt in $x^\alpha = \underset{0}{x}^\alpha$

$$Q_{[23} \ldots Q_{2\varrho\, 2\varrho+1]} \neq 0 \; ; \; Q_{ba} \overset{\text{def}}{=} 2\,\overset{0}{B}{}^\beta_{[b} \, (\partial_{|\beta|} \, \overset{0}{B}{}^\alpha_{a]}) \, Q_\alpha \; ; \; \mathrm{a}, b = 2, \ldots, \mathrm{p} \quad (18)$$

und dies ist die Gleichung (III 7) für das System (14) in X_{n-1}. Aus (3) folgt in $x^\alpha = \underset{0}{x}^\alpha$

$$Q_{[ab} \, P_{b_1 \ldots b_\varrho]} = 0 \, ; \, P_{b_1 \ldots b_\varrho} \overset{\text{def}}{=} \overset{0}{B}{}^{\beta_1}_{b_1} \ldots \overset{0}{B}{}^{\beta_\varrho}_{b_\varrho} \, P_{\beta_1 \ldots \beta_\varrho} \, ; \, a, b, b_1, \ldots, b_\varrho = 2, \ldots, \mathrm{p}, \quad (19)$$

und dies ist die Gleichung (III 9) für das System (14) in X_{n-1}. Aus (12) folgt schliesslich

$$P_{2 \ldots \varrho+1} \neq 0 \; ; \; P_{b_1 \ldots b_\varrho} \overset{\text{def}}{=} \overset{0}{B}{}^{\beta_1}_{b_1} \ldots \overset{0}{B}{}^{\beta_\varrho}_{b_\varrho} \, P_{\beta_1 \ldots \beta_\varrho} \; ; \; b_1, \ldots, b_\varrho = 2, \ldots, \mathrm{p}, \quad (20)$$

und dies ist die Gleichung (III 8) für das System (14) in X_{n-1}. Ausserdem geht aus (III 3) und (1) hervor, dass das System (14) gerade den Halbrang ϱ hat. Wir haben also jetzt ein PFAFFsches System erhalten mit einem um eins geringeren Wert von p und denselben Wert von ϱ. Da wir angenommen haben, dass der Beweis für ein solches System schon geliefert sei, existiert also ein System von $\varrho + 1$ Funktionen $\overset{1}{p}_0, \ldots, \overset{\varrho+1}{p}_0$, die Lösungen sind der Gleichungen

$$\left\| \begin{matrix} \overset{0}{X}_2 \overset{1}{p}_0 & \ldots & \overset{0}{X}_2 \overset{\varrho+1}{p}_0 \\ \vdots & & \vdots \\ \overset{0}{X}_\mathrm{p} \overset{1}{p}_0 & \ldots & \overset{0}{X}_\mathrm{p} \overset{\varrho+1}{p}_0 \end{matrix} \right\| = \text{Matrix vom Range } \varrho \, ; \quad (21)$$

$$\overset{0}{X}_b = \overset{0}{B}{}^\beta_b \, \partial_\beta ; \, b = 2, \ldots, \mathrm{p} ; \, \beta = 2, \ldots, n,$$

und ausserdem in $x^\alpha = \underset{0}{x}^\alpha$ noch den Gleichungen

$$(\partial_{[\beta_1} \overset{1}{p}_0) \ldots (\partial_{\beta_{\varrho+1}]} \overset{\varrho+1}{p}_0) = Q_{[\beta_1} P_{\beta_2 \ldots \beta_{\varrho+1}]} \, ; \, \beta_1, \ldots, \beta_{\varrho+1} = 2, \ldots, n \quad . \quad (22)$$

genügen.

2. *Konstruktion der Funktionen* $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$.

Das System (III 4) ist gleichwertig mit

$$\left.\begin{array}{ll} a) & X_1 \overset{i}{p} = \overset{2}{\sigma} X_2 \overset{i}{p} + \ldots + \overset{\varrho+1}{\sigma} X_{\varrho+1} \overset{i}{p};\; i=1,\ldots,\varrho+1 \\ b) & \theta_k \overset{\text{def}}{=} (X_k \overset{[1}{p})(X_2 \overset{2}{p}) \ldots (X_{\varrho+1} \overset{\varrho+1]}{p}) = 0;\; k=\varrho+2,\ldots,\mathrm{p}. \end{array}\right\} . \quad (23)$$

vorausgesetzt, dass die Ungleichung

$$(X_2 \overset{[i_2}{p}) \ldots (X_{\varrho+1} \overset{i_{\varrho+1}]}{p}) \neq 0;\; i_2, \ldots, i_{\varrho+1} = 1, \ldots, \varrho+1 \; . \; . \; . \quad (24)$$

erfüllt sei.

Sind $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$ Lösungen von (23a), so gilt für die mit diesen Lösungen nach (23*b*, def.) konstruierten Funktionen θ_k

$$X_1 \theta_k = X_2 (\overset{2}{\sigma} \theta_k) + \ldots + X_{\varrho+1} (\overset{\varrho+1}{\sigma} \theta_k) + \zeta_k;\; k=\varrho+2,\ldots,\mathrm{p} \; . \quad (25)$$

wo

$$\zeta_k = (\varrho+1) \{ p_{1[k,2\ldots\varrho+1]} - \overset{2}{\sigma} p_{2[k,2\ldots\varrho+1]} - \ldots - \overset{\varrho+1}{\sigma} p_{\varrho+1[k,2\ldots\varrho+1]} \} \; . \quad (26)$$

und

$$p_{ab,a_1\ldots a_\varrho} \overset{\text{def}}{=} 2\, B^{\omega}_{[a} (\partial_{|\omega|} B^{\lambda}_{b}) B^{\lambda_1}_{a_1} \ldots B^{\lambda_\varrho}_{a_\varrho]} (\partial_{[\lambda} \overset{1}{p}) (\partial_{\lambda_1} \overset{2}{p}) \ldots (\partial_{\lambda_\varrho]} \overset{\varrho+1}{p}) \; . \quad (27)$$

ist. Da die ζ_k für solche Lösungen von (23a), die auch (23*b*) genügen, verschwinden müssen, benutzen wir die Gleichungen

$$\zeta_l = 0;\; l=\varrho+2,\ldots,2\varrho+1 \; . \; . \; . \; . \; . \; . \; . \quad (28)$$

dazu, $\overset{2}{\sigma}, \ldots, \overset{\varrho+1}{\sigma}$ zu lösen. (Wir benutzen also hier nur die ersten ϱ der ζ_k). Für jede dieser Grössen finden wir aus (28) eine rationale Funktion der ersten Ableitungen von $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$ nach $x^2, \ldots, x^n$ (also *nicht* nach x^1), deren Koeffizienten in der Umgebung von $x^\nu = \underset{0}{x^\nu}$ reguläre Funktionen von $x^1, \ldots, x^n$ sind. Der Nenner ist für jedes σ

$$\Delta = \begin{vmatrix} p_{2[\varrho+2,2\ldots\varrho+1]} & \ldots & p_{\varrho+1[\varrho+2,2\ldots\varrho+1]} \\ \vdots & & \vdots \\ p_{2[2\varrho+1,2\ldots\varrho+1]} & \ldots & p_{\varrho+1[2\varrho+1,2\ldots\varrho+1]} \end{vmatrix} \; . \; . \; . \; . \; . \; . \quad (29)$$

Substituiert man diese Werte von $\overset{2}{\sigma}, \ldots, \overset{\varrho+1}{\sigma}$ in (23*a*), dann entstehen nach Umrechnung Gleichungen von der Form

$$\partial_1 \overset{i}{p} = \overset{i}{F} (x^1, \ldots, x^n, \partial_\beta \overset{1}{p}, \ldots, \partial_\beta \overset{\varrho+1}{p});\; \beta=2,\ldots,n;\; i=1,\ldots,\varrho+1 \; . \quad (30)$$

Die $\overset{i}{F}$ sind rationale Funktionen der Variabeln $\partial_\beta \overset{1}{p}, \ldots, \partial_\beta \overset{\varrho+1}{p};\ \beta = 2, \ldots, n$ mit Koeffizienten, die in der Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$ reguläre Funktionen von $x^1, \ldots, x^n$ sind. Der Nenner ist für alle $\overset{i}{F}$ gleich $\triangle$. Wir betrachten diese Funktionen für die Argumentwerte

$$x^\varkappa = \underset{0}{x}^\varkappa;\ \partial_\beta \overset{i}{p} = (\partial_\beta \overset{i}{p}_0)_{x^\alpha = \underset{0}{x}^\alpha};\ \beta = 2, \ldots, n;\ i = 1, \ldots, \varrho + 1. \quad . \quad (31)$$

Infolge (22) und (29) ist für diese Argumentwerte

$$\triangle = (\varrho + 1)^{-\varrho} \begin{vmatrix} Q_{2[\varrho+2} \ P_{2\ldots\varrho+1]} & \ldots & Q_{\varrho+1[\varrho+2} \ P_{2\ldots\varrho+1]} \\ \vdots & & \vdots \\ Q_{2[2\varrho+1} \ P_{2\ldots\varrho+1]} & \ldots & Q_{\varrho+1[2\varrho+1} \ P_{2\ldots\varrho+1]} \end{vmatrix}, \quad . \quad . \quad . \quad (32)$$

und aus dieser Gleichung folgt mit Hilfe der (III S.) angewandten Umrechnung

$$\triangle^2 = (2\varrho + 2)^{-2\varrho} \left(\frac{(2\varrho)!}{\varrho!}\right)^2 (P_{2\ldots\varrho+1})^{2\varrho} (Q_{[23} \ldots Q_{2\varrho\, 2\varrho+1]})^2 \quad . \quad . \quad (33)$$

woraus unter Berücksichtigung von (18) und (20) hervorgeht, dass in der Umgebung dieser Argumentwerte $\triangle \neq 0$ ist. In dieser Umgebung sind also die $\overset{i}{F}$ regulär analytische Funktionen von $x^1, \ldots, x^n;\ \partial_\beta \overset{i}{p};\ \beta = 2, \ldots, n;$ $i = 1, \ldots, \varrho + 1$.

Nach CAUCHY—KOWALEWSKI *existieren also* $\varrho + 1$ *Funktionen* $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$ *der Variabeln* $x^1, \ldots, x^n$, *die* (23a) *erfüllen und für die ausserdem für* $x^1 = \underset{0}{x}^1$ *gilt*

$$\overset{i}{p}(\underset{0}{x}^1, x^2, \ldots, x^n) = \overset{i}{p}_0(x^2, \ldots, x^n);\ i = 1, \ldots, \varrho + 1 \quad . \quad . \quad . \quad (34)$$

In $x^\varkappa = \underset{0}{x}^\varkappa$ ist infolge (22) und (34)

$$(\partial_{[\beta_1} \overset{1}{p}) \ldots (\partial_{\beta_{\varrho+1}]} \overset{\varrho+1}{p}) = Q_{[\beta_1} P_{\beta_2 \ldots \beta_{\varrho+1}]};\ \beta_1, \ldots, \beta_{\varrho+1} = 2, \ldots, n \ . \quad (35)$$

Ueberschiebung mit $B_2^{\beta_2} \ldots B_{\varrho+1}^{\beta_{\varrho+1}}$ ergibt wegen (III 6) in $x^\varkappa = \underset{0}{x}^\varkappa$

$$(\partial_{\beta_1} \overset{[1}{p}) (B_2^{|\beta_2|} \partial_{\beta_2} \overset{2}{p}) \ldots (B_{\varrho+1}^{|\beta_{\varrho+1}|} \partial_{\beta_{\varrho+1}} \overset{\varrho+1]}{p}) = \frac{1}{\varrho + 1} Q_{\beta_1} P_{2\ldots\varrho+1}, \quad . \quad (36)$$

und, da $B_2^1 = 0, \ldots, B_{\varrho+1}^1 = 0$ ist, folgt in $x^\varkappa = \underset{0}{x}^\varkappa$

$$(\partial_{\beta_1} \overset{[1}{p}) (X_2 \overset{2}{p}) \ldots (X_{\varrho+1} \overset{\varrho+1]}{p}) = \frac{1}{\varrho + 1} Q_{\beta_1} P_{2\ldots\varrho+1} \quad . \quad . \quad . \quad (37)$$

Infolge (1) und (11) ist nun das rechte Glied, und somit auch das linke, für $x^{\lambda} = \underset{0}{x^{\lambda}}$ nicht Null. Das linke Glied ist also auch in einer Umgebung von $x^{\lambda} = \underset{0}{x^{\lambda}}$ nicht Null und in dieser Umgebung gilt daher

$$(X_2 \overset{|i_2}{p}) \dots . (X_{\varrho+1} \overset{i_{\varrho+1}|}{p}) \neq 0 ; \; i_2, \dots , i_{\varrho+1} = 1, \dots , \varrho + 1 \quad . \quad . \quad (38)$$

womit bewiesen ist, dass die Lösungen $\overset{1}{p}, \dots , \overset{\varrho+1}{p}$ auch die Gleichungen (24) erfüllen.

3. *Beweis, dass* $\overset{1}{p}, \dots , \overset{\varrho+1}{p}$ *auch* (23*b*), (III 4) *und* (III 5) *erfüllen.*

Wir benutzen die abgekürzte Bezeichnung (vergl. II 8*b*)

$$\Psi_{a,b} \overset{\text{def}}{=} (\varrho + 1)\, p_{a\,[b,2 \dots \varrho+1]} \; ; \quad \mathrm{a}, b = 1, \dots , \mathrm{p} \quad . \quad . \quad . \quad . \quad (39)$$

welcher Ausdruck mit den im vorigen Paragraphen bestimmten $\overset{1}{p}, \dots , \overset{\varrho+1}{p}$ zu bilden ist. Sodann verwenden wir die Gleichungen (26) in der Form

$$\left.\begin{array}{lllll}
\zeta_{\varrho+2} & = \Psi_{1,\varrho+2} & - \overset{2}{\sigma}\, \Psi_{2,\varrho+2} & - \dots . - & \overset{\varrho+1}{\sigma}\, \Psi_{\varrho+1,\varrho+2} \\
\vdots & \vdots & & & \\
\zeta_{2\varrho+1} & = \Psi_{1,2\varrho+1} & - \overset{2}{\sigma}\, \Psi_{2,2\varrho+1} & - \dots . - & \overset{\varrho+1}{\sigma}\, \Psi_{\varrho+1,2\varrho+1} \\
\zeta_j & = \Psi_{1,j} & - \overset{2}{\sigma}\, \Psi_{2,j} & - \dots . - & \overset{\varrho+1}{\sigma}\, \Psi_{\varrho+1,j}
\end{array}\right\} \quad . \quad . \quad (40)$$

wo j einen Index von $2\varrho + 2$ bis p darstellt, um $\overset{2}{\sigma}, \dots , \overset{\varrho+1}{\sigma}$ zu eliminieren. Da $\zeta_{\varrho+2}, \dots , \zeta_{2\varrho+1}$ verschwinden, ergibt sich

$$\triangle\, \zeta_j = \begin{vmatrix}
\Psi_{1,\varrho+2} & \dots . & \Psi_{\varrho+1,\varrho+2} \\
\vdots & & \vdots \\
\Psi_{1,2\varrho+1} & \dots . & \Psi_{\varrho+1,2\varrho+1} \\
\Psi_{1,j} & \dots . & \Psi_{\varrho+1,j}
\end{vmatrix} \overset{\text{def}}{=} D_j \quad . \quad . \quad . \quad . \quad (41)$$

für $j = 2\varrho + 2, \dots , \mathrm{p}$. In der Umgebung von $x^{\lambda} = \underset{0}{x^{\lambda}}$ ist $\triangle \neq 0$. Es gilt die D_j umzuformen.

Dazu bilden wir aus den Lösungen $\overset{1}{p}, \dots , \overset{\varrho+1}{p}$ einen Vektor

$$w_{\lambda} \overset{\text{def}}{=} \underset{1}{\mu}\, \partial_{\lambda} \overset{1}{p} + \dots . + \underset{\varrho+1}{\mu}\, \partial_{\lambda} \overset{\varrho+1}{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

und wählen $\underset{1}{\mu}, \dots , \underset{\varrho+1}{\mu}$ so, dass

$$B_b^{\lambda}\, w_{\lambda} = 0 ; \quad b = 2, \dots . \varrho + 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

ist und nicht alle μ verschwinden. Dann verschwindet auf Grund von (22) und (34) auch w_{λ} nicht.

Wegen (23a), (42) und (43) ist auch

$$B_1^{\lambda}\, w_{\lambda} = 0 \quad . \;.\;.\;.\;.\;.\;.\;.\;. \quad (44)$$

Sodann existiert ein einfacher ϱ-Vektor $\pi_{\lambda_1 \ldots \lambda_\varrho}$, so dass

$$(\partial_{[\lambda_1} \overset{1}{p}) \ldots . (\partial_{\lambda_{\varrho+1}]} \overset{\varrho+1}{p}) = w_{[\lambda_1}\, \pi_{\lambda_2 \ldots \lambda_{\varrho+1}]} \quad . \;.\;.\;.\;. \quad (45)$$

ist, da w_λ Teiler des Gradientproduktes zur Linken ist. Wird diese Gleichung überschoben mit $B_2^{\lambda_2} \ldots B_{\varrho+1}^{\lambda_{\varrho+1}}$, so folgt wegen (43)

$$(\varrho + 1)\, B_2^{\lambda_2} \ldots . B_{\varrho+1}^{\lambda_{\varrho+1}} (\partial_{[\lambda_1} \overset{1}{p}) \ldots . (\partial_{\lambda_{\varrho+1}]} \overset{\varrho+1}{p}) = w_{\lambda_1}\, \pi_{2 \ldots \varrho+1} \quad . \;. \quad (46)$$

wenn wir abkürzend schreiben

$$\pi_{a_1 \ldots a_\varrho} \overset{\text{def}}{=} B_{a_1}^{\lambda_1} \ldots . B_{a_\varrho}^{\lambda_\varrho}\, \pi_{\lambda_1 \ldots \lambda_\varrho};\; \mathrm{a}_1, \ldots ., \mathrm{a}_\varrho = 1, \ldots ., \mathrm{p}. \quad . \;. \;. \quad (47)$$

Nun gilt für die linke Seite von (46)

$$\left.\begin{aligned} & B_2^{\lambda_2} \ldots B_{\varrho+1}^{\lambda_{\varrho+1}} (\partial_{[\lambda_1} \overset{1}{p}) \ldots (\partial_{\lambda_{\varrho+1}]} \overset{\varrho+1}{p}) = \\ & = B_2^{\lambda_2} \ldots B_{\varrho+1}^{\lambda_{\varrho+1}} (\partial_{\lambda_1} \overset{[1}{p}) \ldots (\partial_{\lambda_{\varrho+1}} \overset{\varrho+1]}{p}) = (\partial_{\lambda_1} \overset{[1}{p})\, (X_2 \overset{2}{p}) \ldots X_{\varrho+1} \overset{\varrho+1]}{p}) \end{aligned}\right\} \quad (48)$$

und dieser Ausdruck ist infolge (37) u.f. ungleich Null. Also ist

$$\pi_{2 \ldots \varrho+1} \neq 0. \quad . \;.\;.\;.\;.\;.\;.\;.\;.\;. \quad (49)$$

Aus (23*b*, def.), (43) und (45) folgt

$$\theta_k = \frac{1}{\varrho + 1}\, w_k\, \pi_{2 \ldots \varrho+1} \;;\; w_k \overset{\text{def}}{=} B_k^{\lambda}\, w_\lambda \;;\; k = \varrho + 2, \ldots ., \mathrm{p}. \quad . \;. \quad (50)$$

Aus (39) und (45) folgt

$$\left.\begin{aligned} \Psi_{a,b} & = (\varrho + 1)\, B_{a[b}^{\cdot\cdot\lambda_1} B_2^{\lambda_2} \ldots B_{\varrho+1]}^{\lambda_{\varrho+1}}\, w_{[\lambda_1}\, \pi_{\lambda_2 \ldots \lambda_{\varrho+1}]} = \\ & = w_{a[b}\, \pi_{2 \ldots \varrho+1]}\, (mod.\; w_b) \;;\; w_{ab} \overset{\text{def}}{=} B_{ab}^{\cdot\cdot\lambda}\, w_\lambda \\ & \qquad\qquad B_{ab}^{\cdot\cdot\varkappa} \overset{\text{def}}{=} 2\, B_{[a}^{\eta}\, \partial_{|\eta|}\, B_{b]}^{\varkappa} \\ & \qquad\qquad \mathrm{a} = 1, \ldots ., \mathrm{p} \;;\; b = \varrho + 2, \ldots ., \mathrm{p} \end{aligned}\right\} \quad . \;. \quad (51)$$

und also infolge (49) und (50)

$$\Psi_{a,b} = w_{a[b}\, \pi_{2 \ldots \varrho+1]}\, (mod.\; \theta_b) \;;\; \mathrm{a} = 1, \ldots ., \mathrm{p} \;;\; b = \varrho + 2, \ldots ., \mathrm{p}. \quad (52)$$

Setzen wir diese Ausdrücke in (41) ein, so ergibt sich

$$D_j = \begin{vmatrix} w_{1[\varrho+2}\,\pi_{2\ldots\varrho+1]} & \ldots & w_{\varrho+1[\varrho+2}\,\pi_{2\ldots\varrho+1]} \\ \vdots & & \\ w_{1[2\varrho+1}\,\pi_{2\ldots\varrho+1]} & \ldots & w_{\varrho+1[2\varrho+1}\,\pi_{2\ldots\varrho+1]} \\ w_{1[j}\,\pi_{2\ldots\varrho+1]} & \ldots & w_{\varrho+1[j}\,\pi_{2\ldots\varrho+1]} \end{vmatrix} (mod.\, \theta_{\varrho+2},\ldots,\theta_{2\varrho+1}, \theta_j) \quad (53)$$

Wir betrachten jetzt die Determinante rechts für sich und schreiben diese W_j. W_j ist ein homogenes Polynom in den Variablen $w_{ab} = - w_{ba}$; $a, b = 1, \ldots, 2\varrho + 1, j$. In derselben Weise wie in (III) bei der Ableitung von (III 34) beweist man, dass

$$\left. \begin{array}{r} W_j^2 = A^2 \{ w_{[12} \ldots w_{2\varrho-1\,2\varrho}\, w_{2\varrho+1\,j]} \}^2 \;;\; w_{ab} \overset{\text{def}}{=} B_{ab}^{\cdot\cdot\,\lambda}\, w_\lambda \\ a, b = 1, \ldots, p \end{array} \right\} \quad (54)$$

wo

$$A = (2\varrho + 2)^{-\varrho-1} \frac{(2\varrho+2)!}{(\varrho+1)!} (\pi_{2\ldots\varrho+1})^{\varrho+1} \quad \ldots \quad (55)$$

ist, für alle Werte von w_{ab}; $a, b = 2, \ldots, 2\varrho + 1, j$, die den linearen Gleichungen

$$L_{ab} \overset{\text{def}}{=} w_{[ab}\, \pi_{2\ldots\varrho+1]} = 0 \;;\; a, b = \varrho+2, \ldots, 2\varrho+1, j \quad \ldots \quad (56)$$

genügen. Demzufolge ist

$$\left. \begin{array}{r} W_j^2 - A^2 \{ w_{[12} \ldots w_{2\varrho-1\,2\varrho}\, w_{2\varrho+1\,j]} \}^2 = 0\, (mod.\, L_{ab}) \;; \\ a, b = \varrho+2, \ldots, 2\varrho+1, j \end{array} \right\} \quad \ldots \quad (57)$$

für *beliebige* Werte von w_{ab}; $a, b = 2, \ldots, 2\varrho + 1, j$, und daraus geht hervor, dass es eine Gleichung gibt von der Form

$$W_j = B\, w_{[12} \ldots w_{2\varrho-1\,2\varrho}\, w_{2\varrho+1\,j]}\, (mod.\, L_{ab}) \,;\, a, b = \varrho+2, \ldots, 2\varrho+1, j \quad (58)$$

Insbesondere gilt also

$$\left. \begin{array}{l} D_j = B\, w_{[12} \ldots w_{2\varrho-1\,2\varrho}\, w_{2\varrho+1\,j]}\, (mod.\, \theta_{\varrho+2}, \ldots, \theta_{2\varrho+1}, \theta_j, L_{ab}) = \\ \quad = B\, B_{[12}^{\cdot\cdot\,\lambda_1} \ldots B_{2\varrho-1\,2\varrho}^{\cdot\cdot\,\lambda_\varrho}\, B_{2\varrho+1\,j]}^{\cdot\cdot\,\lambda_{\varrho+1}}\, w_{\lambda_1} \ldots w_{\lambda_{\varrho+1}} (mod.\, \theta_{\varrho+2}, \ldots \\ \qquad \ldots, \theta_{2\varrho+1}, \theta_j, L_{ab}); \quad (a, b = \varrho+2, \ldots, 2\varrho+1. j \end{array} \right\} \quad (59)$$

Wegen der besonderen Wahl (III 10) der B_b^{λ} verschwinden nun die ersten p Bestimmungszahlen $B_{ab}^{\cdot\cdot\,1}, \ldots, B_{ab}^{\cdot\cdot\,p}$ und der Tensor

$$B_{[12}^{\cdot\cdot\,\lambda_1} \ldots B_{2\varrho-1\,2\varrho}^{\cdot\cdot\,\lambda_\varrho}\, B_{2\varrho+1\,j]}^{\cdot\cdot\,\lambda_{\varrho+1}} \quad \ldots \ldots \quad (60)$$

liegt also ganz in dem Gebiet der Massvektoren $\underset{p+1}{e^{\varkappa}}, \ldots, \underset{n}{e^{\varkappa}}$. Wegen (III 10) haben die $B_a^{\varkappa}$ die Form

$$B_a^{\varkappa} \overset{*}{=} \underset{a}{e^{\varkappa}} + B_a^x \underset{x}{e^{\varkappa}} \;;\; \mathrm{a} = 1, \ldots, \mathrm{p} \;;\; x = \mathrm{p}+1, \ldots, \mathrm{n}\ ^{4)} \,. \quad . \quad (61)$$

und durch lineare Transformation der C_λ^x lässt sich also erreichen, dass

$$C_\lambda^x \overset{*}{=} \overset{x}{e}_\lambda - B_a^x \overset{a}{e}_\lambda \;;\; x = \mathrm{p}+1, \ldots, \mathrm{n} \;;\; a = 1, \ldots, \mathrm{p}\ ^{4)} \,. \quad . \quad (62)$$

Infolge (III 3) verschwindet der mit $C_{\lambda_1}^{x_1} \ldots C_{\lambda_{\varrho+1}}^{x_{\varrho+1}}$ überschobene Tensor (60). Da dieser Tensor aber in dem Gebiet der Massvektoren $\underset{p+1}{e^{\varkappa}}, \ldots, \underset{n}{e^{\varkappa}}$ liegt, folgt aus (62) dass derselbe Null sein muss, sodass

$$D_j = 0 \,(mod.\, \theta_{\varrho+2}, \ldots, \theta_{2\varrho+1}, \theta_j, L_{ab}) \quad a, b = \varrho+2, \ldots, 2\varrho+1, j \quad (63)$$

Die L_{ab} sind jetzt noch umzuformen. Infolge (27), (45), (56) ist

$$\left.\begin{aligned} L_{ab} &= w_{[ab}\, \pi_{2\ldots\varrho+1]} = p_{[ab,2\ldots\varrho+1]} = \frac{2}{\varrho+1} X_{[a}\, p_{b2\ldots\varrho+1]} = \\ &= \frac{4}{(\varrho+1)(\varrho+2)} X_{[a}\, \theta_{b]} + \frac{2\varrho}{(\varrho+1)(\varrho+2)} X_{[2}\, p_{3\ldots\varrho+1]ab}. \end{aligned}\right\} \,. \quad (64)$$

wo

$$\left.\begin{aligned} p_{c_2\ldots c_\varrho ab} &= B_{c_2}^{\lambda_1} \ldots B_{c_\varrho}^{\lambda_{\varrho-1}} B_a^{\lambda_\varrho} B_b^{\lambda_{\varrho+1}} (\partial_{[\lambda_1} \overset{1}{p}) \ldots (\partial_{\lambda_{\varrho+1}]} \overset{\varrho+1}{p}); \\ & \mathrm{c}_2, \ldots \mathrm{c}_\varrho = 2 \ldots \varrho+1 \quad ; \quad a, b = \varrho+2, \ldots, 2\varrho+1, j \end{aligned}\right\} \quad (65)$$

ist. (65) geht unter Berücksichtigung von (45) und (43) über in

$$\left.\begin{aligned} p_{c_2\ldots c_\varrho ab} &= \frac{1}{\varrho+1} (w_a\, \pi_{bc_2\ldots c_\varrho} - w_b\, \pi_{ac_2\ldots c_\varrho}); \\ & \mathrm{c}_2, \ldots, \mathrm{c}_\varrho = 2, \ldots, \varrho+1 \quad ; \quad a, b = \varrho+2, \ldots, 2\varrho+1, j \end{aligned}\right\} \quad (66)$$

und daraus geht infolge (50) hervor

$$\left.\begin{aligned} p_{c_2\ldots c_\varrho ab} = 0 \,(mod.\, \theta_a, \theta_b) \quad & a, b = \varrho+2, \ldots, 2\varrho+1, j \\ & \mathrm{c}_2, \ldots, \mathrm{c}_\varrho = 2, \ldots, \varrho+1. \end{aligned}\right\} \,. \quad . \quad . \quad (67)$$

Wird dieses Resultat auf (64) angewandt, so ergibt sich

$$\left.\begin{aligned} L_{ab} = \frac{4}{(\varrho+1)(\varrho+2)} X_{[a}\, \theta_{b]} \,(mod.\, \theta_a, \theta_b, X_2\, \theta_a, \ldots, X_{\varrho+1}\, \theta_a, \\ X_2\, \theta_b, \ldots, X_{\varrho+1}\, \theta_b) \end{aligned}\right\} \quad (68)$$

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[4] Einfachheitshalber haben wir statt $\underset{1}{e^{\varkappa}}, \ldots, \underset{n}{e^{\varkappa}}$ und $\overset{1}{e}_\lambda, \ldots, \overset{n}{e}_\lambda$ in (61) und (62) $\underset{\mathrm{1}}{e^{\varkappa}}, \ldots, \underset{\mathrm{n}}{e^{\varkappa}}$ und $\overset{\mathrm{1}}{e}_\lambda, \ldots, \overset{\mathrm{n}}{e}_\lambda$ geschrieben.

und durch Substitution in (63) ergibt sich schliesslich

$$\left.\begin{array}{r} D_j = 0\,(mod.\,\theta_{\varrho+2},\ldots,\theta_{\mathrm{p}};\,X_{[a}\theta_{b]};\,X_2\theta_a,\ldots X_{\varrho+1}\theta_a;\,X_2\theta_b,\ldots,X_{\varrho+1}\theta_b) \\ a,\,b = \varrho+2,\ldots,2\varrho+1,\,j. \end{array}\right\} \quad (69)$$

Die aus $\overset{1}{p},\ldots,\overset{\varrho+1}{p}$ konstruierten Grössen θ_k, $k = \varrho+2,\ldots,\mathrm{p}$ genügen schon den Gleichungen (25), die infolge (41) übergehen in

$$\left.\begin{array}{r} \partial_1\theta_k = (X_2\overset{2}{\sigma} + \ldots + X_{\varrho+1}\overset{\varrho+1}{\sigma})\,\theta_k + \overset{2}{\sigma}X_2\theta_k + \ldots + \overset{\varrho+1}{\sigma}X_{\varrho+1}\theta_k - \\ - B_1^{p+1}\,\partial_{p+1}\theta_k - \ldots - B_1^n\,\partial_n\theta_k + \dfrac{1}{\triangle}D_k;\,k = \varrho+2,\ldots,\mathrm{p}. \end{array}\right\}. \quad (70)$$

wo $D_{\varrho+2} = 0,\ldots,D_{2\varrho+1} = 0$ zu nehmen ist. Ausserdem gilt infolge (23*b*, def.), (34) und (21)

$$\theta_k\,(\underset{0}{x^1},x^2,\ldots,x^n) = 0;\ k = \varrho+2,\ldots,\mathrm{p} \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Nun gibt es nach CAUCHY-KOWALEWSKI nur ein System von Lösungen $\theta_{\varrho+2},\ldots,\theta_{\mathrm{p}}$ die (70) und (71) erfüllen. Offenbar ist aber $\theta_{\varrho+2} = 0,\ldots,\theta_{\mathrm{p}} = 0$ dieses System, denn aus (69) geht hervor, dass aus $\theta_{\varrho+2} = 0,\ldots,\theta_{\mathrm{p}} = 0$ das Verschwinden von D_j folgt. Damit ist aber bewiesen, dass $\overset{1}{p},\ldots,\overset{\varrho+1}{p}$ auch die Gleichungen (23*b*) erfüllen. $\overset{1}{p},\ldots,\overset{\varrho+1}{p}$ erfüllen also (23*a*, *b*) und (24) und somit auch (III 4).

Der Beweis, dass $\overset{1}{p},\ldots,\overset{\varrho+1}{p}$ auch (III 5) erfüllen ist vollkommen gleichlautend mit dem in (III) gegebenen Beweis (III 36—51) für den Fall $\mathrm{p} = 2\varrho+1$. Man hat nur $u-1$ durch ϱ und in (III 41) p durch $2\varrho+1$ zu ersetzen.

5. *Beweis des Teiles (b) des Haupttheorems.*

Wir gehen aus von einem Vektor Q_λ, der den Bedingungen (III 6) und (III 7) genügt. Infolge (III 3) ist dann Q_{ab} in der E_p der Indizes $1,\ldots,\mathrm{p}$ ein Bivektor vom Range 2ϱ. Diesen Bivektor zerlegen wir in ϱ Blätter und wählen in jedem Blatt einen Vektor $\overset{1}{P}_b,\ldots,\overset{\varrho}{P}_b$. Diese Vektoren sind linear unabhängig. Dazu lassen sich ϱ Vektoren $\overset{1}{P}_\lambda,\ldots,\overset{\varrho}{P}_\lambda$ konstruieren, so dass

$$\overset{h}{P}_b = B_b^\lambda\,\overset{h}{P}_\lambda;\ b = 1,\ldots,\mathrm{p};\,h = 1,\ldots,\varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

ist. Auch diese Vektoren sind linear unabhängig und es existiert also ein ϱ-Vektor

$$P_{\lambda_1\ldots\lambda_\varrho} \overset{\text{def}}{=} \overset{1}{P}_{[\lambda_1}\ldots\overset{\varrho}{P}_{\lambda_\varrho} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

mit der Eigenschaft

$$P_{a_1\ldots a_\varrho} \overset{\text{def}}{=} P_{\lambda_1\ldots\lambda_\varrho} B^{\lambda_1}_{a_1} \ldots B^{\lambda_\varrho}_{a_\varrho} \neq 0 \quad \ldots\ldots \quad (74)$$

Ausserdem folgt aus der Definition der $\overset{h}{P}_b$

$$Q_{[ab} P_{b_1\ldots b_\varrho]} = 0 \quad \ldots\ldots\ldots\ldots \quad (75)$$

Jetzt verfügen wir über ein Q_λ und ein $P_{\lambda_1\ldots\lambda_\varrho}$, die den Bedingungen (III 6, 7, 8, 9) genügen, und nach dem schon bewiesenen Teil (*a*) des Haupttheorems existieren also in einer Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$ Funktionen $\overset{1}{p}, \ldots, \overset{\varrho+1}{p}$ die Lösungen von (III 4) sind und in $x^\varkappa = \underset{0}{x}^\varkappa$ der Gleichung (III 5) genügen. Aus (III 4) folgt aber in dieser Umgebung die Existenz von Funktionen $\underset{1}{\mu}, \ldots, \underset{\varrho+1}{\mu}$ sodass

$$w_\lambda \overset{\text{def}}{=} \underset{1}{\mu}\, \partial_\lambda \overset{1}{p} + \ldots + \underset{\varrho+1}{\mu}\, \partial_\lambda \overset{\varrho+1}{p} \quad \ldots\ldots\ldots \quad (76)$$

den Gleichungen

$$B^\lambda_b\, w_\lambda = 0;\ b = 1, \ldots, \mathrm{p} \quad \ldots\ldots\ldots \quad (77)$$

genügt und also ein Teiler von $w_{\lambda_1\ldots\lambda_q}$ ist. Die μ können nicht alle zugleich verschwinden. Man kann nämlich durch umnumerieren stets erreichen dass (11) gilt und demzufolge in einer Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$ auch (24). Für $\underset{1}{\mu}, \ldots, \underset{\varrho+1}{\mu}$ kann man dann die ϱ-reihigen Determinanten nehmen aus der Matrix, die durch die zweite, dritte bis $(\varrho + 1)$— te Reihe von (III 4) gebildet wird. Diese Determinanten können infolge (24) in $x^\varkappa = \underset{0}{x}^\varkappa$ und in einer Umgebung dieses Punktes nicht alle zugleich verschwinden. Wegen (III 5) sind ferner $\partial_\lambda \overset{1}{p}, \ldots, \partial_\lambda \overset{\varrho+1}{p}$ in $x^\varkappa = \underset{0}{x}^\varkappa$ und in der besagten Umgebung linear unabhängig. Es ist also dort auch $w_\lambda \neq 0$. Aus (III 5) und (76) folgt dass w_λ in $x^\varkappa = \underset{0}{x}^\varkappa$ Teiler von $Q_{[\lambda_1} P_{\lambda_2\ldots\lambda_{\varrho+1}]}$ ist, sodass

$$w_\lambda = \gamma\, Q_\lambda + \underset{1}{\gamma}\, \overset{1}{P}_\lambda + \ldots + \underset{\varrho}{\gamma}\, \overset{\varrho}{P}_\lambda . \quad \ldots\ldots \quad (78)$$

ist, woraus durch Ueberschiebung mit B^λ_b folgt

$$0 = \underset{1}{\gamma}\, \overset{1}{P}_b + \ldots + \underset{\varrho}{\gamma}\, \overset{\varrho}{P}_b \;\; : \;\; b = 1, \ldots, \mathrm{p} \quad \ldots \quad (79)$$

Da aber $\overset{1}{P}_b, \dots, \overset{\varrho}{P}_b$ linear unabhängig sind, müssen alle Koeffizienten $\underset{h}{\gamma}$ in (79) verschwinden, sodass

$$w_\lambda = \gamma\, Q_\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

ist. Da $w_\lambda \neq 0$ ist in $x^\nu = \underset{0}{x^\nu}$, ist auch $\gamma \neq 0$ und durch geeignete Wahl des freien skalaren Faktors in $\underset{1}{\mu}, \dots, \underset{\varrho+1}{\mu}$ kann man erreichen das $w_\lambda = Q_\lambda$ wird.

Die Klasse von w_λ ist infolge (76) höchstens gleich $2\varrho + 1$. Infolge (77) ist aber

$$2\, B^\mu_b\, B^\lambda_a\, \partial_{[\mu}\, w_{\lambda]} = -2\,(B^\omega_{[b}\, \partial_{|\omega|}\, B^\lambda_{a]})\, w_\lambda \quad . \quad . \quad . \quad . \quad . \quad (81)$$

woraus unter Berücksichtigung von (III 7) hervorgeht, dass in $x^\nu = \underset{0}{x^\nu}$

$$-2\, B^\mu_b\, B^\lambda_a\, \partial_{[\mu}\, w_{\lambda]} = Q_{ba} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

ist. Nun ist der Rang von Q_{ba} gleich 2ϱ, der Rang von $\partial_{[\mu}\, w_{\lambda]}$ kann also in $x^\nu = \underset{0}{x^\nu}$ nicht kleiner als 2ϱ sein und zwar gilt dies, infolge (77), *für jede Wahl des freien skalaren Faktors in* w_λ. Daraus geht aber hervor, dass in $x^\nu = \underset{0}{x^\nu}$ und in der Umgebung dieses Punktes

$$w_{[\lambda}\, w_{\mu_1 \lambda_1} \dots \, w_{\mu_\varrho \lambda_\varrho]} \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

ist und w_λ also genau den Rang $2\varrho + 1$ hat.

6. *Beweis des Teiles* (c) *des Haupttheorems.*

Da der Teil (*b*) schon bewiesen ist, bleibt nur noch zu beweisen, dass es stets gelingt q linear unabhängige Vektoren $\overset{p+1}{Q}_\lambda, \dots, \overset{n}{Q}_\lambda$ zu konstruieren, die den Bedingungen (III 6) und (III 7) genügen. Die zu diesen konstruierbaren Teiler $\overset{p+1}{w}_\lambda, \dots, \overset{n}{w}_\lambda$ sind dann linear unabhängig, da sie in $x^\nu = \underset{0}{x^\nu}$ mit den Q_λ zusammenfallen und sie sind alle von der Klasse $2\varrho + 1$.

Wir wählen in $x^\nu = \underset{0}{x^\nu}$ das Koordinatensystem so, dass für die ersten p Massvektoren gilt

$$\underset{b}{e}^\nu \overset{*}{=} B^\nu_b;\ b = 1, \dots, \mathrm{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

Es seien unter den $\binom{p}{2}$ Vektoren $2\, B^\omega_{[b}\, \partial_{|\omega|}\, B^\nu_{a]}$ gerade s linear unabhängige, die infolge der besonderen Wahl (III 10) der B^ν_b auch von der B^ν_b linear unabhängig sind. Diese führen wir in $x^\nu = \underset{0}{x^\nu}$ als s weitere Massvektoren ein

$$\underset{i}{e}^\nu \overset{*}{=} 2\, B^\omega_{[b_i}\, \partial_{|\omega|}\, B^\nu_{a_i]};\ i = \mathrm{p}+1, \dots, \mathrm{p}+\mathrm{s} \quad . \quad . \quad . \quad . \quad (85)$$

Schliesslich wählen wir in $x^\varkappa = \underset{0}{x}{}^\varkappa$ die übrigen, von den vorherigen unabhängigen Massvektoren beliebig und schreiben diese $\underset{t}{e}{}^\varkappa$; $t = \mathrm{p} + \mathrm{s} + 1, \dots, \mathrm{n}$. Infolge dieser besonderen Wahl ist der Ausdruck links in (III 3) für $\sigma = \varrho$ in $x^\varkappa = \underset{0}{x}{}^\varkappa$ von der Form

$$C_{[a_1 b_1}^{\cdot\cdot\cdot\cdot\cdot x_1} \dots C_{a_\varrho b_\varrho]}^{\cdot\cdot\cdot\cdot\cdot x_\varrho} = C_{\lambda_1}^{x_1} \dots C_{\lambda_\varrho}^{x_\varrho} \, a^{i_1 \dots i_\varrho} \, \underset{i_1}{e}{}^{\lambda_1} \dots \underset{i_\varrho}{e}{}^{\lambda_\varrho} \quad . \quad . \quad (86)$$

(denn $C_\lambda^x \underset{b}{e}{}^\lambda = 0$; $x = \mathrm{p} + 1, \dots, \mathrm{n}$; $b = 1, \dots, \mathrm{p}$ in $x^\varkappa = \underset{0}{x}{}^\varkappa$). Infolge (III 3) ist das rechte Glied von (86) nicht Null, und daraus folgt, dass nicht alle $a^{i_1 \dots i_\varrho}$ in $x^\varkappa = \underset{0}{x}{}^\varkappa$ verschwinden. Nun lässt sich (III 6) schreiben

$$Q_b = 0; \; b = 1, \dots, \mathrm{p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

und (III 7)

$$a^{i_1 \dots i_\varrho} Q_{i_1} \dots Q_{i_\varrho} \neq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

In der E_q der Indizes $i = \mathrm{p} + 1, \dots, \mathrm{p} + \mathrm{s}$; $t = \mathrm{p} + \mathrm{s} + 1, \dots, \mathrm{n}$ ist also der Vektor Q mit den Bestimmungszahlen Q_i, Q_t ein beliebiger Vektor, dessen E_{q-1} die Hyperfläche

$$a^{i_1 \dots i_\varrho} Q_{i_1} \dots Q_{i_\varrho} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

nicht berührt. Es gibt sicher q linear unabhängige Vektoren dieser Art. Wählt man dazu $Q_b = 0$, so entstehen in $x^\varkappa = \underset{0}{x}{}^\varkappa$ q linear unabhängige Vektoren $\overset{x}{Q}_\lambda$; $x = \mathrm{p} + 1, \dots, \mathrm{n}$, die den Bedingungen (III 6) und (III 7) genügen.

Die (III, S. 454) aufgestellte Behauptung, dass es nicht möglich wäre q linear unabhängige Teiler, deren Klasse alle kleiner als $2\varrho + 1$ wären, zu konstruieren, ist unrichtig. Das PFAFFsche System

$$\left.\begin{aligned} dx^1 + x^2\, dx^3 &= 0 \\ dx^4 + x^5\, dx^6 &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

bildet ein Gegenbeispiel. Der Rang 2ϱ ist 4 und dennoch gibt es zwei Teiler von Klasse 3 [1]). Die Existenz von q linear unabhängigen Teilern von der Klasse $2\varrho + 1$ ist also keine hinreichende Bedingung dafür, dass der Halbrang von (III 1) gleich ϱ ist. Dagegen beweist man leicht, dass der Halbrang in einem Gebiete, wo $B_b^\varkappa$ und $C_\lambda^\varkappa$ regulär sind, dann und nur dann gleich ϱ ist, wenn es in jedem Punkte des Gebietes Vektoren Q_λ gibt, die in diesem Punkte den Gleichungen (III 6) und (III 7) genügen und ausserdem jeder dieser Vektoren sich zu einem Feld w_λ fortsetzen lässt, das im betrachteten Gebiet Teiler von $w_{\lambda_1 \dots \lambda_q}$ und von der Klasse $2\varrho + 1$ ist.

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[5]) In dem (II S. 108) als Nebenresultat erhaltenen Satz ist demnach unmittelbar nach (47*b*) das „dann und nur dann" zu streichen. Auf die weiteren Resultate hat dies keinen Einfluss.

Physiology. — *On the structure of the synapses "a distance" in the cerebellum and the retina (mossy fibres and glomeruli cerebellosi, and visual cells).* By J. BOEKE.

(Communicated at the meeting of May 25, 1940.)

Several years ago I discussed in these Proceedings (Vol. XXXII, N⁰. 6, 1929) the nature of the interneuronal connections, the synapses. In that paper I came to the conclusion, that, when we have to admit that in the synaptic region the nervous stimulus is not stopped but simply altered, polarised a.s.o., a separation by an intercellular membrane or inert intercellular material seems to be out of the question. The neuronic elements must be connected by a living substance, which is able to conduct the stimulus and connects in a certain way the neurofibrillar structures of the two elements connected. But we certainly have to account for the physiological pecularities of the synapse, and the conclusion was drawn, that we may regard the often finely striated substance of the synapse, which connects the neurofibrillar endloops of the interneuronal fibres with the internal neurofibrillar structure of the nerve cells themselves, as being of the same nature as the periterminal network of the peripheral synapses, the motor endplates and the sensory corpuscles with their protoplasmic core. This would account for their weak staining capacity and the difficulty of demonstrating the striated structure of the interneuronal synapses in a satisfactory manner. It was pointed out by me that if we could regard these interneuronal junctions as of the same nature as the periterminal network of the different peripheral nerve-endings, the physiologic independance of the neurones in relation to drug-action and function, and the peculiar way in which the synapse differs from both the nerve cell and the terminal branches of the nerve fibres, would be accounted for in a satisfactory way, together with the anatomical continuity of structure (and connection by living substance) between them which we have to acknowledge in the light of modern histology (see BOEKE, Problems of Nervous Anatomy, Oxford University Press, 1940).

In the synapse there must be a connection of the two neuronic parts by *living* substance. This is the only way to account for the transmission of the nervous stimulus and its alterations. Even the hypothetical synaptic membrane of SHERRINGTON must be an arrangement of units of the *living* substance, and this arrangement may be present in the periterminal network, not as a real visible membrane, but as a biphasic condition of the living substance itself. If we regard the intervening synaptic substance as a sort of inert, non-living cement substance, sometimes faintly striated, as the classical neuronists do, a connection in which no alteration is possible, we remain in the dark and no real histological basis for the modern conceptions of the function of the nervous system and the nature of the transmission of the nervous stimulus is possible. We have to regard the synaptic substance as a living substance, in which there is present a structure akin to the periterminal network of the peripheral junctions and connecting the neurofibrillar endloops with the neurofibrillar network inside the cell. In this way the synapse may alter its structure as living substance does, it may obtain a more definite linear arrangement of the units composing its connecting fibrils, inside its protoplasmic substance there may be formed a more complex organisation of its linear structure, a transient part of this linear structure may become fixed, in short, inside the synaptic *living* substance the conducting apparatus may become more and better organised (learning, "Bahnung" etc.). Even the structure of the sarkoplasmic base of the motor endplate with its periterminal network may be altered according to its functional condition (cf. BOEKE, l.c. 1940).

In most cases the structure of synaptic junctions nowadays is regarded as consisting of living substance ("protoplasmic continuity is, therefore, the rule at synaptic junctions", GERARD, 1934, in BLOOMs Textbook).

The most difficult case to account for is the synaptic junction in the so-called glomeruli olfactorii and the glomeruli cerebellosi, in which the terminal arborisations of the axons and of the dendrites of the nervous elements of the two different neurons form a very dense entanglement with no evidence whatever of any protoplasmic continuity between them. Indeed, with the exception of HELD (1897) and PENSA (1931) every author follows the footsteps of CAJAL, who described them in 1926 as a mass of inert substance, which presented no definite structure whatever, and in which the telodendria of the different nerve fibres branched and intertwined without coming into contact with each other. After CAJAL, HORNE CRAIGIE (1926), LUGARO, JACOB (1928) and many others the glomeruli form an independant mass of substance (JACOB calls them "Protoplasmainsel") with a sharp outline, in which the nerve branches enter after having lost their myelin sheath. PENSA describes in the glomeruli cerebellosi special nuclei, belonging to oligodendroglia cells, and thus maintains that the groundsubstance is nothing but glious tissue.

According to HELD (1897) the glomeruli cerebellosi form a syncytial mass of protoplasm ("Grundplasma") with neurosomes and in close syncytial connexion with the protoplasm of the surrounding nerve cells. The neurosomes are continuous with the neurosomes of the nerve telodendria entering the glomeruli.

In order to investigate the way in which in these synaptic junctions the connexions between the different nerve fibres and their conducting structure are established, I studied the glomeruli in the cortex of the cerebellum of the cat and of the rabbit and the retina of the horse and of the cat, in silver preparations and after having fixed the tissue as carefully as possible with corrosive sublimate, Hermanns fluid and a mixture of both.

Three questions have to be studied:

a. what is the nature of the mass of the glomeruli? Is it inert material or living protoplasmatic substance? Is it of a glious nature or of nervous origin? Are nuclei present inside the glomerulus?

b. Is this groundsubstance of the glomeruli independant of the surrounding tissue or not?

c. In what manner are the nerve branches entering the glomerulus connected with each other, is there any evidence of a protoplasmic continuity between them?

ad a. In the first place it is interesting to note that in the glomeruli there is a very high metabolic activity, which points to be supposition, that their substance is *living* substance. The glomeruli are surrounded by a close network of capillaries, and everywhere in the brain, wherever synaptic junctions of this type are abundant, the capillaries form a closer mesh than elsewhere. This suggests that these synapses are a region of high metabolic activity, and the same suggestion is made by NOYONS, who found in his wonderfully exact physiological experiments, that the basal metabolism of the cortex of the cerebellum with its extremely numerous synapses is far greater than of the cortex cerebri, where the synaptic junctions are less numerous, much smaller and never form large masses of groundsubstance as in the glomeruli of the cortex cerebelli.

This high metabolism needs a living groundsubstance and not an inert material. Thus JACOB is perfectly right when he calls the glomeruli cerebellosi "Protoplasmainsel", and it is curious to note in this connection, that CAJAL himself has felt the difficulty to regard them simply as consisting of inert cement substance as the neurone theory required. In his drawings as in those of his collaborators (f.i. HORNE CRAIGIE) they appear as sharply defined homogeneous grey spots in which the nerve branches intertwine, but even in 1926 CAJAL in the text compares their substance with the groundsubstance of the motor endplates, which must be able to conduct the nervous stimulus from the terminal arborisation of the motor nerve to the contractile substance of the striated muscle fibre. Now every histologist without any exception regards the groundsubstance of the sole plate as protoplasm, sarcoplasma, as *living substance*, and I may quote here GERARD (in BLOOMs Textbook, 1934): "in the motor end-plate there is obviously protoplasmic continuity between nerve fibre and muscle fibre; but the protoplasm of the end-plate is visibly different from that of both nerve and striated muscle. The physiologic properties of this "junctional tissue" are also unique." GERARD then bases his account of the peculiar physiologic features of synaptic junctions in part upon these motor end-plates "where the junctional tissue is visible" In this protoplasmic junctional tissue of the motor end-plates the periterminal network is described, and even CAJAL himself admits that this periterminal network is of protoplasmic origin and may possibly form a link in the conducting process of the nervous stimulus (CAJAL, 1934). Now, as will be described farther on, the same periterminal network is to be seen in the groundsubstance of the glomeruli, and so we may be convinced that it is of protoplasmic origin and that it is living substance.

The question whether it is of glious nature as was maintained by PENSA (1931), or of nervous nature as was maintained by HELD (1897) is difficult to answer. I have never been able to find the nuclei and the oligodendroglia cells PENSA describes inside the glomeruli. Glious cells and nuclei are to be found everywhere in the granular layer of the cortex cerebelli, but I have never found a single nucleus inside the groundsubstance of the glomeruli. Neurosomes are to be found inside this groundsubstance and when we study carefully-made thin sections we may see, that these granules are in continuous connexion with the granules inside the terminal branches of the nerve fibres and their neurofibrillar expansions. The substance of the glomeruli is everywhere in continuous connexion with the neuroplasm of the entering nerve branches; they lose their myelin sheath (when they are myelinated) as soon as they have entered the substance of the glomerulus, and not before. In my opinion this groundsubstance is of an intermediate character and of protoplasmic origin.

ad b. The groundsubstance of the glomeruli is never independant of the surrounding tissue as it appears in the drawings of CAJAL, HORNE CRAIGIE and other neuronists. It is a part of this tissue and it is everywhere in continuous protoplasmic connexion with the surrounding elements, the granular cells and the mossy fibres. This connexion is only possible when we admit that the groundsubstance is living protoplasm and that it is of nervous origin (and intermediate physiologic character).

ad c. Inside the groundsubstance of the glomeruli a distinct netlike structure, a periterminal network may be made visible, and this network is everywhere in continuous connexion with the neurofibrillar structure of the terminal arborisations of the entering nerve fibres. In figure 1 I have drawn as accurately as possible what my preparations showed me. The network is very delicate, regular with small meshes, exactly as it is described for the periterminal network of the motor end-plates and the sensory corpuscles (BOEKE, Zeitschr. Mikr.-Anat. Forschung, Bd. IV, 1926). It is attached everywhere to the terminal neurofibrillar branches of the nerve fibres and extends throughout the whole of the glomerulus, fading away at its border. The nerve fibres are therefore in evident protoplasmic continuity, but the protoplasm of the glomerulus has an intermediate character as it was stated for the sarcoplasm of the sole-plate with its periterminal network. It conducts the nervous stimulus and appears to possess all the characters of a real synaptic junction.

It is not possible to describe all its features and to discuss them in this short preliminary communication. I hope to give a better and more

J. BOEKE: ON THE STRUCTURE OF THE SYNAPSES "A DISTANCE" IN THE CEREBELLUM AND THE RETINA (MOSSY FIBRES AND GLOMERULI CEREBELLOSI, AND VISUAL CELLS).

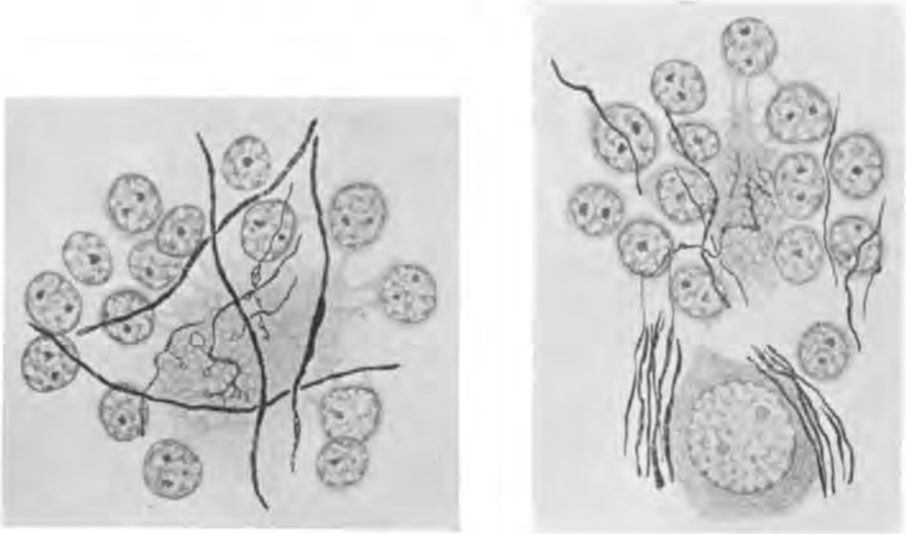

Fig. 1. Glomeruli cerebellosi in the granular layer of the cortex cerebelli of the rabbit with periterminal network and terminal neurofibrillar nerve branches. To the right below a Purkinje-cell.

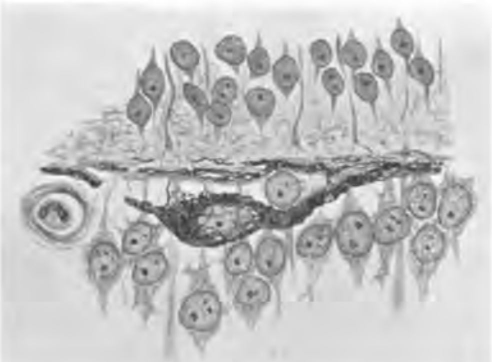

Fig. 2. Cross-section through the retina of a cat in the region between the ends of the visual cells and the cells of the second neuron. The black element is a horizontal cell.

detailed description in the XI. Innervationsstudie in the Zeitschr. f. Mikr.-Anat. Forschung, with more figures and a discussion of the details.

I may add to this description that exactly the same connexions are to be found in the retina between the endknobs of the visual cells and the terminal branches of the second neuron, as it is drawn in fig. 2. Here too we find distinct protoplasmic connexions as it was maintained allready by AKKERINGA (1936) for the ganglion cells of the second and third neuron.

Utrecht, May 1940.

Psychology. — *Occipital forms, facial dimensions, age, intellect and character.* By E. D. WIERSMA.

(Communicated at the meeting of May 25, 1940.)

The investigations on "Facial dimensions, age and intellect" [1]) demonstrated that a connection exists between the dimensions of the face, the age, and the intellectual powers. The following investigation was made with the object of ascertaining in how far the dissimilitude in the form of the occiput is coupled with differences in psychical phenomena (temperament, intellect and character), in bodily build and in physiological functions. The said dissimilitude in the occipital forms concern the prominence or sharp regression of the back of the head. They are frequently distinguishable by the eye. In doubtful cases the distance is measured from the glabella to the vertex, to the middle of the occipital curve and to the limit of the hair growth at the back of the neck. In respect of the prominent occiput, these measurements show greater differences, and in respect of the receding occiput, lesser differences. The material for this investigation was afforded me in various ways.

1. Dr. L. KAISER took photos en profile of 217 pupils (117 male and 100 female ones). In many of these it can be plainly seen whether the occiput is prominent or steeply receding, and in those of the male students it can further be seen whether the prominence is most pronounced in the upper part, the middle part, or lower part of the head.

Not all the photos could be made use of for this investigation, because the hair, especially of the female students, does not allow of the exact form of the back of the head being distinguished in the photos. As Dr. KAISER comes into close contact with her pupils, she was able, in the majority of cases to state the psychical qualities with regard to temperament, intellect and character. I am greatly indebted to Dr. KAISER for this material.

2. In the somato-psychological enquête [2]), I inquired, in question 115, whether the occiput of the individual in question projected or receded. These forms were selected from the enquête material and in both cases it was investigated what differences there were in bodily and mental qualities.

3. In the above-mentioned enquête there was further inquired whether

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[1]) Proc. Kon. Ned. Akad. van Wetensch., Amsterdam, 42, 727 (1939).

[2]) Capita psychopathologica, p. 373—377.
Ter overdenking voor huisarts en student, p. 42, 46—75.
Voor paedagoog en huisarts, p. 56 and following.

the individual in question was shortheaded or longheaded. It is known that in longheaded persons a very large percentage of individuals exhibit a prominet occiput, whereas this is strongly receding in the case of shortheaded persons. The Table below will show this.

Figures of the percentages.

	Longheaded	Shortheaded
Prominent occiput	62.6	17.1
Receding	26.8	42.2

4. The said enquête further supplies information respecting differences in psychical qualities of the fine and slender persons and the thicksets, and teaches that persons with a prominent occiput are more fine and slender and less thickset, while the persons with receding occiput are less fine and slender and more thickset than the average.

Percentage figures.

	Prominent occiput	Receding occiput	Average
Fine and slender persons	43.2	32.8	38.1
Thicksets	9.5	12.4	10.6

5. Measurements were also taken of mentally normal and abnormal persons of widely different ages in the Children's Hospital, in the Surgical Clinic, in the Open Air School, and in the schools for feeble-minded and for imbecile children, at Groningen. My thanks are due to the Heads of these Institutes for permission to enter their institutions, for the assistance afforded me, and for the information supplied. Compasses-measurements ware taken of the distance from the glabella tot the vertex, from glabella to the middle of the occipital curve, from the glabella to the limit of the hair-growth in the neck, and, further, the length of the forehead, of the upper and of the lower jaw, as has been indicated elsewhere [1]).

Below I give the investigation of the above-mentioned groups.

In order to judge of the differences in intellect in her material, Dr. KAISER (group 1) attached value to the existence of special gifts, to adroitness, succinctness and matter of factness in their narration. The enquête material (groups 2, 3 and 4) was judged from the answers to questions as to quickness of grasp and good sense, or as to superficiality and stupidity,

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[1]) Proc. Kon. Ned. Akad. van Wetensch., Amsterdam, **42**, 727 (1939).

Temperamental Qualities.

	Group 1		Group 2			Group 3[1]			Group 4[2]	
	A	n A	A	n A		A	n A		A	n A
Average	57.3	42.7	63.7	37.1	Average	66.7	23.2	Average	66.6	23.4
Prominent occiputs	54.2	45.7	60.5	37.1	Longheaded	63.6	27.1	Fine and slender persons	65.6	25.3
Receding occiputs	63.3	36.7	67.0	32.9	Shortheaded	66.9	20.8	Thicksets	73.0	20.5
	E	n E	E	n E		E	n E		E	n E
Average	61.5	38.5	62.9	37.1	Average	65.3	16.9	Average	65.3	16.9
Prominent occiputs	68.1	31.9	69.1	30.8	Longheaded	64.9	19.2	Fine and slender persons	68.9	18.0
Receding occiputs	46.7	53.3	56.9	43.3	Shortheaded	69.4	16.2	Thicksets	57.1	14.3
	S	P	S	P		S	P		S	P
Average	38.4	61.6	31.9	68.2	Average	40.3	26.6	Average	46.5	24.1
Prominent occiputs	30.6	69.4	33.3	66.6	Longheaded	41.8	25.6	Fine and slender persons	46.9	24.6
Receding occiputs	56.7	43.3	53.2	44.5	Shortheaded	38.3	31.1	Thicksets	53.2	18.8

[1] Zeitschr. f. angewandte Psychologie. B. **38**, p. 151.

[2] Zeitschr. f. angewandte Psychologie. B. **38**, p. 162.

to more or less knowledge of human nature, to broad-mindedness or narrow-mindedness, to individuality or an inclination to imitation, to the greater or lesser powers of observation. Intellectual differences between the persons of group 5 can, for a part, be inferred from the schools which they attend, and were further supplied me by the Heads of the Institutions, which, in many cases, have the I. Q. of the pupils at their disposition.

In forming a judgment of the character the most important point is to ascertain whether the individual in question is chiefly preoccupied with himself, or has always interest in, and a fellow-feeling for, others; whether he thinks in the first place of his own interests or is in the habit of entering into the difficulties of the persons in his environment; in how far thus, egoistic or altruistic motives dominate in his thoughts and actions. Egoism is stamped upon all the lower character traits, on the vital, egophile and the lower abstract tendencies.

For the reasons above-mentioned, in the investigation of group I, I made use only of the photos of the male students. (See table on page 694).

In these four groups persons with a receding occiput possess a greater activity, and persons with a prominent occiput less, than the average. This similarity in the groups exhibits a few exceptions in regard to the emotionality and the psychical after-effects. From the Table below, in which the average values of all the groups are given, it appears that persons with a very receding occiput are more active, less emotional, and of secondary importance than the average, and than the persons with a prominent occiput.

Average	A	n A	E	n E	S	P
Of all occiputs	62.2	31.4	63.7	27.3	46.8	45.1
Of prominent occiputs	60.9	33.8	67.7	24.9	38.1	46.6
Of receding occiputs	67.5	27.7	57.5	31.8	50.3	34.4

Intellect and character.

Intellect and character are in a high degree dependent on the temperament. It is known that mental grasp as well as superficiality is promoted by primary functions; good sense and broad-mindedness, as well as stupidity, by secondary functions; violence and irritability, as well as compassion and helpfulness, by the sensibilities; perseverance and thoughtfulness, as well as suspiciousness and distrust, by the secondary functions; receptivity of new ideas, as well as contradictions in thought and action, by the primary functions. On the grounds of these differences found in all the groups, a survey of the average values is given, as in the temperamental qualities.

In judging the intellect of the persons in the several groups, it was noted whether the above-enumerated symptoms ware evinced or not.

	Group 1		Group 2	
	Intellectual	Not intellectual	Intellectual	Not intellectual
Avarage	63.9	4.2	59.6	9.9
Prominent occiput	62.1	2.5	62.3	9.4
Receding occiput	50.0	3.3	58.1	10.5

The investigation of the longheaded and shortheaded, the fine and slender persons and thicksets, indicates that the longheaded are more secondary, the shortheaded more primary than the average (see Table of the temperamental qualities). In accordance with this, the longheaded are less receptive and more understanding than the shortheaded. The thicksets are more secondary than the fine and slender persons and, accordingly, more understanding.

In the same way the persons in groups 2, 3 and 4 are distinguished in egoistic and altruistic thinking and feeling individuals. The degree of their egoism is evinced in their more ore less indulgence in the satisfying of vital tendencies, in the lessened altruism, their feebler sense of duty and love of truth, their greater suspiciousness and unreliability.

	Group 1		Group 2	
	Egoism	Altr.	Egoism	Altr.
Average	29.9	23.0	18.7	55.1
Prominent occiputs	30.5	20.9	20.1	55.4
Receding occiputs	20.0	30.0	19.7	56.8

	Group 3			Group 4	
	Egoism	Altr.		Egoism	Altr.
Average	22.4	65.7	Average	30.8	47.6
Longheaded	21.6	65.3	Fine and slender	29.3	48.6
Shortheaded	25.7	63.2	Thicksets	27.4	51.7

From the above Table it is apparent that the persons with a very receding occiput stand on a higher moral plane than the persons with a prominent occiput. This holds good for all the groups, except Group 3.

Group 5. The measurements taken in the above-mentioned institutions of 205 persons are expressed in the Table below in percentages of the middle skull-line.

Influence of Age

Age	Number	Upper line	Lowest line	Height of forehead	Length of upper jaw	Length of lower jaw	Length of jaw system
0— 1 y.	21	95.3	97.7	26.1	36.5	15.6	52.1
1— 2 y.	20	97.8	91.2	26.2	36.9	17.0	53.9
2— 5 y.	16	93.8	90.9	29.6	38.3	17.9	56.2
5—12 y.	46	97.3	90.0	26.3	44.1	18.0	62.1
12—18 y.	15	97.7	94.0	28.4	43.1	19.5	62.6
above 18	87	96.2	95.8	29.8	44.4	22.5	66.9

From the above Table it will be seen that the height of the forehead, the length of the upper jaw, of the lower jaw, and of the whole jaw system becomes regularly greater with increasing age in proportion to the middle line of the skull.

Influence of Sex.
Of the 87 adults measured, 47 were males and 40 females.

Sex	Number	Upper skull	Lower skull line	Length of forehead	Length of upper jaw	Length of lower jaw	Length of jaw system
Male	47	96.3	95.8	30.7	44.3	22.1	66.4
Female	40	96.0	95.8	28.9	44.7	21.0	65.7
Male and Female	87	96.2	95.8	29.8	44.4	21.5	65.9

The differences in facial and skull dimensions is of little significance.

Influence of intellectual gifts.

Of 92 pupils in the school for feeble-minded children the I. Q. fluctuated between 60 and 90, of 72 pupils in the school for imbecile children the I. Q. was lower than 60. These feeble-minded and imbeciles are compared in the following Table with normal individuals of about the same age (5—18).

Number	Mental cond.	Upper skull line	Lower skull line	Length of forehead	Length upper jaw	Length lower jaw	Length of jaw system
61	normal	97.5	92.0	27.2	43.6	18.9	62.5
92	feeble-minded	96.0	90.5	23.4	43.8	20.7	64.4
72	imbecile	98.9	86.2	23.6	48.9	19.0	67.9

It will be evident from the above Table that in the imbeciles the upper line of the skull differs most from the middle, the occiput projecting most in the middle. With the normal, the forehead is higher, the upper and the lower jaw and the whole jaw system smaller, in proportion to the middle skull-line than with the feeble-minded and the imbeciles.

In the Open-air School, where children with various bodily infirmities are taught, measurements were taken of 28 children. According to the information from the resident nurse and the teacher, 16 children are intellectually normal and 12 are feeble-minded. The following Table gives a survey of the measurements.

Number	Mental cond.	Upper skull line	Lower skull line	Length of forehead	Length of upper jaw	Length of lower jaw	Length of jaw system
12	feeble-minded	99.4	89.8	24.5	45.0	16.9	
16	normal	96.2	87.8	26.2	43.3	19.4	

This Table, too affords evidence that the forehead of the feeble-minded is smaller than normal size, while the upper jaw is larger than the normal.

Physics. — *The decay of the penetrating cosmic rays.* IV. By E. M. BRUINS. (Communicated by Prof. J. CLAY).

(Communicated at the meeting of May 25, 1940.)

In order to explain the different absorption of penetrating cosmic rays in air and in condensed materials FERMI[1]) suggested that the energyloss of the particles in a condensed material is less then in air. In the present paper[2]) we will compare this hypothesis with that of the decaying electrons. The energyspectrum, being a homogeneous function of degree *s*, no information can be obtained by measuring the distribution of intensity around the vertical, for in *both* cases this has to be

$$I(\vartheta) = I_0 \cos \vartheta^{s-1}.$$

At sealevel, only from a comparison of measurements in one *fixed* direction with different layers of absorbing materials with that in another direction *and* with measurements under the same shield in different directions information about this question can be obtained.[3])

Now the measurements of the variation with zenith-angle ϑ confirmed, in a complete accord with the integral energyspectrum $E^{-1.93} \simeq E^{-2}$ found at greater depths below the sealevel, the $\cos^2 \vartheta$-distribution.

The main difference between the decaying electrons (*a*) and the FERMI-hypothesis (*b*) lies in the fact that in different directions the energy distribution remains *unchanged* in the latter case whereas *the decay changes the energyspectrum* as has been discussed in a former paper (Decay I).

If *t* represents the energyloss expressed in the loss in the atmosphere as a unit we have

a. Decay:
$$I(\vartheta, t=1) = I_0\, C(1) \cos \vartheta^{s-1} = I(\vartheta)$$
$$I(0, t) = I_0\, C(t)\, t^{1-s} = I(t)$$

b. FERMI:
$$I(\vartheta, t=1) = I_0 \cos \vartheta^{s-1} = I(\vartheta)$$
$$I(0, t) = I_0\, t^{1-s} = I(t)$$

where $C(t)$ is the correctionfactor due to the decay which has been calculated by numerical integration (Decay II, III).

a. Comparing the intensity under layers of equal mass per cm² we obtain, when *a* represents the energyloss per gr. per cm² of air and *m* the same

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[1]) E. FERMI, Phys. Rev. **56**, 1242 (1939).

[2]) Comp. Decay I, II, III, Proc. Kon. Ned. Akad. van Wetensch., Amsterdam, **42**, 54, (1939); **42**, 740, (1939); **43**, 75, (1940).

[3]) Comp. J. CLAY. Proc. Kon. Ned. Akad. van Wetensch., Amsterdam, **43**, 437—438, (1940).

quantity per gr. per cm^2 of matter, y, z being the massequivalents of the layers of air and x that of the layer of absorbing material:

a. Decay: $$\frac{I(\vartheta)}{I(t)} = \frac{C(1)}{C(t)} \quad \text{or} \quad \lim_{t \to \infty} \frac{I(\vartheta)}{I(t)} = C(1).$$

The limiting value $1 : C(1) = I(t) : I(\vartheta)$ is given in table I and shown in fig. 1 as a function of the lifetime τ.

b. FERMI: $$\frac{I(\vartheta)}{I(t)} = \left(\frac{ay + mx}{az}\right)^{s-1}$$

or with $m = a - f$ as $z = x + y$

$$\frac{f}{a} = 1 - \sqrt[s-1]{\frac{I(\vartheta)}{I(t)}}.$$

TABLE I.

τ	1	2	4	8	10	$x\,10^{-6}$
$I(t)\ \ I(\vartheta)$	8.00	3.50	2.03	1.49	1.39	

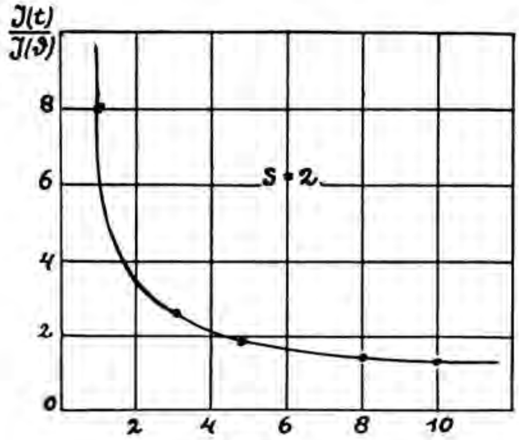

Fig. 1.

β. The relation between ϑ and t for which both intensities reach the same value is

a. Decay: $$\sec\vartheta = \sqrt[s-1]{\frac{C(1)}{C(t)}} \cdot t$$

b. FERMI: $$mx + ay = az = ay \sec\vartheta \quad \text{or}$$

$$(\sec\vartheta - 1) = \frac{m}{a}\,(t - 1).$$

For non decaying particles the *sec* ϑ-*t*-diagram gives a *straight line* the slope of which is determined by $\frac{m}{a} < 1$.

For decaying particles this diagram shows a curve, the slope of which approaches $\sqrt[s-1]{C(1)}$ for great values of t whereas the initial slope can be calculated as $1 - \frac{1}{2}\left(\frac{dl_n\, C(t)}{dt}\right)_{t=1}$

These values are given in table II and the curves for different life-time τ are shown in fig. 2.

TABLE II.

$\tau =$	1	2	4	8	10	∞	$x\, 10^{-6}$
limes slope: $t \to \infty$	0.354	0.535	0.702	0.821	0.848	1	
initial slope $t = 1$	0	0.184	0.427	0.578	0.628	1	

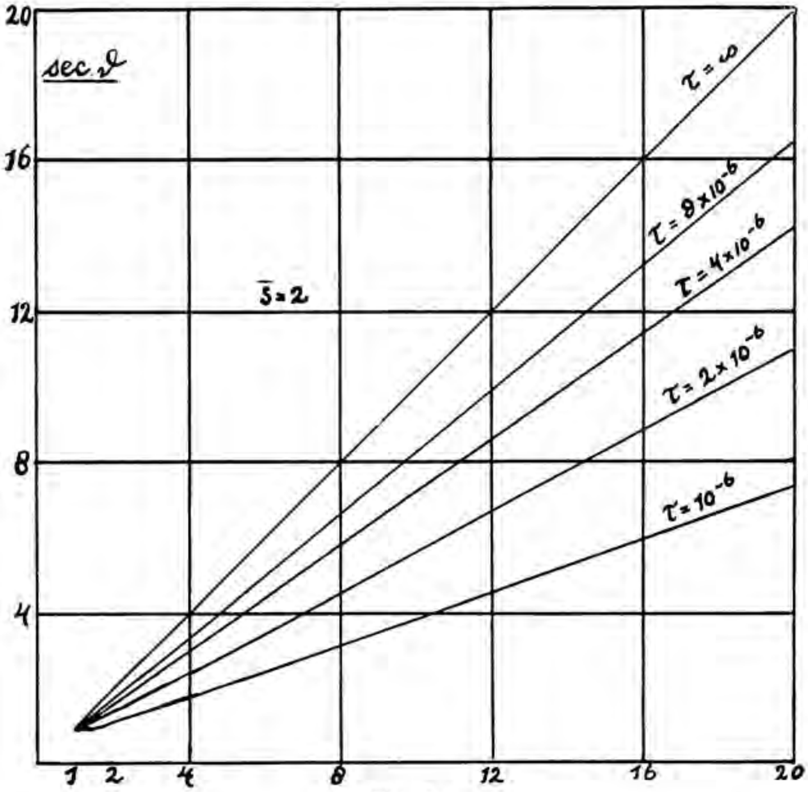

Fig. 2.

A deviation from the straight line through *sec* $\vartheta = 1$, $t = 1$ is in *contradiction* with the hypothesis of FERMI.

Mathematics. — *Ueber eine Erweiterung der* LAPLACE-*Transformation.* (Zweite Mitteilung). Von C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of May 25, 1940.)

Hilfssatz 4. *Ist* $0 < \delta < x$ *und* $\beta > 0$, *so gilt*

$$\lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int_{x}^{x+\delta} K_\nu(st)\, t^{\nu+1}\, dt = \tfrac{1}{2}\pi i x^\nu . \quad . \quad . \quad . \quad (26)$$

und

$$\lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int_{x-\delta}^{x} K_\nu(st)\, t^{\nu+1}\, dt = \tfrac{1}{2}\pi i x^\nu . \quad . \quad . \quad (27)$$

Beweis. Bekanntlich ist [18])

$$\int K_\nu(z)\, z^{\nu+1}\, dz = -z^{\nu+1} K_{\nu+1}(z) + C. \quad . \quad . \quad . \quad . \quad (28)$$

Folglich hat man

$$\lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, s\, ds \int_{a}^{b} K_\nu(st)\, t^{\nu+1}\, dt$$

$$= a^{\nu+1} \lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, K_{\nu+1}(as)\, ds - b^{\nu+1} \lim_{\lambda \to \infty} \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, K_{\nu+1}(bs)\, ds.$$

Die beiden rechtstehenden Integrale können mit Hilfe von (23) und (24) berechnet werden. Nimmt man $a = x$ und $b = x + \delta$, so bekommt man (26); für $a = x - \delta$ und $b = x$ findet man (27).

Hilfssatz 5. *Ist* $0 < \alpha < d$ *und* $\mu \geqq 0$, *so hat man*

$$\left| \int_{\alpha-\mu i}^{\alpha+\mu i} \frac{e^{-\zeta}\, d\zeta}{\zeta} \right| < M, \qquad \left| \int_{\alpha-\mu i}^{\alpha+\mu i} \frac{e^{\zeta}\, d\zeta}{\zeta} \right| < M,$$

wo M *nicht von* α *und* μ, *also nur von* d *abhängt.*

---

[18] WATSON, **[22]**, 79, Formel (3).

Beweis. Für $a > 0$ gilt bekanntlich

$$\int_{\alpha-\infty i}^{\alpha+\infty i} \frac{e^{-\zeta}\, d\zeta}{\zeta} = 0, \quad \int_{\alpha-\infty i}^{\alpha+\infty i} \frac{e^{\zeta}\, d\zeta}{\zeta} = 2\pi i. \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Ferner hat man [19])

$$\int_{\alpha+\mu i}^{\alpha+\infty i} \frac{e^{-\zeta}\, d\zeta}{\zeta} = G\,(a+\mu i), \quad \int_{\alpha-\mu i}^{\alpha-\infty i} \frac{e^{-\zeta}\, d\zeta}{\zeta} = G\,(a-\mu i), \quad . \quad (30)$$

$$\int_{\alpha+\mu i}^{\alpha+\infty i} \frac{e^{\zeta}\, d\zeta}{\zeta} = G\,(-a-\mu i), \quad \int_{\alpha-\mu i}^{\alpha-\infty i} \frac{e^{\zeta}\, d\zeta}{\zeta} = G\,(-a+\mu i); \quad . \quad . \quad (31)$$

hierin ist

$$G\,(z) = -\gamma - \log z - \sum_{r=1}^{\infty} \frac{(-z)^r}{r\,\Gamma(r+1)} = e^{-\frac{1}{2}z}\, z^{-\frac{1}{2}}\, W_{-\frac{1}{2},0}\,(z), \quad . \quad (32)$$

wo γ die EULERsche Konstante und $W_{k,m}\,(z)$ die WHITTAKERsche Funktion bezeichnet.

Aus (29), (30) und (31) ergibt sich nun

$$\int_{\alpha-\mu i}^{\alpha+\mu i} \frac{e^{-\zeta}\, d\zeta}{\zeta} = G\,(a-\mu i) - G\,(a+\mu i), \quad . \quad . \quad . \quad . \quad (33)$$

$$\int_{\alpha-\mu i}^{\alpha+\mu i} \frac{e^{\zeta}\, d\zeta}{\zeta} = 2\pi i + G\,(-a+\mu i) - G\,(-a-\mu i). \quad . \quad . \quad (34)$$

Für die Funktion $G\,(z)$ gilt aber [20]) für festes $\varepsilon > 0$ und für $|z| \to \infty$

$$G\,(z) = e^{-z}\, z^{-1} \{1 + O\,(z^{-1})\} \qquad (-\tfrac{3}{2}\pi + \varepsilon < \arg z < \tfrac{3}{2}\pi - \varepsilon). \quad (35)$$

Die Behauptung des Hilfssatzes ergibt sich jetzt wegen (33) und (34) sofort aus (32) und (35), da $\left|\log\left(\frac{a+\mu i}{a-\mu i}\right)\right| < \pi$ und $\left|\log\left(\frac{-a-\mu i}{-a+\mu i}\right)\right| < 2\pi$ ist.

Hilfssatz 6. *Ist* $0 < c < a < d$ *und* $\mu \geqq 0$, *so hat man*

$$\left| \int_{\alpha-\mu i}^{\alpha} \frac{e^{-\zeta}\, d\zeta}{\zeta} \right| < N, \qquad \left| \int_{\alpha}^{\alpha+\mu i} \frac{e^{-\zeta}\, d\zeta}{\zeta} \right| < N,$$

wo N *nicht von* a *und* μ, *also nur von* c *und* d *abhängt.*

---

[19]) Man vergl. GOLDSTEIN, [**6**], 112.

[20]) Man vergl. WHITTAKER and WATSON, [**23**], § 16.3—16.4.

Beweis. Die Integrale sind stetige Funktionen von α und μ, die bei unbeschränkt wachsendem μ nach einem endlichen Grenzwert streben.

§ 3. Beweis von Satz 1.

Ich zerlege den Beweis in drei Teile[21])

Erster Teil. Für hinreichend kleine positive Werte von δ $(0<\delta<x)$ gilt

$$\frac{1}{\pi i}\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)(xs)^{\frac{1}{2}}\,ds\int_{x}^{x+\delta}K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt=\tfrac{1}{2}F(x+0) \tag{36}$$

und

$$\frac{1}{\pi i}\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)(xs)^{\frac{1}{2}}\,ds\int_{x-\delta}^{x}K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt=\tfrac{1}{2}F(x-0). \tag{37}$$

Zweiter Teil. Ist $0<\delta<x$, so gilt

$$\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)(xs)^{\frac{1}{2}}\,ds\int_{0}^{x-\delta}K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt=0. \quad . \quad . \quad . \tag{38}$$

Dritter Teil. Ist $\delta>0$, so gilt

$$\lim_{\lambda\to\infty}\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)(xs)^{\frac{1}{2}}\,ds\int_{x+\delta}^{\infty}K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt=0. \quad . \quad . \quad . \tag{39}$$

Formel (5) folgt nun sofort aus (36), (37), (38) und (39).

Beweis von (36) und (37). Die Funktion $F(t)$, also auch $t^{-\nu-\frac{1}{2}}F(t)$, ist von beschränkter Variation in der Umgebung des Punktes $t=x$. Für hinreichend kleine positive Werte von τ hat man also im Intervall $x<t\leqq x+\tau$

$$t^{-\nu-\frac{1}{2}}F(t)=x^{-\nu-\frac{1}{2}}F(x+0)+\varphi_1(t)-\varphi_2(t);\, . \quad . \quad . \quad . \tag{40}$$

hierin sind $\varphi_1(t)$ und $\varphi_2(t)$ monoton wachsende Funktionen mit $\lim\limits_{t\to x}\varphi_1(t)=0$, $\lim\limits_{t\to x}\varphi_2(t)=0$.

Bei hinreichend kleinem τ gilt analog im Intervall $x-\tau\leqq t<x$

$$t^{-\nu-\frac{1}{2}}F(t)=x^{-\nu-\frac{1}{2}}F(x-0)+\psi_1(t)-\psi_2(t),\, . \quad . \quad . \quad . \tag{41}$$

---

[21]) Der Beweis von Satz 1 ist mit dem TITCHMARSHschen Beweis der HANKELschen Formel (4) verwandt; man vergl. TITCHMARSH, [**21**], 240—242.

wo $\psi_1(t)$ und $\psi_2(t)$ monoton abnehmende Funktionen mit $\lim\limits_{t\to x}\psi_1(t)=0$, $\lim\limits_{t\to x}\psi_2(t)=0$ bedeuten.

Zu jedem $\varepsilon>0$ kann man nun ein δ $(0<\delta\leqq\tau)$ so wählen, dass

$$\varphi_1(x+\delta)<\varepsilon,\ \varphi_2(x+\delta)<\varepsilon,\ \psi_1(x-\delta)<\varepsilon,\ \psi_2(x-\delta)<\varepsilon \quad . \quad (42)$$

ist.

Nun hat man mit Rücksicht auf (40)

$$\left.\begin{aligned}&\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)(xs)^{\frac{1}{2}}\,ds\int\limits_{x}^{x+\delta} K_\nu(st)(st)^{\frac{1}{2}}F(t)\,dt\\ &\qquad=x^{-\nu}F(x+0)\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,s\,ds\int\limits_{x}^{x+\delta} K_\nu(st)\,t^{\nu+1}\,dt\\ &\qquad+x^{\frac{1}{2}}\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,s\,ds\int\limits_{x}^{x+\delta} K_\nu(st)\,\{\varphi_1(t)-\varphi_2(t)\}\,t^{\nu+1}\,dt.\end{aligned}\right\} \quad . \quad . \quad (43)$$

Das Integral mit $\varphi_1(t)$ lässt sich unter Anwendung des zweiten Mittelwertsatzes folgenderweise reduzieren [22]:

$$\left.\begin{aligned}&x^{\frac{1}{2}}\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,s\,ds\int\limits_{x}^{x+\delta} K_\nu(st)\,\varphi_1(t)\,t^{\nu+1}\,dt\\ &=x^{\frac{1}{2}}\int\limits_{x}^{x+\delta}\varphi_1(t)\,t^{\nu+1}\,dt\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,K_\nu(st)\,s\,ds\\ &=x^{\frac{1}{2}}\varphi_1(x+\delta)\int\limits_{\eta}^{x+\delta} t^{\nu+1}\,dt\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,K_\nu(st)\,s\,ds \qquad (\text{wo } x<\eta<x+\delta)\\ &=x^{\frac{1}{2}}\varphi_1(x+\delta)\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,s\,ds\int\limits_{\eta}^{x+\delta} K_\nu(st)\,t^{\nu+1}\,dt\\ &=x^{\frac{1}{2}}\varphi_1(x+\delta)\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,[\eta^{\nu+1}K_{\nu+1}(\eta s)-(x+\delta)^{\nu+1}K_{\nu+1}\{(x+\delta)s\}]\,ds\end{aligned}\right\} \quad . \quad . \quad (44)$$

wegen (28).

---

[22] Man beachte hierbei, dass das Integral

$$\int\limits_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,K_\nu(st)\,s\,ds$$

rein imaginär ist.

Das Integral mit $\varphi_2(t)$ kann auf dieselbe Weise behandelt werden.

Nun folgt aus (26), dass das erste Integral auf der rechten Seite von (43) gegen $\frac{1}{2}\pi i F(x+0)$ strebt für $\lambda \to \infty$. Der Beweis von (36) ist also geliefert, sobald gezeigt ist, dass die beiden auf der rechten Seite von (43) stehenden Integrale mit $\varphi_1(t)$ bezw. $\varphi_2(t)$ durch geeignete Wahl von δ beliebig klein gemacht werden können. Ich darf mich hierbei auf $\varphi_1(t)$ beschränken.

Nun ist $\varphi_1(x+\delta) < \varepsilon$ für δ hinreichend klein (siehe (42)). Mit Rücksicht auf (44) brauche ich also nur zu zeigen, dass das Integral

$$\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, K_{\nu+1}(\eta s)\, ds$$

beschränkt ist für alle positive Werte von λ und für $x < \eta \leqq x + \delta$, wobei η von λ abhängig sein darf. Nun sieht man leicht ein, wegen (13), (14) und (15), dass ich nur die Integrale

$$\int_{\beta-\lambda i}^{\beta} \frac{e^{-xs-\eta s}\, ds}{s}, \quad \int_{\beta}^{\beta+\lambda i} \frac{e^{-xs-\eta s}\, ds}{s}, \quad \int_{\beta-\lambda i}^{\beta+\lambda i} \frac{e^{xs-\eta s}\, ds}{s}$$

zu betrachten brauche, und diese Integrale sind bezw. gleich

$$\int_{(\beta-\lambda i)(x+\eta)}^{\beta(x+\eta)} \frac{e^{-\zeta}\, d\zeta}{\zeta}, \quad \int_{\beta(x+\eta)}^{(\beta+\lambda i)(x+\eta)} \frac{e^{-\zeta}\, d\zeta}{\zeta}, \quad \int_{(\beta-\lambda i)(\eta-x)}^{(\beta+\lambda i)(\eta-x)} \frac{e^{-\zeta}\, d\zeta}{\zeta},$$

so dass ich mich auf die Hilfssätze 6 und 5 berufen kann.

Hiermit ist (36) bewiesen.

Formel (37) folgt auf analoge Weise aus (41) und (27).

Beweis von (38). Wegen (16) hat man

$$\left.\begin{aligned} &\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\, ds \int_0^{x-\delta} K_\nu(st)\,(st)^{\frac{1}{2}}\, F(t)\, dt \\ &\qquad = x^{\frac{1}{2}} \int_0^{x-\delta} F(t)\, t^{\frac{1}{2}}\, dt \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\, K_\nu(st)\, s\, ds = U_1 + U_2 - U_3 - U_4, \end{aligned}\right\} \quad (45)$$

falls

$$U_1 = x^{\frac{3}{2}}\, (\beta+\lambda i)\, I_{\nu+1}\{(\beta+\lambda i)\, x\} \int_0^{x-\delta} \frac{K_\nu\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2 - t^2}$$

und

$$U_2 = x^{\frac{1}{2}} (\beta + \lambda i)\, I_\nu \{(\beta + \lambda i)\, x\} \int_0^{x-\delta} \frac{K_{\nu+1} \{(\beta + \lambda i)\, t\}\, F(t)\, t^{\frac{3}{2}}\, dt}{x^2 - t^2}$$

gesetzt wird und U_3 und U_4 die analogen Ausdrücke mit $\beta - \lambda i$ statt $\beta + \lambda i$ bezeichnen.

Ich betrachte nun für $\lambda \rightarrow \infty$ den Ausdruck

$$\left\{ \int_0^{\lambda^{-1}} + \int_{\lambda^{-1}}^{x-\delta} \right\} \frac{K_\nu \{(\beta + \lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2 - t^2} \quad . \; . \; . \; . \; . \; . \quad (46)$$

Wegen

$$K_\nu (z) = \frac{\pi}{2 \sin \nu\pi} \{ I_{-\nu}(z) - I_\nu (z) \}$$

gilt für das Integral von 0 bis λ^{-1}

$$\int_0^{\lambda^{-1}} \frac{K_\nu \{(\beta + \lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2 - t^2} = O \left\{ \lambda^{-\nu} \int_0^{\lambda^{-1}} |\, F(t)\, |\, t^{\frac{1}{2}-\nu}\, dt \right\}$$

$$+ O \left\{ \lambda^{\nu} \int_0^{\lambda^{-1}} |F(t)|\, t^{\frac{1}{2}+\nu}\, dt \right\} = O \left\{ \lambda^{-\frac{1}{2}} \int_0^{\lambda^{-1}} |\, F(t)\, |\, dt \right\} = o\,(\lambda^{-\frac{1}{2}}).$$

Für $t \geqq \lambda^{-1}$ benutze ich (man vergl. (13))

$$K_\nu \{(\beta + \lambda i)\, t\} = \frac{\sqrt{\pi}\, e^{-\beta t - i\lambda t}}{\{2\,(\beta + \lambda i)\, t\}^{\frac{1}{2}}} + O\{(\lambda t)^{-\frac{3}{2}}\};$$

mit Hilfe dieser Formel findet man [23]

$$\int_{\lambda^{-1}}^{x-\delta} \frac{K_\nu \{(\beta + \lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2 - t^2} = \left(\frac{\pi}{2\beta + 2\lambda i} \right)^{\frac{1}{2}} \left\{ \int_0^{x-\delta} - \int_0^{\lambda^{-1}} \right\} \frac{e^{-\beta t}\, F(t)\, e^{-i\lambda t}\, dt}{x^2 - t^2}$$

$$+ O \left\{ \lambda^{-\frac{3}{2}} \int_{\lambda^{-1}}^{\lambda^{-\frac{1}{2}}} |\, F(t)\, |\, t^{-1}\, dt \right\} + O \left\{ \lambda^{-\frac{3}{2}} \int_{\lambda^{-\frac{1}{2}}}^{x-\delta} |\, F(t)\, |\, t^{-1}\, dt \right\}$$

$$= o\,(\lambda^{-\frac{1}{2}}) + O \left\{ \lambda^{-\frac{1}{2}} \int_{\lambda^{-1}}^{\lambda^{-\frac{1}{2}}} |F(t)|\, dt \right\} + O \left\{ \lambda^{-1} \int_{\lambda^{-\frac{1}{2}}}^{x-\delta} |F(t)|\, dt \right\} = o\,(\lambda^{-\frac{1}{2}}).$$

---

23) Das Integral $\int_0^{x-\delta} \frac{e^{-\beta t}\, F(t)\, e^{-i\lambda t}\, dt}{x^2 - t^2}$ strebt gegen Null für $\lambda \rightarrow \infty$ wegen des RIEMANN-LEBESGUEschen Lemmas; man vergl. DOETSCH, [**4**], 50; TITCHMARSH, [**21**], 11.

Die beiden Bestandteile des Ausdrucks (46) sind daher $o(\lambda^{-\frac{1}{2}})$. Da wegen (15)

$$(\beta+\lambda i)\, I_{\nu+1}\{(\beta+\lambda i)\, x\} = O(\lambda^{\frac{1}{2}}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

ist, strebt also U_1 gegen Null für $\lambda \to \infty$.

Genau so zeigt man, dass U_2, U_3 und U_4 dem Grenzwert Null zustreben, womit wegen (45) der Beweis von (38) geliefert ist.

Beweis von (39). Im Integral

$$\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\, ds \int_{x+\delta}^{\infty} K_\nu(st)\,(st)^{\frac{1}{2}}\, F(t)\, dt$$

darf man die Integrationsfolge vertauschen; für $t > 0$ und $s = \beta + iy$ mit $-\lambda \leqq y \leqq \lambda$ gilt nämlich [24])

$$|\, K_\nu(st)\,(st)^{\frac{1}{2}}\,| \leqq \sqrt{\frac{\pi}{2}}\, e^{-\beta t},$$

so dass das Integral nach t wegen der zweiten Voraussetzung gleichmässig in s konvergiert. Nach der Vertauschung der Integrationsfolge bekommt man (man vergl. (16) und (45))

$$\int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(xs)\,(xs)^{\frac{1}{2}}\, ds \int_{x+\delta}^{\infty} K_\nu(st)\,(st)^{\frac{1}{2}}\, F(t)\, dt = V_1 + V_2 - V_3 - V_4, \quad (48)$$

falls

$$V_1 = x^{\frac{3}{2}}\,(\beta+\lambda i)\, I_{\nu+1}\{(\beta+\lambda i)\, x\} \int_{x+\delta}^{\infty} \frac{K_\nu\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2-t^2}$$

und

$$V_2 = x^{\frac{1}{2}}\,(\beta+\lambda i)\, I_\nu\{(\beta+\lambda i)\, x\} \int_{x+\delta}^{\infty} \frac{K_{\nu+1}\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{3}{2}}\, dt}{x^2-t^2}$$

gesetzt wird und V_3 und V_4 die analogen Ausdrücke mit $\beta - \lambda i$ statt $\beta + \lambda i$ bezeichnen.

Nun hat man für $\lambda \to \infty$ mit Rücksicht auf (13)

$$\int_{x+\delta}^{\infty} \frac{K_\nu\{(\beta+\lambda i)\, t\}\, F(t)\, t^{\frac{1}{2}}\, dt}{x^2-t^2} = \left(\frac{\pi}{2\beta+2\lambda i}\right)^{\frac{1}{2}} \int_{x+\delta}^{\infty} \frac{e^{-\beta t}\, F(t)\, e^{-i\lambda t}\, dt}{x^2-t^2}$$

$$+\, O\left\{\lambda^{-\frac{3}{2}} \int_{x+\delta}^{\infty} e^{-\beta t}\, |\, F(t)\, |\, t^{-1}\, dt\right\} = o(\lambda^{-\frac{1}{2}})$$

wegen des RIEMANN-LEBESGUEschen Lemmas.

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[24]) Man vergl. Fussnote [3]).

V_1 ist also $o(1)$ (siehe auch (47)). Ebenso zeigt man, dass auch V_2, V_3 und V_4 gegen Null streben für $\lambda \to \infty$, womit wegen (48) der Beweis von (39) erbracht ist.

§ 4. Beweis von Satz 2.

Wegen der Voraussetzung 2 ist

$$\left.\begin{aligned} \sqrt{\frac{2}{\pi}} \int_0^b K_\nu(st)(st)^{\frac{1}{2}} F(t)\,dt &= \frac{1}{\pi i} \int_0^b K_\nu(st)(st)^{\frac{1}{2}}\,dt \lim_{\lambda\to\infty} \int_{\beta-\lambda i}^{\beta+\lambda i} I_\nu(tz)(tz)^{\frac{1}{2}} f(z)\,dz \\ &= \frac{1}{\pi i} \lim_{\lambda\to\infty} \int_{\beta-\lambda i}^{\beta+\lambda i} f(z)\,dz \int_0^b K_\nu(st)(st)^{\frac{1}{2}} I_\nu(tz)(tz)^{\frac{1}{2}}\,dt \\ = \frac{1}{\pi i} \lim_{\lambda\to\infty} \int_{\beta-\lambda i}^{\beta+\lambda i} & \frac{\{z I_{\nu+1}(zb)(zb)^{\frac{1}{2}} K_\nu(sb)(sb)^{\frac{1}{2}} + s I_\nu(zb)(zb)^{\frac{1}{2}} K_{\nu+1}(sb)(sb)^{\frac{1}{2}}\} - s^{\frac{1}{2}-\nu} z^{\frac{1}{2}+\nu}}{z^2-s^2} f(z)\,dz \end{aligned}\right\} \quad (49)$$

wegen (16).

Nun hat man mit Rücksicht auf (13), (14) und (15) für $\Re(s) > \beta > 0$, $z = \beta + iy$ und $b \geqq 1$

$$|K_\nu(sb)(sb)^{\frac{1}{2}}| < De^{-\beta b}, \quad |K_{\nu+1}(sb)(sb)^{\frac{1}{2}}| < De^{-\beta b},$$
$$|I_\nu(zb)(zb)^{\frac{1}{2}}| < De^{\beta b}, \quad |I_{\nu+1}(zb)(zb)^{\frac{1}{2}}| < De^{\beta b},$$

wo D unabhängig von b ist; es gilt daher

$$|z I_{\nu+1}(zb)(zb)^{\frac{1}{2}} K_\nu(sb)(sb)^{\frac{1}{2}} + s I_\nu(zb)(zb)^{\frac{1}{2}} K_{\nu+1}(sb)(sb)^{\frac{1}{2}}| < D^2(|z| + |s|).$$

Der Integrand auf der rechten Seite von (49) ist also [25]) absolut genommen $< \dfrac{E\,|f(z)|}{|z|}$ für alle Werte von $z = \beta + iy$ mit $|y| \geqq L$; hierin sind E und L nicht von b abhängig.

Das auf der rechten Seite von (49) stehende Integral ist daher mit Rücksicht auf der dritten Voraussetzung für alle Werte von $b \geqq 1$ gleichmässig in b konvergent, so dass man b unter dem Integralzeichen gegen ∞ streben lassen darf. Das Resultat wird wegen (13), (14) und (15), da $\Re(s) > \beta$ ist,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty K_\nu(st)(st)^{\frac{1}{2}} F(t)\,dt = \frac{s^{\frac{1}{2}-\nu}}{\pi i} \lim_{\lambda\to\infty} \int_{\beta-\lambda i}^{\beta+\lambda i} \frac{z^{\frac{1}{2}+\nu} f(z)\,dz}{s^2-z^2} \quad . \quad . \quad . \quad (50)$$

Ich betrachte nun das Integral

$$\frac{s^{\frac{1}{2}-\nu}}{\pi i} \int \frac{z^{\frac{1}{2}+\nu} f(z)\,dz}{z^2-s^2},$$

---

[25]) Man beachte auch, dass $\Re(\nu) \leqq \frac{1}{2}$ ist.

im positiven Sinne erstreckt über den Rand des Rechtecks mit den Ecken $\beta+\lambda i$, $\beta-\lambda i$, $\lambda-\lambda i$, $\lambda+\lambda i$. Nach dem Residuensatze besitzt dieses Integral für hinreichend grosses λ den Wert $f(s)$.

Nun kann man beweisen [26] unter Benutzung der vierten und fünften Voraussetzung, dass die auf die Seiten $(\lambda-\lambda i, \lambda+\lambda i)$, $(\lambda+\lambda i, \beta+\lambda i)$ und $(\beta-\lambda i, \lambda-\lambda i)$ bezüglichen Integrale gegen Null streben für $\lambda \to \infty$.

Die rechte Seite von (50) hat daher den Wert $f(s)$, womit der Beweis von Satz 2 geliefert ist.

§ 5. Schlussbemerkung.

Verschiedene Spezialfälle der Relationen (8) und (9) kommen vor in meinen früheren Arbeiten. Ich werde fünf Beispiele anführen:

Erstes Beispiel [27]:

$$H_\nu^{(1)}(z)\,H_\nu^{(2)}(z)=\frac{8z}{\pi}\int_0^\infty K_{2m}(2zu)\,P_{\nu-\frac{1}{2}}^{m}\left(\sqrt{1+u^2}\right)P_{\nu-\frac{1}{2}}^{-m}\left(\sqrt{1+u^2}\right)u\,du$$
$$\left[|\arg z|<\tfrac{1}{2}\pi;\ |\Re(m)|<1\right].$$

$$P_{\nu-\frac{1}{2}}^{m}\left(\sqrt{1+u^2}\right)P_{\nu-\frac{1}{2}}^{-m}\left(\sqrt{1+u^2}\right)=\frac{1}{2i}\int_{\beta-\infty i}^{\beta+\infty i} I_{2m}(2uz)\,H_\nu^{(1)}(z)\,H_\nu^{(2)}(z)\,dz \qquad [u>0].$$

Zweites Beispiel [28]:

$$\left.\begin{aligned}&P_{\nu-\frac{1}{2}}^{m}\left(\sqrt{1+z^2}\right)P_{\nu-\frac{1}{2}}^{-m}\left(\sqrt{1+z^2}\right)=\frac{4\cos\nu\pi}{\pi}\int_0^{\infty e^{-i\arg z}} K_{2\nu}(2v)\,J_m(zv)\,J_{-m}(zv)\,dv\\&\qquad\qquad\left[|\arg z|<\tfrac{1}{2}\pi;\ |\Re(\nu)|<\tfrac{1}{2}\right].\end{aligned}\right\}\ (51)$$

$$\left.\begin{aligned}&J_m(\zeta)\,J_{-m}(\zeta)=\frac{1}{i\cos\nu\pi}\int_{\beta-\infty i}^{\beta+\infty i} I_{2\nu}(2v)\,P_{\nu-\frac{1}{2}}^{m}\left(\sqrt{1+\frac{\zeta^2}{v^2}}\right)P_{\nu-\frac{1}{2}}^{-m}\left(\sqrt{1+\frac{\zeta^2}{v^2}}\right)dv\\&\qquad\qquad\left[\zeta>0;\ \nu\neq\pm\tfrac{1}{2},\pm\tfrac{3}{2},\pm\tfrac{5}{2},\ldots\right].\end{aligned}\right\}\ (52)$$

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[26] Man vergl. einen analogen Beweis in dem entsprechenden DOETSCH-CHURCHILLschen Satz: DOETSCH, [**4**], 127—128; CHURCHILL, [**3**], 570—571.

[27] [**14**], Formel (76) und [**11**], Formel (24).

[28] [**13**], Formel (12) und [**10**], Formel (19).

(51) und (52) gehen über in Formeln der Gestalt (8) und (9), wenn man $z=\frac{1}{s}$ und $v=\frac{1}{2}st$ substituiert in (51) und überdies ζ durch $\frac{1}{2}t$ und v durch $\frac{1}{2}ts$ ersetzt in (52).

Drittes Beispiel [29]:

$$K_{2m}(z\,e^{\frac{1}{2}\pi i})\,K_{2m}(z\,e^{-\frac{1}{2}\pi i})=\frac{1}{z^2}\int\limits_0^{\infty\,e^{2i\arg z}} K_{2k}(v)\,W_{k,m}\left(\frac{z^2}{2v}\right)W_{-k,m}\left(\frac{z^2}{2v}\right)v\,dv, \qquad (53)$$

$$W_{k,m}(\tfrac{1}{2}\zeta)\,W_{-k,m}(\tfrac{1}{2}\zeta)=\frac{\zeta}{\pi i}\int\limits_{\beta-\infty i}^{\beta+\infty i} I_{2k}(v)\,K_{2m}(\zeta^{\frac{1}{2}}v^{\frac{1}{2}}e^{\frac{1}{4}\pi i})\,K_{2m}(\zeta^{\frac{1}{2}}v^{\frac{1}{2}}e^{-\frac{1}{4}\pi i})\,dv. \qquad (54)$$

Viertes Beispiel [30]:

$$G^{4,0}_{0,4}(\tfrac{1}{16}s^2\,|\,a,q,\tfrac{1}{2}\nu,-\tfrac{1}{2}\nu)=2^{3-a-q}\,s^{a+q}\int\limits_0^{\infty} K_{a-q}(st)\,K_\nu\left(\frac{1}{t}\right)t^{a+q-1}\,dt, \qquad (55)$$

$$K_\nu\left(\frac{1}{t}\right)=\frac{2^{a+q-3}\,t^{2-a-q}}{\pi i}\int\limits_{\beta-\infty i}^{\beta+\infty i} I_{a-q}(ts)\,G^{4,0}_{0,4}(\tfrac{1}{16}s^2\,|\,a,q,\tfrac{1}{2}\nu,-\tfrac{1}{2}\nu)\,s^{1-a-q}\,ds. \qquad (56)$$

Das fünfte Beispiel [31] ist ein Spezialfall des vierten $(a=\frac{1}{2}\nu+\frac{1}{2}, q=-\frac{1}{2}\nu+\frac{1}{2})$

$$K_{2\nu}(2s^{\frac{1}{2}})=\frac{s}{\pi}\int\limits_0^{\infty} K_\nu(st)\,K_\nu\left(\frac{1}{t}\right)dt, \;.\;.\;.\;.\;.\; (57)$$

$$K_\nu\left(\frac{1}{t}\right)=\frac{t}{i}\int\limits_{\beta-\infty i}^{\beta+\infty i} I_\nu(ts)\,K_{2\nu}(2s^{\frac{1}{2}})\,ds. \;.\;.\;.\;.\;.\; (58)$$

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[29] **[12]**, 26 und **[9]**, 875, Formel (12). In (53) ist $|\arg z|<\frac{1}{4}\pi$; (54) gilt für $|\arg\zeta|<\frac{1}{2}\pi$.

[30] **[12]**, 24, Satz 8 und **[15]**, Formel (5). Für die Definition der Funktion $G^{4,0}_{0,4}$ vergl. man **[12]**, Formel (6). In (55) und (57) ist $|\arg s|<\frac{1}{2}\pi$; in (56) und (58) wird $t>0$ vorausgesetzt.

[31] Man vergl. HARDY, **[7]**, Formel (4.1) und MEIJER, **[15]**, Formel (35).

Mathematics. — *On* LAMBERT'S *proof for the irrationality of* π. By J. POPKEN. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of May 25, 1940.)

In their beautiful book "An introduction to the theory of numbers" HARDY and WRIGHT state, there is no simple proof for the irrationality of π [1]. Yet, if LAMBERT's classical proof [2] is freed from the continued fraction algorithm, it takes a surprisingly easy form [3]. We only require the following lemma:

Lemma: Let $x \neq 0$. For $h = 0, 1, 2, \ldots$ there are polynominals $p_h(x^{-1})$ and $q_h(x^{-1})$ in x^{-1} with integral coefficients and of degree at most $2h$, so that

$$\left.\begin{aligned} & p_h(x^{-1}) \sin x + q_h(x^{-1}) \cos x = \\ & = (-2)^h \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)\ldots(n+h)}{(2n+2h+1)!} x^{2n+1} \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

Proof: a. $h = 0$. Take $p_0(x^{-1}) \equiv 1$, $q_0(x^{-1}) \equiv 0$, then (1) is identical with

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

b. Supposing (1) to be true for h, then we prove, that (1) also holds for $h + 1$ instead of h.

In fact, if (1) is true, then

$$x^{-1} p_h(x^{-1}) \sin x + x^{-1} q_h(x^{-1}) \cos x =$$

$$= (-2)^h \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)(n+2)\ldots(n+h)}{(2n+2h+1)!} x^{2n}$$

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[1] p. 39 and p. 47.

[2] J. H. LAMBERT, Mémoire sur quelques propriétés remarquables des quantités transcendantes circulaires et logarithmétiques. Histoire Acad. roy. des sciences et belles lettres. Berlin. Année 1761 (1768), p. 265—322.

[3] Nearly the same idea has been applied by HERMITE: Sur quelques approximations algébriques, Oeuvres III, p. 146—149, but HERMITE used an integral.

Differentiation gives

$$\left.\begin{aligned}&\left[\frac{d}{dx}\{x^{-1}\,p_h(x^{-1})\}-x^{-1}\,q_h(x^{-1})\right]\sin x+\\&\qquad\qquad+\left[x^{-1}\,p_h(x^{-1})+\frac{d}{dx}\{x^{-1}\,q_h(x^{-1})\}\right]\cos x\\&=(-2)^h\sum_{n=1}^{\infty}(-1)^n\,\frac{2n\,(n+1)\,(n+2)\ldots(n+h)}{(2n+2h+1)!}\,x^{2n-1}=\\&=(-2)^{h+1}\sum_{\nu=0}^{\infty}(-1)^{\nu}\,\frac{(\nu+1)\,(\nu+2)\ldots(\nu+h+1)}{(2\nu+2h+3)!}\,x^{2\nu+1}\end{aligned}\right\}\quad .\;.\quad(2)$$

Putting

$$\left.\begin{aligned}&\frac{d}{dx}\{x^{-1}\,p_h(x^{-1})\}-x^{-1}\,q_h(x^{-1})=p_{h+1}(x^{-1}),\\&x^{-1}\,p_h(x^{-1})+\frac{d}{dx}\{x^{-1}\,q_h(x^{-1})\}=q_{h+1}(x^{-1}),\end{aligned}\right\}\quad .\;.\;.\;.\;.\quad(3)$$

it is clear, that $p_{h+1}(x^{-1})$ and $q_{h+1}(x^{-1})$ are polynomials in x^{-1} with integral coefficients and of degree at most $2h+2$, since $p_h(x^{-1})$ and $q_h(x^{-1})$ are polynomials, whose coefficients are integers and are of degree $2h$ at most. By (2) and (3)

$$\begin{aligned}&p_{h+1}(x^{-1})\sin x+q_{h+1}(x^{-1})\cos x=\\&\qquad=(-2)^{h+1}\sum_{n=0}^{\infty}(-1)^n\,\frac{(n+1)\,(n+2)\ldots(n+h+1)}{(2n+2h+3)!}\,x^{2n+1}.\end{aligned}$$

Hence part *b.* of our proof is evident.

Proof of LAMBERT's *theorem:* If π is rational, then $\frac{\pi}{4}=\frac{a}{b}$, where a and b are positive integers. Applying the preceding lemma with $x=\frac{\pi}{4}=\frac{a}{b}$, we derive for any integer $h\geqq 0$

$$\begin{aligned}&\left\{p_h\left(\frac{b}{a}\right)+q_h\left(\frac{b}{a}\right)\right\}\tfrac{1}{2}\sqrt{2}=\\&\qquad=(-2)^h\left(\frac{h!}{(2h+1)!}\,\frac{\pi}{4}-\frac{2\,.\,3\ldots(h+1)}{(2h+3)!}\left(\frac{\pi}{4}\right)^3+\ldots\right).\end{aligned}$$

In the alternating series between brackets the terms decrease in absolute value and tend to zero. Hence

$$2^h\left(\frac{h!}{(2h+1)!}\,\frac{\pi}{4}-\frac{2.3\ldots(h+1)}{(2h+3)!}\left(\frac{\pi}{4}\right)^3\right)<$$

$$<\tfrac{1}{2}\sqrt{2}\left|p_h\left(\frac{b}{a}\right)+q_h\left(\frac{b}{a}\right)\right|<2^h\,\frac{h!}{(2h+1)!}\,\frac{\pi}{4}<\frac{2^h}{h!},$$

$$0<\left|a^{2h}\,p_h\left(\frac{b}{a}\right)+a^{2h}\,q_h\left(\frac{b}{a}\right)\right|<\sqrt{2}\,\frac{2^h\,|\,a\,|^{2h}}{h!}.\;.\;.\;.\quad(4)$$

Since $p_h(x^{-1})$ is a polynomial in x^{-1} with integers as coefficients and of degree say g the number $p_h\left(\frac{b}{a}\right)$ can be written as a fraction with denominator a^g. But $g \leqq 2h$, therefore $a^{2h}\,p_h\left(\frac{b}{a}\right)$ is an integer and the same reasoning shows, that $a^{2h}\,q_h\left(\frac{b}{a}\right)$ also is an integer.

Hence for integral $h \geqq 0$

$$\left|a^{2h}\,p_h\left(\frac{b}{a}\right)+a^{2h}\,q_h\left(\frac{b}{a}\right)\right|$$

is a positive integer. But now (4) leads to a contradiction, since

$$\lim_{h\to\infty}\sqrt{2}\,\frac{2^h\,|\,a\,|^{2h}}{h!}=0.$$

Botany. — *Nitrate Assimilation of Aspergillus niger* VAN TIEGHEM. By S. DE BOER. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of May 25, 1940.)

Introduction.

The object of this investigation was the study of nitrate metabolism of Aspergillus niger, for the purpose of ascertaining:

a. whether nitrate, after being taken up, is accumulated by the fungus or is directly used in metabolism, and

b. the influence of the pH on these processes.

Further, the influence of glucose was investigated, as it might be expected that glucose will have a similar effect in the case of Aspergillus as of Avena roots where the nitrate uptake und nitrate reduction is increased by the addition of glucose, as POSTMA (7) has proved.

For this investigation the nitrate had to be quantitatively determined in the fungus as well as in the solution. According to the literature the determination of nitrate by the diphenylamine and phenoldisulphonic acid methods has not yielded satisfactory results. ITZEROTT (5) and BÜNNING (4) used the brucine method in determining the nitrate in a culture solution, whereas in ascertaining the mycelium only qualitative observations were made by the diphenylaminesulphuric acid method. Respecting the brucine method, according to ALTEN and WEILAND (1), account must be taken of great variations (as much as 25 %), and although ITZEROTT and BÜNNING state that they have obtained more favourable results, it will be advisable to employ a better method. The xylenol method, first applied by BLOM and TRESCHOW (3) to soil and plants, is a more exact nitrate determination which has been employed in different researches of recent years. This method, which I followed in determining the nitrate in the culture solution and, in a modified form, also for fungi, yielded satisfactory results.

In physiological experiments concerning nitrate assimilation by fungi, the usual procedure is as follows. The fungi are reared on a medium containing the substance to be examined, in this case nitrate. After some days the quantity still remaining of the nitrate first added to the medium is then ascertained. The difference is caused by the nitrate accumulated in the fungus and the nitrate taken up in metabolism partly for building up new protoplasm. In these circumstances it is not possible to separate these two processes. For this reason I employed fungus mats which, moreover, were starved on water. This method, which has been employed by VAN WAESBERGHE (8) among others, and which suits the fungus, as his experiments show, has the advantage that the growth-phase is separated from the investigation-phase. Moreover, by starving, the reserve substances which might affect the nitrate metabolism, are used up.

§ 1. *Experimental Methods.*

a. *Culture method.*

In a 100 ml Erlemeyer-flask containing 37 ml nitrate-free Raulin-solution, a spore suspension of Aspergillus niger is sown. The fungus is reared for 2 days, after which the mats are starved on water. After one day the water is replaced by a known nitrate solutioa in which, after some time, the nitrate which has not been taken up by the fungus, is determined.

b. *Nitrate determination.*

The nitrate determination in the solution was carried out, with slight modifications, after the directions given by ALTEN et al. (2) and WERR (9):

To 5 ml solution (which must not contain more than 4 mg NO_3) 25 ml 75 vol. % nitrate-free H_2SO_4 is added and, after cooling to about 25°, 3 drops 2 : 4-xylen-1-ol. The flask is closed and well shaken, after which the solution must stand half an hour in the dark, in this time the xylenol is nitrated to nitroxylenol. The solution is then cooled in ice-water and diluted with 100 ml water. The diluted nitroxylenol solution may be kept in the dark till it is distilled with steam. Some nitroxylenol always crystallizes in the condensor; this is removed by shutting off the water of the condensor for some time in order that the crystals may be carried away with the steam. About 70 ml of the distillate is collected in 5 ml 1 % NaOH, in which it forms a yellowish-brown product. Water is added to 100 ml and the extinction of this liquid is determined in a Zeiss Stufenphotometer with S 47 filter.

A standard-curve is made by determining from different nitrate concentrations the respective extinction-values. Between 0.2 and 4.0 mg NO_3 a practically straight line can be obtained. In this way nitrate quantities of 0.2 to 4.0 mg per 5 ml can be determined with 1—2 % of accuracy.

For a nitrate determination in the fungus, it is necessary, according to MCVEY's (6) experiments with meat, to dissolve the nitrate first in water and to remove the protein. This is done in the following way: 2 fungus mats were ground with quartz-sand in a mortar, transferred to a beaker with 20 ml of water, made slightly alkaline with NaOH (indicator litmus) and then heated on a water-bath for an hour. Then this suspension is brought to a volume of 25 ml, and centrifuged. 15 ml of the liquid is acidified with 10 % H_2SO_4 to pH 3.6 (yellow to bromo-cresol green), after which 1 ml 20 % phosphotungstic acid solution is added to precipitate the protein [1]). In a measuring flask this mass was filled up to 25 ml, centrifuged and 5 ml of the liquid was used for the determination of the nitrate. To determine whether the nitrate-quantum has not increased or diminished by these manipulations, a known quantity of nitrate was added to nitrate-free fungus at different stages of the process, and at the end the quantity of nitrate is determined. These values proved to differ from the original by not more than 2 %.

It further appeared that when equal quantities of nitrate are added to fungi all cultivated and starved in the same way, about equal quantities are recovered some days later in the several series (see Table I).

TABLE I. Aspergillus niger, reared for 2 days on a nitrate-free RAULIN-solution, starved for 1 day, thereafter 1 or 2 days on 40 ml phosphate buffer pH 6 with KNO_3. Figures are mg per 1 fungus mat.

Added mg NO_3	25.2	25.2	25.2	25.2
Found in solution after 1 day	23.0	23.8	23.1	23.1
„ „ fungus „ „ „	1.4	1.4	1.4	1.4
„ „ solution „ 2 days	22.8	22.5	22.7	22.7
„ „ fungus „ „ „	0.8	0.6	0.5	0.4

The differences in these four series of 2 fungus mats are slight. This method is therefore well suited for determining nitrate in fungi and their solutions.

§ 2. *Influence of the pH upon the nitrate assimilation.*

a. *With starved fungus mats.*

The fungi, after being starved, are transferred to sterile phosphate buffers, to which nitrate has been added in the form of KNO_3. The pH changes but little and is restored to the original value again by the addition of phosphoric acid or NaOH. The results of the various experiments have been collected in Table II.

The table shows that nitrate has been taken up by the fungus from the solutions. In the fungus there is always a very small quantity of nitrate and thus no question of nitrate accumulation. No distinct connection can be observed between the pH and the quantum NO_3 taken up.

ITZEROTT (5) and BÜNNING (4) concluded from their experiments that the nitrate assimilation is dependent upon the pH, and observed a nitrate accumulation in the cells if the pH of the solution is lower than 3. But, as

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[1] In this case the protein was not precipitated with trichloro-acetic acid, as indicated by GORTER (10), because Cl has a disturbing effect in the xylenol method for nitrate determination.

TABLE II. Aspergillus niger, reared for 2 days on nitrate-free RAULIN-solution, 1 day starved, thereafter 1 or 2 days on 40 ml phosphate buffer of different pH, with nitrate, the original quantity of which is given in mg. The quantities recovered given in percentages of the quantum added. Figures per 1 fungus mat.

Exp. Nr	Duration of experiment after starvation	Original nitrate quantity mg	Nitrate recovered in $^0/_0$ of original quantity	pH 2	pH 4	pH 6	pH 8
26	1 day	45.2	in solution	98.0	91.8	96.5	96.0
			in fungus	0.4	1.8	1.8	1.3
27	1 day	48.3	in solution	98.1	95.8	93.2	
			in fungus	1.8	2.9	5.3	
	2 days		in solution	93.8	97.5	93.8	
			in fungus	2.5	0.2	0.4	
28	2 days	58.5	in solution	84.5	93.2	92.2	
			in fungus	0.8	0.8	0.6	
31	2 days	55.2	in solution	80.0	82.2	86.6	86.1

was remarked above, these authors made no quantitative determinations in the fungus mats, but merely determined nitrate in the buffer solution in experiments with non-starved fungi.

b. *With growing fungi.*

In the case of non-starved, or still growing, fungus mats, nitrate is much more used in metabolism than in the case of starved fungi. Here the pH may have some influence; some experiments were therefore carried out respecting the nitrate assimilation with growing fungi. For this purpose the fungi were reared at least 3 days in Raulin-solutions of different pH to which 5 g KNO_3 per l, had been added. During the growth it was observed that the fungi at high pH formed but very thin mats. After a few days the nitrate quantum in the solutions and fungi, as also the dry-weights, were determined. (See Table III.)

TABLE III. (Exp. nr. 30). Aspergillus niger, reared on 37 ml nitrate containing RAULIN-solution of different pH. Original quantity of nitrate 108.9 mg. Recovered quantities of nitrate in percentages of the original. Figures per 1 fungus mat.

Duration of experiment	Recovered nitrate in $^0/_0$	pH 2	pH 4	pH 6	pH 8
4 days	in solution	54.0	12.0	26.6	75.8
	in fungus	0.0	0.0	0.0	0.0
5 days	in solution	0.0	0.0	0.0	10.5
	in fungus	0.1	0.0	0.0	0.0
4 days	dry weight in mg	184	267	273	117
5 days	dry weight in mg	387	517	457	344

It follows from this experiment that nitrate is quickest taken up by the fungi at pH 4. Nowhere has nitrate been found in the fungus, it is therefore not accumulated. The nitrate when taken up is at once metabolised, and this also takes place most rapidly at pH 4. Likewise the dry-weight is greatest at pH 4. We may thus state that the growth of the fungi is dependent upon the pH and that parallel to this is the nitrate assimilation (the uptake, accumulation and further transformation of nitrate together).

§ 3. *Influence of glucose.*

In contrast with § 2 where either a certain quantum of nitrate was used with starved fungi, or where non-starved fungi were reared on solutions containing nitrate and glucose, in the following experiments both nitrate and glucose were added to the starved fungus mats. By this method it could be ascertained whether carbohydrates are necessary as a source of energy in nitrate assimilation. To bring out the effect of glucose in the two series of experiments given below, different quantities of glucose were used. (See Table IV.)

TABLE IV. Aspergillus niger, reared for 2 days on nitrate-free RAULIN-solution, 1 day starved; thereafter 1 or 2 days on 40 ml phosphate buffer of different pH with nitrate and glucose. The nitrate quantum originally added in mg. The quantities recovered in percentages of the quantum added. Figures per 1 fungus mat.

Exp. Nr	Duration of experiment after starvation	Glucose added mg	Original quantity of nitrate mg	Nitrate recovered in $^0/_0$ of original quantity	pH 2	pH 4	pH 6	pH 8
28	2 days	300	58.5	in solution	47.5	54.9	55.0	
				in fungus	0.6	0.2	0.2	
31	1 day	30	55.2	in solution	83.0	85.7	88.4	86.6
	2 days			in solution	71.5	82.4	82.2	79.1

Compare with these the corresponding values without glucose from Table II, experiment Nrs. 28 and 31.

It appears from these experiments that the addition of glucose promotes the assimilation of nitrate, for in this case more nitrate is taken up and metabolised. There is no accumulation here either, as can be seen from the first experiment. At pH 2 the influence of glucose in these cases is slightly greater.

The dry-weights of the fungi, with or without glucose show at different pH practically no differences. In contrast to the non-starved (§ 2 *b*), these fungi have been reared on a solution of the same pH, and the substances added after starvation have apparently had little influence on their dry-weight.

The influence of glucose is particularly noticeable when a larger quantity is added, for it appears that more glucose has a greater influence in promoting the nitrate metabolism. A similar influence of glucose was seen in the experiments of POSTMA (7) with Avena roots. A determination of the respiration in the case of Aspergillus will have to decide whether here, where the addition of glucose promotes the nitrate assimilation, there also will be an increased respiration.

In conclusion I wish to express my thanks to Dr. A. W. H. VAN HERK, and Mr. A. GORTER for their interest in this research.

Summary.

1. The xylenol method for determining the nitrate in fungi and their culture solutions yields satisfactory results.

2. Nitrate is not accumulated in Aspergillus niger, but is metabolised after uptake.

3. In the case of starved fungus mats, the pH is not seen to have any effect upon the nitrate assimilation.

4. In growing fungi the nitrate assimilation is greatest at pH 4 in connection with their growth.

5. Nitrate assimilation is increased by addition of glucose.

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Amsterdam, Laboratory of plant physiology
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Botany. — *Amino acid Breakdown by Aspergillus niger.* By A. GORTER. (Communicated by Prof. J. C. SCHOUTE.)

(Communicated at the meeting of May 25, 1940.)

Introduction.

Little is known respecting the way in which amino acids are broken down by fungi. VAN WAESBERGHE (11) demonstrated that l(—)aspartic acid can be deaminated by Aspergillus niger, in which process oxygen is taken up, and MOOI (8) found that dl-alanine can also be deaminated by this fungus.

On the other hand, of recent years a number of enzymes have been found, which play a part in the amino acid breakdown by animal tissues and bacteria. The most important of these enzymes are given in Table I.

TABLE I.

Name	Reaction	Experimental conditions	Occurrence	Author
Aspartase I and II	l(—)aspartic acid $\rightleftarrows$ NH_3 + fumaric acid	Opt. pH: 6.0—7.5 Temp.: 37° anaerobic	Various bacteria	VIRTANEN and TARNANEN (10) GALE (4)
d-amino-oxidase	d-amino acid + $\frac{1}{2}$ O_2 $\rightarrow$ NH_3 + keto acid	Opt. pH: 8.5—9.0 Temp.: 37° aerobic	Kidney and liver extracts	KREBS (7) WARBURG and CHRISTIAN (12)
l-amino-oxidase	l-amino acid + $\frac{1}{2}$ O_2 $\rightarrow$ NH_3 + keto acid	Opt. pH: 7.4 Temp.: 37° aerobic	Slices of liver and kidney; bacteria	KREBS (7)
glutamic acid dehydrogenase	l(+)glutamic acid + $H_2O - 2H \rightleftarrows NH_3$ + keto-glutaric acid	Opt. pH: 7.4 Temp.: 30° with as H_2 acceptor: methylene blue	Extr. of animal and plant tissues; bacteria; yeast	VON EULER et al. (3)
aminopherase	l-amino acid' + keto acid' $\rightleftarrows$ keto acid'' + l-amino acid''	Opt. pH: 7.4 Temp.: 37° aerobic and anaerobic	extr. of muscle tissue; bacteria; plant tissue	BRAUNSTEIN and KRITZMANN (1, 2)

I have now investigated whether one or more of these enzymes might be of importance in the breakdown of amino acids by Aspergillus niger. A more detailed survey of the results obtained will be published shortly (5).

§ 1. *Methods.*

All the experiments were carried out with *starved fungus mats.* These are mats which are reared on a culture-solution (RAULIN's solution was used in every case), and then placed on water for 10—20 hours. By this method not only the synthetic reactions were suppressed as far as possible and the breakdown reactions were promoted, but, moreover, starved mats can decompose amino acids more easily than when not starved, as VAN WAESBERGHE (l.c.) has demonstrated.

After the starvation, the water used for this was replaced by a phosphate buffer to which was added the amino acid to be examined, that had been brought to the same pH beforehand.

The culture of the mats, the starvation and the experiment itself was carried out in flasks in which the quantity of oxygen absorbed, and of carbon dioxide produced, could be determined at the same time, during the experiment. For this purpose a certain quantity of alkali was put into the tube A of the flask employed (Fig. 1), in which the carbon dioxide was absorbed which could be determined after the experiment. During the experiment the oxygen uptake caused a decrease in volume, which was measured in a gas-burette connected with the flask.

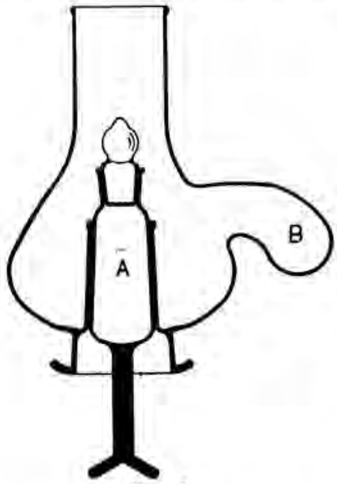

Fig. 1.

The substrate solution could be added when desired from the side-tube B, as is also usual with the Warburg method. The volume of the flasks was about 100 ml; during the experiment they were shaken in a thermostate at 25°, after being filled with O_2. A number of fungus mats was always investigated at the same time in order to eliminate the physiological variability as far as possible.

The starved mats show on a phosphate buffer also a respiration of their own; this must be subtracted from the respiration on phosphate buffer plus substrate in order to obtain the actual O_2 uptake and CO_2 production brought about by the substrate.

§ 2. *Deamination of aspartic acid.*

In the first place I investigated the influence of the pH on the rate of deamination. For this purpose the starved mats were brought on phosphate buffers of different pH, to which l(—)aspartic acid had been added to a final concentration of 0.025 mol. As a measure of the deamination rate, the quantity of NH_3 produced during the first 4 hours was determined. It appears now, from Table 2, that the results of the various experiments differ considerably, but that there is a distinct optimum at pH 2—4. For practical reasons the following experiments were always carried out at pH 4.5—5.0.

TABLE 2. Aspergillus mats reared for 2 days and thereafter starved for 12 hours. For pH 1.0 and 1.5: KCl/HCl buffer; pH 2.0 and 4.0: KH_2PO_4/H_3PO_4 buffer; pH 6.0 and 8.0: Na_2HPO_4/KH_2PO_4 buffer, always 0.1 mol. + 0.025 mol. l(—)aspartic acid. With experiment 63 and 65 10 ml buffer, and shaken in O_2 atmosphere; with experiment 66 20 ml buffer, not shaken, in air atmosphere. Temperature 25°. The figures show mg NH_3-N per mat, formed after 4 hours.

pH / Exper. nr.	1.0	1.5	2.0	4.0	6.0	8.0
63	—	—	0.83	0.59	0.43	0.0
65	0.0	—	1.92	0.91	—	—
66	0.0	0.42	1.40	1.28	0.83	0.21

It is remarkable that the deamination proceeds so rapidly at so low a pH of the buffer solution, whereas the deamination enzymes described, all have a much higher pH optimum (Cf. Table 1). It is true that the pH *in* the cells is different, and probably higher, than in the surrounding buffersolution, while permeability and further transformations of the deamination product also play a part. Yet it is improbable that the pH, *in* the cell, will be near 7.5, for at this pH of the surrounding solution there is hardly any more deamination at all. This is an argument against the activity of the enzymes above-mentioned.

When the deamination and oxygen uptake were measured during the experiment it appeared that at first amino acid disappeared without much O_2 being taken up. Not until later is there a relative increase in the O_2 uptake (Table 3); and after complete deamination approx. 3—4 atoms of oxygen were taken up; VAN WAESBERGHE (l.c.) had already found a value of about 3 atoms. If the breakdown process be continued longer

TABLE 3. Aspergillus mats, 17 hours starved, thereafter on 10 ml phosphate buffer pH 5.0 with 0.024 mol. l(—)aspartic acid. After various lengths of time amino acid determined in solution. O_2 atmosphere. Temperature 25°. Figures mg per fungus mat.

Time in hours	Aspartic acid present mg	Aspartic acid disappeared mg	O_2 uptake ml 0°/76 cm	O_2 uptake in at. p. mol. asp. acid disappeared
0	31.7	—	—	—
2	23.7	8.0	0.53	0.8
4	15.9	15.8	2.24	1.7
6	12.9	18.8	4.37	2.8
10	5.8	25.9	6.67	3.0
24	2.3	29.4	9.82	4.0

the oxygen uptake proceeds but slowly, so that when 3—4 at. of oxygen per mol. aspartic acid has been taken up, the oxidation may be regarded as ended, the CO_2 formed being then approx. 2—2.6 mol. There is thus never complete oxidation. Moreover, the experiment shows that at first deamination proceeds more rapidly that the subsequent oxidation of the deamination product.

§ 3. *Deamination of other amino acids.*

With respect to a number of other amino acids I now investigated whether they were deaminated under the same experimental conditions as aspartic acid, i.e. at pH 4.5—5.0 and in O_2 atmosphere. This proved to be the case with glycocoll, d(—)- and l(+)alanine; dl-aspartic acid; l(+)glutamic acid, dl-leucine and dl-isoleucine; other amino acids were not investigated. None of the amino acids mentioned was oxidised completely, as is shown in Table 4. The deamination itself was complete, except in the case of dl-leucine and dl-isoleucine.

TABLE 4. Starved Aspergillus mats on phosphate buffer pH 4.5 — 5.0 + 0.025 mol. amino acid. O_2 uptake and CO_2 output calculated per mol. transformed amino acid. Averages of a number of experiments.

Substrate	O_2 uptake in at. p. mol. substrate	CO_2 production in mol. p. mol. substrate
l (—) aspartic acid	3.4	2.2
dl-aspartic acid	3.0	—
l (+) alanine	3.4	—
d (—) alanine	3.0	1.5
dl-alanine	3.1	—
glycocoll	1.5	0.8
l (+) glutamic acid	5.3	2.5
dl-leucine	5.1	—
dl-isoleucine	4.3	—

With the amino acids examined the deamination proceeded as a rule in the same way as with aspartic acid, i.e., at first the amino acid disappeared, while but little oxygen is taken up; later the oxygen uptake increased proportionately. The value found for the oxygen uptake and the carbon dioxide output are no indication respecting the way in which the amino acids are broken down or respecting the enzymes which are active thereby.

This last was now further studied by examining what was the first product of the deamination.

§ 4. *The enzymes which play a part in the deamination.*

When *aspartase* plays a part in the aspartic acid breakdown the deamination must also take place anaerobically, and fumaric acid must be demonstrable. This was not the case, however; in nitrogen atmosphere there is no deamination at all. It is possible that the permeability will be much reduced in nitrogen atmosphere and the aspartic acid assimilation therefore retarded, since Aspergillus is a highly aerobic organism. In an aerobic experiment any fumaric acid formed may disappear, however, by oxidation, and therefore Na-arsenite was added to inhibit the oxidation. Fumaric acid was determined according to the method of HAHN and HAARMANN (6) by precipitation as mercuric salt, but in not a single experiment could fumaric acid be demonstrated as a product of the aspartic acid breakdown.

From this it may be concluded that aspartase does not play any part in the breakdown of aspartic acid by Aspergillus.

In the same way I investigated whether the keto acids, which should be produced by the oxidative deamination by means of *amino-oxidases*, could be demonstrated. With aspartic acid the keto acid is oxalo-acetic acid, and with alanine pyruvic acid, for both substances there is a very sensitive colour reaction, as worked out by STRAUB (9). To inhibit a further oxidation of the keto acids Na-arsenite was again used; this substance, according to KREBS (l.c.) does not inhibit the deamination itself. It now appeared that, as a product of the deamination of l(—)aspartic acid, oxalo-acetic acid could never be demonstrated, but pyruvic acid was, although in very small quantities. This was also the case with the inhibited oxidation of dl-alanine; it was not investigated in the case of the other amino acids. Pyruvic acid was also produced, however, with the basal metabolism of the Aspergillus mats, when Na-arsenite was added.

According to these experiments it is thus improbable that an amino-oxidase plays any part, since the keto acids would then be produced in larger quantities. As further argument against the action of an oxidase is that the l- and the d-component of aspartic acid and alanine are deaminated exactly alike (Cf. Table 4), whereas for the oxidation of the d-component another enzyme must be active than for the l-component. The d-amino-oxidase is also active in tissue *extracts*, but it appeared that a ground Aspergillus mat was no more able to deaminate the d-component of aspartic acid and alanine than the l-component. Finally it appeared that glycocoll was easily deaminated, whereas this substance is not affected by the amino-oxidases.

The conclusion from the above experiments is, thus, that an amino-oxidase does not occur in Aspergillus niger, or in any case, does not play any part in the amino acid oxidation under the experimental circumstances.

This oxidation is closely connected with the cell-respiration, for with all the methods by which this latter is inhibited or suppressed (addition of Na-arsenite, grinding, working in N_2 atmosphere) the amino acid oxidation is also inhibited, so that the amino acids are probably oxidised with the help of the ordinary cell-respiration enzymes.

That the *glutamic acid dehydrogenase* might play a part in the oxidation of l(+)glutamic acid is very improbable, for a ground fungus mat cannot oxidise this amino acid with methylene blue as H_2 acceptor. Now, it is very well possible that Aspergillus contains the apodehydrase which is activated by the addition of co-dehydrase I or II, as VON EULER et al. (3) found in a number of animal and plant tissues, but this implies nothing after all regarding the action of the enzyme under my experimental conditions, which are very favourable for the fungus used.

Lastly I investigated whether an aminopherase might be active, that is, whether a "transamination" between amino acid and keto acid might take place by the activity of the fungus. These experiments are not yet completed, but no such transamination has been obtained under various conditions indicated as being favourable. As an example, Table 5 gives the result of an experiment with keto-glutaric acid and alanine, from which it appears that neither glutamic acid nor pyruvic acid was formed at all.

TABLE 5. Starved Aspergillus mats on 10 ml phosphate buffer pH 7.4 in N_2 atmosphere. Addition 1 ml 0.2 % Br-acetate solution. Temp. 25°. Concentration in mol. per mat.

Substrate	l (+) alanine	ketoglutaric acid	l (+) alanine + ketoglutaric acid
Initial conc.	0.172	0.228	0.172 + 0.228
Alanine disappeared	0.024	—	0.009
Pyruvic acid formed	0.000	0.000	0.000
Glutamic acid formed	0.043	—	0.000

Summary.

The various enzymes hitherto known for the amino acid breakdown have no part in the deamination of amino acids by starved mats of Aspergillus niger. With this organism the oxidative deamination proceeds best at pH 2—4 of the surrounding solution. It is closely coupled with the ordinary cell-respiration, and is probably caused by "unspecific" oxidation enzymes.

Amsterdam, Laboratory of Plant Physiology
of the University.

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Biochemistry. — *Tissues of prismatic celloidin cells containing Biocolloids.* **II.** *Coacervation of Gum-Arabic by Toluidin-blue and the phenomena accompanying the dissolution of the coacervate.* By H. G. BUNGENBERG DE JONG and B. KOK. (Communicated by Prof. J. VAN DER HOEVE).

(Communicated at the meeting of April 27, 1940.)

1. *Introduction.*

In an earlier communication[1] we described the accumulation of a basic dye followed by coacervation in the cavities of a collodion membrane containing gum arabic. By the new method of embedding in celloidin membranes, whereby the cells are obtained as tissues[2] this combination was again investigated and extended to the study of the suppression of the coacervate by a very weak salt solution.

2. *Accumulation of Toluidin-blue in the cells containing gum Arabic.*

If a 3 % gum arabic solution be embedded in the celloidin membrane and if, in the cuvette, a greatly diluted Toluidin-blue solution (0.001 %) be allowed to flow over the tissue, in 10 minutes time there will be the beginning of accumulation. The gum arabic solution is coloured violet and this colour will gradually grow deeper.

With respect to the mechanism of the accumulation[3] and the appearance of metachromasia[4], we refer to other publications.

3. *Coacervation in the cell-compartment.*

If 0.01 % Toluidin-blue solution be allowed to flow over the membrane, the accumulation will take place much more rapidly and coacervation very soon follows. In the cells tiny violet-stained drops separate, which at first show a strong Brownian movement. They coalesce gradually into larger drops and sink on to the celloidin wall, which they moisten. After sufficient time of waiting all the coacervate will be found, as metachromatically stained „smear" on the wall of the cell-compartment.

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[1] H. G. BUNGENBERG DE JONG and O. BANK, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 83 (1939).

[2] H. G. BUNGENBERG DE JONG, B. KOK and D. R. KREGER, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 512 (1940).

[3] H. G. BUNGENBERG DE JONG and O. BANK, loc. cit. The theory of the accumulation will be published in more detail in "Protoplasma".

[4] O. BANK and H. G. BUNGENBERG DE JONG, Protoplasma **32**, 489 (1939).

If 0.1 % Toluidin-blue solution be used the same process takes place, only all is much more rapid.

4. *The suppression of the Toluidin-blue-Arabinate-coacervate.*

If, for instance, we obtain coacervation first with 0.1 % Toluidin-blue and wait until the coacervate is entirely spreaded on the wall, and then pour distilled water over the membrane, changes will be seen gradually to take place which indicate that the wetting properties of the coacervate with respect to the wall are changed. Tiny white spots will appear in the smear. These changes will take place much faster if a very weak salt solution (e.g. 5 millimol NaCl) be poured over the membrane. Under these circumstances toluidin-blue will be washed out of the cells, in consequence of which the coacervate will be suppressed after some time, and a stadium results in which toluidin-blue is still accumulated in the cell-compartments. But with further flushing with 5 m. aeq. NaCl the coloration will disappear completely.

The aforesaid phenomena will be comprehensible when one bears in mind that accumulation is less when a neutral salt is present than when it is not.

A constant flow of a weak salt solution over the membrane washes more and more toluidin-blue out of the cells. The first consequence is the suppression of the toluidin-blue arabinate coacervate, after which the diffuse accumulation still left in the cell-compartment disappears. We shall investigate further how the suppression of the coacervate takes place in the cell.

5. *Changes in the wetting properties of the coacervate with respect to the cell-wall.*

The microphoto A represents the final stadium of the coacervation. Whereas it may happen that many cells in the membrane have been injured and many have lost their gum arabic, this is not the case with three cells. In these a large quantity of the coacervate is smeared out more or less evenly on the wall (the white lines are folds in the cell wall). The tissue was now washed with a solution of 5 millimol NaCl per litre. Several consecutive stadia in the changes that occur are shown in the micro-photos B and C. In B we see that the coacervate no longer covers the wall evenly, but has withdrawn materially, wherby rounded masses can be distinguished in the strips still lying on the wall. In C this process has advanced further and there are already coacervate drops in the cell-lumen loosened from the wall (Brownian movement!). After some time there will be no more coacervate on the wall, but merely free coacervate droplets in the lumen of the cells. The morphological changes, as seen in A, B and C, are thus the result of the wetting properties of the coacervate with respect to the wall being greatly altered.

6. *The formation of pulsating vacuoles in the free coacervate droplets.*

Simultaneously with the loosening of the coacervate from the wall, the quantity of coacervate in the cell-compartment gradually diminishes. However the coacervate droplets do not dissolve from outside and thus become gradually smaller in diameter. On the contrary, vacuoles are formed in the coacervate droplets and grow in volume, whereby the droplets even become appreciably larger and transparent in the centre. Many of these vacuolised droplets, some with one, and some with two, vacuoles, can be seen in the micro-photo C. Here we have to do with the same phenomenon as we described previously with respect to the free coacervate droplets, whereby the coacervate gradually dissolves via a stadium with pulsating vacuoles [1]).

In the cell-compartment, too, the vacuole embedded in the droplets increases greatly in volume until the peripheral coacervate-skin breaks, and the vacuole-content is discharged, the coacervate-scale contracting again to a tiny ball, the diameter of which is only 1/3—1/5 of what it was before the central vacuole appeared. This process is repeated several times, whereby the remaining coacervate droplet is smaller and smaller after each break, and at length disappears altogether.

That coacervate gradually dissolves in considerable quantities is also seen in micro-photo C from the increasing even staining in the cell-compartments, for, after the complete dissolving of the coacervate, we are left with a stadium in which there is a strong diffuse accumulation of dye in the cell-compartment.

7. *Influence of the electric field during the stadium of the vacuolising free coacervate droplets.*

We have described previously the formation of hollow spheres in the complex-coacervate of gum arabic-gelatine, i.e. coacervate droplets with a very large central vacuole [2]). We assume now, that the formation of hollow spheres and of coacervate droplets with pulsating vacuoles are virtually dependent upon the same mechanism. In both cases water from outside is carried inside through the coacervate, whereby the only difference is that, with the hollow spheres the coacervate-skin does not break open, as it does in the case of droplets with pulsating vacuoles. With regard to a further investigation as to the mechanism of the formation of hollow spheres, we refer to other publications [3]), and will merely mention here that their formation was solely dependent upon negative coacervate droplets. Thus, in this connection, it was interesting to inquire whether the toluidin-

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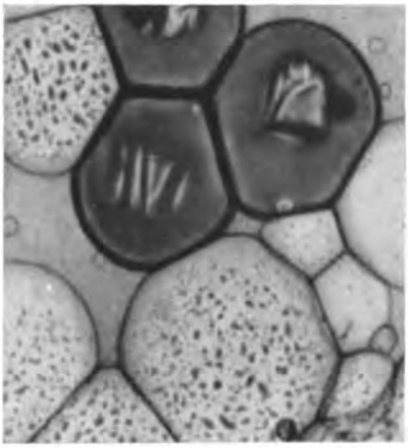
[1]) H. G. BUNGENBERG DE JONG, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 643 (1938).

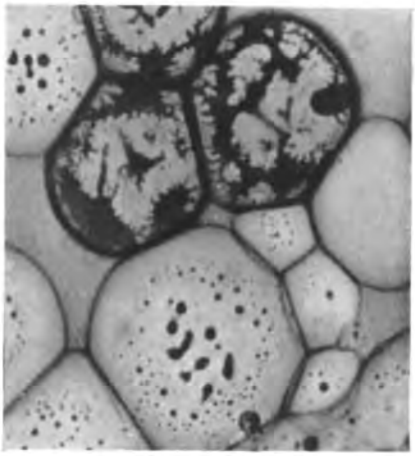
[2]) H. G. BUNGENBERG DE JONG and O. BANK, Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 274 (1939).

[3]) Will be published in "Protoplasma".

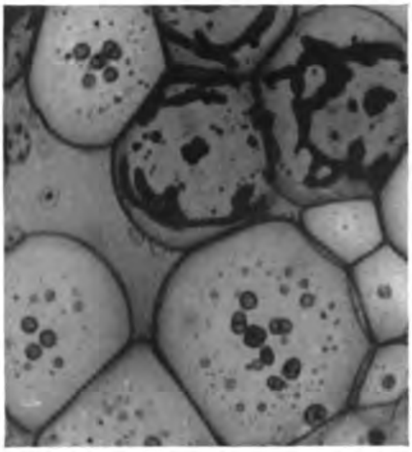
H. G. BUNGENBERG DE JONG and B. KOK: TISSUES OF PRISMATIC CELLOIDIN CELLS CONTAINING BIOCOLLOIDS. II.



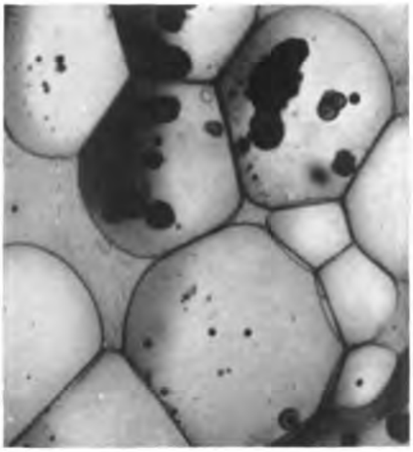
A 190×



B 190×



C 190×



D 190×

blue arabinate in the stadium of the vacuolisation, described sub 6, are also electrophoretically negative. For this reason a direct current field of 30 volt/cM was applied shortly after the micro-photo C was taken. Compare the micro-photo D, where the anode lies on the left and the cathode on the right, and where the photograph was taken a few seconds after the field was applied. The vacuolised coacervate droplets move rapidly to the left, i.e. they are negatively charged.

The micro-photo further shows that all the diffuse staining toluidin-blue in the cell-compartment has shifted to the anode.

At first sight this result appears strange for a dye-cation, but we must take into account that the toluidin-blue-cation is not present in a free state, but adsorbed to the negative arabinate.

In conclusion we would remark that one other notable phenomenon was observed indicative of an electric polarisation in the tissue. In switching on the current the coacervate droplets are seen to move rapidly to the anode. If the current be switched off, they move away slightly in the opposite direction. We shall return to this later.

8. *Effect of distilled water.*

In § 4—7 we discussed the suppression of the coacervate with 5 millimol NaCl. If we repeat the same with distilled water, the same phenomena will be observed in principle. Only in this case all takes place much more slowly. Here, too, the liberated coacervate drops enlarge slowly to hollow spheres, whereas it takes very long before they break. The hollow spheres here are also negatively charged and a distinct polarisation takes place in a direct current field.

March 1940. *Laboratory for Medical Chemistry at Leiden.*

Biochemistry. — *Tissues of prismatic celloidin cells containing biocolloids.* **III.** *a. Behaviour of an embedded complex-coacervate in the electric field. b. Polarisation phenomena.* By H. G. BUNGENBERG DE JONG and D. R. KREGER. (Communicated by Prof. J. VAN DER HOEVE).

(Communicated at the meeting of April 27, 1940.)

1. *Introduction and Methods.*

The method described in a previous communication [1]) for embedding biocolloids in celloidin membranes enables us not only to investigate the coacervation phenomena dependent upon an altered milieu, but also to study the influence of the electric field upon the resulting coacervates.

For this purpose the membranes, spread on a water-surface and fixed on a cuvette, as has been fully described in the earlier communication above-mentioned, were exposed to the influences to be studied and examined under the microscope.

This cuvette has been arranged so that the object-glass can be laid on it with the membrane downwards, and any selected fluid allowed to flow over it and, at the same time, an electric current may be applied. The flushing fluid flows off along the electrodes, so that the electrolysis products are carried away. Fluid and cuvette are so heated that the temperature of the membrane is never lower than 30°.

As objects of study the complex coacervates of gelatine and gum arabic were selected, for reason that this combination, which has been studied exhaustively before [2]), rapidly forms parietal coacervates in the celloidin cells. The stock solution embedded in the tissue was always the same, viz. 6 gr. gum arabic + 5 gr. gelatine + 200 cc distilled water.

In this the coacervation was brought about at different pH, with or without 10 m. aeq. KCl. The pH variation was obtained by allowing acetic acid of different concentrations to flow through the cuvette. Reversal of charge of the complex coacervate at the selected proportion of gelatine and gum arabic is near pH 3.35 (0.01 N acetic acid). The coacervation can be attained at all pH values between pH 4 (0.0005 N acetic acid) and PH 2.65 (0.2 N acetic acid) [3]).

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[1]) See Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, XLIII, 512 (1940).

[2]) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Kolloid Beihefte **43**, 143, 213 (1936).

[3]) Cathaphoretic determination at 36° in a microscopical cuvette yielded pH 3.32 as reversal of charge of the free coacervate drops, and pH 3.28 with 10 m. aeq. KCl.

The influence of the electric field was examined on negative, uncharged as well as on positive coacervates.

For the strength of the direct current field 30 V/cm was taken, as a rule, if there was no KCl in the acetic acid solution used to flush the membrane, and 10 V/cM when 10 m. aeq. KCl was present. The salt accélerates the phenomena very materially, so that, with 10 V/cM and KCl, the process is still appreciably more rapid than with 30 V/cM without the addition of salt.

The phenomena in the electric field were studied on ± 100 membranes. As a rule a new membrane was prepared for each experiment.

The description which here follows of these phenomena has been made as concise as possible, and will be published in more detail at another place (in "Protoplasm").

2. *Phenomena in the electric field.*

These are:

1. If we look through the microscope immediately after the object-glass has been laid on the cuvette, it will be possible, as was earlier described in detail, to see the coacervation taking place.

A current of short duration will show that the droplets and vacuoles in process of formation behave in accordance with the sign of charge of the coacervate, as this is determined by the pH of the flushing acetic acid, i.e. in the case of negative coacervates they migrate to the anode and in the case of positive coacervates to the cathode.

2. When the coacervate has completely formed and has taken up its topographical ultimate position (Fig. 1a), the current being applied, vacuolisation takes place in the parietal coacervate both right and left of the vacuole (Fig. 1 *b*). (See microphoto A.)

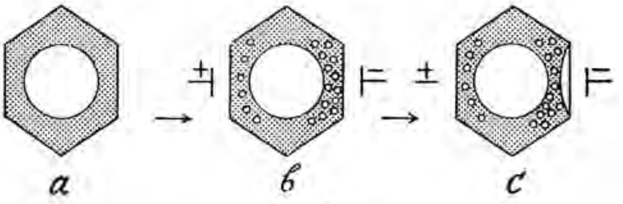


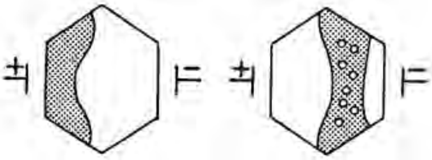
Fig. 1.

The vacuolisation in the coacervate right of the vacuole ("right" is here always the cathode side) is invariably much more marked than that to the left of the vacuole, while, moreover, at the right wall of the cell a secondary vacuole is formed, which expounds to the left (Fig. 1c). This always takes place at the cathode side, irrespective of the sign of charge of the coacervate. See microphoto B.

3. After a short time (± one minute, dependent on the voltage and

presence of KCl), we see in most of the cells containing the positive coacervate, the vacuole on the right and the coacervate on the left (Fig. 2a).

In the case of the negative coacervate, then in most cells the coacervate will be on the right and the vacuole on the left. In this "right" coacervate the vacuolisation described above and the formation of the secondary vacuole continues to take place (Fig. 2b).



Coac. positive   Coac. negative
a   Fig. 2.   b

4. Again, after a short time (± 3 minutes according to the voltage, etc.) a difference will be noticed between the right and left halves of the cell-groups.

Then, in the positive as well as in the negative coacervate, we see in the left half of the cell-groups the coacervate at the right in the cell (Fig. 3a), whereas in the right half of the cell-groups, it will generally be to the left in the cell (Fig. 3 c). See microphoto C. The coacervate standing at the right side (cathode) in the cells (Fig. 3 a) continues vacuolising, especially if KCl is present; the coacervate on the left side (Fig. 3 c) does not do so and is frequently almost vacuole-free.

In the larger cells in the middle of the cell-groups a condition now arises, whereby the coacervate divides the cell, like a partition, into two parts, or stands like two heaps opposite each other. These heaps are often connected by thin coacervate strips along the upper and lower wall of the cell. In Figure 3 b these conditions are seen.

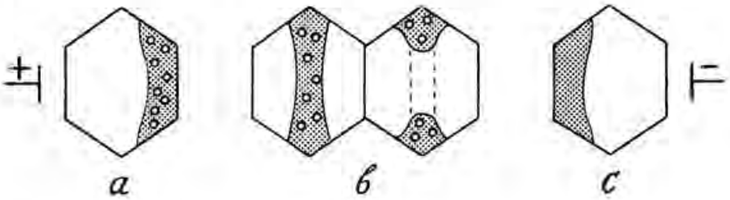
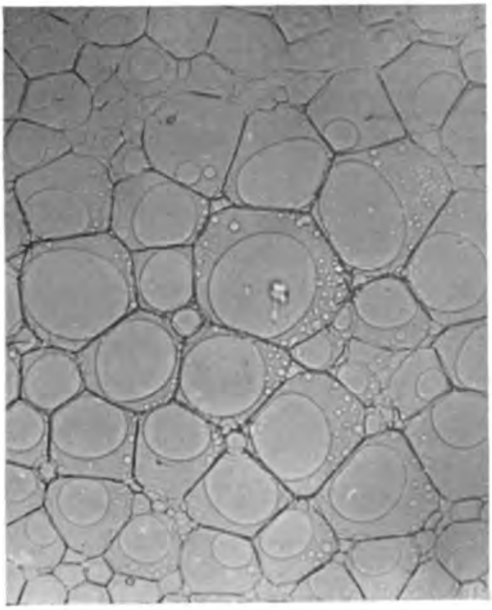


Fig. 3.

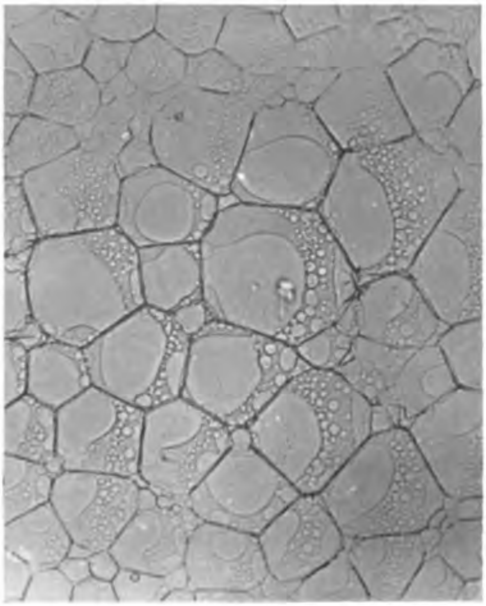
It may be mentioned as a peculiarity that the coacervate partitions are a conspicuous feature of nearly the entire cell-group when the polarity is reversed during the process.

After this, if the current be continued, no fundamental changes take place, except that the coacervate in the cells becomes quantitavely less and less.

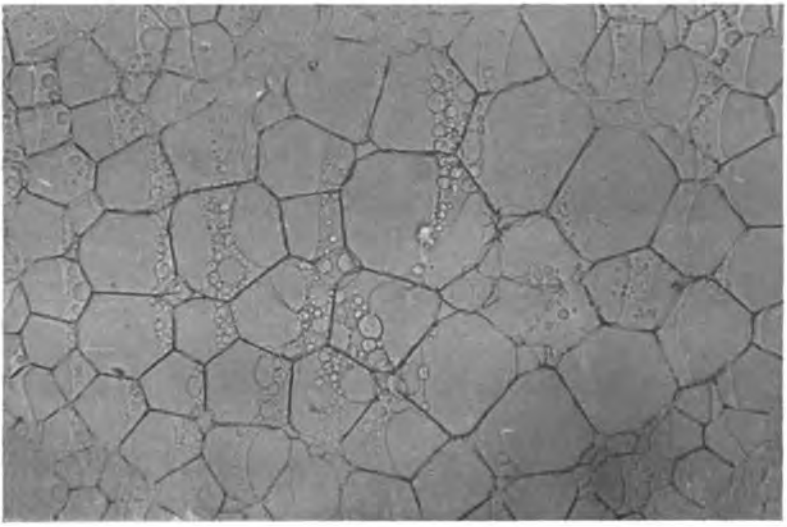
H. G. BUNGENBERG DE JONG and D. R. KREGER: TISSUES OF PRISMATIC CELLOIDIN CELLS CONTAINING BIOCOLLOIDS. III.



A



B



C

Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

3. *Differentiation of the phenomena in the electric field into two groups.*

We have seen in § 2 that the action of the electric field leads to an ultimate condition in which, practically speaking, no further changes occur and in which only the slightest difference, or none at all, between positive and negative coacervate is to be observed. In the course of the process, however, there do appear phenomena which vary with the sign of charge of the coacervate (e.g. the movement of the coacervate droplets and vacuoles during the coacervation process and the position of the coacervate and vacuole with respect to each other in Fig. 2). The phenomena which, in the case of free complex coacervate droplets, occur in the electric field were dealt with fully on a former occasion ¹). It then appeared that in the positive coacervate these are the reverse of those in the negative. Some of the phenomena seen in § 2 may be attributed to this (e.g. the shifting of the vacuoles).

The other phenomena in § 2, and which are the most noticeable ones, are never observed in the case of free coacervate droplets. In this respect it is typical that they are independent of the sign of charge of the coacervate, so that they must be ascribed to the presence of the surrounding celloidin walls.

4. *Working hypothesis concerning the part played by the celloidin walls.*

If we assume that in the electric field concentration differences of the ions, in particular of the H ions, arise between either side of the celloidin walls of the cell-compartiments ²), the principal phenomena (i.e. those whereby the sign of charge of the coacervate has no part) can be explained.

It is obvious that we may assume that in consequence of the said polarisation, the pH will rise on the anode side of the celloidin membrane but fall on the cathode side (confirmation of this hypothesis will be found in the following paragraphs).

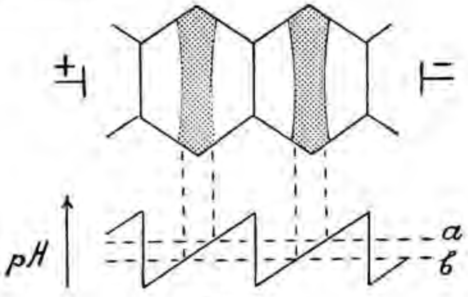


Fig. 4.

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¹) H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, loc. cit. see p. 201—203. See also H. G. BUNGENBERG DE JONG and W. A. L. DEKKER, Biochem. Z. **221**, 403 (1930).

²) BETHE and TOROPOFF, Z. physikal. Chem. **39**, 613 (1902).

Considering some contiguous cells, we might expect in every cell-compartment the pH, consequently, to rise from left to right (see Fig. 4).

In connection with the fact that, with a given ratio of gum arabic and gelatine, the complex coacervation is only possible in a certain pH-range (in Fig. 4 between the lines a and b), the typical position of the coacervate in the middle cells of a group during the ultimate stationary condition (Fig. 3b) is comprehensible.

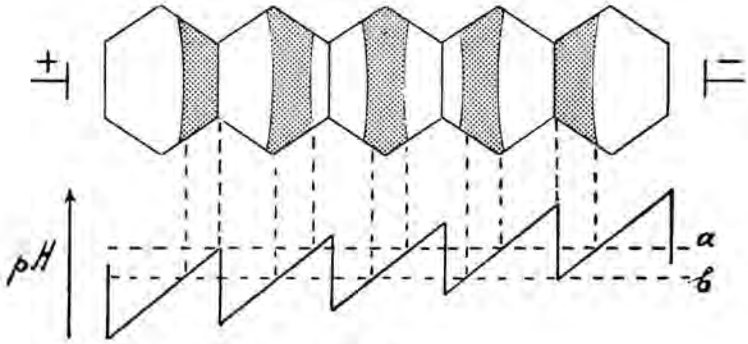

Fig. 5.

In these cells the pH is then too low in the left half, and too high in the right half, and the coacervate can, therefore, exist only in an intermediate zone perpendicular to the field.

If, in the final stationary stadium, we regard, however, not merely the middle cells of a cell-group, but also the cells on either side, we shall come to the conclusion that the average pH in the cells rises from left to right in one cellgroup (left anode, right cathode).

For we notice that in the left half of a cell-group the coacervate is no longer situated in the middle, but at the right side-wall, and similarly that in the right half of a cell-group the coacervate is just at the left side. See fig. *3a*, *3c* and microphoto C.

The pH range in which the coacervate can exist is, thus, in the left cells quite at the right side of the cell, i.e. the average pH is lower here than in the middle cells. Inversely, in the right cells the pH range in which there can be a coacervate, is just quite at the left in the cells, i.e. the average pH is higher here than in the middle cells (see Fig. 5).

We cannot discuss here further the other details of § 2, but will merely state that their explanation may advantageously be based upon the two hypotheses discussed above, viz. in consequence of the polarisation of the celloidin walls there develops a) a pH gradient inside each cell, and b) a gradient of the average pH over the whole cell-group.

In the following paragraphs an account will be given of experiments which confirm both these hypotheses.

5. *Macromodel for the origin of pH differences on both sides of a membrane after applying an electric field over the membrane.*

For this a U-shaped tube was used, one arm of which is divided by two perforated plate-glass slides lying one upon the other (see Fig. 6). The upper one is cemented to the upper half of the tube, the lower to the lower half. When a cellophane membrane has been inserted between the slides, the upper part of the tube can be laid loosely upon the lower half.

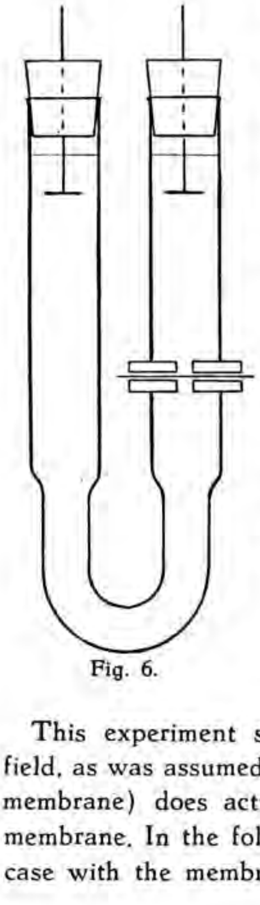

Fig. 6.

The cuvette is filled with a solution of bromthymol-blue in distilled water [1]) up to the upper edge of the slide, the piece of cellophane is laid upon this slide, the upper part then laid down upon it, and the two arms of the tube filled up equally, whereupon 2 electrodes are put into each. To get the potential gradient over the membrane as large as possible, the arms of the cuvette are taken very wide, ± 3.5 cM, and the bore in the slides not more than ± 7 mm in diameter. On the vertical outside edge of both plate-glasses, a spot was made thoroughly transparent, so that any staining in the columns of liquid on either side of the membrane could be observed by looking through the plate-glasses lengthways.

When the current was applied (± 60 V), blue staining appeared in the brownish-yellow fluid on the anode side of the membrane. The same result was obtained with 10 m. aeq. KCl.

This experiment shows that the shifting of the pH in the electric field, as was assumed above in § 4, (the pH rises on the anode side of the membrane) does actually take place, that is to say, with a cellophane membrane. In the following paragraphs we shall see that the same is the case with the membranes of the microscopic cells in the celloidin tissue.

6. *Demonstration of the pH gradient inside the cells and over the entire cell-group by means of bromphenol-blue.*

If the pH differences occurring left and right in the cell, as well as in the cell-group are indeed the cause of the principal phenomena described in § 2, they must be so great that they can be shown with indicators. The difficulty here is to get an indicator with a suitable range in such a high concentration in the cell that this very thin layer of fluid, no thicker than the height of one cell, will allow an observable colour change.

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[1]) The stock solution: 0.2 gr. bromthymol-blue + 3.2 cc NaOH o.1 N dissolved in 500 cc. distilled water, was diluted five times with distilled water.

No results are to be expected unless an indicator is used which, in one of its forms, is accumulated (adsorbed) in the complex-coacervate gum-arabic-gelatine.

For our purpose bromphenol-blue (pH 3.0 yellow, pH 4.6 blue) proved suitable. From a bromphenol-blue solution, even if pH is such as to have an intermediate shade, or a more pronounced blue colour, the yellow form is nevertheless always adsorbed by the complex-coacervate. The yellow form is the non-dissociated form, and the binding must be looked for here in the general affinity of phenol-groups with proteins. This last results in dehydration, and if we follow the behaviour of the complex-coacervate in the cell-membrane in the electric field, after the yellow staining with bromphenol-blue, we shall also see distinct phenomena which indicate a less aqueous, and thereby a more viscous, state of the coacervate. For this experiment the celloidin membrane was flushed with a mixture of 80 vol. % 0.01 N acetic acid + 20 vol. % bromphenol-blue stock solution[1]. When, after 5 minutes a plainly observable staining is seen (light yellowish-green if looked at microscopically in bluelight (clear blue sky) the experiment proper can be proceeded with, either at once, or after a short rinsing of the cuvette with 80 vol. % 0.01N acetic acid + 20 vol. % distilled water.

When the current was applied there was, according to the rule, a strong vacuolisation on the cathode side. The expounding of the secondary vacuoles to the left was very slow and the coacervate made the impression of being very viscous. After some time partitions of green coacervate appeared in nearly all the cells of a shape which differed in some respects from the coacervate partitions, as these have been described above.

About 5 minutes after the current was applied the vacuole fluid in the right half (cathode side) of the cell began to stain blue in some places, especially in the right cell-groups.

In a short time most of the cells in the right cell-group-halves on the cathode side were coloured distinctly blue. This was the case hardly anywhere in the left cell-group halves. The coacervate generally in the middle of the cells remained green, which, especially at the left side of the partitions and in the cells at the left (anode side) of a cell-group became more a yellowish-green.

Herewith the pH gradient, a) inside the cell and, b) over the whole cell-group, has been demonstrated.

7. *Demonstration of the pH gradient inside the cells and over the entire cell-group by means of resorcin.*

As we stated above, phenols exercise in general a dehydrating effect upon proteins. If the charge of the protein is small, phenols cause floccu-

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[1] 0.2 gr. bromidephenol-blue + 2 cc NaOH 0.1 N dissolved in 500 cc distilled water.

lation or coacervation. Also with gelatinsol we find that resorcin brings about coacervation in a fairly limited pH range round about the I.E.P.

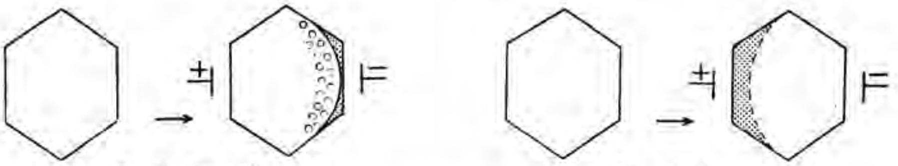
We may make use of this to inquire, in another way than with the aid of indicators, whether the assumed pH shifts actually occur in the electric field.

If gelatine only be enbedded in the membrane and it be flushed with a resorcin solution with a pH which lies outside the pH range in which the coacervation takes place, it should be possible that, in applying the field, the pH on *one* side of the cells is raised or lowered to such a degree that it comes within the coacervation area and we shall see the coacervation take place here.

In consequence of the change of the average cell pH from left to right in a cell-group, this phenomenon will be more distinct again in the one half of the group of cells than in the other.

Such a condition can indeed be realised. For this a 5 % gelatine solution in distilled water is embedded in the membrane.

This was then flushed with a 0.01 N acetic acid solution (pH 3.35), and this solution was followed, without any break, by an acetic acid solution of the same concentration in which, however, 4 % resorcin had been dissolved. This solution was passed through the cuvette for 10 minutes, after which there was not yet any coacervation to be seen in the cells. When the current was applied the coacervation appeared in most of the cells in numberless tiny droplets against the wall on the right side (see Fig. 7 A). In the right half of the cell-group the coacervation extended farther towards the middle of the cells than on the left side, whereas in many of the cells quite to the left in the cell-group there was no coacervation.The pH of the wash was lower than the coacervation range. As has been exposed above, the pH will become higher on the anode side of the membranes, on the cathode side lower, when the current is applied. The appearance of the coacervate in the cells on the right side is thus in accordance with this.



A (pH 3.35)

B (pH ± 6.7)

Fig. 7.

Now, taking a medium which has a pH lying above the coacervation area, we should, in accordance with the same reasoning as above, be able, when the current is applied, to bring about coacervation in the left cell-half, since pH is here lowered.

A pH of ± 7 proved to be the best suited for this, since a buffer solution had already been tried in the preceding experiment to attain a certain pH and had yielded no coacervation, possible owing to the eliminating of the pH differences by the buffer; this pH was obtained by neutralising the acid reaction of the 4 % resorcin solution (per 100 cc 10 cc NaOH 0.1 N was added). With this solution bromthymol-blue was greenish-blue.

This solution was washed for 10 minutes over the membrane with embedded gelatine, after which still no coacervation occurred. When the field was applied, coacervation very quickly occurred (± 20 sec.) as was to be expected, at the left of the cells (Fig. 7 B). Moreover, the difference between anode and cathode sides of a cell-group was again observed, namely on the anode (left) side the phenomenon was stronger than on the right side, so that a change of the average pH for the whole cell-group was evinced here in accordance with the hypothesis in § 4.

When the current was cut off the coacervate disappeared within a few seconds.

March 1940. *Laboratorium for Medical Chemistry at Leiden.*

Anatomy. — *On the cephalization stage of Pithecanthropus erectus and Sinanthropus pekinensis.* By R. BRUMMELKAMP. (Communicated by Prof. C. U. ARIËNS KAPPERS).

(Communicated at the meeting of May 25, 1940.)

There is a difference of opinion among the anthropologists of to-day as to the relation between Pithecanthropus erectus and Sinanthropus pekinensis and the place they occupy in the succession of primates. According to WEIDENREICH [1]) Sinanthropus is more primitive than Pithecanthropus, but this difference is not such that the discussion of the degree of primitiveness of Pithecanthropus and Sinanthropus must be considered of great importance. When compared with other fossil hominids, there is, according to WEIDENREICH, no doubt that *Pithecanthropus and Sinanthropus represent the same general stage of evolution*, the difference existing between the two types having to be considered as regional or racial deviations. A similar view is expressed by BOULE [2]) when he says that the two organisms are very intimately related, if not identical. ARIËNS KAPPERS and BOUMAN [3]), ARIËNS KAPPERS [4]), DUBOIS [5]), KOENIGSWALD

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[1]) WEIDENREICH, FRANZ. Pithecanthropus and Sinanthropus, Nature, Vol. 141, p. 378—379, 1938; Discovery of the Femur and the Humerus of Sinanthropus pekinensis, Nature, Vol. 141, p. 614—617, 1938; The relation of Sinanthropus pekinensis to Pithecanthropus, Javanthropus and Rhodesian Man, Journal of the Royal Anthropological Institute, Vol. LXVII, 1937; The new discoveries of Sinanthropus pekinensis and their bearing on the Sinanthropus and Pithecanthropus problems, Bulletin of the Geological Society of China, Vol. XVI, 1936—1937; Observations on the form and proportions of the endocranial casts of Sinanthropus pekinensis, other hominids and the great apes: a comparative study, Palaeontologica Sinica, Series D, Vol. VII, fasc. 4, 1936.

[2]) BOULE Marcellin. Les fossiles, Paris 1935.

[3]) ARIËNS KAPPERS C. U. and K. H. BOUMAN. Comparison of the endocranial casts of the Pithecanthropus erectus skull found by DUBOIS and VON KOENIGSWALD's Pithecanthropus skull, Kon. Ned. Ak. v. Wet., Proceedings Vol. XLII, No. 1, 1939.

[4]) ARIËNS KAPPERS C. U. De groeven op de lobi frontales van Pithecanthropus erectus DUBOIS, vergeleken met die van den Neanderthalmensch, Homo recens en Chimpanzee, Kon. Ned. Ak. v. Wet. Meded. Dl. XXXVII, No. 10, 1928.

[5]) DUBOIS, EUGÈNE. On the fossil human skull recently described and attributed to Pithecanthropus erectus by G. H. R. VON KOENIGSWALD. Kon. Ak. v. Wet., Proceedings Vol. XLI, No. 4, 1938; Die phylogenetische Grosshirnzunahme autonome Vervollkommnung der animalen Funktionen, Biologia generalis, Bd. VI, pp. 247—292, Wien 1930; On the principal characters of the femur of Pithecanthropus erectus, Kon. Ak. v. Wet., Proceedings Vol. 29, 1926; Racial identity of Homo soloensis Oppenoorth (including Homo modjokertensis VON KOENIGSWALD) and Sinanthropus pekinensis DAVIDSON BLACK. Kon. Ak. v. Wet., Proceedings Vol. 39, No. 10, 1936.

a.o. ¹) hold a different opinion. On the ground of the fissuration of braincasts, general contours, brain capacity and other anatomical details, they believe that *Pithecanthropus erectus and Sinanthropus pekinensis should not be considered as belonging to one species*, but they think that Sinanthropus is nearer to the more advanced Javanthropus soloensis Oppenoorth, or to the Rhodesian-man, while assigning to Pithecanthropus a separate, more primitive place in the succession of primates.

Thus far no use has been made for the above problem, of the *cephalization-rule*. Yet it seems useful to me to pay some attention to this rule, because it establishes a relation between brainweight c.q. skull capacity on the one hand, and body size (expressed in bodyweight, stemlength or certain femur-measurements) on the other, which relation is of the greatest importance for a right appreciation of the significance, which in a certain case is to be ascribed to brainweight, c.q. skull capacity.

Since DUBOIS' publications we know that there is a relation between brainweight E and bodyweight P, according to $E = c.P^r$. When we apply this rule to nearly related animal species, e.g. cat and tiger, it appears that the value of the co-factor c is constant and that of the exponent $r = {}^5/_9$. When this rule is applied to species which are not so nearly related, though belonging to the same order, we find the remarkable fact that the value of the co-factor varies leapwisely, approximately according to the terms of a geometrical progression, whose ratio is $\sqrt{2}$, parallel to the general level of development of the animals in the order examined ²).

Hence the value of the co-factor expressing the degree of development of the cerebrum in so far as it depends on body size, is also an index of the stage in evolution of the animal species concerned.

This co-factor is generally called the cephalization coefficient, and the stage in evolution expressed by it is the cephalization stage or cephalization level.

> Thus we see, for instance, that in the order of the *Ungulates*, of the primitive Tragulus the cephalization coefficient is half, of the domesticated forms of Bovis $\sqrt{2}$ times smaller, of the undomesticated Cervidae $\sqrt{2}$ times greater, and of the highly developed Elephas $2\sqrt{2}$ times greater than that of the majority of animal species belonging to this order. In the same way it appears in the order of *Primates* that in humans the value of factor c is 4 times greater than in Gorilla and Simia satyrus and $2\sqrt{2}$ times greater than in Chimpanzee.

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¹) VON KOENIGSWALD G. H. R. Ein neuer Pithecanthropus-Schädel. Kon. Ned. Ak. v. Wet., Proceedings Vol. 41, 1938.

²) Extensive data concerning the leapwise variation of the co-factor c (cephalization coefficient) are found in a.o. BRUMMELKAMP, Normale en abnormale hersengroei in verband met de cephalisatieleer, N.V. N.H. Uitg. Mij (1937), where an attempt is made to account for this leapwise variation. The probability is set forth of a connection with the leapwise development of the cortex cell nuclei, which is according to a $\sqrt{2}$ fold.

When the cephalization rule is applied to the various human races, the value of the exponent is $^5/_3$ and that of the co-factor is constant, when the *length of the body stem S* (length of head and trunk) is taken as measurement of the body size. The above general formula then becomes $E = c.S^{5/3}$. Written in its logarithmic form, it is $lg\,E = lg\,c + {}^5/_3\,lgS$, i.e. of the first degree and hence may be graphically represented by a straight line. If we put out *lg E* against *lg S* of a number of human races, it appears, according to fig. 1 that indeed the different relation points of these races are on or near a straight line with a $^5/_3$ direction, and so answer to our formula. From this it also appears that, according to the cephalization measure, the stage in evolution of the various human races is the same. The data used in fig. 1 are found in table I.

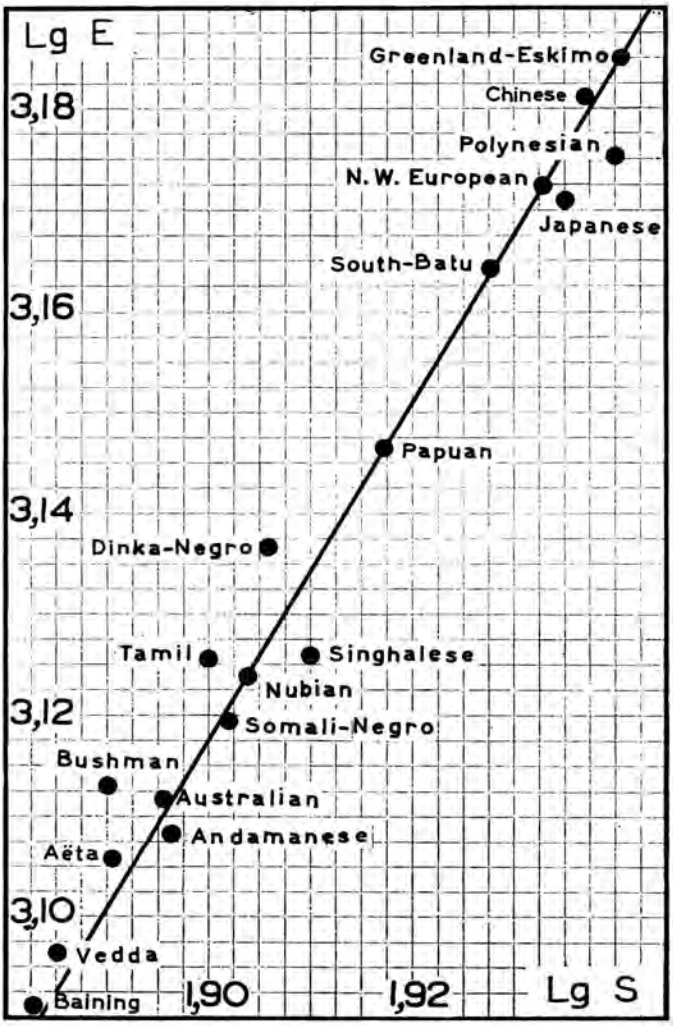

Fig. 1.

TABLE I 1).

Name	Bodylength	Stemlength (S)	Skullcapacity (E)
N. W. European	171.5	85.75	1490
Vedda	153.3	76.65	1250
Singhalese	162.5	81.25	1336
Polynesian	174.0	87.00	1500
Papuan	165.7	82.85	1398
South Batu	169.6	84.80	1460
Bushman	155.2	77.60	1297
Baining	152.9	76.45	1232
Chinese	160.6	86.56	1518
Japanese	159.3	86.02	1485
Eskimo	162.1	87.53	1528
Andamanese	148.6	78.76	1281
Aeta	146.3	77.54	1275
Dinka Negro	173.0	80.45	1389
Somali Negro	171.5	79.75	1315
Australian	165.8	78.73	1295
Nubian	167.0	80.16	1329
Tamil	165.3	79.34	1336

The relation between skull-capacity c.q. brainweight E and stemlength S will prove valuable in calculating the cephalization of fossil hominids and prehominids. In order to make this clear the following:

When we compare two extreme races, e.g. Esquimaux and Australians, we note that in Esquimaux a great skull capacity is attended with a long body stem, and that also their head and trunk exceed those of Australians in length and transverse measurements, and are therefore heavier, which greater weight is carried by shorter, more robust femora. Skull capacity on the one hand, and length of the body stem, weight of trunk and head and carrying capacity of the femora on the other, apparently vary simultaneously in the same direction.

As the carrying capacity of the femur is proportionate to the load to

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1) For these data about bodylength, stemlength and skullcapacity, I refer to my book: Normale en abnormale hersengroei in verband met de cephalisatie-leer, Uitg. N.V. Noord-Hollandsche Uitg. Mij. 1937.

be carried, i.e. the weight of head and trunk, and as this weight is proportionate to the third power of the stemlength, the skull capacity, which in the various races is proportionate to the $^5/_3$ power of the stemlength, will be proportionate to the $^5/_9$ power of the carrying capacity of the femora. Presuming that there is a close relation between shape and function of the femur, we may expect a relation between skull capacity and carrying capacity of the femur as expressed in the measurements of this bone. It is probable that the carrying capacity is proportionate to a certain power of the area of the transverse section in the middle of the femurshaft, and inversely proportionate to a certain power of the femur length. As length we take the maximal length L and as area of the transverse section in the middle of the diaphysis the product of the sagittal and transverse diameter $(d_1 \cdot d_2)$. When, for some reason, the area of the transverse section of the diaphysis middle cannot be applied, we may substitute the quotient of the femur volume (V) divided by the quadrate of the length (L^2). Thus we have at our disposal two formulas, *viz.* $E = f\,(d_1\, d_2/L)$ and $E = f_1(V/L^2)$. On the ground of what has gone before we expect that the two relations will answer to the general form $Y = c.X^r$.

In order to check the correctness of this presumption we set out lgE against $lg\ (d_1 \cdot d_2/L)$ and $lg\ (V/L^2)$ respectively.

In fig. 2 we have set out $lg\ E$ and $lg\ (d_1 \cdot d_2/L)$ of 30 representatives of greatly differing recent and fossil human races. This graph shows that the points thus constructed are approximately in a straight line, which has a $^5/_9$ direction. (In exact figures, calculated by the method of the smallest quadrates, the direction of the line is 0.5403.) The value of the co-factor is 1084, so that this new relation is

$$E = 1084\,(d_1 \cdot d_2/L)^{5/9} \quad \ldots \ldots \ldots \quad (I)$$

By introducing quotient V/L^2 instead of $d_1 \cdot d_2/L$ we obtain fig. 3. It is seen that again the relation-points lie near a line with a $^5/_9$ direction. The value of the co-factor is 38508, so the second relation is

$$E = 38508\,(V/L^2)^{5/9} \quad \ldots \ldots \ldots \quad (II)$$

The data applied in the two graphs are found in tables II and III. *The graphs show that the cephalization stage of fossil hominids does not differ from that of recent hominids.*

We will now use formulas I and II to determine the cephalization stage of Pithecanthropus and Sinanthropus.

Of *Pithecanthropus erectus* both cranium and femur are known. On the ground of the cranium DUBOIS estimates the total capacity of the Pithecanthropus skull at 908 cc "by comparing the remains of the cranial cavity with the corresponding part of the cranial cavities of monkeys which are likest to it as far as their main proportions are concerned (Hylobates, Gorilla, Pongo, Pygathrix, Macacus), taking into consideration the deviations owing to the different size-relations between brain cavity and orbit."

TABLE II. 1)

Name	Femur					Skullcap. (E) (measured)	Skullcap. (calc.)	Error %
	max. length (L)	circumf.	d_1	d_2	$d_{av.}$			
Bavarian, m.	445	88	—	—	28	1490	1485	0.25
American, m.	460.81	—	28.36	29.35	—	1490	1506	1.07
Norvegian, m.	461.8	90.7	—	—	28.87	1490	1505	1.07
Norvegian, f.	417.4	79.7	—	—	25.36	1340	1378	2.84
Eskimo (Sm. S.), m.	397.25	—	24.5	31.5	—	1570	1568	0.13
Japanese, m.	410.7	84.1	—	—	26.77	1485	1477	0.54
Japanese, f.	382.7	72.6	—	—	23.11	1319	1305	1.01
Corean, f.	414	77	—	—	24.51	1345	1333	0.89
Paucarcancha, m.	397.5	75	—	—	23.87	1349.2	1324	1.87
Paucarcancha, f.	371.34	68.2	—	—	21.71	1197.4	1236	3.29
Patallacta, m.	401.66	77.3	—	—	24.6	1401	1361	2.86
Patallacta, f.	370.76	67.5	—	—	21.49	1207.3	1225	1.47
Torontoy, m.	427.8	79.5	—	—	25.31	1357.7	1357	0.05
Torontoy, f.	382.3	68.8	—	—	21.90	1214.4	1230	1.28
Ona-Indian, m.	434.8	85.5	—	—	27.23	1487	1458	1.95
Yaghan-Indian, m.	415.5	84.5	—	—	26.9	1420	1475	3.87
Negro, m.	455	83.7	—	—	26.64	1389	1388	0.08
New-Caledonian, m.	439.6	—	24.75	29.9	—	1440	1448	0.56
New-Caledonian, f.	417.5	—	22.5	26.3	—	1315	1316	0.08
Loyalty-Islander, m.	438.4	—	24.7	29.07	—	1463	1426	2.53
Loyalty-Islander, f.	414.7	—	22.5	26.6	—	1326	1329	0.23
Lapp, m.	412.8	82.8	—	—	26.42	1465	1451	0.96
Lapp, f.	384.2	75.4	—	—	24.0	1305	1357	3.98
Fuegian, m.	414.3	88	—	—	28	1590	1545	2.83
Pěsěchěm, m.	406.5	—	22.25	24.75	—	1299	1283	1.23
Vedda, m.	440.6	—	24.1	25.09	—	1293	1292	0.08
Spy I, m.	420	90	—	—	28.65	1562	1573	0.70
Chapelle aux Snts, m.	430	—	29	31	—	1626	1633	0.43
Combe-Capelle, m.	425	—	25	28.5	—	1435	1445	0.70
Grimaldi, f.	436	—	24.4	25.4	—	1315.5	1318	0.19
Chancelade, m.	408	—	27.5	30.0	—	1595	1603	0.50

[1] Notes to Table II.

Bavarian. Data of skullcapacity (*SC*) and femurmeasurements in MARTIN (1928). *American.* Data of femurm. in N. WILLIAMS INGALLS (1924). He gives the average values of 200 femora of adult men; the value of *SC* equals that of the average N.W. European. *Norwegian.* Data of femurm. in SCHREINER (1935), who gives the data of 521 male and 493 female adult femur-specimens. *SC* as in N.W. European. *Smith-Sound-Eskimo.* Data of femurm. and *SC* taken from the description of an adult male skeleton by HRDLIČKA (1910). *Japanese.* Data of femurm. in FAYUN NASHIZUKA (1926) who describes 10 male and 10 female adult skeletons, but does not mention *SC*. *SC* taken from ADACHI (see MARTIN, 1928). *Corean.* Description of complete skeleton and skull of adult robust woman by KOGANEI (1906). *Peruvian Highlanders,* described by GEORGE GRANT MCCURDY (1926). *Paucarcancha,* male 60 skulls, 51 femora; female 50 skulls, 44 femora; *Patalacta,* male 24 skulls, 18 femora; female 18 skulls, 13 femora; *Torontoy,* male 11 skulls, 15 femora; female 9 skulls, 9 femora. *Indian.* J. VILK HULTKRANZ (1900) describes skulls and skeletons of 3 *Ona Indians* and 2 *Yaghan Indians. Negroes.* Data of femurm. taken from DUBOIS (1926) referring to 26 specimens. Average skullcapacity from the values given by DRONTSCHILOW (1913), ISERLIN (1914) and BENNET BEAN (1906); the latter calculated from brainweight. *New-Caledonian.* SARASIN (1922) gives the average values of 34 male and 27 female skulls and the average values of 11 male and 8 female femora (adult). *Lapp.* SCHREINER (1935) gives the average values of 303 adult male femora and *SC* of 157 adult male skulls; besides of 262 adult female femora and 133 adult female skulls. *Fuegian.* MARTIN (1894) describes 21 skeletons and gives the average *SC* of 25 skulls of adult males. The measurements of the middle of the diaphysis are not given, but "essentially they are not different from ours (European)" (MARTIN). Therefore I used the circumference of the average Bavarian femurdiaphysis. *Pĕsĕchĕm.* Description of a male skeleton by VAN DEN BROEK (1918). On account of the exceptional pilastriform shape of the femurdiaphysis I took the average value of the upper, middle and lower sagittal measurements for the sagittal diameter in the middle of the diaphysis. *Vedda.* SARASIN (1893) mentions the average *SC* of 5 male adult Veddas. The femur is pilastriform, like that of Pĕsĕchĕm-man, therefore I have calculated the sagittal measurements by multiplying the real sagittal diameter in the middle of the diaphysis by the same factor as was done in the Pĕsĕchĕm-man. *Spy I.* FRAIPONT and MAX LOHEST (1887) described two fossil Neanderthal-men. The entire femur of Spy I only was found. In the original article *SC* is not mentioned ("parceque Fraipont a été effrayé des résultats obtenus"). He found namely a capacity of 1562 cc, which was much to high in his opinion, as is mentioned by BOULE (1911—1913) in a footnote in his monograph on the Chapelle-aux-Saints-man. *Chapelle-aux-Saints.* Detailed description of an adult male skeleton of a Neanderthal-man by BOULE (1911—1913). *Combe-Capelle.* Description by KLAATSCH (1909). *SC* was calculated from maximal length, maximal breadth and basion-bregma-height by LEE–PEARSON's formula. KLAATSCH says that the skull is excessively thick, on account of which, after LEE–PEARSON, I have subtracted 50 cc from the amount obtained by their formula. *Grimaldi.* Description by VERNEAU (1906) of the skeleton of an adult woman. *SC* was calculated from maximal length, maximal breadth and basion-bregma-height by LEE–PEARSON's formula. (The second skeleton found at Grimaldi is of a young man, not yet adult and therefore not suitable for this calculation.) *Chancelade.* Description by TESTUT of a skeleton of adult man. *SC* calculated from maximal length, maximal breadth and basion-bregma-height by LEE–PEARSON's formula.

For further information, see my book: Normale en abnormale hersengroei in verband met de cephalisatieleer.

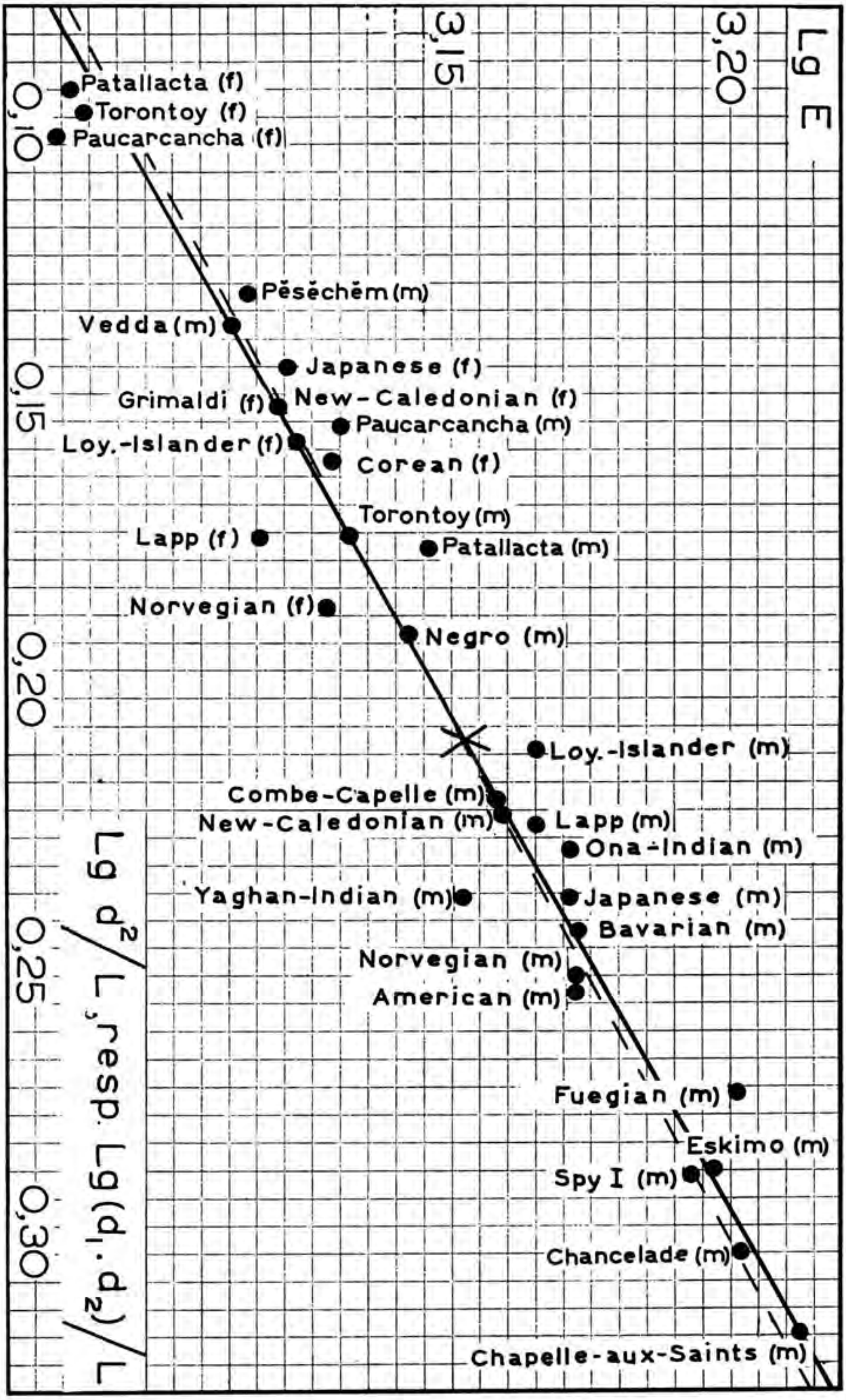
Lg E
3,20
3,15
0,10
0,15
0,20
0,25
0,30
Lg d²/L, resp. Lg(d₁.d₂)/L
Patallacta (f)
Torontoy (f)
Paucarcancha (f)
Pěsěchěm (m)
Vedda (m)
Japanese (f)
Grimaldi (f)
New-Caledonian (f)
Paucarcancha (m)
Loy.-Islander (f)
Corean (f)
Torontoy (m)
Lapp (f)
Patallacta (m)
Norvegian (f)
Negro (m)
Loy.-Islander (m)
Combe-Capelle (m)
New-Caledonian (m)
Lapp (m)
Ona-Indian (m)
Yaghan-Indian (m)
Japanese (m)
Bavarian (m)
Norvegian (m)
American (m)
Fuegian (m)
Eskimo (m)
Spy I (m)
Chancelade (m)
Chapelle-aux-Saints (m)

Fig. 2.

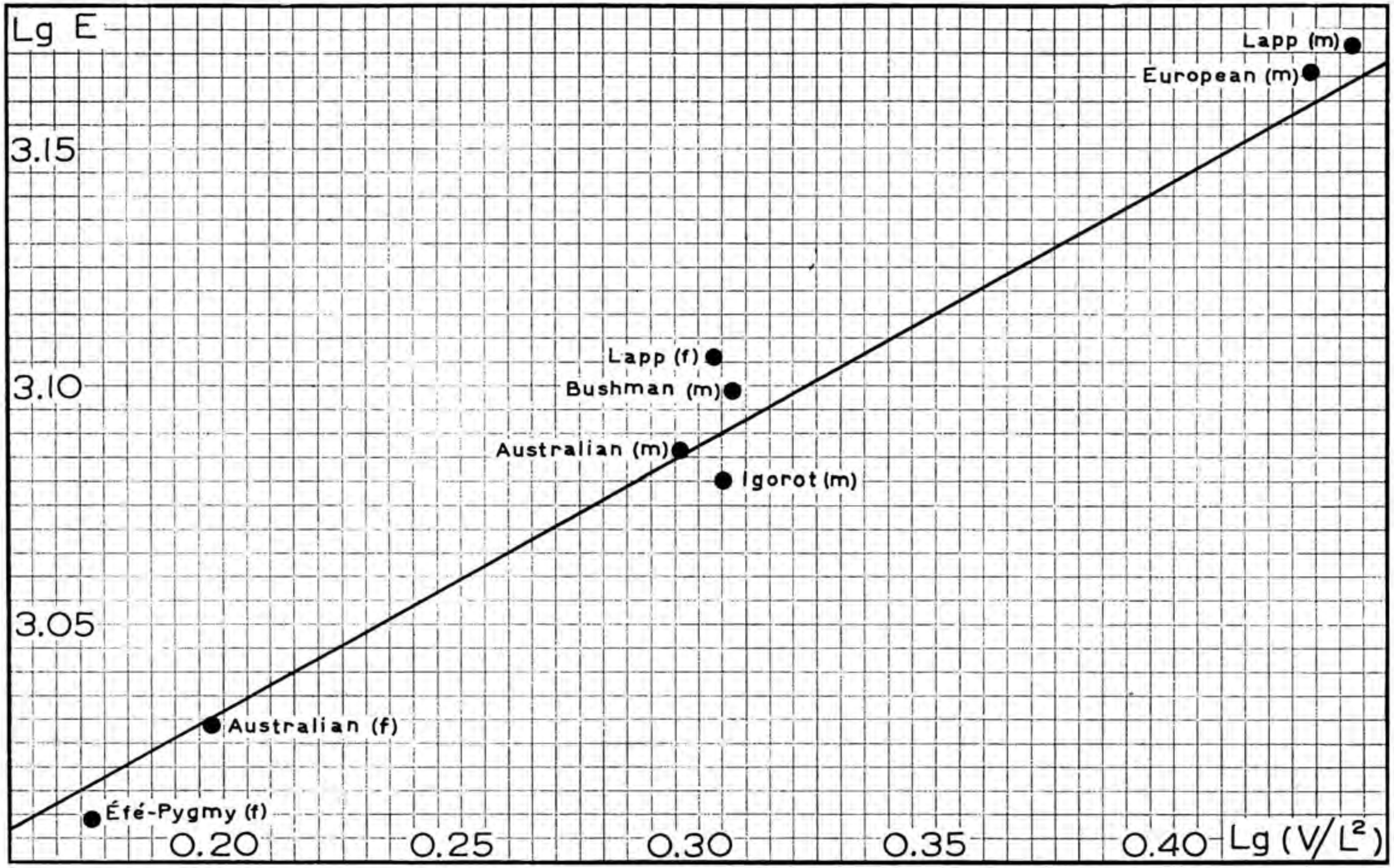
Lg E
3.15
3.10
3.05
Lapp (m)
European (m)
Lapp (f)
Bushman (m)
Australian (m)
Igorot (m)
Australian (f)
Éfé-Pygmy (f)
0.20
0.25
0.30
0.35
0.40
Lg (V/L²)

Fig. 3.

The total skull capacity is 940 cc according to McGREGOR [1]), who, starting from a cast of the internal cranium, constructed a complete encephalon comparing it with the braincast of higher apes and humans. It would seem that McGREGOR's determination, accurately linking up as it does with the actual data, comes nearest to the actual capacity.

TABLE III [2])

Name	Femur maximal length (mm)	Femur volume (cc)	Skullcapacity (measured) (cc)	Skullcapacity (calculated) (cc)	Error in %
Australian, m.	438	384	1219.2	1220	0.07
Australian, f.	401	256	1065.5	1074	0.80
European, m.	431	505	1466	1446	1.36
Lapp, m.	387	415	1490	1461	1.95
Lapp, f.	364	270	1283	1232	3.98
Igorot, m.	386	305	1200	1234	2.83
Bushman, m.	436	391	1260	1238	1.75
Efé-Pygmy, f.	364	200	1020	1043	1.02

The Pithecanthropus femur found by DUBOIS is complete and in excellent condition. Formula I, however, is not applicable here, owing to a slight pathological growth, an inflammatory exostosis, which, as is clearly seen in DUBOIS' figures, has influenced the middle of the diaphysis (periostitis and local thickening of the diaphysis). With formula I a difference of a few mm in the middle of the diaphysis already causes a considerable difference in the skull capacity calculated. By using the femur *volume* the relation becomes much less sensible to such a local deviation in the diaphysis, for which reason we prefer to apply to Pithecanthropus formula II, which is equivalent to formula I. According to DUBOIS' accurate measurements the maximal length of the Pithecanthropus femur is 445 mm, and the volume, without the exostosis, 463 cc.

When we introduce a factor x into the second term of formula II, substituting the above values for skull capacity and femur measurements, (II) becomes $940 = x.\ 38508\ .\ (463/445^2)^{7/8}$, from which we calculate

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[1]) McGREGOR, Recent studies on the skull and brain of Pithecanthropus erectus. Natural History, Vol. 25, 1925.

[2]) All these data are taken from OTTO HAUGER (Der Gehirnreichtum der Australier und anderer Hominiden, beurteilt nach ihrem Skelet. Anatomische Hefte-Merkel und Bonnet, I Abt. Hft 179, Bd. 59, München u. Wiesbaden, 1921), except those of the Efé-Pygmy, which I owe to the kindness of Prof. Dr. A. J. P. v. D. BROEK, Utrecht (personal communication).

$x = 0.7068 = 0.999/\sqrt{2} = 1/\sqrt{2}$. This calculation shows us that the cephalization stage of Pithecanthropus erectus is $1/\sqrt{2}$ times lower than that of recent and fossil humans.

The data of *Sinanthropus pekinensis* are unfoitunately not quite so good as those of Pithecanthropus erectus (one finding place and probably one individual). Yet they allow of an important conclusion.

In the year 1938, Dr. W. C. PEI discovered amond the extensive material recovered from locality 1 at Choukoutien during the excavations of 1936—37, a fragment of a femur (femur *M*) consisting of almost the entire diaphysis. Its length is 312 mm. At the upper end the head and the trochantera are broken of, whereas the entire inferior border of the neck is preserved. The lower end is likewise missing but the upper end of the suprapatellar fossa is still recognizable. Certain parts of the surface are weathered but all the characteristic features are sufficiently preserved to permit a criterion. Its relative completeness makes it possible to determine the general form. Estimated on the basis of the length of the preserved portion of the femur, the total length may have been 400 mm (WEIDENREICH). This femur is considered to belong to a female being on account of the smallness of the two main diameters of the shaft which, in the middle of the diaphysis, are 29.0 and 24.1 respectively. On the same site as femur *M*, according to WEIDENREICH, was found a fragment of a lower jaw with two teeth which also has been attributed to a female individual.

Two years before (1936) three almost complete braincases of Sinanthropus were found in Choukoutien at site *L*. All three skulls represent adult individuals, as proved by the condition of the sutures. According to the size and the general appearance of the skulls (muscle markings, etc.), WEIDENREICH considers skull I and skull III to belong to male individuals and skull II to a female. It is clear that in connection with our problem skull II proves of special interest, as the above mentioned femur also belongs to a female being. According to WEIDENREICH the skull of this female has a capacity of 1050 cc. On the ground of these data we shall now determine the cephalization stage of Sinanthropus, applying formula I. (Formula II cannot be used because of the absence of reliable estimates of the femur volume; on account of the perfect condition of the diaphysis, formula I is eminently suited for our purpose.)

Substituting in (I) 1050 for the skull capacity, 400 for maximal length and 29.9 and 24.1 respectively for the transverse and sagittal measurements in the middle of the diaphysis, and introducing an x-factor into the second term, we get for (I): $1050 = x \,.\, 1084 \,.\, (29.0\ 24.1/400)^{5/9}$, from which we calculate: $x = 0.7105 = 1.008/\sqrt{2} = 1/\sqrt{2}$.

According to these calculations, therefore, the cephalization stages of Pithecanthropus erectus and Sinanthropus pekinensis are the same, and one stage below that of recent and fossil human beings. The agreement in cephalization between the two is striking. On the ground of it we are inclined to consider *Pithecanthropus erectus and Sinanthropus pekinensis as two related types, representing similar stages in evolution.*

The question, whether this relationship is such that the two forms should be considered as belonging to two races of the same species, cannot be answered on the ground of the agreement in cephalization stage. Agreement in cephalization stage does not exclude the possibility of difference in species, and certainly not a difference in race and degree of primitiveness. This is evident from our investigation of recent and fossil human races which differ greatly in primitiveness, while having reached the same stage in cephalization, and also from an investigation of animals, where the same cephalization stage is found in related species (e.g. cat and tiger).

From animal investigation we know that the cephalization stage only indicates the boundary which groups different, nearly related forms together, while separating them from other groups. It is for closer anatomical research to diagnoze, e.g. by studying the fissuration (ARIËNS KAPPERS), the character of this relationship and the greater or lesser primitiveness within this group of related species.

Summary.

The cephalization stages of Pithecanthropus erectus and Sinanthropus pekinensis, measured by the relation existing between skull capacity and femur measurements (formula I and II) are the same; therefore we conclude that these organisms represent a similar stage of evolution.

Anatomy. — *Verdubbeling der fronttanden* (tweede mededeeling). Door TH. E. DE JONGE-COHEN. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of April 27, 1940.)

Toen wij in eene vorige verhandeling in de *Proceedings* opnieuw het probleem der tandverdubbeling ter sprake brachten (1), ontleenden wij onze voorbeelden aan die gevallen, wier vormgenese op transversale klieving van den oorspronkelijken tandaanleg berustte (*schizodontie*) [1].

In aansluiting nu aan deze mededeeling willen wij thans ten tweede male enkele beschouwingen aan hetzelfde onderwerp wijden en een geheel anderen, veel zeldzamer vorm van tandverdubbeling bespreken: het is die, welke als gevolg van longitudinale concrescentie van twee — oorspronkelijk gescheiden — tandkiemen tot stand komt en welken wij uit dien hoofde reeds eerder als *synodontie* bestempelden (2) [2].

Vooraf enkele bijzonderheden over onze gevallen.

* * *

Aantal.

Het totale aantal bedroeg 14, verdeeld over 1 bovenkaak en 11 onderkaken: in 2 daarvan droeg de anomalie een bilateraal-symmetrisch karakter (afb. 5 en 13) en met nadruk onderstrepen wij de waarlijk unieke zeldzaamheid dezer symmetrie!

Verdeeling over de verschillende gebitselementen.

Naast versmelting van medialen met lateralen snijtand kennen wij als tweede mogelijkheid coalescentie van dezen laatsten en hoektand. Zoo blijkt de anomalie de mediaanlijn derhalve niet te overschrijden: synodontie der beide mediale incisivi komt óók in ons materiaal niet voor.

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[1] BOLK (2) had deze reeds vroeger in zijne *Odontologische Studiën* als schizogene variaties beschreven, in deze benaming tevens de formale genese der anomalie tot uitdrukking brengende.

Teneinde echter haar karakter als afwijking *sui generis* scherper tegenover de in deze mededeeling te bespreken dubbeltanden te accentueeren, willen wij haar als *schizodontie* aanduiden.

[2] Aan de omschrijving „*synodontie*" is, gelijk wij destijds reeds betoogden, niet slechts boven de gebruikelijke termini technici het voordeel eener in hare beknoptheid duidelijke omschrijving verbonden, doch ligt evenzeer de gedachte ten grondslag, onze nomenclatuur zoo volkomen mogelijk met de reeds bestaande terminologie in overeenstemming te brengen: c.f. anodontie, syndactylie, sympodie.

Verdeeling over boven- en onderkaak.

Met uitzondering van één specimen in de bovenkaak kwamen al onze gevallen in de onderkaak tot ontwikkeling. Ook onder dit opzicht stemmen onze objecten vrijwel volkomen overeen met de in de literatuur neergelegde vondsten, welke immers eveneens bijkans alle op de onderkaak betrekking hebben.

Geenerlei gevallen daarentegen bij de blijvende boventanden [3]). De verklaring daarvan behoeven wij niet verre te zoeken. Versmelting toch is slechts mogelijk in het allervroegste stadium der tandontwikkeling en dan nog alleen tusschen die elementen, welke in hun tijdstip van aanleg en verkalking niet al te sterk divergeeren. Aldus is in het melkgebit bij alle fronttanden het geval: óók tusschen lateralen snijtand en cuspidatus bestaat ten deze slechts een gering tijdsinterval. Hetzelfde geldt m.m. voor de blijvende serie in de onderkaak: in de bovenkaak daarentegen legt de sterk geretardeerde ontwikkeling van den lateralen incisivus versmelting van dezen met zijne beide synergeten onoverkomelijke moeilijkheden in den weg.

Verdeeling over melkdentitie en blijvende serie.

Terwijl 10 gevallen het melkgebit betroffen, telden wij de anomalie slechts 4 malen in de blijvende rij.

Stellen deze cijfers — in volkomen overeenstemming met de bevindingen van vroegere onderzoekers — het dominante karakter der melkdentitie ten deze in het licht, nog duidelijker komt dit naar voren, wanneer wij, met uitsluiting der gipsafgietsels, onze telling tot de in de mandibulæ aanwezige objecten bepalen. Deze toch zijn alle van de sceletresten afkomstig van twee groote kerkhoven en komen derhalve de verhoudingen *durante vita* het dichtst nabij! Welnu: op een totaal aantal van ongeveer 36.000 schedels vonden wij slechts 1 casus van synodontie in het blijvende gebit tegen 3 in de melkdentitie (v.d. afb. 2, 8, 9 en 3).

Doch ook deze cijfers vormen geen zuiveren maatstaf: immers dienen wij niet uit het oog te verliezen, dat het aantal opgegraven volwassen crania dat der kinderschedeltjes vele malen overtreft. Nemen wij bovendien echter in aanmerking, dat juist van deze laatste een niet zoo heel gering percentage — en dan vermoedelijk juist de voor ons onderzoek meest belangwekkende — ingevolge eene vrij langdurige inhumatieperiode verloren gegaan zijn, dan is de conclusie gewettigd, dat het verhoudingscijfer voor het melkgebit in werkelijkheid nog belangrijk hooger ligt.

Anderzijds echter vormt ons materiaal in zijn uitgesproken voorliefde voor de melktanden een scherp contrast met de gevallen van schizodontie. De morphologische variaties der fronttanden toch maken zich alle zonder

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[3]) Gevallen van secundaire vergroeiing nà beëindigde odontogenese (ingevolge hypertrophie van het cæmentum — apicale parodontitis) vallen buiten het begrip synodontie en blijven in deze mededeeling derhalve *eo ipso* buiten beschouwing.

uitzondering bij voorkeur in de bovenkaak en dan met name in het blijvend gebit kenbaar: in de onderkaak is hunne frequentie aanzienlijk geringer! Identieke verhoudingen vinden wij m.m. in de melkdentitie terug met dien verstande, dat hier de overeenkomstige anomalieën nòg veel zeldzamer zijn dan in het blijvend gebit — m.a.w.: *het laagste percentage bij de melkelementen in de onderkaak*. Ook de schizogene varianten maken op dezen regel geen uitzondering.

Juist omgekeerd nu blijkt de stand van zaken bij synodontie: deze manifesteert zich nagenoeg uitsluitend in de onderkaak en dan bij voorkeur bij de melkelementen!

Dit onderscheid is te opmerkelijk om niet een oogenblik onze aandacht te vragen. Want het accentueert wel op bijzonder fraaie wijze het genetisch verschillend karakter van beide afwijkingen: immers dienen wij geenszins uit het oog te verliezen, dat synodontie — in tegenstelling met schizodontie — niet onder de eigenlijke morphologische gebitsvariaties gerangschikt mag worden. De laatste toch berusten alle op de activeering van latente ontwikkelingspotenties, synodontie daarentegen vormt het substraat eener ontwikkelingsstoornis.

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Het clinisch beeld der synodontie vertoont in zijn wisselend adspect sprekende gelijkenis met dat der schizodontie: deze blijkt niet slechts ectoscopisch aan de kroon waarneembaar, doch wordt in gelijke mate door het RÖNTGENbeeld der wortelformatie bevestigd.

Begrijpelijk derhalve, dat wij in onze vorige mededeeling over dit onderwerp met nadruk betoogden: „dat ook synodontie — eveneens bij voorkeur bij de voortanden — aanleiding kan geven tot de vorming van z.g. tweelingstanden, welke morphotisch alleszins, genetisch geenszins identiek zijn met de gevallen van echte tandverdubbeling" [4]).

Moge deze isomorphie in het licht van onze bovengegeven beschouwingen paradoxaal schijnen, zoodra wij ons rekenschap geven van de ontstaanswijze der dubbeltanden, is het duidelijk, dat wij — ondanks hun verschillend genetisch karakter — èn bij schizodontie èn bij synodontie in feite met eenzelfden ontwikkelingsmodus te doen hebben. Bij schizodontie: longitudinale differentiatie van den oorspronkelijken tandaanleg, daarnaast transversale insnoering met als eindstadium twee *zelfstandige* gebitselementen.

Juist omgekeerd nu verloopt dit proces bij synodontie: hier twee oorspronkelijk gescheiden tandkiemen, die hunne eigene individualiteit — waaraan slechts eene transversale insnoering de herinnering gefixeerd houdt (afb. 8 en 9) — niet wisten te handhaven: longitudinale concentratie en als eindstadium *één* ietwat verbreede tand (afb. 11).

Maar aldus levert in ons dentaal systeem de formale genese der dubbel-

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[4]) l.c. pag. 503—504.

tanden tevens het marquante voorbeeld van een ontwikkelingsgang, die in twee — aan elkander tegengestelde — richtingen verloopen kan.

En wanneer wij in aansluiting aan het bovenstaande tenslotte Uwe aandacht vragen voor den ook in de literatuur een enkele maal beschreven *„drielingstand"*, dan is het duidelijk, dat wij hier te doen hebben met eene wel uitzonderlijk zeldzame combinatie van synodontie met schizodontie: *van onze beide gevallen* (afb. 6 en 7) *manifesteerde één zich in de bovenkaak.*

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Nu zijn wij ons terdege bewust, met de door ons gegeven voorstelling van zaken een van de heerschende opvattingen ten eenenmale afwijkend standpunt in te nemen. Want ofschoon GUILFORD (4) en WEDL (5) reeds nadrukkelijk onderscheid maakten tusschen echte tweelingstanden en tandversmeltingen, houden met name de Duitsche onderzoekers — van welke wij in het bijzonder PEÇKERT (6) vermelden, omdat deze en zijne leerlingen de literatuur met een aantal belangwekkende casuïstische bijdragen verrijken — tot op den huidigen dag nagenoeg algemeen aan de door BUSCH (7) gegeven indeeling vast [5]).

Deze beschouwt niet slechts de echte synodontiegevallen als coalescentieverschijnsel doch ziet óók in de door ons als tweelingstanden beschreven gevallen van schizodontie het substraat eener versmelting — te weten tusschen een normaal en een overtollig element.

In korte samenvatting zouden wij zijne zienswijze aldus kunnen formuleeren: tandverdubbeling berust op samensmelting van twee normale elementen dan wel van een normalen met een overtolligen tand. Daartegenover ligt aan onze zienswijze de gedachtengang ten grondslag, dat wij bij dubbeltanden terdege hebben te onderscheiden tusschen echte tweelingstanden (*schizodontie*) en die vormen, welke door fusie van twee normale gebitselementen (*synodontie*) tot stand komen.

De boven besproken vormovereenkomst verdient echter nog in ander verband onze aandacht. De vraag is: hoe kunnen wij de differentieele diagnose tusschen beide dubbeltandformaties stellen? Vooropstellende, dat moeilijkheden ten deze alleen dan te verwachten zijn, wanneer — met uitsluiting derhalve der cuspidati — de dubbeltand zich binnen het gebied der snijtanden manifesteert, kan bij de beantwoording der vraag de bovenkaak nochtans gevoegelijk buiten beschouwing blijven. In de eerste plaats, wijl — gelijk wij reeds eerder zagen — synodontie hier tot de allergrootste uitzonderingen behoort: slechts zéér enkele gevallen zijn beschreven [6]) en

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[5]) Voor HERBST (8) maken wij nadrukkelijk eene uitzondering.

[6]) Een buitengewoon belangwekkende mededeeling vonden wij bij LEIGH (9). Deze toch verrijkte de literatuur met de beschrijving van een geval van bilaterale synodontie in de bovenkaak: versmelting van de drie melkfronttanden rechts en daarnaast van medialen en lateralen incisivus links achten wij een zóó uitzonderlijk zeldzaam phænomeen, dat wij de vermelding ervan niet achterwege willen laten.

zelfs deze zouden wij niet gaarne alle zonder meer voor onze rekening durven nemen. Daarnaast verlieze men niet uit het oog, dat vorm en grootte der beide incisivi in de bovenkaak gemeenlijk véél sterker divergeeren dan in de onderkaak. Maar tenslotte: reeds vroeger hebben wij er den nadruk op gelegd, dat bij den echten tweelingstand de sulcus, welke zijne beide segmenten onderling afgrenst, vrijwel altijd een smallere mediale zône van een ietwat breedere distale scheidt. Welnu: juist omgekeerde verhoudingen zouden wij in de bovenkaak bij vergroeiing van medialen met lateralen snijtand mogen verwachten!

In de onderkaak daarentegen kan het beeld der synodontie in melkdentitie en blijvend gebit zóó sprekende gelijkenis vertoonen met den echten tweelingstand, dat wij de verleiding niet weerstaan kunnen, twee voorbeelden van dezen laatsten weer te geven (afb. 14 en 15).

De isomorphie der beide dubbeltandformaties blijkt hier zóó volkomen, dat slechts hare relatie tot de omgeving haar morphologisch karakter bepalen kan. Daarbij kan *„Symmetrievergleich"* van niet te onderschatten beteekenis blijken, maar beslissend voor elken casus is het aantal der tanden.

Laten wij die gevallen buiten beschouwing, bij welke de splitsingstendenz haar eindstadium bereikt heeft en de tweelingstand tot twee tanden uitgegroeid is, dan kunnen wij vaststellen, dat bij schizodontie het oorspronkelijke aantal bewaard blijft, terwijl dit bij synodontie uitteraard met één verminderd is — overigens normale gebitsverhoudingen vooropgesteld natuurlijk.

Waar de diagnose derhalve in hoofdzaak *ex juvantibus* gesteld moet worden, zal het practisch vrijwel nooit mogelijk blijken, geïsoleerde gevallen van snijtandsverdubbeling nader te determineeren.

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Tenslotte vraagt ook de causale genese een oogenblik onze aandacht.

Terwijl de ontplooiing eener sluimerende ontwikkelingspotentie het beeld der schizodontie morphologisch het karakter eener progressieve vormdifferentiatie verleent, dragen de gevallen van synodontie alle den stempel eener morphotisch regressieve ontwikkelingstendenz.

Voor hunne totstandkoming zouden wij mèt STURM (10) ruimtegebrek aansprakelijk willen stellen, wel te verstaan in dien zin, dat te geringe afstand tusschen twee tandkiemen zelfstandige ontwikkeling van *beide* in den weg staat (afb. 1).

Aldus een vitium primæ formationis, vormt synodontie een opmerkelijke bijdrage tot de pathologie der ontwikkeling!

SAMENVATTING.

Tandverdubbeling ingevolge synodontie der fronttanden kennen wij bij beide dentities. Bij voorkeur echter manifesteert zij zich bij de ondertanden

van het melkgebit. Naast coalescentie van medialen met lateralen snijtand kennen wij bovendien versmelting van lateralen incisivus met cuspidatus: de mediaanlijn daarentegen schijnt de anomalie niet te overschrijden!

Hare vormgenese berust op longitudinale concrescentie van twee oorspronkelijk gescheiden tandkiemen en verloopt aldus geheel tegengesteld aan die der schizodontie.

ZUSAMMENFASSUNG.

Zahnverdoppelung infolge Synodontie der vorderen Zähne ist bei beiden Dentitionen bekannt. Vorzugsweise tritt sie bei den unteren Zähnen des Milchgebisses in Erscheinung. Neben der Coalescenz des medialen mit dem lateralen Schneidezahn ist uns ausserdem die Verschmelzung des lateralen Incisivus mit dem Cuspidatus bekannt: die Anomalie jedoch überschreitet anscheinend die Medianlinie nicht.

Ihr Formgenese beruht auf längswändiger Verschmelzung zweier ursprünglich geschiedenen Zahnkeimen und verläuft somit ganz entgegengesetzt an der der Schizodontie.

RESUME.

Redoublement des dents à cause de synodontie des incisives est connu des deux dentitions. Il se manifeste de préférence aux dents de lait inférieures. A part de la coalition de la médiale avec l'incisive latérale, nous connaissons également la fusion de l'incisive latérale avec la canine; l'anomalie par contre ne paraît pas dépasser la ligne médiane.

Sa génèse est fondée sur la concrescence longitudinale de deux germes de dents séparées à l'origine et se fait donc tout à fait en sens contraire de celle de la schizodontie.

SUMMARY.

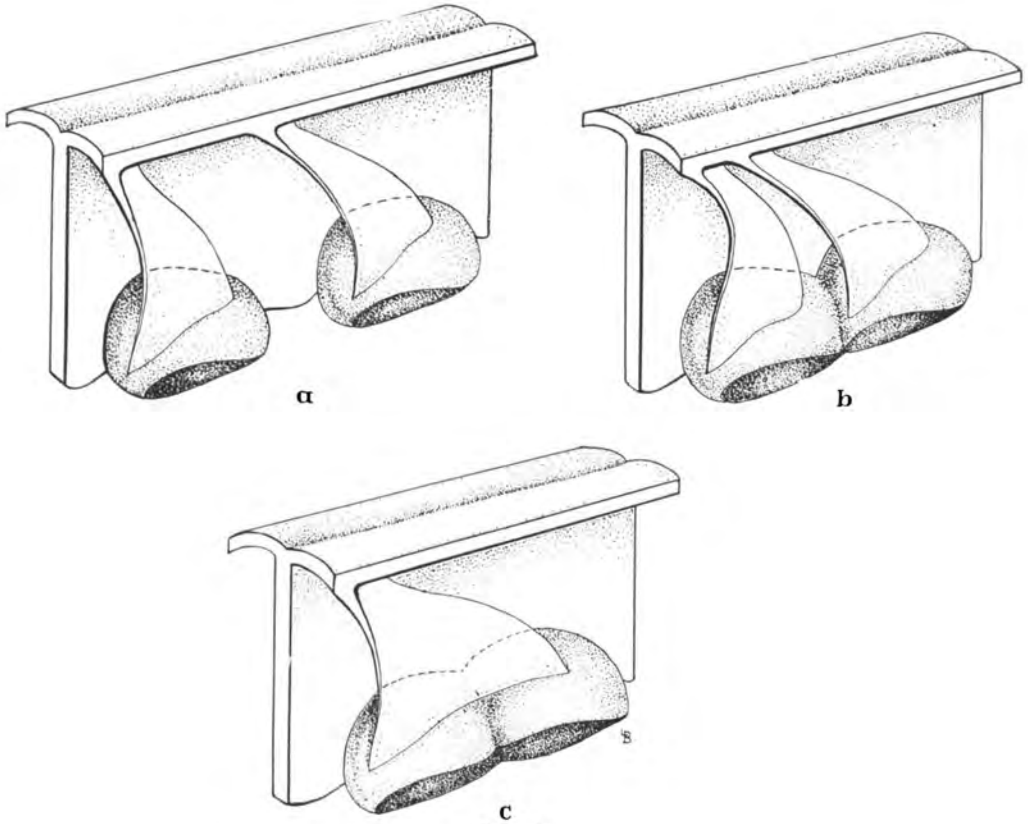
Gemination of the teeth in accordance with synodontia of the frontteeth is wellknown in both dentitions. By preference however gemination appears in the milkteeth of the lower jaw.

Beside coalescence of the medial and lateral incisor we know moreover the fusion of the lateral incisor with cuspidate: the anomaly on the other hand seems not to cross the medianline.

Its genesis is based on longitudinal concrescence of two originally separated toothgerms, and takes place in a sense opposite to that of schizodontia.

Anatomisch-embryologisch Instituut der Universiteit
van Amsterdam. Mauritskade 61, O..

TH. E. DE JONGE-COHEN: VERDUBBELING DER FRONTTANDEN. II.



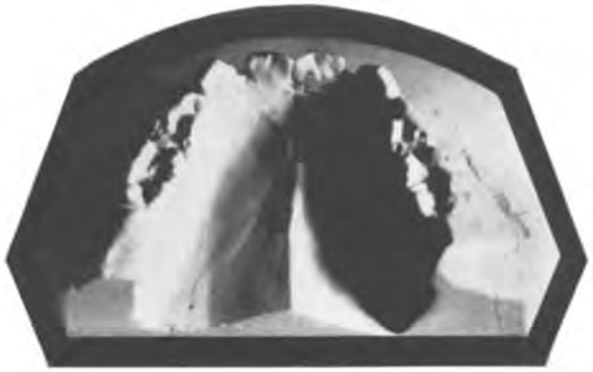
Afb. 1.

Afb. 2a.

Afb. 2b.

Afb. 3.

Afb. 4.



Afb. 5a.

Afb. 5b.

Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

TH. E. DE JONGE-COHEN: Verdubbeling der fronttanden. II


Afb. 6*a*.


Afb. 6*b*.


Afb. 6*c*.


Afb. 7.


Afb. 8*a*.


Afb. 8*b*.


Afb. 9.


Afb. 10.


Afb. 11.


Afb. 12.


Afb. 13 (van oraal gezien).


Afb. 14*a*.


Afb. 14*b*.


Afb. 15.

Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

TOELICHTING DER AFBEELDINGEN.

Afb. 1*a*, *b* en *c* : Drie opeenvolgende phases in de vormgenese der synodontie.

Synodontie van medialen en lateralen incisivus.

Afb. 2*a* en *b* : melkdentitie,
afb. 3 en 4 : blijvend gebit,
afb. 5*a* en *b* : geval van bilateraal-symmetrische synodontie der beide snijtanden,
afb. 6*a*, *b*, *c* en 7 : synodontie der beide elementen in onder- en bovenkaak, gepaard gaande met schizodontie: dens trigeminus!

Synodontie van lateralen incisivus en cuspidatus.

Afb. 8*a* en *b*, 9 en 10: melkdentitie,
afb. 11 en 12 : blijvende serie,
afb. 13 [7]) : geval van bilateraal-symmetrische synodontie.

Schizodontie.

Afb. 14*a* en *b* : lacteale serie (laterale incisivus),
afb. 15 : permanent gebit (mediale incisivus).

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    Zahnanomalien. Ut supra, Jaargang XLI, Afl. 23 (1938).

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[7]) Wij danken dezen casus aan de vriendelijke bereidwilligheid van Dr. STURM te *Gerstungen*, die hem zelf elders reeds beschreef (10).

NEDERL. AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS

VOLUME XLIII

No. 7

President: J. VAN DER HOEVE
Secretary: M. W. WOERDEMAN

CONTENTS

Chemistry. — *On reversible phase-transitions in solid refractory metals.*

By Prof. Dr. F. M. JAEGER.

(Communicated at the meeting of June 29, 1940.)

§ 1. During the last twenty years a great number of systematical investigations have been made in the Laboratory of Inorganic and Physical Chemistry of the Groningen University, concerning the phase-transitions occurring in refractory metals at temperatures between 0° C. and about 1650° C., whilst the accompanying phenomena were studied in detail. The metals here considered all belong to the 4th till 8th group of the periodic table: more especially they bear the atomic numbers: 22 to 28, 40 to 46

4th Group	5th Group	6th Group	7th Group	8th Group				
22	23	24	25	Atomic Number	26	27	28	
Ti	*V*	*Cr*	*Mn*	Element	*Fe*	*Co*	*Ni*	
47.9 (5 isot.)	50.95 (1 isot.)	52.01 (4 isot.)	54.93 (1 isot.)	Atomic Weight	55.84 (4 isot.)	58.94 (2 isot.)	58.69 (5 isot.)	IV th per.
1730°C.	1715°C.	1765°C.	ca. 1245°C.	Meltingpoint	1530°C.	1490°C.	1455°C.	
2 (3)	1	2 (3)	3	Number of Solid States	4 (2)	3 (2)	3 (2)	
40	41	42	43	Atomic Number	44	45	46	
Zr	*Nb*	*Mo*	*Ma*	Element	*Ru*	*Rh*	*Pd*	
91.2 (5 isot.)	92.91 (1 isot.)	95.95 (7 isot.)	(ca. 99) (—)	Atomic Weight	101.7 (6 isot.)	102.91 (2 isot.)	106.7 (6 isot.)	Vth per.
2130°C.	1950°C.	2625°C.	—	Meltingpoint	2450°C.	1966°C.	1554.5° C.	
2 (3)	1	1	—	Number of Solid States	4	2	1	
72	73	74	75	Atomic Number	76	77	78	
Hf	*Ta*	*W*	*Re*	Element	*Os*	*Ir*	*Pt*	
178.4 (5 isot.)	180.88 (1 isot.)	183.92 (4 isot.)	186.31 (2 isot.)	Atomic Weight	191.5 (6 isot.)	193.1 (2 isot.)	195.23 (5 isot.)	VIth per.
2230°C.	3030°C.	3370°C.	3440°C.	Meltingpoint	2700°C.	2454°C.	1773.5° C.	
2	1	1	1	Number of Solid States	1	1	1	

and 72 to 78, including the three series of elements ranging from *titanium* to *nickel*, from *zirconium* to *palladium* and from *hafnium* to *platinum*. Their meltingpoints (under the pressure of 1 atm.) all lie between about 1500° and 3500° C.; many of them belong to the technically most important materials, either in the pure or in the alloyed state.

After the recently ¹) finished investigation of *cobaltum* in this respect, the work which had to be done concerning the 8th group can now be considered as accomplished; so that, therefore, a summarizing review of the results obtained in the latter group, seems not to be out of place here.

§ 2. A few additional remarks may be made here about the methods applied in these investigations. As the purpose was to collect not only the necessary data about the accurate situation of the characteristic transition-temperatures, but also to make sure that no phase-transitions in the solid state could be overlooked, — it appeared from the beginning adviseable to determine those data by *several different* experimental methods in such a way, that the values obtained could be controlled by mutual comparison; moreover, that these values should preferentially not be gained by *separate* observations at *arbitrarily selected* temperatures, but by an experimental device which would allow us to acquire a record of the dependence of the studied properties on *all intermediate* temperatures. For this reason the method of the optically-coupled double-galvanometer, as first designed by SALADIN and LE CHATELIER, in combination with a photographic registration of the curves obtained, was modified in several directions; so that now it allows the direct record on the photographic plate of the curves for the *relative heat-capacity*, the *thermal expansion*, the *electric resistance* and the *thermo-electrical* behaviour of the metals studied in function of the temperature (as abscissa). As soon as in this way the true situation of the remarkable points on the curves is sufficiently ascertained, the accurate measurements of the absolute specific heats by means of the metalblock-calorimeter at carefully selected intervals of the temperature are made in the way that has already often been described ²). In these experiments

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¹) F. M. JAEGER, E. ROSENBOHM and A. J. ZUITHOFF, Rec. d. trav. d. Chim. d. Pays-Bas, **59**, 831 (1940); conf. also: F. M. JAEGER and A. J. ZUITHOFF, these Proceed., **43**, 815 (1940).

²) For these methods, compare e.g.: F. M. JAEGER, E. ROSENBOHM and J. A. BOTTEMA, Rec. Trav. chim. d. Pays-Bas, **52**, 61 (1933); F. M. JAEGER and W. A. VEENSTRA, ibid., **53**, 677 (1934); F. M. JAEGER, E. ROSENBOHM and R. FONTEYNE, ibid., **55**, 615 (1936); E. ROSENBOHM, Physica, **5**, 385 (1938); F. M. JAEGER, E. ROSENBOHM and J. A. BOTTEMA, Rec. d. Trav. d. Chim. d. Pays-Bas, **57**, 1137 (1938); F. M. JAEGER and E. ROSENBOHM, Rec. trav. chim., **47**, 513 (1928); id. Physica **6**, 1123 (1939); F. M. JAEGER and E. ROSENBOHM, Rec. trav. chim. **51**, 1 (1932); F. M. JAEGER and W. A. VEENSTRA, Proc. Kon. Akad. v. Wetensch., Amsterdam, **37**, 280 (1934); M. EWERT, Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, 833 (1936); F. M. JAEGER, E. ROSENBOHM and A. J. ZUITHOFF, Rec. trav. chim. **57**, 1313 (1938); A. J. ZUITHOFF, Rec. trav. chim. **58**, 131 (1939); etc.

the metals were always studied either enclosed in the usual evacuated platinum crucibles of special construction, or within an electrically heated high-vacuum furnace; the metals must beforehand be freed as completely as possible from adherent or absorbed gases.

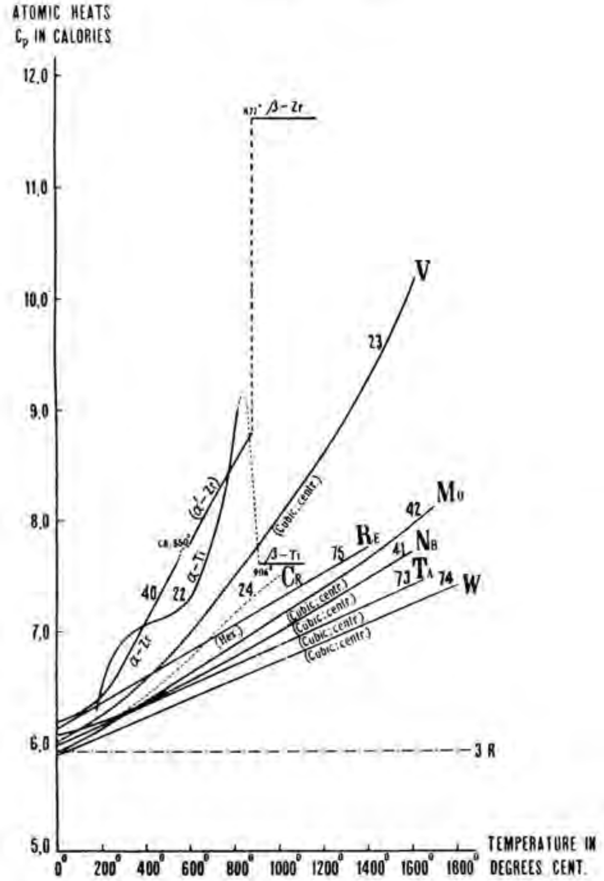

Fig. 1. *The atomic heats C_p of some elements of the 4th, 5th, 6th and 7th group.*

§ 3. With the exception of some elements of groups IV to VII, like *chromium, manganese, masurium* and *hafnium*, — which were not available either in sufficient quantities or in the necessary degree of purity, — the numerical data for all the other metals were finally obtained. It can here be remarked, that the metals of the 5th, 6th and 7th groups, in general proved *not* to occur in more than a *single* crystalline state; thus e.g. in the case of *vanadium, niobium, tantalum, molybdenum, tungsten, rhenium,* etc. (Fig. 1). This fact is contrary to the behaviour of the metals of the 4th group, which all very conspicuously manifest the phenomenon of multiple states. But *all* these elements, — just as is the case with most other metals, — prove to have values for their atomic heats C_p, which, — even at the lower temperatures (50°—100° C.), — are far superior to the theoretical limiting value of $3R$ (= 5,965) calories. —

§ 4. If the elements of the 8th group now are considered here more in detail, — then the general aspect is quite a different one. In the first place it can be stated, that here only the *last tetrade* of elements of this group: *platinum, iridium, osmium* and *palladium, do not manifest the phenomenon of multiple states.* Indeed, they show *no trace* whatsoever of a change of their internal condition between 0° and 1650° C.; they preserve their original structure even at the highest temperatures considered, — as may be seen (Fig. 2) from the perfectly smooth and uninterrupted shape of all curves representing the numerical values of their physical properties in their dependence on the temperature, e.g. of their specific heats, electrical resistance, etc. Especially in the case of *platinum*, — the mean specific heats of which are now known with an accuracy of 0,1 % of their values, — this fact is of the utmost practical importance, because this indispensable metal is used in *all* high-temperature investigations as a *standard*-material for all comparative and absolute measurements; and its different physical properties in function of the temperature must, therefore, everywhere be taken into account. The certitude, that up to the highest temperatures below the meltingpoint all physical properties of *platinum* remain *continuously*

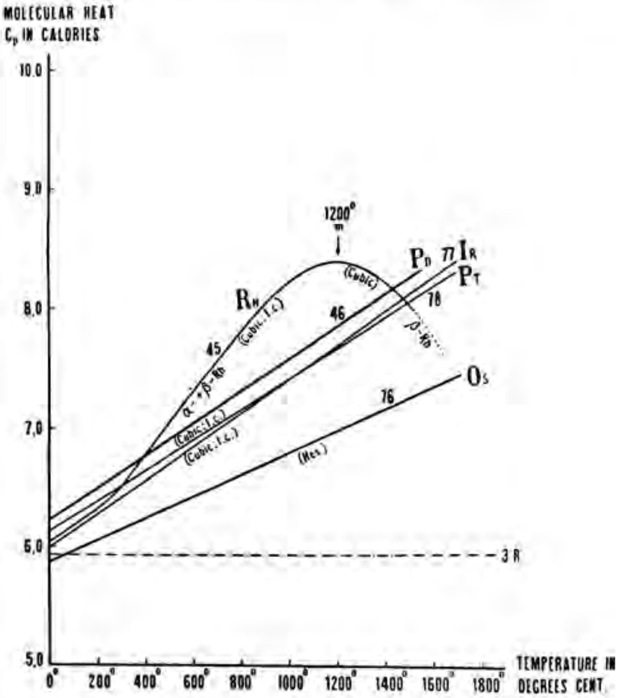

Fig. 2. *The atomic heats C_p of five elements of the 8th group.*

variable with the temperature, is, therefore, of essential interest to the practice of high-temperature experiments.

An apparent, small discrepancy in the c_p-t-curve of palladium, — originally found as a weak maximum — was afterwards proved as *non-existent*, — it only being caused by the adoption of an erroneous figure for the meltingpoint of the metal by DAY and SOSMAN in their work on the nitrogengasthermometer: a redetermination of the meltingpoint of *palladium* in this laboratory yielded the value: 1554°,5 C. instead of the old value: 1549°,2 C. After the corresponding corrections for the indications of our thermocouples had been applied, the deviations of the c_p-t-curve of *palladium* proved to have completely disappeared: *palladium*, up to its meltingpoint, therefore, exhibits a *quite normal* behaviour, — just like the three other platinummetals here considered.

§ 5. In rigorous contradistinction now to the behaviour of the four *latter* elements of the 8th group, is that of the *first tetrade*: *iron, cobaltum, nickel* and *ruthenium*. For these metals all show the phenomenon of multiple phase-transitions in the solid state in a most conspicuous way. Thus *iron* shows four, *cobaltum* three, *nickel* two (or three) and *ruthenium* no less than four, perhaps even five, of such successive reversible changes in the solid state; the field of stability of each of them is, according to the rules of thermodynamical equilibrium, always limited by as many sharply defined temperature-boundaries. In the cases of *iron, cobaltum* and *nickel* there are, in the first place, their "CURIE-points", — i.e. the temperatures at which their *ferromagnetic states* suddenly disappear and make place for weakly-paramagnetic phases; which, in their turn, may at other temperatures be substituted by still different ones. Whilst in the case of *iron* the CURIE-point (760° C.) is, however, *first* reached and *then* the other transformations afterwards occur at 906° C. and 1401° C. successively, — the sequence of these phenomena in *cobaltum* (and probably in *nickel*) is exactly *the reverse*: with *cobaltum* e.g. *ferromagnetic α-cobaltum* proves at 445° C. *first* to be transformed into equally *ferromagnetic β-cobaltum*, with an accompanying change of crystalline structure; subsequently the CURIE-point of the latter is reached at 1125° C., where *β-cobaltum* is changed into paramagnetic *γ-cobaltum*. In all three cases, however, *the transitions at the* CURIE-*point appear never to be accompanied by a change of the crystalline architecture* of the original phase. In the case of *cobaltum* the change at 445° C. (with *nickel* perhaps also at 345°; CURIE-point at 355°—360° C.) is accompanied by a typical hysteresis-effect, — the details of which have been discussed in detail in our paper in the *Recueil* [1]).

Most remarkable with respect to its phase-transitions is *ruthenium*. —

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[1]) F. M. JAEGER, E. ROSENBOHM and A. J. ZUITHOFF, Rec. d. trav. d. Chim. d. Pays-Bas, **59**, 831 (1940); conf. also: F. M. JAEGER and A. J. ZUITHOFF, Proc. Ned. Akad. v. Wetensch., Amsterdam, **33**, 815 (1940).

which in its general points appears to behave in a similar way as *iron*. But *ruthenium* is *paramagnetic* and, therefore, does *not* possess a CURIE-temperature. The metal has three or four transition-points at 310°, 1035°, 1200° and 1500° C. respectively; the nature of the transition-point at 310° C., deduced from the recent measurements of the temperature-coefficient of the electrical resistance, still remains problematic 1). The corresponding transformations at these different temperatures, — as well on heating as on cooling, — all occur *rather rapidly*, without retardation-phenomena of any appreciable significance. Recently this fact was once more qualitatively corroborated by us during a study of the electrical

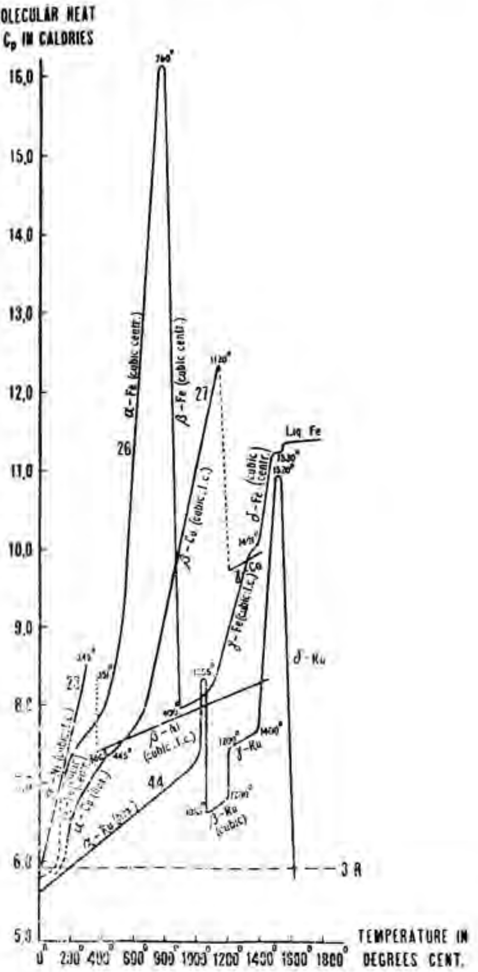

Fig. 3. *The atomic heat of the four first elements of the 8th group.*

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1) There seems *not* to occur a change of the crystalline structure of the metal at this temperature.

resistance of the metal in its dependence on the temperature, which will be published later.

§ 6. Thus it becomes evident, that the *first* tetrade of metals of the 8th group manifests the phenomenon of enantiotropous phase-transitions to a high degree, whilst the *latter* tetrade of this group, does *not* show it at all. Between these two tetrades, however, the element *rhodium* is placed, — like a central link between the two groups. Now *rhodium* shows a c_p-t-curve of an entirely different character, as it consists of *a single, steeply rising, smooth curve* with an unmistakable *maximum* at about 1204° C.; then subsequently the curve at higher temperatures falls-off rather rapidly. As a consequence of a number of experimental facts, earlier investigators have already come to the conclusion, that the internal condition of *rhodium gradually* changes to an appreciable degree, when the metal is heated between 1200° and 1300° C. Thus already in 1908 MENDENHALL and INGERSOLL drew attention to the fact, that at about 1200° C. an abnormal increase in the radiation of the glowing metal is observed; and in 1931 DIXON stated a similar behaviour with respect to the photo-electrical and the thermo-ionic phenomena. In this laboratory an analogous fact was observed in connection with the temperature-coefficient of the electrical resistance, measured in a high vacuum, which also has a maximum between 1200° and 1300° C.

Although the fact, that the inner condition of *rhodium changes* on heating can hardly any longer be doubted, — this change appears to be a *gradual* one. There is nowhere observed any discontinuity in the curves representing the measured properties in function of the temperature. X-ray examinations of heated *rhodium*-wires in a vacuum, however, seem to indicate with much more probability, that metallic *rhodium* at each temperature consists of *a homogeneous mixture* of *two different cubic forms* of the metal in variable proportions, — their ratio being characteristic for each temperature. The quantity of the face-centred, cubic α-form of the metal predominates at the *lower*, the cubic β-form (with a simple cubic elementary cell), on the contrary, prevails at the *higher* temperatures. On the other hand it proves not to be possible to isolate either of the two forms in a completely pure state. At each temperature the metal is a homogeneous mixture of α-*Rh* + β-*Rh*, in which the two forms are in *dynamical equilibrium* with each other at each temperature; so that, when the latter is more or less completely established, the composition of the phase must appear to be continually-variable with the temperature; — in the same way as this seems also to have been observed in the case of the equally cubic α- and β-forms of *manganese*.

In this way, therefore, the metal *rhodium*, with its "dynamic allotropism", evidently represents *the natural transition-term* between the *first* tetrade of elements in the 8th group, — with its conspicuous "static" allotropism —, and the *latter* one with its complete lack of reversible changes in the solid state.

From all this, it thus becomes evident, that also with respect to the phenomena here studied, the eighth group of the periodic system in its present form really constitutes *an entity*, with the rigorous character of true systematical uniformity.

Groningen, Laboratory for Inorganic and Physical
Chemistry of the University.

Anatomy. — *Arrhénoblastomes et changement de sexe.* Par G. KREDIET.

(Communicated at the meeting of June 29, 1940.)

Le changement de sexe était déjà connu du temps des Romains. TACITE mentionne que l'on devait signaler chaque année le nombre des poules changées en coqs. De même, au Moyen Age, ce phénomène était parfaitement connu et on le considérait comme une incarnation de démon; il en résultait que la personne, qui subissait ce changement ou chez laquelle un animal présentait cette métamorphose, était brulée vive. A l'heure actuelle, il y a peu de cultivateurs qui ignorent que les vieilles poules peuvent changer de sexe. On considère cependant encore ce changement comme une espèce de prodige. Est moins connu le fait que la femme et les animaux domestiques femelles peuvent prendre le sexe masculin; les cas ne sont pas rares, en effet dans la littérature on en signale déjà une quantité.

La signification biologique de ce phénomène est restée longtemps ignorée. Grâce à GOLDSCHMIDT les connaissances à ce sujet sont devenues plus étendues. Cet auteur profita de l'expérience des éleveurs-amateurs de papillons, qui avaient observé que par l'accouplement de différentes races de Lymantria dispar on pouvait obtenir des papillons, dont le sexe était intermédiaire entre celui des individus mâles et femelles. Il a systématiquement poussé plus loin ces expériences, les a contrôlées au point de vue génétique et histologique et est arrivé à la conclusion qu'au moment de la fécondation l'intersexualité, ainsi que son degré, sont déjà déterminés. Deux notions importantes se sont fait jour au cours de ces observations, notamment celle de l'épistase et celle du moment où la transformation de sexe doit avoir lieu. Lors de la détermination du sexe des facteurs mâle M et femelle F interviennent. Chez un individu mâle le facteur M est le plus fort et chez un individu femelle c'est le facteur F. Si un individu doit être mâle pur, le facteur M doit être tellement puissant que le facteur F n'est plus en état d'exercer une influence quelconque sur le sexe et ses attributs. Chez un animal femelle c'est l'inverse. Cette prédominance ou épistase doit avoir une certaine grandeur, en d'autres termes doit passer un minimum, pour pouvoir dominer complètement l'autre sexe. Si la différence est plus petite que le minimum épistatique, alors les bases sont jetées pour l'édification d'un intersexué. Il est cependant un fait remarquable, comme l'examen histologique l'a d'ailleurs prouvé, qu'en pareil cas l'évolution du sexe commence par ce que d'après le mécanisme X—2X doit être le sexe, mais qu'à un certain moment de la différenciation cette évolution prend la direction hétérosexuelle et se développe ultérieurement en ce sens. On peut facilement s'en faire une idée avec l'aide du schéma ci-joint. F représente la courbe des facteurs fémelles; M1—M9 les facteurs mâles d'après leurs différentes possibilités. M9—F figure les facteurs de l'animal mâle. Entre M9 et F il y a une prépondérance suffisante de M9, de sorte que les facteurs mâles sont toujours les plus puissants. En sens inverse la même situation se réalise pour M1—F, qui représentent donc les courbes d'un animal femelle. Toutes les autres combinaisons de M avec F figurent la situation chez les intersexués. Aux points a, b, c, d, e, f et g les facteurs M de l'animal originairement femelle deviennent plus puissants et le changement de sexe a donc lieu. A l'endroit M8—F M est plus puissant qu'au point M2—F; l'influence de F est au contraire plus faible.

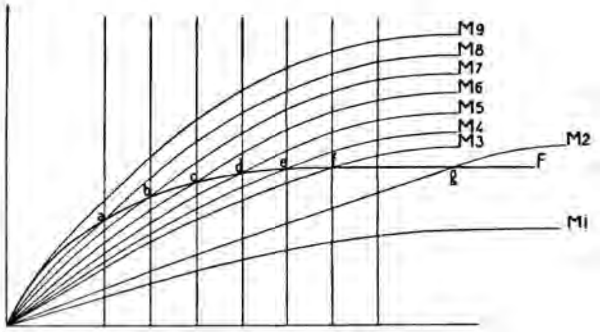

Fig. 1.

A l'aide de ce schéma il est possible d'interpréter les diverses formes que peut revêtir l'intersexualité chez l'homme et chez l'animal. La difficulté de pouvoir rechercher les causes de l'intersexualité chez les animaux supérieurs réside dans le fait que le changement de sexe s'opère dans la plupart des cas pendant la vie intrautérine, donc avant la naissance et qu'il est par conséquent difficile de trouver des individus propres à démontrer le processus de ce changement. Dans la majorité des cas on a du se contenter de déduire des phénomènes observés le stade de l'évolution sexuelle, dans lequel le changement avait eu lieu. A ce point de vue, les cas où le changement a pu être constaté immédiatement après la naissance revêtent une importance particulière. Le plus souvent les symptômes s'y rapportant peuvent être observés aux gonades de porcelets nouveaux-nés, rarement chez la chèvre et exceptionellement chez l'homme (cas de Blair Bell).

A cet égard les oiseaux fournissent un matériel intéressant. Ils montrent de la façon la plus nette le changement de sexe. Il est possible qu'en de pareils cas on ait à faire à des intersexués ordinaires, mais aussi à des animaux porteurs de tumeurs de l'ovaire, du péritoine ou de la couche corticale des glandes surrénales. Chez l'homme on a observé que les cas de changement de sexe, s'opérant après la naissance sont dans la grande majorité des cas caractérisés par la présence de tumeurs de l'ovaire ou de la couche corticale des surrénales. A la suite de cette constatation, le problème de l'intersexualité s'est étendu de façon telle que de nouvelles difficultés d'interprétation ont apparu. On était tenté de croire que les tumeurs pourraient être la cause du changement des caractères sexuels, surtout lorsqu'après leur extirpation, on constata un retour au sexe primitif. Ce raisonnement était admissible, car d'après les expériences mémorables de STEINACH, LIPSCHÜTZ et autres, il a été prouvé, qu'il était possible de masculiniser ou de féminiser un castrat par la transplantation de gonades hétérosexuelles ou par l'injection des hormones hétérosexuelles. L'intersexualité expérimentale est donc un fait acquis; elle serait semblable à celle provoquée par des tumeurs ovariennes ou surrénales, parce qu'on admet que celles-ci sécrètent des hormones hétérosexuelles. Ces tumeurs seraient donc primaires. En opposition avec cette manière de voir, il y en a une autre, qui met sur le même pied les intersexués ordinaires et les cas de masculinisation avec tumeur et qui admet que le changement de sexe est la cause et la tumeur la consequence; cette dernière devant être assimilée à une gonade intersexuee.

Il n'est assurément pas sans intérêt d'examiner lequel des deux points de vue est le plus admissible. Je voudrais essayer par une étude comparative de la masculinisation chez l'homme et chez l'animal, de rechercher s'il s'agit d'intersexués ou de masculinisation par tumeur. Les cas humains ne me sont connus, cela va de soi, que par la littérature; personellement j'ai observé le changement du sexe féminin en masculin chez deux chèvres, quelques poules et canes, un cheval et une vache. Les symptômes furent en résumé les suivants: Une des chèvres avait été en chaleurs et saillie par un bouc. Par après l'animal n'avait plus été en chaleurs, mais n'avait pas non plus été fécondé. La tête changea, reçut un fort revêtement pileux, devint plus large; l'expression des yeux devint plus méchante et, chose curieuse, le pis s'hypertrophia fortement et donna même 1½ Litres de lait normal par jour. Plus tard la sécrétion lactée diminua et le pis devint plus petit. La conduite de l'animal changea fortement. D'abord maniable comme une chèvre, il devint plus tard intraitable et même dangereux. Auprès d'une chèvre, il montrait un vif appétit sexuel mâle; avec des boucs il commençait immédiatement à se battre. A l'autopsie on découvrit deux ovariotestes. La seconde chèvre était encore jeune quand je l'achetai. Elle était en piètre état d'entretien. Du fait que, peu de temps après, l'état général s'améliorait et que le pis commençait à sécréter du lait, le soigneur avait conclu que l'animal avait été fécondé. De même que chez l'animal précédent l'appareil génital externe était resté femelle et le pis se met à sécréter du lait. Cet organe n'atteignit cependant pas les mêmes proportions que chez le premier animal, la sécrétion lactée ne se maintint pas aussi longtemps. Entretemps l'extérieur s'était rapproché de celui d'un bouc; auprès d'une chèvre l'animal ne restait pas indifférent, mais se comportait, surtout auprès de chèvres en chaleurs, d'une façon fort mâle. Vis á vis de boucs, il engageait immédiatement le combat. Ce processus de changement s'accomplit au cours de 4 mois de temps. A l'autopsie, on trouva également deux ovariotestes. Chez aucune de ces deux chèvres, il n'existait une hypertrophie du clitoris.

Le cheval, une jument, fut remarqué dans une infirmerie militaire à cause de ses manières d'étalon, qu'il n'avait pas montrées auparavant. Il saillissait les juments, prenait une attitude d'entier, acquit l'expression de regard typique pour le mâle et laissait l'impression que les ganaches et de ce fait la tête étaient devenues plus lourdes. Le vétérinaire militaire traitant était d'avis que le clitoris était plus développé que normalement; mais à la clinique de la faculté vétérinaire d'Utrecht, où le cheval fut amené, on était d'avis que le développement du clitoris n'avait rien d'anormal. Il est probable que le clitoris était en train de se développer, mais l'animal n'ayant pas été vu antérieurement par celui qui l'examina, celui-ci n'a pas osé conclure que l'organe était trop développé. Au cours de l'opération, on trouva un ovaire gauche normal et un ovaire droit fortement agrandi par une tumeur. Cette tumeur fut examinée par ROBERT MEYER de Berlin, qui diagnostiqua un cystadenome de la partie tubulaire, mais il est à remarquer que de nombreux kystes possédaient une structure rappelant celle d'un follicule, de sorte que le diagnostic folliculome tubulaire aurait également convenu.

Les cas de masculinisation que j'ai observés chez les poules et une fois chez une cane, ont été décrits in extenso dans l'Archiv für Gynaekologie, Bd. 158, 1934, 22—53. J'ai pu les diviser en 4 groupes, présentant en ordre croissant des signes de masculinisation de plus en plus prononcés et allant de pair avec des changements ovariens parallèles. Deux données capitales se dégageaient de cette étude. En premier lieu, que les changements morphologiques externes étaient identiques chez les animaux d'un même groupe, de sorte que seule l'autopsie pouvait déterminer s'il s'agissait d'une tumeur ovarienne ou d'un ovaire évoluant dans les sens d'un ovariotestis ou d'un testicule. En second lieu, que quelle que soit la constatation faite à l'autopsie, les tumeurs aussi bien que les changements testiculoïdes émanaient des mêmes cellules embryonnaires; les cellules-mères de ces néoformations étaient les cellules médullaires et corticales, qui n'interviennent pas dans la formation folliculaire.

La vache, qui changea en taureau, avait été une fois en chaleurs; elle avait été saillie, mais n'avait pas été fécondée. A part les organes génitaux externes, elle acquit tous les attributs de mâle. A l'autopsie, on découvrit deux testicules. En ce qui concerne la genèse de la masculinisation, les recherches effectuées ne fournirent pas les résultats qu'on aurait pu espérer chez un animal qui avait été suivi de si près.

A part ces cas, que j'ai pu observer au cours d'une vingtaine d'années, il m'a été possible d'examiner quelquefois des préparations microscopiques de gonades de porcs intersexués en bas âge, qui montraient un processus de changement de sexe encore en cours. Presque sans exception, la transformation s'opérait du sexe féminin vers le sexe masculin.

Aussi bien dans les cas signalés dans la littérature, que dans les cas mentionnés plus haut, il est possible de distinguer deux groupes, que j'ai désignés sous les noms de *intersexués physiologiques* et *intersexués patho-*

logiques. Les derniers sont des individus, chez lesquels la masculinisation s'accompagne de la formation d'une tumeur, les premiers étaient des animaux, chez lesquels la glande sexuelle subit des changements dans le sens du sexe opposé.

Les études les plus étendues dans le domaine des tumeurs masculinisantes, se rapportent à tous les cas observés chez l'homme. En principe les symptômes sont identiques à ceux présentés par les animaux, de même les tumeurs peuvent, en ce qui concerne leur genèse, être mises sur le même pied que celles des animaux, de sorte que les éléments nécessaires à une étude comparée existent.

De toutes les tumeurs trouvées en cours des années dans les ovaires, celles qui sont en rapport avec la masculinisation ne constituent qu'une infime minorité. Elles ne sont pas caractérisées par une structure spéciale, de sorte que les différents chercheurs ont posé les diagnostics les plus divers. ROBERT MEIJER a eu le grand mérite de les avoir regardées toutes sous un angle unique et de les avoir désignées par un seul nom. Il les nomme *arrhénoblastomes*, mais il se trouve obligé, vu leurs aspects divers, de faire différents groupes, qui ont cependant de commun que ce sont *des tumeurs épithéliales et qu'elles peuvent être ramenées à des cellules génitales à potentiel embryonnaire*. En dehors des arrhénoblastomes, il existe encore d'autres tumeurs satisfaisant à ces exigences. Ce sont les tumeurs à cellules de granuleuse ou folliculomes et les carcinomes à grandes cellules, séminomes ou disgerminomes, comme les appelle ROBERT MEIJER. D'après ce chercheur, toutes les deux n'ont cependant rien à voir avec la masculinisation et elles constituent donc des tumeurs possédant d'autres propriétés; ainsi les premières p. ex. conduiraient à l'hyperféminisation.

Les arrhénoblastomes sont subdivisés comme suit:

a. l'adénome testiculaire tubulaire de l'ovaire (PICK), qui par sa structure rappelle le plus celle du testicule et qui, chose étrange, ne serait pas toujours accompagné de masculinisation

b. les formes atypiques caractérisés par des proliférations épithéliales diffuses, des cordons et des canalicules; elles peuvent contenir des kystes, même des kystes muqueux et ont une tendance aux hémorrhagies, aux liquéfactions et à la nécrose

c. les formes intermédiaires entre *a* et *b*, qui possèdent des masses épithéliales atypiques, peuvent renfermer les canalicules et des cordons évoluant dans le sens testiculaire.

Les tumeurs à cellules de granuleuse sont considérées par ROBERT MEIJER comme étant spécifiquement féminines et pouvant donner lieu à l'accroissement et même à la sécrétion des mamelles. On connait néanmoins dans la littérature des cas où elles sont considérées comme étant la cause de la masculinisation qui les accompagne. BERGSTRAND a été le premier à attirer l'attention sur ce fait. Il signale six cas, dont 3 se rapportaient à des tumeurs ovariennes composées. Un de ces 6 cas a été reconnu par BERNER comme étant un adénome testiculaire tubulaire, alors que d'autre part

ROBERT MEIJER a porté pour un autre le diagnostic d'arrhénoblastome. SCHILLER a également examiné un cas de masculinisation, où il trouva une tumeur à cellules de granuleuse, plus tard cependant il est revenu sur ce diagnostic. SOETIDJO HARDJOSOEKATMO mentionne dans „Geneeskundig Tijdschrift voor Nederlands Indië", au cours de la description d'un cas d'arrhénoblastome malin de l'ovaire, que EERLAND et VOS ont posé le diagnostic de tumeur à cellules de granuleuse dans un cas, qui auparavant avait été admis comme arrhénoblastome. La masculinisation que j'ai observée chez une jument était accompagnée d'une tumeur de l'ovaire que ROBERT MEIJER appela cystadénoma partim tubulare et dans laquelle il y avait des grands et des petits kystes, dont la paroi rappelle celle d'un follicule et de laquelle SALOMON, qui travaille dans le laboratoire de PEYRON, dit qu'elle ressemble aux tumeurs microkystiques relativement fréquentes chez la jument, considérées d'ordinaire comme des folliculomes (les cas de masculinisation sont cependant très rares chez le cheval). A l'opinion de ROBERT MEIJER, d'après laquelle les tumeurs à cellules de granuleuse sont spécifiquement féminines, on doit donc apporter ce correctif qu'il peut exister dans les cas de masculinisation des circonstances, dans lesquelles elles peuvent être rencontrées. On ne peut dénier que de la sorte la spécificité de l'arrhénoblastome est battue en brèche. Quelle est à ce point la situation des carcinomes à grandes cellules? Ces tumeurs se rencontrent chez des animaux aussi bien mâles, que femelles et intersexués. Elles sont connues comme étant des tumeurs des gonades chez les intersexués. Elles portent encore d'autres noms, dont les plus connus sont: séminome, épithéliome séminifère, sarcome alvéolaire, disgerminome. MASSON et avec lui CHENOT fut le premier à reconnaître un épithéliome séminifère dans un ovaire. ROBERT MEIJER était d'avis que l'appellation disgerminome était adéquate, parce que la tumeur ne procéderait pas des cellules des tubes séminifères, mais bien d'un épithélium germinatif neutre, qui serait de qualité inférieure (unterwertig), donc disgerminatif. Ce que cet épithélium aurait perdu en potentiel, il l'aurait gagné en puissance proliférative. H. O. NEUMANN affirme que ces tumeurs se rencontrent le plus souvent dans les gonades d'intersexués, alors que MOSZKOWICZ fait remarquer que si l'on ne les rencontre pas chez des intersexués, ce sera du moins chez des individus présentant des signes d'hypogénitalisme.

Les carcinomes à grandes cellules seraient donc des tumeurs tout autres que les arrhénoblastomes. ROBERT MEIJER signale les différences suivantes: „Während diese auf ein spezifisch männlich gerichtete Anlage oder Verwandlung der Zellen durch ihre besondere Wirkung hin erweisen, so kommt ähnliches nicht bei den grosszelligen Carcinomen vor. Sie erscheinen nur als Begleiter der angeborenen Zwittrigkeit, während umgekehrt die Arrhenoblastomen nur bei Personen bekannt sind, die von Haus aus nicht zwittrig erscheinen, sondern es im gewissen Grade erst werden und zwar unter dem Einfluss der Geschwulst." D'après le même auteur, les carcinomes à grandes cellules n'ont, au point de vue morphogénétique rien de commun avec une „zwittrige Anlage", „sondern nur in der Verbindung mit der Abartung von sexueller Epithelanlage, also mit Asexualität der Zellen und zwar in einer frühembryonalen Zeit, bevor die normale geschlechtliche Differenzierung der Keimepithelien erfolgt. Dieses ist wahrscheinlich, weil sonst kaum eine so weitgehende Aehnlichkeit, ja Gleichheit der Geschwulst bei beiden Geschlechtern sich ergeben würde."

Il est difficile de concilier cette opinion avec la réalité de la formation du sexe. Ces tumeurs seraient donc issues de cellules embryonnaires asexuées, rien que parce qu'elles sont les mêmes chez les individus à sexe normal et chez les intersexués. Pour ce motif ces cellules devraient être originaires d'une époque précédant la différenciation des épithéliums germinatifs (cellules-mères des cellules germinatives). En réalité les choses ne se passent pas ainsi. Il n'existe pas de stade asexuel, car lors de la fécondation le sexe (éventuellement l'intersexe) est déterminé et toutes les cellules de l'individu, aussi bien les somatiques que les germinatives, possèderont ce même sexe. Le fait qu'il existe un stade embryonnaire, où le sexe de l'individu ne peut pas encore être reconnu, est uniquement la conséquence de l'insuffisance de différentiation des organes génitaux et cela ne saurait rien changer au fait que l'embryon avec toutes ces cellules possède déjà un sexe déterminé. L'interprétation donnée par ROBERT MEIJER de la morphogenèse des carcinomes à grandes cellules, n'est pas exacte en ce qui concerne les relations de ces tumeurs avec le sexe de l'individu qui en est porteur.

La différence entre les carcinomes à grandes cellules et les arrhénoblastomes résidérait dans le fait que les derniers sont à réduire à une prédisposition ou à une transformation de cellules embryonnaires dans le sens spécifiquement masculin, tandis que les premiers ne constitueraient qu'un phénomène accompagnant l'intersexualité congénitale. Qu'est-ce qu'une prédisposition à sens masculin chez un individu femelle? Est-ce une partie du corps ou de l'ovaire qui a changé ou qui est en train de changer de sexe? Ou est-ce une partie issue d'une cellule qui a perdu un des X-chromosomes? On sait que les animaux peuvent changer de sexe; cela est la conséquence de la détermination lors de la fécondation dans le cas d'un intersexué, qui ne peut valoir jamais pour une partie de l'individu. En ce qui concerne la perte d'un X-chromosome, on peut faire remarquer que la masculinisation est également connue chez les oiseaux, de telle sorte que l'on devrait parler ici d'augmentation d'un X-chromosome! La prédisposition à sens masculin, aussi bien que l'asexualité des cellules représentent deux notions, qui n'ont aucune signification pour la genèse des arrhénoblastomes et des carcinomes à grandes cellules.

Si l'on examine de près les arrhénoblastomes, on arrive à la conclusion qu'il ne s'agit pas d'une tumeur à structure spécifique. Ceci est confirmé par la classification en groupes différents, dans lesquels sont rangées des tumeurs de forme et de structure fort diverses. Toutes ces tumeurs ont 3 propriétés de commun:

a. on les trouve dans les cas de masculinisation
b. elles sont de nature épithéliale
c. elles procèdent de cellules à potentiel embryonnaire et génital.

Une tumeur à cellules de granuleuse, qui serait spécifiquement féminine, mais qu'on rencontrerait chez un individu masculinisé répondrait aussi à ces trois conditions et constituerait donc également un arrhénoblastome. BENECKE (cité par HARDJOSOEKATMO), du laboratoire de ROBERT MEIJER range les carcinomes à grandes cellules parmi les tumeurs masculinisantes, de sorte qu'on peut se demander s'il n'est pas préférable de mettre une fin à cette confusion en nommant arrhénoblastomes toutes les tumeurs rencontrées dans les cas de masculinisation ou se trouvant en rapport avec ce processus. A ce groupe, représentant une notion fonctionnelle, peuvent alors appartenir toutes les tumeurs qui répondent aux trois conditions précitées, quel que soit leur caractère morphologique ou quel que soit le diagnostic que l'anatomo-pathologiste pense devoir établir.

En présence de la multiplicité des tumeurs, on peut se poser la question si ces tumeurs peuvent bien être une cause de masculinisation. Il est tellement difficile d'admettre que des structures si dissemblables auraient toutes la même signification. Ne vaudrait-il pas mieux de considérer les tumeurs comme des phénomènes d'accompagnement et de rechercher la cause de la masculinisation chez l'individu même? Quand on raisonne de cette façon, on est frappé par la grande ressemblance que présentent les intersexués physiologiques et les intersexués pathologiques. Chez aucune espèce animale cette ressemblance est aussi nette que chez les oiseaux. Les intersexués physiologiques mammifères ont accompli leur transformation sexuelle au stade intrautérin; on ne peut guère l'observer, mais on peut la supposer avec de grandes chances de probabilité. S'agit-il au contraire d'intersexués pathologiques, alors ce changement est constaté dans le courant de la vie. Chez les oiseaux ce processus s'accomplit dans les deux cas après l'éclosion et il est caractérisé par des phénomènes tout à fait identiques, de sorte que ce n'est qu'à l'autopsie qu'on peut s'assurer s'il s'agit de l'une ou de l'autre espèce d'intersexués. Dans les deux cas on trouve dans l'ovaire des néoformations, qui procèdent des mêmes cellules embryonnaires. Il s'en suit que le parenchyme tumoral et les tubes testiculaires, dans lesquels peut même s'accomplir une spermatogenèse parfaite, ont la même origine, de sorte que les ovariotestes, eventuellement les testicules et les arrhénoblastomes sont équivalents. En ce qui concerne la morphogenèse, il n'existe donc pas de différences de principe entre les intersexués physiologiques et pathologiques. Seul le produit des cellules embryonnaires à potentiel génital est différent. On peut donc conclure des faits observés que dans les deux cas le changement de sexe est la cause des néoformations, qui se produisent dans l'ovaire, mais on ne saurait dire pourquoi les cellules latentes embryonnaires, qui ont été poussées à se multiplier, donnent naissance dans l'un cas à des éléments plutôt normaux et dans l'autre à des éléments anormaux. Il apparait de toute évidence qu'ici aussi il faut penser à une cause d'ordre général, parce que non seulement les cellules médullaires masculines primitives, mais aussi les cellules corticales, considérées comme féminines, participent au processus.

Dans cet ordre d'idées, la direction dans laquelle s'engage l'évolution de la tumeur, n'a guère d'importance. Cette direction dépend de facteurs inconnus jusqu'à présent et qui vraisemblablement sont en rapport étroit avec le processus de changement de sexe et en dépendent peut-être. Je voudrais considérer ces facteurs, aussi bien que les tumeurs comme des phénomènes d'accompagnement, de sorte que l'on ne doit pas rechercher l'élément primaire dans la tumeur ou dans la cause qui préside à sa formation, mais bien dans le changement de sexe.

A la rigueur de pareilles tumeurs ne pourraient pas être appelées arrhénoblastomes et l'on pourrait supprimer cette dénomination ou la remplacer par une autre, mais si l'on veut quand même conserver ce terme, proposé par ROBERT MEIJER, en raison de grand mérite de ses études sur les cas de masculinisation avec tumeur, dans ce cas il faut qu'on y attache un sens quelque peu différent. Le terme arrhénoblastome devrait donc désigner les tumeurs formées lors de masculinisation par des cellules de l'ovaire à potentiel embryonnaire et génital; la notion étiologique doit donc être totalement supprimée.

Tout le monde ne se déclarera pas d'accord avec cette manière de voir, parce qu'en démontrant une morphogenèse identique des néoformations de l'ovaire chez les intersexuès physiologiques et pathologiques on n'a pas détruit tous les arguments en faveur de la théorie étiologique des arrhénoblastomes. Il existe d'ailleurs encore deux arguments qui à première vue semblent fortement plaider en faveur de cette opinion. Le premier se base sur le fait que chez les femmes, auxquelles on enlève la tumeur, les caractères féminins réapparaissent. On connaît même des femmes pareilles qui par après ont encore eu des enfants. Le second argument réside dans la présence de tumeurs de la couche corticale des surrénales dans des cas de masculinisation. De tels blastomes ne peuvent pourtant pas être issus des cellules médullaires ou corticales de l'ovaire, en sorte que les considérations émises plus haut ne sauraient y être appliquées.

Et malgré tout, je ne pense pas que ces deux arguments sont en contradiction avec l'opinion que le changement de sexe est primaire et que la tumeur est secondaire.

L'opinion d'après laquelle le retour des caractères féminins serait la conséquence de disparition des hormones hétérosexuelles après l'enlèvement de la tumeur, est fort attrayante, parce que tellement simple et s'accordant si facilement avec les faits observés. Je ne nierai pas que la sécrétion interne de la tumeur joue un rôle important, mais, tout aussi bien que la tumeur, cette sécrétion n'est qu'une conséquence. A mon avis le défaut de cette opinion réside dans une figuration erronnée de l'apparition des caractères sexuels secondaires. Ceux-ci ne sont pas uniquement le résultat de l'influence des hormones gonadales. Il existe encore une autre influence qu'on néglige d'ordinaire. Il faut prendre comme point de départ le moment de la fécondation où le sexe est déterminé avec tous ses attributs. Chaque cellule procédant de l'ovule fécondé a le même équipement chromosomique que celui-ci et est donc du même sexe. On appelle cela le sexe cellulaire ou somatique, parce que tout le corps le possède. Si un individu naît sans glandes génitales, il possède néanmoins un type masculin ou féminin. Pareil animal n'est pas asexué, comme on le pense souvent, mais il lui manque les témoins du sexe sous forme de glandes génitales. Les raisons de ce sexe somatique résident dans les facteurs chromosomiques, présents dans chaque cellule et exerçant leur influence toute la vie durant. Elles sont primaires et sont responsables de l'ébauche des caractères sexuels secondaires. A la puberté se fait sentir la grande influence des hormones gonadales, qui réalisent en peu de temps le parachèvement total du type sexuel et le maintien de cet état pendant toute la durée de la vie sexuelle. Lors de la castration ces influences sont supprimées, mais l'action des facteurs chromosomiques persiste. Après avoir été châtré, un animal n'est pas devenu asexué, il est uniquement un individu sans glandes génitales.

Par le fait qu'un arrhénoblastome peut provoquer si rapidement l'évolution des caractères secondaires mâles, on a toutes les raisons d'admettre qu'il secrète des hormones hétérosexuelles, capables d'extérioriser les caractères masculins pouvant encore apparaître dans un individu jusque là féminin. Après extirpation de la tumeur, la sécrétion interne est supprimée, les caractères masculins formés rapidement disparaissent tout aussi vite et les attributs féminins réapparaissent. Les caractères masculins, qui ne peuvent complètement rétrocéder, se maintiennent, p. ex. la pomme d'Adam trop développée. Souvent aussi il est fait mention d'une persistance partielle de la voix grave, vestige typique de la masculinisation.

Après avoir extirpé la tumeur en même temps que l'ovaire, qui en est le porteur, il persiste encore l'ovaire de l'autre côté et l'on peut se demander quelle sera son influence sur l'organisme. Pour ceux qui attachent une signification primaire à la tumeur, la réponse est simple. L'individu est à nouveau devenu féminin et l'a d'ailleurs toujours été; donc l'ovaire continuera son activité en ce sens. Pour ceux qui considèrent ces patients comme des intersexués, la question se pose autrement; ils doivent en effet tâcher de trouver une solution au moyen de la théorie de GOLDSCHMIDT. A ce propos, j'attire l'attention sur les facteurs M et F qui déterminent le sexe. Le cas sus-visé avec changement de sexe après la naissance est déterminé par M2—F (fig. 1). Depuis longtemps F et M2 s'en vont très peu l'un de l'autre et même après le changement de sexe, la différence entre les deux n'est pas très grande. Il n'existe qu'une faible intersexualité; même après que M2 est devenu le plus fort, F exercera encore une grande influence dans l'organisme. Le renforcement de M2 par les hormones de la tumeur disparaît après l'extirpation.

L'ovaire restant est une glande sexuelle dans un organisme intersexué.

Afin de pouvoir juger de la signification des gonades dans ces circonstances extraordinaires, je me suis occupé, il y a une dizaine d'années, d'enlever les gonades à un grand nombre d'intersexués. J'ai pu ainsi observer quels sont les caractères disparaissant de ce qui restait de l'appareil génital. Il m'a semblé qu'on pouvait mettre les résultats obtenus sur le même pied que ceux obtenus après castration d'animaux normaux au point de vue sexuel. Quelles que fussent les gonades enlevées chez les intersexués (ovariotestes ou testicules), il est certain qu'elles assuraient le développement sexuel de l'appareil génital pendant la vie sexuelle. D'ordinaire cet appareil génital se compose de parties mâles et femelles, de telle sorte que p. ex. les testicules assuraient le développement total de la matrice. On a supposé qu'une hormone intersexuelle assurait le développement du caractère sexuel chez les hermaphrodites. De REGT n'est pas parvenu à mettre en évidence la présence de cette hormone gonadale dans l'urine des porcs intersexués, mais il trouva des hormones mâles et femelles dans des proportions tout autres que chez des individus normaux. BUTENANDT est parvenu plus tard à préparer une hormone intersexuelle.

Il est donc possible que l'ovaire de l'individu masculinisé se comporte comme la glande génitale d'un intersexué et qu'il sécrète donc des hormones testiculaires et ovariennes dans des proportions autres qu'à l'état normal. Je ne puis pas cependant attacher une grande importance à ces faits, car KOCH a démontré qu'au cours d'une période de quatre semaines, il y a chez une femme des époques où elle excrète dans l'urine des quantités plus grandes d'hormone masculine que d'hormone féminine.

L'observation suivante semble être plus importante. Il s'agit notamment de la comparaison que j'ai faite entre les possibilités de fécondation chez un animal femelle et chez un animal intersexué. Chez le premier les possibilités dépendent des chaleurs, de l'ovulation et de la capacité pour l'appareil génital de recevoir les spermatozoides et de préparer la voie vers l'ovule arrivé à maturité. Chez le second les organes génitaux ne sont d'ordinaire pas à même de remplir la dernière condition, de sorte que la comparaison doit se limiter aux chaleurs et à l'ovulation. Chez des intersexués, j'ai pu constater quelquefois des chaleurs manifestés alternant souvent avec une excitation sexuelle mâle. J'ai aussi trouvé deux générations successives de corps jaunes; chez un animal dans deux ovariotestes et chez un autre dans un ovariotestis et dans un ovaire. Ils étaient tout à fait identiques à ceux qu'on trouve dans un ovaire normal.

S'il était impossible pour un intersexué de produire de l'ovulation et d'entrer en chaleurs, ces deux propriétés seraient caractéristiques pour un individu femelle; mais comme tel n'est pas le cas, la thèse d'après laquelle sont intersexués les individus se masculinisant tout en présentant un développement de tumeur, n'est pas démolie. Par conséquent l'argument du retour des caractères féminins après enlèvement de la tumeur ne saurait être jeté dans la balance en défaveur de cette thèse et cela avec d'autant plus de raison que ces individus possèdent une faible intersexualité (M2—F).

La seconde objection contre l'état intersexué des individus, dont la masculinisation est caractérisée par le développement d'une tumeur, se rapporte à ceux qui sont porteurs d'une tumeur de la couche corticale de surrénale, formée de cellules procédant de cette substance corticale. A première vue, il n'existe pas ici de cellules embryonnaires à potentiel génital, comme les cellules médullaires et corticales de l'ovaire. On a cependant affaire ici au blastome d'un tissu fournissant une sécrétion interne et qui, comme on le sait d'ailleurs, peut après la castration sécréter d'une façon vicariante les hormones produites à l'état normal par la gonade. On peut donc admettre, sans risquer de se tromper, que la sécrétion interne de la tumeur est à même de réaliser la transformation des caractères sexuels secondaires.

Afin de pouvoir rechercher la nature et la signification des tumeurs de la couche corticale des surrénales et de pouvoir les comparer avec les arrhénoblastomes, il est d'importance capitale de connaître les propriétés des cellules dont précèdent les tumeurs. Il est surtout important de s'enquérir de l'existence possible dans ces cellules de la couche corticale du potentiel présent dans les cellules ovariennes embryonnaires.

Le développement de cet organe doit fournir les indications requises. On sait que la surrénale procède de deux ébauches. La partie médullaire (organe suprarénal) dérive du système nerveux autonome, des sympathogonies procédant des cellules du cordon-limite. La partie corticale (organe interrénal) a son origine dans la partie orale de l'épithélium péritonéal qui donne naissance aux gonades. La partie corticale de la surrénale et les gonades possèdent donc un blastème commun. La différence entre les deux réside dans le fait que l'épithélium péritonéal, dont sont issus les testicules et les ovaires sous l'influence des gonocytes primaires, devient épithélium germinatif, tandis que cette transformation ne s'opère pas pour le tissu-mère de la couche corticale de la surrénale. Il est possible que des gonocytes primaires extrarégionaux puissent jouer un rôle ici, mais il est alors difficile de concevoir pourquoi l'épithélium péritonéal ne devient pas épithélium germinatif et pourquoi l'organe qui en est issu ne devient pas une glande sexuelle. Pour ce motif il est préférable de rechercher quelles sont les propriétés des cellules de l'épithélium péritonéal. Au point de vue phylogénétique, c'est un tissu remplissant 3 fonctions fondamentales: fonction de revêtement, fonction sécrétoire et fonction de formation de cellules germinatives. La première de ces fonctions est conservée chez les animaux supérieurs, les deux autres ont été transmises à des organes distincts: reins et glandes génitales, mais il y a des indications montrant que la fonction génitale n'est pas entièrement abolie. Je ne vise pas ici en premier lieu les expériences d'ovariotomie chez des oiseaux, à la suite desquelles un nouvel ovaire se serait formé à partir de l'épithélium péritonéal, parce que d'autres expériences contredisent ces résultats. En de pareils cas, il est toujours possible qu'un reliquat d'ovaire ait été le point de départ du nouvel organe. Mais je voudrais attirer l'attention sur quelques animaux caractérisés par la présence de testicules multiples disséminés sur tout le péritoine. Moi-même j'ai eu l'occasion d'examiner pareil porc. Il n'est pas impossible que des gonocytes primaires erratiques ont d'abord changé ici l'épithélium péritonéal en épithélium germinatif. Le fait que cette transformation s'est opérée uniquement dans le péritoine et pas en d'autres endroits où des gonocytes errants auront probablement aussi échoué, plaide en faveur de la prédilection du péritoine et du maintien de ses propriétés primitives. Je veux au surplus rappeler le cas d'une poule masculinisée qui possédait une multitude de tumeurs, des papillo-cysto-carcinomes issus tous de l'épithélium péritonéal. J'insiste tout spécialement sur le cas de ce porc et de cette poule, parce qu'ils constituent des expériences de la nature, ayant comme point de départ les propriétés de l'épithélium péritonéal. On ne sait pas sous quelles influences le potentiel latent se manifeste à nouveau, mais ces deux cas prouvent que ce potentiel continue à exister.

Ces tumeurs de la poule issues de l'épithélium péritonéal ont comme origine des cellules à potentiel embryonnaire génital, sont rencontrées dans une cas de masculinisation et répondent par conséquent à la définition des arrhénoblastomes. Il n'y a donc aucune raison pour ne pas classer ces papillo-cysto-carcinomes parmi cette catégorie.

Il existe donc une parenté cellulaire entre les gonades et la couche corticale de la surrénale, donc également entre les tumeurs procédant des cellules médullaires et corticales de l'ovaire et celles issues de la couche corticale de la surrénale. Il devient donc évident que les tumeurs de la couche corticale de la surrénale dans des cas de masculinisation se rapprochent sensiblement, en ce qui concerne leur genèse, des arrhénoblastomes de l'ovaire. Dans ces conditions, il n'est pas étonnant qu'elles puissent posséder semblable sécrétion interne. Dans ces deux cas, aussi bien que dans les gonades, cette sécrétion est la conséquence des facteurs génitaux qui priment dans l'organisme, tout en tenant évidemment compte de la différence minime entre F et M dans les cas où la masculinisation se constate dans le courant de la vie (cas M2—F). La sécrétion hormonale d'une tumeur de la couche corticale de la surrénale, prenant la direction masculine après le changement de sexe, n'empêche pas d'admettre que non plus la tumeur soit primaire, mais bien le changement de sexe.

Il me semble que toutes les objections soulevées contre la nature intersexuelle des individus changeant de sexe au cours de l'évolution des tumeurs de l'ovaire ou de la couche corticale de la surrénale, tombent quand on analyse la genèse des tumeurs et qu'on tient compte de la formation de l'intersexe. Je partage l'avis d'HALBAN, qui subdivisait les ovules en féminins, masculins et hermaphrodites et qui proclamait que les femmes devenant homme devaient posséder une prédisposition hermaphroditique. C'est avec raison que cet auteur attribue un rôle protecteur à la sécrétion interne influençant le développement des caractères sexuels secondaires.

Physics. — *On the buckling of a thin-walled circular tube loaded by pure bending.* I. By C. B. BIEZENO and J. J. KOCH.

(Communicated at the meeting of June 29, 1940.)

1. *Introduction.* If a thin-walled circular tube in his end-sections is loaded by two equal and opposite bending moments M, it may be stated that its cross section alters its circular shape into an oval one, owing to the fact, that, apart from the normal bending stresses in the cross section of the tube, there arise tangential bending-stresses in its meridional sections. A closer examination of the fact learns, that the curvature $\frac{1}{\varrho}$ of the "axis" of the tube does not increase proportionally with the loading moment M.

If "a" be the radius of the tube, h its thickness, E the elasticity-modulus of the material and ν the reciprocal value of POISSON's coefficient, the following relation between M and ϱ exists:

$$M = \pi a^3 h \left[\frac{1}{\varrho} - \frac{3}{2} (1 - \nu^2) \frac{a^4}{\varrho^3 h^2} \right] \quad . \quad . \quad . \quad . \quad . \quad (1)$$

the graphical representation of which is shown qualitatively in fig. 1.

It is seen from this figure that there exists a critical value M_{crit} of M, characterized by the fact, that no increase of M occurs, if $1/\varrho$ increases. Hence a break-down of the tube is to be expected.

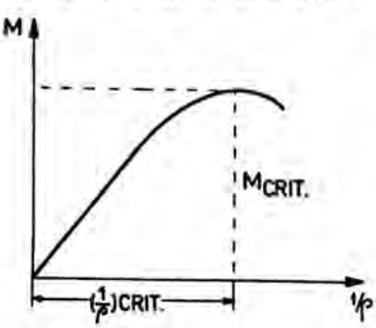

Fig. 1.

The phenomenon here described has been studied at great length by BRAZIER[1]).

In the present paper quite another phenomenon is studied, which occurs for certain critical values of M, and is characterized by the simultaneous appearance of longitudinal and circumferential waves in the cylindrical shape of the tube. We assume, that, — if unloaded —, the tube possesses such initial curvature, that under the action of the buckling moment it is straight, and — in cross-section — circular and of constant thickness h. Hereby our buckling problem relates to

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[1]) Comp. BRAZIER, Aeronautical Research Committee, Reports and Memoranda No. 1081, M 49.

a *straight* circular tube, loaded at his ends by linear changing bending stresses.

Preliminary we shall have to solve some detail-questions (sections 2—6). In section 2 the formulae for the displacements and stresses of a cylindrical tube, submitted to prescribed radial, tangential and axial stresses, R, Φ and Z are reproduced. In section 3 a system of particular loads B is defined and calculated, which plays a fundamental role in our proper buckling problem. The loads B will be called *"elementary normal loads"*, the corresponding deformations, "elementary normal deformations". Section 4 deals with the differential equations of the buckling problem; more particularly it is shown that the differential equations for the displacements obtained in section 2 may be looked upon as the required buckling-equations, provided that R, Φ and Z be replaced by adequate "would-be" forces. The so-obtained differential equations are homogeneous and linear in the displacements and therefore only admit solutions (different from zero) for special values of the loading moment $M = \mu \overline{M}$ ($\overline{M}$ = unit of bending moment). The values μ, for which buckling of the tube is possible, will be called the "total characteristic values" of our problem, the corresponding deformations T of the tube "total normal deformations". Section 5 bears on the development of the "total normal deformations" into a series of the "elementary normal deformations". Finally it is shown in section 6, how by iteration the smallest characteristic value μ can approximately be calculated.

2. *The cylindrical circular tube of constant thickness under a prescribed load-system.* As shown in fig. 2 the position of an arbitrary point of

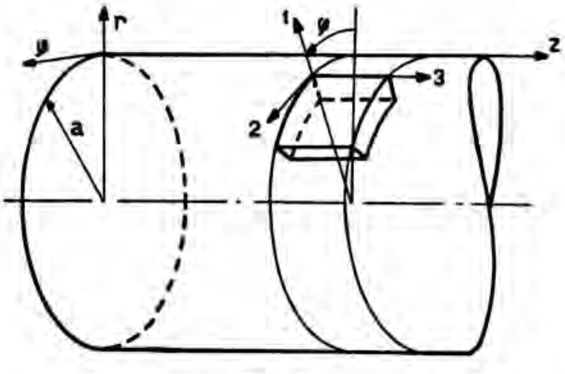

Fig. 2.

the middle-surface of the tube is given by its "cylinder"-coordinates a, φ, and z. The radial, tangential and axial displacements of such a point are called u, v, w; the components of the external load of the tube with reference to the unit of surface, and taken in radial, tangential and axial direction are designed by R, Φ, and Z. Fig. 3 shows the nomenclature of the so-called "internal" forces and moments, all of which refer to the unit of length of their corresponding section.

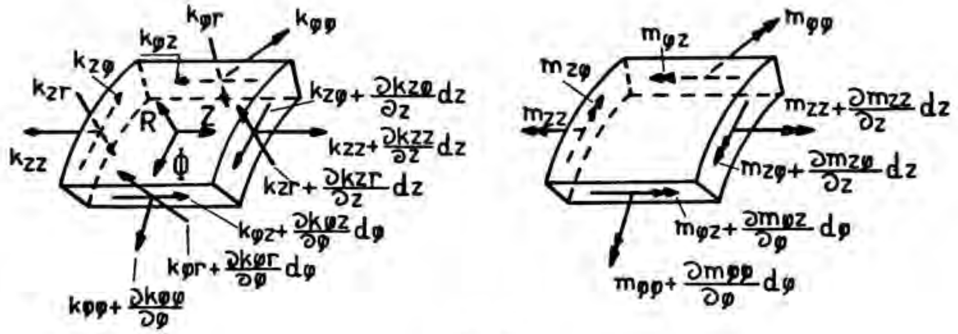

Fig. 3.

If, for abbreviation, we put

$$A^* = \frac{Eh^3}{12\,(1-\nu^2)}, \quad B = \frac{Eh}{1-\nu^2} \quad \left(\nu = \frac{1}{m}\right) \quad . \quad . \quad . \quad . \quad (1)$$

and for later purposes

$$\frac{A^*}{Ba^2} = \frac{h^2}{12a} = k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1^*)$$

the equations for u, v, w may be written as follows:

$$\left.\begin{aligned}
&\frac{1}{a}\left(\frac{u}{a} + \frac{1}{a}\frac{\partial v}{\partial \varphi} + \nu\frac{\partial w}{\partial z}\right) + \frac{A^*}{B}\left(\frac{1}{a^4}\frac{\partial^4 u}{\partial \varphi^4} + \frac{2}{a^2}\frac{\partial^4 u}{\partial \varphi^2\,\partial z^2} + \frac{\partial^4 u}{\partial z^4} + \right.\\
&\left. + \frac{2}{a^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{u}{a^4} - \frac{3-\nu}{2a^2}\frac{\partial^3 v}{\partial \varphi\,\partial z^2} + \frac{1-\nu}{2a^3}\frac{\partial^3 w}{\partial \varphi^2\,\partial z} - \frac{1}{a}\frac{\partial^3 w}{\partial z^3}\right) - \frac{R}{B} = 0\\
&\frac{1}{a^2}\frac{\partial u}{\partial \varphi} + \frac{1}{a^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial z^2} + \frac{1+\nu}{2a}\frac{\partial^2 w}{\partial \varphi\,\partial z} + \frac{A^*}{B}\left(-\frac{3-\nu}{2a^2}\frac{\partial^3 u}{\partial \varphi\,\partial z^2} + \right.\\
&\left. + 3\,\frac{1-\nu}{2a^2}\frac{\partial^2 v}{\partial z^2}\right) + \frac{\Phi}{B} = 0\\
&\frac{\nu}{a}\frac{\partial u}{\partial z} + \frac{\nu+1}{2a}\frac{\partial^2 v}{\partial \varphi\,\partial z} + \frac{1-\nu}{2a^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2} + \frac{A^*}{B}\left(\frac{1-\nu}{2a^3}\frac{\partial^3 u}{\partial \varphi^2\,\partial z} - \right.\\
&\left. - \frac{1}{a}\frac{\partial^3 u}{\partial z^3} + \frac{1-\nu}{2a^4}\frac{\partial^2 w}{\partial \varphi^2}\right) + \frac{Z}{B} = 0.
\end{aligned}\right\} . \quad (2)$$

The internal forces and moments can be calculated from u, v, w, by the relations (3):

$$\left.\begin{aligned}
k_{\phi\phi} &= \frac{A^*}{a^3}\left(\frac{\partial^2 u}{\partial\varphi^2}+u\right)+B\left(\frac{u}{a}+\frac{1}{a}\frac{\partial v}{\partial\varphi}+\nu\frac{\partial w}{\partial z}\right)\\
k_{\phi z} &= \frac{(1-\nu)\,A^*}{2a^2}\left(\frac{\partial^2 u}{\partial\varphi\,\partial z}+\frac{1}{a}\frac{\partial w}{\partial\varphi}\right)+\frac{(1-\nu)\,B}{2}\left(\frac{\partial v}{\partial z}+\frac{1}{a}\frac{\partial w}{\partial\varphi}\right)\\
m_{\phi\phi} &= \frac{(1-\nu)\,A^*}{2a}\left(2\frac{\partial^2 u}{\partial\varphi\,\partial z}-\frac{\partial v}{\partial z}+\frac{1}{a}\frac{\partial w}{\partial\varphi}\right)\\
k_{z\phi} &= \frac{(1-\nu)\,A^*}{2a^2}\left(-\frac{\partial^2 u}{\partial\varphi\,\partial z}+\frac{\partial v}{\partial z}\right)+\frac{(1-\nu)\,B}{2}\left(\frac{\partial v}{\partial z}+\frac{1}{a}\frac{\partial w}{\partial\varphi}\right)\\
k_{zz} &= -\frac{A^*}{a}\frac{\partial^2 u}{\partial z^2}+B\left(\nu\frac{u}{a}+\frac{\nu}{a}\frac{\partial v}{\partial\varphi}+\frac{\partial w}{\partial z}\right)\\
m_{z\phi} &= A^*\left(\frac{\nu}{a^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{\partial^2 u}{\partial z^2}-\frac{\nu}{a^2}\frac{\partial v}{\partial\varphi}-\frac{1}{a}\frac{\partial w}{\partial z}\right)\\
m_{zz} &= \frac{(1-\nu)\,A^*}{a}\left(-\frac{\partial^2 u}{\partial\varphi\,\partial z}+\frac{\partial v}{\partial z}\right)
\end{aligned}\right\} . \qquad (3)$$

We see at once, that for the special loads

$$\left.\begin{aligned}
R &= a_{pq}\cos p\varphi\sin\lambda\frac{z}{a}\\
\Phi &= b_{pq}\sin p\varphi\sin\lambda\frac{z}{a}\\
Z &= 0
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4)$$

— where p and q design arbitrary positive, integer numbers and λ stands for

$$\lambda=\frac{\pi\,q\,a}{l} \quad \ldots\ldots\ldots\ldots \quad (5)$$

(l being the length of the cylinder) — the equations (2) admit solutions of the following type

$$\left.\begin{aligned}
u &= u_{pq}\cos p\varphi\sin\lambda\frac{z}{a}\\
v &= v_{pq}\sin p\varphi\sin\lambda\frac{z}{a}\\
w &= w_{pq}\cos p\varphi\cos\lambda\frac{z}{a}
\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (6)$$

u_{pq}, v_{pq} and w_{pq} representing functions of p and q, which are to be calculated from the equations (7)

$$\left.\begin{aligned}
&[1+k\,(p^4+2p^2\lambda^2+\lambda^4-2p^2+1)]\,u_{pq}+\left[p+\frac{3-\nu}{2}\,kp\lambda^2\right]v_{pq}+\\
&\qquad+\left[-\nu\lambda+k\left(\frac{1-\nu}{2}\,p^2\lambda-\lambda^3\right)\right]w_{pq}=\frac{\mathrm{a}^2}{B}\,a_{pq}\\
&\left[p+\frac{3-\nu}{2}\,kp\lambda^2\right]u_{pq}+\left[p^2+\frac{1-\nu}{2}\,(1+3k)\,\lambda^2\right]v_{pq}-\\
&\qquad-\frac{1+\nu}{2}\,p\lambda w_{pq}=\frac{\mathrm{a}^2}{B}\,b_{pq}\\
&\left[-\nu\,\lambda+k\left(\frac{1-\nu}{2}\,p^2\,\lambda-\lambda^3\right)\right]u_{pq}-\frac{1+\nu}{2}\,p\,\lambda\,v_{pq}+\\
&\qquad+\left[\lambda^2+\frac{1-\nu}{2}\,(1+k)\,p^2\right]w_{pq}=0
\end{aligned}\right\}. \quad (7)$$

(It may be noted, that for all solutions (7), the displacements u and v are zero for $z=0$ and $z=l$, whereas w is different from zero at the ends of the cylinder). We restrict ourselves to solving u_{pq} and v_{pq} and find, under the essential condition, that the thickness h of the tube be small enough to neglect all terms which contain the parameter k (comp. 1*) in higher than the first degree

$$\left.\begin{aligned}
u_{pq}&=\alpha_{pq}\,a_{pq}+\beta_{pq}\,b_{pq}\\
v_{pq}&=\beta_{pq}\,a_{pq}+\gamma_{pq}\,b_{pq}
\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

α_{pq}, β_{pq} and γ_{pq} standing for:

$$\alpha_{pq}=\left(\frac{T_1}{N}\right)_{pq}\frac{\mathrm{a}^2}{B}\,;\;\beta_{pq}\equiv-\left(\frac{T_2}{N}\right)_{pq}\frac{\mathrm{a}^2}{B}\,;\;\gamma_{pq}=\left(\frac{T_3}{N}\right)_{pq}\frac{\mathrm{a}^2}{B}\;\cdot\;\cdot \quad (9)$$

T_1, T_2, T_3, N themselves standing for:

$$\left.\begin{aligned}
&T_1=(1+k)\,p^4+[2\,\lambda^2+2\,(1-\nu)\,\lambda^2\,k]\,p^2+(1+3\,k)\,\lambda^4\\
&T_2=p\,[[1+k\,(2\,\lambda^2+1)]\,p^2+(\nu+2)\,\lambda^2+2\,k\,\lambda^4]\\
&T_3=kp^2\,(p^2-1)^2+k\,\frac{2\,(2-\nu)}{1-\nu}\,\lambda^2\,p^4+\left[1+\frac{5-\nu}{1-\nu}\,k\lambda^4+\right.\\
&\qquad\left.+\frac{2\,(\nu-\nu^2-2)}{1-\nu}\,k\,\lambda^2+k\right]p^2+2\,(1+\nu)\,\lambda^2+\frac{2}{1-\nu}\,k\,(\lambda^6-2\,\nu\lambda^4)\\
&N=k\,p^8+k\,[4\,\lambda^2-2]\,p^6+k\,[6\,\lambda^4-2\,(4-\nu)\,\lambda^2+1]\,p^4+k\,[4\,\lambda^6-\\
&\qquad-6\,\lambda^4+2\,(2-\nu)\,\lambda^2]\,p^2+(1-\nu^2)\,\lambda^4+k\lambda^8-2\,\nu\lambda^6\,k+(4-3\,\nu^2)\,\lambda^4\,k.
\end{aligned}\right\}. \quad (10)$$

3. *The "elementary normal loads" B, and the corresponding "elementary normal deformations" D.* As "elementary normal loads" B and

corresponding "elementary normal deformations" D we define such loads R_{pq}, Φ_{pq} (comp. 2, 4) and such corresponding displacements u_{pq}, v_{pq}, for which

$$\frac{R_{pq}}{u_{pq}} = \frac{\Phi_{pq}}{v_{pq}} = \omega \, . \quad \ldots \ldots \ldots \quad (1)$$

Condition (1) is identical with the requirement, that the equations

$$\left.\begin{aligned} a_{pq} &= \omega\,(\alpha_{pq}\,a_{pq} + \beta_{pq}\,b_{pq}) \\ b_{pq} &= \omega\,(\beta_{pq}\,a_{pq} + \gamma_{pq}\,b_{pq}) \end{aligned}\right\} \text{ or } \left.\begin{aligned} (\omega\,\alpha_{pq} - 1)\,a_{pq} + \omega\,\beta_{pq}\,b_{pq} &= 0 \\ \omega\,\beta_{pq}\,a_{pq} + (\omega\,\gamma_{pq} - 1)\,b_{pq} &= 0 \end{aligned}\right\} . \quad (2)$$

admit solutions a_{pq}, b_{pq} different from zero. (Comp. 2, 8 and 2, 4).

Therefore the equation

$$\begin{vmatrix} \omega\,\alpha_{pq} - 1 & \omega\,\beta_{pq} \\ \omega\,\beta_{pq} & \omega\,\gamma_{pq} - 1 \end{vmatrix} = 0 \; . \quad \ldots \ldots \ldots \quad (3)$$

has to be satisfied by ω. The two (real) roots of this equation are:

$$\begin{matrix} 1/\omega^{*}_{pq} \\ \text{resp.} \\ 1/\omega^{**}_{pq} \end{matrix} = \frac{\alpha_{pq} + \gamma_{pq}}{2} \pm \sqrt{\left(\frac{\alpha_{pq} - \gamma_{pq}}{2}\right)^{2} + \beta_{pq}{}^{2}} \; . \quad \ldots \ldots \quad (4)$$

The corresponding loads and displacements — which of course are definite except for a factor of proportionality $\varkappa$ — may readily be calculated from the equations (3).

With:

$$\left.\begin{aligned} E^{*}_{pq} &\equiv -\frac{\alpha_{pq} - \gamma_{pq}}{2\,\beta_{pq}} + \sqrt{\left(\frac{\alpha_{pq} - \gamma_{pq}}{2\,\beta_{pq}}\right)^{2} + 1}\,; \; E^{**}_{pq} \equiv -\frac{\alpha_{pq} - \gamma_{pq}}{2\,\beta_{pq}} - \sqrt{\left(\frac{\alpha_{pq} - \gamma_{pq}}{2\,\beta_{pq}}\right)^{2} + 1} \\ F^{*}_{pq} &\equiv \frac{\alpha_{pq} + \gamma_{pq}}{2} + \sqrt{\left(\frac{\alpha_{pq} - \gamma_{pq}}{2}\right)^{2} + \beta^{2}_{pq}}\,; \; F^{**}_{pq} \equiv \frac{\alpha_{pq} + \gamma_{pq}}{2} - \sqrt{\left(\frac{\alpha_{pq} - \gamma_{pq}}{2}\right)^{2} + \beta^{2}_{pq}} \\ G^{*}_{pq} &= E^{*}_{pq}\,F^{*}_{pq}\,; \qquad\qquad\qquad\qquad G^{**}_{pq} = E^{**}_{pq}\,F^{**}_{pq} \end{aligned}\right\} (5)$$

they can be written as follows:

$$\left.\begin{aligned} R^{*} &= \varkappa^{*} \cos p\varphi \sin \lambda \frac{z}{a} & R^{**} &= \varkappa^{**} \cos p\varphi \sin \lambda \frac{z}{a} \\ \Phi^{*} &= \varkappa^{*}\,E^{*}_{pq} \sin p\varphi \sin \lambda \frac{z}{a} & \Phi^{**} &= \varkappa^{**}\,E^{**}_{pq} \sin p\varphi \sin \lambda \frac{z}{a} \\ u^{*} &= \varkappa^{*}\,F^{*}_{pq} \cos p\varphi \sin \lambda \frac{z}{a} & u^{**} &= \varkappa^{**}\,F^{**}_{pq} \cos p\varphi \sin \lambda \frac{z}{a} \\ v^{*} &= \varkappa^{*}\,G^{*}_{pq} \sin p\varphi \sin \lambda \frac{z}{a} & v^{**} &= \varkappa^{**}\,G^{**}_{pq} \sin p\varphi \sin \lambda \frac{z}{a} \end{aligned}\right\} . \quad (6)$$

the first solution obviously belonging to ω^{*}, the second one to ω^{**}. To get rid of the undeterminate factors $\varkappa^{*}$ and $\varkappa^{**}$ we standardize our solutions by the condition of standardization

$$\int\limits_{0}^{l}\int\limits_{0}^{2\pi}(u^{*}R^{*}+v^{*}\Phi^{*})\,ad\varphi\,dz=\frac{\pi\,al}{2}\text{ resp. }\int\limits_{0}^{l}\int\limits_{0}^{2\pi}(u^{**}R^{**}+v^{**}\Phi^{**})\,ad\varphi\,dz=\frac{\pi\,al}{2}\quad(7)$$

and find hereby:

$$\varkappa^{*}=\frac{1}{\sqrt{F^{*}_{pq}+E^{*}_{pq}\,G^{*}_{pq}}}\text{ resp. }\varkappa^{**}=\frac{1}{\sqrt{F^{**}_{pq}+E^{**}_{pq}\,G^{**}_{pq}}}\quad .\quad .\quad .\quad(8)$$

Henceforth the expression "elementary normal function" will be used in the same sense as "standardized elementary normal function", so that from now the elementary normal loads and deformations will be represented by the eqs. (6) and (8).

In the following sections we shall have to deal with systems of elementary normal functions, belonging to a fixed value of the second affix q. In such case this affix will be suppressed. It can easily be proved, that, under this understanding, the following so-called "orthogonality"-relations exist between the functions (6):

$$\left.\left.\begin{array}{ll}\left.\begin{array}{l}\int\limits_{0}^{l}\int\limits_{0}^{2\pi}[\mathrm{R}^{*}_{k}u^{*}_{l}+\Phi^{*}_{k}v^{*}_{l}]\,ad\varphi dz=0\\ \int\limits_{0}^{l}\int\limits_{0}^{2\pi}[R^{**}_{k}u^{**}_{l}+\Phi^{**}_{k}v^{**}_{l}]\,ad\varphi dz=0\end{array}\right\}& k\neq l\\ \left.\begin{array}{l}\int\limits_{0}^{l}\int\limits_{0}^{2\pi}[R^{*}_{k}u^{**}_{l}+\Phi^{*}_{k}v^{**}_{l}]\,ad\varphi dz=0\\ \int\limits_{0}^{l}\int\limits_{0}^{2\pi}[R^{**}_{k}u^{*}_{l}+\Phi^{**}_{k}v^{*}_{l}]\,ad\varphi dz=0\end{array}\right\}&\begin{array}{l}\text{for }\mathit{all}\\ \text{values}\\ \text{of }k\\ \text{and }l.\end{array}\end{array}\right\}\begin{array}{l}k\text{ and }l\text{ denoting}\\ \text{values of the}\\ \text{first affix }p;\\ \text{the second affix}\\ q\text{ being fixed.}\end{array}\right\}\quad .\quad .\quad(9)$$

4. *The differential-equations of the buckling tube.* If only the load components R, Φ and Z and the displacements u, v, w be adequately interpreted, the equations (2, 2) may be looked upon as the differential-equations of the buckling tube, loaded by axial (and linearly changing) bending stresses at its endsections. As stated in section 1, we assume that, — thanks to its initial curvature — the tube, under influence of its buckling moments $M=\mu\overline{M}$, can be regarded as straight and of constant thickness. The displacements u, v, w of any point of the middle-surface, *from this initial state of stress and strain* (II) replace from this moment the quantities, designed in the same way in equation 2, 2, denoting originally the displacements from the *unstressed state* (I).

One could be inclined, in applying the equations (2, 2) to the buckling problem, to put R, Φ and Z equal to zero, due to the fact that these quantities evidently now stand for the *supplementary* loads, which eventually arise with the transition from state II to the (indefinitely) neighbouring buckling-configuration (II′). The following fact, however, has to be observed. An arbitrary element of the tube, as represented in fig. 4, being in state II, finds itself in equilibrium, under the axial forces $Q\, ad\varphi$, Q representing the external force pro unit of circumferential length exerted on the end-sections of the tube in the points $(z = 0, \varphi)$ and $(z = l, \varphi)$.

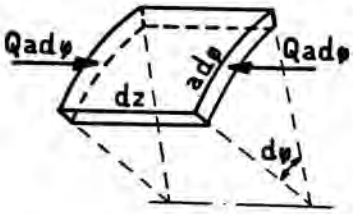

Fig. 4.

If now the transition takes place from state II to state II′ these forces retain their magnitude (because the external load of the tube does not alter at all), but they change in direction in accordance with the change in shape and curvature of the surface-elements on which they act. Therefore these forces (though unchanged in magnitude) produce components in the directions r, φ and z, which, divided by the surface $ad\varphi\, dz$, play the role of R, Φ and Z in the equations (2, 2).

It can be shown [1]), that in our case R, Φ, Z amount to

$$R = -Q\frac{\partial^2 u}{\partial z^2}, \quad \Phi = -Q\frac{\partial^2 v}{\partial z^2}, \quad Z = 0 \quad . \quad . \quad . \quad . \quad (1)$$

and therefore the differential-equations of our buckling problem run as follows:

$$\left.\begin{aligned}
&\frac{1}{a}\left(\frac{u}{a}+\frac{1}{a}\frac{\partial v}{\partial\varphi}+\nu\frac{\partial w}{\partial z}\right)+\frac{A^*}{B}\left(\frac{1}{a^4}\frac{\partial^4 u}{\partial\varphi^4}+\frac{2}{a^2}\frac{\partial^4 u}{\partial\varphi^2\partial z^2}+\frac{\partial^4 u}{\partial z^4}+\right.\\
&\qquad\left.+\frac{2}{a^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{u}{a^4}-\frac{3-\nu}{2a^2}\frac{\partial^3 v}{\partial\varphi\,\partial z^2}+\frac{1-\nu}{2a^3}\frac{\partial^3 w}{\partial\varphi^2\partial z}-\frac{1}{a}\frac{\partial^3 w}{\partial z^3}\right)+\\
&\qquad+\frac{Q}{B}\frac{\partial^2 u}{\partial z^2}=0\\
&\frac{1}{a^2}\frac{\partial u}{\partial\varphi}+\frac{1}{a^2}\frac{\partial^2 v}{\partial\varphi^2}+\frac{1-\nu}{2}\frac{\partial^2 v}{\partial z^2}+\frac{1+\nu}{2a}\frac{\partial^2 w}{\partial\varphi\,\partial z}+\frac{A^*}{B}\left(-\frac{3-\nu}{2a^2}\frac{\partial^3 u}{\partial\varphi\partial z^2}+\right.\\
&\qquad\left.+3\,\frac{1-\nu}{2a^2}\frac{\partial^2 v}{6z^2}\right)-\frac{Q}{B}\frac{\partial^2 v}{\partial z^2}=0\\
&\frac{\nu}{a}\frac{\partial u}{\partial z}+\frac{1+\nu}{2a}\frac{\partial^2 v}{\partial\varphi\,\partial z}+\frac{1-\nu}{2a^2}\frac{\partial^2 w}{\partial\varphi^2}+\frac{\partial^2 w}{\partial z^2}+\frac{A^*}{B}\left(\frac{1-\nu}{2a^3}\frac{\partial^3 u}{\partial\varphi^2\partial z}-\right.\\
&\qquad\left.-\frac{1}{a}\frac{\partial^3 u}{\partial z^3}\right)+\frac{1-\nu}{2a^4}\frac{\partial^2 w}{\partial\varphi^2}=0
\end{aligned}\right\} \quad . \quad (2)$$

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[1]) Comp. W. FLÜGGE, Statik und Dynamik der Schalen; Springer, Berlin 1934. C. B. BIEZENO and R. GRAMMEL, Technische Dynamik; Springer, Berlin 1939.

There only remains to express $\frac{Q}{B}$ in terms of the loading moment $M = \mu \overline{M}$. We find (comp. fig. 5 and 2, 1):

$$\frac{Q}{B} = \frac{M\,a \cos\varphi}{\pi a^3 h}\, h . \frac{(1-\nu^2)}{E h} = \frac{\mu \overline{M} (1-\nu^2)}{\pi a^2 E h} \cos\varphi \quad . \quad . \quad . \quad . \quad (3)$$

It has already been stated in section 1, that the eqs. (2) are homogeneous in u, v, w and their derivatives, and therefore only have solutions, different from $u = v = w = 0$, for distinct values of the parameter μ, respectively for distinct values of the loading moment $M = \mu \overline{M}$. Those values will be called the "total characteristic numbers" respectively the "total characteristic moments" of our problem; the corresponding deformations (T) the "total characteristic deformations". From now it is our subject to calculate the *smallest* characteristic moment.

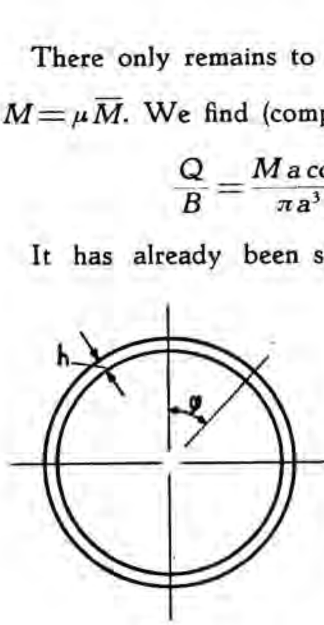

Fig. 5.

5. *The expansion of a "total characteristic deformation" T into a series of elementary normal deformations D. The influence numbers a_{ij}.* The main result of the preceding section lays in the fact, that in case of buckling the external loading moments M of the tube give rise to a "would-be" surface load R, Φ, Z given by

$$R = -Q \frac{\partial^2 u}{\partial z^2} = -\frac{M}{\pi a^2} \frac{\partial^2 u}{\partial z^2} \cos\varphi, \; \Phi = -Q \frac{\partial^2 v}{\partial z^2} = -\frac{M}{\pi a^2} \frac{\partial^2 v}{\partial z^2} \cos\varphi, \; Z = 0 \quad (1)$$

the magnitude of which depends upon the total characteristic deformation u, v, w, that corresponds with M. We learn from (1) that — if this total deformation be decomposed in a set of other deformations $u_1, v_1, w_1; u_2, v_2, w_2 \ldots$ —, such that $u = u_1 + u_2 + \ldots$, $v = v_1 + v_2 + \ldots$, $w = w_1 + w_2 + \ldots$, the loadsystem (1) may be decomposed in a set of other loadsystems, each of which is calculable by (1) from u_1, v_1, w_1, u_2, v_2, w_2 etc.

If therefore the total characteristic deformation T be expanded in a (infinite) series of "elementary normal deformations" D:

$$T = \sum_{i=1}^{\infty} d_i D_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and if the loadsystem, derived from D_i with the aid of (1) be called $\overline{B}_i$ then the "would-be" loadsystem belonging to the total characteristic deformation under consideration, may be written as:

$$\Sigma\, d_i \overline{B}_i .$$

Each system $\overline{B}_i$, on its part, can be expanded in a (finite or infinite) series of elementary normal loadsystems B, so that, at the end the would-be loadsystem (1), produced by the total characteristic deformation under consideration, can be regarded upon as the sum of an infinite number of groups of a (finite or infinite) number of elementary normal loads B.

Two remarks of some importance are here to be made. Firstly it has been silently assumed, that all deformations and load systems, introduced in this section belong to the *same* parameter λ (comp. (2, 5)), which is characteristic for the total deformation T, so that all deformations and loads, considered here, have the same number of longitudinal waves. In accordance with a remark, already made in section 3, we therefore denote the functions T, B and D by one single suffix p, relating to the number of circumferential waves.

Secondly, we could — of course — have developed *directly* the load system (1) into a series of the elementary normal loads. We did not, however, proceed in this way in view of the introduction of a system of so-called influence-numbers a_{ij}, which will now be defined.

If (artificially) the tube be given the elementary normal deformation D, then we can — by the aid of (1) — formally calculate the "would-be" loadsystems roused by two unit bending moments $\overline{M}$, acting at the ends of the tube. As stated before, this loadsystem can be developed in a (finite or infinite) series of elementary, normal functions B. The coefficient a_{ij}, which in this expansion belongs to the elementary normal function B_i is called "the influence number *of* the elementary normal deformation D_j *with respect to* the elementary normal load B_i.

This formal definition provides us with an expedient to obtain system of homogeneous linear equations for the coefficients d_i in the expression (2). Indeed, if a deformation D_j provokes a "would-be" load, which contains a_{ij} times B_i, (assumed that the tube be charged by unit bending moments $\overline{M}$), then a deformation $d_j D_j$ provokes a would-be load, which contains $\mu d_j a_{ij}$ times B_i, assumed that the tube be charged by two bending moments $M = \mu \overline{M}$.

The total characteristic deformation $T = \sum\limits_{i=1}^{\infty} d_i D_i$ therefore provokes a would-be load, which contains B_i

$$\sum_{j=1}^{\infty} \mu\, d_j\, a_{ij} \text{ times.}$$

On the other hand it was stated in section 2, that the deformation $T = \sum\limits_{i=1}^{\infty} d_i D_i$ only can be maintained by the load $\sum\limits_{i=1}^{\infty} d_i B_i$, and therefore the system of equations

$$d_i = \sum_{j=1} \mu\, d_j\, a_{ij} \quad i = 1, 2 \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

must hold.

It only has solutions different from $d_i = 0$ $(i = 1, 2 \ldots)$ if $\frac{1}{\mu}$ satisfies the equation:

$$\begin{vmatrix} a_{11} - \frac{1}{\mu} & a_{12} & a_{13} \cdots\cdots \\ a_{21} & a_{22} - \frac{1}{\mu} & a_{23} \cdots\cdots \\ a_{31} & a_{32} & a_{33} - \frac{1}{\mu} \cdots \\ \vdots & \vdots & \vdots \quad \ddots \end{vmatrix} = 0 \quad . \quad . \quad . \quad (4)$$

which formally can be considered as representing our problem.

Now we proceed by proving, that — if only the elementary normal functions be suitably numbered — the reciprocal relation $a_{ij} = a_{ji}$ holds, so that equation (4) is a secular one, possessing only real roots. To this end we calculate by using (1) the would-be loads R, Φ — to be denoted by $\overline{R}^*, \overline{\Phi}^*$, resp. $\overline{R}^{**}, \overline{\Phi}^{**}$ — belonging to the elementary normal deformations u^*, v^*, resp. u^{**}, v^{**} defined by (3, 6) and (3, 8). They are — if the "order" of the underlying elementary deformation be indicated by p —

$$\left.\begin{aligned} \overline{R}^*_p &= \frac{\lambda^2 F^*_p}{2\pi a^2 \sqrt{F^*_p + E^*_p\, G^*_p}} \left[\cos(p+1)\,\varphi + \cos(p-1)\,\varphi\right] \sin\frac{\lambda z}{a} \\ \overline{\Phi}^*_p &= \frac{\lambda^2 G^*_p}{2\pi a^2 \sqrt{F^*_p + E^*_p\, G^*_p}} \left[\sin(p+1)\,\varphi + \sin(p-1)\,\varphi\right] \sin\frac{\lambda z}{a} \\ \overline{R}^{**}_p &= \frac{\lambda^2 F^{**}_p}{2\pi a^2 \sqrt{F^{**}_p + E^{**}_p\, G^{**}_p}} \left[\cos(p+1)\,\varphi + \cos(p-1)\,\varphi\right] \sin\frac{\lambda z}{a} \\ \overline{\Phi}^{**}_p &= \frac{\lambda^2 G^{**}_p}{2\pi a^2 \sqrt{F^{**}_p + E^{**}_p\, G^{**}_p}} \left[\sin(p+1)\,\varphi + \sin(p-1)\,\varphi\right] \sin\frac{\lambda z}{a} \end{aligned}\right\} \quad . \quad . \quad (5)$$

Both load-systems $\overline{R}^*_p$, $\overline{\Phi}^*_p$, and $\overline{R}^{**}_p$, $\overline{\Phi}^{**}_p$ can be linearly expressed in the elementary load-systems

$$B^*_{p-1} = (R^*_{p-1}, \Phi^*_{p-1}); \quad B^*_{p+1} = (R^*_{p+1}, \Phi^*_{p+1}); \quad B^{**}_{p-1} = (R^{**}_{p-1}, \Phi^{**}_{p-1});$$
$$B^{**}_{p+1} = (R^{**}_{p+1}, \Phi^{**}_{p+1}).$$

If we restrict ourselves to the system $\overline{R}^{*}_{p}, \overline{\Phi}^{*}_{p}$, and if we put

$$\left.\begin{aligned}\overline{R}^{*}_{p} &= \delta^{1}_{p}\, R^{*}_{p-1} + \delta^{2}_{p}\, R^{**}_{p-1} + \delta^{3}_{p}\, R^{*}_{p+1} + \delta^{4}_{p}\, R^{**}_{p+1} \\ \overline{\Phi}^{*}_{p} &= \delta^{1}_{p}\, \Phi^{*}_{p-1} + \delta^{2}_{p}\, \Phi^{**}_{p-1} + \delta^{3}_{p}\, \Phi^{*}_{p+1} + \delta^{4}_{p}\, \Phi^{**}_{p+1}\end{aligned}\right\} \quad . \; . \; . \; (6)$$

the coefficients δ^{i}_{p} $(i = 1, 2 \ldots 4)$ can be found by using the relations

$$\left.\begin{aligned}\frac{\pi a l}{2}\,\delta^{1}_{p} &= \int_{0}^{l}\int_{0}^{2\pi} [\overline{R}^{*}_{p}\, u^{*}_{p-1} + \overline{\Phi}^{*}_{p}\, v^{*}_{p-1}]\, a d\varphi\, dz \\ \frac{\pi a l}{2}\,\delta^{2}_{p} &= \int_{0}^{l}\int_{0}^{2\pi} [\overline{R}^{*}_{p}\, u^{**}_{p-1} + \overline{\Phi}^{*}_{p}\, v^{**}_{p-1}]\, a d\varphi\, dz \\ \frac{\pi a l}{2}\,\delta^{3}_{p} &= \int_{0}^{l}\int_{0}^{2\pi} [\overline{R}^{*}_{p}\, u^{*}_{p+1} + \overline{\Phi}^{*}_{p}\, v^{*}_{p+1}]\, a d\varphi\, dz \\ \frac{\pi a l}{2}\,\delta^{4}_{p} &= \int_{0}^{l}\int_{0}^{2\pi} [\overline{R}^{*}_{p}\, u^{**}_{p+1} + \overline{\Phi}^{*}_{p}\, v^{**}_{p+1}]\, a d\varphi\, dz\end{aligned}\right| \quad . \; . \; . \; (7)$$

which can be verified by substituting the expressions (6) in the right-hand members and by taking into account the relations of orthogonality established in section 3 (comp. 3, 9). Substitution of the explicite (5) of $\overline{R}^{*}_{p}, \overline{\Phi}^{*}_{p}, \overline{R}^{**}_{p}, \overline{\Phi}^{**}_{p}$ into (7) gives:

$$\left.\begin{aligned}\delta^{1}_{p} &= \frac{\lambda^{2}}{2\pi a^{2}}\left[\frac{F^{*}_{p} F^{*}_{p-1} + G^{*}_{p} G^{*}_{p-1}}{\sqrt{F^{*}_{p} + E^{*}_{p} G^{*}_{p}}\,\sqrt{F^{*}_{p-1} + E^{*}_{p-1} G^{*}_{p-1}}}\right] \\ \delta^{2}_{p} &= \frac{\lambda^{2}}{2\pi a^{2}}\left[\frac{F^{*}_{p} F^{**}_{p-1} + G^{*}_{p} G^{**}_{p-1}}{\sqrt{F^{*}_{p} + E^{*}_{p} G^{*}_{p}}\,\sqrt{F^{**}_{p-1} + E^{**}_{p-1} G^{**}_{p-1}}}\right] \\ \delta^{3}_{p} &= \frac{\lambda^{2}}{2\pi a^{2}}\left[\frac{F^{*}_{p} F^{*}_{p+1} + G^{*}_{p} G^{*}_{p+1}}{\sqrt{F^{*}_{p} + E^{*}_{p} G^{*}_{p}}\,\sqrt{F^{*}_{p+1} + E^{*}_{p+1} G^{*}_{p+1}}}\right] \\ \delta^{4}_{p} &= \frac{\lambda^{2}}{2\pi a^{2}}\left[\frac{F_{p} F_{p+1} + G_{p} G_{p+1}}{\sqrt{F^{*}_{p} + E^{*}_{p} G^{*}_{p}}\,\sqrt{F^{**}_{p+1} + E^{**}_{p+1} G^{**}_{p+1}}}\right]\end{aligned}\right\} \quad . \; . \; (8)$$

If again we put

$$\left.\begin{aligned}\overline{R}^{**}_{p} &= \varepsilon^{1}_{p}\, R^{*}_{p-1} + \varepsilon^{2}_{p}\, R^{**}_{p-1} + \varepsilon^{3}_{p}\, R^{*}_{p+1} + \varepsilon^{4}_{p}\, R^{**}_{p+1} \\ \overline{\Phi}^{**}_{p} &= \varepsilon^{1}_{p}\, \Phi^{*}_{p-1} + \varepsilon^{2}_{p}\, \Phi^{**}_{p-1} + \varepsilon^{3}_{p}\, \Phi^{*}_{p+1} + \varepsilon^{4}_{p}\, \Phi^{**}_{p+1}\end{aligned}\right\} \quad . \; . \; . \; (9)$$

we find in an analogous way

$$\left.\begin{aligned}
\varepsilon_p^1 &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F_p^{**}\,F_{p-1}^{*}+G_p^{**}\,G_{p-1}^{*}}{\sqrt{F_p^{**}+E_p^{**}\,G_p^{**}}\,\sqrt{F_{p-1}^{*}+E_{p-1}^{*}\,G_{p-1}^{*}}}\right]\\
\varepsilon_p^2 &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F_p^{**}\,F_{p-1}^{**}+G_p^{**}\,G_{p-1}^{**}}{\sqrt{F_p^{**}+E_p^{**}\,G_p^{**}}\,\sqrt{F_{p-1}^{**}+E_{p-1}^{**}\,G_{p-1}^{**}}}\right]\\
\varepsilon_p^3 &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F_p^{**}\,F_{p+1}^{*}+G_p^{**}\,G_{p+1}^{*}}{\sqrt{F_p^{**}+E_p^{**}\,G_p^{**}}\,\sqrt{F_{p+1}^{*}+E_{p+1}^{*}\,G_{p+1}^{*}}}\right]\\
\varepsilon_p^4 &= \frac{\lambda^2}{2\pi a^2}\left[\frac{F_p^{**}\,F_{p+1}^{**}+G_p^{**}\,G_{p+1}^{**}}{\sqrt{F_p^{**}+E_p^{**}\,G_p^{**}}\,\sqrt{F_{p+1}^{**}+E_{p+1}^{**}\,G_{p+1}^{**}}}\right]
\end{aligned}\right\}\quad . \quad . \quad (10)$$

It goes without saying that each of the coefficients δ_p^i and ε_p^i $(i=1,2\dots4)$ represents an influence-number a_{ij} in the previously defined sense. Furthermore it is evident, that no other influence-coefficients exist except those represented by (8) and (10).

The question, which suffixes must be ascribed to the coefficient a, to let it represent a given δ or ε, depends upon the way in which the elementary normal deformations $D_i\equiv(u_i^*,v_i^*)$ and $D_i^{**}\equiv(u_i^{**},v_i^{**})$ are arranged. We fix, that

$D_0^*\equiv 0$ will be indicated by D_i $(i=0)$ and in consequence B_0^* by B_i $(i=0)$
D_0^{**} ,, ,, ,, ,, D_i $(i=1)$,, ,, ,, B_0^{**} by B_i $(i=1)$
D_1^* ,, ,, ,, ,, D_i $(i=2)$,, ,, ,, B_1^* by B_i $(i=2)$
D_1^{**} ,, ,, ,, ,, D_i $(i=3)$,, ,, ,, B_1^{**} by B_i $(i=3)$
a. s. o.

Bearing in mind that δ_p^1 represents the influence-coefficient of the elementary normal load B_{p-1}^* with respect to the elementary normal deformation D_p^*, and that B_{p-1}^* and D_p^* in the just defined sequences of normal loads B_i and normal deformations D_i have the numbers $i=2(p-1)$, resp. $i=2p$, then it is obvious that δ_p^1 has to be called $a_{2(p-1),2p}$, and that the significance of the coefficients δ_p^i, ε_p^i in general can be derived from the following scheme

$$\left.\begin{aligned}
\delta_p^1 &\equiv a_{2(p-1),2p} & \varepsilon_p^1 &= a_{2(p-1),2p+1}\\
\delta_p^2 &\equiv a_{2(p-1)+1,2p} & \varepsilon_p^2 &= a_{2(p-1)+1,2p+1}\\
\delta_p^3 &\equiv a_{2(p+1),2p} & \varepsilon_p^3 &= a_{2(p+1),2p+1}\\
\delta_p^4 &\equiv a_{2(p+1)+1,2p} & \varepsilon_p^4 &= a_{2(p+1)+1,2p+1}
\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (11)$$

From the equations (8) and (10) we deduce

$$\delta_p^1 = \delta_{p-1}^3 \ , \ \varepsilon_p^1 = \delta_{p-1}^4 \ , \ \delta_p^2 = \varepsilon_{p-1}^3 \ , \ \varepsilon_p^2 = \varepsilon_{p-1}^4 ,$$

and consequently from equations (11)

$$\left.\begin{array}{l} a_{2(p-1),2p} = a_{2p,2(p-1)} \\ a_{2(p-1),2p+1} = a_{2p+1,2(p-1)} \\ a_{2(p-1)+1,2p} = a_{2p,2(p-1)+1} \\ a_{2(p-1)+1,2p+1} = a_{2p+1,2(p-1)+1} \end{array}\right\} \quad . \ . \ . \ . \ . \ . \ . \quad (12)$$

These equations obviously can be contracted in the single relation of reciprocity:

$$a_{ij} = a_{ji}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (13)$$

The numbering of normal functions, introduced in this section, therefore make the left-hand side of (4) to a symmetrical determinant. If furthermore we give attention to the fact, that in our new nomenclature influence-coefficients of the type a_{ii} do not occur, and that a great number of the other coefficients is zero, equation (4) can be replaced by:

$$\begin{vmatrix} \frac{1}{\mu} & a_{12} & a_{13} & 0 & 0 & 0 & 0 \dots \\ a_{21} & -\frac{1}{\mu} & 0 & a_{24} & a_{25} & 0 & 0 \dots \\ a_{31} & 0 & -\frac{1}{\mu} & a_{34} & a_{35} & 0 & 0 \dots \\ 0 & a_{42} & a_{43} & -\frac{1}{\mu} & 0 & a_{46} & a_{47} \dots \\ 0 & a_{52} & a_{53} & 0 & -\frac{1}{\mu} & a_{56} & a_{57} \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix} = 0 \quad (14)$$

with $a_{ij} = a_{ji}$.

After a well-known theorem the roots of this equation are one and all real. If the columns 2, 3, 6, 7, 10, 11, ... and the rows 1, 4, 5, 8, 9, ... of the determinant are multiplied by -1, no alteration takes place in its general shape, except that all terms $\frac{1}{\mu}$ change their sign. Therefore it can be stated beforehand, that all roots of equation (14) occur in pairs of equal magnitude and opposite sign.

Every root μ_k of (13) corresponds to a total characteristic deformation T_k

$$T_k = \sum_{i=1}^{\infty} d_{ki} \, D_i,$$

the coefficients d_i satisfying the equations:

$$d_{ki} = \sum_{j=1}^{n} \mu_k \, a_{ij} \, d_{kj} \ (i = 1, 2 \dots). \quad . \ . \ . \ . \ . \ . \quad (15)$$

Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im R_4.* (Vierte Mitteilung). Von R. WEITZENBÖCK.

(Communicated at the meeting of June 29, 1940.)

Ich ermittle in dieser Mitteilung die einfachsten Differentialinvarianten einer Regelfläche F im R_4. Neben der Invariante Q mit den Gewichten 9 und 5 stellen sich zwei Invarianten A und B vom φ'-Gewichte 20 und vom λ-Gewicht 10 ein. Eine lineare Kombination derselben, $U = -36A + 24B$ ergibt sich durch geometrische Betrachtungen.

§ 12.

Die Ausdrücke

$$J_\varrho = h_{ik,\,rs} = (h\,i^2 k^2)\,(h\,r^2 s^2) = (a_h\,a_i{}^2\,a_k{}^2)\,(a_h\,a_r{}^2\,a_s{}^2)$$

sind vom φ'-Gewichte

$$\varrho = h + i + k + r + s,$$

d.h. bei den Transformationen (7) hat man

$$\widetilde{J}_\varrho = \widetilde{h}_{\widetilde{ik},\,\widetilde{rs}} = J_\varrho\,.\,(\varphi')^\varrho + J' \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

wo J' weitere Glieder mit $\varphi', \varphi'', \ldots$ andeutet. Bei einer *Invariante* J_ϱ ist $J' = 0$.

Da $0_{12,\,rs}$ und $0_{13,\,22}$ verschwinden, müssen wir $\varrho \geqq 9$ haben. Man sieht leicht, dass sich bei $\varrho = 9$ nur eine einzige Invariante J_9 ergibt, nämlich

$$Q = 0_{13,\,22}$$

(Vgl. die auf Gleichung (41) folgenden Gleichungen).

Bei $\varrho = 10$ haben wir zunächst mit $h = 0$ die folgenden Ausdrücke J_{10}, die nicht unmittelbar nach den Gleichungen (1) reduzierbar sind:

$$0_{13,\,33} = R \quad , \quad 0_{13,\,24} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

Bei $h = 1$, also in $1_{ik,\,rs}$ kann man den Fall $1_{0k,\,rs}$ weglassen, da er wegen

$$1_{0k,\,rs} = -0_{1k,\,rs} - k_{01,\,rs} = -0_{1k,\,rs}$$

auf $h = 0$ zurückführt. Es bleibt daher bei $\varrho = 10$ mit $h = 1$ nur

$$1_{13,\,23} = -\tfrac{1}{2}\,3_{11,\,23} = +\tfrac{1}{2}\,3_{02,\,23} = -\tfrac{1}{2}\,2_{03,\,23}\,.$$

sodass wir also $h > 1$ annehmen können. Bei $h = 2$ ergeben sich die Ausdrücke

$$2_{03,23}\,,\ 2_{03,05}\,.$$

Hier ist der zweite bereits durch Gleichung (82) durch Q' und R ausgedrückt:

$$2_{03,05} = \tfrac{5}{6}\,\mathrm{R} - 5\,Q' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

Beim ersten haben wir vermöge der zyklischen Symmetrie:

$$2_{03,23} = -\tfrac{1}{2}\,3_{03,22} = +\tfrac{1}{4}\,0_{33,22} = -\tfrac{1}{4}\,0_{22,33} = +\tfrac{1}{3}\,0_{13,33} = \tfrac{1}{3}\,R \quad . \quad (103)$$

Die Fälle $h > 2$ lassen sich alle auf (102) und (101) reduzieren.

Für $0_{13,24}$ haben wir bereits oben (vgl. die Gleichung vor Gleichung (82)) gefunden

$$0_{13,24} = -\tfrac{5}{6}\,R + Q'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

Hiermit sind alle J_{10} auf R und Q' reduziert.

Bei $\varrho = 11$ ergeben sich für $h = 0$ zunächst wieder die Ausdrücke

$$0_{13,34}\,,\ 0_{13,25}\,,\ 0_{23,33} = S,\ 0_{23,14} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

Wenn wir $R = 0_{13,33}$ differenzieren, so entsteht

$$R' = 1_{13,33} + 0_{23,33} + 0_{14,33} + 2.0_{13,34}\,.$$

Hier ist der erste Term rechts Null wegen

$$1_{13,33} = -\tfrac{1}{2}\,3_{11,33} = 0;$$

der zweite Term ist S und beim dritten Term drücken wir 14 nach (1) aus:

$$0_{14,33} = -2.0_{23,33} = -2S.$$

Also kommt

$$0_{13,34} = \tfrac{1}{2}\,(R' + S). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

Damit ist der erste der Ausdrücke (105) auf R' und S reduziert.

Bei $h = 1$ gibt 1_{0i} wieder Reduktion auf (105) und alle anderen Ansätze können auf $h > 1$ zurückgebracht werden. Da ferners in $h_{ik,rs}$ bei $h > 3$ $i + k \leqq 3$ vorausgesetzt werden darf, sind nur noch die Fälle $h = 2$ und $h = 3$ zu untersuchen. Aber auch von $h = 3$ kann man hier absehen, da sich alle Ansätze $3_{ik,rs}$ auf $h = 2$ reduzieren lassen. Bei $h = 2$ hat man, da $2_{02,rs}$ auf (105) führt, die folgenden Ausdrücke zu untersuchen:

$$\left.\begin{array}{llll} 2_{03,33} & 2_{03,24} & 2_{03,15} & (i+k=3) \\ 2_{04,23} & 2_{04,14} & & (i+k=4) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (107)$$

Hier wird nun der erste dieser Ausdrücke

$$2_{03,33} = -0_{23,33} = -S; \quad \dots\dots \quad (108)$$

beim zweiten hat man

$$2_{03,24} = -\tfrac{1}{2}\, 4_{03,22} = \tfrac{1}{2}\, 0_{34,22} + \tfrac{1}{2}\, 3_{04,22}\,.$$

oder wegen

$$0_{34,22} = -\tfrac{4}{3}\, 0_{34,13} = \tfrac{4}{3}\, 0_{13,34}\,.$$

also nach Gleichung (106)

$$0_{34,22} = \tfrac{2}{3}\,(R' + S). \quad \dots\dots \quad (109)$$

und wegen

$$3_{04,22} = -\tfrac{4}{3}\, 3_{04,13} = +\tfrac{2}{3}\, 1_{04,33} = -\tfrac{2}{3}\, 0_{14,33} = \tfrac{4}{3}\, 0_{23,33} = \tfrac{4}{3}\, S:$$

$$2_{03,24} = \tfrac{1}{3}\, R' + S\,. \quad \dots\dots \quad (110)$$

Beim dritten der Ausdrücke (107) wird

$$2_{03,15} = -1_{03,25} - 5_{03,12} = +0_{13,25} \quad \dots\dots \quad (111)$$

also Reduktion auf (105).

Weiters ist

$$2_{04,23} = -\tfrac{1}{2}\, 3_{04,22} = +\tfrac{2}{3}\, 3_{04,13} = -\tfrac{1}{3}\, 1_{04,33} = +\tfrac{1}{3}\, 0_{14,33} = -\tfrac{2}{3}\, 0_{23,33} = -\tfrac{2}{3}\, S. \quad (112)$$

und schliesslich

$$2_{04,14} = -0_{24,14} - 4_{02,14} = 0_{14,24} = -2.0_{23,24} \quad \dots\dots \quad (113)$$

da

$$4_{02,14} = -\tfrac{1}{2}\, 1_{02,44} = +\tfrac{1}{2}\, 1_{11,44} = 0$$

wird.

Hiermit sind alle Ausdrücke (107) auf *R'*, *S* und (105) reduziert.

Differenzieren wir (104), so entsteht

$$-\tfrac{5}{6}\, R' + Q'' = 1_{13,24} + 0_{23,24} + 0_{14,24} + 0_{13,34} + 0_{13,25} \quad \dots \quad (114)$$

Hier wird der erste Term rechts:

$$1_{13,24} = -2_{13,14} - 4_{13,12} = +\tfrac{1}{4}\, 2_{04,14} - 4_{13,12}\,,$$

also nach (113):

$$1_{13,24} = -\tfrac{1}{2}\, 0_{23,24} - 4_{13,12}\,.$$

Für den letzten Ausdruck rechter Hand haben wir:

$$4_{13,12} = -\tfrac{1}{3}\, 4_{13,03} = +\tfrac{1}{3}\, 0_{13,34} + \tfrac{1}{3}\, 3_{13,04} = \tfrac{1}{3}\, 0_{13,34} - \tfrac{1}{6}\, 1_{33,04} =$$

$$= \tfrac{1}{3}\, 0_{13,34} + \tfrac{1}{6}\, 0_{33,14} = \tfrac{1}{3}\, 0_{13,34} - \tfrac{1}{3}\, 0_{33,23}\,.$$

also nach (106):

$$4_{13,12} = \tfrac{1}{6} R' + \tfrac{1}{2} S.$$

Damit wird

$$1_{13,24} = -\tfrac{1}{2}\, 0_{23,24} - \tfrac{1}{6} R' - \tfrac{1}{2} S. \quad \ldots \quad (115)$$

Beim dritten Term in (114) erhält man

$$0_{14,24} = -2.0_{23,24}\,.$$

Damit kommt schliesslich statt (114):

$$0_{23,24} = \tfrac{2}{3}\, 0_{13,25} + \tfrac{7}{10} R' - \tfrac{2}{3} Q''. \quad \ldots \quad (116)$$

Damit sind alle Ausdrücke J_{11} reduziert auf S, R', Q'' und $0_{13,25}$, wobei statt $0_{13,25}$ auch $0_{23,24}$ genommen werden kann.

§ 13.

Die Resultate des vorigen § können wir wie folgt formulieren: Jedes J_{10} hat die Gestalt

$$J_{10} = aR + bQ' = \alpha\,.\,0_{13,33} + \beta\,.\,0_{13,24} \quad \ldots \quad (117)$$

und jedes J_{11} die Gestalt

$$J_{11} = aS + bR' + cQ'' + d.0_{13,25} = \alpha.0_{23,33} + \beta.0_{13,34} + \gamma.0_{13,25} + \delta.0_{23,24} \quad (118)$$

mit konstanten Koeffizienten a, b, c, d bzw. α, β, γ und δ.

Darüber hinaus ist leicht zu beweisen, dass die angeschriebenen Ausdrücke für J_{10} und J_{11} nur dann verschwinden (für alle Parameterwerte t), wenn alle Koeffizienten a bis d Null sind. Das bedeutet die lineare Unabhängigkeit von R und Q' und die von S, R', Q'' und $0_{13,25}$. Zum Beweise kann man von $0_{ik} = (yz)_{ik}$ ausgehen, was wir aber hier nicht weiter ausführen wollen.

Für die in (118) rechter Hand stehenden Ausdrücke ergeben sich dagegen bei Benutzung der Identität (91) quadratische Beziehungen. Man erhält:

$$0_{13,34}\,.\,0_{23,33} = -0_{13,23}\,.\,0_{33,34} - 0_{13,33}\,.\,0_{34,23}$$

$$\tfrac{1}{2}(R' + S)\,.\,S = -Q\,.\,0_{33,34} + R\,.\,0_{23,34} \quad \ldots \quad (119)$$

$$0_{13,34}\,.\,0_{23,24} = -0_{13,23}\,.\,0_{24,34} - 0_{13,24}\,.\,0_{34,23}$$

$$\tfrac{1}{2}(R' + S)\,.\,0_{23,24} = -Q\,.\,0_{24,34} + (Q' - \tfrac{5}{6} R)\,.\,0_{23,34} \quad \ldots \quad (120)$$

$$0_{13,25}\,.\,0_{23,33} = -0_{13,23}\,.\,0_{33,25} - 0_{13,33}\,.\,0_{25,23}$$

$$S\,.\,0_{13,25} = -Q\,.\,0_{33,25} + R\,.\,0_{23,25} \quad \ldots \quad (121)$$

$$0_{13,25}\,.\,0_{23,24} = -0_{13,23}\,.\,0_{24,25} - 0_{13,24}\,.\,0_{25,23}$$

$$0_{13,25}\,.\,0_{23,24} = -Q\,.\,0_{24,25} + (Q' - \tfrac{5}{6} R)\,.\,0_{23,25} \quad \ldots \quad (121a)$$

Nach (116) kann man dieser letzten Gleichung die Gestalt geben

$$\tfrac{2}{3}(0_{13,25})^2 - 0_{13,25} \,.\, (\tfrac{2}{3} Q'' - \tfrac{7}{9} R') = - Q \,.\, 0_{24,25} + (Q' - \tfrac{5}{6} R) \,.\, 0_{23,25} \quad (121b)$$

Aus (119) folgt: Sind Q und $R \equiv 0$, so ist auch $S \equiv 0$. Ferner aus (121b): $Q \equiv 0$, $R \equiv 0$ zieht $0_{13,25} \equiv 0$ und damit nach (116) auch $0_{23,24} \equiv 0$ nach sich. *Also verschwinden dann alle Ausdrücke J_{10} und J_{11}.*

§ 14.

In Gleichung (43) wird die Invariante U durch Formen J_{10} und J_{11} ausgedrückt:

$$\left.\begin{aligned} U = (\tfrac{1}{2} R + 3 Q') \,.\, (0_{22,33} + 3 \,.\, 0_{22,24} + 3 \,.\, 0_{22,15}) + \\ + 4 Q \,.\, (2_{03,33} + 3 \,.\, 2_{03,24} + 3 \,.\, 2_{03,15} + 2_{03,06}) - \\ - 12 Q \,.\, (0_{23,33} + 3 \,.\, 0_{23,24} + 3 \,.\, 0_{23,15}) \end{aligned}\right\} \,.\quad (122)$$

Nach § 12 erhalten wir hier:

$$0_{22,33} = -\frac{4}{3} R,\ 0_{22,24} = -\frac{4}{3} \,.\, 0_{13,24} = \frac{10}{9} R - \frac{4}{3} Q',\ 0_{22,15} = -\frac{5}{9} R + \frac{10}{3} Q' \quad (123)$$

$$2_{03,33} = -S,\ 2_{03,24} = \frac{1}{3} R' + S,\ 2_{03,15} = 0_{13,25},\ 2_{03,06} = -5 R' - 5 S - 6 . 0_{13,25} \quad (124)$$

$$\left.\begin{aligned} 0_{23,33} = S \quad , \quad 0_{23,24} = \frac{7}{9} R' - \frac{2}{3} Q'' + \frac{2}{3} \,.\, 0_{13,25} \\ 0_{23,15} = -\frac{5}{3} S - \frac{35}{18} R' + \frac{5}{3} Q'' - \frac{5}{3} \,.\, 0_{13,25}. \end{aligned}\right\} \quad . \quad . \quad (125)$$

Setzt man dies alles in (122) ein, so kommt:

$$U = \frac{1}{6} R^2 + 4 Q' R + 18 Q'^2 - 36 Q Q'' + 26 Q R' + 36 Q S + 24 Q \,.\, 0_{13,25}. \quad (126)$$

Diese Invariante hat die Gewichte $\varrho = 20$, $\sigma = 10$. Sie ist (nach Q) aber nicht die einfachste Differentialinvariante, wie wir jetzt beweisen wollen.

Zu diesem Zwecke schreiben wir mit Hilfe der Gleichungen (12) und (14) die zu den Transformationen

$$t = \varphi(\widetilde{t}) \ , \ \widehat{a_{ik}} = \lambda(t) \,.\, a_{ik}$$

gehörigen Transformationsgleichungen für die in (126) rechts stehenden Ausdrücke

$$Q,\ Q',\ Q'',\ R,\ R',\ S \text{ und } S_0 = 0_{13,25}$$

auf. Was die zuletzt genannte Grösse S_0 betrifft, hat man vorerst die Gleichungen (12) und (14) zu ergänzen mit den beiden folgenden

$$\left.\begin{aligned} \widetilde{M}'_{25} = {} & M'_{25} \,.\, \varphi'^7 - M'_{33} \,.\, \tfrac{5}{3}\, \varphi'^5 \varphi'' + M'_{24} \,.\, \tfrac{15}{2}\, \varphi'^5 \varphi'' - M_{06} \,.\, \tfrac{1}{6}\, \varphi'^5 \varphi'' + \\ & + M'_{23} \,.\, (-5\, \varphi'^3 \varphi''^2 + 10\, \varphi'^4 \varphi''') - \\ & - M'_{05} \,.\, 2\, \varphi'^3 \varphi''^2 + M'_{13} \,.\, (15\, \varphi' \varphi''^3 - \tfrac{10}{3}\, \varphi'^2 \varphi'' \varphi''') - \\ & - M'_{04} \,.\, (\tfrac{10}{3}\, \varphi'^2 \varphi'' \varphi''' + \tfrac{5}{3}\, \varphi'^3 \varphi^{IV}) - \\ & - M'_{03} \,.\, \tfrac{1}{3}\, (\varphi'^2 \varphi^{V} + 10\, \varphi''^2 \varphi''' + 5\, \varphi' \varphi'' \varphi^{IV}) - M'_{02} \,.\, \varphi'' \varphi^{V}. \end{aligned}\right\} \quad (127)$$

$$\left.\begin{aligned} \widehat{M}'_{25} = {} & M'_{25} \,.\, \lambda^2 - M'_{33} \,.\, \tfrac{10}{3}\, \lambda\lambda' - M'_{06} \,.\, \tfrac{1}{3}\, \lambda\lambda' + M'_{23} \,.\, (10\, \lambda\lambda'' - 20\, \lambda'^2) + \\ & + M'_{05}\, (\lambda\lambda'' - 2\, \lambda'^2) + \\ & + M'_{13} \,.\, (20\, \lambda' \lambda'' - \tfrac{40}{3}\, \lambda\lambda''') + M'_{04} \,.\, (5\, \lambda' \lambda'' - \tfrac{10}{3}\, \lambda\lambda''') + \\ & + M'_{03} \,.\, (10\, \lambda''^2 - \tfrac{5}{3}\, \lambda\lambda^{IV} - \tfrac{20}{3}\, \lambda' \lambda''') + \\ & + M'_{02} \,.\, (10\, \lambda'' \lambda''' - 10\, \lambda' \lambda^{IV} + \lambda\lambda^{V}). \end{aligned}\right\} \quad (128)$$

Wir haben dann bei $t \rightarrow \widetilde{t}$:

$$\left.\begin{aligned} \widetilde{Q}'' &= Q'' \,.\, \varphi'^{11} + Q' \,.\, 19\, \varphi'^9 \varphi'' + Q \,.\, (72\, \varphi'^7 \varphi''^2 + 9\, \varphi'^8 \varphi''') \\ \widetilde{R}' &= R' \,.\, \varphi'^{11} + R \,.\, 10\, \varphi'^9 \varphi'' + Q' \,.\, 6\, \varphi'^9 \varphi'' + Q\, (48\, \varphi'^7 \varphi''^2 + 6\, \varphi'^8 \varphi''') \\ \widetilde{S} &= S \,.\, \varphi'^{11} - R \,.\, 3\, \varphi'^9 \varphi'' - Q \,.\, (6\, \varphi'^7 \varphi''^2 + 2\, \varphi'^8 \varphi''') \\ \widetilde{S}_0 &= S_0 \,.\, \varphi'^{11} - R \,.\, \tfrac{95}{12}\, \varphi'^9 \varphi'' + Q' \,.\, \tfrac{15}{2}\, \varphi'^9 \,.\, \varphi'' + \\ & \qquad + Q \,.\, (-5\, \varphi'^7 \varphi''^2 + 10\, \varphi'^8 \varphi''') \\ \widetilde{R}^2 &= R^2 \,.\, \varphi'^{20} + RQ \,.\, 12\, \varphi'^{18} \varphi'' + Q^2 \,.\, 36\, \varphi'^{16} \varphi''^2 \\ \widetilde{Q}' \widetilde{R} &= Q' R \,.\, \varphi'^{20} + RQ \,.\, 9\, \varphi'^{18} \varphi'' + QQ' \,.\, 6\, \varphi'^{18} \varphi'' + Q^2 \,.\, 54\, \varphi'^{16} \varphi''^2 \\ \widetilde{Q}'^2 &= Q'^2 \,.\, \varphi'^{20} + QQ' \,.\, 18\, \varphi'^{18} \varphi'' + Q^2 \,.\, 81\, \varphi'^{16} \varphi''^2 \end{aligned}\right\} \quad (129)$$

und ebenso bei $a_{ik} \rightarrow \widehat{a_{ik}}$:

$$\left.\begin{aligned} \widehat{Q}'' &= Q'' \,.\, \lambda^5 + Q' \,.\, 10\, \lambda^4 \lambda' + Q \,.\, (20\, \lambda^3 \lambda'^2 + 5\, \lambda^4 \lambda'') \\ \widehat{R}' &= R' \,.\, \lambda^5 + R \,.\, 5\, \lambda^4 \lambda' + Q' \,.\, 6\, \lambda^4 \lambda' + Q\, (24\, \lambda^3 \lambda'^2 + 6\, \lambda^4 \lambda'') \\ \widehat{S} &= S \,.\, \lambda^5 - R \,.\, 2\, \lambda^4 \lambda' - Q \,.\, 6\, \lambda^4 \lambda'' \\ \widehat{S}_0 &= S_0 \,.\, \lambda^5 - R \,.\, \tfrac{10}{3}\, \lambda^4 \lambda' + Q \,.\, (10\, \lambda^4 \lambda'' - 20\, \lambda'^2) \\ \widehat{R^2} &= R^2 \,.\, \lambda^{10} + RQ \,.\, 12\, \lambda^9 \lambda' + Q^2 \,.\, 36\, \lambda^8 \lambda'^2 \\ \widehat{Q}' \widehat{R} &= Q' R \,.\, \lambda^{10} + RQ \,.\, 5\, \lambda^9 \lambda' + QQ' \,.\, 6\, \lambda^9 \lambda' + Q^2 \,.\, 30\, \lambda^8 \lambda'^2 \\ \widehat{Q}'^2 &= Q'^2 \,.\, \lambda^{10} + QQ' \,.\, 10\, \lambda^9 \lambda' + Q^2 \,.\, 25\, \lambda^8 \lambda^{12} \end{aligned}\right\} \quad (130)$$

Jede Differentialinvariante vom φ'-Gewichte elf entsteht aus den ersten vier der Gleichungen (129) durch Elimination von φ''' und φ'' und φ''^2, und zwar durch lineare Kombination mit konstanten Koeffizienten x, y, z und s, d.h.

$$\widetilde{J} = x\widetilde{Q}'' + y\widetilde{R}' + z\widetilde{S} + s\widetilde{S}_0$$

muss ohne φ'', φ''^2 und φ''' sein. Ebenso darf

$$\widehat{J} = x\widehat{Q}'' + y\widehat{R}' + z\widehat{S} + s\widehat{S}_0$$

kein λ' und λ'' enthalten. Diese Forderungen führen auf ein System linearer Gleichungen für x, y, z und s, das nur die Lösung 0, 0, 0, 0 hat.

Der nächst-einfachere Fall bringt dieselben Ueberlegungen für etwaige Invarianten vom φ'-Gewichte 20. Hier sind dann sieben homogene Konstanten x, y, z, s, a, b und c zu ermitteln, sodass

$$\widetilde{J} = x\widetilde{Q}\widetilde{Q}'' + y\widetilde{Q}\widetilde{R}' + z\widetilde{Q}\widetilde{S} + s\widetilde{Q}\widetilde{S}_0 + a\widetilde{R}^2 + b\widetilde{Q}'\widetilde{R} + c\widetilde{Q}'^2 \quad (131)$$

kein φ'', φ''^2 und φ''' enthält und analog $\widehat{J}$ ohne λ' und λ'' ist. Die Koeffizienten dieser Grössen finden wir in der aus (129) und (130) entnommenen Tabelle

	$\varphi'^{18}\varphi''$	$\varphi'^{16}\varphi''^2$	$\varphi'^{17}\varphi'''$		$\lambda^9\lambda'$	$\lambda^8\lambda'^2$	$\lambda^9\lambda''$
QQ''	$19\,QQ'$	$72\,Q^2$	$9\,Q^2$	x	$10\,QQ'$	$20\,Q^2$	$5\,Q^2$
QR'	$10\,QR + 6\,QQ'$	$48\,Q^2$	$6\,Q^2$	y	$5\,RQ + 6\,QQ'$	$24\,Q^2$	$6\,Q^2$
QS	$-3\,RQ$	$-6\,Q^2$	$-2\,Q^2$	z	$-2\,RQ$	0	$-6\,Q^2$
QS_0	$-\frac{95}{12}RQ + \frac{15}{2}QQ'$	$-5Q^2$	$10\,Q^2$	s	$-\frac{10}{3}RQ$	$-20\,Q^2$	$10\,Q^2$
R^2	$12\,RQ$	$36\,Q^2$	0	a	$12\,RQ$	$36\,Q^2$	0
RQ'	$9\,RQ + 6\,QQ'$	$54\,Q^2$	0	b	$5\,RQ + 6\,QQ'$	$30\,Q^2$	0
Q'^2	$18\,QQ'$	$81\,Q^2$	0	c	$10\,QQ'$	$25\,Q^2$	0

Hieraus ergeben sich die Gleichungen

$$\left.\begin{array}{ll}
1a)\; 19x+6y+\frac{15}{2}s+6b+18c=0 & 2)\; 72x+48y-6z-5s+36a+54b+81c=0 \\
1b)\; 10y-3z-\frac{95}{12}s+12a+9b=0 & 3)\; 9x+6y-2z+10s=0 \\
4a)\; 5y-2z-\frac{10}{3}s+12a+5b=0 & 5)\; 20x+24y-20s+36a+30b+25c=0 \\
4b)\; 10x+6y+6b+10c=0 & 6)\; 5x+6y-6z+10s=0
\end{array}\right\} (132)$$

Berechnen wir hier aus 1*a*), 1*b*) und 4*b*) die Grössen *a*, *b* und *c* und setzen die gefundenen Ausdrücke in 4*a*), 2) und 5) ein, so ergeben sich die Gleichungen 3) und 6). Alle acht Gleichungen sind demnach äquivalent mit

$$11x + 6y + 30s = 0 \quad \text{und} \quad z = -x.$$

Damit nimmt die allgemeinste, aus (131) entspringende Invariante J die Gestalt an

$$\left.\begin{aligned} J_{20} = \left(-\frac{73}{288}R^2 + \frac{49}{24}Q'R - \frac{9}{8}Q'^2 + QQ'' - \frac{11}{6}QR' - QS\right).x + \\ + \left(-\frac{215}{576}R^2 + \frac{155}{48}Q'R - \frac{15}{16}Q'^2 - \frac{5}{3}QR' + QS_0\right).s \end{aligned}\right\} \quad (133)$$

wobei x und s beliebige Zahlenkoeffizienten bedeuten.

Wir setzen

$$J_{20} = x.A + s.B \quad \ldots\ldots\ldots \quad (134)$$

wo A und B aus (133) zu entnehmen sind.

Man beweist leicht, dass weder A noch B identisch Null und dass beide linear-unabhängig von einander sind.

Den Werten $x = -36$ und $s = 24$ entspricht die Invariante U von (126):

$$U = -36A + 24B. \quad \ldots\ldots\ldots \quad (135)$$

Mathematics. — *Zur projektiven Differentialgeometrie der Regelflächen im* R_4. (Fünfte Mitteilung). Von R. WEITZENBÖCK.

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Mit Hilfe des Heftpunktes H und des invarianten Punktes F lässt sich für die Erzeugende 0_{ik} der Fläche eine invariante Darstellung geben. Dies ist auch möglich, wenn man den Punkt M und dessen Ableitungen benutzt. Hier erhält man dann auch invariante Darstellungen, wenn man die gewöhnlichen Ableitungen durch kovariante Ableitungen ersetzt. Dies führt im Falle einer allgemeinen Fläche zur Beantwortung der Frage nach einem *wesentlichen System von Differentialinvarianten.* Man erhält deren sieben, von denen bei invarianter Normierung noch zwei in Wegfall kommen.

§ 15.

Wenn wir den durch $(xM'_{02}) = 0$ gegebenen Tangentialraum kurz durch die Reihe M' andeuten, also

$$(xM'_{02}) = (x0^2\, 2^2) = (xM') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)$$

setzen, und wenn die Ableitungen von M' durch

$$M'_1 = \frac{dM'}{dt}, \quad M'_2 = \frac{d^2M'}{dt^2}, \ldots$$

festgehalten werden, so ist nach (1):

$$(xM'_1) = (xM'_{12}) + (xM'_{03}) = \tfrac{2}{3}\,(xM'_{03})$$
$$(xM'_2) = \tfrac{2}{3}\,[(xM'_{13}) + (xM'_{04})], \text{ u. s. w.}$$

Die Heftebene m_{ijk} erscheint dann nach (32) dargestellt durch

$$m'_{ik} = (M'_{02}\, M'_{03})_{ik} = \tfrac{3}{2}\,(M'\, M'_1)_{ik}. \quad . \quad . \quad . \quad . \quad . \quad (137)$$

Die Gerade m_{ik} von (37) ist die Schnittgerade dreier aufeinanderfolgender Tangentialräume und erhält jetzt die Gestalt

$$m_{rs} = -\tfrac{3}{4}\,(M'\, M'_1\, M'_2)_{ijk}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

Den Schnittpunkt von vier konsekutiven Tangentialräumen haben wir M genannt (vgl. in § 5 die Gleichungen (39) und (42)). Mit Hilfe der Reihe M' haben wir

$$(Mu') = \tfrac{9}{8}\,(u'\, M'\, M'_1\, M'_2\, M'_3). \quad . \quad . \quad . \quad . \quad . \quad (139)$$

wobei (Mu') nach (42) und (82) gegeben ist durch

$$(Mu') = (\tfrac{1}{2} R + 3 Q') \,.\, (Hu') + 4 Q \,.\, [(2 u')\, 2_{03} - 3 \,.\, (0 u')\, 0_{23}]. \quad (140)$$

Setzen wir hier

$$u' = M'_4 = \tfrac{8}{9} (M'_{33} + 3 M'_{24} + 3 M'_{15} + M'_{06}),$$

so entsteht nach (43) die Invariante U. Wir haben

$$\triangle_{M'} = (M' \, M'_1 \, M'_2 \, M'_3 \, M'_4) = \tfrac{8}{9} (MM'_4) = \tfrac{16}{27} U. \quad . \quad . \quad (141)$$

Geht man umgekehrt vom Punkte M aus und setzt

$$M_1 = \frac{dM}{dt}, \quad M_2 = \frac{d^2 M}{dt^2}, \ldots$$

so findet man:

$$(MM_1)_{ik} = \tfrac{3}{4} U \,.\, (M' \, M'_1 \, M'_2)_{rst} = -U \,.\, m_{ik} \quad . \quad . \quad . \quad (142)$$

$$(MM_1 \, M_2)_{ijk} = \tfrac{1}{2} U^2 \,.\, (M' \, M'_1)_{rs} \quad . \quad . \quad . \quad . \quad . \quad (143)$$

$$(MM_1 \, M_2 \, M_3)_i = -\tfrac{1}{3} U^3 \,.\, M'_i \quad . \quad . \quad . \quad . \quad . \quad (144)$$

$$\triangle_M = (MM_1 \, M_2 \, M_3 \, M_4) = \tfrac{2}{9} U^4 = \frac{3^{10}}{2^{15}} \,.\, \triangle^4_{M'} \quad . \quad . \quad . \quad (145)$$

Diese Formeln zeigen: Bei $U \neq 0$ ist es für die Untersuchung der Kurve C_M = Ort der Punkte M gleichgiltig ob man C_M als Punktort oder als Ort der Schmieg-R_3 M' auffasst.

§ 16.

Nach (142) ist die Gerade, die die Punkte M und $M_1 = \frac{dM}{dt}$ verbindet, mit m_{ik} identisch, geht also durch den Heftpunkt H. Es muss also zwischen den drei Punkten M, M_1 und H eine lineare Relation bestehen, die wir jetzt ableiten wollen.

Im Ansatze

$$\tau \,.\, (Mu') + (M_1 u') \equiv \varepsilon \,.\, (Hu') \qquad \text{(identisch in } u'\text{)} \quad . \quad . \quad (146)$$

ermitteln wir zuerst τ durch die Substitution $u' = M'_{13}$. Führt man diese Substitution in (140) und der daraus durch Differentiation entstehenden Gleichung für $(M_1 u')$ aus, so ergibt sich nach leichter Rechnung

$$(MM'_{13}) = -16 \, Q^2 \,, \quad (HM'_{13}) = 0$$

$$(M_1 M'_{13}) = -24 \, QQ' - \tfrac{4}{3} \, QR. \quad . \quad . \quad . \quad . \quad . \quad (147)$$

Also wird bei $Q \neq 0$:

$$4\,Q\,.\,\tau = -6\,Q' - \tfrac{1}{3}\,R,$$

d.h. (146) bekommt die Gestalt

$$-(6\,Q' + \tfrac{1}{3}\,R)\,.\,M + 4\,Q\,.\,M_1 \equiv \varepsilon\,.\,4\,Q\,.\,H.$$

Zur Bestimmung von ε setzen wir jetzt $u' = M'_{23}$. Es wird

$$\left.\begin{aligned} (MM'_{23}) = \tfrac{4}{3}\,Q\,(-\tfrac{1}{2}\,R + 3\,Q')\ ,\ (HM'_{23}) = \tfrac{4}{3}\,Q \\ (M_1M'_{23}) = (\tfrac{1}{2}\,R' + 3\,Q'')\,.\,\tfrac{4}{3}\,Q - \tfrac{4}{3}\,Q'\,R - \\ -12\,QS - 12\,Q\,0_{23,24} - \tfrac{8}{9}\,R\,.\,(\tfrac{1}{2}\,R + 3\,Q'). \end{aligned}\right\} \quad .\quad .\quad (148)$$

Dies gibt nach (125) und (126)

$$4\,Q\,\varepsilon = -U$$

und damit erhalten wir schliesslich für (146):

$$(6\,Q' + \tfrac{1}{3}\,R)\,.\,M - 4\,Q\,.\,M_1 = U\,.\,H. \quad .\quad .\quad .\quad .\quad (149)$$

Nach Gleichung (143) liegt, wieder für $U \neq 0$, der Punkt M_2 in der Heftebene m'_{rs}. Es müssen sich also die Punkte H_1 (Gleichung (54)), G (Gleichung (95)) und F (Gleichung (96)) als Linearkombinationen von M, M_1 und M_2 darstellen lassen. Wir sahen am Ende des § 11, dass die Gerade MG die Tangente h_{ik} im Punkte $M + 3G$ schneidet. Daher müssen sich α und β so bestimmen lassen, dass

$$(Mu') + 3\,(Gu') \equiv \alpha\,.\,(Hu') + \beta\,.\,(H_1u')$$

wird. Setzt man hier wieder für u' die Reihen M'_{13} und M'_{23} ein, wobei man für G den Ausdruck (95) benutzt, so entsteht

$$2\,M + 6\,G = (6\,Q' + 7\,R)\,.\,H - 24\,Q\,.\,H_1. \quad .\quad .\quad .\quad .\quad (150)$$

Differenzieren wir jetzt (149), so kommt:

$$(6\,Q'' + \tfrac{1}{3}\,R')\,.\,M + (2\,Q' + \tfrac{1}{3}\,R)\,.\,M_1 - 4\,QM_2 = U'\,.\,H + U\,.\,H_1. \quad (151)$$

Hier kann man mit Hilfe von (149) H eliminieren und erhält

$$\left.\begin{aligned} U^2\,.\,H_1 = (6\,UQ'' + \tfrac{1}{3}\,UR' - 6\,U'\,Q' - \tfrac{1}{3}\,U'\,R)\,.\,M + \\ + (2\,Q'\,U + 4\,Q\,U' + \tfrac{1}{3}\,UR)\,.\,M_1 - 4\,QU\,.\,M_2. \end{aligned}\right\}\,. \quad (152)$$

Aus (150) findet man hiermit

$$\left.\begin{aligned} U^2\,.\,G = [-\tfrac{1}{3}\,U^2 + U\,(6\,Q' + 7\,R)\,(Q' + \tfrac{1}{18}\,R) - \\ -4\,Q\,(6\,UQ'' + \tfrac{1}{3}\,UR' - 6\,U'\,Q' - \tfrac{1}{3}\,U'\,R)]\,.\,M - \\ -(12\,QQ'\,U + 16\,Q^2\,U' + 6\,QRU)\,.\,M_1 + 16\,Q^2\,U\,.\,M_2 \end{aligned}\right\} \quad (153)$$

Wegen $F = M + G$ folgt hieraus

$$\left.\begin{aligned} U^2 . F = [\tfrac{8}{3} U^2 + U(6Q' + 7R)(Q' + \tfrac{1}{18} R) - \\ -4Q(6UQ'' + \tfrac{1}{3} UR' - 6U'Q' - \tfrac{1}{3} U'R)] . M - \\ -(12QQ'U + 16Q^2U' + 6QRU) . M_1 + 16Q^2U . M_2. \end{aligned}\right\} \quad (154)$$

Der Punkt F liegt auf der Erzeugenden 0_{ik}. Aus

$$F = (\tfrac{3}{2} R + 3Q') . H - 16Q . (0u') 0_{23} \qquad H = (0u') 0_{22}$$

finden wir dann

$$(HF)_{ik} = -16Q(0\dot{0})_{ik} 0_{22} \dot{0}_{23} = -16Q . 0_{ik} . -\tfrac{4}{3} Q,$$

$$(HF)_{ik} = \frac{64}{3} Q^2 . 0_{ik}. \quad . \; . \; . \; . \; . \; . \; . \quad (155)$$

Berechnen wir dann $(HF)_{ik}$ nach (149) und (154), so erscheint 0_{ik} als Gerade der Heftebene (MM_1M_2) wie folgt dargestellt:

$$\left.\begin{aligned} \frac{64}{3} U^3 Q . 0_{ik} = -(M_1M_2)_{ik} . 64Q^2U - (M_2M)_{ik} . 16QU(6Q' + \tfrac{1}{3} R) + \\ + (MM_1)_{ik} . 2\{-(6Q'U + 8QU' + 3RU)(6Q' + \tfrac{1}{3} R) + \\ + 2 . [\tfrac{8}{3} U^2 + U(6Q' + 7R)(Q' + \tfrac{1}{18} R)] - 8Q[6Q''U + \\ + \tfrac{1}{3} R'U - 6Q'U' - \tfrac{1}{3} RU']\} \end{aligned}\right\} \quad (156)$$

§ 17.

Die Ausdrücke M_1 und M_2 sind keine projektiven Differentialkontravarianten; demzufolge sind die Koeffizienten von $(M_2M)_{ik}$ und von $(MM_1)_{ik}$ in der letzten Gleichung keine Invarianten. Um eine invariante Darstellung von 0_{ik} in der Heftebene zu erhalten, müssen wir statt M_1 und M_2 invariante Punkte $M_{(1)}$ und $M_{(2)}$ wählen. Dies kommt darauf hinaus: die gewöhnlichen Ableitungen nach dem Parameter t sind zu ersetzen durch kovariante Ableitungen.

Dies kann man auf verschiedene Arten tun. Auf der Hand liegt, die Formel (18) zu benutzen. $M = (Mu')$ ist vom φ'-Gewichte 14 und vom λ-Gewichte 8. Sind dann J_1, J_2 zwei Differentialinvarianten mit den Gewichten (ϱ_1, σ_1) bzw. (ϱ_2, σ_2), so wird

$$\{M, J_1, J_2\} = \begin{vmatrix} M_1 & J_1' & J_2' \\ 14M & \varrho_1 J_1 & \varrho_2 J_2 \\ 8M & \sigma_1 J_1 & \sigma_2 J_2 \end{vmatrix} = M . [(14\sigma_1 - 8\varrho_1) J_1 J_2' - (14\sigma_2 - 8\varrho_2) J_1' J_2] + \\ + M_1 . (\varrho_1 \sigma_2 - \varrho_2 \sigma_1) J_1 J_2$$

ein invarianter Punkt der Geraden MM_1.

Wählt man hier z.B. als einfachste Invarianten J:

$$J_1 = Q \text{ mit } \varrho_1 = 9 \quad , \quad \sigma_1 = 5 \text{ und } J_2 = U \text{ mit } \varrho_2 = 20 \quad , \quad \sigma_2 = 10,$$

so ergibt sich aus einer Komitante $K(\varrho, \sigma)$ die „kovariante Ableitung" K_I:

$$K_I = \{K, Q, U\} = -10\,QUK_1 + [(5\varrho - 9\sigma)\,QU' - (10\varrho - 20\sigma)\,Q'U]\,.\,K. \quad (157)$$

Dies gibt für $K = M$ den Punkt

$$M_I = (20\,Q'U - 2\,QU')\,.\,M - 10\,QU\,.\,M_1. \quad . \quad . \quad . \quad (158)$$

Es geht aber noch einfacher. Wir haben nämlich die Transformationsgleichungen

$$\begin{aligned} \widetilde{M} &= M\,.\,\varphi'^{14} & \widehat{M} &= M\,.\,\lambda^8 \\ \widetilde{M}_1 &= M_1\,.\,\varphi'^{15} + M\,.\,14\,\varphi'^{13}\,\varphi'' & \widehat{M}_1 &= M_1\,.\,\lambda^8 + M\,.\,8\,\lambda^7\,\lambda' \end{aligned}$$

Nehmen wir hierzu:

$$\begin{aligned} \widetilde{Q}' &= Q'\,.\,\varphi'^{10} + Q\,.\,9\,\varphi'^{8}\,\varphi'' & \widehat{Q}' &= Q'\,.\,\lambda^5 + Q\,.\,5\,\lambda^4\,\lambda' \\ \widetilde{R} &= R\,.\,\varphi'^{10} + Q\,.\,6\,\varphi'^{8}\,\varphi'' & \widehat{R} &= R\,.\,\lambda^5 + Q\,.\,6\,\lambda^4\,\lambda' \end{aligned}$$

dann lassen sich in

$$M_{(1)} = QM_1 + (aR + bQ')\,.\,M$$

die Konstanten a und b so wählen, dass $\widetilde{M}_{(1)}$ frei von φ'' und $\widehat{M}_1$ frei von λ' wird. Man erhält

$$M_{(1)} = 24\,Q\,.\,M_1 - 2\,(R + 18\,Q')\,.\,M \quad . \quad . \quad . \quad . \quad (159)$$

und auch dies kann man eine „kovariante Ableitung" nennen.

Allgemein erhält man auf diese Weise bei einer Komitante K mit den Gewichten ϱ und σ die kovariante Ableitung

$$K_{(1)} = 24\,Q\,.\,K_1 + [(5\varrho - 9\sigma)\,R - 6\,(\varrho - \sigma)\,Q']\,.\,K; \quad . \quad . \quad (160)$$

$K_{(1)}$ hat die Gewichte $\varrho + 10$ und $\sigma + 5$.

Vergleichen wir (159) mit (149), so finden wir

$$M_{(1)} = -(36\,Q' + 2\,R)\,.\,M + 24\,Q\,.\,M_1 = -6\,U\,.\,H, \quad . \quad (161)$$

d. h. $M_{(1)}$ führt auf den Punkt H.

Die kovariante Ableitung $Q_{(1)}$ von Q selbst verschwindet identisch, wie man aus (160) für $Q = K$ sofort feststellt. Bei der Invariante U dagegen erhält man

$$U_{(1)} = 24\,QU' + (10\,R - 60\,Q')\,U = 24\,V. \quad . \quad . \quad . \quad (162)$$

Ist $K = J \,.\, L$, so gilt nach (160) die Differentiationsregel

$$(J \,.\, L)_{(1)} = J_{(1)} L + J L_{(1)}.$$

Wir übertragen nun die Gleichung (156) auf die Punkte M, $M_{(1)}$ und $M_{(2)}$, wobei $M_{(2)} = (M_{(1)})_{(1)}$ gesetzt ist („zweite" kovariante Ableitung). Wir haben nach (160) und (150)

$$H_{(1)} = 24\,Q \,.\, H_1 - (7\,R + 6\,Q') \,.\, H = -2\,M - 6\,G \quad . \quad . \quad (163)$$

und aus (161):

$$M_{(2)} = -6\,U_{(1)} \,.\, H - 6\,U \,.\, H_{(1)}.$$

Also wird, wenn wir $H_{(1)}$ eliminieren, analog mit (153):

$$36\,U \,.\, G = -12\,U \,.\, M + 6\,U_{(1)} \,.\, H + M_{(2)},$$

somit nach (161):

$$36\,U^2 \,.\, G = -12\,U^2 \,.\, M + U \,.\, M_{(2)} - U_{(1)} \,.\, M_{(1)}. \quad . \quad . \quad . \quad (164)$$

Hieraus, analog zu (154):

$$36\,U^2 \,.\, F = 24\,U^2 \,.\, M + U \,.\, M_{(2)} - U_{(1)} \,.\, M_{(1)}, \quad . \quad . \quad . \quad . \quad (165)$$

und hieraus schliesslich nach (155) bei $U \neq 0$:

$$72 \,.\, 64 \,.\, Q^2\, U^2 \,.\, 0_{ik} = 24\,U \,.\, (M M_{(1)})_{ik} - (M_{(1)} M_{(2)})_{ik} \,. \quad . \quad (166)$$

Wir benötigen auch die entsprechende Gleichung für die durch römische Ziffern festgehaltenen kovarianten Ableitungen M_I, $M_{II}, \ldots$ (Vgl. Gleichung (158)). Zunächst bemerken wir, dass nach (157)

$$Q_I = 0 \quad \text{und} \quad U_I = 0$$

gilt. Dann wird, wenn wir M_1 aus (158) und (161) eliminieren:

$$M_I = \tfrac{5}{2}\,U^2 \,.\, H - (2\,QU' - 5\,Q'U + \tfrac{5}{6}\,RU) \,.\, M, \quad . \quad . \quad . \quad (167)$$

oder auch, nach (162):

$$M_I = \tfrac{5}{2}\,U^2 \,.\, H - 2\,V \,.\, M \text{ mit } V = QU' + \tfrac{5}{12}\,RU - \tfrac{5}{2}\,Q'U. \quad (167\text{a})$$

Nach (157) haben wir für $K = H$:

$$H_I = -10\,QU \,.\, H_1 + (20\,Q'U - 7\,QU') \,.\, H. \,. \quad . \quad . \quad . \quad (168)$$

Eliminieren wir mit Hilfe von (163) H_1, so kommt

$$H_I = \tfrac{5}{6}\,U \,.\, M + \tfrac{5}{2}\,U \,.\, G - 7\,V \,.\, H. \,. \quad . \quad . \quad . \quad . \quad . \quad (169)$$

Weiters ergibt sich durch kovariantes Differentieren aus (167*a*):

$$M_{II} = \tfrac{5}{2} U^2 . H_I - 2 V_I . M - 2 V . M_I .$$

Setzen wir hier (169) ein, so entsteht

$$\frac{25}{4} U^3 . G = M_{II} + 2 V . M_I + \left(2 V_I - \frac{25}{12} U^3 \right) . M + \frac{35}{2} U^2 V . H. \quad (170)$$

Hierin ist nach (157) für $K = V$:

$$V_I = 5 Q (3 U' V - 2 U V') = 5 Q . W . \quad . \quad . \quad . \quad . \quad (171)$$

d. h. V_I zerfällt in Q und die Invariante

$$W = 3 U' V - 2 U V' ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

W hat die Gewichte $\varrho = 51$ und $\sigma = 25$. Statt (170) haben wir dann auch

$$\frac{25}{4} U^3 . G = M_{II} + 2 V . M_I + \left(10 Q W - \frac{25}{12} U^3 \right) . M + \frac{35}{2} U^2 V . H. \quad (173)$$

Hieraus weiter für $F = M + G$:

$$\frac{25}{4} U^3 . F = M_{II} + 2 V . M_I + \left(10 Q W + \frac{25}{6} U^3 \right) . M + \frac{35}{2} U^2 V . H. \quad (174)$$

Mit Hilfe von (155) finden wir also schliesslich nach (167*a*):

$$\left. \begin{aligned} \frac{10^3}{3} Q^2 U^5 . 0_{ik} = (M_I M_{II})_{ik} - 2 V . (M_{II} M)_{ik} + \\ + \left[4 V^2 - 10 Q W - \frac{25}{6} U^3 \right] . (M M_I)_{ik} . \end{aligned} \right\} \quad . \quad . \quad . \quad (175)$$

Hiermit ist also die Erzeugende 0_{ik} auch durch die drei Seiten des Dreieckes $M M_I M_{II}$ ausgedrückt.

§ 18.

Die Gleichung (175) gestattet es, auf die Frage nach den *wesentlichen Differentialinvarianten* eine Antwort zu geben, wobei wir den sogenannten allgemeinen Fall voraussetzen, in dem keine der Invarianten Q und U identisch Null ist. Ein wesentliches System von Differentialinvarianten $J_1, J_2, \ldots, J_m$ liegt bekanntlich vor, wenn jede weitere Differentialinvariante von F durch diese Invarianten J_s und durch deren Ableitungen $\frac{d J_s}{d t}, \frac{d^2 J_s}{d t^2}, \ldots$ ausgedrückt werden kann.

Auf Grund der quinären Identität

$$(01234) . 5 = (12345) . 0 - (02345) . 1 + (01345) . 2 - (01245) . 3 + (01235) . 4 \quad (176a)$$

oder kürzer

$$C_5 . 5 = C_0 . 0 - C_1 . 1 + C_2 . 2 - C_3 . 3 + C_4 . 4 \quad . \quad . \quad (176b)$$

kann bei $C_5 \neq 0$ der Punkt 5 als Linearkombination der Punkte 0, 1, 2, 3, 4 dargestellt werden. Wir wählen für 0, 1, 2, ... die Punkte $M, M_I, M_{II}, \ldots$ und haben dann

$$\left.\begin{array}{l} C_0 = (M_I \, M_{II} \, M_{III} \, M_{IV} \, M_V) \;, \; C_1 = (M M_{II} \, M_{III} \, M_{IV} \, M_V), \\ C_2 = (M M_I \, M_{III} \, M_{IV} \, M_V) \;, \; C_3 = (M M_I \, M_{II} \, M_{IV} \, M_V), \\ C_4 = (M M_I \, M_{II} \, M_{III} \, M_V) \;, \; C_5 = (M M_I \, M_{II} \, M_{III} \, M_{IV}). \end{array}\right\} (177)$$

Wenn wir Gleichung (158) wiederholt kovariant differentieren, so finden wir

$$C_5 = (M M_I \, M_{II} \, M_{III} \, M_{IV}) = (-10\, QU)^{10} . (M M_1 \, M_2 \, M_3 \, M_4),$$

also nach (145):

$$C_5 = \tfrac{8}{9}\, U^4 (-10\, QU)^{10}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178)$$

Ferner ist nach (157), da C_5 die Gewichte 370 und 190 hat,

$$C_4 = (C_5)_I = -10\, QUC_5' + (140\, QU' + 100\, Q'U) . C_5 = 0. \quad (179)$$

Dass man auch C_3 weglassen kann, wenn man dafür die durch (133) gegebene Invariante

$$A = -\frac{73}{288} R^2 + \frac{49}{24} Q' R - \frac{9}{8} Q'^2 + QQ'' - \frac{11}{6} QR' - QS \quad (180)$$

zum System rechnet, werden wir sogleich zeigen.

Es bleiben dann für das wesentliche System:

$$Q, U, V, A, C_0, C_1, C_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

Sind diese Invarianten als Funktionen des Parameters t gegeben, so ist die Regelfläche F bis auf projektive Transformationen des Raumes R_4 bestimmt.

Beweis. Man kennt dann die Funktionen C_i von (177), kann also M_V und damit alle höheren $M_{VI}, M_{VII}, \ldots$ linear durch die fünf Punkte $M, M_I, \ldots, M_{IV}$ ausdrücken. Dabei ist das vierdimensionale Simplex

$M\,M_I\,M_{II}\,M_{III}\,M_{IV}$ willkürlich gewählt. Aus (175) erhält man weiters durch fortgesetzte kovariante Differentiation Ausdrücke für $0_{ik}, 1_{ik}, 2_{ik}, \ldots$ in denen nur die Invarianten (181) und deren Ableitungen und nur die Punkte M bis M_{IV} vorkommen. Also liegen dann die Werte von $0_{ik}, 1_{ik}, \ldots$ für $t=0$ fest, womit auch $0_{ik}(t)$, also F gegeben ist.

Wir haben jetzt noch den Beweis dafür nach zu tragen, dass man im wesentlichen System die Invariante C_3 durch A ersetzen kann. Zu diesem Zwecke vereinfachen wir uns die Rechnung durch eine Normierung, die sich durch die Gleichungen

$$Q \equiv 1 \;,\; U \equiv -\tfrac{1}{10} \quad \ldots\ldots\ldots \quad (182)$$

ausdrücken lässt. Man erreicht dies durch Wahl eines projektiv-invarianten Parameters $\widetilde{t}$ und eines Faktors $\lambda(t)$ für welche

$$Q \,.\, \varphi'^{\,9} \,.\, \lambda^5 \equiv 1 \text{ und } U \,.\, \varphi'^{\,20} \,.\, \lambda^{10} \equiv -\tfrac{1}{10}$$

wird. Es gilt dann $Q' \equiv 0$, $U' \equiv 0$ und nach (157) wird

$$K_I = K_1 = \frac{dK}{dt},$$

d. h. die kovariante Ableitung geht in die gewöhnliche über.

Wir finden bei dieser Normierung weiters, dass nach (167a)

$$V = -\tfrac{1}{24} R \quad \ldots\ldots\ldots \quad (183)$$

und nach (172)

$$W = -\tfrac{1}{120} R' \quad \ldots\ldots\ldots \quad (184)$$

wird. Demzufolge geht jetzt (175) über in

$$0_{ik} = -(MM_1)_{ik} \,.\, (\tfrac{25}{12} R^2 + 25 R' + \tfrac{5}{4}) + 25 R \,.\, (MM_2)_{ik} - 300 \,.\, (M_1 M_2)_{ik}. \quad (185)$$

Setzen wir

$$\omega = \tfrac{25}{12} R^2 + 25 R' + \tfrac{5}{4}$$

und schreiben (rs) statt $(M_r\, M_s)_{ik}$, so ergeben sich aus (185) die Gleichungen

$$\left.\begin{aligned}
0_{ik} &= -\,\omega\,(01) + 25R\,(02) - 300\,(12)\\
1_{ik} &= -\,\omega'\,(01) + (25R' - \omega)\,(02) + 25R\,(12) + 25R\,(03) - 300\,(13)\\
2_{ik} &= -\,\omega''(01) + (25R'' - 2\,\omega')(02) + (50R' - \omega)(12) + (50R' - \omega)(03) +\\
&\qquad + 50R(13) + 25R(04) - 300(23) - 300(14)\\
3_{ik} &= -\,\omega'''(01) + (25R''' - 3\,\omega'')(02) + (75R'' - 3\,\omega')(12) +\\
&\quad + (75R'' - 3\,\omega')(03) + (150R' - 2\,\omega)(13) + (75R' - \omega)(04) +\\
&\quad + 50R(23) + 75R(14) + 25R(05) - 600(24) - 300(15).
\end{aligned}\right\} \quad (186)$$

Mit diesen Ausdrücken kann man $S = 0_{23,33}$ ermitteln. Eine längere Rechnung ergibt, da jetzt nach (178)

$$C_5 = \tfrac{2}{5} \,.\, 10^{-4} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (187)$$

ist:

$$S = f(R, R') + 3 \,.\, 100^2 \,.\, C_3, \; . \; . \; . \; . \; . \; . \; . \quad (188)$$

d. h. C_3 kann durch S und nach (180) also durch A ersetzt werden.

Wir bemerken noch, dass auch die Punkte $H, H_1, H_2, \ldots$ beim Aufbau eines wesentlichen Systems Verwendung finden können. So haben wir bei obiger Normierung z. B. nach (167*a*)

$$H = -\frac{10}{3} RM + 40\, M_1 \quad . \; . \; . \; . \; . \; . \; . \quad (189)$$

Daraus findet man durch Differentiation Ausdrücke für $H_1, H_2, \ldots$ und kann dann die Invariante $\triangle_H$ von (88) durch $R, R', \ldots$ und die C_i von (177) ausdrücken.

Chemistry. — *The Exact Measurement of the Specific Heat of Metals at High Temperatures.* **XXXI.** *The Mean Specific Heat of Cobaltum in Connection with the Granular Size of its Crystalline Structure.*

By F. M. JAEGER and A. J. ZUITHOFF.

(Communicated at the meeting of June 29, 1940.)

§ 1. The contents of this paper can be considered as a sequel to our previous publication recently issued on the properties of metallic *cobaltum* [1]. Moreover, we wish to draw attention in it to some phenomena of more general interest, because they prove to allow of a clearer insight with respect to the influence which the special coarseness of the crystalline structure of metals seemingly has on their physical properties, — more especially on the numerical values of their mean specific heats. The latter fact has previously been emphasized by us in the course of the same kind of measurements made with other metals in this laboratory [2]; and the present results evidently corroborate our earlier conclusions in a particularly striking way.

In the paper referred to, we made use of two samples of *cobaltum*: one massive sample of very pure cobaltum, consisting of *large* crystallites and furnished by Dr. DE BOER of the PHILIPS' WORKS in Eindhoven and in this paper designated as *cobaltum-A*; and a second specimen from the *Union minière du Haut Katanga* in *Brussels*, — here indicated as cobal*tum-B*, — which was electrolytically deposed and which proved to be composed of *extremely small* crystallites, which were hardly perceptible in a microscopical enlargement of 600—800 times. The two samples were very hard, sample-*B* being the hardest of the two. Whilst *cobaltum-A*, when brought into dilute hydrochloric acid in excess and heated on the waterbath, proved to dissolve *readily* within a few days, an equal weight of *cobaltum-B* treated in the same way did not or only *extremely slowly* dissolve even after several weeks, — notwithstanding the fact, that it contained somewhat more, — although very little, — cobaltous oxide and about 1 % less of metallic cobaltum than the other sample. Only after preliminary heating of *cobaltum-B* at about 1000° C. during a long time,

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[1] F. M. JAEGER, E. ROSENBOHM and A. J. ZUITHOFF, Rec. d. trav. d. chim. d. Pays-Bas, **59**, 831, (1940).

[2] F. M. JAEGER and E. ROSENBOHM, ibid., **53**, 456 a. f. (beryllium), (1934); F. M. JAEGER and W. A. VEENSTRA, ibid., **53**, 924, 925 a. f., (*zirconium-B* and -*C*), (1934); T. J. POPPEMA and F. M. JAEGER, ibid., **38**, 824, (1935). Conf. also: F. M. JAEGER, E. ROSENBOHM and J. A. BOTTEMA, Rec. d. trav. d. Chim. d. Pays-Bas, **52**, 61, (1933).

the solubility proved to have increased to that of the other sample and on microscopical examination it now appeared to have obtained a rather coarser texture, composed of readily discernable grains.

This strange behaviour induced us to compare its mean specific heat c_p, — starting at low temperatures, — with that of *cobaltum-A*. The results of these investigations are communicated in the following sections.

§ 2. The measurements of $\bar{c}_p$ were made in the usual way, the metal being enclosed in a vacuum platinum crucible of the ordinary type. Starting at 588° C., the results of these determinations at increasing temperatures are recorded sub 1—9 in the Table I.

Evidently the whole curve thus determined is situated *beneath* the curve for sample-*A*: the $\bar{c}_p$-values at the same temperature are at 588° C. about 2 %, at 738° C., about 1,8 % and between 797° C. and 1181° C. about 0,6 % (mean values) *lower* than in the case of sample-*A*. The transition-point (1125° C.), however, has remained *un*changed; and, as the two curves have about the same course, also the *true* specific heats C_p of the two samples are practically identical. After this heating at 1182° C. the values of $\bar{c}_p$ were once more determined, now starting at 649° C., until 1296° C. was reached. This time all values of $\bar{c}_p$, up to 1000° C., *were seen practically to co-incide* with those of the curve for sample-*A*; but at temperatures surpassing 1000° C. the curve takes a course *intermediate* between that for *A* and the curve for *B* obtained by measurements 1—9;

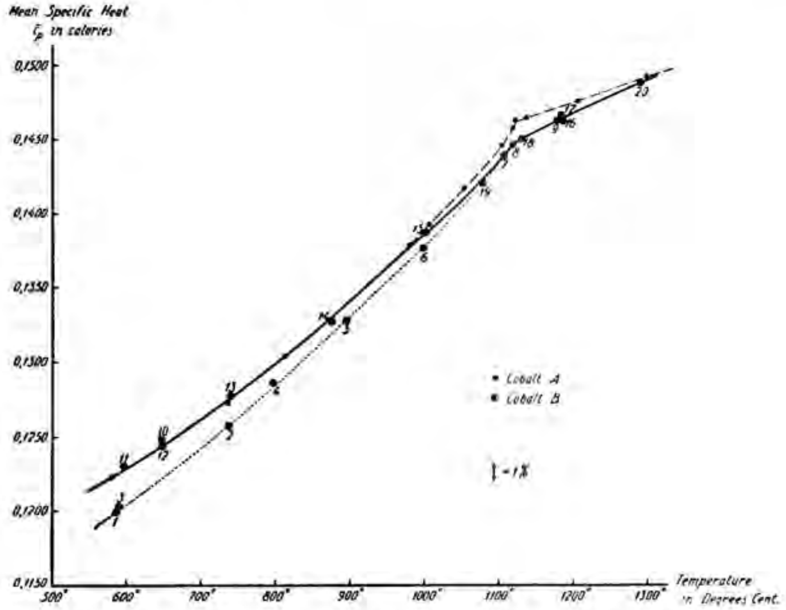

Fig. 1. *The mean specific heats of Cobaltum-A and of Cobaltum-B between 580° and 1350° C.*

TABLE I. Mean Specific Heats $\bar{c}_p$ of Cobaltum-*B* (Katanga) between 100° and 1400° C.

Sequence-number of Measurement:	Temperature t in Cent.:	Final temperature t' of the Calorimeter:	Increase of the temperature of the Calorimeter $\triangle t$ in mikrovolts:	Quantity of Heat Q developed by 1 Gr. between t° and 25° C in cal.:	*B* Mean Specific Heat $\bar{c}_p$ between t and t' in cal.:	*A* Mean specific Heat $\bar{c}_p$ of sample-*A*:	Differences $(A-B)$:	Differences $\triangle$ in percent:
1	587°.7	21°.45	318.72	67.48^{8}	0.1199	0.1224	+0.0025	Mean: 1.8%
2	738.2	22.76	419.19	89.65^{6}	0.1257	0.1275	0.0018	
3	589.7	21.66^{7}	320.64	67.97^{2}	0.1204	0.1226	0 0022	
4	797.5	23.04^{5}	462.41	99.30^{0}	0.1285	0.1296	0.0011	Mean: 0.6%
5	895.1	23.98	534.95	115.57^{4}	0.1328	0.1332	0.0004	
6	1000.1	23.08	618.19	134.13^{4}	0.1376	0.1383	0 0007	
7	1106.4	24.16	709.92	155.69^{3}	0.1440	0.1444	0.0004	
8	1120.1	23.76	723.99	158.41^{1}	0.1447	0.1458	0.0011	
9	1181.2	25.50	771.12	169.16^{5}	0.1463	0.1472	0.0009	
10	649.1	24.14	363.02	77.77^{2}	0.1246	0.1243	−0.0003	till 1000° C practically 0%
11	596.3	23.07	329.07	70.24^{4}	0.1230	0.1228	−0.0002	
12	648.8	22.98	362.97	77.59^{6}	0.1244	0.1246	+0.0002	
13	739.9	23.36	425.07	91.16^{8}	0.1277	0.1276	−0.0001	
14	876.8	22.42	523.73	112.96^{0}	0.1327	0.1322	−0.0005	
15	1002.2	24.75	623.07	135.67^{2}	0.1388	0.1386	−0.0002	
16	1184.8	24.08	774.37	169.70^{0}	0.1463	0.1476	+0.0013	Mean: 0.8%
17	1185.4	25.65	775.64	170.27^{0}	0.1467	0.1478	+0.0011	
18	1130.5	24.96	731.53	160.28^{2}	0.1450	0.1462	+0.0012	
19	1079.9	24.42	686.09	149.78^{6}	0.1420	0.1431	+0.0011	
20	1295.9	24.75	861.40	189.11^{1}	0.1488	0.1490	+0.0002	0.1%

the transition-point is again *not altered* and about 1125° C., the divergence of the higher and lower curves remains 0,8 %, but then gradually approaches the curve for the β-modification of sample-*A*, so as to *completely* coincide with it at about 1320° C. The deviation of the two curves is greatest at the CURIE-point and decreases afterwards up to 1320° C., — where it becomes zero. Evidently the heat of transformation still makes itself apparent *above* the CURIE-point.

A special controlling experiment was made, moreover, for the purpose of ascertaining that the values obtained do indeed correspond to *real* equilibria: in experiment 16 the normal time (45 minutes) of heating at a constant temperature was increased to 60 minutes, in 17 to 90 minutes; and in this way it was found that the results of experiments 16, 17 and 9, — all at 1181° — 1185° C., — were in full accordance with each other.

The data of Table I and those of Table II of our previous paper on cobaltum (loco cit.) are graphically represented in Fig. 1 besides each other.

§ 3. By these experiments it has been clearly established, that the mean specific heats $\bar{c}_p$ of *cobaltum* are found *to be so much lower, the more its composing crystallites are smaller:* a fact in plain agreement with that previously stated in the case of other metals studied in this laboratory, — where in general it was found, that the values of c_p observed were *variable with the degree of division* of the materials considered. Thus in the case of *antimony* [1]) c_p of the metal as small granules proved to be about 2 % lower than that of *antimony* in the form of a compact lump; and as well from MAGNUS and HOLTZMANN's experiments with *beryllium* [2]), as from our investigations [3]) on the same metal, it must be concluded, that c_p depends on the size of the grains constituting the metal. With *zirconium* and *beryllium* [2]) the values of $\bar{c}_p$ for powdered metals appeared *higher* than those for massive bars; but the fact, that the deviations here considered can occasionally be in the *one*, or, with other instances, in the *opposite* direction, — was also clearly stated during our studies [5]) on the *influence of prealable cold working* of such metals on the values of c_p measured. For in these processes of crushing and drawing, also a change in the *size* of the particles in the metals doubtlessly occurs to a larger or smaller extent. Another most remarkable circumstance is, that the occasionally observed "retardation"-phenomena at, or in the vicinity of, the transition-points in the massive lumps of such metals, are seen strongly to diminish and even totally to disappear, when the same materials are

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[1]) T. J. POPPEMA and F. M. JAEGER, loco cit.

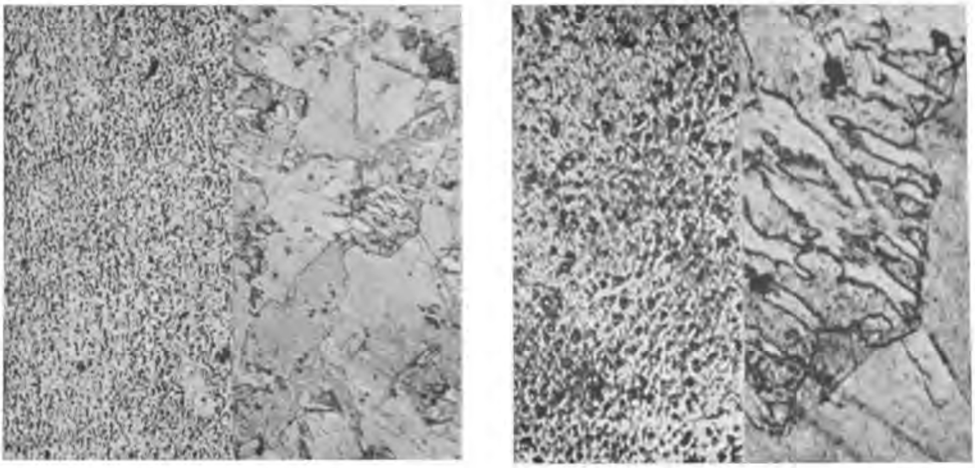
[2]) A. MAGNUS and H. HOLTZMANN, Ann. d. Phys., **3**, 585, (1929).

[3]) F. M. JAEGER and E. ROSENBOHM, Proc. Kon. Akad. v. Wetensch., Amsterdam, **37**, 67, (1934); Recueil, loco cit.

[4]) F. M. JAEGER and W. A. VEENSTRA, loco cit.

[5]) F. M. JAEGER, E. ROSENBOHM and J. A. BOTTEMA, loco cit.; conf. p. 84.

F. M. JAEGER and A. J. ZUITHOFF: THE EXACT MEASUREMENT OF THE SPECIFIC HEAT OF METALS AT HIGH TEMPERATURES. XXXI. THE MEAN SPECIFIC HEAT OF COBALTUM IN CONNECTION WITH THE GRANULAR SIZE OF ITS CRYSTALLINE STRUCTURE.



a.
Cobaltum-B

before heating, attacked with aqua regia. | after repeated heating, attacked with aqua regia (25 %)

Enlargement: 60 times.

b.
Cobaltum-B

The same as in *a*, but enlargement: 275 times.

Fig. 2. *Microphotographs of Cobaltum-B before and after heating.*

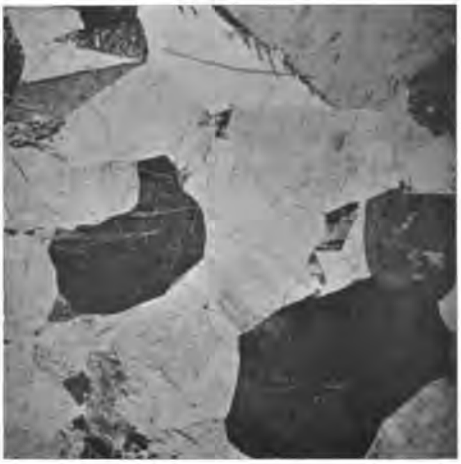


Fig. 3. *Cobaltum-A*

without preliminary heating; etched with 5 % aqua regia.

Enlargement: 7,5 times.

Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

finely powdered, — the degree of division of the crystallites thus being considerably increased. In the case of *zirconium* it could even be demonstrated, that *sintering* of the powdered material into a coherent mass caused the hysteresis-phenomena once more to return (2).

In the case of *cobaltum* doubtlessly the "recrystallisation" of the metal after heating at higher temperatures must be the cause of the fact, that the values of c_p of both *cobaltum-B* and *cobaltum-A* become identical again between 500° and 1000° C. But then the other fact still remains, that about 125° *below* and till 200° *above* the CURIE-point, *the original deviations persist* and that *they reach a maximum value at this transition-point itself*. It must be evident, that here the influence of the *heat of transition* at this point must be the cause of the fact, that curve *A* still remains higher situated than curve-*B*. It is this heat of transition corresponding to the transformation at the CURIE-point, which, — making itself already perceptible some 125° below this temperature, — appears to be scattered over a considerable range of temperatures before the CURIE-point is reached, — just as in the case of pure *iron* (6). This particular interpretation in the case of the finely-crystallized *cobaltum-B* seems to be justified to a high degree, if it may only be supposed, that the transition-heat at 1125° C. now was *appreciably smaller*, than in the case of the coarse-crystalline *cobaltum-A*. The situation of the transition-temperature itself, however, is seen *not* to be altered by this fact.

§ 4. Finally, as a result of the microscopical examinations, we here insert fig. 2, which shows a microphotograph *a* of the original preparation at room-temperature and after heating to 1200° C.; and a micro-photograph *b* of the same samples at an enlargement of 275 times. The increase in size of the crystallites by the gradually achieved recrystallisation is quite evident. The *heated* sample proved, moreover, now to have become readily soluble in dilute hydrochloric acid. For the purpose of comparison we add (in fig. 3) a microphotograph of *cobaltum-A*, which was *not* previously heated; here an enlargement was used of only 7—8 times.

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(6) F. M. JAEGER, E. ROSENBOHM and A. J. ZUITHOFF, Rec. d. trav. d. chim. d. Pays-Bas, **57**, 1322, (1938).

*Groningen, Laboratory for Inorganic and Physical
Chemistry of the University.*

Geology. — *New data on the smaller islands North of Venezuela.* By L. RUTTEN.

(Communicated at the meeting of June 29, 1940.)

In 1931 I reviewed (4) the geological literature of the Venezuelan islands between Bonaire and Trinidad and gave some data on the base of the study of rock-samples, collected by R. LUDWIG and by P. R. LOWE. In 1936 Mr. P. WAGENAAR HUMMELINCK visited these islands on a biological voyage; he made at the same time collections of rocks which he kindly gave to my institute. The following additional data on the geology of the islands are the result of the study of these collections.

Los Frailes. Nothing seems to have been published on these small cliffs which lie quite near to the northeast-coast of Margarita [1]). There are samples from Puerto Real and from La Peche.

No. **156**, Puerto Real, SW-slope of Morro Grande, alt. 40 m, is a diabasic diorite. The main components are albitized, idiomorphic plagioclase and pyroxene, with ophitic texture; large spots of chlorite with some epidote probably originated from biotite. Some green hornblende, octahedral ore and many very slender needles of apatite; moreover traces of micropegmatite, occurring as interstitial material between the main components.

No. **157** and **158**, same locality, sampled from isolated pieces of rock; are diabases. The first is very coarse, the second is fine-grained and fluidal; in both, the plagioclases are more or less albitized.

No. **159**, found at the top of Morro Grande in large blocks, is a coarse quartz-epidote-chlorite rock. It contains octahedral limonitized ore and slender needles of apatite. Both epidote and chlorite may occur in spherulites. It is very probably a postmagmatic rock.

No. **160** from the SW part of La Peche is probably a quartzdioriteporphyrite which, however, has some characteristics of a diabase. It contains rare phenocrysts of decalcified plagioclase partly filled with epidote, and a groundmass with an ophitic texture, consisting of laths of albite and small crystals of a green biotite which partly have changed into spherulitical chlorite. There is some interstitial material of quartz and micro-albite-pegmatite.

No. **161**, same locality, is a green rock, consisting almost entirely of uralite and albite, with, locally, some prehnite. It is probably a metamorphic porphyrite-tuff. According to Mr. HUMMELINCK's field-notes sample **160** occurs as a "lightcoloured ledge" in **161**.

No. **162** is, according to HUMMELINCK, the contact-zone between **160** and **161**. It is a dirty-green rock, partly coarse-grained, partly fine-grained. The coarse-grained part is

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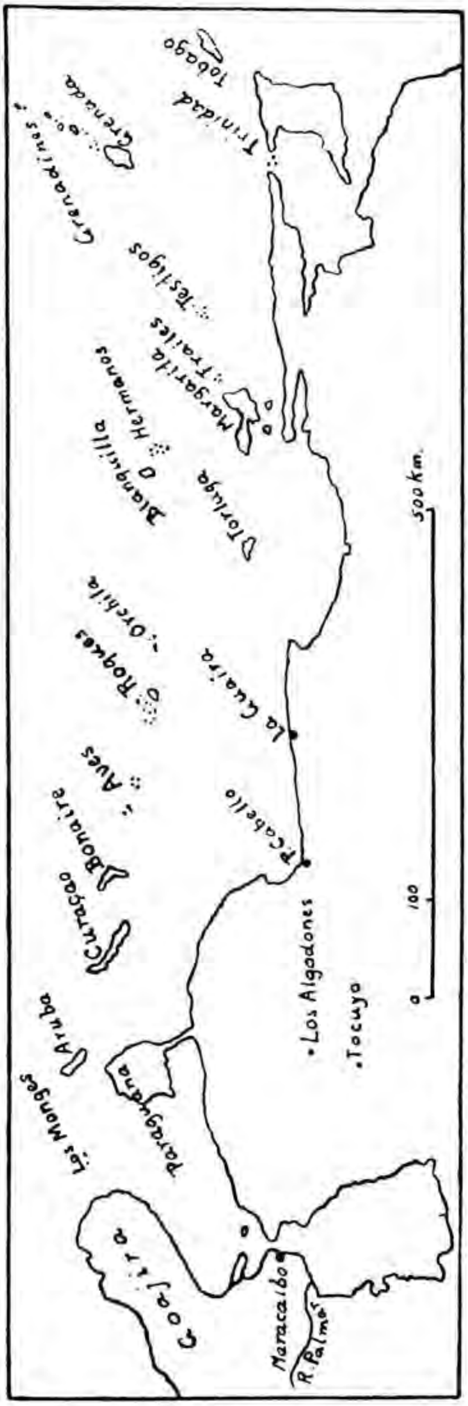
[1]) In the academical thesis of Mr. HUMMELINCK (1*a*) published after the communication of this paper, some notes (p. 16) and a sketchmap (p. 44) are to be found. The latter shows that the islands are arranged "en échelon" with a WNW-strike.

a metamorphosed porphyrite-breccia, each grain being a piece of porphyrite, whilst the intermediate substance consists mainly of uralite. The fine-grained part is a metamorphic porphyrite-tuff, containing equally much uralite and a veinlet of prehnite.



No. **163**, from the N-hill of La Peche is a coarse uralite-diabase with large crystals of labradorite and uralitic hornblende, which often includes rests of pyroxene. There are many skeleton-crystals of ore, slender apatite needles and a vein of prehnite.

No. **164**, from the basal part of the S-hill of La Peche, is a strongly uralitized porphyrite-tuff: it is a quartz-uralite-albite rock with remains of porphyritic texture.

No. **165**, from the depression between the N- and the S-hill of La Peche, is equally a metamorphic, uralitized, porphyritic tuff; it consists of a fibrous fabric of uralite and albite with, locally, remains of plagioclase-phenocrysts, and with an undeniable piece of porphyrite which, although being metamorphosed, shows still clearly the texture of a groundmass. It contains a vein of prehnite.

There is an enormous difference between the magmatic rocks of Los Frailes which have not suffered from crushing forces and the schistose rocks of the near Margarita (5) which have suffered strongly from crushing. On the other hand the rocks of Los Frailes fit in very well with those of the island-row Aruba-Los Testigos, the numbers **157**, **158**, **161**, **162**, **163**, **164** and **165** resembling the rocks of the cretaceous, volcanic basement of some of these islands, and more especially the contactmetamorphic tuffs and diabases from Aruba (7). It is not certain whether the samples **156**, **159** and **160** belong to the volcanic-basement-series or to the younger group of intruding rocks, so well known from Aruba, but also from other islands. They have on the one side "dioritic relations", on the other side "diabasic relations". We suppose that, in any case, an intrusive rock must occur in the underground, near to Los Frailes, given the strong uralitization of some porphyritic tuffs and breccias.

Los Testigos. I could state in 1931 (l.c.) the presence of hornblende-granodiorite and of spessartite on these islands. Mr. HUMMELINCK took samples on Morro de Iguana, Chiwu, Angoletta, Tamarindo (Testigo Grande) and Isla de Conejo (see 1*a*, p. 15, 44). They enlarge greatly our knowledge of this group of cliffs.

No. **138**, top of Morro de Iguana, is related to the granodiorite, described in 1931. It differs from it by its femic minerals which are chlorite-epidote (ex: biotite) and pyroxene with a marginal zone of amphibole; the leucocratic components in both rocks are the same.

No. **139**, from an elevated place at the E. coast of Morro de Iguana, differs from the foregoing by the presence of more femic components (augite, bronzite, biotite, with skeleton crystals of ore). The leucocratic minerals are idiomorphic, finely twinned, zonally built andesines with more acid margins, and a filling-mass of rather coarse orthoclase and quartz. The rock may be called a biotite-pyroxene-quartzmonzonite.

No. **140**, sampled from isolated fragments of rock on Morro de Iguana is a quartz-chlorite-epidote-rock with crystals of limonitized magnetite; it is very probably a postmagmatic rock.

No. **141**, from Chiwu, a cliff, measuring about 300 × 80 m, is again a granodiorite. It contains large, porphyritic plagioclases, passing gradually into smaller ones, all idiomorphic, finely twinned and zonal. The femic minerals, probably biotite, have entirely been changed into masses of chlorite and carbonate. There is an abundant filling mass of coarsely crystalline orthoclase and quartz.

No. **142** from Chiwu is a, probably postmagmatical, quartz-epidote-zoisite rock.

No. **143** and **145**, from the small island of Angoletta (it measures, according to Mr. HUMMELINCK, only 80 × 30 m) are uralitic quartzdioriteporphyrites. They contain phenocrysts of plagioclase and phenocryst-like spots of uralite. In both rocks the texture of the groundmass is granitic-isodiametric, and it consists of plagioclase, quartz and uralite.

No. **144** is a quartzitic rock, probably a sample from a quartz-vein. It occurs as a dike in no. **143**.

No. **146** from top of Morro Grande, Tamarindo, is a granodiorite, resembling **138**.

No. **147** which occurs as a white dike of 20 cm within the former rock, is a plagiaplite, consisting of an aplitic intergrowth of quartz and dusty, acid plagioclase (n : 1.540); it contains an extremely thin veinlet of epidote.

No. **148**, from Playa Guzman, on the E. side of Morro Grande, Tamarindo, is again a granodiorite, resembling **138** and **146**, but with only a few femic components (hornblende).

No. **149**, from Tamarindo, found in isolated blocks, is a quartz-sand, cemented by phosphorite.

No. **150**, from NW-edge of Isla de Conejo, is a porphyrite with fine phenocrysts and splinters of plagioclase and with some chlorite-spots (?ex femic phenocrysts); the groundmass is very fine-grained and fluidal. The slide contains some inclusions of coarse-grained porphyrite.

No. **151**, from the same locality, is a totally epidotized and silicified porphyrite: the well-recognizable phenocrysts of plagioclase have been replaced entirely by epidote-aggregates; the groundmass, of which the fluidal texture is still visible, consists of epidote and quartz.

No. **152**, from the same locality, is a phosphorite with splinters of quartz.

No. **153**, from top of Isla de Conejo, is a porphyrite, strongly resembling **150**; so is also **154**, from the S-coast of Isla de Conejo, in which some chlorite-spots may be recognized by their form as having originated from amphibole crystals.

No. **155**, from the same locality, is again an epidotized and silicified porphyrite, entirely comparable with **151**.

The rocks of Morro de Iguana, Chiwu, Tamarindo are so alike the rocks of the batholithic sequence of Aruba (7) that they might have been collected on that island; the uralitic quartzdioriteporphyrite from Angoletta with its quartz veins belongs very probably to the same sequence. The rocks from Isla de Conejo, on the other hand, are comparable with the rocks of the volcanic basement of different islands, and there is no reason, why they should not belong to it. The Testigos islands are, as the Frailes, arranged "en échelon" with NW-strike (1a); Isla del Conejo is clearly separated from the rest of the group.

Los Hermanos (1a, p. 17, 18, 44). I could describe in 1931 (l.c.) a hornblendegabbro, sampled by P. R. LOWE, from these islands. The character of the rocks, collected by Mr. HUMMELINCK, is not so clear as that of the rocks from Los Frailes and Los Testigos.

There are four rocks which, although not typical, fit in rather well with the batholithic rocks in Aruba etc.

No. **254**, from N. slope of Morro Fondeadero, is a hornblendegabbro with hooibergite-habit. The very basic plagioclases (labrador-bytownite) are somewhat zonal; the amphibole is normal, green hornblende.

No. **256**, from the NW-side of Morro Fondeadere at sealevel, is quartzhornblendediorite with much apatite in coarse prisms. The partly idiomorphic plagioclases are somewhat zonal; they have the composition of andesine; there is a few quartz, occurring as filling-mass.

No. **258**, from N-top of Morro Pando, is an atypical quartzdiorite. The chief components are andesine and quartz without distinct crystallization-sequence, the quartz being, however, more clearly xenomorphic than the feldspar. The femic elements are hornblende, somewhat fibrous and very light-coloured, and small biotite-crystals. The quartz shows feeble undulatory extinction.

No. **261**, equally from N-top of Morro Pando, is a quartz-biotite-diorite with acid plagioclase (n : 1.540), sometimes strongly epidotized, quartz and streaks of biotite.

Four other rocks may belong to the batholithic sequence; they might, however, also belong to an older basement.

No. **252**, from top of Morro Fondeadero, is a coarse amphibolite; the plagioclases are labrador-andesine. At one side in the slide the amphibolite has been covered by phosphorite.

No. **257**, from NW-side of Morro Fondeadero, at sealevel (same locality as **256**) is an amphibolite with pyroxene.

Nos. **259** and **260** from N-top of Morro Pando, are heavy, black-and-white-spotted rocks with basic plagioclases (labrador-bytownite), green hornblende, some quartz and magnetite. They may be related with the hooibergites of Aruba, but the ill-developed crystallization-sequence and the strongly basic plagioclases make it questionable whether they belong to them.

Blanquilla. The only rock, known until now from Blanquilla, is a biotitegranodiorite (4). Mr. HUMMELINCK took samples at Valuchu, in the SE-part and at Puerto Llaque, in the SW-part of the island. The samples from Valuchu are partly young limestones, partly corals, of which Prof. GERTH kindly identified Favia fragum Esp., Porites astreoides Lam., Acropora muricata L., Orbicella acropora (L.), and Platygyra (Maeandra) viridis (Le Sueur), partly quaternary molluscs, which are now studied by Miss T. VAN BENTHEM JUTTING, partly quartzdioritic rocks.

Nos. **262**, **269**, **275** are quartzdiorites which differ from the earlier described granodiorite chiefly by the absence (or extreme scarceness) of orthoclase. The biotite has been partly chloritized; the idiomorphic, zonal and finely twinned plagioclases are oligoclase-andesine to andesine; the quartz which shows undulatory extinction, occurs as filling mass, together with some albite.

No. **270** is an aplitic quartzdiorite with very scarce chloritized biotite.

Nos. **263—268** are young limestones, all containing clastic material of quartzdiorites and not containing any other clastic material; remains of organisms are rare; **270** has been partly phosphatized.

Nos. **271**, **274** are young limestones, full of organic structures.

No. **276** is a young conglomerate with calcitic cement; the grains of the conglomerate are exclusively of quartzdioritic nature; the cement contains very fine Amphiroa's.

The samples of Puerto Llaque are for the most part granitic and quartzdioritic rocks; one sample is a conglomerate.

No. **277**, comparable with **276**, is a conglomerate of quartzdioritic material with calcitic cement, wherein Amphiroa and remains of Lamellibranchiata and Echinodermata.

Nos. **278** and **281** are biotitegranodiorites; the biotite has been partly chloritized; the plagioclases occur in two generations; the slides present a filling-mass of quartz, orthoclase and myrmekite; accessories are titanite, apatite and zircon. The rocks have suffered from crushing: the quartzes show rather strong undulatory extinction.

No. **279** is a typical quartz-hornblende-biotite-diorite.

No. **280** is a granite-aplite with some large sub-idiomorphic crystals of oligoclase, with large perthitic orthoclases and with abundant, undulatory extinguishing quartzes. Part of the slide shows a micropegmatitic intergrowth of quartz and orthoclase.

No. **282** is a crushed aplite, containing some acid plagioclases and a xenomorphic intergrowth of microcline, microclineperthite, micropegmatite and quartz.

It will be clear from the foregoing that the basement of Blanquilla exists of rocks which are quite comparable with those of the batholith of Aruba: all the rocks of the basement are typical representants of this batholith, and even the clastic material in different young conglomerates and limestones belongs to the same sequence.

Orchila. Two years ago M. ROST (3) has published a geological map of Orchila, showing that the island has a basement of crystalline schists with some granitic-dioritic rocks in the W-part of the island. Different masses of "basalt" have been indicated in the centre of the island; the greatest part of the island is covered by young coral-limestones. Mr. HUMMELINCK has taken samples in the W-part of the island, but I do not think that their detailed description is of any worth, as certainly the collections of Mr. ROST which have not yet been studied will prove to be of more importance. I should only like to indicate that the schists of Orchila which have suffered very strongly from crushing, present a great relationship to those of Margarita, that mylonitized granitic aplites from Orchila seem to be equally related to those of Margarita, and that young limestones contain very fine Amphiroa's.

Los Roques. I concluded in 1931 that the rocks of Gran Roque are closely related to the batholithic rocks of Aruba. Since then two descriptions of the island have appeared. S. E. AGUERREVERE and V. M. LÓPEZ (1) have given a detailed description with a fine geological map; M. ROST has equally given a description with a geological sketchmap (3). In his description, ROST presents a "semi-magmatical" theory on the genesis of the phosphorites of the island, which seems to me to be absolutely erroneous; it is, however, here not the place to criticize it. Mr. HUMMELINCK has sampled on Gran Roque, at the side of phosphatic rocks, many magmatic rocks which confirm absolutely the views held by me in 1931. It would be unreasonable to describe the whole collection in detail; the following is an enumeration of the types of rocks in the collection of Mr. HUMMELINCK: hornblendegabbro, uralitic gabbro, uralitic gabbro-diabase, quartzbiotitediorite, quartzamphiboledioriteporphyrite, biotitegranodiorite, biotitegranite, strongly crushed plagiaplite, quartzaplitepegmatite, granitemicropegmatite and amphibolite. With the exception of the amphibolites the samples fit in very well with the batholithic rocks of Aruba; the fine-grained amphibolites are comparable with the amphibolites that have been met-with in Aruba in the contact-zone around the batholithe (7).

On low islands of the Los Roques-group Mr. HUMMELINCK has collected some young calcareous rocks: a sandstone, consisting entirely of rounded fragments of calcite on Isla Larga (Cayo Grande) and fine-grained, partly organic limestone-breccias on Cayo de Agua.

Los Aves. On Ave de Barlovento Mr. HUMMELINCK sampled two porous limestones with organic remains, among which Amphiroa.

Tortuga. SIEVERS (6) has described Tortuga as a low, calcareous island.

Mr. HUMMELINCK took samples (nos. **363—369**) of the limestones; moreover he collected molluscs, which are studied by Miss. T. VAN BENTHEM JUTTING and corals, of which Prof. H. GERTH kindly identified the following species: Pocillopora crassoramosa Dunc., Siderastrea siderea Ell. a. Soll., Acropora muricata (L.), Orbicella acropora (L.), Madracis decactis (Lym.), Colpophyllia gyrosa Edw. a. H. and Eusmilia sp. The limestones are all very porous rocks with mostly abundant grains of quartz. They contain Amphiroa and Amphistegina at the side of ill-preserved other organic remains.

Centinela (75 km W. of Tortuga). A sample of a very fine-grained siliceous rock (nos. **361**, **362**) with veinlets of quartz and with larger ones of phosphate has been sampled on this cliff. The rock is probably a (?cretaceous) chert.

Summary.

1. The study of the new collection of rocks from the islands between Los Aves and Los Testigos confirms the conclusions, at which I arrived in 1931.

2. These conclusions are extended in so far as: a. the small group of Los Frailes has been proved to belong equally to this row, b. at the side of batholithic rocks there have been found also rocks of the volcanic basement, viz. on Los Frailes and Los Testigos.

3. The metamorphic basement of Orchila presents great resemblance with the basement of Margarita.

4. The young, quaternary, capping limestones in this group are everywhere characterized by the presence of Amphiroa which occurs equally in great abundance in quaternary limestones of Curaçao (2).

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Utrecht, June 1940.

Geology. — *On the Geology of Margarita, Cubagua and Coche (Venezuela).* By L. RUTTEN.

(Communicated at the meeting of June 29, 1940.)

Mr. P. WAGENAAR HUMMELINCK, of the Utrecht University, made in 1936 a voyage to the North Coast of Venezuela and to the islands North of Venezuela. His purpose was in the first place to make zoological collections; at the side of this he did also geological observations and collected rocks, which he put kindly at my disposal. The following is the result of the study of these rock-collections as far as regards the large island of Margarita and the small islands Cubagua and Coche.

The following data on these islands are to be found in the literature. DAUXION DE LAVAYSSE describes Margarita as being very dry and unfertil; the Macanao mountains are said to consist of micaschists (2). WALL (11) indicates the existence of his "Caribbean System" of metamorphic rocks and of younger sandstones of unknown age with a maximal dip of 35°; these sandstones crop out between Porlamar and Pampatar. W. SIEVERS (10) mentions, on the base of the diary and the collections of R. LUDWIG, the occurrence of gneiss-micaschists, graphite schists and phyllites, all considered as archaean rocks, of tertiary limestone in the neighbourhood of San Juan Griego, and of serpentine. C. MAURY (5), without giving details, says that there occur probably sediments of Midway-age on the island. R. A. LIDDLE (4), in his well-known book on Venezuela and Trinidad, is rather explicit on Margarita. He gives (p. 72) a N—S section, wherein he indicates: 1? Palaeozoic or Basal Cretaceous (schists and metamorphic limestone, shale and sandstone), 2. Basal Cretaceous? (partly altered limestone and sandstone capping mountains), 3. ?Lower Cretaceous (sandstone, shale and limestone), 6. granite (mica-hornblende granite, schistose at edges of mass., magnesite deposits at granite and schist contact). (1) and (2) are separated from (3) by a fault near Porlamar. LIDDLE's text (p. 73—74) is, unhappily, not in agreement with his section. In his section he calls the sandstone-shale formation of Porlamar "Lower Cretaceous", in his text "Upper Cretaceous". Moreover he states that it is impossible to differentiate the schistose rocks of the island into Precretaceous and Cretaceous, and he does not at all prove that part of the schists is of cretaceous age. He only says that he regards the marbles of the island as metamorphosed "El Cantil limestone", and that the "shiny mica schists which comprise the greater part of the island ...... are occasionally capped by a few meters of this marmolized limestone". HESS (3) mentions the occurrence of serpentines and says that in the

West Indies "the serpentines intrude Upper Cretaceous rocks for the most part" (p. 85). In recent time a special paper on Margarita has been published by P. I. AGUERREVERE (1); it is accompanied by a geological map. He states that metamorphic rocks (micaschists, gneiss, serpentine and crystalline limestone) are predominant; of these, the marbles are thought to be the youngest and of cretaceous age. A coarse granite intrudes the metamorphic rocks near Paraguachi; the existence of basic eruptive rocks is possible. A strongly folded formation, regarded as eocene, is found in the neighbourhood of Porlamar and Pampatar. A rock at the W.hill of the "Tetas de Maria Guevar" is considered to be a black lava; the description, however, is not very convincing: "una lava de color casi negro, excepto sus cristales de cuarzo que son blancos o sin color" (p. 402). Data about miocene and pliocene strata are very vague.

Cubagua and Coche are low islands (LIDDLE); SIEVERS supposes that Cubagua consists of archaean rocks; AGUERREVERE mentions from East Coche metamorphic rocks and variegated shales, capped by a ?pliocene conglomerate, containing pebbles of quartz and marble. These pebbles indicate that, formerly, Coche must have belonged to a greater land-area [1]).

The rocks of the three islands in the collection HUMMELINCK belong to:

1. The metamorphic basement of Margarita and Coche,
2. The Eocene of Margarita,
3. The young "capping" formations (?Pleistocene) of the three islands.

They will be discussed in this order. For localities, see the accompanying map.

Basement of Margarita. A. Gneisses. no. **5**, **6**, **7**, **8**, **10**, **13**, **25**, **36**, **194**, **195**, **240**, **371**, **372**, **373**, **375**. (No. **36** from an isolated rock-fragment; the others all outcropping). The gneisses have been found exclusively in the eastern part of the island; it must, however, been taken in mind that the W.part has been very poorly sampled. In the literature gneisses have been only mentioned by AGUERREVERE (1); microscopical descriptions are absolutely lacking.

The rocks **5** and **6** are sericite-albitegneisses [2]). Both contain porphyroblasts of albite (often with inclusions of sericite), and a mosaic of small quartzes and albites with some sericite, somewhat differentiated into layers. The twinning lamellae of the large albites are bent and sometimes broken. Both rocks have been exposed to crushing forces after their gneissification. The samples **36**, **371** and **375** are equally sericite- or muscovite-albitegneisses; **194** and **195** differ from the foregoing by strong

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[1]) Some geological data on the three islands are to be found in the academical thesis of Mr. HUMMELINCK (3a. p. 16, 17, 45, 46, 47) which was published after the communication of this paper.

[2]) In the following I shall call "albite" the plagioclases with a refraction index of less than 1.540. These acid plagioclases, which are very common in the Margarita-rocks include albite s. str. and albite-oligoclase.

mylonitization. Quartz and albite have been crushed along more or less parallel zones, and the remaining larger crystals show strong undulatory extinction; **7** and **10** are analogous.

372 and **373** are albitegneisses with garnet; **372** contains moreover muscovite, chlorite, some biotite and tourmaline, **373** contains sericite and zoisite. **240** is a fine-grained, schistous albitegneiss with chlorite and zoisite.

Sample **8** has the habit of a coarse, schistose muscovitegneiss. It contains: a. some large, perthitic orthoclases, b. rather many large albites, partly with inclusions of sericite and idiomorphic epidote, c. small crystalls of (a) and (b) which probably originated — by crushing — from larger ones, d. large spots of strongly undulatory and cataclastic quartz, e. streaks with epidote, muscovite and some biotite, f. some large titanites, g. some fibrous sillimannite. Clearly, there is a large difference between this sample and the foregoing gneisses. Possibly this difference is related with the fact that no. **8** has been found at the same locality as a granitic rock.

No. **13** has a character of its own. The leucocratic minerals — quartz and albite — form a minority; the bulk of the finely schistous rock being composed of muscovite, epidote (mostly idiomorphic) and hornblende (in idiomorphic prisms; pleochroism yellow — greenblue). Accessories are rutile, chlorite, garnet and ore. This rock is from Puerto Manzanillo, where very basic rocks predominate.

We see that most of the gneisses are albite-gneisses with no great variation; only two samples are very different from the rest: the albite-orthoclase-gneiss no. 8 and the melanocratic albite-gneiss no. 13. Many rocks present signs of strong, crushing dynamometamorphosis posterior to the gneissification.

B. Micaschists. Samples: **2**, **45**, **54**, **55**, **56**, **59**, **69**, **70**, **72**, **73**, **74**, **75**, **79**, **80**, **81**, **82**, **97**, **187**, **192**, **193**, **235**, **236**, **244**, **247**, **248**, **251**, **370**, **374**, **377** (no. **72** and **244** from isolated rock fragments; the other ones from outcropping rock). The micaschists have been found at numerous localities throughout the island. They have formerly been mentioned by DAUXION DE LAVAYSSE (2), WALL (11), SIEVERS (10), LIDDLE (4) and AGUERREVERE (1); descriptions of the rocks are entirely lacking.

Most rocks are muscovite- and sericite-quartz-schists; biotite is extremely rare. Generally the schistosity is well developed in consequence of the concentration of mica and quartz in alternating layers. The quartzes are often interlocked. In many cases (f.i. **74**, **244**) the quartzes are cataclastic at their periphery, indicating that the rocks have suffered from crushing after their metamorphosis. Rarely, some plagioclases are found (**74**, **248**), the rocks passing then into gneissic micaschists. The following accessoria are found.

Chlorite occurs frequently (f.i. **2**, **45**, **75**, **192**, **248**); it may even be present in equal quantity as mica.

Many samples are rich in graphite (**2**, **45**, **72**, **82**, **97**, **187**, **235**, **247**,

251); they become very dark and even black, f.i. at the Tetas de Maria Guevar.

Garnet (**54**, **56**, **59**, **192**, **193**, **244**, **247**, **248**, **370**) is found frequently, mostly in isodiametric, xenomorphic grains with sieve-structure. All the garnet-containing rocks are relatively coarse.

Tourmaline (pleochroism: lightbrown — colourless) in idiomorphic prisms occurs in the samples **187**, **192**, **236**, **248**.

Rutile has been found in **79**, **248**, **251**; Zircon and Pyrite are common.

At the side of true micaschists there occur rarely quartzschists (**55**), whilst **97** is a quartz-graphite schist, in which the graphite has been concentrated in thin layers; the rock gives the impression of having originated from a chert.

There is one sample which deserves special description (**192**). It is a quartz-muscovite-schist with garnet and tourmaline, and with many crystals of a blue mineral with the following characteristics. Refraction high; pleochroism lightyellow-blue; one good cleavage; optical angle small and optical character positive, which can be stated in sections parallel to the cleavage; angle of extinction with regard to the cleavage maxim. 20°. The mineral is chloritoid, agreeing absolutely with chloritoids from Piemont, Wallis and Rhode Island.

C. Various Basic Schists. Samples: **37**, **38**, **39**, **40**, **41**, **49**, **50**, **51**, **52**, **92**. No data in the literature.

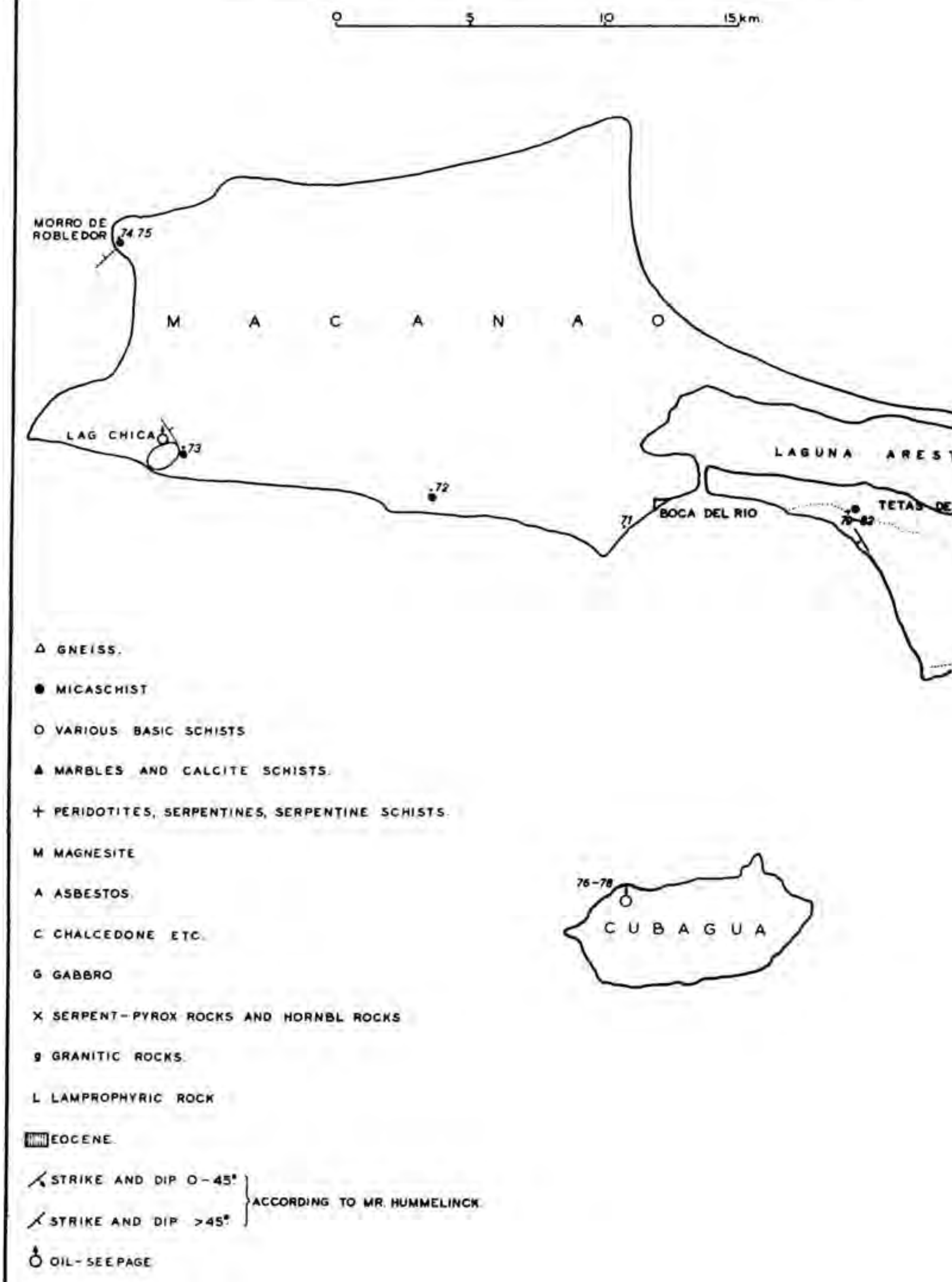
The samples **37**—**41** are all from Punta Ausente, where no other basement-rocks have been collected (**42**—**44** are subrecent limestones). No. **37** is an amphibole-eclogite with fine prisms of amphibole (yellow-greenblue), garnet (with inclusions of quartz), grains and idiomorphic crystals of epidote-zoisite, many plates of muscovite, grains of rutile, and with some quartz as matrix. No. **38** is a subschistose green rock with predominating hornblende and epidote, less muscovite, and with accessory quartz, chlorite and rutile; it is an amphibole-epidote-zoisite-rock. Nos. **39** and **40** are poorly schistose; they are composed of large, clear albites with many inclusions of blue-green hornblende and epidote-zoisite, and of large crystals of hornblende and epidote. Rutile is also present. The rocks are epidote-amphibole-albite-rocks. No. **41** has the same habit as the foregoing; it is, however, an epidote-muscovite-albite-rock. The samples **37**—**41** are very similar to inclusions, which are found in the serpentine-massives of Santa Clara province, Cuba (8a).

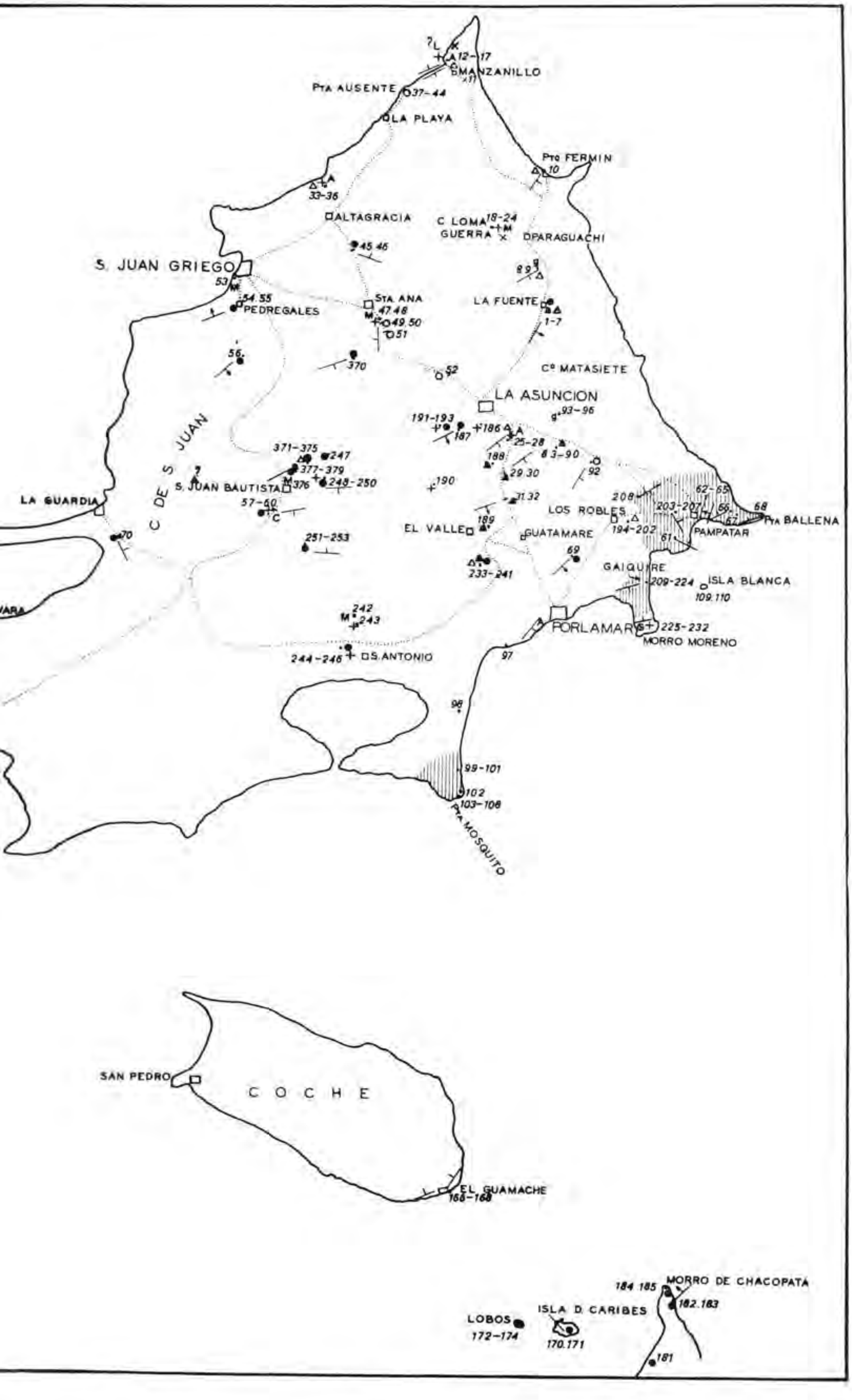
The samples **49**—**52** have been found on the road from Sta. Ana to La Asunción; they are fine-grained, schistose amphibolites, containing amphibole, albite and epidote-zoisite; in **51** and **52** there is, moreover, some mica.

The last sample (**92**) is an actinolite-schist with accessory zoisite and plagioclase.

An albite-chlorite-schist (**25**) may be regarded as an intermediate between the micaschists and the basic schists.

ISLAND MARGARITA




MANZANILLO
12-17
11
PTA AUSENTE
37-44
LA PLAYA
PTO FERMIN
10
33-36
ALTAGRACIA
C. LOMA GUERRA
18-24
PARAGUACHI
45.46
S. JUAN GRIEGO
53
54.55
PEDREGALES
STA ANA
47.48
49.50
51
8.9
LA FUENTE
1-7
56
370
52
Cº MATASIETE
LA ASUNCION
93-96
191-193
187
186
25-28
83-90
188
92
371-375
247
377-379
29.30
S. JUAN BAUTISTA
376
248-250
190
C DE S. JUAN
31.32
LA GUARDIA
57-60
C
189
LOS ROBLES
208
62-65
203-207
66
67
68
PTA BALLENA
EL VALLE
GUATAMARE
194-202
PAMPATAR
61
70
251-253
69
233-241
GAIQUIRE
209-224
ISLA BLANCA
109.110
242
243
PORLAMAR
225-232
MORRO MORENO
244-246
S. ANTONIO
97
98
99-101
102
103-108
PTA MOSQUITO
SAN PEDRO
COCHE
EL GUAMACHE
166-168
184 185
MORRO DE CHACOPATA
182.183
LOBOS
172-174
ISLA D. CARIBES
170.171
181

D. Crystalline Limestones.

Samples **1**, **3**, **4**, alternating with micaschists and gneiss, and, according to the field-notes of HUMMELINCK, also with quartzites.

Samples **188**, **29**, **30**, **31**, **32**, **189**, **83—90**, **233**, **234**, **237**, **241** apparently belong to a broad zone with ENE-strike and S-dip. According to the field-notes of HUMMELINCK the marbles are forming entire hills; at **31**, **32** schists, marbles and quartzites are alternating; at the locality **83—90** there is equally an alternation of marbles and schists; from **188** schists with strata of marble are mentioned; in the marble of **241** is a large cave (Cueba del Piache, 3a, p. 54).

Samples **378**, **379**, from large, isolated blocks. The people of San Juan told Mr. HUMMELINCK that much marble occurs in the Cerros de San Juan.

Evidently the marbles have been sampled in three districts: the area S. and E. of La Asunción, the surroundings of S. Juan Bautista, and the area around La Fuente [1]).

Crystalline limestones from Margarita have been mentioned by LIDDLE (4) and by AGUERREVERE (1). As is to be seen from the foregoing and on the map the marbles are often alternating with other metamorphic rocks, whilst the whole zone with crystalline limestone S. of La Asunción apparently lies between other schists. It is certainly not true that the marbles are "capping mountains" (LIDDLE), nor has it been proved that they are the youngest element of the basement-rocks (AGUERREVERE). As far as can be judged they form normal intercalations within the schist-series.

The rocks under consideration are sometimes without schistosity, white or grey-white (**29**, **30**, **84**, **85**, **87**, **88**, **90**, **189**, **233**, **234**, **237**, **241**, **378**, **379**) or they are schistose, sometimes hardly recognizable as calcite-schists, and with colours: white, green-white, grey, black-grey (**1**, **3**, **4**, **31**, **83**, **86**, **89**, **188**). In the first group calcite is always strongly predominating; in the second group the accessories may equal or even surpass the calcite. In some cases there are hardly any accessoria (**237**, **378**, **379**). Where present, the accessories are: quartz, albite, sericite-muscovite; more rarely epidote-zoisite, chlorite, biotite; very rarely tourmaline (**3**, **84**), garnet (**86**), titanite in grains (**4**), pyrite (**84**), and graphite-dust. Tourmaline and epidote may be idiomorphic; quartz occurs mostly as "drops"; it may, however, also occur in thin layers with interlocked crystals.

In the schistose samples the calcite-crystals are always elongated in the planes of the schistosity; in the true marbles they may be isodiametric.

As in the case of the gneisses and the micaschists, it can sometimes be observed that the marbles have suffered from crushing forces after their

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[1]) According to the map of AGUERREVERE marbles are to be found also in the W. part of the island.

crystallization, a cataclastic zone having developed between the large grains of calcite.

E. Peridotites, Serpentines, Asbestos, Gabbros, Magnesite etc.

Samples: Serpentinized peridotites **18**, **191**, **225**, **227**, **230**; Serpentines and Serpentine-schists, often with many accessories **16**, **20**, **21**, **26**, **28**, **33**, **34**, **48**, **58**, **60**, **186**, **190**, **229**, **232**, **243**, **246**, **249**, **250**; Magnesite **19**, **47**, **53**, **242**, **376**; Asbestos **14**, **15**, **27**, **35**; Serpentine-pyroxene-rock **17**, **22**, **23**; Hornblende-rock **24**; Gabbro **226**, **228**, **231**; Chalcedone etc. **57**.

According to the field-notes of Mr. HUMMELINCK the samples **14—17** have been collected in an old asbestos-mine; **18—24** proceed from the old magnesite-mine of Cerro Loma Guerra; **23**, **24**, **57**, **243**, **246** and **376** have been taken from isolated fragments of rock. I am not certain whether **53** has been really collected at the locality indicated on the map.

The localities are evenly distributed over the whole eastern part of the island; the western part has not furnished samples, possibly because it has been visited only very cursorily.

SIEVERS (10), AGUERREVERE (1) and HESS (3) mention the occurrence of serpentine on the island. LIDDLE (4) mentions the magnesites, which he regards as contact-products from the periphery of a granite-massif. This is certainly mistaken: the magnesites are found — as elsewhere in the world — in relation with peridotites or serpentines. There are no microscopical descriptions of this group of rocks in the literature.

Five samples are peridotites-in-serpentinization or serpentines, which clearly originated from peridotites. In (**18**) there is still much olivine, each grain with its net-work of serpentine. It is interesting that the original olivines, whose circumferences are recognizable by simultaneous extinction, *have very elongated forms*, which proves that the rock must have crystallized under the influence of thrust-powers. In **191** a considerable content of olivine remains; in this rock, the crystals were isodiametric. At the side of serpentine with net-structure there occur large plates of antigorite which show no trace of orientation. The forming of magnesite has already begun. Samples **225**, **227** and **230** are from the same locality and intimately related inter se. They are serpentines which, by their mash-structure, by the presence of chromite and of bastite (**230**), prove that they originated from peridotites. They contain abundant iron-ore; in 225 veins of opal are present.

As indicated above, the collection contains a large number of serpentines and serpentine-schists. Some of these will be mentioned separately. The rocks **16**, **20**, **21**, **33**, **34**, **48** are coarse antigorite-schists with accessory crystals of magnetite, which are sometimes limonitized. Nos. **26**, **28**, **58** are antigorite-chlorite-schists; **26** contains moreover talc; **58** shows a vein with chalcedone, quartz and opal. In these rocks occur crystalplates, consisting partly of antigorite, partly of chlorite: thus, these two minerals have the faculty of growing in parallel superposition. No. **190** is an antigorite-jadeite-schist; the jadeite is partly so coarse that pyroxene-cleavage may be recognized, partly it is finely-fibrous, and recognizable only by the great angle of exinction. The antigorite-chlorite schist **186** contains equally some finely-fibrous jadeite. The samples **229**, **232**, **246**, **249**, **250** are ordinary serpentines and serpentine-schists; nos. **229** and **232** very probably originated from a peridotite. No. **60** is a serpentine with advanced formation of limonite and quartz.

Four somewhat abnormal rocks have been found in the northernmost part of the island. No. **17** is a poorly schistose, emerald-green rock. It consists for the greatest part of monoclinic pyroxene which may pass marginally into uralite; there is a matrix of serpentine and magnetite: it is a pyroxene-serpentine-rock. No. **22** is a serpentine-pyroxene-schist with "stream-lined" serpentine and less pyroxene. Still poorer in pyroxene is the serpentine-pyroxene-schist **23** which contains accesorily magnetite; the sections present some veins with opal. Quite another rock is **24** which consists almost entirely of large crystals of bluegreen hornblende with magnetite as an accessory. It has been found as an isolated rock-fragment and may originate from an intrusive dike.

The magnesites **19**, **47**, **53**, **242**, **376**, and the quartz-chalcedone-rock **57** belong to the weathering-products of the serpentines. The magnesites are dirty-white or light-green rocks, often somewhat concretionary. In one of the samples Dr. W. VAN TONGEREN determined the presence of 47 % magnesite. No. **47** presents still, at the side of magnesite, some antigorite. The sample **57** consists of quartz and chalcedone, with remains of serpentine-schist.

The asbestos-containing rocks of Margarita are certainly intimately related with the serpentines. At Manzanillo a beautiful tremolite-asbestos (**14**), an asbestos-schist with antigorite (**15**) and an antigorite-schist (**16**) have been sampled; at the locality **25—28** a chlorite-containing tremolite-asbestos has been found together with antigorite-chlorite-schists. The serpentine-schist **34** and the tremolite-asbestos **35** proceed equally from the same locality. In the field-notes of Mr. HUMMELINCK asbestos-schists have been mentioned from a point S. from Porlamar; there is no sample from this locality.

At the Morro Moreno three gabbroid rocks (**226**, **228**, **231**) have been sampled in intimate relation with serpentines. The rocks, which have suffered from strong pressure, are saussurite-gabbros.

F. Acid intrusive rocks. Samples: **9**, **93**, **94**, **95**, **96**. These rocks are very rare, and their character is not absolutely certain. LIDDLE (4), in his section, and AGUERREVERE (1) mention granitic rocks from Paraguachi, without giving details.

No. **9** is a white, non-schistose rock, containing large feldspars. The slide presents: 1. various large microclines with "quartz-drops", 2. some large orthoclase-perthites, 3. many large albites, frequently filled with sericite. These three minerals may show traces of idiomorphism. At the side of them there is a fabric of quartz, acid plagioclase and microcline in smaller crystals; the intergrowth may be aplitic. The quartzes are strongly undulatory, at many points even strongly cataclastic and crushed along shear-planes. The rock seems to be a dynamomorphic aplite. It was found near Paraguachi, from where LIDDLE and AGUERREVERE mentioned their "granites".

Samples **93—96**. The rocks are poorly schistous. They show in the slides: 1. porphyritic albites with traces of idiomorphism, often strongly bent, broken and crushed at the margins, 2. quartz, partly in large crystals with undulatory extinction and with marginal crushing zones, partly also in totally crushed zones along shear planes, 3. sericite, epidote, zoisite and chlorite concentrated in small green streaks. The rocks are strongly dynamometamorphic albite-aplites.

G. Lamprophyric rocks. Sample 12 is possibly a strongly altered biotite – amphibole – lamprophyre.

Basement of Coche. AGUERREVERE has mentioned from the NE. part of Coche "una roca cristalina verdosa, probablemente metamorfica" (1, p. 400). HUMMELINCK has visited only the SE. part of the island, the area of Guamache, where he sampled three sericite-quartzites of the basement (**166**, **167**, **168**), covered by a young conglomerate.

General considerations on the Basement-Rocks of Margarita and Coche. Most of the rocks of the basement of Maragarita are para-schists. As such may be regarded all the mica-schists, most, if not all of the gneisses, and all the crystalline limestones and calcite-bearing schists. Intrusive rocks and ortho-schists are, of course, the aplites, the problematic lamprophyre, the gabbros, the peridotites and those serpentines which clearly have originated from peridotites. For the other serpentines and related rocks, and for the "various basic schists" it is impossible to say, whether they are ortho- or para-schists, although it is probable that many of them are ortho-rocks. Thus, we can distinguish in the basement of Margarita a series of older rocks, the para-schists, and a series of (somewhat) younger rocks, the (intrusive) ortho-rocks. The series of para-schists seems to be quite homogeneous and not to comprise rocks of different cycles: the different kinds of rock are clearly linked by transitional types. Gneiss, micaschist, calcite-bearing schists and quartzite (not sampled) are alternating at locality **1—7**; marbles, different schists and quartzites (not sampled) are intimately connected in the region S. of La Asunción. But also the ultrabasic schists, part of which must have ortho-character, seem to be related intimately with the foregoing. This holds good for the association gneiss-serpentines at P. Manzanillo and in the area S and SW from La Asunción. The only rock which does not fit well into the metamorphic complex is the graphite-bearing quartzite no. **97** (?ex chert), S. of Porlamar, which might belong to a younger cycle.

Most of the rocks of the basement of Margarita are also known from the Caribbean Coast-Range of Venezuela. Gneisses and micaschists, comparable with those of Margarita, have been described from the roads La Guaira-Caracas and Puerto Cabello-Valencia (6). Mr. HUMMELINCK has collected samples at Esmeralda and Puerto Santo near Carupano, of which I mention: a micaschist with chlorite, a gneissic micaschist, an albite-gneiss and dynamometamorphic albitites. Rocks comparable with the marbles and the calcite-schists have been found along the road La Guaira-Caracas (6) and at Puerto Santo: coarse marbles without accessories, schistose marbles with graphite, quartz, albite, epidote and chlorite, and basic schists with calcite-layers. Serpentine-schists are equally known from the road La Guaira-Caracas, and from the mainland, S. of Margarita, where, at Manglillo, samples of a serpentine-talc-rock, of non-schistose, somewhat opalized antigorite-serpentine, of quartz-chalcedone-rock and of altered gabbro have been collected by Mr. HUMMELINCK.

I am not certain whether the few rocks from Coche, all sericite-quartzites, belong to the same cycle as the basement rocks of Margarita, where this type is practically absent. Only the southernmost sample from Margarita (97) is an (aberrant) graphite-quartzite. On the very small islands S. of Coche (Lobos, Isla de Caribes) and at Chacopata the same rocks as on Coche have been collected: sericite- and muscovite-quartzites (**170**, **171**, **173**, **174**, **181**, **183**, **184**, **185**), whilst rocks of the "Margarita-type" (serpentines and micaschists) are found somewhat more to the S. at Manglillo. It is evident that the sericite-quartzites occupy a well-defined area, where other basementrocks seem to be lacking.

I am inclined to regard the quartzitic rocks of Coche and of Chacopata-Lobos as metamorphic Cretaceous and the other rocks of the metamorphic basement as older. The quartzites are not materially different from lower cretaceous sandstones in different parts of Venezuela (RUTTEN, 8, p. 345); the basement-rocks of Margarita, with their many basic intercalations (and/or intrusions) and their lack of what might be called "indicative cretaceous stratigraphy" (i.e. the existence of a sequence: sandstones-limestones-shales) would be older. It is certain that the basement rocks are all pre-eocene, as the non-metamorphic Eocene of Margarita contains pebbles of the basement rocks. If the age, contributed to gabbroid rocks of Lara by RUTTEN (7) and to serpentines of the Serranía del Interiór by SCHÜRMANN (9) is right, basic and ultrabasic rocks of different age must exist in North Venezuela and Margarita.

In E. Margarita Mr. HUMMELINCK has measured a considerable number of strikes and dips. A glance at the map shows that by far the most strikes are NE, the dips SE. From the field-notes it can be concluded that plication is frequently visible in the outcrops (f.i. Manzanillo, P. Ausente, Tetas de Maria Guevar, Morro de Robledor). Thus we may not regard the SE-dipping beds as the SE-wing of a gigantic fold, *but certainly we have to do with isoclinal folds*, the details of which can, however, not been unravelled with the available data. It must be observed that the NE-strike which is the oldest visible tectonic direction in Margarita and Coche tends to cross the much younger direction which is indicated by the row of islands Aruba-Los Testigos (7). *It is, therefore, quite probable that the metamorphic basement continues unto unknown distances below the Caribbean Sea.*

In relation with the tectonic processes two facts must once more be mentioned: first that one of the peridotites has crystallized under the influence of thrusting forces, second that *almost all the rocks of the basement show signs of crushing metamorphosis, posterior to the crystallization metamorphosis and, of course, prior to the deposition of the Eocene*

Eocene of Margarita. Sediments in the region of Pampatar and Porlamar, already known to WALL (11), have been called Cretaceous by LIDDLE (4), Midway by MAURY (5) and Eocene by AGUERREVERE (1). Mr. HUMMELINCK has collected samples in Pampatar, Gaiquire and Punta Mosquito. The rocks are sandstones, greywacke-sandstones, shales, conglomerates and some limestone; the age is eocene. There is, however, a rather marked difference between the rocks of the three localities, and they will therefore be described separately.

The rocks of Pampatar and Punta Ballena (**61—68**) are grey or green, brown-weathering, platy, calcite-bearing sandstones and shales. Almost all of them contain Globigerina's and ?Radiolaria; no. **62** contains a fragment of a Lepidocyclina. The clastic minerals are: quartz, plagioclase, muscovite, biotite, serpentine and chloritic minerals. There are in the sandstones fragments of phyllite, quartzite, quartz-muscovite-schist, intergrowths of quartz and albite and cherts. The samples **203—208** proceed from the region W. of Pampatar. The calcite-bearing sandstones (**203, 206, 207**) contain very rare, smaller Foraminifera, quartz, plagioclase, muscovite, serpentine; **207** contains grains of schists and of porphyritic material. **204** is a marl, **208** a limestone without fossils. No. **205** is a conglomerate with a very interesting association of pebbles: black chert, many porphyritic fragments, plagiaplite and quartzepidote-rocks which are very similar to rocks, associated in Aruba with the quartzdiorites.

On Gaiquire, which is a small island within a lagoon N. of Morro Moreno Mr. HUMMELINCK collected the sandstones **211, 212, 213, 219** (with clastic: quartz, plagioclase, muscovite, serpentine, quartzite, chert, porphyrites and fine-grained schist) and the conglomerates **216** and **223** (with pebbles of porphyrites, cherts, quartzite and sericite-quartzite). The numbers **214, 217, 218, 220—222** and **224** are pebbles from the conglomerate; they are porphyrites, amygdaloidal porphyrites and cherts. Sample **215** is a dark limestone. Sample **210** is a conglomerate with Orbitoids, probably belonging to the genus Discocyclina.

The samples **99—108** proceed from Punta Mosquito. According to the fieldnotes of Mr. HUMMELINCK there occur here conglomerates with rapidly changing thickness, sandstones and limestone. At the coast fine, small anticlines are to be seen. The pebbles in the conglomerates are quartzite, schisty quartzite, black chert, true radiolarite, quartzdiorite and porphyrite. The fossils are Corallinaceae, Globigerinae, small Camerinidae, Lepidocyclinae and Discocyclinae. It was possible to isolate a lot of Orbitoids from a sample and to recognize the presence of Discocyclina georgiana Cushman and of Lepidocyclina trinitatis H. Douv., proving the eocene age of the deposit.

To the Eocene belong probably also the samples **109** and **110** from Isla Blanca, S. of Pampatar, being a phosphatized marl and a phosphatized sandy limestone with Globigerina.

The rocks of the Eocene are in the first place of interest by the clastic material which they contain. Part of it has been clearly derived from Margarita; many components must, however, proceed from elsewhere. The cherts and radiolarites which very probably are of cretaceous age, the porphyrites and the quartzdioritic rocks can not come from Margarita. The porphyrites and quartzdiorites may come from the North: we find similar rocks on Los Frailes and Los Testigos; the origin of the cherts is unknown.

The eocene rocks are rather strongly folded; in the area of Pampatar they seem to form a syncline; it is quite probable that they have been separated from the basement of Margarita — as indicated in LIDDLE's section (4) — by a fault.

The youngest "capping" deposits. It seems that the whole of Cubagua and a large part of Coche is covered by young marine, partly detritical deposits [1]). Their elevation above the sea to about 200 ft must have occurred in subrecent times. An elevated plateau with detritical, marine deposits covers a large part of Western Margarita; in the midst of it arises the mountaineous massive of Macanao (American Admiralty Chart 2035). According to the field-notes of HUMMELINCK 15—25 m of detritical deposits are to be seen to the West of Boca del Rio. He is of the opinion that, farther in the interior, this plateau rises to about 200 m; it is dissected by deep canyons. Slightly elevated marine deposits are found S. of the Laguna Arestinga, at the coast near San Juan Griego, at Punta Ausente and N. of Punta Mosquito. It is clear from the foregoing that different parts of Margarita have been elevated in subrecent time to different heights.

The petroleum seepages in Margarita and Cubagua. A rather impor-

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[1]) A small collection of quaternary molluscs of Cubagua will be studied by Miss T. VAN BENTHEM JUTTING in Amsterdam; Prof. GERTH, of Amsterdam was so kind as to determine the following corals from the island: Millepora alcicornis (L), Orbicella acropora (L), Oculina diffusa (L), Siderastraea radians Pall.

tant seepage of oil must exist to the NW. of Laguna Chica in Western Margarita; Mr. HUMMELINCK got a bottle of the heavy brown oil, but did not visit the locality. From NW. Cubagua he collected some samples of sand, strongly impregnated with oil. We may presume that in the underground tertiary or cretaceous strata are to be found and that there exist important faults along which the oil has migrated to the surface. The presence of tertiary strata in the underground would be in accordance with the occurrence of Eocene on Margarita; the occurrence of cretaceous strata would not be strange, given the occurrence of clastic cretaceous material in the Magdalena Eocene. *The situation of the oil-bearing strata in the strike of the large oilbearing geosyncline of Lara-Falcon is quite natural.*

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Palæontology. — *The fossil human remains discovered in Java by* Dr. G. H. R. VON KOENIGSWALD *and attributed by him to* Pithecanthropus erectus, *in reality remains of* Homo sapiens soloensis [1]) *Continuation.* By Prof. EUG. DUBOIS.

(Communicated at the meeting of June 29, 1940.)

In order to give a still somewhat better idea of the dentition of Homo wadjakensis II, I reproduce in Plates III and IV, natural size, the telephotographic outlines of the fossil maxilla and mandibula, published ½ natural size in 1920 (l.c.), before all the fossil pieces found were united, especially the right ramus with the corpus mandibulae, and 6 teeth, dropped on the spot, were inserted. Clearly these accurate outlines of the original fossils, including the occlusal views, (together with the two views of the casts of the completed upper and lower jaw, reproduced in Plate II) show the important fact that there did not exist any diastema, and that the upper canine did not penetrate between the lower canine and first premolar. In my eyes VON KOENIGSWALD's and WEIDENREICH's photographic reproductions of the upper jaw of "Pithecanthropus" skull IV, of Sangiran, January 1939 [2]), reproduced in natural size in the annexed Plate V, shows the same thing. The small seeming diastema of the authors can be put to account of the damaged and incomplete alveoli, the said penetration of the canine to account of its sagging.

It is interesting to compare with the dentition of VON KOENIGSWALD's finds, and the Solo-man skulls, a similar but elaborately described find of another fossil man related to the present Australian race: the skull of Talgai in Queensland, Australia, which, discovered in 1884, was elaborately described in 1918 [3]). This skull of a "male youth" (for m^3 was still

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[1]) Corresponding to this alteration of the title, the following corrections in the text of my paper of March 30, 1940 (Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam **43**, 494—496 (1940)) may be inserted here:

P. 494, line 13 from below, instead of **is identical** read **belongs to the same group**
„ 495, „ 10 „ above, cancel **Homo wadjakensis II and**
„ 495, „ 25 „ „ read **an exact Solo-man skull.**
„ 495, „ 3 „ below, instead of **not** read **beit much**
„ 495, „ 2 „ „ place a note [1]) behind **II**: Concerning Sinanthropus pekinensis, see FRANS WEIDENREICH, The mandibles of Sinanthropus pekinensis: a comparative study. Palæontologia Sinica. Ser. D, Vol. VII, Fasc. 3, p. 33.

[2]) G. H. R. VON KOENIGSWALD und FRANZ WEIDENREICH, The relationship between Pithecanthropus and Sinanthropus. "Nature", vol. 144, pp. 926—929, Dec. 2, 1939.

[3]) STEWART ARTHUR SMITH, The fossil human Skull found at Talgai, Queensland. Philosophical Transactions of the Royal Society of London. Series B, Vol. 208, pp. 351—387. [Plates 12—18]. 1918. — See also EUG. DUBOIS, The proto-australian fossil man of Wadjak, Java. Proc. Kon. Akad. v. Wetensch., Amsterdam **23**, 1013—1051 (1920), p. 1028.

EUG. DUBOIS: *The fossil human remains discovered in Java by Dr.* G. H. R. VON KOENIGSWALD *and attributed by him to* Pithecanthropus erectus, *in reality remains of* Homo sapiens soloensis. *Continuation.*

PLATE III. Homo wadjakensis II.

Nat. size.

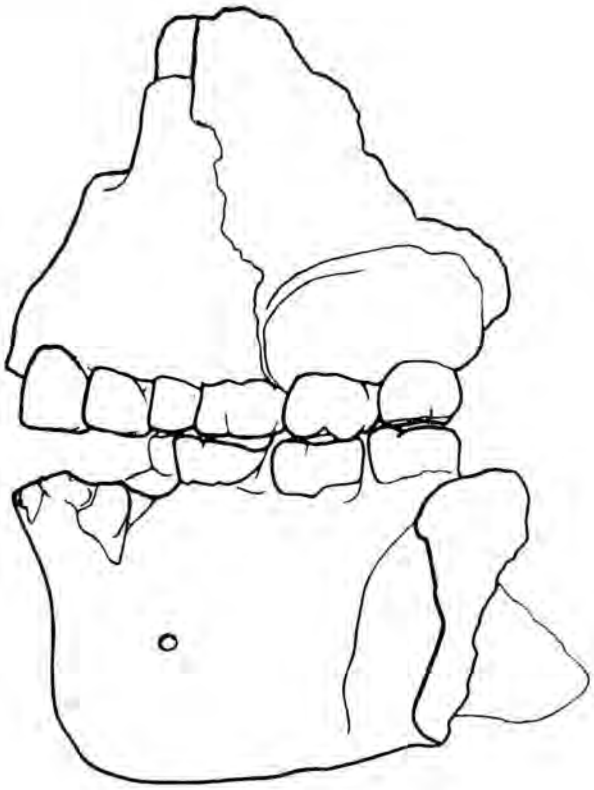

Fig. 1.
Left-side view of Upper and Lower Jaw.

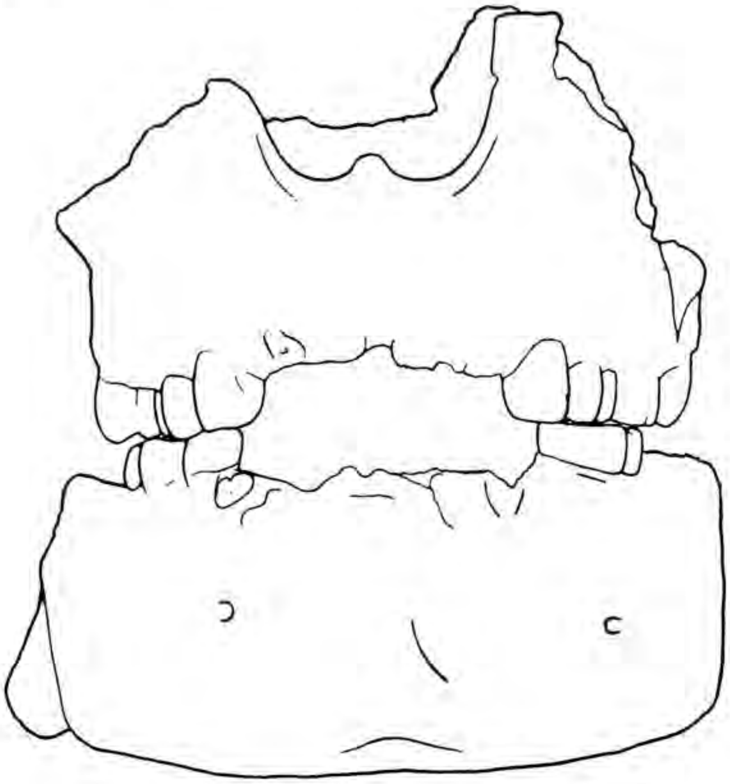

Fig. 2.
Front view of Upper and Lower Jaw.

PLATE IV. Homo wadjakensis II.

Nat. size.

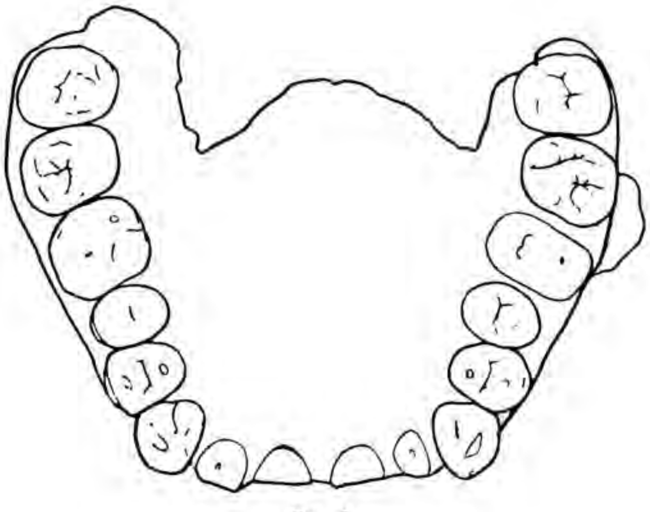

Fig. 3.
Palatal view of Upper Jaw.

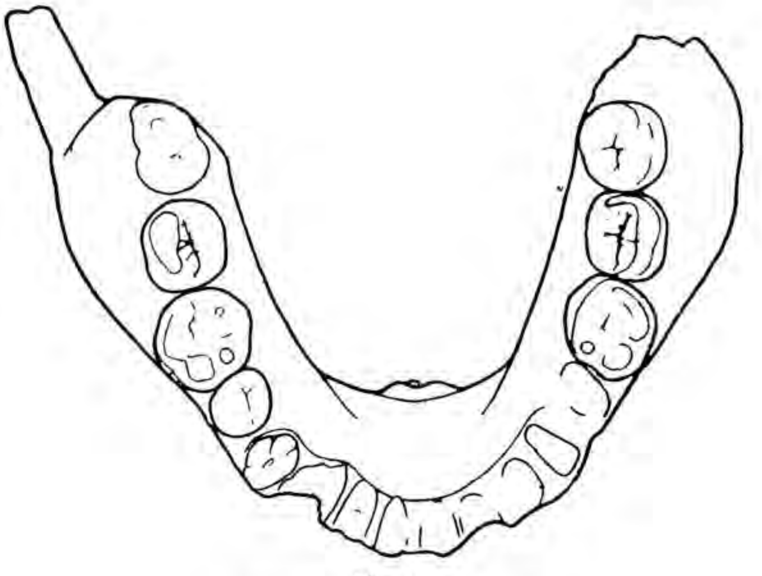

Fig. 4.
Lower Jaw, seen from above.

unerupted), though cracked *in situ* into numerous fragments, more or less considerably dislocated, but held in position by thin layers of calcareous earthy matrix cementing them together, the condition resembling a coarse mosaic, can yet be clearly recognized as not deviating in its general features from the present aboriginal Australian skull. The cranium as a whole, and the palatum, however, hardly admit of any reliable measurements. These can still be made at the tooth-crowns, each in itself, but most of them have more or less receded from each other; the apparent palatal area thus considerably exceeds the real, which, in my opinion, was no larger than that of the Australian native of present times. SMITH supposes that the upper canine, in an analogous way as in the dentition of the Apes, though without a true diastema in the maxilla, penetrated, almost ape-like, with its apex between the lower canine and the lower first premolar.

In my opinion there is reason to doubt this, on the ground of a comparison with the teeth of Wadjak II, especially on account of identified contact facets on the Talgai and the Wadjak upper canine. In his reconstruction, SMITH lowers the upper canine till the upper border of its crown comes very nearly on a level with the upper border of the crown of the premolar. Erroneously, for the crown-border of such a large upper canine as the Talgai one is always considerably above the level of the crown-border of the first upper premolar; in the maxilla of Wadjak the distance is 3 mm. This upper canine, therefore, cannot, in life, have projected so far downward. The canine of Wadjak II, which strikingly resembles that of Talgai, is also of equal breadth as the latter, and if its wear were as little advanced as that of the canine of the boy of Talgai, it would no doubt be as pointed and little shorter than the latter. For these reasons I cannot agree with SMITH in ascribing to the fossil skull of Queensland, which he too considers as typically Australian, "a human dentition in which these anthropoid characters are manifested in a manner quite unknown in man, except for the single example of *Eoanthropus*".

Now, twenty-two years after STEWART ARTHUR SMITH's publication, we observe, in a strikingly similar case, VON KOENIGSWALD and WEIDENREICH led astray to the same mistake, by the same belief in evolution from Ape to Man through gradual transformation of parts of the body. This belief, in matters of evolution, results from almost exclusive morphological consideration of the organisms. Hence the apparent inconsiderate supposition of those investigators, that the large, bestial, canines of the Apes are mainly organs of defence, and became gradually smaller and less apelike in the course of Man's phylogenetic development. Apparently, however, according to studies of the animals in nature, the large canines of the Apes are not principally organs of defence, in contrast to the canines of Man, but are specially adapted to the kind of food of those animals. Indeed, many years ago, EMIL SELENKA expressed this opinion with regard to the Orang Utan, studied in Borneo. If that is true of the Anthropoids in general, as probably also of the Monkeys, and even, after all, of the Carnivores, such a gradual transformation from Ape to Man is hardly conceivable.

In the same communication on the relationship between *Pithecanthropus* and *Sinanthropus* ("Nature", l.c., pp. 928—929), VON KOENIGSWALD and WEIDENREICH give clear expression to their implicit belief in human evolution by gradual transformation, in the lines I may quote here:

"Considered from the general point of view of human evolution, Pithecanthropus and Sinanthropus, the two representatives of the Prehominid stage, are related to each other in the same way as two different races of present mankind, which may also display certain variations in the degree of their advancement.

"The Prehominids are separated from the Neanderthal group by a considerable gap. On the other hand, an apparently close relationship exists between Pithecanthropus and *Homo soloensis*, the skulls of the latter appearing like an enlarged form of the former. Certain peculiarities of Pithecanthropus reappear in exactly the same form in *Homo soloensis*. Those traits which suggest an already more advanced type, like the greater cranial capacity, and several other structural features, can be derived directly from Pithecanthropus, and correspond to the condition in the Neanderthal stage already attained by *Homo soloensis*. The two available fragments of the tibia of *Homo soloensis* show no special particularities, with the exception of a pronounced platymeria, exhibiting only recent human characters in their general form and in details.

"The finds reported herein show that Java has become the most important centre for the study of Prehominid forms. Not only Prehominids, but also the following evolutionary stage, *Homo soloensis*, are represented there. Furthermore, we know that the Wadjak man of Java represents another early form of recent man, whose upper jaw (Wadjak II) displays in some respects a most surprising resemblance to the Pithecanthropus upper jaw."

In reading this, we may not lose sight of the insufficiency, in geological as well as in morphological respect, of what we learned from the descriptions about the true nature of the fossils, as they were found by the collectors of VON KOENIGSWALD, or himself. This insufficiency, indeed, concerns the exact geological localization of the finds, as well as the original condition of the fossils before the restoration. Apparently, VON KOENIGSWALD, in his opinion about the true nature of his finds, does rely more on (insufficient) stratigraphical data than on unprejudiced examination of the fossil remains themselves. No wonder that his implicit belief in human evolution from Ape to Man, by gradual transformation of parts of the body, according to the Darwinian point of view, would lead him astray about the true nature of his finds, and wrongly guide the hand which made the restorations.

Indeed, on closer examination, the fossil human remains attributed by VON KOENIGSWALD to Pithecanthropus, appear all of them to be, owing to their morphological character, without any doubt remains of *Homo soloensis*.

This result of morphological examination does not fit in with VON KOENIGSWALD's opinion that his fossils belong to the Trinil layers, or even (the Modjokerto child skull) an older layer than these.

But circumstances of regional geology, as well as facts concerning the character and state of fossilization and the external appearance of the fossils, properties which they have in common with the Talgai skull, evidence that the skulls and jaws of VON KOENIGSWALD's "Pithecanthropus" belong to a relatively much later geological age than do the layers in which (in the years 1890, 1891, 1892, 1897 and 1899) I found real *Pithecanthropus erectus* fossils. In the regions where the remains of *Homo soloensis* were found, at Ngandong, and of VON KOENIGSWALD's "Pithecanthropus", at Sangiran, limestones of older formations come to the surface, to absolute higher levels than the Trinil layers. These limestones actually contributed to the "matrix" (composed for the greater part of detritus from Trinil rock) of those human fossils, and determined character and state of the fossilization, which, as observed on the fossils of Sangiran and Talgai, were unable to resist the influences of wear in such an effective manner as did Trinil fossils.

Thus, also in geological respect VON KOENIGSWALD's finds are different from the real Pithecanthropus fossils. They are, doubtless, remains of *Homo soloensis*.

Concerning *Homo wadjakensis* and *Sinanthropus pekinensis*, we may remark that, although the character of fossilization is not different from that of *Homo soloensis*, the conditions of preservation at Wadjak and Choukoutien were better.

Naturally, I perfectly agree with VON KOENIGSWALD and WEIDENREICH in regarding *Homo wadjakensis* and *Homo soloensis* both as early forms of recent man (*Homo sapiens*). These forms, however, differ in some important morphological characters. Indeed, the Wadjak skulls lack the platycephaly, the torus supraorbitalis, the strongly expressed occipital slant, those conspicuous features of every Solo man, as also of "Pithecanthropus" Skull IV. On the other hand, Wadjak man Skull II exhibits a very pronounced chin, whereas, apparently, *Homo soloensis* possessed only a rudimental one, comparable with the trigonum mentale, which WEIDENREICH (l.c.) found in lower jaws of *Sinanthropus pekinensis*. The upper jaw, however, was probably of similar form in Solo man and Wadjak man; therefore that part of the skull of Wadjak man II displays "a most surprising resemblance" to the upper jaw of "Pithecanthropus" Skull IV, which really is a *Homo soloensis* skull.

Nevertheless, we may regard the two early forms of *Homo sapiens*, protoaustralians of Java, as nearly related (they may even, as it seems, geologically belong to quite the same age), because those distinctive morphological characters, as THOMAS HENRY HUXLEY [1]) said in the case of the Neanderthal cranium, "did not extend deep into the organization" — thereby, undoubtedly, meaning that they did not directly concern the animal organization.

Therefore, the morphological characters, especially of the supraorbital region and the occiput of the Wadjak II Skull, in contrast to the Wadjak I Skull, approach somewhat Ngandong skulls (*Homo soloensis* I and V), though the first Wadjak skull had come to light near the spot where in the next year the second skull was dug out, so that they are certainly geologically contemporaneous.

In this second Wadjak skull, the very prominent and low-lying arcus superciliares meet with a likewise swollen glabella, forming with it a much broader protuberance than the proper glabellar one, and melt together with the medial part of the orbital arch, as far as about the incisura supraorbitalis. The shape of the orbit is thereby made rectangular. Accordingly, there is above that broad protuberance, in the same breadth of ample 60 mm, a real sulcus supraglabellaris, the beginning of a sulcus supratoralis. There is, however, no fossa supraglabellaris, which is so characteristic a feature of the Neanderthalian supraorbital region.

The same form of the supraorbital region of Wadjak Skull II, now, is of frequent occurrence in Australian-aborigine skulls. Some of them approach in this respect the Neanderthal type still more, as far as to develop a real torus supraorbitalis reaching to the processus zygomaticus.

Of the Ngandong skulls (*Homo soloensis*) I and V certainly do this. Ngandong Skull VI, on the contrary, exhibits a transversally depressed glabella, thus entirely lacking the proper glabellar protuberance of Wadjak Skull II, and also of Wadjak Skull I. Nevertheless, Ngandong Skull VI possesses a moderately pronounced but real torus supraorbitalis, although this is interrupted in the middle. In many present-Australian skulls, also, the glabella is somewhat depressed transversally.

Apparently the different conditions of the supraorbital region in skulls of Australian aboriginals [2]) are represented, more or less separately

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[1]) Evidence as to Man's Place in Nature. London 1863. P. 157: "So large a mass of brain as this, would alone suggest that the pithecoid tendencies, indicated by this skull, did not extend deep into the organization."

[2]) HERMANN KLAATSCH, The Skull of the Australian Aboriginal. Reports from the Pathological Laboratory of the Lunacy Department. New South Wales Government. Vol. I. Part. III. Sydney 1908.

With this paper KLAATSCH, initiating investigations "based on principles of evolution", intended dealing with the skulls of Australian aboriginals, with special reference to specimens obtained by Dr. W. E. ROTH from different parts of Queensland. These skulls, comprising those of about ninety individuals, and deposited in the Australian Museum, Sydney, are of very great value in that their full-blood origin is assured. With a view to giving a general outline of the variations met with throughout a large number of the series, KLAATSCH has found it necessary to restrict special descriptions and (selectively), in the Wadjak and Ngandong skulls, most strongly in the latter. The same we observe concerning the development of (1) the slanting nuchal plane; (2) the deviation of the dental arch from the horse-shoe form and resemblance to the form of the upper dental arch of Homo wadjakensis II (Plate V, Fig. 3), this being only relatively much broader — length-index **74** (identical with that of Rhodesian man), in contradistinction to **90** of KLAATSCH's Australian R. 69 —; (3) the very pronounced chin. The existence of a trigonal mental prominence is the rule in the mandibles of Australian aboriginals [1]); (4) the platycephaly.

Abundant evidence about this varied skull morphology of Australian aboriginals is available in the collections of the Australian Museum, Sydney, of which KLAATSCH has made an elaborate study [2]).

The development of these features — in the opinion of the majority of anatomists — is a consequence of their physiological significance. According to the excellent investigations of TOLDT [3]), this development is directly correlated with that of the jaws in general, and the different parts of the apparatus concerned in mastication in particular. Accordingly they develop only at the same time with those parts, during the ontogeny. The pressure which, in the act of masticating, is exercised by the teeth on the upper jaw bone, is transferred to the frontal bone by three pillars: a nasal one and two jugal pillars. If the inferior frontal region is situated approximately in the direction of this pressure, it remains unchanged; with receding forehead and flattened skull (platycephaly), however, reinforcements in that frontal cross-beam come into being, especially in the case of a strong dentition.

Concerning the platycephaly, we observe that, amongst recent races of man, the Australian aboriginal possesses the minimum of calvaria-

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illustrations to certain specimens which showed differences sufficiently great to warrant them being considered typical. —

Compare also the studies of D. J. CUNNINGHAM, The Evolution of the Eyebrow Region of the Forehead, with special reference to the excessive supraorbital development in the Neanderthal race. Transactions Roy. Society of Edinburgh. Vol. 46, Part II, No. 12, 1908. — Furthermore, the earlier studies of G. SCHWALBE: Studien über Pithecanthropus erectus. Zeitschrift für Morphologie und Anthropologie. Band I. 1899; also: Der Neanderthalschädel. Bonner Jahrbücher, Heft 106, 1901, and diverse other papers.

[1]) Concerning the Trigonum mentale, see: HANS VIRCHOW, Die menschlichen Skeletreste aus dem Kämpfe'schen Bruch im Travertin von Ehringsdorf bei Weimar, Jena 1920, pp. 51—64: Kinngegend. — Concerning the Australian aboriginal's dentition I may quote: T. D. CAMPBELL, Dentition and Palate of the Australian Aboriginal. Thesis, University of Adelaide, 1925. See particularly illustrations of **squarish front portion** of the dental arch, and of **parallel arch form**; the first feature, with a deviation from the horse-shoe form, is met with in the upper dental arch of Wadjak II; the parallel arch form is the extreme development of that deviation.

[2]) See note 2 of foregoing page.

[3]) C. TOLDT, Brauenwülste, Tori supraorbitales, und Brauenbögen, Arcus superciliares, und ihre mechanische Bedeutung. Wien 1914. Mitteilungen der Anthropologischen Gesellschaft in Wien. Band 44.

height-index in relation to the glabella-inion line, (SCHWALBE's Kalottenhöhe-Index), 45, according to the reports of BERRY, ROBERTSON and STUART CROSS [1]), for 100 Australians, which minimum thus is below the index, 52, of Wadjak Skull I, whereas, according to the same reports, the medium in the Australian aboriginal race is 53 and the maximum 62. Wadjak Skull II, judging from the skull fragments, was possibly somewhat more flattened than Wadjak Skull I. Ngandong Skull VI, the only one of Homo soloensis which is so well preserved as to allow accurate measurements, in this respect, has an index 42; the index of the less well preserved Ngandong skulls I and V could not, apparently, differ much from this. Thus, it appears that the flattening of the skull in *Homo sapiens soloensis* was equal to that in *Homo neanderthalensis* — with calvaria-height-indices from 40 to 44.

Similar observations obtain with regard to a feature in the occipital region of the human skull, of frequent occurrence but different development: the torus occipitalis — being a swelling of the field between the lineae nuchae superiores (lineae nuchales terminales) and the lineae nuchae supremae (lineae nuchales supraterminales) — in recent races, especially again, in the Australian aboriginal, and the fossil races in consideration, comparatively the fossil species *Homo neanderthalensis*. It is, moreover, noteworthy that in Australian aboriginal skulls, alike in Ngandong Skull VI, only the medial part of each torus occipitalis half is developed.

Obviously, all those differences of features and conditions of homologic parts of the skull, in the Australian aboriginal, in Wadjak man and Ngandong man, comparatively in *Homo neanderthalensis,* have only mechanical (direct physiological) significance; they do not correspond to different stages of human evolution.

As Peking man (*Sinanthropus pekinensis*) has so many peculiarities in common with Ngandong man (*Homo* or *Javanthropus soloensis*) [2]) that both must be considered unquestionable members of the same proto-australian group, differences of features and conditions of those homologic parts of the skull neither correspond here to a different stage of human evolution. These morphological differences refer principally to a still more pronounced development of the torus supraorbitalis than in *Homo neanderthalensis,* and a correspondingly excessive flattening of the skull cap — the calvaria-height-index, in relation to the glabella-inion line, being so

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[1]) See quotations in: EUG. DUBOIS, The proto-australian fossil man of Wadjak, Java. Proc. Kon. Akad. v. Wetensch., Amsterdam **23**, 1018 (1920).

[2]) To the arguments given in my papers, quoted on p. 496 of the first part of this communication, may be added the demonstration of the similarity of the endocranial casts by C. U. ARIËNS KAPPERS in "The endocranial casts of the Ehringsdorf and *Homo soloensis* skulls". Journal of Anatomy. Vol. **71**, Part I, October 1936. Cambridge University Press. pp. 67—75.

low as 36 to 41, according to my measurements on the published representations of the skulls.

Another proof of Peking man belonging to one and the same group of proto-australians with Ngandong man and Wadjak man, is given again by the upper and the lower jaw. Up to the present there is only one upper jaw of Sinanthropus pekinensis available, but this, though slightly damaged, shows the features in question with sufficient distinctness. It is described and figured in WEIDENREICH's elaborate treatise on the dentition of Sinanthropus pekinensis (1937) [1], Text, p. 136; Atlas, Fig. 345. Here, unmistakably again, appears a striking resemblance to the upper jaw of Wadjak man II. For comparison of the dentition in the lower jaw, there is hardly anything more available of Sinanthropus than the alveolar arch of one specimen of sufficient lower jaw. But this shows at least probable similarity in the very large breadth-index of the dentition. [2] It is unnecessary to repeat the occurrence of the trigonum mentale in Sinanthropus, characterizing *Homo sapiens*.

Concerning the bearing on human evolution of the supposed similarity of the skulls of *Pithecanthropus* of Trinil and the (first) *Sinanthropus* of Choukoutien, implying near relationship, nay generic identity, it may now be observed that the real morphologic evidence is completely inconsistent with this supposition, which, indeed, is only based on the overlooking of this evidence.

Whereas, from the first, the gibbon-like features of the *Pithecanthropus* skull has always drawn the attention of those unbiassed by the preconceived opinion of human evolution by gradual transformation of parts of the body, others — and they were not a few — would not see those features, even their suspicion of prejudice fell on them who could not deny the existence of the features in question.

In reality the morphologic similarity of the large skull cap, excavated from a volcanic tuff, at Trinil, in 1891, and described in 1894 as *Pithecanthropus erectus*, to the homologous part of the skull of a small *Hylobates* species is most striking. Not only the median contour, but also the special form of the torus supraorbitalis and of the lower tabular part of the occipital bone, such as it was before it was naturally damaged in the fossil state, is quite gibbon-like. An absolutely distinctive gibbon-like feature, however, is preserved, just in that damaged region of the planum nuchale. One feature in this region the high situation of the linea nuchae inferior (linea plani nuchalis), in the Gibbons and the other Anthropomorpha, is very different from that in Man. Another feature, on the contrary, distinguishes the gibbons — *Hylobates* and *Symphalangus* — from the large anthropomorpha; this is the development, in the gibbons, of a deep

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1) FRANZ WEIDENREICH, The dentition of *Sinanthropus pekinensis*. Palæontologia Sinica. New Series. D, No. 1. Peiping 1937. Text and Atlas.

2) FRANZ WEIDENREICH, The mandibles of Sinanthropus pekinensis. Ibid. Series D, Vol. 7, Fasc. 3. Peiping 1936. See especially p. 100, Table XV.

depression below about the inner third of the linea nuchae inferior (linea plani nuchalis), that part of the linea, and the depression below it, serving for the insertion of the musculus rectus capitis dorsalis minor. In this respect Chimpanzee and Orang-utan are different, even more than Man. In *Hylobates* and *Symphalangus* the linea plani nuchalis is very much nearer the linea nuchalis terminalis than in Man; however, in *Symphalangus* it deviates relatively more at the crista occipitalis externa than laterally. This medial deviation is more considerable in Chimpanzee; but this feature there is variable, the deviation may be over a longer extent, as it is regularly in the Orang-utan, here with varying distance from the linea terminalis.

Now, the skull cap of *Pithecanthropus erectus* having lost in this region much of the superficial bone substance, with the crista occipitalis externa and all those lines, it has, nevertheless, preserved a large and deep united depression, corresponding to the place of insertion of the musculi recti capitis dorsales minores. The entire region has, undoubtedly, once been quite gibbon-like, but all the features had dimensions, the double of those of a present Siamang.

Needless to insist on the gibbon-like appearance of *Pithecanthropus erectus*, and on the unfoundedness of the conception of human evolution by gradual transformation of Chimpanzee-like ancestors.

The most important conclusion of a close examination of the Trinil skull cap is that the so-called *Sinanthropus pekinensis*, an indubitable *Homo*, is no near relation of *Pithecanthropus*, to say nothing of identity. The resemblance in the median sagittal contours of the two crania, which have about equal capacity, cannot surprise us, taking into consideration the mechanical significance of platycephaly. There are significant differences, as, above all, concerning the parietal vertex of the brain, a human distinctive which *Pithecanthropus* lacks, as do the Anthropomorpha and Monkeys.

As to the now known limb bones of *Sinanthropus*, they are, also in the eyes of WEIDENREICH, quite human; particularly the femur is in every respect of the *Homo sapiens* type, and the dimensions are such as correspond to the small cranial capacities. The six thigh-bones which we now have of *Pithecanthropus erectus*, all of them showing in the preserved parts the same morphologic deviations from the human femur, are, on the contrary, large in proportion to the cranial capacity. This does not fit in with the opinion that they might be human thigh-bones.

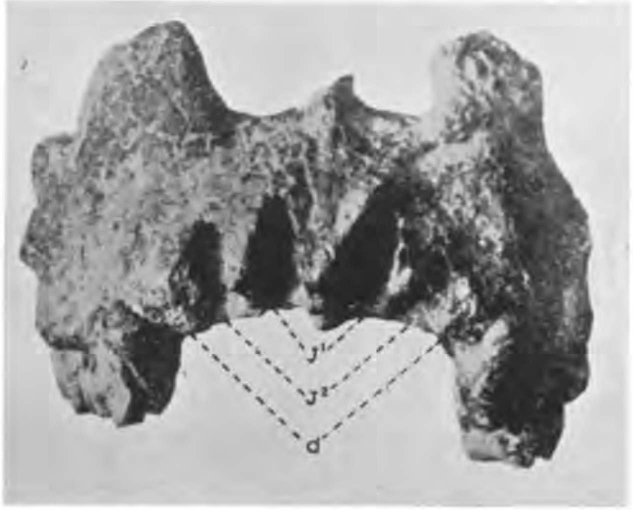
Concerning the teeth attributed to *Pithecanthropus erectus* one may refer, for the first lower premolar, to the resemblance to the homologous tooth of the gibbon-like *Propliopithecus* from the Oligocene of Egypt. As to the upper molars, it may be remarked that the shape of their crown not only resembles that of the homologous molars in the Orang-utan, but also, in some cases, in the Chimpanzee, and even in the Siamang, whereas the pattern on the crowns, as it appears to me, does not really resemble that of any orang-utan molar. On the other hand, I confess not to feel at liberty to attribute, with VON KOENIGSWALD and others, some

EUG. DUBOIS: *The fossil human remains discovered in Java by Dr.* G. H. R. VON KOENIGSWALD *and attributed by him to* Pithecanthropus erectus, *in reality remains of* Homo sapiens soloensis. *Continuation.*

PLATE V.

a

c



(*a*) Upper jaw of a male Pithecanthropus (Sangiran, January 1939), viewed from the right side. *d*, diastema; *J*¹, alveolus of *J*¹; *J*², alveolus of *J*². nat. size. (*c*) The same as (*a*) but viewed from in front. nat. size.

From "Nature" Dec. 2, 1939, Vol. 144, p. 927.
The figures one and a half times enlarged.

EUG. DUBOIS: *The fossil human remains discovered in Java by Dr.* G. H. R. VON KOENIGSWALD *and attributed by him to* Pithecanthropus erectus, *in reality remains of* Homo sapiens soloensis. *Continuation.*

PLATE VI.

Brains, natural size, viewed from above, of a Polecat (*Putorius putorius*) (left figure) and a Stone Marten (*Martes foina*) (right figure), both animals full-grown; the species are of equal medium size of body and reckoned belonging to the same family.

fossil molars they found in Java and China, to fossil orang-utan; the pattern on the crowns showing more resemblance to the proto-australians of Java and China.

The insufficiency of the endeavours of WEIDENREICH, VON KOENIGSWALD, and other able investigators, to prove in China and Java the reality of human evolution in Darwinian way, by gradual, slow and steady transformation of parts of the body, is a consequence of their neglecting the holismic or *Ganzheits* conception of the organisms, firmly established by the researches in experimental embryology, on which conception, although having their origin early and spontaneously in another starting-point, are also based my own and LOUIS LAPICQUE's researches on phylogenetic progress of the brain, especially the cerebrum, in the Vertebrates.

Concerning the principal result of these researches I may here refer to the annex Plate VI, and, further, to my paper "Die phylogenetische Grosshirnzunahme autonome Vervollkommnung der animalen Funktionen" [1].

Obviously then, Man did not come into being in the Darwinian way, by gradual transformations required by the outer world power, but by inner world, autonomous, power. Nevertheless, Man is nearly allied to organisms even two autonomous degrees lower.

This — near relationship to the Anthropomorpha, living and extinct — is admirably proved by comparative anatomy and palæontology; however, those sciences were not able to teach us more concerning anthropogeny than their own insufficiency. I refer particularly to the important and excellent researches in South Africa. Much other meritorious endeavour, in the same way, has had no other results.

In finishing, I may call attention to the fact that the proto-australians we now know: Wadjak and Ngandong man, Peking man, also Rhodesian man, all of them, have the same absolute and relative low cranial capacity in common with the Australian aboriginal. The race has apparently neither progressed nor altered in this respect.

Furthermore, be it observed that, in accordance with the higher degree of autonomous brain progression, *Pithecanthropus erectus* possesses a calvaria-height-index equal to that of the small *Hylobates* species, not of the large *Symphalangus*, indeed the index of a *Hylobates agilis* being 35.5 and that of a *Symphalangus syndactylus* (both animals from the wild and full-grown) 24.5. The index of *Pithecanthropus erectus* was about 33.5.

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[1] Biologia Generalis, Band 6, pp. 247—292. Wien und Leipzig 1930.

(To be continued.)

Physics. — *The mechanism of emulsion formation in turbulent flow.* I. Experimental Part. By P. H. CLAY. (Laboratorium voor Technische Physica van de Technische Hoogeschool te Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of June 29, 1940.)

Zusammenfassung.

Der Mechanismus der mechanischen Bereitung von groben Emulsionen mit Hilfe der Wirkung eines turbulenten Strömungfeldes wird experimentell und theoretisch untersucht mit Rücksicht auf technische Anwendungen. Man kann zwei elementare Prozesse unterscheiden, die entgegengesetzte Wirkungen auf den Dispersionszustand ausüben: das Zerbrechen und das Zusammenfliessen der Tröpfchen. Die Wirkungen dieser Prozesse führen nach längerer Zeit zu einem Gleichgewicht im Dispersionszustand.

Zwei Apparate werden beschrieben, in denen die Tröpfchen der groben Emulsion im turbulenten Bewegungszustand, direkt bei sehr kurzer Belichtungszeit und Vergrösserung bis 40 ×, photographiert werden. Im ersten, technischen, Apparat wird die Turbulenz bei Strömung durch ein Rohr mit 10 cm Durchmesser benutzt; im zweiten Modelapparat die Turbulenz zwischen zwei coaxialen Zylindern, wobei der innere gedreht wird. Eine Methode zur Vermittlung der Grössenverteilung der Tröpfchen durch Ausmessen der photographischen Aufnahmen wird beschrieben. Messergebnisse werden gegeben. Es stellt sich heraus, dass die Tröpfchen auch während der heftigen Bewegung ihre Kugelform in sehr grosser Annäherung beibehalten.

1. *Introduction.*

In the normal procedure for the mechanical preparation of an emulsion, a coarse dispersion of the two fluids, which do not mix, is brought into a laminar or turbulent state of motion. The hydrodynamic field affects the state of dispersion of the emulsion in two different ways: a droplet of the dispersed phase may be broken up into two or more smaller droplets; on the other hand a droplet may meet another droplet with a certain chance of coalescence into one larger droplet. Our knowledge about the nature of these elementary processes and about the conditions under which they occur is still very poor, though we know much more about the physicochemical properties of more or less stable emulsions at rest. We refer to: CLAYTON, *The theory of emulsions and their technical treatment* (Oxford 1935, Clarendon Press) p. 289.

The conditions under which these two elementary processes will occur depend on the properties of the hydrodynamic field used, as well as on various chemical and physical properties of the two fluids. The physical properties in question are the densities and viscosities of both liquid phases (ϱ and η for the exterior phase, ϱ' and η' for the dispersed phase), the volume percentage occupied by the dispersed phase (c %) and the interfacial tension (γ). The chemical properties will on the whole be the same as those considered in the theory of emulsions, consequently they concern chiefly the surface layer between the two liquids. Their influence will increase as the droplets become smaller. If the droplets are very small the electrical charge of the droplets may in special cases play a part. (The electrical conductivity of the exterior phase then has to fulfill certain special conditions.)

The state of dispersion which arises from the action of the elementary processes is usually characterized by the (differential) distribution of sizes of the droplets $n(a)da$ (in droplets per cm^3 emulsion); here a means the radius of the droplet if it were spherical. This state of dispersion is generally due to the action of the elementary processes in the mechanical mixer as well as to the original state of dispersion of the emulsion when it entered the mixer. When, however, a pair of liquids in a mixer is kept in the same state of motion long enough, the state of dispersion becomes stationary. Then the influences of the elementary processes which always counteract each other have led to an equilibrium in the state of dispersion. In this way it is possible that an emulsion which is not stable at rest, is stabilized by a state of motion. Such a stationary state of dispersion is free from the influence of the original state of dispersion at the entrance of the mixer.

In the technical applications of the mechanical preparation of emulsions, two possibilities are to be distinguished.

a. Sometimes the emulsion is the product wanted; then the emulsion has to meet some distinct requirements of stability etc. Generally the dispersion has to be sufficiently fine. In this case a is usually under 20 μ. The distribution of sizes of the droplets is in this case of some interest, but usually it is not of critical importance. During the mechanical preparation the elementary process of coalescence plays a secondary part; it is, however, of primary importance for the stability of the product and therefore it is studied amply, technically as well as from the purely scientific point of view.

b. In other cases the emulsion is but an intermediate stage in a certain technological physical or chemical process.

Two liquids, for instance, must be made to interact chemically or physically; one of the liquids is dispersed into the other in a mixing apparatus, usually by means of turbulent motion, untill a rather coarse emulsion is formed in which the large surface of contact promotes the interaction. After that the emulsion has to be separated into its components;

usually this is done in a settling tank or in a centrifuge. An example is the physical or chemical extraction process. In this case the mean size of the droplets of the temporary emulsion is of large technical and economical interest, because e.g. fine droplets promote the interaction, but impede the separation. With a given apparatus we may expect a rather sharp minimum in the total time required for interaction and separation together to occur at a certain critical mean size of the droplets. In this case both elementary processes play their part, while the emulsions remain so coarse that the physical properties of the liquids will have a preponderant influence. So far researches on this case are in an incipient stage only.

Here we give a report of the first results of some investigations which were projected in connection with the technical problems which arise in case *b*. They were carried out under the guidance of Professor W. J. D. VAN DIJCK in the Laboratory of Technical Physics of the Technical University in Delft. In two simple arrangements we studied the equilibrium in the state of dispersion mentioned above, in dependence of the physical properties of the liquids and of the properties of the field of flow. Moreover for some emulsions we considered the velocity with which this equilibrium was obtained, by producing a temporary preponderance of the process of breaking or a temporary preponderance of the process of coalescence. We chose such combinations of two liquids as are not or nearly not inclined to form a stable emulsion, and we considered the turbulent case only.

For the fields of flow we chose in the first place the turbulent field of flow through a 4" circular tube at REYNOLDS numbers Re varying from 50.000 to 500.000, because this case of turbulence normally occurs in practice. Here, $Re = 2U_m R\varrho/\eta$, where U_m is the mean velocity of flow through the tube with radius R. The apparatus is described in No. 2. Besides we designed a model apparatus as described in No. 3, in order to have a check on the general validity of the results.

We studied the state of dispersion by direct photography of the emulsion in turbulent motion, as we wanted to be sure that we would obtain the undisturbed state of dispersion.

It was possible to take photographs because our emulsions remained rather coarse and because we restricted ourselves to small volume percentages of the dispersed phase (see No. 5).

The optical arrangement is described in No. 2. The photographs directly show us something about the elementary processes (see No. 6). By measuring the sharp images of the droplets on the photographs we could calculate the actual size distribution of the droplets. The procedure used is described in No. 5.

We give a discussion of the results of the experiments in No.'s 10 and 11, after a treatment of the elementary processes in No.'s 8 and 9. The properties of the turbulent field in our apparatus, which must be known for this treatment, are discussed in No. 7.

2. *The technical apparatus.*

In order to realize the stationary state of dispersion which is characteristic for turbulent flow through 4″ tube, we built a closed circuit of 4″ "linepipe" (see Fig. 1), in which the emulsion is kept in circulation by means of a centrifugal pump. The circuit consists of two straight parallel tubes, about 700 cm long, connected at one side by a piece of curved tube bent over 180°, with a radius of curvature of 86 cm. At the other end the lower tube is connected with the low pressure side of the centrifugal pump; the high pressure side of the pump is connected to the other end of the upper tube by means of two curved tubes of 90°, with a radius of curvature of 60 cm. The inner wall of the tubes remains fluent at all flange couplings of the circuit; the pump is so chosen, that the circumferential velocity of the rotor is lowest for the required supply and pressure drop, in order to disturb the emulsion as little as possible. Hydrodynamic data for the circuit are given below Fig. 1. From the resistance coefficient of $1{,}90 \,.\, 10^{-2}$ for the straight tubes in a very large range of Re we conclude that the grain of the wall is fine, but very inhomogeneous. In the curved tubes the average coefficient of resistance is $4{,}26 \,.\, 10^{-2}$. This much higher value is caused by the vortices which always are formed in curved tubes, as well as by a larger roughness of the wall. As the total length of the curved tubes is 453 cm, this high coefficient of resistance is of great importance for the emulsion formation in the circuit (see No. 9 and No. 10). The results of the experiments make it highly probable that the hydrodynamic field in the pump does not disturb the emulsifying properties of the circuit as caused by the turbulent flow through the tubes only (see No. 10).

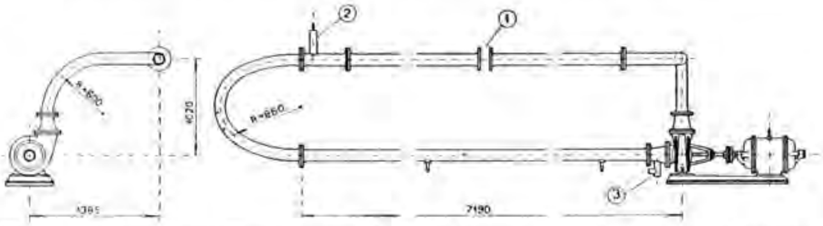

Fig. 1. The technical arrangement. At 1 the brass ring with the optical arrangement is mounted; 2 is the waste and de-aeration pipe; 3 is the inlet pipe. Hydrodynamic data for $Re = 4{,}5 \,.\, 10^4$ to $5{,}5 \,.\, 10^4$: the circuit consists of: 1376,5 cm straight tube of radius 5,1 cm and resistance coefficient $1.90 \,.\, 10^{-2}$; 453 cm curved tube of the same radius (radius of curvature 60 and 86 cm) and mean resistance coefficient $4{,}26 \,.\, 10^{-2}$. The radius of the rotor in the centrifugal pump is 22,5 cm; its circumferential velocity is $210\ (U_m/100)^{1{,}04}$ cm/sec (U_m in cm/sec).

The optical arrangement for photographing the droplets of the emulsion is mounted in a 10 cm broad brass ring, which is fitted in the middle of the straight tube at the output side of the pump (see 1, Fig. 1). As the emulsion circulates very fast, there is no objection against taking photographs at this place in the circuit. Applying direct photography to the study of the emulsion droplets the following difficulties arose, which determined the choice of the arrangement.

a. The emulsion is rather untransparent.

b. The magnification should be sufficiently high, while the object is moving very fast. This requires a very short time of illumination, which gives difficulties with the intensity. If we wish our photographs sharp within 0,1 mm at a magnification 50 and if the mean velocity of flow is 500 cm/sec ($Re \approx 500.000$ in water) the time of illumination should be less then 2.10^{-7} sec.

c. The layer of emulsion droplets between the focussed layer and the object-glass should be thin, in order not to disturb the image formation (see No. 5).

d. The field of flow in the neighbourhood of the photographed droplets and also the general flow should be disturbed as little as possible.

The arrangement represented in Fig. 2 had the following characteristics.

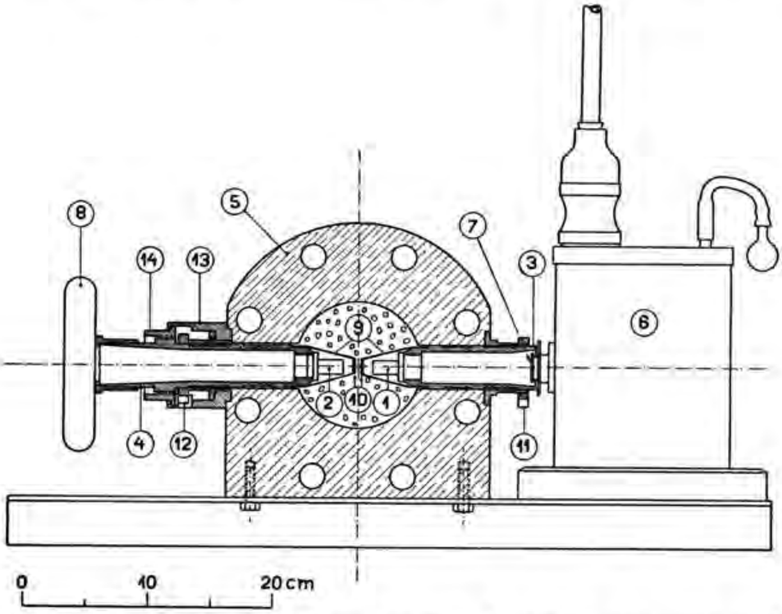

Fig. 2. A section through the brass ring with the optical arrangement in the circuit, as described in No. 2. The tubes 3 and 4 which pass water tight through ring 5, contain the object-glasses 1 and 2. They are made of stainless steel; their positions are secured by means of the clamping rings 11 and 12. Tube 4 can be adjusted with the aid of the threaded ring 14.

The emulsion in the centre of the tube is photographed with transmitted illumination. The difference in the refractive indices of the two liquid phases must produce the necessary contrast. Object-glass 2 and condensor 1 (see Fig. 2), both of Zeiss type *A*, are placed as in an ordinary microscope. The image formed by the object-glass falls directly on the sensitive material of a camera 8, in order to be able to come out with a low intensity of illumination (Leica camera with Agfa FK isochrome film).

In connection with the points *a* and *c* the distance between the condenser and the object-glass is chosen so small as is consistent with point *d*. To this purpose the dry object-glasses 1 and 2 which are used as object-glass and condenser are roofed by little waterproof caps 9 with small plan parallel glass windows 10. The emulsion flows around these caps and through the slit of about 4 mm width between the windows, where it is illuminated through the condenser window; the droplets in the middle of the slit are focussed through the object-glass window on the sensitive material. Thus an emulsion layer of only about 2 mm thickness disturbs the illumination and another of the same thickness disturbs the image formation. The direct magnification is 12,6; the negatives are magnified in printing. The highest total magnification is 100.

The apparatus for the illumination 6, Fig. 2, consists of a multiplate condenser of about 500 $\mu\mu F$, with a very small inner selfinduction. Directly attached to it is a spark gap consisting of two tapered magnesium electrodes at a distance of 1,5 à 2 mm. The whole is placed in limpid transformer oil; the spark passes at 15 à 20 kV. The circuit, formed by the condenser and the spark gap, has a calculated resonance frequency of 3.10^7. An image of the spark is formed by a magnifying glass 7 in the first focal plane of the condenser. Always the photographs were sufficiently clear and sharp provided the focussing was right. Fig. 3 gives a photograph of the optical arrangement; it also shows an influence machine of Hommelsdorf used as the source of tension for the spark.

3. *The model arrangement.*

The second apparatus was designed for orientating investigations. For the turbulent field we chose the field of flow in the space between two coaxial cylinders with radii R_1 and R_2 ($R_1 > R_2$), the inner one rotating at an angular velocity ω. This arrangement is simple, and disturbances as caused by the pump and by the curved parts in the circuit do not occur.

As the dimensions of the model are small, an optical arrangement which protrudes in the emulsion, as is used in the circuit, would disturb the flow too much.

For this reason we have chosen the arrangement shown in Fig. 4. A nickle plated brass cylinder 2 rotates with angular velocity ω in a cylindrical glass dish 1. The dish 1 is covered and closed by means of glass plate 4 and a cork packing ring 3. The (annular) space between the dish 1 and the cylinder 2 is occupied by the emulsion. The axis of the cylinder 2 is led through plate 4 by means of a liquid trap 5 and is driven by the gears 9. The illuminating light passes axially through the whole system. For the rest the optical arrangement is similar to the arrangement described in No. 2. The layer which is sharply in focus lies about 5 mm below the glass plate in the emulsion. The same spark gap as in No. 2 is used as source of illumination (15 on Fig. 4). Of course the illuminating light rays are rather badly scattered and weakened by absorption before they reach the focal plane in this optical arrangement. Therefore the magnification must be lower than in the circuit (the direct magnification was 3,44; the highest total magnification was 20). For the same reason we are more restricted in the volume percentage, occupied by the dispersed phase in the model (see No. 5), than is the case in the circuit.

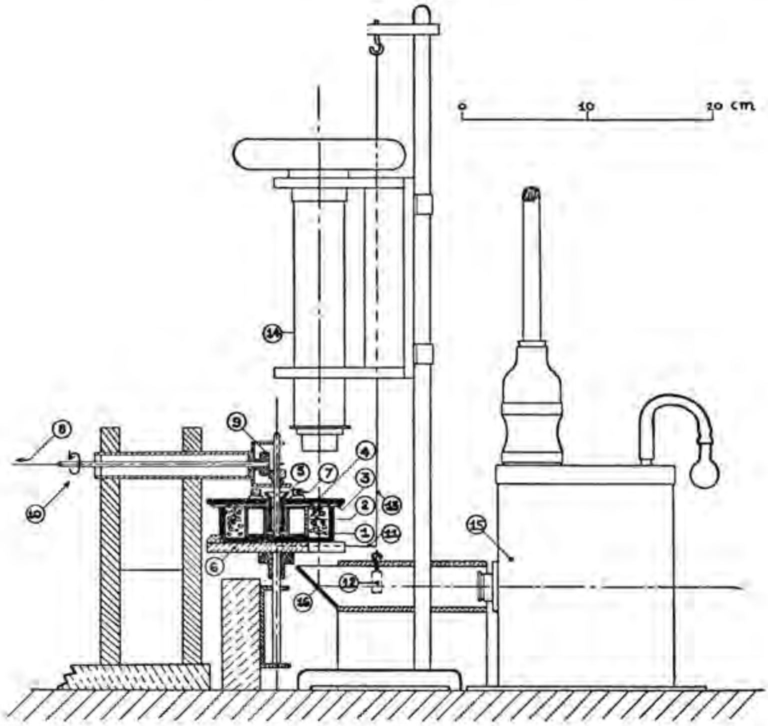

Fig. 4. A section through the model arrangement as described in No. 3. The couple measurements are made with the aid of the threads 11 and 13 and the weight 12; 3 is a cork packing ring; the small strips 7 produce a pressure upon the glass plate; 16 is a mirror.

Our knowledge about the field of flow between long coaxial rotating cylinders is extensively treated in a recent compilatory work of S. GOLDSTEIN[1]. There it is proved that the field of flow may be unstable. When the REYNOLDS number, defined by $Re = \varrho\omega R_1 (R_1 - R_2)/\eta$ rises above a certain critical value, dependent on $(R_1 - R_2)/R_1$, annular vortices appear.

In the model the axial length is limited and this probably has some influence on the field of flow, which also may be disturbed by the slits at the bottom of the disk, at the glass plate and in the liquid trap. The results obtained with the model, which are discussed in No. 10, contain no

Fig. 3. The optical arrangement in the circuit, seen from the camera side.

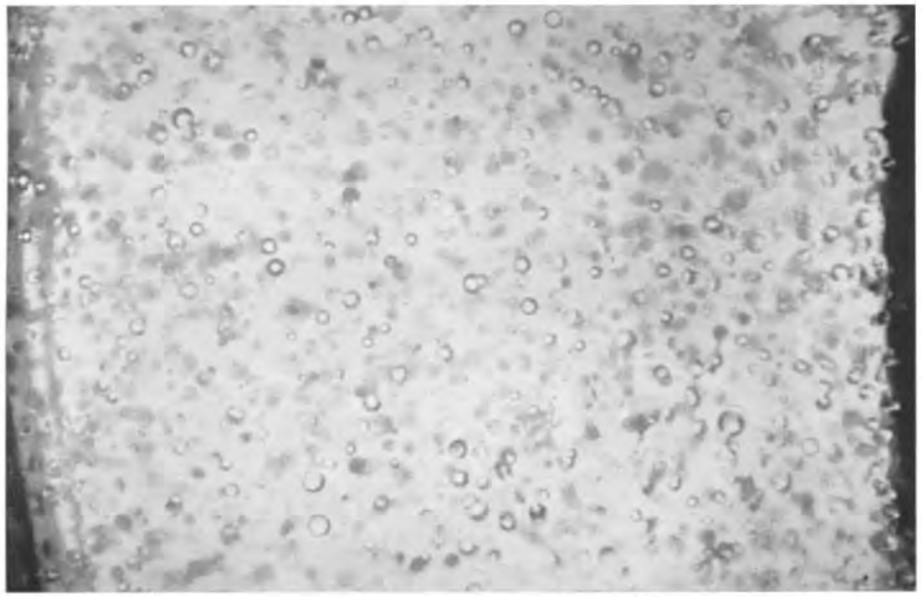

Fig. 7. Photograph (1,18) series M in the model. Magnification 7,7; the distance between the place where u_s/U_τ has reached its maximum value and the inner wall is about 1,5 cm on the picture.

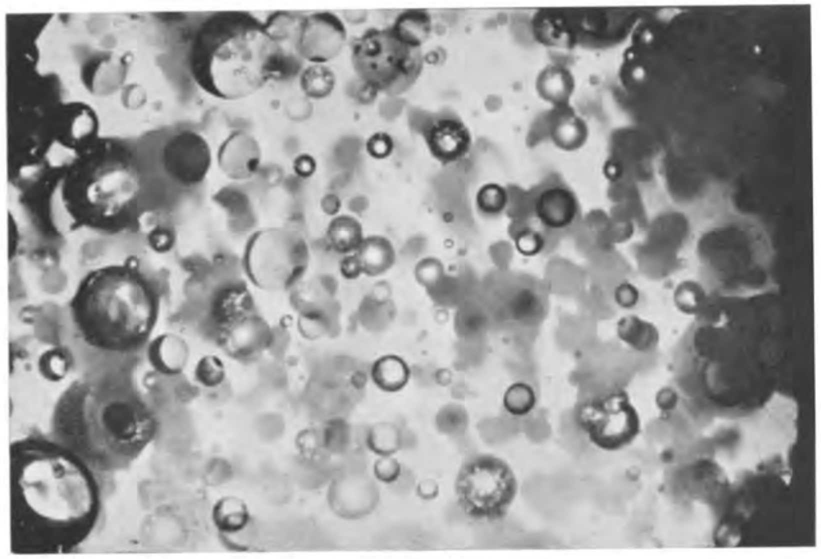

Fig. 8. Photograph (16,3) series R in the model. Magnification 10. Many droplets are seen clinging together.

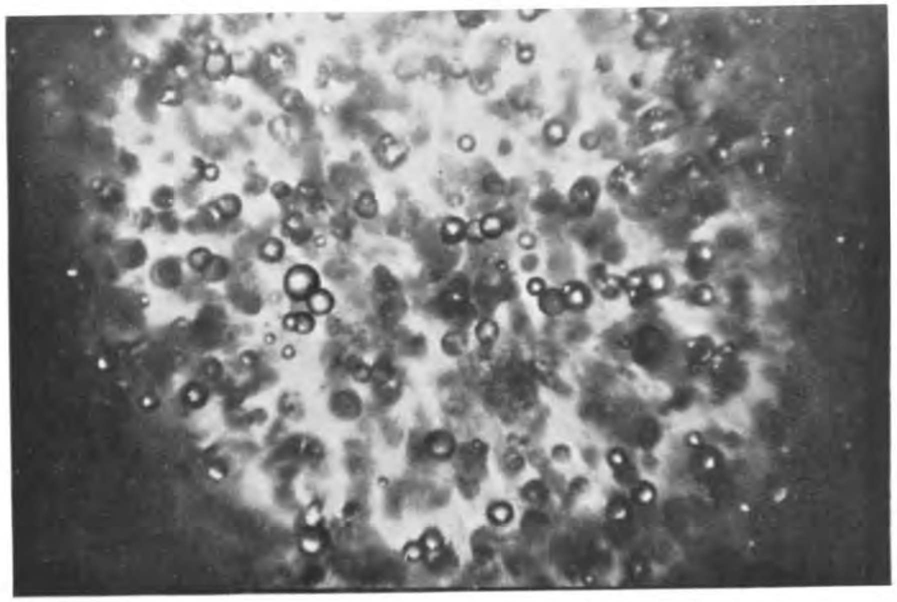

Fig. 9. Photograph (19,7) series T in the circuit. Magnification 40.

evidence for such disturbing influences, however. We always had two annular vortices.

In order to obtain some information about the hydrodynamic field in this model arrangement, we measured the couple which the fluid exerts on the dish. For this purpose the dish is placed on the little table 6 (fig. 4), which can rotate freely around the common vertical axis of the cylinders. If we neglect the couples exerted on the bottom *), on the glass plate and in the liquid trap, we can calculate the mean shearing stress at the walls of the cylinders (τ_1 at R_1; $\tau_2 = (R_1/R_2)\ \tau_1$ at R_2) and derive a resistance coefficient $\tau_1/\varrho\omega^2 R_1{}^2$. The result gives Fig. 5, together with some curves calculated from measurements of G. I. TAYLOR at other values of $(R_1—R_2)/R_1$. A comparison of the three curves justifies our neglections sufficiently.

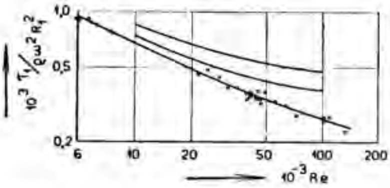

Fig. 5. The curves show the resistance coefficient $\tau_1/\varrho\omega^2 R_1{}^2$ as a function of the REYNOLDS number $Re = \varrho\omega R_1 (R_1—R_2)/\eta$ for the flow between coaxial cylinders with the inner cylinder rotating.

Curve 1 for the model arrangement: $(R_1—R_2)/R_1 = 0{,}40$
Curve 2 for long cylinders : $(R_1—R_2)/R_1 = 0{,}21$
Curve 3 for long cylinders : $(R_1—R_2)/R_1 = 0{,}15$

The curves 2 and 3 are given by GOLDSTEIN; the data are taken from TAYLOR.

4. *The experiments.*

The emulsions must be homogeneous, separation by settling must be prevented.

This gives no difficulties in the circuit where the turbulence is intense, where the dispersion is fine and where the turbulent diffusion is very active. In the model arrangement the annular vortices take care of the mixing if the turbulent diffusion may be deficient. In executing the experiments, groups of 3 to 6 photographs were taken of the emulsions under different selected circumstances; this we call a "measurement". By measuring the sharp images of droplets on the photographs (see No. 5) we find the size distribution of the emulsion droplets. The photographs of one measurement were taken with a difference in time of 5 to 20 seconds.

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*) We obtained an estimation of the couple exerted on the bottom of the dish only by filling the dish to a height of but 3 mm above the bottom. The couple measured in this case was always less than 10 % of the couple measured in the normal case, even at the lowest REYNOLDS numbers.

We call the measurements made in succession with one combination of liquids a "series" (indicated it by a capital letter).

The following types of measurements were made:

1. Measurements with an emulsion in equilibrium, i.e. an emulsion which was kept in the same state of motion during 20 to 60 minutes. (This period appears usually to be long enough; see the data and also No. 11).

2. Beginning with a measured emulsion, in equilibrium at $Re = Re_1$, we suddenly changed Re_1 into Re_2 and made a measurement after an interval of time T. If $Re_2 > Re_1$, the breaking process has predominated during the time T, and a comparison of the original state of dispersion with the new one gives an impression of the activity of this process of breaking at $Re = Re_2$ during the time T. If $Re_2 < Re_1$, we obtain an impression of the process of coalescence (see No. 11).

The measurements made with the model arrangement are given in table A, those made with the circuit are given in table B.

The liquids which we used in the model were acquired from a pharmaceutical chemist; besides we used distilled water and very pure kerosene. During the model measurements the temperature always remained sufficiently constant (about 18° C).

In the circuit we used commercial liquids and tapwater. The liquids were polluted somewhat during a series of measurements because rust particles etc. left the wall. The temperature always rose gradually during the execution of the measurements of one series which was due to the energy dissipation in the circulating emulsion. We have corrected the properties of the liquid phases for the change of temperature.

The number of measurements made in the circuit is still small. In the model arrangement we made measurements at $\varrho = \varrho'$ with η from 0,6 cP till 10 cP, η'/η from 0,3 till 30, γ from 1 dyne/cm till 49 dyne/cm, c till 4 % and Re from 5000 till 100.000; and some measurements at $\varrho' > \varrho$.

5. *The determination of the size distributions.*

The photographs show images of droplets in any degree of sharpness. This makes the measuring out of the photographs more difficult. However, it is a general and remarkable phenomenon that the sharp images of droplets on the photographs make it certain, that nearly all droplets in the turbulent emulsion retain the spherical form notwithstanding the violent motion. This fact (see No. 6) makes it possible to measure out the photographs on a reasonable basis, because the sharpness of the images of spherical particles obeys sufficiently simple relations.

If we put a lower limit to the sharpness of the images of the spherical droplets in the photographs of one measurement, we restrict ourselves to droplets with their centres in a certain region or layer of sharp image formation. This layer will be limited in first approximation by two parallel planes, perpendicular to the optical axis on both sides of the focussed plane, at a mutual distance which we call the depth of focus s. In a given optical system s may depend on the refractive indices of the droplets and of the exterior phase, and also on the droplet radius. We cannot predict this dependence theoretically.

From the size distribution of the droplets with sharp images on the photographs of

TABLE A. Measurements made with the model arrangement.

Measurement and series	Exterior Phase			Dispersed Phase								Dispersion					
	ϱ	η	L	ϱ'	η'	L	γ dyne/	c	T	Re	τ_2 dyne/	a_{95} 10^{-3}	a_{95}/a_{50}	a_{10}/a_{50}	C_1	C_2	Rapidity of coalescence
	g/cm³	cP		g/cm³	cP		cm	%	min.	$\times 10^3$	cm²	cm			$\times 10^{-2}$		
1— 3, 2 A	1.00	10.1	*cde*	1.00	1.07	*a*	44	2.1	20	4.9	147	82	1.61	0.41	1.13	7.2	slow
4— 6, 2 A									20	4.1	115	120	1.40	0.56	1.10	6.7	
7— 9, 2 A									20	5.6	180	64	1.31	0.56	1.10	7.5	
1— 3, 3 B	1.17	10.8	*de*	1.17	11.6	*ab*	32	2.1	20	4.7	145	81	1.91	0.65	0.90	5.6	very slow
4— 6, 3 B									20	6.1	213	30	1.47	0.70	1.42	10.2	
1— 2, 4 C	1.00	1.77	*ce*	1.00	1.07	*a*	47	0.9	20	29.0	72	87	1.47	0.55	1.03	15.5	moderate
4— 5, 4 C									20	50	176	41	2.16	0.69	0.73	13.1	
8, 4 C									20	65	273	15.6	1.63	0.41	1.14	24.3	
9—11, 4 C									20	39	119	44	1.78	0.49			
1— 3, 5 D	1.19	10.2	*de*	1.19	37	*ab*	17.2	0.9	20	7.0	228	27	1.44	0.68	0.77	5.8	very slow
10—12, 5 E	1.00	10.1	*cde*	1.00	1.07	*a*	44	0.9	20	7.6	298	43	1.59	0.48	0.84	6.7	very slow
4— 6, 9 F	1.00	1.77	*ce*	1.00	1.07	*a*	47	3.3	16	29	72	86	1.69	0.38	1.05	15.2	slow
7— 8, 9 F									5	52	187	32	1.99	0.50			
11—12, 9 F									20	52	187	24	1.92	0.51			
1— 2, 10 F									40	53	197	19.1	1.56	0.42	1.38	25.5	
2— 4, 13 G	1.00	1.07	*a*	1.00	30	*de*	41	3.8	20	51	67	117	—	—	0.68	12.2	very slow
6— 7, 13 G									5	86	159	74	1.44	0.48			
8— 9, 13 G									20	86	159	53	2.07	0.45			
11—12, 13 G									60	86	159	56	3.30	0.54	0.41	9.6	
1— 2, 14 H	1.00	1.07	*a*	1.00	30	*de*	41	1.5	20	58	98	110	—	—	0.50	7.8	very slow
3— 4, 14 H									5	94	178	52	1.42	0.33			
5— 6, 14 H									60	94	178	14.9	1.46	0.64	1.38	31	
7— 8, 14 I	1.00	1.07	*a*	1.00	1.77	*ce*	47	3.8	20	57	77	117	—	—	0.55	10.5	very slow
9—10, 14 I									5	78	136	59	1.51	0.35			
11—12, 14 I									20	78	136	26	1.60	0.55	1.30	27	
1, 15 J*)	1.00	1.20	*a*	1.00	3.98	*ef*	5.5	3.8	20	26	28.0	49	2.13	0.51	0.58	8.3	slow
3— 4, 15 J									3	36	47	22	1.46	0.63			
5— 6, 15 J									60	38	51	16.2	1.51	0.75	0.83	13.6	
7— 8, 15 J									3	29	33	17.8	1.61	0.75			
9—10, 15 J									20	31	38	20	1.61	0.65			
1—2, 16 K*)	1.00	1.20	*a*	1.00	3.98	*ef*	5.5	1.5	20	24	24.3	35	1.51	0.61	0.94	12.8	slow
3, 16 K									3	35	45	29	1.47	0.61			
5— 6, 16 K									57	36	46	19.1	1.28	0.59	0.81	12.7	
7— 8, 16 K									6	23	22.5	22	1.54	0.60			
3, 17 L*)	0.78	0.65	*g*	0.74	0.58	*h*	1.05**)	<1	20	25	9.2	18.2	1.40	0.64	0.89	12.1	moderate
6— 7,17 M*)	0.78	0.65	*g*	0.74	0.58	*h*	1.05**)	>2	20	15.1	4.4	26	1.47	0.61	1.68	18.6	fast
8, 17 M									3	23	8.7	19.2	1.41	0.59			
11, 17 M									58	26	10.5	15.9	1.51	0.54	0.92	12.8	
1— 2, 18 M									1	13.1	3.4	17.5	1.35	0.60			
3— 4, 18 M									4	13.3	3.4	26	1.54	0.63			
9—10, 18 N	1.00	1.77	*ce****)	1.00	1.07	*a*	20	3.8	20	30	77	37	1.70	0.57	0.98	14.6	no
28—33, 1 O	0.77	1.50	*c*	1.00	1.07	*a*	49	4.8	29	49	162	70	1.82	0.36	0.49	8.7	fast
35—38, 1 O									3	38	108	88	2.38	0.35			
16—20, 2 O									20	57	200	55	2.12	0.56	0.47	8.8	
22—26, 2 O									4	46	147	70	1.90	0.37			
28—31, 2 P*)	0.83	4.62	*f*	1.00	1.20	*a*	4.0	<4.8	41	6.9	70	19.7	1.40	0.75	0.82	6.2	moderate
33—36, 2 P									6	5.8	56	19.8	1.26	0.73			
0— 5, 3 Q*)	0.83	4.62	*f*	1.00	1.20	*a*	4.0	4.8	44	6.0	56	32	1.62	0.63	0.65	4.6	moderate
8—12, 3 Q									7	4.9	41	38	1.70	0.67			
14—17, 3 R	0.77	1.50	*c*	1.00	1.07	*a*	49	1.9	20	51	167	60	2.31	0.52	0.53	9.9	fast
24—27, 3 R									20	63	252	36	1.76	0.62	0.57	10.8	
30—32, 3 R									6	53	179	51	2.07	0.59			

TABLE B. Measurements made with the technical arrangement.

Measurement and series	Exterior Phase			Dispersed Phase									Dispersion			
	ϱ	η	L	ϱ'	η'	L	γ dyne/	c	*temp*	T	Re	$\bar{\tau}_{ok}$ dyne/	a_{95} 10^{-3}	a_{95}/a_{50}	a_{10}/a_{50}	C_k	Rapidity of coalescence
	g/cm³	cP		g/cm³	cP		cm	%	°C	min.	$\times 10^4$	cm²	cm				
6—10, 6 S	1.00	0.95	*a*	0.88	~60	*d*	46	12	23	50	24	247	19.9	1.96	0.75	18.7	fast
12—15, 6 S									24	4	31	420	12.2	1.22	0.63		
45— 4, 7 T	1.00	0.94	*a*	0.88	~60	*d*	46	3.7	23	30	31	413	8.3	1.60	0.62	27	fast
6—10, 7 T		0.94		0.88					23	4	47	970	6.2	1.42	0.67		
15—19, 7 T		0.87		0.88					25	21	51	970	7.0	1.64	0.69	13.5	
21—25, 7 T		0.76		0.87					29	22	65	1220	5.6	1.57	0.69	13.4	
28—31, 7 T		0.72		0.87					30	5	42	525	7.8	1.51	0.66		
45— 7, 9 U*)	0.99	1.4	*a*	0.83	6.9	*i*	4.9	≈2	16.2	20	5.1	26.4	9.7	1.26	0.41	38	very slow
8—12, 9 U		1.4			6.8				16.5	7	8.4	71.0	5.9	1.59	0.50		
12—15, 9 U		1.4			6.8				16.5	32	8.4	71.0	6.1	1.89	0.54	22	
16—18, 9 U		1.3			6.7				16.7	20	11	111	3.1	1.63	0.55	28	
28—30, 9 U		1.3			6.7				16.8	20	4.8	21.1	3.7	1.82	0.55		
32—34, 9 V*)		1.3	*a*		6.7	*i*	4.9	5.5	16.9	30	6.1	32.7	11.1	2.42	0.51	27	very slow

In table A: $C_1 = \frac{2\gamma}{a_{95}\varrho\omega^2 R_1^2}$; $C_2 = \frac{2\gamma}{a_{95}\tau_2}$. In Table B: $C_k = \frac{2\gamma}{a_{95}\tau_{ok}}$.

The values of these quantities are calculated for stationary states of dipersion only.

In both tables *L* denotes the composition of the exterior phase, resp. of the dispersed phase: *a* means water, *b* glycerol, *c* kerosene, *d* liquid paraffine, *e* pentachloro ethane, *f* propanol, *g* methanol, *h* heptane, *i* fusel oil (70 % propanol).

*) The liquids used in these series are partly mutually soluble. Before the measurements were made the equilibrium was established.

**) We are not quite certain of this value for the interfacial tension.

***) To this exterior phase we added 1 % "Palsgaard" emulsifier.

one measurement, we can derive the real size distribution of the droplets in the emulsion, provided we know s as a function of a for the case in question. The applicability of this procedure depends on the reproducibility of the criterion of sharpness and on the fact whether we can measure s as a function of a.

The criterion of sharpness must be a subjective one. We chose the subjectively judged image sharpness of the circumference of the droplets on the photographs and measured $s = s(a)$ with the aid of the following indirect method.

In an emulsion with c % of the volume occupied by the dispersed phase and with $n(a)\,da$ droplets per cm³ with a radius between a and $a + da$, the following relation must be fulfilled:

$$\int_0^\infty n(a)\,.\,{}^4/_3\,\pi a^3\,da = \frac{c}{100}.$$

If we measure out a surface O on the photographs at a magnification V, we obtain a size distribution of the droplets in the region of sharp image formation given by $n'(a)\,da$.

$$\text{Now } n(a) = n'(a)\frac{V^2}{O\,s(a)}, \text{ and therefore } \frac{V^2}{O}\int_0^\infty \frac{n'(a)}{s(a)}\,.\,{}^4/_3\,\pi a^3\,da = \frac{c}{100}. \quad (5, 1)$$

It appears that in the emulsions investigated the huge part of the dispersed volume is occupied by droplets of a limited size region which may be characterized by a radius a_1. (see No. 6, remark *b1*). Therefore (5,1) can be approximated by:

$$\frac{V^2}{O\,s(a_1)}\int_0^\infty n'(a)\,.\,{}^4/_3\,\pi a^3\,da = \frac{c}{100}.$$

From this equation we can calculate $s(a_1)$. Using different measurements of one series we obtain s as a function of a for that series. The method can be refined a little bit, but we have to extrapolate to the largest and to the smallest droplets. In practice we work with size classes for the droplets (see Figs. 10, 11 and 12).

All photographs have been measured out by the same person and for him we determined the regions of sharp image formation. In the circuit we found the same relation between s and a with both emulsions; consequently the refractive indices had no noticeable influence. With the model arrangement the result is the same, but here c appears to have some influence on the depth of focus s, what may be explained by the primitive illuminating optics. The results for the circuit and for the model (at low values of c) are given in Fig. 6.

We cannot expect a large accuracy in this procedure: the clear photographs fulfilled (5,1) with a maximum deviation of 20 %. Per measurement we had a number of sharp images of droplets varying from 100 to 300. This result is connected with the form of the distribution curves (see No. 6, remark *b* 1).

The sharp images of the droplets covered the surface of the photographs for 14 % at most. Finally we remark that with the optical system of the circuit $s(a)/a$ lies between 1 and 2. With that of the model arrangement it lies between 3 and 10, as can be deduced from Fig. 6.

The region of applicability of the photographic method is limited by the disturbance of the image formation by the droplets of the emulsion layer between the layer in focus and the object glass. This disturbance depends on the optical system used and on the refractive indices. In first approximation it will be proportional to the droplet surface per cm^3 ($\sim c/a_S$ if $a_S{}^2$ is used to denote the average value of a^2), and proportional to the thickness of the disturbing layer.

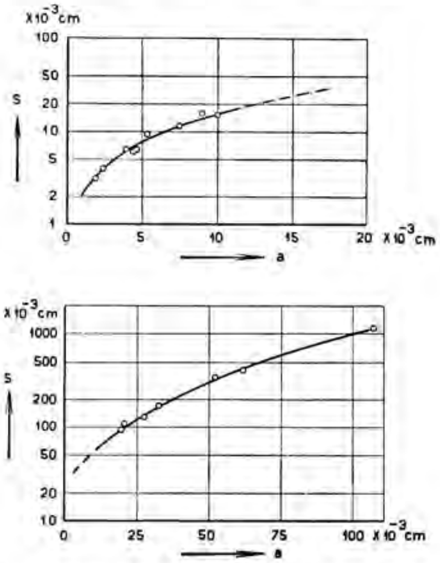

Fig. 6. The depth of focus s as a function of the radius a of the droplet for the technical arrangement (upper curve, data from series S, T, U and V) and for the model arrangement (lower curve, data from series H and K).

6. *Results. General remarks.*

The result of the investigations is embodied in photographs and in size distributions of the states of dispersion in the investigated emulsions.

We first consider the photographs (see Figs. 7, 8 and 9). The model photographs show a section through the whole turbulent field (see No. 9). The circuit photographs give the situation in the centre of the tube only, which may be disturbed moreover by the optical arrangement. Nevertheless all the photographs give the same general impression, in which the following pecularities call for attention.

*a*1. Nearly all the droplets are spherical, also those near to the wall in the model.

*a*2. Breaking processes are seen only seldom. Some droplets show local deformations in the form of tiny protuberances, which probably will ultimately lead to the separation of a very small droplet. These details on the photographs could not be reproduced.

*a*3. On the photographs we often see droplets which apparently cling together. Especially we often see that one or more very small droplets adhere to one larger droplet (see Fig. 8). Certain emulsions show this phenomenon very often (see No. 8).

The small number of breaking processes indicates that these must have a very rapid course.

Some distributions of sizes are given in Figs. 10, 11 and 12.

The distributions of sizes can be treated in two different ways. We can judge the qualitatively according to their general form and compare them.

Then the following points are observed:

*b*1. All curves decline steeply towards the side of the larger droplets (see No. 5). This is in contrast with the distributions of sizes, measured

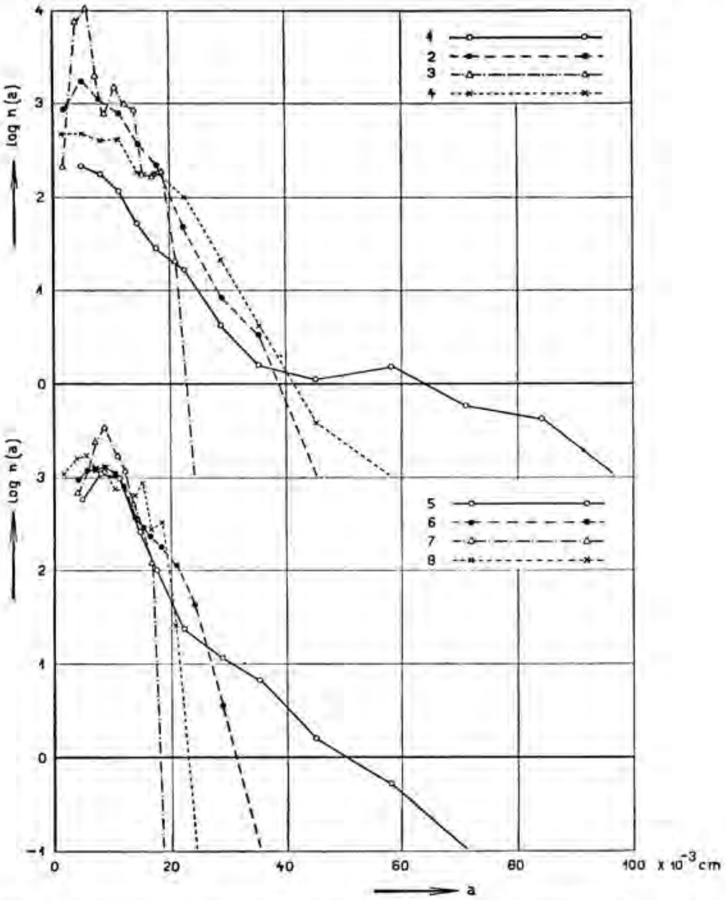

Fig. 10. Size frequency curves of the droplets of some emulsions made in the model:
Curve (1) is (4—6,9) series *F*; (2) is (7—8,9) series *F*; (3) is (1—2,10) series *F*; (4) is (9—10,18) series *N*; (5) is (11—12,13) series *G*; (6) is (11—12,14) series *I*; (7) is (5—6,15) series *J*; (8) is (5—6,16) series *K*. The scale for $n(a)$ is logarithmic*).

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*) The logarithmic scale for $n(a)$ in Figs. 10 and 11 is taken in such a way, that $\log\ n(a) = 0$ corresponds to $n(a)\,da = 0{,}118$ droplets per cm^3 per vol. % of the dispersed phase, if da is measured in 10^{-3} cm. In Fig. 12 in the same way $n(a)\,da = 127$ droplets per cm^3 per vol. % of the dispersed phase. This allows direct comparison of the various curves.

with rain drops or fog drops, and with older emulsions (which were not homogenized very rigorously), and in which during a long time only

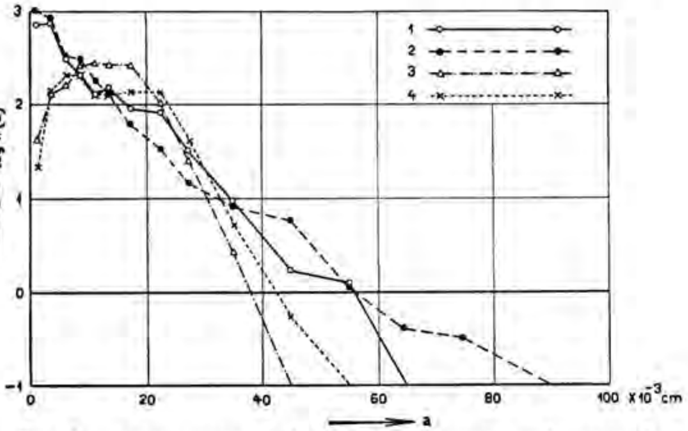

Fig. 11. Size frequency curves of the droplets of some emulsions made in the model. Curve (1) is (16—20,2) series *O*; (2) is (22—26,2) series *O*; (3) is (0—5,3) series *Q*; (4) is (8—12,5) series *Q*. The scale for $n(a)$ is logarithmic*).

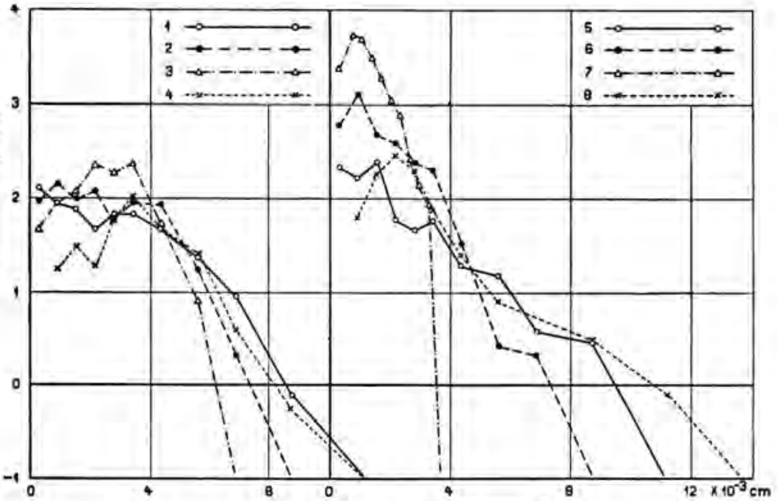

Fig. 12. Size frequency curves of the droplets of some emulsions made in the circuit. Curve (1) is (45—4,7) series *T*; (2) is (6—10,7) series *T*; (3) is (21—25,7) series *T*; (4) is (28—31,7) series *T*; (5) is (45—7,9) series *U*; (6) is (12—15,9) series *U*; (7) is (16—18,9) series *U*; (8) is (32—24,9) series *V*. The scale for $n(a)$ is logarithmic*).

coalescence was in action. So the steep decline is characteristic for the action of the breaking process (see No.'s 9 and 10).

b2. To the side of the smaller droplets the size distributions show

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*) See note p. 863.

various forms: some are flat, some rise, other ones have a distinct maximum at a certain value of a.

c. We can also consider the states of dispersion on a quantitative basis. To that purpose from the size distribution of the emulsion droplets we calculate the integral distributions of the dispersed volume, and from these we take some characteristic radii: a_{95}, a_{50} and a_{10}. The droplets with $a \leqslant a_{95}$ contain 95 % of the dispersed volume, those with $a \leqslant a_{50}$ contain 50 % and those with $a \leqslant a_{10}$ contain 10 %. In connection with *b*1 we may consider a_{95} as the radius of the largest droplets in our emulsions. The tables give the values of a_{95}, a_{95}/a_{50} and a_{10}/a_{50} for all the emulsions investigated.

From the values of the last quantities, it follows that the integral distributions of the dispersed volume are usually nearly the same for the greater part of the emulsions. These quantitative data are very important especially in technical applications.

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Mathematics. — *Ueber affine Invarianten bei quadratischen Formen.* Von O. BOTTEMA. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of June 29, 1940.)

1. In zwei in diesen Proceedings erschienenen Arbeiten hat WEITZENBÖCK [1]) die Invarianten einer quadratischen Form und einer Linearform bei der speziellen affinen Gruppe bzw. in der Ebene und in R_3 untersucht und einige geometrische Anwendungen gegeben.

Wir machen im Folgenden einige Bemerkungen über die von WEITZENBÖCK gebildeten absoluten Invarianten und über die von ihm für $n = 2$ und $n = 3$ gelöste geometrische Frage, welche wir für allgemeines n erledigen.

Wir beschränken uns dabei vorläufig auf den Fall der Ebene. WEITZENBÖCK zeigt dass eine ternäre quadratische Form und eine lineare Form in homogenen Punktkoordinaten $x_1 : x_2 : x_3$,

$$f = \Sigma\, a_{ik}\, x_i\, x_k \;,\; (v' x) = \Sigma\, v'_i\, x_i \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

ein kleinstes volles System von ganz-rationalen affinen Invarianten besitzen das aus fünf Bildungen besteht, welche nicht-symbolisch geschrieben folgendermassen aussehen:

$$\left.\begin{array}{l}
D = \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad\qquad C = a_{11}\, a_{22} - a_{12}^2 \\[2ex]
\Phi = - \begin{vmatrix} a_{11} & a_{12} & a_{13} & v'_1 \\ a_{21} & a_{22} & a_{23} & v'_2 \\ a_{31} & a_{32} & a_{33} & v'_3 \\ v'_1 & v'_2 & v'_3 & 0 \end{vmatrix} \quad U = a_{11}\, v_1'^2 - 2\, a_{12}\, v'_1\, v'_2 + a_{22}\, v_2'^2 \\[2ex]
M = v'_1\,(a_{12}\, a_{23} - a_{13}\, a_{22}) + v'_2\,(a_{13}\, a_{12} - a_{11}\, a_{23}) + v'_3\,(a_{11}\, a_{22} - a_{12}^2).
\end{array}\right\} \quad . \quad (2)$$

Zwischen diesen Komitanten besteht eine einzige Syzygie

$$C\,\Phi = D\,U + M^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Auf Grund dieser Gleichung wird nun bei der Bestimmung der absoluten Invarianten D fortgelassen. WEITZENBÖCK bildet die absoluten Invarianten

$$a = \frac{M^2}{C\,\Phi} \quad , \quad \beta = \frac{C\,U^2}{\Phi^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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[1]) WEITZENBÖCK, Ueber affine Invarianten bei Kegelschnitten, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 159—167; Zur Affingeometrie der F_2 in R_3, 168—178, (1940).

und gibt eine geometrische Deutung der Invarianten R und S, welchen den Bedingungen

$$R^2 = \frac{-1}{\alpha\beta} \quad , \quad S^2 = \frac{-\alpha}{\beta} \quad \ldots\ldots\ldots \quad (5)$$

genügen.

2. Wir möchten nun den Invarianten α und β ein anderes Paar von absoluten Invarianten gegenüberstellen, durch die ebenfalls alle weiteren absoluten Invarianten ausdrückbar sind und welche invariantentheoretisch einen gewissen Vorzug haben.

Wenn wir statt D die Komitante U ausser Acht lassen, so gibt der Ansatz

$$J = D^{a_1} C^{a_2} \Phi^{a_3} M^{a_4}$$

da J vom Grade Null in den Koeffizienten a_{ik} und in den Linienkoordinaten v_i' sein muss, für die Exponenten die Gleichungen

$$3a_1 + 2a_2 + 2a_3 + 2a_4 = 0 \quad , \quad 2a_3 + a_4 = 0$$

woraus

$$a_4 = -2a_3 \quad , \quad a_2 = -\tfrac{3}{2}a_1 + a_3$$

also

$$J = (D\,C^{-\frac{3}{2}})^{a_1} (C\,\Phi\,M^{-2})^{a_3}$$

folgt.

Es sind also

$$k = D^2 C^{-3} \quad , \quad l = C\,\Phi\,M^{-2} \quad \ldots\ldots\ldots \quad (6)$$

ebenfalls zwei unabhängige rationale absolute Invarianten. Den Zusammenhang mit α und β zeigen folgende Gleichungen:

$$k = \frac{(1-\alpha)^2}{\beta} \quad , \quad l = \frac{1}{\alpha} \quad \ldots\ldots\ldots \quad (7)$$

und

$$a = \frac{1}{l} \quad , \quad \beta = \frac{(l-1)^2}{k\,l^2} \quad \ldots\ldots\ldots \quad (8)$$

Die Invarianten k bzw. l haben nun folgende Eigenschaften: die für das aus einer quadratischen Form und einer Linearform gebildete System abgeleitete absolute Invariante k der speziellen affinen Gruppe, ist auch eine absolute Invariante für die quadratische Form allein; die für das aus einer quadratischen Form und einer Linearform gebildete System abgeleitete absolute Invariante l der speziellen affinen Gruppe, ist auch eine absolute Invariante der allgemeinen affinen Gruppe.

Die erste Eigenschaft folgt unmittelbar aus der Definition von k; in D^2C^{-3} treten nur die Koeffizienten der quadratischen Form auf.

Die zweite Eigenschaft geht hervor aus der Bemerkung, dass C, Φ und M Invarianten der allgemeinen affinen Gruppe sind und je das Gewicht 2 haben; $C\Phi M^{-2}$ ist also eine absolute Invariante dieser Gruppe. Man zeigt leicht, dass sie die einzige absolute Invariante ist.

Was die WEITZENBÖCKschen Invarianten betrifft, so ist offenbar α eine absolute Invariante gegenüber der allgemeinen affinen Gruppe, während β, R und S diese Eigenschaft nicht zukommt.

Von den Invarianten R und S hat WEITZENBÖCK eine geometrische Deutung gegeben. Wir geben eine solche von k und l (und also von α).

Was k betrifft, die eine absolute, speziell-affine Invariante des durch $f=0$ dargestellten Kegelschnittes ist, so zeigt man leicht:

$$k=\varrho^3 \quad \ldots \ldots \ldots \ldots \quad (9)$$

wo ϱ der affine Krümmungshalbmesser des Kegelschnittes ist, also der reziproke Wert der Affinkrümmung ²). Für $k>0$, $k=0$, $k<0$ ist der Kegelschnitt bzw. eine Ellipse, eine Parabel, eine Hyperbel. Im ersten Fall hat man z.B. noch die einfache Deutung ³)

$$k=\left(\frac{F}{\pi}\right)^2 \quad \ldots \ldots \ldots \ldots \quad (10)$$

wo F den Flächeninhalt der Ellipse bedeutet; auch bei der Hyperbel könnte man eine analoge Bedeutung angeben.

Was die zweite Invariante l betrifft, wollen wir eine geometrische Bedeutung angeben nicht von l selbst, sondern von dem Ausdruck

$$l'=\frac{l}{l-1}=\frac{C\Phi}{C\Phi-M^2}=\frac{C\Phi}{DU} \quad \ldots \ldots \ldots \quad (11)$$

welche von WEITZENBÖCK in der 2. Arbeit genannt und mit δ bezeichnet worden ist (S. 170). Wir gehen dabei folgendermassen vor.

Die Schnittpunkte P und Q des Kegelschnittes $f=0$ und der Geraden

$$v'_1 x_1 + v'_2 x_2 + v'_3 x_3 = 0$$

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²) BLASCHKE, Differentialgeometrie, II, S. 18 (1923).

³) BLASCHKE, l.c. S. 18.

haben bzw. die Koordinaten (p_1, p_2, p_3) und (q_1, q_2, q_3). Wenn wir aus $f=0$ und $(v'x)=0$ die Koordinate x_2 mittels

$$x_2 = \frac{-v'_1 x_1 - v'_3 x_3}{v'_2}$$

eliminieren, so entsteht eine quadratische Gleichung für $x = \frac{x_1}{x_3}$, welche so aussieht:

$$\left.\begin{aligned} U x^2 + 2x(-a_{23} v'_1 v'_2 + a_{22} v'_1 v'_3 + a_{13} v'^2_2 - a_{12} v'_2 v'_3) + \\ + (a_{33} v'_2 - 2 a_{23} v'_2 v'_3 + a_{22} v'^2_3) = 0. \end{aligned}\right\} \quad . \qquad (12)$$

Die Wurzeln dieser Gleichung sind offenbar $\frac{p_1}{p_3}$ und $\frac{q_1}{q_3}$. Nun ist die Diskriminante $\triangle$ dieser Gleichung Null, wenn die Gerade $(v'x)=0$ den Kegelschnitt berührt, das heisst wenn $\Phi = 0$; Φ ist also ein Faktor von $\triangle$ und man findet leicht

$$\triangle = - v'^2_2 \Phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Für das Quadrat der Wurzeldifferenz hat man also:

$$\left(\frac{p_1}{p_3} - \frac{q_1}{q_3}\right)^2 = \frac{-4 v'^2_2 \Phi}{U^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

Wir betrachten jetzt die mit $(v'x)=0$ parallele Gerade, welche durch den Mittelpunkt des Kegelschnittes geht. Sie hat die Gleichung

$$v'_1 x_1 + v'_2 x_2 + v'_3 (1+p) x_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

Die Schnittpunkte dieser Geraden mit dem Kegelschnitt werden mit $A \equiv (a_1, a_2, a_3)$ und $B \equiv (b_1, b_2, b_3)$ bezeichnet. Man hat also

$$\left(\frac{a_1}{a_3} - \frac{b_1}{b_3}\right)^2 = \frac{-4 v'^2_2 \Phi_1}{U_1^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

wenn Φ_1 und U_1 die Funktionen sind, welche man erhält indem man in Φ und U die Variabele v'_3 durch $(1+p)v'_3$ ersetzt. Diese Substitution gibt uns aber:

$$U_1 = U \quad , \quad \Phi_1 = \Phi + 2 p v'_3 M + p^2 v'_3 C \quad . \quad . \quad . \quad . \qquad (17)$$

während die Bedingung, dasz die parallele Gerade durch den Mittelpunkt gehe die Gleichung

$$M + p v'_3 C = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

ergibt. Wir haben also

$$\Phi_1 = \Phi - \frac{M^2}{C} = \frac{DU}{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

Aus (14), (16) und (19) geht nun hervor:

$$\frac{\left(\frac{p_1}{p_3}-\frac{q_1}{q_3}\right)^2}{\left(\frac{a_1}{a_3}-\frac{b_1}{b_3}\right)^2}=\frac{\Phi}{\Phi_1}=\frac{C\Phi}{DU}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Wir finden somit:

Sind P und Q die Schnittpunkte der Geraden $(v'x)=0$ und des Kegelschnittes $f=0$, A und B die Schnittpunkte des Kegelschnittes und der mit $(v'x)=0$ parallelen Geraden durch den Mittelpunkt des Kegelschnittes, so ist

$$l'=\left(\frac{PQ}{AB}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Das Quadrat eines Streckenverhältnisses auf parallelen (oder zusammenfallenden) Geraden ist augenscheinlich eine absolute Invariante für die Gruppe der allgemeinen Affinitäten. Die abgeleiteten Eigenschaften sind offenbar gültig für jeden Mittelpunktskegelschnitt.

3. Ist $f=0$ eine reelle Ellipse, die von der Geraden $(v'x)=0$ in zwei reellen Punkten geschnitten wird, so zerteilt die Gerade die Ellipse in zwei Segmente. WEITZENBÖCK hat gezeigt, wie sich die Flächeninhalte F' und F'' in den absoluten affinen Invarianten ausdrücken lassen. Die Berechnung kann man vereinfachen, indem man die Figur auf ein geeignet gewähltes Koordinatensystem bezieht.

Wird der Ursprung im Mittelpunkt der Ellipse genommen, die x_1-Achse mit der Geraden $(v'x)=0$ parallel und ist die x_2-Achse im Bezug auf die Ellipse mit der x_1-Achse konjugiert, so erhalten die Ellipse und die Gerade, nachdem man noch eine geeignete speziell-affine Transformation ausführt, die Gleichungen:

$$x_1^2+x_2^2-R^2x_3^2=0 \quad , \quad x_2-p\,x_3=0. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Man hat dann:

$$D=-R^2 \;,\; C=1 \;,\; \Phi=-(R^2-p^2) \;,\; M=-p \;,\; U=1$$

und also:

$$k=R^4 \;,\; l=\frac{p^2-R^2}{p^2} \;,\; l'=\frac{R^2-p^2}{R^2}$$

womit die angegebene Deutung der Invariante l' bestätigt wird. Führte man eine euklidische Massbestimmung ein, wobei die isotropen Punkte in den uneigentlichen Punkten der Ellipse gewählt werden, so hätte man

$$l'=\sin^2\alpha \;,\; l=-\operatorname{tg}^2\alpha$$

wo 2α einer der Bogen PQ der nun zum Kreis gewordenen Ellipse ist. Die genannten Flächeninhalte kann man jetzt elementargeometrisch bestimmen. Für das (allgemein affin-invariante) Verhältnis dieser Zahlen findet man:

$$\left.\begin{aligned} F' : F'' &= \left(\frac{\pi}{2} - \operatorname{arc\,cotg} \sqrt{-l} - \frac{\sqrt{-l}}{l-1}\right) : \left(\frac{\pi}{2} + \operatorname{arc\,cotg} \sqrt{-l} + \frac{\sqrt{-l}}{l-1}\right) \\ &= \left(\frac{\pi}{2} - \arccos \sqrt{l'} - \sqrt{l'(1-l')}\right) : \left(\frac{\pi}{2} + \arccos \sqrt{l'} + \sqrt{l'(1-l')}\right) \end{aligned}\right\} \quad (23)$$

während

$$F' + F'' = \pi \sqrt{k} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

Mittels (5) und (8) werden die WEITZENBÖCKschen Resultate zurückgewonnen[4]. Auch die für die Parabel und die Hyperbel gegebenen Formeln können mit der Methode des kanonischen Koordinatensystems abgeleitet werden.

4. Die oben gewonnenen Eigenschaften kann man auf den Raum von n Dimensionen übertragen. In R_n haben eine quadratische Form $f = \sum\limits_{i,k=1}^{n+1} a_{ik}\, x_i\, x_k$ und eine Linearform $(v'x) = \sum\limits_{i=1}^{n+1} v'_i\, x_i$ folgende fünf Komitanten:

$$\left.\begin{aligned} D &= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1,n+1} \\ a_{21} & a_{22} & \dots & a_{2,n+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,n+1} \end{vmatrix}, \quad C = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \\ \Phi &= -\begin{vmatrix} a_{11} & \dots & a_{1,n+1} & v'_1 \\ \cdot & \cdot & \cdot & v'_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1,1} & \dots & a_{n+1,n+1} & v'_{n+1} \\ v'_1\, v'_2 & \dots & v'_{n+1} & 0 \end{vmatrix}, \quad M = -\begin{vmatrix} a_{11} & \dots & a_{1,n+1} & v'_1 \\ \cdot & \cdot & \cdot & v'_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1,1} & \dots & a_{n+1,n+1} & v'_{n+1} \\ 0\;\; 0 & & -1 & 0 \end{vmatrix} \\ U &= -\begin{vmatrix} a_{11} & \dots & a_{1n} & v'_1 \\ \cdot & \cdot & \cdot & v'_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \dots & a_{nn} & v'_n \\ v'_1 & \dots & v'_n & 0 \end{vmatrix} \end{aligned}\right\} \quad (25)$$

welche der Bedingung $C\Phi = M^2 + DU$ genügen.

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[4] S. 166 steht im letzten Glied von (32) $\left|\begin{matrix} R \\ S \end{matrix}\right.$ statt $\left|\begin{matrix} S \\ R \end{matrix}\right.$.

Als absolute Invarianten kann man wählen

$$k_n = D^n \, C^{-(n+1)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

und $l_n = \frac{C\Phi}{M^2}$ oder

$$l_n' = \frac{l_n}{l_n - 1} = \frac{C\Phi}{DU} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

k_n ist dabei wieder eine Invariante nur von f; l_n bzw. l_n' sind Invarianten für die allgemeine affine Gruppe. k_n hängt für ein Hyperellipsoid in einfacher Weise mit dem Volumen F_n zusammen. Wie man leicht nachgeht hat man [5])

$$\left.\begin{aligned} k_{2m} &= (m\,!)^2 \, \pi^{-2m} \, F_{2m}^2 \\ k_{2m+1} &= -1^2 \, . \, 3^2 \, . \, 5^2 \, . \, . \, . \, . \, (2m+1)^2 \, . \, 4^{-m-1} \, \pi^{-2m} \, F_{2m+1}^2 \end{aligned}\right\} \quad . \quad (28)$$

Die Invariante l' hat eine geometrische Bedeutung, welche mit der für den Fall $n = 2$ gefundenen analog ist. Ist $f = 0$ ein Hyperellipsoid, aus welchem $(v'x) = 0$ die quadratische Varietät $V((n-2)$-ter Dimension) und der mit $(v'x) = 0$ paralleler Raum durch den Mittelpunkt die quadratische Varietät V_1 ausschneidet, so sind V und V_1 homothetische Figuren; die Homothetie ist bekanntlich ein für die allgemeine affine Gruppe invarianter Begriff. Die Homothetie-Zentren liegen auf demjenigen Durchmesser des Ellipsoids, welcher mit $(v'x) = 0$ konjugiert ist. Sind g bzw. $-g$ die Faktoren derjenigen Homothetien, welche V_1 in V überführen, so ist

$$l' = g^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Der Beweis dieser Behauptung kann man, wenn man auf die Benutzung eines bestimmten Koordinatensystems verzichten will, in einer Weise führen, welche mit der von uns für $n = 2$ gegeben wesentlich übereinstimmt. Wählt man ein geeignetes Koordinatensystem, so hat man für f und $(v'x)$ bzw. folgende Formen:

$$x_1^2 + x_2^2 + \ldots . \, x_n^2 - R^2 \, x_{n+1}^2 \quad \text{und} \quad x_n - p \, x_{n+1}$$

woraus hervorgeht

$$D = -R^2, \, C = 1, \, \Phi = p^2 - R^2, \, M = p, \, U = 1, \, k = (-1)^n \, R^{2n}, \, l' = \frac{R^2 - p^2}{R^2}$$

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[5]) Für $n = 3$ BLASCHKE, l.c. S. 112.

Die Volumen F'_n und F''_n der beiden Teile, worin das Ellipsoid von dem Raum $(v'x)=0$ zerteilt wird, können mittels bekannter Formeln für das Volumen eines Hyperkugelsegmentes[6]) niedergeschrieben werden.

Für $n=3$ hat man im Besonderen:

$$\left.\begin{aligned} F'_3 : F''_3 &= \{2-(2+l')\sqrt{1-l'}\} : \{2+(2+l')\sqrt{1-l'}\} \\ F'_3 + F''_3 &= \tfrac{4}{3}\,\pi\sqrt{-k} \end{aligned}\right\} \qquad (30)$$

welche Resultate mit den WEITZENBÖCKschen (S. 177, (25) und (26)) nach einer Substitution übereinstimmen.

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[6]) SCHOUTE, Mehrdimensionale Geometrie II, S. 290 (1905).

Chemistry. — *On geometric isomerism in Luteo-Cobaltic-salts.* By I. LIFSCHITZ and K. M. DIJKEMA. (Communicated by Prof. F. M. JAEGER.)

(Communicated at the meeting of June 29, 1940.)

§ 1. According to the classical co-ordination theory, it might be expected, that complex compounds of trivalent metals, such as, for instance: **I.** $[Met.^{III}A_3]$; **II.** $[Met.A\text{-}B\text{-}X_2]$; **III.** $[Met.A_2X_2]$, etc., in which A and B are co-ordinatively-bivalent asymmetrical substituents, X being a co-ordinatively-monovalent substituent, — wil occur in geometrically isomeric forms. *Luteo*-salts I, for instance, of the ion: $[Met.^{III}(pn)_3]^{\cdots}$ and *triacidotriammino*-salts *I*, for instance: $[Met.^{III}(glyc)_3]$, would be obtainable in *two*, some *diacidotetrammino*-salts *III*, for instance, of the ion: *flavo*-$[Met.^{III}(pn)_2\ (NO_2)_2]^{\cdot}$, even in *three* isomeric forms. Experimentally such isomerism has been observed only in cobaltic compounds and even there only in a few cases. In the first place geometrically isomeric forms were found in complex cobaltic compounds with 3 mol. α-amino-acid [1]. A. WERNER could prepare the isomeric forms of *flavo*-$[Co\,(en)(pn)(NO_2)]_2X$. [2]. But even *flavo*-$[Co\,(pn)_2(NO_2)_2]X$, of which three isomerides might theoretically be expected, appears only to occur in *one* single form [3].

The reasons why the geometric isomerism predicted by WERNER is so rarely observed, will not be discussed here. In the first place it must experimentally be investigated, whether a greater number of geometrically isomeric salts *can* be obtained.

Apart from this, however, the preparation of such isomerides in an optically-active form would be of interest with respect to the study of rotatory power and rotatory dispersion in general.

Since geometrically isomeric substances differ almost exclusively with respect to the *degree of symmetry* of their molecules, an investigation of

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[1] C.f. I. LIFSCHITZ, Proc. Kon. Akad. v. Wetensch., Amsterdam **27**, 721 (1924); **39**, 1192 (1936); **42**, 173 (1939).

[2] A. WERNER, Helv. Acta I, 5 (1918).

[3] The statements of MIKLOSICH, who thought that he had isolated the 3 forms expected, are incorrect. MIKLOSICH in his experiments started from *commercial* propylenediamine, which nearly always contains ethylenediamine, as was proved by the experiments of one of us (LI). As a matter of fact., H. E. WATTS (Diss. Zürich 1912) and H. HÜRLIMANN (Diss. Zürich 1918) failed to obtain such isomerides in using *pure* propylenediamine.

their rotatory power may yield important data concerning the influence of differences in symmetry on rotatory power [1]).

§ 2. In an investigation of the complex-formation of *phenylated ethylenediamines*, it was found that the *luteo*-cobaltic-salts of *monophenyl-ethylenediamine* can rather easily be obtained in geometrically isomeric forms. This is the first case of geometric isomerism in the *luteo*-series having been demonstrated and of the polarimetric investigation of such isomers having become possible.

Phenyl-ethylenediamine: $C_6H_5CH(NH_2)—CH_2(NH_2)$ (*phenen*) was obtained in the racemic and in the optically-active forms by catalytic hydrogenation of *α-acetaminophenyl-acetic-acid-nitril*:

$$C_6H_5CH\,.\,(NHCOCH_3)\,.\,CN$$

according to H. REIHLEN's method [2]). In preparing the *luteo*-salts, the following prescription may be followed as well in the case of the racemic as of the optically-active bases.

20,4 g. phenyl-ethylenediamine and 4,2 cc HCl (s.g. 1.19) are added to a solution of 11.9 g $CoCl_2\,.\,6H_2O$ in 350—400 cc methylalcohol and a current of air is blown through it by means of a gasdispersion-tube. Very soon a yellow salt crystallizes, whilst the solution becomes dark brown. After ca. 20 hours the solid salt (L_1) is sucked off, boiled three times with methyl-alcohol (each time with ca. 100 cc.) and, after complete cooling, sucked off.

The salt thus obtained is then fractionated by crystallisation from water, whilst some cc. of HCl (1,4) are added to the filtered and slightly cooled solution [3]).

Besides some luteo-salt L_1 the deep-brown methyl-alcoholic mother liquor of L_1 contains a deep-brown, occasionally somewhat greyish-coloured salt, which, after evaporation, is left as a syrupy substance, which on cooling solidifies into a crystalline mass. It probably consists of a polynuclear salt, which, however, even by re-crystallization from water [4]), could not be obtained in a pure state.

When this salt is dissolved in hot water, to which some cc hydrochloric acid (1 : 4) are added, and the solution is filtered and cooled, the colour changes to a yellowish-brown and a yellowish-brown salt (L_2) crystallizes from it; the latter can be purified by fractionated crystallisation from very dilute *HCl* in the way previously described.

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[1]) I. LIFSCHITZ, Rec. Trav. Chim. Pays-Bas **58**, 785 (1939); more extensive literature is given there.

[2]) H. REIHLEN, Ann. **493**, 20 (1932); **494**, 157 (1932).

[3]) L_1 is considerably less soluble in dilute HCl than in pure water; moreover, in this way a brown colouring matter is removed (see following pages).

[4]) After re-crystallization from water, one finds, for instance, $Cl = 14{,}80\,\%$; $14{,}75\,\%$; N (micro) $= 13{,}01\,\%$; $13{,}06\,\%$; $H_2O = 6{,}9\,\%$ so that $N : Cl = 2{,}24 : 1$.

This salt appears to be the *luteo*-salt L_2, which is geometrically isomeric with L_1. The dry solid salts differ but little in colour, — L_2 having a little darker yellow hue. Analysis yielded the following values:

	Co:	Cl:	N:	H_2O:
L_1	9.72 %; 9.78 %	17.51 %; 17.56 %	13.77 %; 13.67 %	5.0 %
L_2	9.39 %; 9.36 %	16.89 %; 16.83 %	13.44 %; 13.31 %	8.6 %; 9.2 %
calculated for $[Co(phenen)_3]Cl_3 . 3\,H_2O$:	9.39	16.98	13.39	8.6 %
calculated for $[Co(phenen)_3Cl_3] . 2\,H_2O$:	9.67	17.48	13.78	5.7 %.

Thus L_1 appears to be a *di*hydrate, L_2 a *tri*hydrate: A determination of the molecular weight of the dried salts in phenol, yielded the following values:

	gr. salt	gr. phenol	Δt:	M (found):	
L_1	0.293	17.25	0°.210	591	
L_1	0.290	16.23	0°.222	596	M (calculated) = 573
L_2	0.391	16.07	0°.290	607	

Both salts, therefore, are *mononuclear* ones and have *the same* molecular weight. The difference of colour of the solid salts is also noticeable in aqueous solutions: solutions of the same concentration by means of a PULFRICH-photometer proved to show an absorption of different extent. In methyl-alcohol L_1 dissolves with difficulty, L_2 rather easily.

The saturated aqueous solution of L_1 at 25° contains 1.16 gr. that of L_2 2.47 gr. of salt in 100 cc of water. In hot, boiling water both salts readily dissolve, L_2 melting under water to a brown liquid mass, whereas L_1 remains solid, until complete solution has occurred.

On heating in a drying room, the two salts become green, L_2 at ca. 140°, L_1 at 150°, under partial conversion into *praseo*-salts.

The data concerning the rotation-curves of the two isomerides are to be found in Fig. 1, (curves *I* and *II*). It is noteworthy that, — in contradistinction to the case of *l*-stilbenediamine, — *l*-phenylethylenediamine yields *levo*-rotating *luteo*- and *praseo*-salts. The rotation-curves of the two isomerides show the same qualitative course, but they strongly differ in regard to the *absolute* magnitude of their rotations. Both isomerides possess a considerably higher maximal rotatory power than $[Co(l\text{-}stien)_3]Cl$. As the curve shows, the *l*-base gives a *positive* contribution to the rotation [1]); the partial racemic complexes: $[Co(rac.\ phenen)_3]X_3$ therefore, ought to possess a much higher rotatory power. The polarimetric investigation of the geometrically isomeric *luteo*-salts fully confirms the rules [2]) previously stated by one of the authors (LI) with other geometrically isomeric series:

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[1]) In the red part of the spectrum, where the luteo-complexes practically have *no* rotation, the said contribution is very evident, see curve II.

[2]) I. LIFSCHITZ, l.c. Proc. Ned. Akad. v. Wetensch. Amsterdam, l.c.

1. Geometrically isomeric substances show a rotation which is *qualitatively quite analogous*, but *quantitatively very different*.

2. A *lower* degree of symmetry of the molecule causes a *higher* rotation: $[Co(l\text{-}stien)_3]X_3$ rotates much less than $[Co(l\text{-}phenen)_3]X_3$ and practically as much as $[Co(en)_3]\ X_3$ ¹).

3. From this, the working hypothesis obtains fresh support ²), that of two geometric isomerides the *less symmetrical* one shows the *higher* rotation.

§ 3. Besides *luteo*-salts, we also prepared the *praseo*-compounds; theoretically these might also occur in *two* isomeric forms, namely in a *cis*- and a *trans*-form. We, indeed, got the impression that *two* praseo-chlorides $[Co(phenen)_2Cl_2]Cl$ are formed; but they were only little stable and, therefore, they could up till now not be isolated in the pure state with complete certainty.

For their preparation the following prescription is recommended:

7.8 gr. of $CoCl_2 \,.\, 6\,H_2O$ are dissolved in 300 cc alcohol, 8.1 gr. base and then 10 cc HCl (s.g. 1.19) are added and the mixture then is oxydized with a current of air during about 20 hours. A green salt soon begins to crystallize, the solution assuming a blackish-green colour.

The crystallized *praseo*-salt (P_1) is sucked off and the mother liquor strongly concentrated; on cooling, a considerably darker salt (P_2) then begins to crystallize. Both salts, after recrystallizing them from alcoholic hydrochloric acid and drying at 130°, are, however, still impure. A pure *praseo-nitrate* could finally be obtained by mixing a solution of P_1 in methyl-alcohol with an aqueous solution of KNO_3. This salt, which dissolves in water with great difficulty and slightly better in alcohol, gave the following values:

$[Co(phenen)_2Cl_2]NO_3$ Calc.: $Cl = 15{,}31\ \%$; $N = 15{,}09\ \%$; found: $Cl = 15{,}44\ \%$; $15{,}54\ \%$; $15{,}26\ \%$; $N = 14{,}83\ \%$; $14{,}88\ \%$. The values for nitrogen, which are a little too low, are caused by the presence of Cl in the molecule ³). The data concerning the rotation-curve of the *praseo*-salt, which shows a course perfectly analogous to that of the curves of other *prasea*-compounds ⁴), is given in Fig. 1 by curve *III*. By treating the *praseo-chlorides* with potassium- or silver-oxalate, an *oxalo-salt*: $[Co(phenen)_2C_2O_4)]Cl$ finally also could be obtained. The salt crystallizes from water in red little prisms, containing 2 molecules of water: Calc.: $N = 11{,}42\ \%$; $Cl = 7{,}2\ \%$; found: $N = 11{,}33\ \%$ and $11{,}40\ \%$; Cl 7,2 and 7,0 %. Curve *IV* in Fig. 1 graphically represents the rotation of this salt. It can be remarked, that the *oxalo*-salt:

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¹) I. LIFSCHITZ and J. G. BOS; this paper will soon appear in Rec. trav. chim. d. Pays-Bas, **59**, (1940).

²) I. LIFSCHITZ, Rec. trav. chim. d. Pays-Bas, **58**, 785 (1939).

³) I. LIFSCHITZ and J. G. BOS, Rec. trav. chim. **58**, 795 (1939).

⁴) F. M. JAEGER and H. BLUMENDAL, Z. anorg. allg. chem. **175**, 191 (1928).

$[Co(phenen)_2C_2O_4]Cl$ may occur in *three* geometrically isomeric forms. An isomeric salt, however, was in our experiments only found once, in a minimal quantity. Because of the lack of material, for the present we have abstained from more extensive investigations. The determination of N of this salt yielded: 11,40 and 11,50 % N; calculated for the compound: $[Co(phenen)_2C_2O_4]Cl \cdot 2\,H_2O$: 11,42 %.

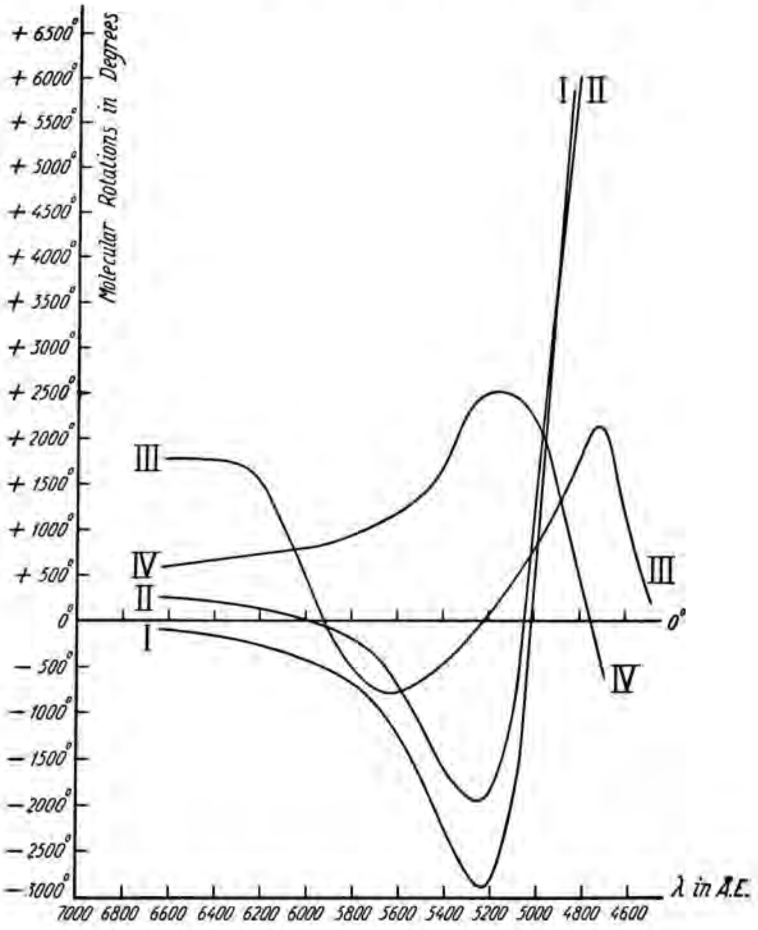

Fig. 1. Rotatory Dispersion of:

I. Luteo-salt L_1 in water, (c = 0,2252 gr. in 100 cc).
II. Luteo-salt L_2 in water, (c = 0,2244 gr. in 100 cc).
III. Praseo-salt: $[Co(phenen)_2Cl_2]\,NO_3$ in alcohol + 5 % conc. HCl (c = 0,1300 gr. in 100 cc).
IV. Oxalo-salt: $[Co(phenen)_2C_2O_4]Cl$ in water (c = 0,1180 gr. in 100 cc).

Groningen, Laboratory for Inorganic and
Physical Chemistry of the University.

Plantkunde. — *Snelle bloei van Iris Wedgwood.* (with summary). (Mededeeling N⁰. 64 van het Laboratorium voor Plantenphysiologisch Onderzoek te Wageningen). Door ANNIE M. HARTSEMA en IDA LUYTEN. (Communicated by Prof. A. H. BLAAUW).

(Communicated at the meeting of June 29, 1940.)

Nadat wij in het voorjaar van 1934 voor het eerst bol-irissen hadden vervroegd, n.l. de uit Marokko in Zuid-Frankrijk geïmporteerde *Iris tingitana* (Med. N⁰. 40, 1934), werden in den zomer van 1934 ook proeven ingezet over het vervroegen van den bloei van enkele Hollandsche Irissen, o.a. de variëteiten Imperator en Wedgwood. Over de resultaten van onze proeven met Iris Imperator verschenen reeds 2 publicaties, Med. N⁰. 48, 1936 en Med. N⁰. 57, 1939. Thans zullen wij de proeven met Iris Wedgwood beschrijven.

Zooals in Med. N⁰. 62, 1939, werd vermeld, ontstond de variëteit Wedgwood door kruisingen met of van de Marokkaansche *Iris tingitana*. In 1934/35 werden voor 't eerst proeven genomen om den bloei van Wedgwood te vervroegen o.a. met de behandeling die wij voor *Iris tingitana* hebben beschreven, n.l. 3 weken 28° gevolgd door 9°. In geen van deze proeven werd echter bloei bereikt.

In 1935/36 werd daarom met Wedgwood een groot aantal proeven genomen om te kunnen vaststellen, in hoeverre andere temperatuurbehandelingen betere resultaten zouden geven. De bollen van Wedgwood van de maat 10—11 cm (omtrek) waren op 7 Augustus 1935 ontvangen. Voor iedere proef werden 2 × 10 stuks gebruikt, bestemd voor 2 kistjes. De bollen werden uitgezocht op een gewicht van 22—33 gram, 10 stuks wogen 260 gram. Ten deele werden de bollen direct geplant in 5°, 7°, 9° of 13° C.; de andere groepen van 20 stuks werden eerst gedurende 1, 3 of 5 weken bij een hoogere temperatuur gelegd, en daarna eveneens bij 5°, 7°, 9° of 13° C. geplant. Als hoogere vóórtemperatuur was, evenals bij de proeven met Imperator 23°, 28° en 31° C. gekozen. De zin van deze warme *vóórtemperatuur* is in Med. 48 p. 605 uiteengezet. De lage temperaturen (5°, 7°, 9° en 13°) waarbij geplant werd, noemden wij *prepareer-temperaturen*. Pas nadat de planten een spruit van een zekere lengte ontwikkeld hebben, komen ze in de *trektemperatuur*, n.l. in een matig-warme kas van 15° C., waarin de strekking van de spruiten verder voortgezet wordt en tenslotte de bloemen zich ontplooien.

Nu bleek achterna, dat niet in alle bollen een bloem aangelegd was en dat 't al of niet aanleggen van de bloem samenhing met de toegepaste temperatuur-behandeling. In tabel 1 wordt aangegeven hoeveel bloemen per groep van 20 bollen gevormd waren.

TABEL 1. 1935/36.

vóórtemperatuur:		—	23°			28°			31°
gedurende		—	1 wk.	3 wk.	5 wk.	1 wk.	3 wk.	5 wk.	1 wk.
prepareer-temperatuur	5°	0	5	15	19	3	16	19	5
	7°	0	8	17	18	11	18	19	9
	9°	1	15	20	20	13	19	20	14
	13°	17	18	19	20	20	20	20	20

Uit deze tabel blijkt wel dat voor Wedgwood een voorbehandeling met een hoogere temperatuur noodzakelijk is voor het slagen van den bloemaanleg, daar zonder deze voorbehandeling (zie 1e kolom) bij geen der gebruikte prepareer-temperaturen alle 20 bloemen aangelegd werden. Bovendien blijkt, dat de inwerking van deze hoogere temperatuur gedurende 1 week in het algemeen te kort is, omdat alleen in 13° na 1 week 28° of 1 week 31° alle bloemen gevormd werden. Naarmate de voorbehandeling bij 23° of 28° langer geduurd heeft, zien wij dat meer bloemen aangelegd worden; 31° werd in dit jaar niet langer dan 1 week toegepast. Maar zelfs na 5 weken 23° of 28° werden in 5° en 7° nog niet alle bloemen aangelegd. Dit was uitsluitend het geval in 9° en 13°.

Het was niet te verwachten, dat de aangelegde bloemen in deze eerste proeven ook alle tot bloei zouden komen. Evenals bij de eerste proeven met Imperator, werden de kistjes bij het bereiken van een neuslengte van tenminste 6 cm (dwz. dat alle of bijna alle spruiten 6 cm of meer buiten den bol steken) uit de prepareertemperatuur overgebracht naar een kas van 15°. In tabel 2 geven wij alleen de resultaten van die proeven, waarbij alle bloemen aangelegd waren. Ook in enkele andere proeven ontplooiden zich bloemen, maar het heeft geen zin, deze hier te vermelden. Zouden wij alle

TABEL 2. 1935/36.

behandeling	6 cm bereikt	aantal dagen na planten	1e bloem open	aantal dagen na 6 cm	aantal dagen na planten	totaal aantal bloemen
3 w. 23° — 9°	31 Oct.	65	7 Jan.	68	133	3
5 w. 23° — 9°	6 Nov.	57	—	—	—	0
5 w. 28° — 9°	11 Nov.	62	14 Jan.	64	126	9
5 w. 23° —13°	21 Oct.	41	10 Jan.	81	122	14
1 w. 28° —13°	1 Oct.	49	18 Dec.	78	127	4
3 w. 28° —13°	9 Oct.	43	26 Dec.	78	121	4
5 w. 28° —13°	21 Oct.	41	3 Jan.	74	115	8
1 w. 31° —13°	30 Sept.	49	—	—	—	0

toegepaste behandelingen weergeven, dan zou daaruit blijken dat er evenals bij Imperator (vergelijk tabel 1 en 2 van Med. 48) een verband bestaat tusschen de gebruikte prepareer-temperaturen en de tijden die noodig zijn om na het planten 6 cm neuslengte te bereiken en evenzeer tusschen deze temperaturen en de tijden, die na het bereiken van 6 cm neuslengte tot het begin van den bloei verloopen.

Ook uit bovenstaande tabel 2 blijkt, dat na planten in 13° eerder 6 cm bereikt wordt dan na planten in 9°, terwijl omgekeerd de tijd tusschen het bereiken van 6 cm en het begin van den bloei bij 9° korter is dan bij 13°. In tegenstelling met onze ervaringen bij Imperator is bij Wedgwood de som van de genoemde tijden na planten bij 9° *niet korter* dan na planten in 13°. Voor snellen bloei van Wedgwood is 13° dan ook te verkiezen boven 9°. Overigens zijn de bloei-resultaten van de hier genoemde proeven nog ongunstig. Slechts in één geval werd een behoorlijk bloei-percentage, nl. 70 % bereikt, en wel na een voorbehandeling van 5 weken 23°, gevolgd door planten in 13°. Opvallend is dat na 5 weken 23° bij planten in 9° geen enkele bloem opengegaan is. In het algemeen blijkt uit tabel 2 wel, dat ook voor den bloei een langer-durende voorbehandeling te verkiezen is boven korter-durende. Men verliest daardoor echter zooveel tijd, dat de bloei van Wedgwood niet veel eerder begint dan die van Imperator (vergelijk Med. 48 blz. 608, waar 80 % bloei bereikt kon worden na een behandeling met 1 week 31°, gevolgd door planten bij 9°; begin van den bloei reeds op 14 Januari.)

Daarom moest gezocht worden naar methoden om beteren en zoo mogelijk ook vroegeren bloei te verkrijgen. In de eerste plaats werd op aanwijzing van kweekers niet direct na de warme voorbehandeling in de lage prepareertemperatuur geplant, doch pas 6 weken later. Gedurende deze 6 weken werden de bollen droog bewaard bij de prepareertemperatuur. Naar wij vernamen had men daarmee in de practijk goede resultaten bereikt. Ook GRIFFITHS (1936) vermeldt, dat men vroegen bloei van Wedgwood kan verkrijgen door de bollen na een voorbehandeling met 80° F. (26½° C.) gedurende de maand Juli (de bollen werden reeds begin Juli gerooid) minstens 4 weken bij 50° F. (9° C.) te bewaren.

In de tweede plaats werden, uit dezelfde overwegingen als wij in Med. 57 reeds vermeldden, de kistjes niet direct bij het bereiken van tenminste 6 cm neuslengte overgebracht naar de kas van 15°. Het tijdstip van dit overbrengen bij 6 cm was immers willekeurig gekozen en uit andere onderzoekingen bij Iris Imperator was gebleken, dat de bloemen pas vrij laat en meestal 't snelst bij lage temperatuur (7 à 9° C.) gevormd worden. Het leek daarom gunstiger de bloemen eerst grootendeels te laten afmaken in de lage prepareertemperatuur en het tijdstip van overbrengen niet te laten bepalen door de lengte van de spruit, doch door het stadium van de bloem.

In 1936/37 werd als prepareer-temperatuur uitsluitend 9° C. gekozen, terwijl de bollen voorbehandeld werden met 31° C. gedurende 1 week, overeenstemmend met de voor snellen bloei van Imperator vastgestelde behandeling. Verder werden dus de bollen 6 weken droog bewaard bij 9° en daarna geplant bij dezelfde temperatuur. Na het bereiken van minstens 6 cm neuslengte werden de kistjes nog gedurende 2 of 4 weken bij 9° gelaten en wel in een koude kas, waar de spruiten dus in het licht stonden. Na deze 2 of 4 weken werd van iedere groep een kistje gefixeerd en werden 2 kistjes naar kas 15° C. overgebracht. Zooals te verwachten was, waren ook ditmaal niet in alle bollen bloemen aangelegd, van 30 gefixeerde bollen hadden 5 geen bloem gevormd. Uit de fixaties bleek dat bij het bereiken van 6 cm de bloemen reeds in stadium IV tot V verkeerden, 2 weken later was stadium V tot VI bereikt, en 4 weken later waren alle bloemen geheel klaar. Daar bij geen dezer proeven 50 % bloei werd bereikt, zullen wij de resultaten hier niet vermelden.

In 1937/38 werd overgegaan tot langere voorbehandelingen en wel 3 weken 31° of 5 weken 28°. Als prepareer-temperatuur werd behalve 9° ook 11° gebruikt; ook nu bleven de bollen na de voorbehandeling nog 6 weken droog bewaard bij de prepareer-temperatuur voordat ze geplant werden. De bollen waren ontvangen op 13 Augustus 1937 en ze werden uitgezocht tusschen 22 en 26 gram (gewicht per 10 stuks 233 gram). Voor iedere proef werden ditmaal 3 kistjes van 10 stuks gebruikt. Voorzichtigheidshalve werd ook nu niet direct bij het bereiken van 6 cm neuslengte overgebracht naar een kas van 15°, doch eerst nog 3 weken naar een kas van 9° resp. 11° C. De resultaten vindt men in tabel 3 vermeld.

TABEL 3. 1937/38.

voorbehandeling	ongeplant bewaard	6 cm bereikt (in 9° of 11°)	over n. kas 15°	1e bloem open	totaal aantal bloemen
3 w. 31°	6 w. 9°	15 Nov.	6 Dec.	21 Jan.	21 : 30
3 w. 31°	6 w. 11°	8 Nov.	29 Nov.	11 Jan.	17 : 30
5 w. 28°	6 w. 9°	25 Nov.	16 Dec.	30 Jan.	17 : 30
5 w. 28°	6 w. 11°	18 Nov.	9 Dec.	22 Jan.	16 : 30

Ook in dit geval begon de bloei vrij laat; het beste resultaat, 70 % bloei, werd bereikt bij de eerste groep: 3 w. 31 ° en vervolgens 9°. De 2e groep: 3 w. 31° gevolgd door 11° begon echter 10 dagen eerder te bloeien. Bij deze proeven bleken alle bloemen te zijn aangelegd. Dit is weer een bewijs hoe noodzakelijk de voorwarmte is. Verder werd ook hier van iedere groep een kistje gefixeerd op het oogenblik van overbrengen naar kas 15°; vastgesteld kon worden dat de bloemen toen reeds geheel gereed waren. Wij komen hierop nog nader terug.

In 1938 werden al onze Iris-proeven in plaats van in het tot nu toe gebruikte zuivere duinzand uitgeplant in voedzamen grond, samengesteld uit $^3/_5$ bladgrond, $^1/_5$ verteerde koemest en $^1/_5$ scherp zand, vermengd met

2 kg Thomasslakkenmeel per m^3; aan dit grondmengsel werd toegevoegd ¼ volume fijn gemalen baksteen (gravel) om het luchtgehalte te verhoogen. Aanleiding daartoe waren de resultaten van onze proeven met Iris Imperator in 1937/38 (zie Med. N⁰. 57) waarbij geconstateerd werd, dat extra voeding vóór of gedurende het trekseizoen, een zeer gunstigen invloed op het bloeipercentage had. Door een technische vergissing bij deze serie was het resultaat van den voedzamen grond in dit jaar te slecht om te beoordeelen.

Daarom moesten deze proeven in 1939/40 herhaald worden, waarbij deze fout vermeden werd. De bollen werden in den zomer van 1939 pas op 21 Augustus gerooid en konden niet eerder dan op 23 Augustus bij 31° resp. 23° geplaatst worden. Ditmaal werden als voorbehandelingen gekozen 3 weken 31° en 5 weken 23°. Ook nu werden de bollen na de warme voorbehandeling nog gedurende 6 weken droog bewaard en wel bij 9°, 11° en 13°. Ter contrôle werd één groep direct bij 9° geplant, terwijl een andere contrôle-groep niet in voedzamen grond vermengd met gravel, doch in zuiver zand, eveneens vermengd met gravel, geplant werd. De bollen werden ditmaal uitgezocht tusschen 25 en 30 gram, gewicht per 10 stuks 259 gram. Voor iedere proef werden 2 × 10 stuks gebruikt. Wij zullen nu eerst het resultaat van de contrôleproeven vermelden in vergelijking met een overeenkomstige proef van een der later te bespreken series. Uit tabel 4 blijkt, dat de bloei van de 6 weken droog bewaarde bollen enkele dagen later begon dan die van de direct-geplante (beide na dezelfde voorbehandeling). Het gebruik van voedzamen grond in plaats van zuiver duinzand gaf enkele dagen vervroeging van den bloei maar bovendien een veel krachtiger gewas. De stengels waren in de zandproeven niet alleen dunner maar ook merkbaar korter, de gemiddelde lengte was 65 cm in plaats van 70 cm.

TABEL 4. 1939/40.

behandeling	grondsoort	6 cm bereikt	1e bloem open	totaal aantal bloeiend	aantal dagen 1e tot laatste bloem
3 w. 31° — 9°	voedz. grond	13 Nov.	24 Jan.	20	9
3 w. 31° + 6 w. 9°	voedz. grond	20 Nov.	26 Jan.	20	8
3 w. 31° + 6 w. 9°	zand	21 Nov.	29 Jan.	20	7

Bij deze proeven kwamen steeds alle bloemen open, wat waarschijnlijk toegeschreven moet worden aan het zeer gunstige jaar, zooals ook uit de andere tabellen zal blijken.

Alle andere proeven werden dus geplant in voedzamen grond vermengd met gravel. De eerste serie werd bij het bereiken van 6 cm neuslengte direct overgebracht naar kas 15°. De 2e en 3e serie brachten wij na 6 cm nog gedurende 10 dagen resp. 3 weken in een kas van dezelfde temperatuur als waarbij geplant was; de 4e serie werd evenals de 3e gedurende 3 weken in een koelere kas gehouden, vervolgens in een kas van 15° geplaatst en tenslotte overgebracht naar een kas van 17° op het moment dat bij de meeste planten het 5e loofblad zichtbaar was. Wij zullen deze vier series afzonderlijk in tabellen weergeven en bespreken (tabel 5—8).

TABEL 5. 1e serie 1939/40.

voorbehandeling	ongeplant bewaard	6 cm bereikt (in 9°, 11° of 13°)	aantal dagen na 't planten	1e bloem open	aantal dagen na 6 cm	totaal aantal bloeiend	aantal dagen 1e tot laatste bloem
3 w. 31°	6 w. 9°	20 Nov.	25	16 Jan.	57	20	7
3 w. 31°	6 w. 11°	13 Nov.	18	11 Jan.	59	19 (1)	10
3 w. 31°	6 w. 13°	11 Nov.	16	10 Jan.	60	18 (2)	6
5 w. 23°	6 w. 9°	30 Nov.	22	5 Febr.	67	19	6
5 w. 23°	6 w. 11°	25 Nov.	17	29 Jan.	65	20	7
5 w. 23°	6 w. 13°	23 Nov.	15	30 Jan.	68	19 (1)	7

Zooals te verwachten was, werd 6 cm neuslengte eerder bereikt na planten in 13° dan na planten in 9° of 11°. Toch is het verschil tusschen 9° en 13° niet zoo groot als bijv. in tabel 2. Vergelijken wij de met 5 weken 23° voorbehandelde groepen, dan blijkt uit tabel 2, dat in 13° 6 cm neuslengte 16 dagen eerder bereikt werd dan in 9°, terwijl in tabel 5 dit verschil slechts 7 dagen bedraagt. Nog duidelijker was de voorsprong in tabel 2 na 5 weken 28°: in 13° 21 dagen eerder dan in 9°! Bij het in bloei komen was deze voorsprong verminderd tot 10 dagen. In tabel 5 zien we de voorsprong zelfs geheel verdwijnen, maar toch komen de in 13° en 11° geplante bollen nog eerder in bloei dan de in 9° geplante, in tegenstelling met onze ervaringen bij Imperator, waar 9° of 7° den snelsten bloei geven.

In tabel 6 worden de proeven van de 2e serie samengevat. Het aantal dagen dat na het planten verloopt totdat 6 cm neuslengte bereikt wordt, is in al deze proeven hetzelfde en wordt dus niet herhaald.

TABEL 6. 2e serie 1939/40.

voorbehandeling	ongeplant bewaard	over n. kas 15° (10 dg. na 6 cm)	1e bloem open	aantal dagen na 6 cm	totaal aantal bloeiend	aantal dagen 1e tot laatste bloem
3 w. 31°	6 w. 9°	30 Nov.	22 Jan.	63	19	7
3 w. 31°	6 w. 11°	23 Nov.	15 Jan.	63	19	6
3 w. 31°	6 w. 13°	21 Nov.	13 Jan.	63	18 (1)	6
5 w. 23°	6 w. 9°	10 Dec.	5 Febr.	67	19	6
5 w. 23°	6 w. 11°	5 Dec.	30 Jan.	66	19 (1)	6
5 w. 23°	6 w. 13°	4 Dec.	29 Jan.	67	19 (1)	11

Vergelijken we de data van het opengaan der 1e bloemen in deze tabel met die van het bereiken van 6 cm, dan blijkt dat er in beide groepen vrijwel een constant aantal dagen tusschen ligt. Dat de bloei niet gelijktijdig begint is uitsluitend te wijten aan het verschil dat reeds bestond bij het bereiken van 6 cm. Bij de met 3 weken 31° voorbehandelde groepen begon de bloei iets later dan in tabel 5, bij de andere groepen ongeveer gelijktijdig.

In tabel 7 vindt men de proeven van de 3e serie. Vooral de met 3 weken 31° voorbehandelde proeven zijn hier weer enkele dagen in bloei verlaat door het later overbrengen naar kas 15°; van de met 5 weken 23° voorbehandelde is alleen de in 9° geplante merkbaar verlaat.

TABEL 7. 3e serie 1939/40.

voor-behandeling	ongeplant bewaard	over n. kas 15° (3 wk. na 6 cm)	1e bloem open	aantal dagen na 6 cm	totaal aantal bloeiend	aantal dagen 1e tot laatste bloem
3 w. 31°	6 w. 9°	11 Dec.	26 Jan.	67	20	6
3 w. 31°	6 w. 11°	4 Dec.	18 Jan.	66	18	7
3 w. 31°	6 w. 13°	2 Dec.	14 Jan.	64	19	7
5 w. 23°	6 w. 9°	21 Dec.	10 Febr.	72	20	5
5 w. 23°	6 w. 11°	16 Dec.	1 Febr.	67	20	6
5 w. 23°	6 w. 13°	14 Dec.	1 Febr.	69	19 (1)	9

In alle drie deze proevenseries (tabel 5—7) *bloeien de met slechts 3 weken 31° voorbehandelde groepen eerder dan de met 5 weken 23° voorbehandelde.* Dit is begrijpelijk omdat de hooge voor-temperatuur uitsluitend een gunstige nawerking heeft op de bloemvorming en geen directe blad- of bloemvorming toelaat. *De vroegste bloei werd hier bereikt na de behandeling 3 weken 31° + 6 weken 13°, geplant in 13° en overgebracht naar kas 15° bij 6 cm neuslengte.* (Begin van den bloei 10 Januari, zie tabel 5). Bij deze behandeling maakt het niet veel verschil of men langer wacht met het overbrengen naar kas 15°, 3 weken later overbrengen geeft slechts 4 dagen verlating van den bloei. Bij de in 9° en 11° geplante groepen is het tijdstip van overbrengen iets meer van invloed op het begin van den bloei, hoe langer men wacht met overbrengen, hoe later de bloei begint.

Verder ziet men uit de tabellen 5—7, dat na al deze behandelingen 90—100 % bloei verkregen werd. In de kolommen, die het aantal bloemen aangeven, vindt men tusschen haakjes het aantal verdroogde bloemen vermeld. De andere niet-bloeiende bollen waren meer of minder ziek. Hoewel slechts weinig bloemen verdroogden, wat waarschijnlijk aan het zeer gunstige jaar te danken is, is het toch opvallend dat deze verdrogingen vooral bij de in 13° geplante groepen voorkomen. Verder volgt uit de tabellen 5—7, dat het voor het slagen van den bloei niet noodzakelijk is om langer dan tot 6 cm neuslengte te wachten met het overbrengen naar kas 15°. Ook de voorbehandeling blijkt geen invloed te hebben op het bloeipercentage.

Het is nu de vraag of men den bloei nog meer vervroegen kan door in de plaats van een kas van 15° een kas van 17° te gebruiken. Deze proef werd in de 4e serie genomen, doch alleen met groepen die te vergelijken zijn met die van de 3e serie (tabel 7). Het overbrengen naar kas 17° geschiedde pas bij het zichtbaar worden van het 5e loofblad. Uit tabel 8 ziet men dat dit bij de behandeling 3 weken 31° + 6 weken 13° reeds 3 dagen na het overbrengen naar kas 15° mogelijk was, terwijl er bij de andere groepen wat langer gewacht moest worden.

TABEL 8. 4e serie 1939/40.

voorbehandeling	ongeplant bewaard	over n. kas 15°	over n. kas 17°	1e bloem open	aantal dagen na 6 cm	totaal aantal bloeiend	aantal dagen 1e tot laatste bloem
3 w. 31°	6 w. 9°	11 Dec.	25 Dec.	18 Jan.	59	20	9
3 w. 31°	6 w. 11°	4 Dec.	16 Dec.	12 Jan.	60	19	6
3 w. 31°	6 w. 13°	2 Dec.	5 Dec.	4 Jan.	54	16 (4)	7
5 w. 23°	6 w. 9°	21 Dec.	9 Jan.	3 Febr.	65	19 (1)	4
5 w. 23°	6 w. 11°	16 Dec.	27 Dec.	25 Jan.	61	20	3
5 w. 23°	6 w. 13°	14 Dec.	21 Dec.	21 Jan.	59	16 (1)	4

Hoewel het overbrengen naar kas 17° pas laat geschiedde, blijkt uit de vergelijking van tabel 7 en 8, dat hierdoor nog een duidelijke vervroeging kon worden bereikt. De snelstbloeiende groep van de vorige series (tabel 5): 3 weken 31° + 6 weken 13°, begon nu nog eenige dagen eerder te bloeien, n.l. reeds op 4 Januari, maar er verdroogden daarbij 4 van de 20 bloemen, zooals uit de tusschen haakjes geplaatste cijfers blijkt. Het is te verwachten dat het aantal verdroogde bloemen zou toenemen, indien men zou trachten den bloei nog meer te vervroegen, door bijv. reeds bij het bereiken van 6 cm naar kas 17° over te brengen. Wij zullen dit nader moeten onderzoeken.

Uit de laatste kolom van de tabellen 5—8 ziet men, dat in bijna alle proeven binnen een week na het opengaan van de 1e bloem, de beide kistjes geheel in bloei stonden. Opvallend snel was dit bij de 3 laatste groepen van tabel 8, waar in 3 tot 4 dagen alle bloemen opengingen.

De foto geeft een beeld van een bloeiende groep op 16 Januari 1940, na de behandeling: 3 weken 31° + 6 weken 11° (zie tabel 8, 2e groep). De bloei begon op 12 Januari, op de foto ziet men in het kistje links alle 10 bloemen reeds open; rechts zijn nog drie bloemen in knop, terwijl 1 zieke bol geen spruit ontwikkeld heeft. De stengels zijn zeer krachtig en stevig, de lengte gemeten van den bol tot de bloem bedroeg 65—72 cm (gemiddeld 70 cm). De kleur van de bloemen was mooi lichtblauw, had volstrekt niet geleden door het verblijf in de warmere kas van 17°. Een nadeel van de

Fig. 1. *Iris Wedgwood.* 3 wk. 31° + 6 wk. 11°; geplant bij 11°, bij 6 cm naar kas 11° gedurende 3 weken, daarna kas 15° en op 16 Dec. naar kas 17°; foto 16 Januari 1940.

bloemen van Wedgwood is dat deze, zelfs in een kas van 15°, reeds 4 à 5 dagen na het opengaan beginnen te verwelken, terwijl de bloemen van Imperator onder dezelfde omstandigheden tot 10 dagen goed blijven.

Hoewel onze bollen afkomstig waren van een op mozaïek-ziekte geselecteerde partij, merkten wij op de loof- en spatha-bladen nog wel kleine mozaïek-vlekken op. In enkele groepen kwamen ook planten voor, die duidelijker mozaïek-ziekte vertoonden; de stengel van zoo'n zieke plant bleef vaak korter, de bloem kwam langzamer open en verdroogde soms geheel of gedeeltelijk. Vlekken in de bloemen kwamen bij deze zieke planten nooit voor.

Van alle hier genoemde proeven werd zoowel bij het bereiken van 6 cm als 3 weken later telkens een kistje gefixeerd, om te kunnen nagaan, hoe het met den bloemaanleg stond. Ook nu weer bleek, dat de bloemaanleg van Wedgwood eerder begint dan die van Imperator (vergelijk ook Med. N⁰. 62), want bij het bereiken van 6 cm neuslengte, was de bloem reeds grootendeels gevormd. 3 weken later bleken alle bloemen geheel af te zijn, zoowel bij de groepen, die gedurende deze 3 weken in de kassen van lagere temperatuur waren blijven staan, als bij die welke reeds bij 6 cm naar kas 15° waren overgebracht. In het laatste geval was de strekking van het loof en de jonge bloemstengel duidelijk het sterkst. In tabel 9 geven wij alleen de stadia der hoofdbloemen bij het bereiken van 6 cm.

TABEL 9. Toestand bij het bereiken van 6 cm neuslengte.

voorbehandeling	ongeplant bewaard	fixatiedatum (6 cm neuslengte)	gem. aantal loofbl.	V−	V	V+	V—VI	VI−	VI
3 w. 31°	6 w. 9°	20 Nov.	5.0	—	1	—	8	1	—
3 w. 31°	6 w. 11°	13 Nov.	5.3	—	1	—	2	7	—
3 w. 31°	6 w. 13°	11 Nov.	5.9	—	2	—	5	1	1
5 w. 23°	6 w. 9°	30 Nov.	5.0	—	1	—	7	2	—
5 w. 23°	6 w. 11°	25 Nov.	5.0	—	1	—	1	7	—
5 w. 23°	6 w. 13°	23 Nov.	5.8	1	3	1	2	2	1

Uit tabel 9 blijkt dat reeds bij 6 cm neuslengte buiten den bol, alle bloemen in stadium V of verder zijn, d.w.z. dat alleen de vruchtbladen nog gevormd moeten worden.

Verder valt het op dat de in 11° gevormde bloemen het verst ontwikkeld zijn, terwijl de in 13° gevormde juist het minst ver zijn. Dit laatste is vooral daarom merkwaardig, omdat in 13° het eerst 6 cm wordt bereikt. Dit zou er op kunnen wijzen, dat 13° wel de snelste strekking geeft, maar niet de vlugste ontwikkeling van de bloemen. Deze conclusie is echter voorbarig, want uit de fixaties blijkt, dat er in 13° gemiddeld een loofblad meer gevormd is, dan in 9° of 11° (zie 3e kolom van tabel 9). Wij zullen nog nader moeten onderzoeken, of in 13° tengevolge van het grooter aantal loofbladen de bloemvorming al dan niet later begint dan in 9° of 11°. Bij het begin der proeven op 24 Augustus, bedroeg het aantal loofbladen gemiddeld 4.1, zoodat dus in 9° en 11° gemiddeld slechts 1 loofblad gevormd werd na 24 Augustus en in 13° 2 loofbladen.

Er is dus wel een groot verschil met de bloemvorming in de trekproeven van Iris Imperator, die bij het bereiken van 6 cm nog maar pas begonnen is (zie Med. N⁰. 57, tabel 1). Ook de 2e bloem van Wedgwood was bij het bereiken van 6 cm neuslengte reeds vrij ver ontwikkeld, n.l. in stadium III tot IV; 3 weken later was in 15° deze 2e bloem geheel klaar, in de andere temperaturen in stadium V tot VI. In onze trekproeven ontplooiden deze tweede bloemen zich slechts in enkele gevallen, in totaal bij 23 van de 480 bollen. Verder kwam eenmaal een z.g. derde bloem voor in den oksel van het hoogste loofblad, de tweede bloem ontplooide zich daarbij echter niet. Bij het gefixeerde materiaal werd deze z.g. derde bloem (zie Med. N⁰. 62 blz. 810) vrij vaak aangetroffen, vooral in het materiaal, dat 3 weken na 6 cm gefixeerd was.

De bloei van de in 1939 opgezette proeven was dus zeer bevredigend, n.l. 90 tot 100 %, behalve in één proef (de snelste) van tabel 8 (80 %). Voor een deel is dit zeker te danken aan de betere voeding gedurende de ontwikkeling in de kistjes. Maar het feit dat ook de proef met zuiver duinzand (tabel 4) dit jaar 100 % bloei gaf, tegenover 70 % in 1937 (tabel 3, 1e regel), wijst er reeds op, dat 1939 een bijzonder gunstig bloei-jaar is geweest. Waaraan dit toe te schrijven is, weten wij niet. Wij hopen echter door voortzetting van deze trekproeven tenslotte te kunnen vaststellen, welke factoren hierbij een rol spelen. Wij denken vooral aan de temperatuur in den grond gedurende de laatste weken vóór het rooien. De gegevens, die wij tot nu toe daarover verzameld hebben, blijken voor conclusies nog niet voldoende.

Samenvatting der resultaten en advies voor de practijk.

1. Voor goeden en vroegen bloei van Iris Wedgwood is een *warme voorbehandeling noodzakelijk.* Men kieze daarvoor 3 weken 31° of 5 weken 23° à 28° C.

2. De *snelste bloei* wordt bereikt door *bewaren en planten in 13° na de voorbehandeling van 3 weken 31°.* Zoodra alle spruiten 6 cm buiten den bol steken, kan overgebracht worden naar een matig warme kas van 15°. Beter is het nog 3 weken te wachten, omdat daardoor de kans op verdroging minder wordt, hoewel men iets in snelheid verliest.

3. Overbrenging naar een warme kas van 17° geeft nog enkele dagen vervroeging van den bloei; dit overbrengen geschiedde in onze proeven pas bij het zichtbaar-worden van het 5e loofblad, maar het ging gepaard met een vermindering van het bloei-percentage van 90 à 100 % tot 80 %.

4. Bewaren en planten in 11° in plaats van in 13° na de voorbehandeling van 3 weken 31° geeft slechts enkele dagen verlating van den bloei.

Bewaren en planten in 9° vertraagt den bloei van Wedgwood veel sterker. De kans op verdrogen der bloemen is in 9° en 11° minder dan in 13°.

5. Na de warme voorbehandeling kunnen de bollen zonder bezwaar nog gedurende 6 weken *droog bewaard worden* bij 13° (resp. 11° of 9°). Dit vertraagt den bloei slechts weinig en geeft verschillende voordeelen voor de practijk.

6. Uitplanten van de bollen in kistjes met *voedzamen grond vermengd met gravel* (verhouding 4 : 1) gaf in onze proeven mooiere resultaten dan uitplanten in zuiver zand.

Wageningen, Mei 1940.

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SUMMARY.

Early flowering of Dutch Irises var. Wedgwood.

Iris Wedgwood, originated from crosses of the Moroccan Iris tingitana, is one of the most favourable Dutch Irises by the colour of its flower and its tendency to early flowering.

By various experiments it could first of all be stated that the flower-origination of Wedgwood is favourably influenced by a hot preliminary treatment, e.g. at 23° or 28° during 3 to 5 weeks (see table 1).

After the preliminary treatment the bulbs are planted at a lower temperature. The quickest extension was found after planting in 13° (see table 2), after this temperature also the earliest flowering occurred. In the first experiments the plants were conveyed to a greenhouse of 15° when they had reached a nose-length of 6 cm. Later the low tempe-

rature treatment was continued for a few weeks, without yielding better results (table 3). This was confirmed by experiments in 1939/40, in which the plants were conveyed to a greenhouse of 15° as soon as they were 6 cm long (table 5), after 10 days (table 6) and after 3 weeks (table 7). By fixations it could be stated that when a nose-length of 6 cm has been reached, the flower of Wedgwood is pretty well finished, except the carpels, which have not yet formed then (see table 9). Here also the flower-origination of Wedgwood proved to be earlier than that of Imperator (cf. also Comm. 57 and 62).

The method usual in practice, by which the bulbs are not at once planted after the hot preliminary treatment, but are first kept dry for a few weeks at the lower temperature, proved satisfactory also in our experiments. Flowering is not appreciably retarded by it (see table 4). Also GRIFFITHS (1936) indicates a hot preliminary treatment at 80° F. (26½° C.) during the month of July (in America the bulbs are already dug in the beginning of July), after which they are kept at 50° F. (9° C.) during August and not planted until the beginning of September.

The quickest and at the same time good bloom was in our experiments of 1939 obtained after a preliminary treatment of 3 weeks at 31°, followed by 6 weeks at 13° and then planting at 13°; when a length of 6 cm was reached the boxes were removed to a greenhouse of 15°, where flowering started on January 10, 1940 (see table 5). Flowering started sooner still with a batch which, after the same preliminary treatment, was only put in the greenhouse of 15° 3 weeks after attaining 6 cm, and 3 days later, when the 5th foliage-leaf became visible, in the greenhouse of 17°. Here flowering started already on January 4 (see table 8), but 4 of the 20 flowers shrivelled.

The very good results of the series of experiments in 1939/40 (table 5—8) are ascribed to the use of a nourishing soil, mixed with gravel. It should be borne in mind, however, that this year was a very favourable flowering-year generally.

Anatomy. — *The anatomy of the mammary gland in mice with regard to the degree of its disposition for cancer* [1]. By P. J. VAN GULIK and R. KORTEWEG. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of June 29, 1940.)

I. *Introduction.*

In 1933 LITTLE c.s. in America and KORTEWEG in the Netherlands proved that the disposition for cancer of the mammary gland in female mice is largely determined by an influence (a so-called extrachromosomal factor) on the young ones, emanating from the mother, but not from the father. (An indication in this direction which, owing to the, at that time still imperfect technique, could not be elaborated, had already been given by LATHROP and LOEB as early as 1918).

As in our breeding-colony the father also invariably stays with the young ones, only the following influences, exclusively working from the mother's side, are taken into consideration:

1. the composition of the cytoplasma of the ovum;
2. the influence emanating from the mother-animal on the young ones while staying in utero;
3. the composition of the milk sucked by the young ones.

That however also a purely genetic (chromosomal) factor makes its influence felt in determining the degree of this disposition was emphatically argued by KORTEWEG in 1936 (1).

In 1936 BITTNER demonstrated that the extrachromosomal factor might be identical with the influence emanating from the milk on the young ones (2). As soon as possible after their birth BITTNER had separated the young ones of a cancer-strain from their mothers, and had put them with suckling mice of a non-cancer-strain. Later on most of these foster-nursed mice did not get cancer in spite of all expectations. Similar experiments on a larger scale were made by BITTNER, ANDERVONT and KORTEWEG. This paper relates to these experiments and to an extensive anatomical investigation of the mice involved.

II. *Materials.*

Our experiments were made on specimens of three different inbred strains of mice and on hybrids of these strains:

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[1] From the Laboratory of the Antoni van Leeuwenhoekhuis (Netherlands Institute for Cancer Research).

1. the dilute brown strain MURRAY—LITTLE (called K-strain by us);
2. the black strain C 57 LITTLE (by us called G-strain);
3. the O 20 Leeuwenhoekhuis-strain (bred by us for 20 generations by means of brother-to-sister mating, and called by us O-strain).

We exclusively used virginal female mice (in male mice cancer of the mammary gland practically does not occur). By "cancer" we mean in this paper "cancer of the mammary gland".

In the K-strain 85 %, in the O-strain 1 % of the virginal females get cancer of the mammary gland. In the G-strain this type of cancer was only once found in several hundreds of animals.

III. *Influence of the milk and of the genetic composition on the frequency of cancer of the mammary gland.*

A. *Influence of the milk.*

Table I shows our results with mice nursed by their own mother or foster-nursed. Of K-females, nursed by their own mother, 85 % get cancer later on; if K-females are brought up by a foster-mother of the non-cancer strain G, this percentage falls to 9 %. Whilst the mother-nursed G-females practically never get cancer later on, 13 % of them get cancer when brought up on the milk of a K-strain foster-mother.

TABLE I.

Type of mice	Normal or controls			Foster-nursed		
	Number of mice	Number developing breast cancer	Tumor incidence (percent)	Number of mice	Number developing breast cancer	Tumor incidence (percent)
K	168	142	85 ±2.8	110	10	9 ± 2.7
G	194	0	0	67	9	13 ± 4.1
F_1 K ♀ × G ♂	335	232	69 ±2.5	47	2	4 ± 2.9
F_1 G ♀ × K ♂	722	11	1.5±0.4	91	42	46 ± 5.2
O_{20}	272	3	1 ±1.6	—	—	—
π_{10}	101	1	1	—	—	—
$-\pi_{10}$	95	1	1	—	—	—

Of the F_1-hybrids between these strains, whose mothers belong to the K-strain, 69 % respectively 4 % get cancer, according to their having been suckled by their own mother or by a G-foster-mother. Of the reciprocal F_1 hybrids bred by mothers belonging to the G-strain, 1½ % get cancer, and 46 % of those fostered by a K-foster-mother.

These results obtained by us, mainly corroborate BITTNERS (3) and ANDERVONTS (4), and they convincingly prove the very great influence which the milk, on which the mice are bred, exercises on the degree of their disposition for cancer of the mammary gland.

B. *Influence of the genetic composition.*

When comparing the normal K-females with the F_1-females K♀×G♂, which two types of mice had a K-mother, we see that the cancer percentage is significantly higher when also the father is a K-animal (85 % ± 2,8) than when the latter belongs to the G-non-cancer strain (69 % ± 2,5). Here the influence of the father on the disposition for cancer is proved; this influence cannot be but a purely genetic one. The percentages we found in our foster-nursed females of the same groups: 9 % ± 2,7 and 4 % ± 2,9 are not significantly different, although pointing in the same direction.

On comparing the fostered G-females with the fostered F_1-females whose mothers belonged to the G-strain, we found 13 % ± 4,1 resp. 46 % ± 5,2, a statistically significant difference, from which also the influence of the paternal, consequently purely genetic factor, appears.

In backcrosses from one strain to the other we found the following facts:

Starting from K-females and regularly crossing them and their female descendants in the succeeding generations with G-males, it appeared that of 101 consequently 11 times backcrossed females (by us called π_{10}) only one got cancer of the mammary gland. After having been backcrossed so repeatedly these mice have practically got the genetic composition of G-mice. The influence of the extrachromosomal factor on the disposition for cancer has been entirely lost in them.

In backcrossings of G-females to the K-strain, only one out of the 95, eleven times backcrossed, females (called by us -π_{10}) appeared to have got cancer of the mammary gland. Although these mice have practically obtained the genetic composition of the cancer-strain, yet the disposition for cancer of the mammary gland is very low in them.

The genetic composition for the disposition for cancer of the mammary gland is apparently of little importance if not one or more extrachromosomal factors are active too. In case of presence of these extrachromosomal factors there is no doubt about the significance of the genetic composition.

In the K-strain in which the extrachromosomal factor and the chromosomal factor exercise their influence together again and again, the disposition for cancer of the mammary gland, as a result of these factors, has remained very high for more than 80 generations.

In backcrossing G-females to the K-strain, the disposition remains low, even after 11 backcrossings (our -π_{10} animals), though genetically these animals practically belong to the K-strain. From this we may conclude that the extrachromosomal factor is not in the last instance a product of the genetic composition, but that the latter is independent of it.

In backcrossing K-females to the G-strain the disposition has, after 11 backcrossings (our π_{10} animals) become very low, even lower than in our G-mice fostered by a K-female. From this we may conclude that the extrachromosomal factor becomes inactive after a number of generations when the chromosomal factor is not present at the same time.

IV. *Architecture of the mammary gland in our mice.*

After the method of VINTEMBERGER (5), slightly modified by the authors, the second mammary gland on one side was dissected, stained in hemalum and mounted in balsam of Canada. (The corresponding second mammary gland on the other side and the inguinal mammary glands were embedded for histological examination. The authors will report on this later on). Of every dissected mammary gland an enlarged drawing was made with a projection-apparatus.

A. *Normal mice.*

Fig. 1 presents the mammary glands of a G-strain and of a K-strain mouse of 8 months old. When killed both mice were in the same stage of the di-estrus. The architecture of the mammary gland of the G-strain mouse recalls the picture of a tree in winter; the mammary gland of the K-strain mouse resembles a budding tree in spring. The primary duct in the G-strain mouse (fig. 2) merely shows widening and buds in the second half, starting from the teat, whereas the beginnings of the secondary ducts show buds nor widening hardly anywhere. In contrast with this we find in the K-strain mouse (fig. 3) similar widening and buds along the entire length of the primary duct and also at the beginning of the secondary ducts.

Fig. 1. Drawing of the projected second mammary gland of normal mice. Some ducts penetrate pads of fat, which are indicated by a dotted line. In these pads of fat the milk ducts are not dissected.

a = mammary gland of a G-strain mouse; resembles a tree in wintertime.
b = mammary gland of a K-strain mouse; resembles a tree in springtime.

P. J. VAN GULIK and R. KORTEWEG: THE ANATOMY OF THE MAMMARY GLAND IN MICE WITH REGARD TO THE DEGREE OF ITS DISPOSITION FOR CANCER.

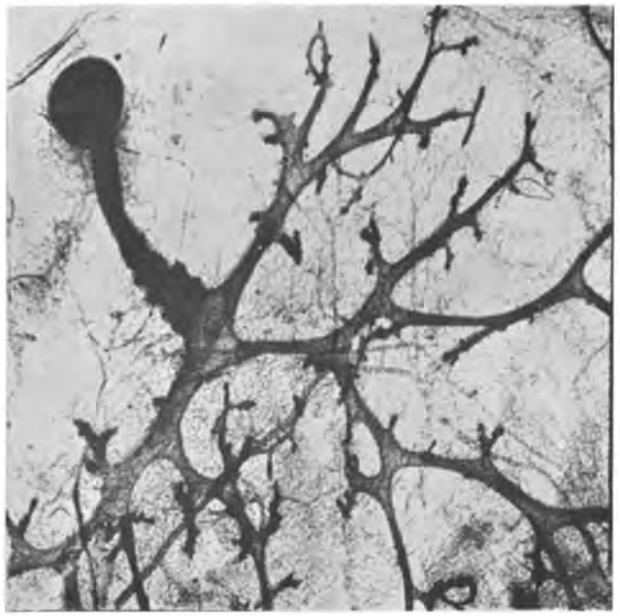

Fig. 2. Primary duct and beginning of secondary ducts of second mammary gland of a normal G-strain mouse.

Fig. 3. Primary duct and beginning of secondary ducts of second mammary gland of a normal K-strain mouse. The primary duct has buds all over its total length. Note the striking difference with fig. 2.

Fig. 5. Part of second mammary gland of a normal K-strain mouse with nodules of hyperplastic breasttissue (GARDNER).

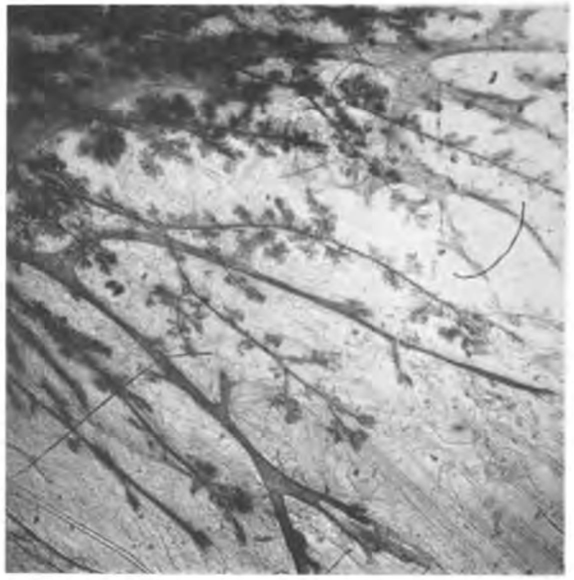

Fig. 8. Part of second mammary gland of a G-female foster-nursed by K-female. In this photo two different types of milkducts are to be seen, viz. K-milkducts inoculated on G-milkducts.

Fig. 10. Part of second mammary gland of F_1 G♀ × K♂ hybrid foster-nursed by K-female. Note the appearance of nodules in inoculated K-milkducts on G-gland-tree.

Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

Fig. 4 presents a drawing of the second mammary gland of a nine months old O-strain mouse in the di-estrus stage. The architecture of its gland-tree is between that of the G-strain and the K-strain. The primary duct is longer and shows many small buds along its entire length; the beginning of the secondary ducts is nearly devoid of similar buds.

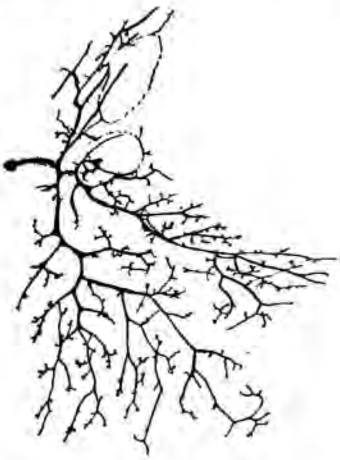

Fig. 4. Second mammary gland of a normal O-strain mouse.

In a great number of specimens of these three strains the above anatomical characteristics appeared to be constantly present. The characteristic strain-differences of the primary duct were already typically present in new-born mice.

Our study being in progress, GARDNER and his co-workers (6) published an interesting paper. They examined the second mammary gland of non-virginal mice of different strains. The authors mentioned noticed in the mammary gland of cancer-strain mice (the same strain which we examined), when the animals were getting older, nodules of hyperplastic mammary tissue. The frequency of these nodules, which in a sense may be considered a preliminary stage of cancer, increases as the mice have advanced further into the "cancer-age". On the basis of our material we can, as far as virginal mice are concerned, fully confirm GARDNERs discovery. (Fig. 5.)

GARDNER c.s. did not notice the differences in architecture of the mammary gland in the various strains; yet these differences are distinctly visible in the photos reproduced by them.

In the normal F_1-hybrids of which the mother belonged to the K-strain, the gland-tree principally represents the K-type but it also slightly resembles the G-type (Fig. 6 b). In the normal F_1-hybrids of which the mother belonged to the G-strain the gland-tree shows the G-type, slightly modified in the direction of the K-type (Fig. 6 a).

In these F_1-hybrids the primary duct represents the type characteristic of the strain to which the mother belonged [1].

B. *Foster-nursed mice.*

In the K-females fostered by G-females (Fig. 7 b), the primary duct shows the normal K-type; the gland-tree however possesses the G-aspect.

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[1] In both species of backcrossed mice (π_{10} and $-\pi_{10}$) the gland-tree represents the G-type, in π_{10} mice, started from a K-female, slightly modified in the direction of the K-type. The study of these mice is still in progress.

In the G-females fostered by K-females (Fig. 7 a) the primary duct presents the normal G-type, but in the gland-tree a very remarkable change is to be observed. On the whole the architecture of the gland-tree is of the G-type, but at the same time some ducts spring from the ducts, which, in architecture are identical with that of the K-type. It looks as if here ducts of the K-type had been inoculated on a gland-tree of the G-type (Fig. 8). In these same inoculated milkducts we found the nodules described by GARDNER.

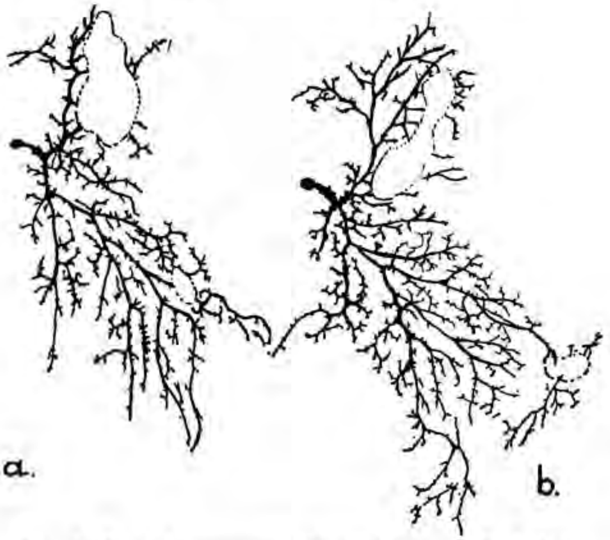

Fig. 6. Second mammary gland of normal F_1 hybrids.
a = F_1 G♀ × K♂ .: b = F_1 K♀ × G♂ .

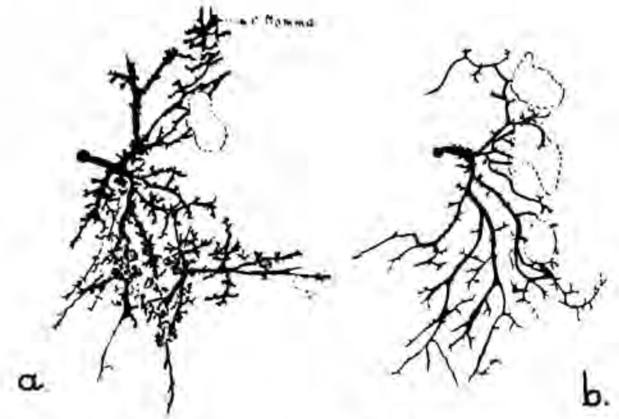

Fig. 7. Second mammary gland of foster-nursed mice.
a = G-female foster-nursed by K-female. K-branches inoculated on a G-tree. In these inoculated K-branches nodules (indicated by hatched areas).
b = K-female foster-nursed by G-female. In this gland-tree the milkducts are all alike.

In the fostered F_1-hybrids the primary duct presents the type belonging to the mother (Fig. 9). In both types of hybrids the gland-tree is mainly of the G-type, but in the G-females fostered by K-females we again find a similar "inoculating" of milkducts of the K-type on a gland-tree of the G-type (Fig. 10).

Fig. 9. Second mammary gland of foster-nursed F_1 hybrids.
a = F_1 G♀ × K♂ foster-nursed by K-female. Here again inoculating of K-milkducts on a G-gland-tree. In these grafted milkducts formation of nodules.
b = F_1 K♀ × G♂ foster-nursed by G-female. No inoculating at all.

In table II we have collected the various data.

The first column gives the types of mice, the second column the manner

TABLE II.

1	2	3	4	5	6	7	8	9
Type of mice	Nursed by:	Genetic formula	Extrachromosomal factors:			Architecture of mamm. gland		Tumor incidence
			Ovular plasma	Uterine influence	Milk-factor	Primary duct	Gland-tree	
K	Own mother	K K	K	K	K	K	K	85 %
G	” ”	G G	G	G	G	G	G	0 %
F_1 K ♀ × G ♂	” ”	K G	K	K	K	K	K → G	69 %
F_1 G ♀ × K ♂	” ”	G K	G	G	G	G	G → K	1.5 %
K	G foster-mother	K K	K	K	G	K	G	9 %
G	K-foster-mother	G G	G	G	K	G	G + K	13 %
F_1 K ♀ × G ♂	G-foster-mother	K G	K	K	G	K	G → K	4 %
F_1 G ♀ × K ♂	K-foster-mother	G K	G	G	K	G	G + K	46 %

of nursing, the third the genetic formula, in other words the chromosomal factors which may be of importance for the degree of the disposition. The extrachromosomal factors: plasma of the ovum, intra-uterine influence and kind of the milk are given in the 4th, 5th and 6th columns, K resp. G meaning that the factor relating to it, originates in a K- resp. G-mouse.

In the 7th and 8th column we give the architecture of the mammary gland: in the 7th column the architecture of the primary duct with the beginning of the first ramification and in the 8th column that of the gland-tree. In the latter case the first letter indicates the principal type; $G \rightarrow K$ meaning G-type modified slightly in the direction of the K-type; $G + K$ means G-type, in which as a foreign element, sharply distinguished from it, some milkducts of the K-type are found. In the last column the frequency of cancer of the mammary gland is given for each of the various types of mice.

From this table it appears that the primary duct presents the pure type of the maternal animal: the reciprocal F_1-hybrids possess a different type, although their genetic formulae are identical. From the fact that the nature of the milk is immaterial for the architecture of the primary duct (foster-nursing makes no difference) it follows that the influence of plasma and uterus, or one of them must be responsible for this architecture. (As these two factors, according to the method applied by us, are invariably both derived from the same animal, the significance of each of these factors cannot be settled by us).

The architecture of the gland-tree in pure K-mice differs from that of F_1-hybrids K♀ × G♂. In these two the genetic composition differs, whilst the nature of the extrachromosomal factors is the same. A similar difference in architecture exists between pure G-mice and the F_1-hybrids G♀ × K♂. From this it follows that the architecture of the gland-tree is determined, at least partly, by purely genetic influences.

From our table it appears however that the influence of the extrachromosomal factors on the architecture of the gland-tree is of much greater importance: in K-mice, fostered by G-females, the K-type to be expected, under the influence of the G-milk, has been altered into the G-type. Only in case all three extrachromosomal factors are derived from a K-mouse, does the gland-tree show a genuine K-type: if only the milk or only plasma + uterine influence of the K-animal have been at work, the gland-tree shows a — possibly more or less modified — G-type (compare normal K with fostered K, normal F_1 K♀ × G♂ with fostered F_1 K♀ × G♂ and normal F_1 K♀ × G♂ with fostered F_1 G♀ × K♂). Both the kind of milk and the nature of plasma- and uterusfactor consequently influence the architecture of the gland-tree.

In the two types foster-nursed by a K-mouse (G by K and F_1 G♀ × K♂ by K) inoculation of K-ducts on the G-gland-tree has taken place, as a distinctly foreign element.

V. *The anatomical architecture of the mammary gland as an explanation for the degree of its disposition for cancer.*

From table II it appears that the only gland-tree showing the genuine K-type (normal K-females) possesses the highest disposition for cancer by far. Next comes, concerning the degree of this disposition, the only other gland-tree of the K-type already slightly modified however in the direction of the G-type (normal F_1 K♀ × G♂ hybrids). Hereupon follow both gland-trees of the G-type which show inoculated genuine K-ducts (G-females foster-nursed by K-females; F_1 G♀ × K♂ hybrids foster-nursed by K-females). The disposition of the G-gland-trees, which have been slightly modified in the direction of the K-type but still possess no genuine K-type ducts is still lower (F_1 K♀ × G♂ hybrids foster-nursed by G-females; normal F_1 G♀ × K♂ hybrids).

The disposition of the gland-tree of the genuine G-type is extremely low. (In K-females fostered by G-females, with 9 % cancer, we found a gland-tree of the G-type. This apparent contradiction may be explained by the fact that of these mice only a very small number have been examined).

From our material it most convincingly appears that the degree of the disposition for cancer of the mammary gland is closely connected with the anatomical architecture of this organ.

VI. *Summary.*

The factors determining the degree of disposition for cancer of the mammary gland in the mouse are discussed on the basis of the result of extensive breeding experiments. In a great number of mice the anatomical architecture of the mammary gland was examined. The relation existing between this architecture and the disposition for cancer of the mammary gland is discussed.

VII. *Conclusions.*

1. There exist typical differences in architecture of the mammary gland between the mice-strains examined by the authors.
2. This architecture is influenced both by genetic and extrachromosomal factors.
3. The architecture of the primary duct is determined by the plasma- and (or) uterus-factor.
4. The architecture of the gland-tree is determined both by the genetic and by the extrachromosomal factors: milk, plasma- and uterusfactor.
5. There is a strong positive correlation between the architecture of the mammary gland in different types of mice with regard to the degree of its disposition for cancer.

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Neurology. — *Further observations about the cervical form of position-nystagmus and its anatomical base.* By A. BIEMOND, chef-de-clinique of the Neurological Clinic, Wilhelminagasthuis, Amsterdam. (From the Neurological Clinic, Director Prof. B. BROUWER, and the Otological Clinic, Director Prof. A. DE KLEYN). (Communicated by Prof. B. BROUWER).

(Communicated at the meeting of June 29, 1940.)

In 1939 the author of this article published a communication on a new form of experimental position-nystagmus in rabbits and its clinical significance for man [1]). The nystagmus could be elicited in 8 of the 10 cases by cutting the second cervical posterior root of the animals. This nystagmus which was described in detail, occurred immediately after the cutting, appeared only in both lateral positions, was nearly always of an associated character and could be elicited only during a restricted time (½ to ¾ of an hour). In the same article three clinical observations were recorded: the patients, suffering from a neuritis, resp. radiculitis cervico-brachialis, developed in the course of this disease a severe dizziness. Examination revealed a typical position-nystagmus, whereas the vestibular reactivity was normal. In these cases the dizziness and the position-nystagmus disappeared almost simultaneously with the cure of the neuritic symptoms.

It was taken for granted that here the position-nystagmus was of peripheral (cervical) origin. This idea was strongly supported by the experiments mentioned above.

The author pointed to the necessity of enlarging the experimental investigation, partly by cutting the lower cervical roots, partly by anatomical examination of the retrograde degeneration in the spinal cord and in the medulla oblongata.

In the following pages this investigation, performed in the laboratories of Prof. DE KLEYN and Prof. BROUWER, is described. After that some new observations of cervical position-nystagmus in man are communicated.

Cutting of the 3rd, resp. 4th or 5th cervical posterior root in rabbits.

This operation presented great technical difficulties as the third and following cervical roots ramify directly after its emergence from the intervertebral foramen. Therefore cutting of these roots can only be performed after opening of the vertebral canal. However, a severe venous hemorrhage which is usually fatal, always follows: in the first series of experiments (cutting of the third cervical root) all animals died before the roots were cut. The big amount of experimental animals, placed at our disposal together with the continuous improvement of the operative technic, enabled us, however, to examine a few rabbits in the desired way. The results were as follows:

In 3 experiments we were able to cut the third left root. All three animals showed a distinct position-nystagmus, which especially occurred when the head of the animal was turned to the right. At first the nystagmus was of a rotatory character (whereby in the quick phase the upper pole of the left eye was turned in temporal direction) but soon passed off into a horizontal nystagmus. Also in the left lateral position a distinct nystagmus could be seen but less marked than in the right lateral position. All three animals died some minutes after cutting of the cervical root.

In another series of 3 rabbits we could cut the fourth left cervical root. In these three experiments a typical position-nystagmus occurred with the same symptoms as described above and also more marked in the right lateral position. Here also the animals could be kept alive only a few minutes. In two cases the fifth cervical root was cut: here not even the slightest sign of a position-nystagmus developed, no more than in one case where the sixth cervical root was cut.

Further experiments of cutting the 2nd cervical root.

As mentioned above, in our first paper 10 experiments of cutting the 2nd cervical root were recorded, of which in 8 cases a typical position-nystagmus was observed. We thought it necessary to investigate why in two cases the cutting had a negative result. For this investigation we made use of a larger series of rabbits (25) in which the same operation was performed. Special attention was given to moments which possibly could have a positive or negative influence. From this new series 18 rabbits developed a distinct position-nystagmus. We had the impression that a quickly performed operation could easily damage the 2nd cervical root or one of its principal ramifications before the cutting was done. In this way the experiment could possibly be influenced unfavorably and this could be responsible for the absence of the position-nystagmus in the minor part of the cases. In fact it could be demonstrated that 5 rabbits, in which the 2nd cervical root was first freed carefully under conscientious evitation of any premature irritation, and thereafter cut, *all* developed a distinct position-nystagmus.

Summarizing we can conclude that extravertebral cutting of the 2nd cervical posterior root in rabbits causes in the large majority of cases (31 of 40 experiments) a typical position-nystagmus; this nystagmus was elicited in *all* cases (two series of 3 experiments) by intravertebral cutting of the 3rd or 4th cervical posterior root. It seems impossible to produce this phenomenon by cutting the lower cervical roots.

A. BIEMOND: FURTHER OBSERVATIONS ABOUT THE CERVICAL FORM OF POSITION-NYSTAGMUS AND ITS ANATOMICAL BASE.

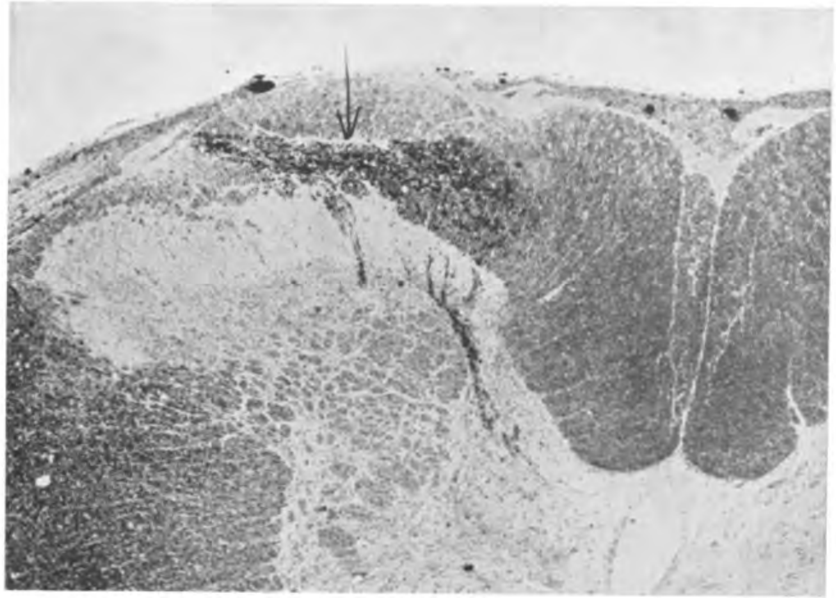

Fig. I

Fig. II

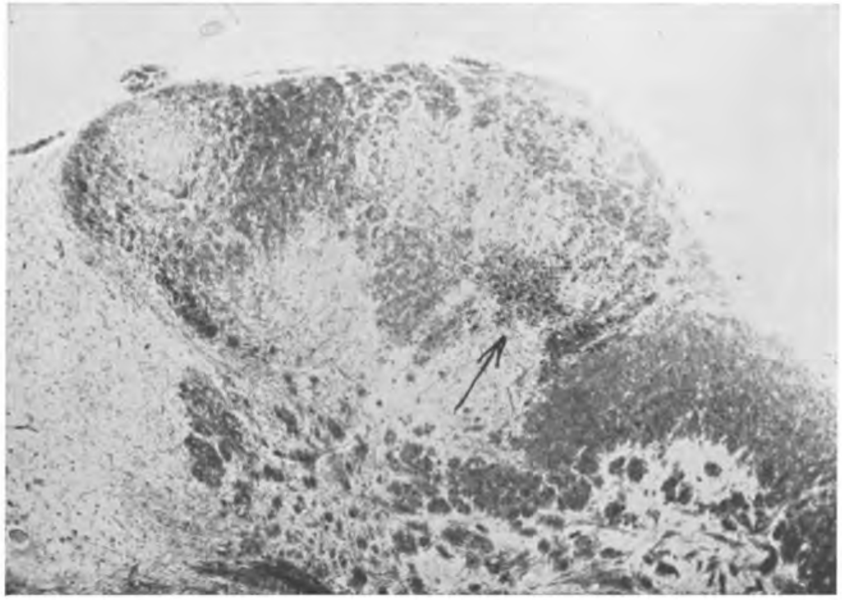

Fig III

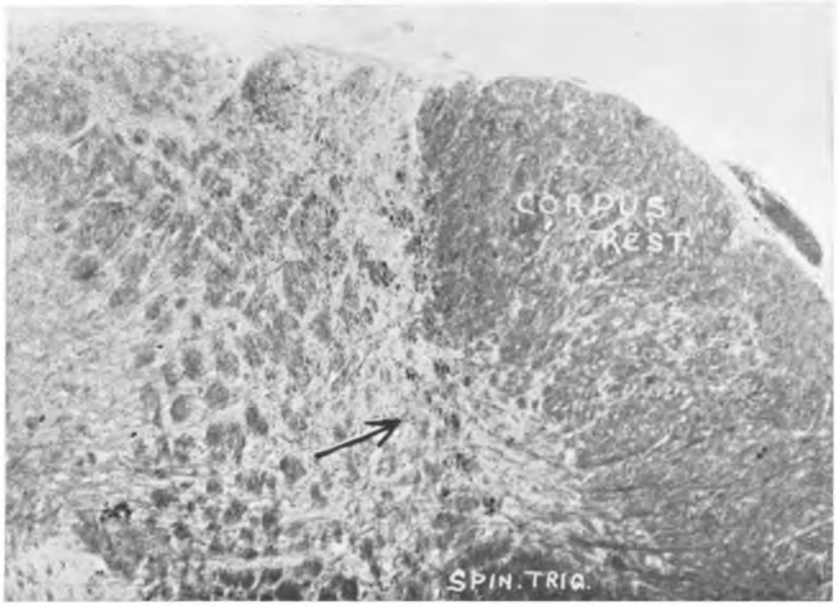

Fig. IV

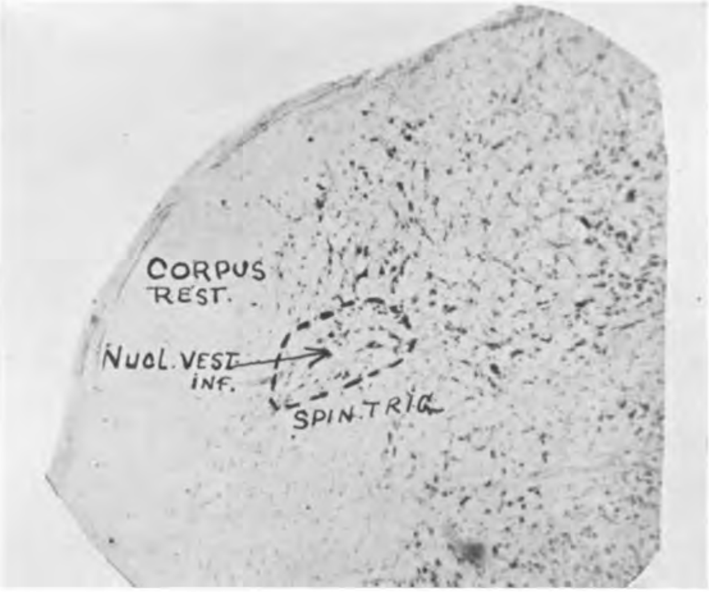

Fig. V. Nucleus vestibularis inferior (dotted line) in a normal Nissl-coupe (rabbit).

Anatomical research.

Two rabbits in which the left second cervical posterior root was cut, were killed after two weeks: the cervical cord and the medulla oblongata were cut serially and stained by the Marchi-method. At the same time we disposed of serial sections of the same parts of the central nervous system of three rabbits in which respectively C III, C IV and C V were cut. These last series originated of previous experiments of dr. TEN CATE, performed for Prof. BROUWER for other purposes. These sections could be excellently used for our present investigation. In discussing the C II sections the Marchi preparations reveal that in both cases a completely specific secondary degeneration developed, which can be followed systematically from the cervical cord to the medulla oblongata. At a level under the cutting of the root, the Marchi granules occupy a certain area in the funiculus of Burdach. Here it is noticable that the granules lay not closely packed but are scattered in a rather large cross-area. It is evident that here a degeneration in the descending fibres of the posterior root exists. On the level of the cut posterior root, the degeneration of this root and the localized degeneration of the posterior columns forms a single unintermittent area of Marchi granules. Above the level of the cut root the Marchi granules, which here are closely packed and form a circumscript area, take a very definite position in the lateral part of the funiculus of Burdach (fig. I). At a more higher level in the series the pyramidal crossing is seen and at the same time the nuclei of Goll are found. It is evident that the degeneration of Marchi does not have any contact with the nuclei of Goll. At a still higher level we can see the hypoglossus nucleus and hypoglossus roots; here also the vagus nucleus becomes visible. On this level the nucleus of Burdach is distinctly seen laterally of the nucleus of Goll. The area of Marchi granules changes its form; it now has the appearance of a comma with its convexity directed to medial. The biggest (upper) part of the comma comes directly into contact with the nucleus of Burdach and, in following sections, begins to merge itself gradually into it (fig. II and III). The smaller lower part of the comma now enters into contact with the nucleus of von Monakow, which appears laterally under the nucleus of Burdach. Some degenerated fibres also pass into the nucleus of von Monakow. On a much higher level, where the nuclei of Goll, Burdach and von Monakow disappear, a small area of Marchi granules is still present, which merges into the corner formed between the spinal trigeminus root and the corpus restiforme. Comparition with Nissl-coupes of different levels, just before and after the end of the nuclei of the posterior column, show that the Marchi-granules (at least a small amount) prolong further than the last rest of the nuclei of the posterior column. It also appears that the most proximal part of the degenerated fibres merge into an area which is occupied by the *lower vestibular nucleus* (nucleus vestibularis inferior, nucleus radix descendens VIII) (fig. IV and V).

Examination of the other series (resp. cutting of C III, C IV or C V) reveals the following. Also here a degeneration in the posterior column, occupying a more or less circumscript area, could be followed regularly from below to above. It was evident that here too part of the degenerated fibres merge gradually into the nucleus of BURDACH and that another part passed into the nucleus of VON MONAKOW. Here also, at least in the series where C III or C IV were cut, a thin band of fibres was seen which extended above the level of the last rests of the nucleus of the posterior column and passed into the above described area of the nucleus vestibularis inferior. Here attention must be drawn to the fact that the mentioned area of fibres in which C IV was cut, appeared to be of minor caliber than in the series where C III was cut. In the C V-series it could not be concluded with sufficient certainty that an extention of the degeneration above the level of the nuclei of BURDACH and VON MONAKOW was present.

From this anatomical examination it has become evident that cutting of C II, resp. of C III or C IV in rabbits, causes a secondary degeneration in the homolateral posterior column, which not only merges into the nuclei of BURDACH and VON MONAKOW, but at the same time partly passes into the nucleus vestibularis inferior, which, as known, shows a gradual transition into the nucleus of DEITERS which is found more laterally *).

The above described new form of experimental position-nystagmus in rabbits, caused by cutting resp. of C II, C III or C IV thus gets an anatomical base. It was namely possible to conclude that the same roots send a certain contingent of their fibres to the lower vestibular nucleus. It may be considered reasonable that cutting of these roots causes in this nucleus-area a stimulation-condition resulting in a position-nystagmus. The fact that this experimental position-nystagmus is only of short duration completely corresponds to this conception. It is, however, still inexplicable why this experimental position-nystagmus asserted itself more in the contralateral than in the homolateral position.

Significance for the neurological clinic.

The general clinical significance of position-nystagmus is discussed briefly in our last publication. A more detailed survey is given in a recent summarizing article of MEYER zum GOTTESBERGE [4]). However, in this paper the new cervical form of position-nystagmus is not mentioned.

The above described, experimentally and anatomically founded, form of

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*) How far this also holds true for other mammals must further be investigated. FERRARO and BARRERA [2]) who examined the secondary degeneration after lesion of the cervical posterior roots in macacus rhesus, saw that this degeneration ended only in the nucleus cuneatus and in the nucleus of Von Monakow. The ventrolateral part of the last nucleus, however, in which especially the higher cervical roots should end, nearly comes into contact with the nucleus vestibularis inferior. KAPPERS, HUBER and CROSBY have pointed to the fact that the nucleus cuneatus externus receives descending vestibular fibres but do not mention a connection in retrograde direction [3]).

position-nystagmus will be interpreted as the *cervical* form of position-nystagmus. As far as known it has special clinical significance in those cases of neuritis (resp. radiculitis) cervico-brachialis, where dizziness and position-nystagmus are associated. As mentioned above, in our first paper three of such cases are recorded. Last year we were able to observe again two patients suffering from this combined affection. Both were middle-aged women. In the first case a left-sided neuritis brachialis with position-nystagmus to the left was present when the patient was brought in the left lateral position. The second case was a right-side neuritis brachialis with position-nystagmus to the right, present in the right lateral position. In both cases the caloric vestibular reactions were normal. All symptoms disappeared completely under the usual treatment of the neuritis.

It may be expected that extra- and intrameduallar processes, especially tumors, will give rise to a typical position-nystagmus. However, as the systematical investigation on position-nystagmus is of recent date, further particulars are not available.

In literature some observations are recorded in which attention is drawn to the occurrence of nystagmus and dizziness in processes of the cervical cord (HELSMOORTEL and V. BOGAERT [5]). A satisfactory explanation could not be given.

Position-nystagmus after cutting of the 2nd and 3rd cervical root in man.

We are able to communicate one observation in man, which may be considered as a physiological equivalent to the experiment in animal.

A man, 25 years of age, suffering for 9 months of a severe torticollis to the right, was admitted to the Neurological Clinic on November 3, 1939. Neither medical, nor electrical therapy had any effect: this was the reason why on April 19, 1940 laminectomy with intradural cutting of the left 2nd and 3rd cervical posterior root was performed. Immediately after this operation (which was done with local anesthesia) a marked nystagmus to the left developed if the patient turned his head to the right or to the left. Thus here a typical position-nystagmus was present which existed for at least 20 minutes. During this period the surgeon finished the operation in the usual way. After the operation the patient was examined in the left and right lateral position for about 40 minutes. It now appeared that only in the right lateral position a slight nystagmus was present to the left; in the left lateral position this nystagmus was absent.

One hour after the cutting the position-nystagmus could no longer be elicited.

The torticollis was greatly improved by the operation.

Summary and conclusions.

1. Cutting of the 2nd, 3rd or 4th cervical posterior root in rabbits elicited a position-nystagmus. Also in man this phenomenon could be seen after cutting of the 2nd and 3rd cervical posterior root.

2. Examination of the Marchi-degeneration in rabbits, in which such cutting of a posterior root was performed, revealed that part of the fibres of the 2nd, 3rd and 4th cervical posterior root terminate in the vestibular nucleus.

3. These physiological and anatomical data may form the base for a new neurological syndrome consisting of: cervical lesion of the posterior root with position-nystagmus and position-dizziness. Of this syndrome some cases are recorded.

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Medicine. — *Is thyrotropic hormone consumed, when acting upon the thyroid gland?* By F. GLASER and J. A. COHEN. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of June 29, 1940.)

Concerning the nature of hormonal activity great uncertainty still prevails. There are data which are indicative of a *consumption* (decay, chemical linking, adsorption, accumulation) at the seat of activity (e.g. the observations of GAILLARD and DE JONGH on the uterus explantates in vitro) (1).

Other experiments would seem to point rather in the direction of a hormonal activity, solely dependent upon the *concentration* of the hormones in the tissue fluids and thus independent of the quantity of reacting tissue. A certain quantity of testosterone, for instance, has the same effect on the *intact* male genital tract as upon the remainder after partial removal. The same holds good for the effect of oestrone upon the female tractus (HERZ c.s.) (2).

We endeavoured to solve this problem by an investigation upon adult hypophysectomized rats, a number of which also underwent extirpation of one thyroid, in order to ascertain in how far there was any difference in both groups in the activity of the thyroid gland (judged from the histological aspect) after injection of a certain dose of thyrotropic hormone. Thyrotropic hormone was chosen, since apart from the thyroid gland, few seats of activity are known. (This not being the case for the influence of oestrone and testosterone on the female and male genital tract respectively, the correctness of HERZ' conclusion, quoted above, might be questioned.)

Technique:

The hypophysis of adult male rats was removed. In some animals of each series, simultaneously experimented upon, unilateral thyroidectomy was performed at the same time. Six days later subcutaneous injections with thyrotropic hormone were begun (10 U Ambinon Organon per rat daily). The day after the last injection, the animals were autopsied and the activity of the thyroid gland (or glands) was estimated histologically according to the criteria of HEYL and LAQUEUR (3). Only those animals are recorded below, the sellae of which had proved to contain no pituitary remainders.

Results:

Group I (Control group, not injected, autopsy 7 days after the operation).

Number of animal	Weight at operation	Weight at autopsy	Thyroid activity
A. *Hypophysectomy.*			
B 6122	169	160	P
B 6123	150	143	P
B 6124	163	154	P
B. *Hypophysectomy and unilateral thyroidectomy.*			
B 6125	167	160	P
B 6126	153	152	P
B 6127	146	141	P

No difference between groups A and B.

Group II. Six days after the operation the animals were injected subcutaneously with 10 U thyr. hormone daily, on the 6th and 7th day after operation. Autopsy the 8th day.

Number of animal	Weight at operation	Weight at autopsy	Thyroid activity
A. *Hypophysectomy.*			
B 6110	136	124	r
B 6111	161	148	s
B 6112	156	154	s
B 6114	176	170	r—s
B 6115	183	171	r—s
B. *Hypophysectomy and unilateral thyroidectomy.*			
B 6116	160	147	r
B 6117	154	152	r—s
B 6118	176	156	r—s
B 6119	161	150	s—t
B 6120	171	158	r—s
B 6121	143	116	r—s

No significant difference between groups A and B.

Group III. From at least 6 days after the operation daily subcutaneous injection with 10 U thyrotropic hormone, for 7 days.

Number of animal	Weight at operation	Weight at autopsy	Thyroid activity
A. *Hypophysectomy.*			
6891	190	158	r
6892	150	130	r
6893	170	142	r
6895	156	148	s
6898	175	164	q
5411	263	220	q
5418	222	190	r
5419	182	165	s
B. *Hypophysectomy and unilateral thyroidectomy.*			
B 6889	168	138	r
B 6890	150	150	q—r
B 6896	154	134	q—s
B 6897	152	124	q—r ?
B 6899	170	146	q—r
B 6900	194	161	q
B 6901	188	254	r
B 5410	219	200	r—s
B 5413	245	235	r
B 5415	206	184	s
B 5416	194	180	r—s

Again no difference of any importance was observed.

An unequal inhibition of the thyroids by the forming of thyroxine in different quantities, of itself a conceivable source of error, would only have caused difficulty in the event of shown differences; we may therefore pass over this knotty point without comment.

Summary and Conclusions.

The effect of injections of thyrotropic hormone in hypophysectomized adult rats upon the histologically judged activity of the thyroid gland is no greater when one of both thyroids has also been previously extirpated, than in animals in which both glands are present, by a treatment with thyroid hormone of short duration no more than with one of longer duration.

From this it follows that it is solely the concentration of this hormone in the body which is of importance for its function, and that there is no consumption.

After terminating the above investigation we received the publication of SELYE (4), who, in a similar research as our own came to the same conclusion as regards the action of gonadotropic hormone upon the ovarium.

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Medicine. — *Hypophysis and bloodpicture* VI. *Direct arguments for the blooddemolition-theory.* By G. A. OVERBEEK and P. RUITINGA Jr. (From the department of Pharmacology of the University of Leiden (Holland). Director: Prof. Dr. S. E. DE JONGH). (Communicated by Prof. J. VAN DER HOEVE).

(Communicated at the meeting of June 29, 1940.)

It has become clear from earlier investigations, described in this series of papers, (1. 2. 3. 4. 5.,) that the influence of the hypophysis on the red blood picture is not caused by a direct stimulation of the bone marrow but rather by an augmentation of the processes of erythrocyt-destruction. The arguments given in favour of this theory, though strong enough, always bore an indirect character except the one about the influence on the urobilin excretion. The latter investigation, however, was seriously hindered by technical difficulties, so that the results were less impressive than we hoped them to be. Therefore, we sought and found further direct arguments for our blooddemolition-theory, which will be described in this paper. They are: I. the production of anemia and II. an influence on the bilirubin excretion with the bile after the injection of pituitary extract.

I. If pituitary extracts should contain a substance, promoting blood destruction, one might expect an anemia after treatment with high dosages. In our earlier work we found indications that our injected animals had less erythrocytes than the controls, especially if these controls were injected with boiled pituitary extract, which tends to augment the number of erythrocytes.

After isoelectrical purification of crude alkaline anterior lobe extract (for technical details see the preparation of Y extract, RUITINGA (5)) normal rabbits as well as hypophysectomized rats could be rendered anemic by injection of the new product (table I). This anemia was always accompanied by a strong reticulocytosis. As the number of reticulocytes was much higher than obtained by the injection of a crude extract and often correlated with the seriousness of the anemia (in many cases the anemia became manifest, when the reticulocyte level surmounted 10 %), we think it quite clear that the influence on the reticulocytes must be seen as the consequence of a diminution of the erythrocytes.

One might ask, what could be the cause of the augmented activity of the purified extract. We supposed that the effect was caused by the elimination

TABLE I.

Anemia in rabbits and rats after the injection of pituitary extract.

Number of animal	Erythrocytes mill./mm^3		Reticulocytes $^0/_{00}$		Remarks
	Initial value	Lowest value	Initial value	Highest value	
					Rabbit
D 632	—	1.6	—	320	1×2.5 cc Y extr. for 5 d.
D 633	5.1	3.7	2	60	ibid. 8
—	5.0	4.1	6	6	ibid. 10
—	5.5	4.0	4	40	ibid. 13
					Hyp. ect. rat
B 5484	6.9	4.1	2	220	2×0.5 cc Y-extr. for 14 d.
B 5485	6.9	4.0	8	100	ibid.
B 5487	6.6	5.2	2	120	ibid.
B 6049	6.5	6.2	72	120	ibid.
B 6050	6.6	3.9	48	400	ibid.
B 6051	5.9	5.9	28	72	ibid.
B 6052	7.3	5.7	24	104	ibid.

Note: In some other hypophysectomized rats erythrocyte values were observed of 3.0; 3.7; 4.3 and 3.8 mill./mm^3. As we have no initial values of these animals at our disposition, they are not included in the table, though they may be considered as definitely anemic.

of a factor influencing blood *formation*. Such a factor was described by FLAKS et al. (6). These investigators succeeded in producing a polycythemia in normal rats by the *oral* administration of anterior lobe extract. They never observed any effect after the *injection* of their extracts. As it is not very likely that this substance would show its activity after *oral* administration only, and knowing that crude extracts stimulate blood destruction when *injected*, it becomes probable that the substances described by FLAKS and by us are both present in crude extract. The activity of these crude extracts cannot be very great in either respect, the substances counteracting each others activity.

If this were true, it should be possible to produce a polycythemia by *injecting* the fraction remaining after the preparation of the Y extract.

Although the results were not overwhelming, sometimes a slight polycythemia could be shown after the administration of this extract. In two out of nine rats the erythrocytes increased from 8 million/mm^3 to more than

10 millions/mm^3, this being the level above which FLAKS et al. consider a polycythemia to be present.

In other animals increases to more than nine millions/mm^3 were observed. This result therefore indicates the existence of two substances in the anterior lobe of the hypophysis both stimulating the bone marrow, but acting in a different way. One of them acts in a direct way (FLAKS) the other indirectly by promoting blood destruction.

II. The excretion of bilirubin with the bile was studied in the following way:

A rat was anaesthetized with urethane, and the duodenum was ligated by the pylorus after washing with about 5 cc. of saline. The duodenum was emptied by gentle digital pressure and a second ligature was made about 7 cm more distally. In this way the bile-duct opens into the ligated intestinal loop. The peritoneum and the skin were closed, and the animal was put away for one hour. Consecutively the ligated loop was taken from the animal and the contents brought into a small centrifuging glass. After centrifuging, the clear yellow liquid was sucked up in a syringe, the amount was measured and the whole diluted with water to 10 cc.

The yellow colour was compared with the colour of diluted bile of a normal rat arbitrarily chosen as a standard (100). The yellow colour must originate from bilirubin only, as it was impossible to bring the yellow substance in chloroform, unless the mixture was acidified with strong hydrochloric acid, after which the yellow substance passed *quantitatively* into the chloroform. We used this rather primitive test method for the estimation of bilirubin because the usual methods failed with rats' bile (HYMANS VAN DEN BERGH test, method of SABATINI). The latter method which was designed for the estimation of bilirubin in urine was also tried after addition of diluted urine to the bile. Some colour was obtained, but for our purpose the method was not satisfactory in a quantitative sense, so that we sticked to the primitive method described above.

The use of a rat as a "standard rat" made it impossible to compare the results directly. Therefore we calculated the percentage of difference between each hypophysectomized rat and each normal rat in every group, in this way eliminating the influence of the accidental choice of the "standard rat". These percentages of difference could be averaged for the different groups.

In a first experimental series, hypophysectomized rats (about 10 days after the operation) were compared with normal rats. Table II shows that in all operated animals the reticulocyte level was low. In almost all instances the bilirubin excretion was diminished, although the amount of bile was not significantly different. The average percentage of decrease, calculated as mentioned before, amounted to 31.7 %.

In a second series hypophysectomized rats treated with crude pituitary extract were compared with normal rats. Table III proves that the number of reticulocytes was normal or above normal and that the excretion of bilirubin was certainly not less than in the normal rats. There even was an average increase of 18.5 % compared with normal.

Thus it could be shown that hypophysectomized rats excrete less bilirubin than normal rats, and that this failure can be corrected by the injection of crude pituitary extract.

These results are in complete accordance with the theory that blood-demolition is diminished in the absence of the hypophysis.

TABLE II.

Bilirubin excretion in normal and hypophysectomizid rats.

Number of rat	Body weight on day of experiment	Retic. $^0/_{00}$ on day of experiment	cc of bile	Total bilirub.	Days after operation	Remarks
B 7153	95	0	0.35	60	9	Hyp. ect.
B 7154	126	4	0.60	73	9	„ „
B 7155	111	36	0.40	105	—	Normal
B 715o	93	68	0.45	100	—	„
B 7157	120	2	0.40	35	10	Hyp. ect.
B 7158	116	4	0.60	64	10	„ „
B 7159	142	36	0.36	94	—	Normal
B 7160	110	20	0.40	100	—	„
B 7161	123	0	0.25	33.5	10	Hyp. ect.
B 7162	99	2	0.22	36.5	10	„ „
B 7163	126	32	0.25	100	—	Normal
B 7164	95	72	0.22	46	—	„
B 7165	112	0	0.25	70	13	Hyp. ect.
B 7166	132	0	0.22	65	13	„ „
B 7242	112	2	0.25	66	13	„ „
B 7243	115	44	0.33	61	—	Normal
B 7244	133	60	0.52	100	—	„
B 7245	134	0	0.56	86	13	Hyp. ect.
B 7246	100	0	0.36	61	13	„ „
B 7247	94	48	0.35	104	—	Normal
B 7248	96	72	0.28	100	—	„

TABLE III.

Bilirubin excretion in normal and hypophysectomized rats treated with pituitary extract

Number of rat	Body weight on day of experiment	Retic. $^0/_{00}$ on day of experiment	cc of bile	Total bilirub.	Days after operation	Remarks
B 7102	186	108	0.60	185	15	Hyp. ect.
B 7103	176	100	0.35	115	15	„ „
—	140	44	0.50	100	—	Normal
—	180	100	0.40	115	—	„
B 7223	131	44	0.33	108	9	Hyp. ect.
B 7224	114	56	0.39	119	9	„ „
B 7225	135	64	0.42	117	9	„ „
B 7261	146	76	0.30	100	—	Normal
B 7262	114	52	0.16	86	—	„
B 7263	106	120	0.20	79	—	„
B 7226	132	72	0.60	64	11	Hyp. ect.
B 7227	131	136	0.40	66	11	„ „
B 7228	142	92	0.58	76	11	„ „
B 7264	154	60	0.31	100	—	Normal
B 7265	163	68	0.31	86	—	„
B 7266	150	32	0.35	49	—	„

The hypophysectomized rats were daily subcutaneously injected with 2 × 0.5 cc of crude alkaline anterior lobe extract.

Summary:

1. Crude alkaline pituitary extracts can be separated in two fractions, one promoting blood destruction and causing a hemolytic anemia, the other causing sometimes a slight polycythemia.
2. Hypophysectomized rats excrete less bilirubin with the bile than normal rats. After the injection of pituitary extract normal quantities are excreted.

These facts are *direct* arguments for the soundness of our theory that the hypophysis contains some factor promoting erythrocyte destruction.

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Medicine. — *Kompendium aus „Seh-Hören"*. Von Lothar Goldschmidt.
(Communicated by Prof. A. de Kleyn.)

(Communicated at the meeting of June 29, 1940.)

Durch einen vor ca. 15 Jahren erlittenen Eisenbahnunfall verlor ich — zum Teil — mein früheres normales Hörvermögen und wurde dadurch veranlasst, mich eingehender mit der Gebärdensprache zu befassen. In diesem Zusammenhang will ich nur darauf hinweisen, dass bereits bei den Völkern des Altertums Zeichensprachen bestanden, und dass das Interesse für die Zeichen- und Gebärdensprache sich bis zum heutigen Tage nicht verminderte. (Siehe W. Wundt's Völkerpsychologie, G. Mallery's Sign Language among North American Indians, neuerdings M. Critchley: „The Language of Gesture", die zur Zeit angewandten Versuche durch Sir R. Paget, Royal Academy of Great Britain, usf.)

Meine Arbeiten, die sich auf ein grosses empirisches Material gründen, werden durch die bekannte Fingersprache der Taubstummen nicht ersetzt. Bei meinen Methoden habe ich besonders die Hörenden berücksichtigt, die sich bei der Berufsarbeit unter ungünstigen Umständen mit der Lautsprache schwer verständigen können, während mit den von mir beschriebenen Systemen die Schwierigkeiten aufgehoben werden können.

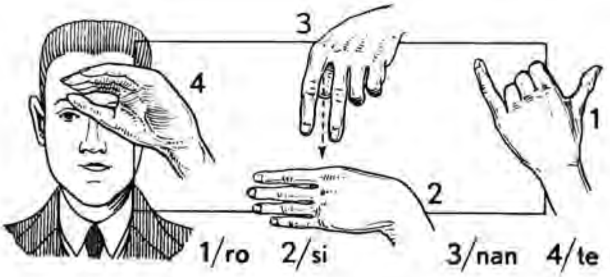
a. „Gesto" = Gebärdenstenographie.
(Auf hieroglyphischer Grundlage.)
Einhand = Sprache für alle Sprachen.

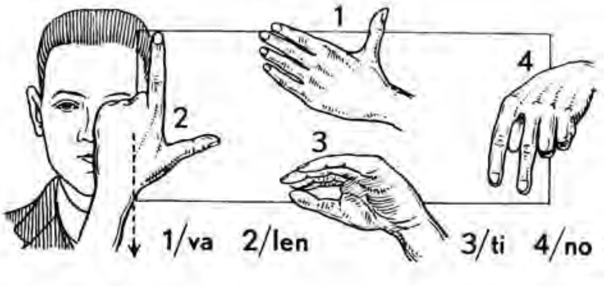
b. „Langest" = Weltgebärdensprache.
(Möglichkeit einer relativ leicht fassbaren internationalen Verständigung mit einfachster Grammatik und Wörterbuch.)

c. „Insig" = Internationale Handsignale.
(Analog den Flaggensignalen der Marine.)

„Gebärdenstenographie". „Gesto" *„Gebaren-Stenografie".*
„Gesture-Shorthand". *„Sténographie des gestes".*

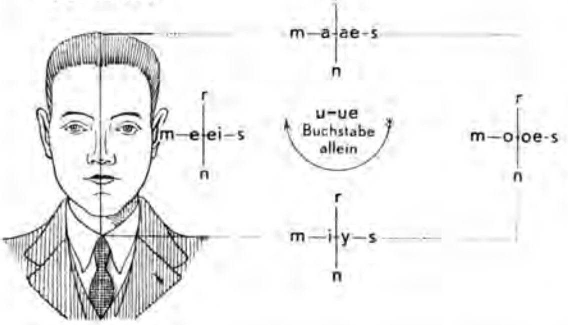
Die Ausführung: erfolgt mit einer Hand, der Einheitlichkeit halber mit der linken, und damit evtl. die rechte gleichzeitig zum Schreiben, Sortieren, ect. benutzt werden kann. Der Zeigeflächenumfang ist aus Schema ersichtlich. Automatisch folgt dem jeweilig an einer der 4 Seiten gezeigten Handbuchstaben der Vokal. Also: oben = *a*, links seitwärts = *o*, unten = *i*, in Nasennähe = *e*. Wird der Handbuchstabe beim Zeigen dabei etwas gedreht, ist es: *ae*, *oe*, *ei*, und unten: *y*. U wird falls nicht allein, durch Halbkreisbewegung in der Mitte der Zeigefläche gezeigt. Ebenfalls dort die allein stehenden Buchstaben. Eine kl. Drehung am Ende der U = Kreisbewegung ergibt: *ue*. Steht Vokal vor dem Konsonanten bezw. am Silbenbeginn oder Ende, genügt Hinweis nach der betreffenden Seite. Folgt dem Vokal ein *n*, macht man eine kurze Bewegung nach unten, bei *r* = nach oben, *s* = zur Seite, bei *m* = zum Gesicht. (s. Schema). Wiederholung des vorhergehenden Buchstabens ergibt sich aus kurzer Hin- und Zurückbewegung. Wie bei der Stenographie können nicht hörbare sowie Doppel-Laute und -Konsonanten nur einmal gezeigt werden. (= phonetische Wiedergabe). Weitere Kürzungen ergibt die Praxis und evtl. Kombination mit der Gebärdensprache. („Langest".)





Beispiele: „garage" = *g*, *r*, – oben, *g* – in Nasennähe. — „malaga" = *m*, *l*, *g*, – oben. — „faust" = *f* von oben in U-Halbkreis der in *st* endigt. „se(he)n, ste(he)n, we(he)n, ge(he)n" usw. = den betreffenden Buchstaben von Nasennähe abwärts (etwa Kinn) zeigen. — „schein, mein, dein, rein, sein, fein, wein, bein" usw. = Buchstabe in Nasennähe mit Drehung und Abwärtsbewegung. (wie oben.) — „valentino" (s. Skizze) = *v* – oben, *l* – Nasennähe mit Abwärtsbewegung, *t* – unten, *n* – seitwärts. — „rosinante" (s. Skizze) *r* – seitwärts, *s* – unten, *n* – oben mit Abwärtsbewegung, *t* – Nasennähe.

SCHEMA:



Vokalfolge-Grundstellung:
ist Linienschnittpunkt auf Schema.

Stopzeichen:
Kl. Luftquerstrich an jedem Wortende.

Abkürzungen für das Handbuchstabenalphabet:
H = Handfläche, Z = Zeigefinger, D = Daumen, F = Finger, K = Kleine Finger, M = Mittelfinger.

„Weltgebärdensprache" „Langest". „Wereld-gebarentaal"
„Language of gestures of the world" „Langue des gestes de la monde"

Die durch den Film, Sport, Foto, Reklame usw. weltverbreitete natürliche Gebärde ist heute Gemeingut aller Menschen und formt sich wie die Lautsprache nach der jeweiligen Umgebung ständig neu. Man versetze sich in die Lage eines Menschen, der in ein fremdes Land kommt dessen Sprache er nicht versteht. — Für die Satzbildung wird hier erstmalig eine einfache Grammatik gebildet. ebenfalls als Hinweis ein Gebärdenwörterbuch.

Gebärdengrammatik: (Auszug)

Artikel:	Fürwort:	
	ich, mir, mich, =	auf sich zeigen
	du, dir,	auf Gegenüber zeigen
der	er (Mann)	Seitenstrich, Handfläche nach innen
die	sie (Frau)	Halbkreis
das, ein		Handfläche nach unten
	wir,	Kreis
	Jhr	2 × vorwärts zeigen
die (Mehrzahl)	sie	2 Seitenstriche
man		Hand auf Schulter

Besitzanzeigende Fürworte:	Desgleichen + entsprechender Hinweis m. Daumen.
haben (mit Abarten)	Hand auf Brust
sein (mit Abarten)	Hand von Brust abwärts (= kl. Körperstrich)
Vergangenheit	Hand rückwärts über Schulter
Zukunft	Hand vorwärts
Frageform	Hand jeweils drehen (? = Daumen auf Handfläche)
Befehlsform	Faust vorwärts
Ohne Zeitangabe	gilt Gegenwart, mit Artikel: Substantiv.

Beispiele: „ich habe gehabt" = auf sich + rückwärts + auf Brust zeigen. „sie wird sein" = Halbkreis + vorwärts + Körperstrich. „sie würden haben" = 2 Seitenstriche + Vorwärtsdrehung + Brust. „wir haben" = Kreis + Brust. — „wir sind" = Kreis + Körperstrich. — „würden wir sein?" = Hand vorw. drehen + Kreis + Körperstrich + ? (s. oben) „sie hatte kein Geld" = Halbkreis + rückw. + Hand schütteln + Geldhand (= Daumen u. Zeigef. reiben).

„Internationale Handsignale" „Insig". „Internationale Handseinen"
„International Signs of the hand" „Signales de la main internationales".

Die hierfür angelegte Sammlung scheint notwendig zu sein. Denn durch die Gleichart der Maschinen, Arbeitsmethoden, Sport usw. sind sie auf allen Gebieten gleichartig, in denen die Lautspracheverständigung durch Lärm, Entfernung usw. erschwert oder wie beim Radiosender, Tonfilmstudio, Verkehrsregelung usw. unmöglich ist. Ferner erfordern die vielen durch Lärm, Staub, Ausdünstung, (Vergiftungsgefahr, Berufsschwerhörigkeit, Berufskrankheiten des Ohres und der oberen Luftwege usw.) gesundheitsgefährdende Arbeitsstätten, Abdichtung der betreffenden Organe, was wie psychologisch verständlich, ohne Zeichenverständigungsmöglichkeit immer unterbleiben wird. „Insig" wird auf Gebieten angewandt deren grosser Umfang garnicht genug bekannt ist. Sie sind international notwendig und sollten in den Gewerbe- und Handelsschulen gelehrt werden. Die „Technische-Hochschule, Charlottenburg" und der „Verein Deutscher Ingenieure", Berlin, haben erst jetzt wieder das Bestehen einer derartigen Sammlung verneint. Das Interesse der Chefingenieure der Werften, Webereien, Fabriken Krananlagen, Rotations-Schiffs-Maschinenräume (Pfadfinder-Luftschutz-Gasmaskenträger) usw. bestätigt die Zweckmässigkeit einer derartigen Sammlung.

Vorliegende bereits erprobte 3 Systeme (evtl. auch kombiniert angewandt) sind ein neuer Beitrag der Psychotechnik im Dienst der Betriebswirtschaft ausser der sonstigen universalen Bedeutung. Ein Erlernen dieser — neuen körperlich = geistigen Eigenschaft — ist kaum nötig. Die Absicht des Verfassers ist, nicht nur Schwerhörigen eine leichte (auch evtl. „ablesbare") Verständigung ohne Apparate — die, gleichviel welcher Konstruktion, nicht wie es beim Auge bei herabgesetztem Sehvermögen das jeweilige Glas tut, das Hörfähigkeitsminimum ausgleichen können — zu bieten sondern auch die bisher noch nicht vorhandenen aber notwendigen international verständlichen für *Normalhörende geeignete Gebärdensysteme* gefunden zu haben.

NEDERL. AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS

VOLUME XLIII

No. 8

President: J. VAN DER HOEVE
Secretary: M. W. WOERDEMAN

CONTENTS

Physics. — *On the buckling of a thin-walled circular tube loaded by pure bending.* II. By C. B. BIEZENO and J. J. KOCH.

(Communicated at the meeting of September 28, 1940.)

6. *Determination of the smallest total characteristic number μ by iteration, the number q of the longitudinal waves, occurring in the corresponding total deformation, being presumed as fixed.*

The most direct way to attack our problem, consisting in solving equation (14) of the preceding section, practically seems impossible. Therefore we have recourse to a generalisation of an iterative method which by the second author has been established in his doctor-thesis[1]).

We consider the following system of linear homogeneous equations:

$$d_i = \sum_{j=1}^{n} \mu\, a_{ij}\, d_j \quad (i = 1, 2 \ldots n) \quad \ldots \ldots \quad (1)$$

which only under the condition

$$\left.\begin{vmatrix} a_{11} - \dfrac{1}{\mu} & a_{12} & \ldots\; a_{1n} \\ a_{21} & a_{22} - \dfrac{1}{\mu} & \ldots\; a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} & \ldots\; a_{nn} - \dfrac{1}{\mu} \end{vmatrix} = 0 \right\} \quad \ldots \ldots \quad (2)$$

admits solutions $d_i\,(i = 1 \ldots n)$ different from zero. It be fixed that $a_{ij} = a_{ji}$, so that all roots of (2) are real. Furthermore it may be assumed that all roots $\mu_1 \ldots \mu_n$ be different, and that

$$|\mu_1| < |\mu_2| < |\mu_3| \ldots . < |\mu_n| . \quad \ldots \ldots \quad (3)$$

To every μ_k corresponds a solution $d_{ki}\ (i = 1, 2 \ldots n)$, which may be normalized by the condition

$$\Sigma\, d_{ki}^2 = 1 \quad \ldots \ldots \ldots \quad (4)$$

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[1]) Comp. J. J. KOCH, Eenige toepassingen van de leer der eigenfuncties op vraagstukken uit de Toegepaste Mechanica, 1929, Waltman, Delft.

Also: C. B. BIEZENO und R. GRAMMEL, Technische Dynamik, Chapter III, 1939. Springer, Berlin.

It can easily be proved, that — for $k \neq l$ —

$$\sum_{i=1}^{n} d_{ki}\, d_{li} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Indeed we find, making use of (1)

$$\sum_{i=1}^{n} d_{ki}\, d_{li} = \mu_k \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, d_{kj}\, d_{li} = \mu_l \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, d_{lj}\, d_{ki}$$

$$= \mu_l \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji}\, d_{li}\, d_{kj} = \mu_l \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, d_{kj}\, d_{li}.$$

Comparison of the second and last member of this linked equation, leads — on account of $\mu_l \neq \mu_k$ — to the value zero for all members and especially to $\Sigma\, d_{ki}\, d_{li} = 0$.

Another statement, we make use of in arguing the method of iteration to be explained a few lines further on, is the following one:

An arbitrary set of values ϑ_{1i} $(i = 1, 2 \ldots n)$ can always be expressed as a linear sum of the n solutions d_{ki} $(i = 1, 2 \ldots n)$ corresponding to the n-roots μ_k, so that ϑ_{1i} may be written as:

$$\vartheta_{1i} = \mathrm{a}_1\, d_{1i} + \mathrm{a}_2\, d_{2i} + \ldots + \mathrm{a}_n\, d_{ni} \;\; (i = 1, 2 \ldots n) \quad . \quad . \quad . \quad (6)$$

Indeed, if we sum up the equations (6) after having multiplied each equation with the number d_{ki}, whose second suffix corresponds with the number i of the equation, we find

$$\sum_{i=1}^{n} d_{ki}\, \vartheta_{1i} = \mathrm{a}_1 \sum_{i=1}^{n} d_{ki}\, d_{1i} + \mathrm{a}_2 \sum_{i=1}^{n} d_{ki}\, d_{2i} + \ldots + \mathrm{a}_k \sum_{i=1}^{n} d_{ki}^2 +$$

$$+ \ldots + \mathrm{a}_n \sum_{i=1}^{n} d_{ki}\, d_{ni}$$

or, with regard to (4) and (5):

$$\mathrm{a}_k = \sum_{i=1}^{n} d_{ki}\, \vartheta_{1i} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The iteration, mentioned before, consists in deriving from an arbitrary set of quantities ϑ_{1i} another set ϑ_{2i}, defined by

$$\vartheta_{2i} = \sum_{j=1}^{n} a_{ij}\, \vartheta_{1j} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If we substitute in (8) the expression, which for ϑ_{1j} follows from (6), we find

$$\vartheta_{2i} = \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{n} \mathrm{a}_k\, d_{kj}\right) = \sum_{k=1}^{n} \mathrm{a}_k \sum_{j=1}^{n} a_{ij}\, d_{kj} . \quad . \quad . \quad . \quad . \quad (9)$$

Taking into account, that

$$d_{ki} = \sum_{j=1}^{n} \mu_k a_{ij} d_{kj} \quad \text{resp.} \quad \sum_{j=1}^{n} a_{ij} d_{kj} = \frac{d_{ki}}{\mu_k}$$

(comp. (1)), the first iteration ϑ_{2i}, derived from ϑ_{1i} becomes:

$$\vartheta_{2i} = \frac{a_1}{\mu_1} d_{1i} + \frac{a_2}{\mu_2} d_{2i} + \ldots + \frac{a_n}{\mu_n} d_{ni} \quad (i = 1, 2 \ldots n). \quad . \quad . \quad (10)$$

We see at once, that, if the iterative process be repeated s times, we get:

$$\vartheta_{s+1,i} = \frac{a_1}{\mu_1^s} d_{1i} + \frac{a_2}{\mu_2^s} d_{2i} + \ldots + \frac{a_n}{\mu_n^s} d_{ni} \quad (i = 1, 2 \ldots n). \quad . \quad . \quad (11)$$

and consequently:

$$\frac{\vartheta_{s,i}}{\vartheta_{s+1,i}} = \frac{d_{1i} + \left(\frac{\mu_1}{\mu_2}\right)^{s-1} \frac{a_2}{a_1} d_{2i} + \ldots + \left(\frac{\mu_1}{\mu_n}\right)^{s-1} \frac{a_n}{a_1} d_{ni}}{d_{1i} + \left(\frac{\mu_1}{\mu_2}\right)^{s} \frac{a_2}{a_1} d_{2i} + \ldots + \left(\frac{\mu_1}{\mu_n}\right)^{s} \frac{a_n}{a_1} d_{ni}} \mu_1 \quad . . \quad (12)$$

The fractions $\left(\frac{\mu_1}{\mu_i}\right)$ being all < 1 (comp. (3)), the second member of this equation tends, with ever-increasing s, to μ_1, so that two conclusions are to be drawn:

1. two consecutive iterations ϑ_{si} and $\vartheta_{s+1,i}$ $(i = 1, 2 \ldots n)$ tend — with increasing s — to proportionality;

2. the "factor of proportionality" represents the smallest characteristic number μ_1.

Theoretically one has to repeat indefinitely the iteration process to obtain the exact-value of μ_1. For technical purposes, which we have in mind here, a relative small number of iterations will do to obtain a sufficient degree of proportionality and to look on the "average" factor of proportionality $\bar{\mu}_1$ as the approximative value of μ_1. It may be emphasized here, that the numerical work which has to be done to arrive at an acceptible approximative value $\bar{\mu}_1$ can remarkably be reduced by using a formula given by KOCH [1]) which, applied to our problem, runs

$$\bar{\mu}_1 = \frac{\sum_{i=1}^{n} \vartheta_{1i} \vartheta_{2i}}{\sum_{i=1}^{n} \vartheta_{2i}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

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[1]) J. J. KOCH, Bestimmung höherer kritischer Drehzahlen schnellaufender Wellen, Verhandlungen des 2ten internationalen Kongresses für technische Mechanik, Zürich 1926.

If in the right hand member we substitute ϑ_{1i} and ϑ_{2i} from (6) and (8) we find — paying attention to (4) and (5) —

$$\bar{\mu}_1 = \frac{1 + \frac{a_2^2}{a_1^2}\frac{\mu_1}{\mu_2} + \dots + \frac{a_n^2}{a_1^2}\frac{\mu_1}{\mu_n}}{1 + \frac{a_2^2}{a_1^2}\left(\frac{\mu_1}{\mu_2}\right)^2 + \dots + \frac{a_n^2}{a_1^2}\left(\frac{\mu_1}{\mu_n}\right)^2}\,\mu_1 \quad . \quad . \quad . \quad . \quad (14)$$

The degree of approximation of μ_1 by $\bar{\mu}_1$, depends evidently on the choice of the initial set of values ϑ_{1i}. If this choice was an unfavourable one, it may occur, that the discrepancy between $\bar{\mu}_1$ and μ_1 is greater than admittable. If, in a particular case, there should be some doubt about the accuracy of $\bar{\mu}_1$, the iteration can be repeated once more and μ_1 is then approximated more closely by

$$\bar{\mu}_1 = \frac{\sum_{i=1}^{n} \vartheta_{2i}\,\vartheta_{3i}}{\sum_{i=1}^{n} \vartheta_{3i}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If $\bar{\bar{\mu}}$ should considerably differ from μ_1, there is some reason, to go on with a third iteration. At the first sight it seems, that no data are available, to judge "a priori" whether the first set of ϑ_{1i} is a suitable one or not. And it must be admitted that mathematically spoken, such data do not exist. It may be noticed, however, — without going into details here — that technical problems in most cases possess some intrinsic feature, which indeed enables a suitable choice.

In the following section we intend to apply the foregoing results to equation (14) and the corresponding set of equations (15) of section **5**. Two remarks have to be made in this connection. The first one bears upon the fact, that the roots of (**5**, 14) occur in pairs of same absolute magnitude and opposite sign. This requires a slight modification in our procedure to approximate μ_1.

Returning to equation (8), let us consider the case that $\mu_2 = -\mu_1$, then $\vartheta_{s+1,i}$ becomes:

$$\vartheta_{s+1,i} = \frac{a_1}{\mu_1^s}\,d_{1i} + (-1)^s\,\frac{a_2}{\mu_1^s}\,d_{2i} + \frac{a_3}{\mu_3^s}\,d_{3i} + \dots + \frac{a_n}{\mu_n^s}\,d_{ni} \quad . \quad . \quad (16)$$

Accordingly $\vartheta_{s-1,i}$ is given by

$$\vartheta_{s-1,i} = \frac{a_1}{\mu_1^{s-2}}\,d_{1i} + (-1)^{s-2}\,\frac{a_2}{\mu_1^{s-2}}\,d_{2i} + \frac{a_3}{\mu_3^s}\,d_{3i} + \dots + \frac{a_n}{\mu_n^s}\,d_{ni}.$$

Therefore the quotient $\vartheta_{s-1,i} : \vartheta_{s+1,i}$ amounts to:

$$\frac{\vartheta_{s-1,i}}{\vartheta_{s+1,i}} = \frac{\left[d_{1i} + (-1)^s \frac{a_2}{a_1} d_{2i} \right] + \left(\frac{\mu_1}{\mu_3}\right)^{s-2} \frac{a_3}{a_1} d_{3i} + \ldots + \left(\frac{\mu_1}{\mu_n}\right)^{s-2} \frac{a_n}{a_1} d_{ni}}{\left[d_{1i} + (-1)^s \frac{a_2}{a_1} d_{2i} \right] + \left(\frac{\mu_1}{\mu_3}\right)^{s} \frac{a_3}{a_1} d_{3i} + \ldots + \left(\frac{\mu_1}{\mu_n}\right)^{s} \frac{a_n}{a_1} d_{ni}} \cdot \mu_1^2 \quad . \quad (17)$$

and it is seen that, with increasing s, the quotient of two ϑ's, whose index-numbers differ by the amount 2, now tend to the *square* of the characteristic number μ of smallest absolute magnitude.

Similarly the approximating formula (13) has to be replaced by

$$\mu_1{}^2 = \frac{\sum\limits_{i=1}^{n} \vartheta_{1i}\, \vartheta_{3i}}{\sum\limits_{i=1}^{n} \vartheta_{3i}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Secondly it must be stated explicitely, that the application of the results obtained in this section, to equations (14) and (15) of section 5, entails a transition to $n = \infty$. We take it for granted, that this transition is admissible and do not enter into discussions, which from a mathematical standpoint are indispensable.

7. *Continuation. Determination of the smallest total characteristic number μ by iteration, the number q of the longitudinal waves, occurring in the corresponding total deformation being presumed as fixed.*

The practical application to our buckling problem of the just described iteration involves the knowledge of the influence-numbers a_{ij}. It will be shown now that the calculation of these influence-numbers, (which, though theoretically possible, would lead to elaborate ciphering work) can be omitted by a mechanical interpretation of the law of iteration. To this end we start again from the arbitrary set of numbers ϑ_{1i}, and define, by the aid of these numbers a deformation D^1, composed of a linear sum of the elementary normal deformations D_i:

$$D^1 = \vartheta_{11}\, D_1 + \vartheta_{12}\, D_2 + \vartheta_{13}\, D_3 + \ldots \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We artificially submit our tube to this deformation and *in this fixed shape*, subject it to the action of two unit moments M, working at its ends. If then we ask the "would-be" load B^1 formally produced by two moments M, we find — comp. the definition of the influence-numbers in section 5 —

$$B^1 = \vartheta_1 \sum_i a_{i1}\, B_i + \vartheta_{12} \sum_i a_{i2}\, B_i + \vartheta_{13} \sum_i a_{i3}\, B_i + \ldots = \sum_i \left(\sum_j \vartheta_{1j}\, a_{ij}\right) B_i .$$

If this load should be placed on the *unloaded* and undeformed tube, it would produce a deformation D^2, given by

$$D^2 = \sum_i \left(\sum_j \vartheta_{1j}\, a_{ij}\right) B_i$$

or, with regard to (6, 8)

$$D^2 = \vartheta_{21}\, D_1 + \vartheta_{22}\, D_2 + \vartheta_{23}\, D_3 + \dots \qquad (2)$$

If therefore, we start from an arbitrary deformation D^1, subject the tube (artificially) to this deformation, calculate (formally) the "would-be" load to which this deformation gives rise if the tube — so to say frozen in his shape — is subjected to the action of two unit-moments $\overline{M}$, derive from this "would-be" load the deformation D^2 produced by it if it acts on the undeformed tube, start from D^2 to obtain in the same manner a deformation D^3 a. s. o., then the sets of coefficients ϑ_{ki} $(k = 2, 3 \dots)$ in the expansions of D^2, D^3, D^k with respect to the elementary normal functions D_i, play the role of the iterative sets, deduced from the set ϑ_{1i} after the law of iteration, given in section **6**.

It will be observed that this mechanical interpretation of our iterative process is free from mentioning the influence-numbers a_{ij}; the only thing we have to do is to start with an arbitrary deformation D^1 and to deduct from this deformation, in a well defined manner, other deformations D^2, D^3 a. s. o. This initial deformation however, can be considered either as a linear sum of the elementary normal deformations D_i, or as a sum of the special functions (2, 6). We will adopt the latter conception, because it facilitates the practical handling of our mechanical iteration process. Accordingly we define from this moment the initial deformation D^1, by [1])

$$\left.\begin{aligned} u^1 &= \sin \lambda \frac{z}{a} \sum_{p=0}^{\infty} u^1_p \cos p\varphi \\ v^1 &= \sin \lambda \frac{z}{a} \sum_{p=0}^{\infty} v^1_p \sin p\varphi \end{aligned}\right\} \qquad (3)$$

Written in this form, it is an easy matter to derive from it the first "would-be" load B^1, for it can be calculated, (using (5, 1)), that for $M = \overline{M}$ (= unit moment) the partial load corresponding to

$$u^1_p \cos p\varphi \sin \lambda \frac{z}{a},\ v^1_p \sin p\varphi \sin \lambda \frac{z}{a} \qquad (4)$$

---

[1]) It may be remembered once more, that all deductions of the previous and the present section refer to the possibility of buckling, corresponding to a *fixed* value of q. Therefore all suffixes q can be suppressed.

amounts to:

$$\left.\begin{aligned}
R &= -\frac{1}{\pi a^2}\,\frac{\partial^2 u_p^0 \cos p\varphi \sin\lambda \frac{z}{a}}{\partial z^2}\cos\varphi = \\
&= \frac{\lambda^2 \overline{M}}{2\pi a^4}\, u_p^0 \left[\cos(p-1)\varphi + \cos(p+1)\varphi\right]\sin\lambda\frac{z}{a} \\
\Phi &= -\frac{1}{\pi a^2}\,\frac{\partial^2 v_p^0 \sin p\varphi \sin\lambda \frac{z}{a}}{\partial z^2}\cos\varphi = \\
&= \frac{\lambda^2 \overline{M}}{2\pi a^4}\, v_p^0 \left[\sin(p-1)\varphi + \sin(p+1)\varphi\right]\sin\lambda\frac{z}{a}
\end{aligned}\right\} \quad . \quad . \quad (5)$$

The first iterated deformation, produced by such a load can be found with the aid of formulae (2, 4), (2, 8) and (2, 9), expressing that a load

$$\left.\begin{aligned}
R_p &= a_p \cos p\varphi \sin\lambda\frac{z}{a} \\
\Phi_p &= b_p \sin p\varphi \sin\lambda\frac{z}{a}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

gives rise to a deformation $u_p \cos p\varphi \sin\lambda\frac{z}{a}$, $v_p \sin p\varphi \sin\lambda\frac{z}{a}$, u_p and v_p being related with a_p and b_p by

$$\left.\begin{aligned}
u_p &= \alpha_p a_p + \beta_p b_p \\
v_p &= \beta_p a_p + \gamma_p b_p
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Herewith the data are at our disposal to develop a scheme of practical iteration. It follows from (5), that a deformation D^1, characterized by the numbers

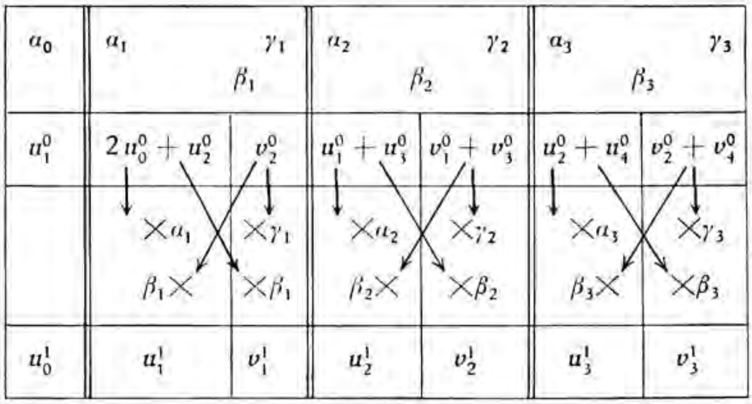
$$u_0^0,\, u_1^0,\, v_1^0,\, u_2^0,\, v_2^0, \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(where v_0^0 is suppressed in consequence of the fact, that the corresponding deformation $\sin 0.\varphi \sin\frac{\lambda z}{a}$ is identically zero), involves — except for the factor of multiplication $\frac{\lambda^2 \overline{M}}{2a}$ — a "would-be" load B^1, — characterized by the coefficients a_p and b_p, — the magnitude of which can be read from the last line of schedule (9)

first row	u_0^0	u_1^0	v_1^0	u_2^0	v_2^0	u_3^0	v_3^0
second row	u_1^0	u_2^0	v_2^0	u_3^0	v_3^0	u_4^0	v_4^0
third row	—	$2u_0^0$	—	u_1^0	v_1^0	u_2^0	v_2^0
sum up	u_1^0	$2u_0^0+u_2^0$	v_2^0	$u_1^0+u_3^0$	$v_1^0+v_3^0$	$u_2^0+u_4^0$	$v_2^0+v_4^0$
fourth row	$\equiv a_0$	$\equiv a_1$	$\equiv b_1$	$\equiv a_2$	$\equiv b_2$	$\equiv a_3$	$\equiv b_3$

(9)

Apart from a single irregularity at the beginning, it is seen that the second and third row of this table are obtained, by shifting the first row to places to the left, resp. to the right. It can readily be seen from schedule (10), — in which the first row, denoted as fourth row is identical with the fourth row from schedule (9) — how the first iteration



	α_0	α_1 β_1	γ_1	α_2 β_2	γ_2	α_3 β_3	γ_3
fourth row	u_1^0	$2u_0^0+u_2^0$	v_2^0	$u_1^0+u_3^0$	$v_1^0+v_3^0$	$u_2^0+u_4^0$	$v_2^0+v_4^0$
fifth row		$\times\alpha_1$	$\times\gamma_1$	$\times\alpha_2$	$\times\gamma_2$	$\times\alpha_3$	$\times\gamma_3$
sixth row		$\beta_1\times$	$\times\beta_1$	$\beta_2\times$	$\times\beta_2$	$\beta_3\times$	$\times\beta_3$
seventh row	u_0^1	u_1^1	v_1^1	u_2^1	v_2^1	u_3^1	v_3^1

(10)

$u_0^1, u_1^1, v_1^1, \ldots$, except for a factor $\dfrac{a^2}{B}$, tabularly can be calculated from $a_0, a_1, b_1, \ldots$ In practice, the two schedules (9) and (10) naturally will be united in a single one; the multiplication-factors α, β, γ from schedule (10) then being placed at the top of the whole table. It will be remarked, that the only numbers, which in behalf of the iteration have to be calculated beforehand, are the coefficients $\alpha_p, \beta_p, \gamma_p$, given by (**3**, 9) and (**3**, 10). From the set (u_i^1, v_i^1) a new set (u_i^2, v_i^2) is derived in the same way a. s. o., and it is evident that with an increasing number (n) of iterations, proportionality will be reached between (u_i^n, v_i^n) and (u_i^{n+2}, v_i^{n+2}), both sets being characteristic for the deformations D^n and D^{n+2}, the proportionality of which — for ever-increasing n — has been proved in section **6**.

One must hold in mind that, $u_0^1, u_1^1, v_1^1 \dots$ represent the first iteration but for a factor $\frac{\lambda^2 M}{2\pi a^2 B}$.

Therefore if the process is to be carried on to the n^{th} and $(n+2)^{\text{th}}$ iteration to obtain sufficient proportionality between u_i^n and u_i^{n+2}, then the (approximate) factor of proportionality

$$\mu^* = \frac{u_i^n}{u_i^{n+2}} \quad \dots\dots\dots\dots \quad (11)$$

has to be multiplied by $\left(\frac{\lambda^2 M}{2\pi a^2 B}\right)^{-2}$ to represent (approximatively) the squared value μ_1.

Consequently the required *smallest* buckling moment, corresponding to the assumed value q (or λ) is given by

$$M = \mu \bar{M} = \frac{2\pi a^2 B \sqrt{\mu^*}}{\lambda^2} \quad \dots\dots\dots \quad (12)$$

8. *The critical buckling moment.* In the heading of the two preceding sections, as well as in the foregoing sentence it has been explicitely expressed, that the iteration process relates to a fixed value of the parameter q (or λ), i. e. to a fixed number of longitudinal waves in the buckling-deformation. Therefore if the *minimum* or critical buckling moment is desired, the iteration process has to be repeated for a great number of values λ. It cannot be denied that this involves a rather tiresome business, as, above all, the calculation of the coefficients $\alpha_p, \beta_p, \gamma_p$ from (3, 9) and (3, 10) requires a great amount of ciphering.

Therefore it is of valuable help, that for large values of λ a prediction can be made about the relation between M and λ^2.

From examples, calculated for technical purposes, the authors have learnt, that for thin-walled and long tubes and for large values of λ^2 the coefficients β and γ are so small, that they scarcely influence the numerical end-result of the iteration.

Moreover, under the same conditions, α_p may be approximated by

$$\alpha_p = \frac{(p^2+\lambda^2)^2}{(1-\nu^2)\lambda^4 + k\lambda^8} \quad \dots\dots\dots \quad (1)$$

A rough estimate makes it acceptable that $\sqrt{\mu^*}$ will be influenced only by the iterative numbers connected with low index-numbers p, and furthermore that $\sqrt{\mu^*}$ itself may be roughly estimated by

$$\sqrt{\mu^*} \backsim \frac{1}{2\alpha_p} \quad \dots\dots\dots\dots \quad (2)$$

so that for values of λ, high with respect to p, one may write approximately:

$$\sqrt{\mu^*} = \frac{1-\nu^2}{2} + \frac{k}{2}\,\lambda^4 \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

and consequently (comp. 7, (12))

$$M = 2\pi a^2 B \left\{ \frac{1-\nu^2}{\lambda^2} + k\,\lambda^2 \right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

This approximation — rough as it is — is of considerable importance. If for a practical case, the iteration has to be carried out for a number of values λ, and if one starts with small, but gradually increasing values of λ^2, one will find that M seems to be a monotonously *decreasing* function of λ^2. It is, of course, quite certain, that a minimum of M exists, but it is rather tiresome, that no data are at hand to estimate for which value of λ^2 this minimum is to be expected. Now it can be seen from (4), that a minimum of M exists, say for $\lambda^2 = \lambda_c^2$, and especially that for $\lambda^2 > \lambda_c^2$, M increases very rapidly, so that the character of the (M, λ^2)-curve is a hyperbolic one.

With this fact in view the iterative work can be considerably abbreviated by approximating the (M, λ^2)-curve by the formula

$$M = \frac{\alpha}{\lambda^2} + \beta\,\lambda^2 \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Having roughly estimated λ_c^2 by using (4) as a first approximation, one shall take care to iterate M for two values of λ^2, one of which lies considerably *under* the value λ_c^2, the other lying *above* this limit, but in its neighbourhood. The two sets of corresponding values (M_1, λ_1^2) and (M_2, λ_2^2) enable the determination of the coefficients α and β in (5). The minimum value of M, then to be obtained from this formula, represents with great accuracy the critical value of M.

9. *The skew-symmetrical buckling.* Up to now only such buckling-deformations have been considered, which are symmetrical with respect to the plane in which the moments M act. However, still another type of deformations exists, represented by

$$\left.\begin{aligned} u &= \sin\lambda\frac{z}{a}\sum_{p=1}^{\infty} u_p \sin p\varphi \\ v &= \sin\lambda\frac{z}{a}\sum_{p=0}^{\infty} -v_p \cos p\varphi \\ w &= \cos\lambda\frac{z}{a}\sum_{p=1}^{\infty} w_p \sin p\varphi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

It can be investigated along the same line as the symmetrical type, and therefore we restrain ourselves here to an abbreviated treatment.

Firstly we substitute the expressions (1) in the equations (4, 2), taking account of (4, 1), and obtain:

$$\left.\begin{aligned}
&\sum_{p=0}^{\infty}\Big\{u_p[1+k(\lambda^4+2\lambda^2p^2+p^4-2p^2+1)]+v_p\left[p+k\frac{3-\nu}{2}\lambda^2p\right]-\\
&\quad-w_p\left[\nu\lambda+k\left(\lambda^3-\frac{1-\nu}{2}\lambda p^2\right)\right]\Big\}\sin p\varphi=\frac{M\lambda^2}{\pi a^2 B}\sum_{p=0}^{\infty}u_p\sin p\varphi\cos\varphi\\
&\sum_{p=0}^{\infty}\Big\{+u_p\left[p+k\frac{3-\nu}{2}\lambda^2p\right]+v_p\left[p^2+\frac{1-\nu}{2}\lambda^2+k\frac{3(1-\nu)}{2}\lambda^2\right]-\\
&\quad-w_p\left[\frac{1+\nu}{2}\lambda p\right]\Big\}\cos p\varphi=\frac{M\lambda^2}{\pi a^2 B}\sum_{p=0}^{\infty}v_p\cos p\varphi\cos\varphi\\
&\sum_{p=0}^{\infty}\Big\{-u_p\left[\mu\lambda+k\left(\lambda^3-\frac{1-\nu}{2}\lambda p^2\right)\right]-v_p\left[\frac{1+\nu}{2}\lambda_p\right]+w_p\left[\lambda^2+\right.\\
&\quad\left.+\frac{1-\nu}{2}p^2+k\frac{1-\nu}{2}p^2\right]\Big\}\sin p\varphi=0.
\end{aligned}\right\} . \quad (2)$$

Representing the coefficients of u_p, v_p, w_p in the left members of these equations by a_{11}, a_{12}, a_{13}; $a_{21}, a_{22}, a_{23}\ldots$ and replacing the products of goniometrical functions, occurring in the right members by sums of such functions, we find:

$$\left.\begin{aligned}
&\sum_{p=0}^{\infty}[a_{11}u_p+a_{12}v_p+a_{13}w_p]\sin p\varphi=\\
&\qquad=\frac{M\lambda^2}{\pi a^2 B}\sum_{p=0}^{\infty}u_p[\sin(p+1)\varphi+\sin(p-1)\varphi]\\
&\sum_{p=0}^{\infty}[a_{21}u_p+a_{22}v_p+a_{23}w_p]\cos p\varphi=\\
&\qquad=\frac{M\lambda^2}{\pi a^2 B}\sum_{p=0}^{\infty}v_p[\cos(p+1)\varphi+\cos(p-1)\varphi]\\
&\sum_{p=0}^{\infty}[a_{31}u_p+a_{32}v_p+a_{33}w_p]\sin p\varphi=0.
\end{aligned}\right\} \quad .\;.\;. \quad (3)$$

If then these equations are ordered with respect to *sin* $p\varphi$, and *cos* $p\varphi$, the following system of equations for u_p, v_p, w_p arises if all coefficients of these functions are put equal to zero:

$$\left.\begin{array}{l}
p=0 \quad a^0_{22}\, v_0 = \dfrac{M\lambda^2}{2\pi a^2 B}\, v_1 \\[2ex]
p=1 \left\{\begin{array}{l}
a^1_{11}\, u_1 + a^1_{12}\, v_1 + a^1_{13}\, w_1 = \dfrac{M\lambda^2}{2\pi a^2 B}\, u_2 \\[2ex]
a^1_{21}\, u_1 + a^1_{22}\, v_1 + a^1_{23}\, w_1 = \dfrac{M\lambda^2}{2\pi a^2 B}\,(2v_0 + v_2) \\[2ex]
a^1_{31}\, u_1 + a^1_{32}\, v_1 + a^1_{33}\, w_1 = 0
\end{array}\right. \\[6ex]
\text{and, in general:} \\[2ex]
\begin{array}{l}\text{for arbitrary} \\ \text{integer } p\end{array} \left\{\begin{array}{l}
a^p_{11}\, u_p + a^p_{12}\, v_p + a^p_{13}\, w_p = \dfrac{M\lambda^2}{2\pi a^2 B}\,(u_{p-1} + u_{p+1}) \\[2ex]
a^p_{21}\, u_p + a^p_{22}\, v_p + a^p_{23}\, w_p = \dfrac{M\lambda^2}{2\pi a^2 B}\,(v_{p-1} + v_{p+1}) \\[2ex]
a^p_{31}\, u_p + a^p_{32}\, v_p + a^p_{33}\, w_p = 0
\end{array}\right.
\end{array}\right\} . \quad (4)$$

Putting

$$\triangle^p \equiv \begin{vmatrix} a^p_{11} & a^p_{12} & a^p_{13} \\[2ex] a^p_{21} & a^p_{22} & a^p_{23} \\[2ex] a^p_{31} & a^p_{32} & a^p_{33} \end{vmatrix} \quad \text{and} \left.\begin{array}{r} a^p = \dfrac{A^p_{11}}{\triangle^p} \\[2ex] b^p = \dfrac{A^p_{21}}{\triangle^p} \\[2ex] c^p = \dfrac{A^p_{22}}{\triangle^p} \end{array}\right\} \quad . \quad . \quad . \quad . \quad (5)$$

(A^p_{ij} designing the minor determinant of the element a_{ij} in $\triangle^p$) we find

$$\left.\begin{array}{l}
u_0 = 0 \\[2ex]
v_0 = \dfrac{M\lambda^2}{2\pi a^2 B}\,\dfrac{v_1}{a^0_{2n}} = \dfrac{M\lambda^2}{2\pi a^2 B}\,\dfrac{v_1}{\frac{1-\nu}{2}\,\lambda^2\,(1+3k)} \\[3ex]
u_1 = \dfrac{M\lambda^2}{2\pi a^2 B}\,\dfrac{[u_2 A^1_{11} + (2v_0 + v_2)\, A^1_{21}]}{\triangle^1} \\[3ex]
v_1 = \dfrac{M\lambda^2}{2\pi a^2 B}\,\dfrac{[u_2 A^1_{12} + (2v_0 + v_2)\, A^1_{22}]}{\triangle^1} \\[3ex]
u_p = \dfrac{M\lambda^2}{2\pi a^2 B}\,[a^p\,(u_{p-1} + u_{p+1}) + b^p\,(v_{p-1} + v_{p+1})] \\[3ex]
v_p = \dfrac{M\lambda^2}{2\pi a^2 B}\,[b^p\,(u_{p-1} + u_{p+1}) + c^p\,(v_{p-1} + v_{p+1})]
\end{array}\right\} \quad . \quad . \quad . \quad (6)$$

This again is a system of homogeneous linear equations, which only for special values of the bending moment $M = \mu \overline{M}$ admits a solution u_p, v_p.

Obviously from now the determination of the total characteristic numbers μ is identical with that, treated in the previous sections, and therefore we restrict ourselves in reproducing the scheme of iteration:

c^0	a^1	c^1	a^2	c^2	a^3	c^3
	b^1		b^2		b^3	
v_0^0	u_1^0	v_1^0	u_2^0	v_2^0	u_3^0	v_3^0
v_1^0	u_2^0	v_2^0	u_3^0	v_3^0	u_4^0	v_4^0
—	—	$2\,v_0^0$	u_1^0	v_1^0	u_2^0	v_2^0
v_1^0	u_2^0	$2\,v_0^0+v_2^0$	$u_1^0+u_3^0$	$v_1^0+v_3^0$	$u_2^0+u_4^0$	$v_2^0+v_4^0$
$c^0\,v_1^0$	$a^1\,u_2^0$	$c^1\,(2\,v_0^0+v_2^0)$	$a^2\,(u_1^0+u_3^0)$	$c^2\,(v_1^0+v_3^0)$	$a^3\,(u_2^0+u_4^0)$	$c^3\,(v_2^0+v_4^0)$
	$b^1\,(2\,v_0^0+v_2^0)$	$b^1\,u_2^0$	$b^2\,(v_1^0+v_3^0)$	$b^2\,(u_1^0+u_3^0)$	$b^3\,(v_2^0+v_4^0)$	$b^3\,(u_2^0+u_4^0)$
v_0^1	u_1^1	v_1^1	u_2^1	v_2^1	u_3^1	v_3^1

(7)

Calculations, made in connection with a technical problem, proved that the critical value of M for the anti-symmetrical case exceeds that of the buckling moment, which produces symmetrical deformation.

It may be taken for granted that this statement has general validity.

Hydrodynamics. — *On the application of statistical mechanics to the theory of turbulent fluid motion. A hypothesis which can serve as a basis for a statistical treatment of some mathematical model systems.* I. By J. M. BURGERS. (Mededeeling N⁰. 39 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft).

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1. In a previous paper certain systems of mathematical equations have been investigated, which served to illustrate various relations occurring in the theory of turbulent fluid motion [1]. It was hoped that the investigation of these systems might throw light in particular on those features which are of importance for the development of a statistical theory. No success, however, was attained in this respect: it was mentioned that the statistical method, proposed in an earlier series of papers, when adapted to the systems considered did not lead to satisfactory results [2]. In particular it failed to lead to a "spectrum" of elementary components, fulfilling the requirement that it should consist of a part in which the mean square amplitude $\overline{\zeta_n \zeta_n^*}$ should be of the form c/n^2, followed by a "tail" in which $\overline{\zeta_n \zeta_n^*}$ should decrease much more rapidly.

The conclusion was drawn that the assumption of equal a priori weights for all possible values of the amplitudes of the elementary components, combined with the usual form of the equation for the dissipation of energy, in which the non linear terms of the equations describing the system have no influence, should be considered as unsuitable.

It is the object of the present paper to show that the difficulty encountered here can be removed by introducing an additional hypothesis, which may be regarded as describing — be it in a schematical way — the influence of the non linear terms of the equations upon the distribution of weights over the phase space.

2. The solutions of the systems of equations considered in the previous paper, owing to the presence of non linear terms in these equations, show a tendency towards the development of *discontinuities*.

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[1] J. M. BURGERS, Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, Verhand. Kon. Nederl. Akad. v. Wetenschappen, Afd. Natuurkunde (1e sectie), **17**, No. 2 (1939).

[2] *L.c.* p. 47, under III.

On looking more closely, however, it appeared that true (*i.e.* infinitely sharp) discontinuities are not formed: the presence in the equations of certain terms comparable to the frictional terms of hydrodynamics, prevents that the derivative of the function becomes infinite in an apparent discontinuity, the actual value being roughly proportional to the square of the amplitude of the discontinuity and inversely proportional to the "coefficient of friction" ν. The "breadth" of the discontinuity at the same time is directly proportional to ν and inversely proportional to the amplitude [3].

These results have the consequence that with decreasing amplitude a discontinuity more and more loses its character: a discontinuity of small amplitude cannot at the same time be a sharp one, and gradually merges into a shape devoid of any peculiarities. Another consequence is that the dissipation in a discontinuity always has a finite value, which appears to be roughly proportional to the third power of the amplitude of the discontinuity, whereas it is independent of the coefficient ν. [4]

When a solution $\psi(y, t)$, where y is restricted to the domain $0 \leqslant y \leqslant 1$, is developed into a series of the form $\Sigma \zeta_n \sin \pi ny$, the magnitude of the coefficients of higher orders is mainly dependent upon the character of the discontinuities present in the function ψ. By way of example we might assume that in the discontinuities ψ can be approximated by an expression of the form: $\psi = B_i - A_i^2 (y - y_i)/\nu$, where y_i represents the centre of the discontinuity, A_i its (half) amplitude, while its extent is defined by the inequalities: $-\nu/A_i \leqslant y - y_i \leqslant +\nu/A_i$. It is then found that the value of ζ_n becomes:

$$\zeta_n = -\frac{1}{\pi^2 n^2 \nu} \sum_i 4 A_i^2 \sin(\pi n \nu / A_i) \cos(\pi n y_i) + 0(n^{-2}) \quad . \quad . \quad (1)$$

When a different supposition is made, *e.g.* that in the discontinuities ψ can be approximated by $\psi = B_i - A_i \, tgh \, \{A_i (y - y_i)/\nu\}$, the value of ζ_n becomes:

$$\zeta_n = -2\pi\nu \sum_i \frac{\cos(\pi n y_i)}{\sinh(\pi^2 n\nu / 2A_i)} + 0(n^{-2}). \quad . \quad . \quad . \quad . \quad (2)$$

In both formulae $0(n^{-2})$ denotes a term which decreases at least as rapidly as n^{-2}, and possibly even more rapidly [5].

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[3] *L.c.* pp. 40—43; see also p. 23.

[4] Compare *l.c.* p. 43, eq. (19.10), and also pp. 25, 26, eqs. (13.2) and (13.4).

That this property of a discontinuity in the systems considered has an analogy in a property of fluid motion, has been pointed out in these Proceedings **43** (1940), p. 11, footnote 11.

[5] The results given in form. (1) and (2) can be obtained by means of a method indicated by E. T. WHITTAKER and G. N. WATSON, Modern Analysis (3rd Ed., Cambridge 1920), p. 167, § 9.3 (it is stated there that this method originally is due to STOKES).

It is of interest now to consider the dependence of ζ_n upon the values of the A_i. So long as the A_i are large, so that $\pi^2 n \nu / 2 A_i$ is small compared with unity, both formulae give:

$$\zeta_n \cong -\frac{4}{\pi n} \sum_i A_i \cos(\pi n y_i) + 0\,(n^{-2}) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When $A_i = \pi^2 n \nu / 4$, the corresponding term in the expression for ζ_n becomes:

$$-(\pi^2 \nu/4)\, sin\,(4/\pi) \cos(\pi n y_i) \cong -2{,}4\, \nu \cos(\pi n y_i),$$

$$\text{resp. } -(2\pi\nu/sinh\, 2) \cos(\pi n y_i) \cong -1{,}7\, \nu \cos(\pi n y_i).$$

For values of A_i decreasing below $n\nu$ these terms decrease very rapidly, at least proportionally to A_i^2.

It must be observed that in the first type of discontinuity considered above the circumstance that $\partial\psi/\partial y$ changes abruptly at the ends of the discontinuity introduces an unnatural feature. When $\partial\psi/\partial y$ and the higher derivatives are continuous functions, it is found that the behaviour of ζ_n approaches much more to that described by form. (2), while also the quantity denoted by $0(n^{-2})$ decreases much more rapidly than n^{-2}.

3. The results of the preceding section have shown that the values of the coefficients ζ_n, once they have diminished to an amount of the order ν, decrease very rapidly upon a further decrease of the amplitude of the discontinuities. We may express this result in another form by saying that finite values of the ζ_n below an amount of the order ν have a very small chance of appearance, although, of course, there is no restriction to the appearance of the value $\zeta_n = 0$.

This result must be considered as describing an influence exerted by the non linear terms in the equations of the system upon the relative probabilities of the values which can be taken by the ζ_n. It must be expected that this influence will have an effect upon the statistical properties of the system.

It is true that the result as stated here is still of a rather vague character. In order to be able to make use of it in developing a statistical theory we shall replace it by the following more concrete hypothesis: — it will be assumed that the absolute values of the coefficients ζ_n occurring in the FOURIER series cannot take values between zero and a finite amount, a "threshold value", provisionally to be written: $\delta = \beta\nu$.

It has been shown by EHRENFEST that the introduction of such an assumption into a statistical theory has an important influence upon the final results of the theory, and that in particular it is sufficient to ensure that there shall be no "equipartition" along the spectrum down to the members of the highest orders[6]). EHRENFEST demonstrated this theorem for the case of radiation, where the system considered is a conservative one, and where the distribution of the energy over the spectrum is investigated. In the present case we have to do with a non conservative system; instead of the equation expressing the conservation of energy here the equation expressing that the energy dissipated in the system is equal to the energy delivered to it by exterior forces, forms the condition upon which the statistical theory is built, and the problem is to find the distribution of the dissipation over the spectrum. Whereas in the statistical considerations developed in a former series of papers, where no assumption was introduced of the type indicated above and where all values of the ζ_n were considered as equally probable, the difficulty was encountered that there appeared to be "equipartition of dissipation" along the whole spectrum, this difficulty is now found to disappear, and a natural "limit" to the spectrum is obtained.

The hypothesis in the form given above is not yet sufficiently definite, as an assumption still is necessary concerning the a priori probabilities of values of the $|\zeta_n|$ above the threshold value δ. In order to make the mathematical calculations not too complicated, we shall introduce the further assumption that the real and imaginary parts of the coefficients $\zeta_n = \xi_n + i\eta_n$ only may take values which are of the form $k\delta$, where the k are integers. It cannot be said that there is a necessity for the introduction of this further restriction upon the possible values of the ζ_n; possibly the assumption that all absolute values greater than δ should possess the same probability, might serve as well, but it is probable that this will lead to much more cumbersome formulae, without greatly affecting the general character of the results.

4. In order to build up a statistical theory of the behaviour of the system, we avail ourselves of a method already applied in some previous papers[7]). An instantaneous state of the secondary motion of the system can be described by giving the course of the function $\psi(y, t)$ at that instant; instead of this function we can also give the values of the FOURIER coefficients $\zeta_n = \xi_n + i\eta_n$. When a multidimensional space is introduced having the ξ_n and η_n as coordinates, the instantaneous state of the system can be represented in this space by a single point. As, according to the hypothesis introduced above, the values of the ξ_n and η_n should be integer multiples of δ, the representative point must belong to a lattice of points, formed with the spacing δ in the direction of every coordinate. The lattice points form an enumerable set; they will be indicated by numbers 0, 1, 2 . . . . m, ...

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[6]) P. EHRENFEST, Welche Züge der Lichtquantenhypothese spielen in der Theorie der Wärmestrahlung eine wesentliche Rolle? Ann. d. Physik (IV) **36**, 91—118 (1911).

[7]) Compare J. M. BURGERS, these Proceedings **32**, 822 (1929), and **36**, 278 (1933).

When the state of the system is observed at a large number (M) of instants, separated by equal intervals of time (sufficiently large in order that we may neglect any correlation between the instantaneous states of the system at consecutive instants), the history of the system can be described "microscopically" by giving the sequence of the M representative points in the phase space. For a "statistical" description of the properties of the system the exact order of these points is of no importance: it is sufficient to know the number of times that the system is at each of the points 0, 1, 2, ... m, ... of the lattice. The number of times that the system is in the instantaneous state represented by the point numbered m will be written: f_m. The behaviour of the system then is statistically determined by the numbers f_0, f_1, f_2, ... f_m, ...

Various statistical types of behaviour of the system can be imagined; these types will be distinguished by an index s. Each type is characterized by a set of numbers f_{0s}, f_{1s}, f_{2s}, ... f_{ms} .... These numbers fulfill the relation:

$$\sum_m f_{ms} = M. \quad \ldots \ldots \ldots \quad (4)$$

To each such a set of numbers there will belong

$$W_s = M!/(f_{0s}!\, f_{1s}!\, f_{2s}! \ldots) \quad \ldots \ldots \ldots \quad (5)$$

different forms of arrangement of the sequence of instantaneous states, *i.e.* different "microscopical" histories of the system.

In each instantaneous state m of the system the dissipation of energy by the secondary motion in unit time has a definite value, given by the expression: [8]

$$\varepsilon_m = \tfrac{1}{2}\, \nu \pi^2 \sum_n n^2 \left(\xi_{nm}^2 + \eta_{nm}^2\right) \quad \ldots \ldots \ldots \quad (6)$$

When a sufficiently large number M of instantaneous states has been observed, the average dissipation in unit time must be equal to the energy transferred in unit time to the secondary motion, which latter quantity with a sufficient approximation for our purpose is given by PU. Hence the following equation must be fulfilled by every sequence of instantaneous states:

$$\left(\sum_m f_{ms}\, \varepsilon_m\right)/M = PU \quad \ldots \ldots \ldots \quad (7)$$

We now assume that all sequences of M instantaneous states, for which the numbers f_{ms} fulfill both eq. (4) and eq. (7), are equally

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[8] Compare the paper mentioned in footnote 1), p. 44, eq. (20.10), in which the term $\frac{1}{2}\nu\pi^2 \sum n^2 \zeta_n \zeta_n^*$ describes the dissipation of energy by the secondary motion.

probable. Consequently when a great number of sequences are observed, every statistically described state, *i.e.* every set of numbers f_{ms}, will occur with a frequency proportional to the number W_s for that set. Hence the *average statistical state* of the system will be defined by a set of numbers $\bar{f}_m$, determined by the equation:

$$\bar{f}_m = (\sum_s f_{ms} W_s)/(\sum_s W_s) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where the summation with respect to s extends over all sets of numbers f_{ms}, fulfilling (4) and (7).

The further work now to be done is the evaluation of various mean values characterizing the system, making use of the quantities $\bar{f}_m$ defined by (8).

5. The evaluation of the sums occurring in eq. (8) can be done with the aid of FOWLER's method of calculating averages by introducing a "partition function" [9]. We put:

$$F(\sigma) = \sum_m \sigma^{\varepsilon_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where σ represents a (complex) auxiliary variable; ε_m is given by (6), and the summation extends over all lattice points of the phase space. The sum can be assumed to be convergent so long as $|\sigma| < 1$; the function F becomes infinite at the point $\sigma = 1$.

It can be shown that the sum $\sum_s W_s$ which occurs in (8) is equal to the coefficient of σ^{PUM} in the development of F^M in a power series according to σ; and that $\sum_s f_{ms} W_s$ is equal to M times the coefficient of $\sigma^{PUM-\varepsilon_m}$ in the development of F^{M-1}. These coefficients can be obtained by means of the following integrals:

$$\sum_s W_s = \frac{1}{2\pi i} \int_\gamma \frac{d\sigma}{\sigma} \frac{F^M}{\sigma^{PUM}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

$$\sum_s f_{ms} W_s = \frac{1}{2\pi i} \int_\gamma \frac{d\sigma}{\sigma} \frac{M F^{M-1} \sigma^{\varepsilon_m}}{\sigma^{PUM}} \quad . \quad . \quad . \quad . \quad . \quad (10b)$$

where γ denotes a contour circulating counter clockwise round $\sigma = 0$.

As the integrand F^M/σ^{PUM} becomes infinite both at $\sigma = 0$ and at $\sigma = 1$,

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[9] See C. G. DARWIN and R. H. FOWLER, Philos. Magaz. (6) **44**, 450 (1922); R. H. FOWLER, Statistical Mechanics (1st Ed., Cambridge 1929), p. 22 seqq.

there will be a minimum on the real axis at some point σ^*, which is the solution of the equation:

$$\frac{d}{d\sigma}\left(F^M/\sigma^{PUM}\right)=0,$$

or:

$$\sigma\,(dF/d\sigma)=PUF \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

When the contour γ is arranged so as to pass through the point σ^* in the direction of the imaginary axis, the integrals (10*a*), (10*b*) can be approximated by applying the method of "steepest descent". This leads to the result that, apart from a factor depending upon the second derivative of (F^M/σ^{PUM}), which factor is the same for both integrals, the values of the expressions (10*a*), (10*b*) are practically determined by the magnitude of the integrand at the point σ^*. It is not necessary to write down the formulae in full, as by division we at once arrive at the result:

$$\left(\sum_s f_{ms}\,W_s\right)/\left(\sum_s W_s\right)=M\,\sigma^{\varepsilon_m}/F,$$

from which, by (8):

$$\bar{f}_m/M=\sigma^{\varepsilon_m}/F. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

To simplify notation σ has been written for σ^*; henceforward σ will always denote the (real) root of equation (11).

6. Having obtained an expression for $\bar{f}_m$ we now are in a position which makes it possible to calculate all kinds of average values pertaining to the system. The most important average values are those of ξ_n^2 and η_n^2.

We have:

$$\overline{\xi_n^2}=\sum_m(\bar{f}_m/M)\,\xi_{nm}^2=\left(\sum_m\sigma^{\varepsilon_m}\,\xi_{nm}^2\right)/F. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Now the values of ξ and η must be integer multiples of δ. Writing:

$$\xi_{nm}=x_{nm}\,\delta\,;\ \eta_{nm}=y_{nm}\,\delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

we have:

$$\varepsilon_m=\tfrac{1}{2}\,\nu\,\pi^2\,\delta^2\sum_n n^2\left(x_{nm}^2+y_{nm}^2\right).$$

We put:

$$\theta=\sigma^{\nu\pi^2\delta^2/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

so that θ, like σ, is a real number between 0 and 1. We then have:

$$\sigma^{\varepsilon_m} = \theta^{\Sigma n^2 \left(x_{nm}^2 + y_{nm}^2\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where the summation in the exponent refers to all values of n, *i.e.* to all coordinates x_{nm}, y_{nm} of a single point m of the lattice.

According to eq. (9) the value of the function F is obtained by summing this expression over all points m of the lattice in the phase space, which means that all the x_{nm} as well as the y_{nm} must run consecutively through all integer numbers, positive and negative. We therefore have, *e.g.*:

$$\sum_m \theta^{\Sigma n^2 x_{nm}^2} = \prod_n \left(\sum_k \theta^{n^2 k^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

as the general term of the sum in the left hand member is the same as the general term of the product in the right hand member. In a similar way we have:

$$\sum_m x_{nm}^2 \, \theta^{\Sigma n^2 x_{nm}^2} = \left(\sum_k k^2 \theta^{n^2 k^2}\right) \prod_{n_1} \left(\sum_k \theta^{n_1^2 k^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where the product with respect to n_1 refers to all possible values of n_1 with the exception of $n_1 = n$. The summation with respect to k both in (17) and in (18) refers to all positive and negative integer values.

As the y_{nm} in the exponent of (16) can be treated in a similar way as the x_{nm}, it is possible to transform eq. (13) into the following simpler expression:

$$\overline{\xi_n^2} = \delta^2 \left(\sum_k k^2 \theta^{n^2 k^2}\right) / \left(\sum_k \theta^{n^2 k^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Writing:

$$\theta^{n^2} = \theta_n; \quad \varphi = \sum_k \theta_n^{k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

we finally obtain:

$$\overline{\xi_n^2} = \delta^2 \, \theta_n \, d\,(\ln \varphi)/d\theta_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where the notation *ln* has been used to denote the natural logarithm.

7. The function φ introduced in (20) is a Theta-function; in fact: [10]

$$\varphi = \vartheta_3 (z, q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

with $z = 0$ and $q = \theta_n$. Whereas the series given in (20) is useful when θ_n is a small quantity, the theory of the Theta-functions makes it possible

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[10] See E. T. WHITTAKER and G. N. WATSON, Modern Analysis (3rd Ed., Cambridge 1920), p. 464.

to obtain an approximation also for the case where θ_n is near to unity[11]. In this way we have:

$$\varphi = 1 + 2\,\theta_n + 2\,\theta_n^4 + \ldots \qquad (23a)$$

and:

$$\varphi = \sqrt{\frac{\pi}{-\ln\theta_n}}\,(1 + e^{\pi^2/\ln\theta_n} + \ldots) \qquad (23b)$$

When a precision of 0,001 is considered necessary the first formula, as far as written down, can be used for $\theta_n \leqslant 0{,}25$, and the second one, as far as written down, for $\theta_n \geqslant 0{,}25$. When a precision of 0,01 is sufficient, the last terms of both formulae can be omitted.

Application of equation (21) now gives:

when θ_n is small:

$$\overline{\xi_n^2} \simeq 2\,\delta^2\,\theta_n = 2\,\delta^2\,\sigma^{\nu\pi^2 n^2 \delta^2/2} \qquad (24a)$$

when θ_n is near unity:

$$\overline{\xi_n^2} \simeq \frac{\delta^2}{-2\ln\theta_n} = \frac{1}{\nu\,\pi^2\,n^2\,.\,(-\ln\sigma)} \qquad (24b)$$

The same expressions are obtained for $\overline{\eta_n^2}$.

These results show that for small values of n the mean values of ξ_n and η_n, and thus also that of $|\zeta_n|$, are inversely proportional to n, so that here we have "equipartition of dissipation", whereas for large values of n, where eq. (24a) applies, the mean values of ξ_n etc. decrease more rapidly.

It remains to find the value of $(-\ln\sigma)$ and that of the total dissipation.

8. In general it may be assumed that with increasing values of n the expression (24*b*) decreases so rapidly, that we may write:

$$\sum_n \overline{\xi_n^2} = \sum_n \overline{\eta_n^2} \simeq \frac{1}{6\,\nu\,.\,(-\ln\sigma)}. \qquad (25)$$

Application of eq. (20.5) of the paper mentioned in footnote 1) (in which $dU/dt = 0$, for a stationary state of the system, while νU can be neglected) gives:

$$P = \tfrac{1}{2}\sum_n(\overline{\xi_n^2} + \overline{\eta_n^2}) = \frac{1}{6\,\nu\,.\,(-\ln\sigma)} \qquad (26)$$

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[11] *Ibid.* p. 474, § 21.51.

so that:

$$-\ln \sigma = 1/6\, \nu P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Equation (24*b*) then gives:

$$\overline{\xi_n^2} = \overline{\eta_n^2} = 6\, P/\pi^2 n^2 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

for *n* not too great.

In order to find the total dissipation, it is necessary to calculate $\Sigma n^2 (\overline{\xi_n^2} + \overline{\eta_n^2})$, which sum now is convergent[12]. Now eq. (24*b*) and eq. (28) approximately can be applied until a number N_1, determined by $(\nu \pi^2 N_1{}^2 \delta^2/2) \,.\, \ln \sigma = \ln 0{,}25$, from which, making use of (27): $N_1 \cong 1{,}3\, P/\delta$. Consequently we may write:

$$\sum_{n=1}^{N_1} n^2 (\overline{\xi_n^2} + \overline{\eta_n^2}) \simeq \frac{12\, N_1\, P}{\pi^2} \simeq 1{,}6\, P^{3/2}/\delta.$$

For values of $n > N_1$ we must have recourse to form. (24*a*). This formula is somewhat less exact than (24*b*), and it gives a value slightly too high. It can be applied to obtain an upper limit of the amount which must be added to the sum already found. We will omit the details of the calculation, and mention as a rough approximation that the value of $\Sigma n^2 (\overline{\xi_n^2} + \overline{\eta_n^2})$ for $n > N_1$ will lie between: $1{,}30\, \delta^2 N_1{}^3$ and $1{,}15\, \delta^2 N_1{}^3$, *i.e.* between: $2{,}9\, P^{3/2}/\delta$ and $2{,}5\, P^{3/2}/\delta$. Possibly the value 2,6 may be taken as a tolerable approximation for the coefficient.

Substitution of these results into the expression for the total dissipation of energy by the secondary motion gives as a rough approximation:

$$\tfrac{1}{2}\, \nu \pi^2 \sum_n n^2 (\overline{\xi_n^2} + \overline{\eta_n^2}) \simeq 21\, \nu P^{3/2}/\delta = PU \quad . \quad . \quad . \quad . \quad (29)$$

from which: $U \simeq 21\, \nu \sqrt{P}/\delta$.

Now according to the hypothesis formulated in section 3, δ should be equal to $\beta \nu$, where β is a number, presumably of the order unity. Hence we finally obtain:

$$U \simeq (21/\beta) \sqrt{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

It is hoped that in a further paper we may come back to the meaning that can be attached to this result.

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[12] The following calculation in reality is identical with the working out of eq. (11) of the text above in order to find the connection between σ and PU. As a connection between σ and P has already been established in eq. (27), the result to be obtained will enable us to express U as a function of the other quantities, and thus to find the relation between U and P. — A direct calculation of the function F and its derivative would bring great difficulties.

Mathematics. — *Ueber vier Gerade in* R_4 [1]. Von W. VAN DER WOUDE.

(Communicated at the meeting of September 28, 1940.)

Es ist bekannt, dass vier Gerade in R_4 durch eine fünfte zu *fünf assoziierten Geraden* ergänzt werden können, d.h. zu fünf Geraden mit der Eigenschaft:
jede Ebene, die vier dieser Geraden schneidet, schneidet auch die fünfte.

Hierbei wird angenommen, dass die vier gegebenen Geraden allgemeine Lage haben.

Es ist fraglos der Mühe wert, die an ihre Lage gestellten Anforderungen näher zu präzisieren. Diese sind:

a) Es liegen keine drei Gerade in demselben R_3;

b) Alle vier schneiden nicht dieselbe Gerade;

c) Es gibt keine Ebene durch eine der vier Geraden, die die drei anderen schneidet.

Nachstehend betrachte ich die Figur von vier Geraden, bei welcher diesen Anforderungen *nicht* entsprochen wird. Zuvor sei bemerkt, dass ich immer *eine* dieser Anforderungen fallen lasse, jedoch in umgekehrter Reihenfolge, wie sie hier angegeben sind. Es wird sich zeigen, dass der Fall, bei welchem der Anforderung *c*) nicht genügt wird (unter I) nur ein *besonderer Fall* von dem allgemeinen ist, während die beiden anderen Fälle besser als *Ausnahmefälle* bezeichnet werden können. Aus diesem Grunde vermelde ich in I nur das Resultat. In II und III findet man statt einer fünften assoziierten eine Regelfläche. In dem oben als (*a*) bezeichneten Falle III ist dies sogleich einzusehen. Weniger einfach ist der Fall II; die Behandlung desselben ist der Hauptzweck dieses Artikels.

Ferner wird angenommen dass die vier gegebenen Geraden, abgesehen von den ihnen auferlegten Bedingungen, allgemeine Lage haben, d.h. dass zwischen ihren Koordinaten keine anderen algebraischen Beziehungen bestehen, als diejenigen, die durch diese Bedingungen ausgedrückt werden.

I.

§ 1. *Gegeben sind vier Gerade* a_1, a_2, a_3, a_4; *eine Ebene durch* a_4 *schneidet* a_1, a_2, a_3.

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[1] Die Figur von vier Geraden in R_4 wird sehr eingehend behandelt in
H. F. BAKER, Principles of Geometry IV; Cambridge, University Press;
siehe auch:
R. WEITZENBÖCK, Die projektiven Invarianten von vier und fünf Geraden in R_4. Proc. Kon. Akad. v. Wetensch, Amsterdam XLII; Indagationes Mathematicae I, Fasc. 2.
W. VAN DER WOUDE, On certain linear constructions. Proc. Kon. Akad. v. Wetensch., Amsterdam, XLI.

Die von BAKER l.c. für a_5 angegebene Konstruktion bleibt völlig eindeutig bestehen. Jedoch fallen einige dabei auftretende Punkte zusammen, ebenfalls einige Gerade; schliesslich fällt a_5 mit a_4 zusammen. Es wird nicht erforderlich sein, hierauf einzugehen.

II.

§ 2. *Die Geraden a_1, a_2, a_3, a_4 schneiden dieselbe Gerade b in den Punkten T_1, T_2, T_3, T_4.*

Im Verfolge werden wir einen Punkt T durch die Gleichung (in Raumkoordinaten) $T = 0$ darstellen.

Wir wählen — was auf drei verschiedene Weisen möglich ist — zwei Punkte, A und B, so, dass wir durch eine geeignete Wahl des in jeder Gleichung vorkommenden konstanten Faktors schreiben können

$$T_1 = A + B,\ T_2 = A - B,\ T_3 = A + kB,\ T_4 = A - kB \quad (k \neq 0,\ k \neq \pm 1).$$

P möge ein allgemeiner Punkt in R_4 sein. Durch Projizieren der vier Geraden aus P auf einen beliebigen R_3 sehen wir sofort:

Ausser der Ebene (P, b) geht noch eine Ebene durch P, welche die vier Geraden a schneidet. Wir nennen die Schnittpunkte C, D, E, F, und sorgen, dass die lineare Beziehung zwischen diesen Punkten ausgedrückt wird durch

$$C + D + E + F = 0. \quad \ldots\ldots\ldots \quad (1)$$

Eine andere identische lineare Beziehung zwischen A, B, C, D, E und F besteht nicht.

Wir stellen nun die Geraden a_i folgendermassen dar

$$\left.\begin{aligned} a_1 \ldots\ldots \alpha(A + B) + C &= 0 \\ a_2 \ldots\ldots \beta(A - B) + D &= 0 \\ a_3 \ldots\ldots \gamma(A + kB) + E &= 0 \\ a_4 \ldots\ldots \delta(A - kB) + F &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (2)$$

Dadurch, dass α ein bestimmter Wert gegeben wird, ist ein Punkt von a_1 bestimmt. Wir fragen nun erst:

Welche Beziehungen müssen zwischen $\alpha, \beta, \gamma, \delta$ bestehen, damit die hierdurch bestimmten Punkte P, Q, R, S, bezw. auf $a_1, \ldots a_4$, in derselben Ebene liegen?

Dazu ist erforderlich und genügend, dass vier Konstante a, b, c, d gefunden werden können, sodass identisch

$$aP + bQ + cR + dS = 0.$$

Diese identische Beziehung, die zu einer identischen Beziehung zwischen A, B, C, D, E, F zurückgeführt wird, kann jedoch keine andere sein als (1). Also ist

$$a = b = c = d;$$

Ferner zeigt sich

$$\left.\begin{aligned}\alpha+\beta+\gamma+\delta&=0\\ \alpha-\beta+k\,(\gamma-\delta)&=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Die Ebenen, welche die vier Geraden schneiden, zerfallen in 2 Systeme, von je ∞^2 Ebenen:

1. alle Ebenen durch *b*;
2. die ferner genannten Ebenen: diejenigen des *zweiten* Systems, bestimmt durch die Punkte *P*, *Q*, *R*, *S*,

$$\left.\begin{aligned}P=\alpha\,(A+B)+C,\ Q=\beta\,(A-B)+D,\\ R=\gamma\,(A+kB)+E,\ S=\delta\,(A-kB)+F\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4)$$

wobei $\alpha, \beta, \gamma, \delta$ (3) genügen.

§ 3. Betrachten wir dieses letzte Ebenensystem näher. Es leuchtet ein, dass ein solches System das dualistische Analogon einer geradlinigen dreidimensionalen Punktvarietät (∞^2 Gerade) in R_4 ist. Dort bestimmen wir gewöhnlich zuerst die Punktgleichung des Raumes, der durch die Punkte jener Geraden gebildet wird, hier also die tangentiale Gleichung, welcher die Räume durch diese Ebenen genügen.

Ersetzen wir $A=0$ durch $\xi_1=0$,
$B=0$ durch $\xi_2=0$,
$C, D, E, F=0$ durch $\xi_3, \xi_4, \xi_5, \xi_6=0$
(wobei $\xi_3+\xi_4+\xi_5+\xi_6=0$), ... (1$^+$),

dann finden wir jene Gleichung dadurch, dass $\alpha, \beta, \gamma, \delta$ und ξ_6 eliminiert wird aus

$$\left.\begin{aligned}\alpha\,(\xi_1+\xi_2)+\xi_3&=0\\ \beta\,(\xi_1-\xi_2)+\xi_4&=0\\ \gamma\,(\xi_1+k\,\xi_2)+\xi_5&=0\\ (\xi_1-k\,\xi_2)+\xi_6&=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4^+)$$

mit (1$^+$) und (3).

Die gesuchte Gleichung ist dann diejenige einer Varietät dritter Klasse:

$$(1+k)\,(\xi_1+k\,\xi_2)\,\{\xi_3\,(\xi_1-\xi_2)+\xi_4\,(\xi_1+\xi_2)\}-2\,k\,\xi_5\,(\xi_1^2-\xi_2^2)=0.$$

§ 4. Durch einen allgemeinen Punkt in R_4 geht eine Ebene dieses zweiten Ebenensystems. Jedes Paar dieses Systems schneidet einander mindestens in einem Punkt und nur in einem Punkt. Hätten nämlich zwei jener Ebenen eine Gerade gemein, dann würden sie in dem selben R_3 liegen und in diesem R_3 lägen dann mindestens drei der Geraden a_i, was streitig mit unserer Voraussetzung sein würde. Da durch einen allgemeinen Punkt nur eine Ebene geht, ergibt sich:

Schneiden zwei Ebenen des zweiten Systems einander in einem Punkte Q, dann ist Q der Schnittpunkt unendlich vieler dieser Ebenen.

Hier erheben sich also folgende Fragen: zu bestimmen

1) den geometrischen Ort von Q;

2) den geometrischen Ort der Ebenen durch Q, wenn Q festgehalten wird.

Beide Fragen hängen eng mit einer dritten zusammen: zu bestimmen

3) den geometrischen Ort der Geraden, die alle Ebenen beider Systeme (siehe § 2) schneiden; mit anderen Worten der Geraden, welche die fünfte assoziierte der vier Geraden a_i ersetzen.

Die dritte Frage behandeln wir zuerst.

Wir kehren zu diesem Zwecke zurück zu den Gleichungen (4):

$$P = \alpha(A+B) + C, \; Q = \beta(A-B) + D, \; R = \gamma(A+kB) + E.$$

Durch P, Q, R ist eine Ebene des zweiten Systems bestimmt, wenn nach (3) die Beziehung besteht

$$(k+1)\alpha + (k-1)\beta + 2k\gamma = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

Wir können diese Ebene darstellen durch einen Punkt U:

$$U = pP + qQ + rR = (\alpha p + \beta q + \gamma r)A + (\alpha p - \beta q + k\gamma r)B + pC + qD + rE. \quad (6)$$

worin p, q, r Parameter sind. Betrachten wir aber p, q, r als Konstante und α, β, γ als Parameter, zwischen denen die Beziehung (5) besteht, dann stellt dieselbe Gleichung eine Ebene des *ersten* Systems dar.

Wir können nun versuchen, eine Beziehung zwischen den Konstanten p, q, r zu finden, sodass (6) nicht eine Ebene, sondern eine Gerade darstellt. Die beiden Koeffizienten von A und B werden nur von einem Parameter abhängen, wenn

$$\begin{vmatrix} p & q & r \\ p & -q & kr \\ k+1 & k-1 & 2k \end{vmatrix} = 0,$$

d.h wenn

$$4kpq = (k+1)^2 qr - (k-1)^2 pr \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

Eliminieren wir γ und r aus (5), (6) und (7), dann finden wir

$$\left.\begin{aligned} U = \{\alpha(k-1)p + \beta(k+1)q\}[\{-(k-1)p + (k+1)q\}A - \\ - \{(k-1)p + (k+1)q\}B] + \{(k+1)^2 q - (k-1)^2 p\}(pC + qD) + 4kpqE \end{aligned}\right\}. \quad (8)$$

Mit p, q als Konstante stellt (8) eine Gerade dar, die alle Ebenen des zweiten Systems schneidet und auch, da sie b schneidet, alle Ebenen des ersten Systems. Wenn wir hiernach p und q (oder lieber ihr Verhältnis) verändern, dann zeigt sich:

Der gesuchte Ort ist eine Regelfläche F von einer Geraden beschrieben, die von einem Parameter abhängt; jede Gerade schneidet alle Ebenen beider Systeme. Durch jeden Punkt von b geht eine Gerade dieser Regelfläge.

Führen wir nochmals einen Koordinatensimplex mit A, B, C, D, E als Eckpunkte ein, dann können wir diese Regelfläche bestimmen durch

$$\left.\begin{aligned} x_1 &= \{-(k-1)\,p+(k+1)\,q\}\ \{(k-1)\,p\alpha+(k+1)\,q\beta\} \\ x_2 &= -\{(k-1)\,p+(k+1)\,q\}\ \{(k-1)\,p\alpha+(k+1)\,q\beta\} \\ x_3 &= p\ \{(k+1)^2\,q-(k-1)^2\,p\} \\ x_4 &= q\ \{(k+1)^2\,q-(k-1)^2\,p\} \\ x_5 &= 4\,k\,p\,q \end{aligned}\right\} \quad . \quad . \quad (9)$$

Wir können sie auch als den teilweisen Durchschnitt zweier quadratischen Räume betrachten, z.B. von

$$\begin{gathered} (k-1)\,(x_1-x_2)\,x_3+(k+1)\,(x_1+x_2)\,x_4=0 \\ 2\,k\,(x_1-x_2)\,x_3-(k+1)\,x_5\,(kx_1-x_2)=0, \end{gathered}$$

wobei man die in beiden Räumen liegende Ebene

$$x_1=x_2=0$$

nicht mitzählt.

Die Regelfläche F ist also eine solche dritten Grades. Jede Ebene (α,β) schneidet F nach einem Kegelschnitt.

§ 5. Die Koordinaten eines allgemeinen Punktes U — siehe (6) — können wir schreiben

$$\left.\begin{aligned} x_1 &= \alpha p+\beta q+\gamma r \\ x_2 &= \alpha p-\beta q+\gamma kr \\ x_3 &= p \\ x_4 &= q \\ x_5 &= r \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

wobei

$$(k+1)\,\alpha+(k-1)\,\beta+2\,k\gamma=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5$$

Durch p, q, r, α, β, γ ist ein Punkt bestimmt. Umgekehrt bestimmt ein allgemeiner Punkt U $(x_1, x_2, x_3, x_4, x_5)$ die Verhältnisse von p, q, r und weiter, mit (5), α, β, γ; geometrisch ausgedrückt: Durch einen allgemeinen Punkt geht eine Ebene jedes Systems. Jedoch werden unendlich viele Ebenen des zweiten Systems durch einen Punkt Q (y) gehen, wenn wir y_1, y_2, y_3, y_4, y_5 — oder y_1, y_2, p, q, r — solche Werte geben, dass α, β und γ

aus den beiden ersten Gleichungen (10) und (5) nicht eindeutig aufgelöst werden können. Dafür ist erforderlich

$$\begin{vmatrix} p & q & r \\ p & -q & kr \\ k+1 & k-1 & 2k \end{vmatrix} = 0.$$

So finden wir (7) wieder. Der geometrische Ort von Q (y) wird also aus (10), (4) und (7) gefunden. Dies sind aber diesselben Gleichungen, wodurch F bestimmt wurde. Daher:

Die Regelfläche F ist der geometrische Ort des Punktes Q, in welchem unendlich viele Ebenen des zweiten Systems einander schneiden.

§ 6. Es möge eine Ebene des zweiten Systems durch α, β, γ, wozwischen (5) besteht, gegeben sein. Wir können — siehe (10) — jene Ebene darstellen durch

$$\begin{aligned} x_1 &= \alpha x_3 + \beta x_4 + \gamma x_5 \\ x_2 &= \alpha x_3 - \beta x_4 + \gamma k x_5 \end{aligned}$$

wobei (5) entsprochen ist,
oder durch

$$\begin{aligned} x_1 &= \alpha \left(x_3 - \frac{k+1}{2k} x_5 \right) + \beta \left(x_4 - \frac{k-1}{2k} x_5 \right) \\ x_2 &= \alpha \left(x_3 - \frac{k+1}{2} x_5 \right) - \beta \left(x_4 + \frac{k-1}{2} x_5 \right). \end{aligned}$$

Die Bedingungen, welche ausdrücken, dass eine Ebene des zweiten Systems durch einen Punkt Q (y) geht, werden also ausgedrückt durch

$$\left\| \begin{matrix} x_1 & x_3 - \frac{k+1}{2k} x_5 & x_4 - \frac{k-1}{2k} x_5 \\ x_2 & x_3 - \frac{k+1}{2} & -x_4 - \frac{k-1}{2} x_5 \\ y_1 & y_3 - \frac{k+1}{2k} y_5 & y_4 - \frac{k-1}{2k} y_5 \\ y_2 & y_3 - \frac{k+1}{2} y_5 & -y_4 - \frac{k-1}{2} y_5 \end{matrix} \right\| = 0. \quad . \quad . \quad . \quad . \quad (11)$$

d.h. dadurch, dass alle Determinanten von drei Reihen und Kolonnen aus dieser Matrix gleich Null gestellt werden. Für einen Punkt, der nicht auf F liegt, genügt es natürlich, die beiden Gleichungen ersten Grades aus (11) zu nehmen. Aber wir wissen, dass für einen Punkt von F, jene Ebene unbestimmt ist. Das will sagen, dass ein Punkt Q (y) von F entspricht:

$$\left\| \begin{array}{ccc} y_1 & y_3 - \frac{k+1}{2k} y_5 & y_4 - \frac{k-1}{2k} y_5 \\ y_2 & y_3 - \frac{k+1}{2} y_5 & -y_4 - \frac{k-1}{2} y_5 \end{array} \right\| = 0. \quad . \quad . \quad (12)$$

Die aus (12) erhaltenen Gleichungen, die dadurch bekommen werden, dass jede Determinante zweiten Grades gleich Null gesetzt wird, ergeben also aufs neue eine Darstellung von F, die mit dem Schluss von § 4 in Uebereinstimmung ist.

Dann können wir aus (11) eine Kolonne weglassen und haben den geometrischen Ort der Punkte x, die in den Ebenen des zweiten Systems durch Q (y) liegen.

Die Ebenen des zweiten Systems durch einen Punkt Q (y) der Regelfläche F bilden einen quadratischen Kegelraum mit Q als Spitze, dargestellt durch

$$\left\| \begin{array}{ccc} x_1 & x_3 - \frac{k+1}{2k} x_5 & x_4 - \frac{k-1}{2k} x_5 \\ x_2 & x_3 - \frac{k+1}{2} x_5 & -x_4 - \frac{k-1}{2} x_5 \\ y_1 & y_3 - \frac{k+1}{2k} y_5 & y_4 - \frac{k-1}{2k} x_5 \end{array} \right\| = 0. \quad . \quad . \quad . \quad (13)$$

Nach einem Satze WEITZENBÖCKS ist der geometrische Ort der Spitze eines quadratischen Kegels durch vier allgemein liegende Gerade die fünfte assoziierte. Es zeigt sich hier, dass jener geometrische Ort in unserem Falle — die vier Geraden schneiden eine Gerade b — die Regelfläche F ist, die also auch in dieser Hinsicht die fünfte assoziierte ersetzt.

Ferner erhellt aus (11) und (12), dass F ganz auf K liegt. Also liegen auf K alle Ebenen, die durch Projektion der Erzeugenden von F aus Q entstehen.

Die Regelfläche F liegt ganz auf dem Kegelraum K. Die beiden Ebenensysteme von K, deren jedes von einem Parameter abhängt, sind

1. *die Ebenen durch Q, welche die Erzeugenden von F projizieren;*
2. *die Ebenen durch Q des zweiparametrigen Ebenensystems, das wir bisher immer „das zweite" nannten (§ 2).*

§ 6. *Man kann die Erzeugenden von F als das System gemeinschaftlicher Sehnen von ∞^4 Kurven des vierten Grades betrachten.* Setzt man in (9)

$$\frac{p}{q} = \lambda, \; (k-1) \frac{p}{q} \alpha + (k+1) \beta = \mu,$$

dann wird F folgendermassen dargestellt:

$$\left.\begin{aligned} x_1 &= \{-(k-1)\lambda + k + 1\}\,\mu \\ x_2 &= -\{(k-1)\lambda + k + 1\}\,\mu \\ x_3 &= \lambda\,\{-(k-1)^2\lambda + (k+1)^2 \\ x_4 &= -(k-1)^2\lambda + (k+1)^2 \\ x_5 &= 4k\lambda \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (14)$$

Eine erzeugende l von F ist dadurch bestimmt, dass λ ein konstanter Wert gegeben wird; nehmen wir dann auch μ konstant an, dann ist auf l ein Punkt bestimmt.

Setzen wir in (14) noch

$$\lambda = \frac{\varphi(\mu)}{\psi(\mu)}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

worin φ und ψ Formen des zweiten und des ersten Grades sind, dann ist durch (14) und (15) ein System von ∞^4 Kurven $C_4^{(4)}$ des vierten Grades bestimmt. Wählen wir z.B.

$$\lambda = \mu^2,$$

dann haben wir ein $C_4^{(4)}$, worauf die Erzeugenden von F eine quadratische Involution bestimmen. Es ist bekannt, dass die Sehnen dann alle dieselbe Gerade, die man die *Involutionsachse* nennen kann — hier also b —, schneiden. Man begegnet also einer bekannten Figur, aus welcher leicht verschiedene vorher besprochene Eigenschaften abgeleitet werden können [1]). In Zusammenhang mit dem Folgenden vermelde ich noch:

Keine zwei Erzeugenden von F (Sehnen von $C_4^{(4)}$) schneiden einander; keine drei Erzeugenden von F liegen in demselben R_3.

§ 7. Wir gingen aus von vier Geraden a_1, a_2, a_3, a_4, die eine Gerade b schneiden und ferner allgemeine Lage haben. Hierdurch sind die Ebenen des zweiten Systems und daraus ist F völlig bestimmt. Wir weisen nun noch nach, dass dieses Ebenensystem und F invariant sind, wenn wir die Geraden a_i $(i = 1, \ldots 4)$ durch vier andere beliebig gewählte Erzeugende von F ersetzen. Hierfür genügt es, zu zeigen, dass wir eine dieser Geraden, z.B. a_4, durch eine andere Erzeugende a_4^+ ersetzen können. Wir wissen schon, dass jede Ebene, die nicht durch b geht und a_1, a_2, a_3, a_4 schneidet, auch a_4^+ schneiden wird; wir müssen beweisen, dass jede Ebene die nicht durch b geht und a_1, a_2, a_3, a_4^+ schneidet, auch a_4 schneiden wird.

Wir wollen also von einer nicht durch b gehenden Ebene ausgehen, die

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[1]) J. F. DE VRIES. Analytische behandeling van de kromme van den vierden graad in R_4; Dissertation Leiden, 1922.

[2]) H. G. TELLING. The Rational Quartic Curve in Space of three and four Dimensions. Cambridge Tracts in Mathematics, No. 34, 1936.

a_1, a_2, a_3, a_4^+ schneidet, nacheinander in P^+, Q^+, R^+, S^+. Wir können durch P^+ und Q^+ auch eine Ebene unseres früheren zweiten Systems bringen, die also a_1, a_2, a_3, a_4 und a_4^+ schneidet. Nehmen wir an, dass diese Ebene nicht mit der vorigen zusammenfällt, dann liegen a_3 und a_4^+ in demselben Raum mit diesen beiden Ebenen und mithin auch mit ihrer Schnittlinie P^+Q^+. Anders gesagt: Der Raum durch a_3 und a_4^+ schneidet a_1 und a_2 in P^+ und Q^+. Der Raum durch a_3 und a_4^+ schneidet aber alle Erzeugenden von F in Punkten von b; P^+ und Q^+ sind also auf b genommen, gegen die Verabredung.

Die Ebene durch P^+, Q^+, R^+, S^+ fällt also mit einer unserer früheren Ebenen des zweiten Systems zusammen. *Beim Uebergang von* a_i $(i = 1, \ldots 4)$ *auf vier andere Gerade von* F *sind das zweite Ebenensystem und* F *invariant.*

III.

§ 8. *Drei Gerade* a_1, a_2, a_3 *liegen in einem* R_3, *der von* a_4 *in einem Punkt* P *geschnitten wird.*

Möge $[\alpha]$ die quadratische Regelschar sein, wozu a_1, a_2, a_3 gehören und $[\beta]$ die komplementäre Regelschar von $[\alpha]$. Man sieht sofort, dass das Ebenensystem, das die vier Geraden a_1 schneidet, besteht aus:

1. dem Ebenenbündel P in R_3;
2. ∞^1 Ebenenbüscheln; Träger eines solchen Büschels ist jede Gerade b von $[\beta]$; der Büschel liegt im dreidimensionalen Raum durch b und a_4.

Schliesslich leuchtet es ein, dass jede unter 1) und 2) genannte Ebene von jeder Geraden aus $[\alpha]$ geschnitten wird. *Die Geraden der Schar* $[\alpha]$ *treten hier also an die Stelle der fünften assoziierten* a_5.

Mathematics. — *Ueber algebraische Systeme von partiellen Differentialgleichungen erster Ordnung. I. Gleichungen mit einer Unbekannten.*

Vollständige Mannigfaltigkeiten und vollständige Ideale.

Von J. A. SCHOUTEN und W. VAN DER KULK.

(Communicated at the meeting of September 28, 1940.)

1. *Einleitung.* In vorigen Mitteilungen [1]) haben wir bewiesen, dass jedes System von q PFAFFschen Gleichungen mit einem Halbrang ϱ: $(2\varrho \leqq n-q)$; gleichwertig ist mit einem System von q Gleichungen, deren Klasse höchstens gleich $2\varrho+1$ ist. Hiermit ist also eine obere Grenze festgelegt, die man nicht zu überschreiten braucht. Eine weitere Frage wäre nun das System zu ersetzen durch $q=\Sigma q_i$ Gleichungen von der Klasse i; $i=1, 3, \ldots, 2\varrho+1$, so dass die Partialsummen q_1, q_1+q_2, $q_1+q_2+q_3$, u.s.w. alle den höchst möglichen Wert bekommen.

Der Weg zur Bestimmung von q_1 ist bekannt. Das adjungierte System von $p=n-q$ linearen partiellen Differentialgleichungen

$$\underset{b}{v}^{\mu} w_\mu = 0 \;;\; b=1,\ldots,\mathrm{p} \;;\; p=n-q \quad . \quad . \quad . \quad . \quad . \quad (1)$$

mit der Nebenbedingung

$$w_{\mu\lambda} \overset{\text{def}}{=} 2\,\partial_{[\mu}\, w_{\lambda]} = 0 \ (\text{oder } w_\lambda = \partial_\lambda\, s) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

wird mit Hilfe der LIE'schen Klammerausdrücke verlängert

$$\underset{ab}{v}^{\mu} w_\mu \overset{\text{def}}{=} 2\, \underset{[a}{v}^{\omega} (\partial_{|\omega|}\, \underset{b]}{v}^{\mu})\, w_\mu = 2\, \underset{a}{v}^{\omega}\, \underset{b}{v}^{\mu}\, w_{\mu\omega} = 0 \;;\; a, b = 1, \ldots, \mathrm{p} \quad . \quad (3)$$

und es entstehen so $\binom{p}{2}$ neue lineare Gleichungen. Das Verfahren wird fortgesetzt bis man zu einem vollständigen, d. i. nicht mehr in dieser Weise verlängerungsfähigen System gelangt. Schliesslich folgt der Existenzbeweis, der dartut, dass jeder Vektor w_λ, der in *einem* Punkt den dort algebraischen Gleichungen eines vollständigen Systems genügt, sich zu einem Gradientfeld fortsetzen lässt, dessen Skalar f dann eine Lösung von (1, 2) darstellt. Damit ist das Problem schon gelöst, d.h. es ist eine volkommene Uebersicht über die möglichen Lösungen von (1, 2) gewonnen.

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[1]) J. A. SCHOUTEN und W. VAN DER KULK, Beiträge zur Theorie der Systeme PFAFFscher Gleichungen, Proc. Kon. Ned. Akad. v. Wet. 43, 18—31, 179—188, 453—462, 674—686 (1940).

Dass das Resultat sich auch so aussprechen lässt, dass es genau $n-p^*$ unabhängige Lösungen von (1, 2) gibt, wenn das zugehörige vollständige System p^* linear unabhängige Gleichungen enthält und dass eine beliebige Funktion dieser $n-p^*$ Lösungen die allgemeine Lösung von (1, 2) darstellt, ist eine in diesem einfachen Falle erfreulicherweis mögliche elegante Umformulierung des Resultates, die aber nichts wesentliches mehr hinzufügt.

Wünscht man nun $q_1 + q_2$ zu bestimmen, so kommt statt (2) die Bedingung

$$w_{[\omega\mu}\, w_{\lambda\varkappa]} = 0 \ ; \ (\text{oder } w_\lambda = \partial_\lambda s_1 + s_2\, \partial_\lambda s_3) \quad . \quad . \quad . \quad . \quad (4)$$

und bei der Verlängerung entstehen zunächst $\binom{p}{4}$ Gleichungen zweiten Grades in w_λ

$$\underset{[ab}{v^\mu}\ \underset{cd]}{v^\lambda}\, w_\mu\, w_\lambda = 4\, \underset{[a}{v^\omega}\, \underset{b}{v^\mu}\, \underset{c}{v^\lambda}\, \underset{d]}{v^\varkappa}\, w_{[\omega\mu}\, w_{\lambda\varkappa]} = 0 \ , \ a, b, c, d = 1, \ldots, \text{p.} \quad . \quad (5)$$

Fortsetzung dieses Verfahrens mit Hilfe der LIE'schen Klammerausdrücke ist jetzt nicht mehr möglich. Statt dessen kann man aber die Differentialkomitanten [2]) verwenden, die sich stets aus zwei beliebigen kontravarianten Grössen bilden lassen und die eine unmittelbare Verallgemeinerung der Komitante im LIE'schen Differential darstellen. Mit Hilfe dieser Komitanten lässt sich ein Verfahren ausbilden zur Verlängerung eines gegebenen Systems und dieses Verfahren lässt sich dann ebenfalls solange fortsetzen, bis man zu einem nicht mehr in dieser Weise verlängerungsfähigen System gelangt. Sodann wäre zu beweisen, dass sich jede nicht ungeschickt gewählte Lösung dieses in *einem* Punkte algebraischen Systems zu einem Feld der Klasse 3 fortsetzen lässt [3]). Mit diesem Existenzbeweis wäre das Problem im Wesen gelöst. $q_1 + q_2$ ist die Anzahl der linear unabhängigen Lösungen des erwähnten algebraischen Systems. Nähere Angaben über die Bildungsmöglichkeiten von s_1, s_2 und s_3 blieben erwünscht, können aber nichts wesentlich neues mehr hinzufügen.

Die Bestimmung von $q_1 + q_2 + q_3$ u.s.w. geschieht nach dem selben Schema. Stets entstehen Systeme von partiellen Differentialgleichungen erster Ordnung, deren linke Seiten homogene ganze rationale Funktionen der Bestimmungszahlen eines Vektors $w_\lambda = \partial_\lambda s_1 + s_2\, \partial_\lambda s_3 + \ldots$ sind, wo $s_1, s_2, s_3, \ldots$ die unbekannten Funktionen sind. Solche Systeme wollen wir *algebraische Systeme von partiellen Differentialgleichungen erster Ordnung* nennen.

Von diesen Systemen sollen nun zunächst die mit *einer* Unbekannten

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[2]) J. A. SCHOUTEN, Ueber Differentialkomitanten zweier kontravarianter Grössen, Proc. Kon. Ned. Akad. v. Wet. **43**, 449—452 (1940).

[3]) Ist $p = 3$, so ist das gegebene System schon nicht mehr erweiterungsfähig. Dann ist aber $\varrho \leqq 1$ und die Lösung des Problems ist schon in dem obenerwähnten Haupttheorem enthalten.

betrachtet werden, also solche mit $w_\lambda = \partial_\lambda s$, deren Gleichungen aber von beliebigem Grade in w_λ sind. Für die Bestimmung der q_i ist dies nicht unmittelbar zu verwenden, da die Bestimmung von q_1 schon erledigt ist und die von q_2 die Betrachtung von Gleichungen mit drei Unbekannten erfordert. Diese erste Mitteilung bringt eine Anzahl von Sätzen über „vollständige" algebraische Mannigfaltigkeiten und „vollständige" Ideale im „Raum" der $\lfloor w_\lambda \rfloor$.

2. *Algebraische Mannigfaltigkeiten und zugehörige Ideale.*

Wir betrachten ein System von N partiellen Differentialgleichungen erster Ordnung von der Form

$$\underset{i}{P}^{\mu\lambda_1\ldots\lambda_{a_i}} (\partial_\mu s)(\partial_{\lambda_1} s)\ldots(\partial_{\lambda_{a_i}} s) = 0 \;;\; i = 1, \ldots, N. \quad . \quad . \quad . \quad (6)$$

wo die $\underset{i}{P}^{\mu\lambda_1\ldots\lambda_{a_i}}$ Tensoren (symmetrische Affinoren) sind mit Bestimmungszahlen, die analytisch [4]) sind in $x^\varkappa$ in der Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$. Ist $\mathfrak{M}$ die Menge aller kovarianter Pseudovektorfelder $\lfloor w_\lambda \rfloor$ [5]), die den Gleichungen

$$\underset{i}{P}^{\mu\lambda_1\ldots\lambda_{a_i}} w_\mu w_{\lambda_1} \ldots w_{\lambda_{a_i}} = 0 \;;\; i = 1, \ldots, N \quad . \quad . \quad . \quad . \quad . \quad (7)$$

genügen, so bedeutet lösen von (6) die Bestimmung aller Felder von Klasse 1 (d.s. Gradientfelder bis auf einen skalaren Faktor) in $\mathfrak{M}$. Jedes Pseudovektorfeld $\lfloor w_\lambda \rfloor$ lässt sich auffassen als Punkt eines $(n-1)$-dimensionalen projektiven Raumes P_{n-1}, wenn man als Grundkörper die meromorphen [6]) Funktionen der $x^\varkappa$ in der Umgebung von $x^\varkappa = \underset{0}{x}^\varkappa$ wählt. Wir denken uns diesen Grundkörper algebraisch abgeschlossen. Ist das Feld $\lfloor w_\lambda \rfloor$ von der Klasse k, so nennen wir auch den Punkt von der Klasse k. Von der zu konstruierenden Funktion s verlangen wir natürlich nicht nur dass sie meromorph sei sondern ausserdem noch dass sie in der besagten Umgebung analytisch ist. In P_{n-1} ist $\mathfrak{M}$ eine algebraische Mannigfaltigkeit [7]). Für $N = 1$ ist $\mathfrak{M}$ stets eine Hyperfläche, d. i. eine reine $(n-2)$-dimensionale Mannigfaltigkeit [8]) und P heisst ein Tensor der Hyperfläche.

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[4]) $f(x^\varkappa)$ heisst *analytisch* in dem Gebiet $|x^\varkappa - \underset{0}{x}^\varkappa| < a$, $(a > 0)$, wenn diese Funktion sich daselbst darstellen lässt durch eine konvergente Potenzreihe nach steigenden positiven ganzen Potenzen der $x^\varkappa - \underset{0}{x}^\varkappa$; $\varkappa = 1, \ldots, n$. Vgl. H. BEHNKE und P. THULLEN, Theorie der Funktionen mehrerer komplexer Veränderlicher, S. 15.

[5]) $\lfloor\ \rfloor$ bedeutet „bis auf einen beliebigen skalaren Faktor". Ein Pseudovektorfeld ist ein Vektorfeld, das nur bis auf einen skalaren Faktor gegeben ist.

[6]) $f(x^\varkappa)$ heisst *meromorph* in dem Gebiet $|x^\varkappa - \underset{0}{x}^\varkappa| < a$, $(a > 0)$ wenn diese Funktion sich daselbst darstellen lässt als Quotient zweier analytischer Funktionen. Vgl. H. BEHNKE und P. THULLEN a.a.O. S. 15.

[7]) B. L. V. D. WAERDEN, Einf. in die alg. Geometrie (weiter zitiert als E. a. G.), § 28, S. 107 u. f.

[8]) E. a. G., S. 114.

Ist jeder Punkt von $\mathfrak{M}$ ein Punkt der Hyperfläche

$$P^{\mu\lambda_1\ldots\lambda_a} w_\mu w_{\lambda_1} \ldots w_{\lambda_a} = 0, \quad \ldots \ldots \ldots \quad (8)$$

so sagt man, dass die Hyperfläche (8) *durch* $\mathfrak{M}$ *geht* oder auch $\mathfrak{M}$ *enthält*, und dass der Tensor P *auf* $\mathfrak{M}$ *verschwindet* oder auch *in* $\mathfrak{M}$, falls $\mathfrak{M}$ sich zu einem einzigen Punkt reduziert. Die Gesamtheit aller Tensoren oder auch aller Polynome der Form (8) die auf (in) $\mathfrak{M}$ verschwinden, bilden das *zu* $\mathfrak{M}$ *gehörige Ideal* [9]). Dieses Ideal besitzt bekanntlich eine endliche Basis und man kann die Gleichungen (7) durch die mit Hilfe dieser Basis gebildeten Gleichungen ersetzen. Wir wählen das Gebiet, in welchem wir die Lösungen von (6) suchen, nun stets so, dass die Koeffizienten der Polynome dieser Basis daselbst analytische Funktionen der $x^\varkappa$ sind.

3. *Vollständige Mannigfaltigkeiten und zugehörige Ideale.*

Sind $P^{\mu\lambda_1\ldots\lambda_a}$ und $Q^{\mu\lambda_1\ldots\lambda_b}$ zwei Tensoren, so gibt es eine Differentialkomitante [10])

$$\left.\begin{aligned} \{D(P;Q)\}^{\lambda_1\ldots\lambda_{a+b+1}} \overset{\text{def}}{=} & (a+1)\, P^{\mu(\lambda_1\ldots\lambda_a}\, \partial_\mu\, Q^{\lambda_{a+1}\ldots\lambda_{a+b+1})} - \\ & - (b+1)\, Q^{\mu(\lambda_1\ldots\lambda_b}\, \partial_\mu\, P^{\lambda_{b+1}\ldots\lambda_{a+b+1})} \end{aligned}\right\} \quad \ldots \quad (9)$$

die ebenfalls ein Tensor ist. Nun gilt der Satz

Sind P und Q zwei Tensoren, die auf $\mathfrak{M}$ verschwinden, so liegen die Punkte der Klasse 1 von $\mathfrak{M}$ auf der durch $D(P;Q)$ bestimmten Hyperfläche.

Der Beweis folgt unmittelbar aus der Formel (10) auf S. 452 der zitierten Arbeit:

$$\left.\begin{aligned} \{D(P;Q)\}^{\mu\lambda_1\ldots\lambda_{a+b}}\, w_\mu w_{\lambda_1} \ldots w_{\lambda_{a+b}} = 2\,(a+1)\,(b+1)\, & P^{\lambda\varkappa_1\ldots\varkappa_a}\, Q^{\mu\nu_1\ldots\nu_b} \\ & w_{\varkappa_1} \ldots w_{\varkappa_a}\, w_{\nu_1} \ldots w_{\nu_b}\, \partial_{[\mu}\, w_{\lambda]} \end{aligned}\right\} (10)$$

Man bilde nun alle Tensoren von der Form $D(P;Q)$, wo P und Q zwei beliebige Tensoren sind, die auf $\mathfrak{M}$ verschwinden. Die in $\mathfrak{M}$ gelegenen Nullstellen dieser Tensoren bilden eine algebraische Mannigfaltigkeit $\mathfrak{M}_1$ und es ist $\mathfrak{M} \supseteqq \mathfrak{M}_1$. Ist $\mathfrak{M}_1$ leer, so hat $\mathfrak{M}$ keine Punkte der Klasse 1 und (6) also keine Lösungen. Ist $\mathfrak{M}_1$ nicht leer, so kann man aus $\mathfrak{M}_1$ in derselben Weise eine algebraische Mannigfaltigkeit $\mathfrak{M}_2$ bilden, $\mathfrak{M}_1 \supseteqq \mathfrak{M}_2$, u.s.w. Für diese Mannigfaltigkeiten ist

$$\mathfrak{M} \supseteqq \mathfrak{M}_1 \supseteqq \mathfrak{M}_2 \supseteqq \ldots \quad \ldots \ldots \ldots \quad (11)$$

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[9]) B. L. V. D. WAERDEN, Moderne Algebra (weiter zitiert als M. A.) II, S. 52.

[10]) J. A. SCHOUTEN, l.c. Fussnote 2).

und für die zugehörigen Ideale also

$$\mathfrak{A} \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \ldots \qquad \ldots\ldots\ldots \quad (12)$$

Ausserdem folgt aus dem Bildungsprozess der $\mathfrak{M}_i$, dass ein Gleichheitszeichen, das in diesen Reihen auftritt nur von Gleichheitszeichen gefolgt werden kann. Nach dem Teilerkettensatz für Ideale [11]) muss es nun eine kleinste ganze Zahl ν geben, so, dass $\mathfrak{A}_\nu = \mathfrak{A}_{\nu+1} = \mathfrak{A}_{\nu+2} = \ldots$ und also auch $\mathfrak{M}_\nu = \mathfrak{M}_{\nu+1} = \mathfrak{M}_{\nu+2} = \ldots$, und daraus geht hervor, dass sich $\mathfrak{M}_\nu$ mit Hilfe des beschriebenen Prozesses nicht mehr verkleinern lässt. Eine solche Mannigfaltigkeit nennen wir *vollständig* und das zugehörige Ideal ein *vollständiges Ideal*. Eine Mannigfaltigkeit heisst also vollständig, wenn aus dem Verschwinden von P und Q auf der Mannigfaltigkeit das Verschwinden von $D(P;Q)$ daselbst folgt [12]). Eine algebraische Hyperfläche ist demnach stets vollständig da sie durch eine einzige homogene Gleichung gegeben werden kann [13]). Man beweist leicht, dass ein zugehöriges Ideal $\mathfrak{A}$ dann und nur dann vollständig ist wenn aus $P \equiv 0\,(\mathfrak{A})$ und $Q \equiv 0\,(\mathfrak{A})$ folgt $D(P;Q) \equiv 0\,(\mathfrak{A})$. Die Bestimmung der Punkte von Klasse 1 einer Mannigfaltigkeit ist jetzt zurück geführt auf die Bestimmung der Punkte von Klasse 1 einer vollständigen Mannigfaltigkeit.

Ist $\mathfrak{M}'$ vollständig und $\mathfrak{M}' \subseteq \mathfrak{M}$, so zeigt man leicht, dass auch $\mathfrak{M}' \subseteq \mathfrak{M}_\nu$ ist. Also enthält $\mathfrak{M}_\nu$ sämtliche in $\mathfrak{M}$ gelegene vollständige Mannigfaltigkeiten.

Es soll jetzt eine Methode ausgebildet werden um aus dem zu $\mathfrak{M}_k$ gehörigen Ideal $\mathfrak{M}_{k+1}$ zu bestimmen. Es sei

$$\underset{i}{P}^{\mu\varkappa_1\ldots\varkappa_{a_i}} \;;\; i = 1, \ldots, s \quad \ldots\ldots\ldots \quad (13)$$

eine Basis des zu $\mathfrak{M}_k$ gehörigen Ideals. Sodann bilden wir die $\binom{s}{2}$ Tensoren $D(\underset{i}{P};\underset{j}{P});\, i\,j = 1, \ldots, s$ und die in $\mathfrak{M}_k$ gelegenen Nullpunkte dieser Tensoren. Jedenfalls liegt $\mathfrak{M}_{k+1}$ in der Mannigfaltigkeit dieser Nullpunkte. Sind nun R und S zwei beliebige Tensoren, die auf $\mathfrak{M}_k$ verschwinden, dann sind sie in dem zugehörigen Ideal von $\mathfrak{M}_k$ enthalten. Da die $\underset{i}{P}$ eine Basis dieses Ideals bilden ist also

$$\left.\begin{aligned} R &\equiv 0\,(mod\, \underset{i}{P}) \\ S &\equiv 0\,(mod\, \underset{i}{P}) \end{aligned}\right\} ;\; i = 1, \ldots s \,. \quad \ldots\ldots \quad (14)$$

Daraus geht aber hervor, dass

$$D(R;S) \equiv 0\,(mod\, \underset{i}{P},\, D(\underset{i}{P};\underset{j}{P}));\, i, j = 1, \ldots, s \quad \ldots \quad (15)$$

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[11]) M. A. II, S. 25.

[12]) Man beweist leicht dass n. u. h. ist dass $D(P,Q)$ stets verschwindet wenn P und Q zu einer bestimmt gewählten Basis des zugehörigen Ideals gehören.

[13]) E. a. G., S. 113.

ist, und dass also die zu $D(R;S)$ gehörige Hyperfläche die oben gebildete Mannigfaltigkeit der Nullpunkte der Tensoren $D(\underset{i}{P};\underset{j}{P})$, die in $\mathfrak{M}_k$ liegen, enthält. Also enthält $\mathfrak{M}_{k+1}$ diese Mannigfaltigkeit, während $\mathfrak{M}_{k+1}$ anderseits in derselben enthalten ist. Die konstruierte Nullpunktmannigfaltigkeit ist also gerade $\mathfrak{M}_{k+1}$.

4. *Irreduzibele Teile einer vollständigen Mannigfaltigkeit.*

Hilfssatz I. Sind $\mathfrak{M}_1$ *und* $\mathfrak{M}_2$ *vollstandig, so ist auch die Vereinigung* $\mathfrak{M}=(\mathfrak{M}_1, \mathfrak{M}_2)$ *vollständig.*

Beweis: Verschwinden die Tensoren P und Q auf $\mathfrak{M}$, so verschwinden sie auf $\mathfrak{M}_1$ und auf $\mathfrak{M}_2$. Da aber $\mathfrak{M}_1$ und $\mathfrak{M}_2$ vollständig sind verschwindet auch $D(P:Q)$ auf $\mathfrak{M}_1$ und auf $\mathfrak{M}_2$ und daher auch auf $\mathfrak{M}$. Also ist $\mathfrak{M}$ vollständig.

Hilfssatz II. $\mathfrak{M}_2$ *sei beliebig und* $\mathfrak{M}_1$ *irreduzibel aber kein Teil von* $\mathfrak{M}_2$. *Ferner sei die Vereinigung* $\mathfrak{M}=(\mathfrak{M}_1, \mathfrak{M}_2)$ *vollständig. Dann ist auch* $\mathfrak{M}_1$ *vollständig.*

Beweis: P und Q seien Tensoren, die auf $\mathfrak{M}_1$ verschwinden. Da $\mathfrak{M}_1$ nicht Teil von $\mathfrak{M}_2$ ist, gibt es sicher einen Tensor R, der auf $\mathfrak{M}_2$ verschwindet, aber nicht auf $\mathfrak{M}_1$. Also sind die symmetrischen Produkte PR und QR Tensoren, die auf $\mathfrak{M}$ verschwinden. Da $\mathfrak{M}$ aber vollständig ist verschwindet demnach auch $D(PR;QR)$ auf $\mathfrak{M}$. Nun ist aber

$$D(PR;QR)=RR\,D(P;Q)+P\,D(R;QR)+Q\,D(PR;R), \quad (16)$$

in welcher Formel alle Produkte symmetrische Produkte sind. Ist nun $\mathfrak{P}$ ein beliebiger Punkt von $\mathfrak{M}_1$ in welchem R nicht verschwindet, so verschwinden sowohl P als Q als auch $D(PR;QR)$ in $\mathfrak{P}$ und infolge (16) also auch $D(P;Q)$. Die Hyperfläche $D(P;Q)=0$ enthält also jedenfalls alle Punkte von $\mathfrak{M}_1$ wo R nicht verschwindet. Es sei nun $\mathfrak{M}'$ der Durchschnitt von $\mathfrak{M}_1$ und der Hyperfläche $D(P;Q)=0$ und $\mathfrak{M}''$ der Durchschnitt von $\mathfrak{M}_1$ und der Hyperfläche $R=0$. Genügt nun ein Punkt von $\mathfrak{M}_1$ der Gleichung $R=0$, so ist er auch Punkt von $\mathfrak{M}''$. Genügt er aber der Gleichung $R=0$ nicht, so genügt er, wie oben bewiesen wurde, der Gleichung $D(P;Q)=0$ und ist somit Punkt von $\mathfrak{M}'$. Daraus folgt, dass $\mathfrak{M}_1$ die Vereinigung von $\mathfrak{M}'$ und $\mathfrak{M}''$ ist. Da jedoch R nicht Null ist auf $\mathfrak{M}_1$ muss $\mathfrak{M}''$ ein echter Teil von $\mathfrak{M}_1$ sein, woraus, wegen der Irreduzibilität von $\mathfrak{M}_1$, hervorgeht, dass $\mathfrak{M}'=\mathfrak{M}_1$ ist und $D(P;Q)$ Null ist auf $\mathfrak{M}$.

Jede algebraische Mannigfaltigkeit lässt sich eindeutig darstellen als die Vereinigung einer endlichen Anzahl von irreduzibelen algebraischen Mannigfaltigkeiten von denen keine in der Vereinigung der übrigen enthalten ist[14]. Wir beweisen jetzt den

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[14] E. a. G. § 28 und M. A. II, S. 65.

Satz: *Eine algebraische Mannigfaltigkeit* $\mathfrak{M}$ *in* P_{n-1} *ist dann und nur dann vollständig, wenn jeder ihrer irreduzibelen Teile vollständig ist* [15]).

Beweis: a. Sind die Teile alle vollständig, so folgt die Behauptung durch wiederholte Anwendung des Hilfssatzes I.

b. Es seien $\mathfrak{M}_1, \dots, \mathfrak{M}_p$ die irreduzibelen Teile und es sei $\mathfrak{M}'$ die Vereinigung von $\mathfrak{M}_2, \dots, \mathfrak{M}_p$. Dann ist $\mathfrak{M} = (\mathfrak{M}_1, \mathfrak{M}')$ und $\mathfrak{M}_1$ ist irreduzibel und kein Teil von $\mathfrak{M}'$. Nach dem Hilfssatze II ist also $\mathfrak{M}_1$ vollständig. In derselben Weise folgt die Vollständigkeit der anderen irreduzibelen Teile.

Die Bestimmung der Punkte von der Klasse 1 einer Mannigfaltigkeit ist jetzt zurückgeführt auf die Bestimmung der Punkte von der Klasse 1 einer irreduzibelen vollständigen Mannigfaltigkeit.

5. *Projizierende Mannigfaltigkeiten*.

Ist $\mathfrak{M}$ eine algebraische Mannigfaltigkeit in P_{n-1} und $\mathfrak{P}$ ein Punkt in P_{n-1}, so bildet die Gesamtheit aller Geraden, die die Punkte von $\mathfrak{M}$ mit $\mathfrak{P}$ verbinden eine neue algebraische Mannigfaltigkeit, die wir die *projizierende Mannigfaltigkeit von* $\mathfrak{M}$ *i.b. auf* $\mathfrak{P}$ nennen. Wir beweisen den

Satz: *Ist* $\mathfrak{M}$ *vollständig und* $\mathfrak{P}$ *ein Punkt von der Klasse* 1, *so ist die projizierende Mannigfaltigkeit* $\mathfrak{M}_1$ *von* $\mathfrak{M}$ *i.b. auf* $\mathfrak{P}$ *ebenfalls vollständig* [16]).

Beweis: Das Bezugssystem sei so gewählt, dass $\mathfrak{P}$ der Massgradientvektor $\overset{1}{e}_\lambda$ ist. Ist sodann w_λ ein beliebiger Punkt von $\mathfrak{M}$, so ist $w_\lambda + \beta \overset{1}{e}_\lambda$ ein Punkt von $\mathfrak{M}_1$ wenn β ein beliebiges Skalarfeld ist. Es mögen nun die Tensoren $\underset{i}{P}^{\mu\lambda_1 \dots \lambda_{a_i}}$; $i = 1, \dots, N$ eine Basis des zugehörigen Ideals von $\mathfrak{M}_1$ bilden. Dann ist

$$\underset{i}{F}(x^\varkappa, w_\lambda) \overset{\text{def}}{=} \underset{i}{P}^{\mu\lambda_1 \dots \lambda_{a_i}} w_\mu w_{\lambda_1} \dots w_{\lambda_{a_i}} = 0; i = 1, \dots, N \quad . \quad . \quad (17)$$

aber auch

$$\underset{i}{F}(x^\varkappa, w_\lambda + \beta \overset{1}{e}_\lambda) = 0 \; ; \; i = 1, \dots, N \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Schreibt man nun

$$\underset{i}{P} = \underset{i0}{P} + \underset{i1}{P} \underset{1}{e} + \underset{i2}{P} \underset{1}{e} \underset{1}{e} + \dots + \underset{i(a_i+1)}{P} \underset{1}{e} \dots \underset{1}{e}, \quad . \quad . \quad . \quad (19)$$

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[15]) Mit Hilfe dieses Satzes zeigt man leicht, dass, im Falle wo $\mathfrak{M}$ vollständig ist und aus einer endlichen Anzahl von Punkten besteht, diese Punkte alle von der Klasse 1 sind.

[16]) Dieser Satz lässt sich verallgemeinern. Sind $\mathfrak{M}_1$ und $\mathfrak{M}_2$ vollständig, so ist die Mannigfaltigkeit, die von den Verbindingsgeraden aller Punkte von $\mathfrak{M}_1$ mit allen Punkten von $\mathfrak{M}_2$ gebildet wird, ebenfalls vollständig.

(es sind stets symmetrische Produkte gemeint) so lässt sich aus den Tensoren $\underset{ij}{P}$; $i=1,\dots,N$; $j=0,\dots,a_i+1$, die alle $\underset{1}{e}$ nicht mehr enthalten, und deren Bestimmungszahlen also alle Null sind, sofern sie einen oder mehrere Indizes *1* tragen, wiederum eine Basis des zugehörigen Ideals von $\mathfrak{M}_1$ bilden, da alle diese Tensoren auf $\mathfrak{M}_1$ verschwinden. Sind Q und R zwei beliebige Tensoren aus dieser Basis, so sind sie beide Null auf $\mathfrak{M}$ und es ist auch $D(Q;R)$ Null auf $\mathfrak{M}$, da $\mathfrak{M}$ vollständig ist. Da aber weder Q noch R Bestimmungszahlen mit einem Index *1* besitzen, besitzt auch $D(Q;R)$ infolge der Definitionsgleichung (9) keine solche Bestimmungszahlen und aus dem Verschwinden auf $\mathfrak{M}$ folgt also das Verschwinden auf $\mathfrak{M}_1$. Also ist $\mathfrak{M}_1$ vollständig.

Zu bemerken ist, dass $\mathfrak{M}_1$ vollständig bleibt, wenn man in den Gleichungen von $\mathfrak{M}_1$ für x^1 einen festen Wert $\underset{0}{x}^1$ einsetzt. Denn in $D(Q;R)$ treten ja keine Differentiationen nach x^1 auf.

Sodann beweisen wir den

Satz$^{1}_{1}$: *Ist* $\mathfrak{M}$ *irreduzibel und* $\mathfrak{P}$ *ein nicht zu* $\mathfrak{M}$ *gehöriger Punkt von beliebiger Klasse, so ist auch die projizierende Mannigfaltigkeit* $\mathfrak{M}_1$ *von* $\mathfrak{M}$ *i.b. auf* $\mathfrak{P}$ *irreduzibel.*

Beweis: Wäre $\mathfrak{M}_1$ reduzibel, so gäbe es (erstes Kriterium der Irreduzibilität)[17] zwei Tensoren P und Q, die beide nicht auf $\mathfrak{M}_1$ verschwinden, während ihr symmetrisches Produkt auf $\mathfrak{M}_1$ verschwindet. Wir wählen nun ein im allgemeinen anholonomes Bezugssystem so, dass $\mathfrak{P}$ der Massvektor $\overset{1}{e}_\lambda$ ist, der also kein Gradientvektor zu sein braucht, und schreiben

$$\left.\begin{aligned} P &= \underset{0}{P} + \underset{1}{P}\,\underset{1}{e} + \underset{2}{P}\,\underset{1}{e}\,\underset{1}{e} + \dots \\ Q &= \underset{0}{Q} + \underset{1}{Q}\,\underset{1}{e} + \underset{2}{Q}\,\underset{1}{e}\,\underset{1}{e} + \dots \end{aligned}\right\} \quad \dots\dots \quad (20)$$

(es sind stets symmetrische Produkte gemeint) so, dass die Tensoren $\underset{0}{P}, \underset{0}{Q}, \underset{1}{P}, \underset{1}{Q}, \dots$ alle $\underset{1}{e}$ nicht mehr enthalten. Die auf $\mathfrak{M}_1$ nicht verschwindenden Tensoren höchster Valenz in diesen Reihen seien $\underset{a}{P}$ und $\underset{b}{Q}$. Sodann sind

$$\left.\begin{aligned} P' &= \underset{0}{P} + \underset{1}{P}\,\underset{1}{e} + \dots + \underset{a}{P}\,\underset{1}{e}\dots\underset{1}{e} \\ Q' &= \underset{0}{Q} + \underset{1}{Q}\,\underset{1}{e} + \dots + \underset{b}{Q}\,\underset{1}{e}\dots\underset{1}{e} \end{aligned}\right\} \quad \dots\dots \quad (21)$$

zwei Tensoren, die beide nicht auf $\mathfrak{M}_1$ verschwinden, während ihr symmetrisches Produkt daselbst verschwindet. Das selbe gilt also auch

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[17] E. a. G., S. 109.

von den beiden Tensoren $\underset{a}{P}$ und $\underset{b}{Q}$, die beide $\underset{1}{e}$ nicht mehr enthalten. Da aber $\mathfrak{M}$ irreduzibel ist und $\underset{a}{P}\underset{b}{Q}$ auf $\mathfrak{M}$ verschwindet, muss entweder $\underset{a}{P}$ oder $\underset{b}{Q}$ auf $\mathfrak{M}$ verschwinden. Dies würde aber, da in beiden $\underset{1}{e}$ fehlt, zur Folge haben, dass entweder $\underset{a}{P}$ oder $\underset{b}{Q}$ auch $\mathfrak{M}_1$ verschwindet, entgegen der Voraussetzung.

Plantkunde. — *Snelle bloei van Hollandsche Irissen („Imperator") III.* (Mededeeling N⁰. 65 van het Laboratorium voor Plantenphysiologisch Onderzoek te Wageningen.) Door A. H. BLAAUW, IDA LUYTEN en ANNIE M. HARTSEMA.

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Over het forceeren van de Hollandsche Bol-iris „Imperator" hebben wij destijds 2 mededeelingen gepubliceerd (zie Meded. N⁰. 48 in 1936 en N⁰. 57 in 1938), waarnaar wij hier verwijzen. Deze onderzoekingen werden voortgezet in 1938—'39 en 1939—'40 om tot verdere zekerheid te komen. De proeven van '38—'39 — met 27 verschillende behandelingen — zijn bijna alle mislukt door een technische vergissing bij het sproeien, waardoor grond en wortels onder in de kistjes langen tijd droog bleven. Slechts een paar proeven die laat waren, na 5 weken 23° als voortemperatuur, gevolgd door 9° en 7°, konden nog, met 26 en 28 bloemen op 30 bollen, tot hun recht komen. Wij zullen hier dus direct overgaan tot de beschrijving van de 46 proeven, die in het seizoen 1939—'40 werden genomen.

Tot dusver was gevonden, dat de vroegste bloei verkregen werd door slechts 1 week 31° te geven gevolgd door 9°; dat deze snelle forceering vrij gewaagd is, maar dat een gunstige voeding te voren (bollen van klei beter dan van zand) of tijdens de proef (met Neon-licht beter dan zonder), in dit geval het resultaat zeer verbetert; — dat verder een langere voortemperatuur van 5 weken 23° tot 28° gevolgd door 9° of 7° een zeer bevredigenden bloei geeft met groote zekerheid, waarbij dan zandbollen even gunstig zijn als kleibollen; — dat het gewenscht scheen bij den overgang uit de koude naar de trek-temperatuur niet zoozeer de looflengte (bijv. 6 cm) als maat te nemen, maar liever het stadium van de bloemvorming en dat het beter zou zijn de bollen niet te spoedig uit de kou in 15° te plaatsen; daartoe waren in alle proeven van 1937/'38 de kistjes na het bereiken van 6 cm looflengte nog 3 weken in 9° of 7° gelaten, opdat de bloemvorming een eind verder voortgeschreden zou zijn.

Omtrent dit laatste punt bleef echter nog onzekerheid bestaan, of het werkelijk noodig was, de bloemvorming langer in de koude te laten plaats vinden, — of deze na 9° niet even goed in 15° C. kan voortgezet worden; want de bloei werd een paar weken verlaat, wanneer de planten na 6 cm loof nog 3 weken in koude in plaats van in 15° C. stonden.

Reeds bij de tenslotte mislukte proeven van 1938—'39 werden daartoe bollen gefixeerd als 6 cm looflengte in 9° bereikt was en vervolgens nadat de bollen nog 10 en 21 dagen in de kas van 9°, 13° en 15° C. gestaan hadden.

Tabel 1 geeft de stadia aan, waarin de bollen op die tijdstippen na die verschillende temperaturen verkeeren. Ook na 30 dagen 15° werden nog 10 bollen gefixeerd. Daaruit blijkt reeds dat de bloemvorming, die bij 6 cm op 27 October zelfs nog niet begonnen was (stadium I) in 13° en 15° C. vlot plaats heeft en wel in 15° C. 't vlugst. Vooral in 9° C. geschiedt dit vrij wat langzamer. Natuurlijk is de snelle aanleg, zooals in 15° C., volstrekt nog geen waarborg, dat de bloemen evengoed zullen strekken en uitkomen als na een langzamer aanleg. Maar bewezen was hiermee wel reeds, dat de bloemaanleg zelf, — waarvan vroeger alleen het voortschrijden in 9° C., na 6 cm, beschreven was (zie Meded. N⁰. 57, Tabel 1 en 2), — in 15° C. goed en sneller verloopt, zelfs wanneer die aanleg bij den overgang van 9° naar 15° C. uiterlijk nog niet of nauwelijks begonnen was.

TABEL I. Verloop der bloemvorming in 9°, 13° en 15° (1938—1939).

	Stadium:	I	I—II	II−	II	II+	II—III	III−	III	III+	III—IV	IV−	IV	IV+	IV—V	V−	V	V+	V—VI	VI−	VI
1 week 31° daarna 9° tot 6 cm (27 Oct.)	toestand:	9																			
Gevolgd door:	+11 d. 9°		4	3	2																
	+10 d. 13°	1	2	3	3	1															
	+11 d. 15°			1	4	2		2	1												
	+21 d. 9°			1	2	4	1	1	1												
	+21 d. 13°									1	3	1	4	1							
	+20 d. 15°									1	1	1		2	3	1					
	+30 d. 15°																	1	6	3	

In 1939—'40 werd niet meer met de overgangstemperatuur 13° gewerkt, maar alleen met 9° (of 7°) gevolgd door 15° C. Na 2—3 of 4 weken 31° als voor-temperatuur, volgde het prepareeren in 9° of 7° C., totdat 6 cm bereikt was. Dan werd een 10-tal gefixeerd. Nu ging een deel der bollen direct in de kas van 15°, terwijl andere eerst nog in een kas van 9° C. bleven en pas na 10 dagen of na 21 dagen naar 15° C. gingen. Weer werd daarbij het verloop van de bloemvorming bij 9° en bij 15° C. vergeleken door fixaties. De uitkomst van deze fixaties geeft Tabel 2 voor de bollen die 9° C. als prepareertemperatuur hebben gehad, en Tabel 3 voor die welke met 7° C. geprepareerd zijn. De laatste zijn alleen met 2 of 4 weken 31° voorbehandeld.

Deze belangrijke fixaties van 1938 en 1939 geven ons nog beter dan vroeger een kijk op hetgeen er gebeurt bij het begin van het trekken:

1°. Als 6 cm looflengte bereikt is staat de bloemvorming in de meeste jaren in het allereerste begin na 9° prepareertemperatuur. Wordt 7° gebruikt, dan wordt 6 cm looflengte daarin twee weken later bereikt dan in 9°, maar de bloemaanleg is een weinig verder op gang dan na 9°. 2°. In 1936—'37 en 1937—'38 werd alleen bewezen, dat door de bollen 2, 3 en 4 weken langer in 9° te laten na 6 cm looflengte, de bloemvorming zeer regelmatig voortgang had (zie tabel 1 en 2 in Meded. N⁰. 57). Zoo verkeerden we in de meening, dat het voor het trekken daarom wellicht gunstiger zou zijn de bloemen eerst wat verder te doen vormen in 9° alvorens naar een hoogere trektemperatuur van 15° (eventueel 13° C.) over te gaan. Deze nieuwe fixaties van 1938—'39 en 1939—'40 bewijzen echter, dat de bloemvorming even goed verloopt in 13° en 15° als in 9° en bovendien sneller. Uit deze fixaties valt dus af te leiden, dat er voor de bloemvorming zelve allerminst bezwaar schijnt te bestaan om na 6 cm direct in de trektemperatuur van 15° C. over te gaan.

Het bloem-vormen heeft dus pas volop plaats als het forceeren in de kas al begint, iets wat bij het trekken van gewassen waarschijnlijk maar zeer zelden voorkomt. In elk geval verklaart dit voldoende waarom hooge

TABEL 2. Verloop der bloemvorming in 9° en 15° C.
na prepareering in 9° (1939—1940).

	Stadium:	I	I—II	II⁻	II	II+	II—III	III⁻	III	III+	III—IV	IV⁻	IV	IV+	IV—V	V⁻	V	V+	V—VI	VI⁻	VI
2 wk. 31° — 9° tot 6 cm (30 Oct.)	toestand:	3	7																		
	+ 10 d. 9°		5		2	3															
	10 d. 15°		2	1	2	2	2		1												
	21 d. 9°						4	2	1	1	2										
	21 d. 15°									1		1	1	2	1		4				
3 wk. 31° — 9° tot 6 cm (7 Nov.)	toestand:	1	9																		
	+ 10 d. 9°		9		1																
	10 d. 15°				3	1	2	3		1											
	21 d. 9°						3	3	3												
	21 d. 15°											1	2	3	1		3				
4 wk. 31° — 9° tot 6 cm (13 Nov.)	toestand:	7	3																		
	+ 10 d. 9°		1	7	1		1														
	10 d. 15°			1	7	1	1														
	21 d. 9°				3		2	1	1	3											
	21 d. 15°										1	3	4		1		1				

TABEL 3. Verloop der bloemvorming in 9° en 15° C. na prepareeren in 7° C. (1939—1940).

	Stadium:	I	I—II	II^-	II	II+	II—III	III^-	III	III+	III—IV	IV^-	IV	IV+	IV—V	V^-	V	V+	V—VI	VI^-	VI
2 wk. 31° — 7° tot 6 cm (14 Nov.)	toestand:			4	5	1															
	+10 d. 9°								1	7	1	1									
	+10 d. 15°									1		2	1	4	1	1					
	+21 d. 9°												2	1	1	2	4				
	+21 d. 15°																			10	
4 wk. 31° — 7° tot 6 cm (28 Nov.)	toestand:		2	6	2																
	+10 d. 9°								6	4											
	+10 d. 15°										8	1	1								
	+21 d. 9°											3	3	4							
	+21 d. 15°																			6	4

trek-temperaturen bij de Bol-iris onbruikbaar zijn, want andere, later te publiceeren proeven hebben ons geleerd, dat deze bollen reeds in 20° C. geen bloem meer kunnen vormen. 3°. Ook na het prepareeren in 7° C. (Tabel 3) verloopt de bloemvorming in kas 15° zeer vlot en vlugger dan in kas 9°.

Of het nu voor het goed in bloei komen beter is na 6 cm direct in 15° C. over te brengen of nog korten tijd in 9° C. te laten, is hiermee nog niet uitgemaakt en zullen we hieronder bij den uitslag der proeven bespreken.

De proeven in 1939 werden genomen met bollen, die op zandgrond waren gegroeid en die bij een omtrek van ongeveer 9—10 cm werden uitgewogen tusschen 16 en 24 gram per stuk, waarbij elke 10 stuks 181,5 gram woog. Dat is iets zwaarder dan in de vorige jaren (14—20 of 21 gram). Zij werden allen, op één proef na, geplant in voedzamen tuingrond gemengd met fijne gravel. Eén proef werd in gewoon duinzand geplant zonder extra voeding.

Voorbehandeld werd gedurende 2, 3 en 4 weken met 31° en 3, 4 en 5 weken met 28° of 23° C. De vrij riskante korte voorbehandeling met 1 week 31° C. werd dus door 2 weken 31° C. vervangen. Geplant werd direct na deze warmte en vervolgens geprepareerd in de kistjes bij 9° en de meeste proeven ook bij 7° C.

Na deze behandelingen werden de kistjes òf bij 6 cm, òf 10 dagen, òf 21 dagen later naar 15° C. gebracht.

Alle proeven begonnen op 17 Augustus 1939. De uitkomsten zijn weergegeven en verdeeld over 3 tabellen, die elk weer uit 5 × 3 soorten van behandeling bestaan. In de laatste kolom is opgegeven het aantal bollen, dat volkomen geslaagde bloemen gaf van de 30 proefplanten. Het aantal zieke planten was steeds gering, bedroeg meestal 0, 1 of 2, slechts drie keer 3 en één keer 4, en vertoonde geenerlei samenhang met de soort behandeling. Het niet-slagen der bloemen (30 minus het vermelde aantal bloeiers), is dus slechts zelden aan ziekte te wijten (3½ % bij de 1380 proefplanten). Het uit zich in het vroeger of later zitten blijven der bloemen.

In de tabellen 4, 5 en 6 is de verdeeling zoodanig gemaakt, dat de verschillen in behandeling en de uitkomsten zoo overzichtelijk mogelijk zijn. Voor de 45 soorten van behandeling verwijzen we dus eenvoudig naar die tabellen. De uitkomsten zullen hier nu vergelijkenderwijs besproken worden, waarbij 1°. het meer of minder vroeg bloeien en 2°. het grooter of kleiner aantal geslaagde bloemen die uitkomst bepaalt.

A. Is het nu beter *direct* bij 6 cm looflengte uit 9° of 7° C. in 15° C. over te brengen, of heeft het waarde nog 10 of 21 dagen in een kas bij 9° C. de bollen te laten en dus de bloem in de kou eerst verder te laten aanleggen (zie tabel 1—3)?

Overziet men alle proeven dan wordt het *begin van den bloeitijd* door 10 dagen langer in 9° C. inplaats van 15° C. te laten slechts weinige dagen (van 0 tot 5 dagen, gem. 2,3 d.) verschoven, — door 21 dagen langer in 9° te laten tot hoogstens 8 à 10 dagen verschoven, dikwijls veel minder (gem. 5,7 d.). Wat dit punt betreft, ontstaat er dus meestal slechts *een geringe verlating.*

Hoe staat het met het slagen der bloemen, waar het bovenal op aankomt? Vergelijkt men telkens de drietallen in de tabellen, dan lijkt bij 2 w. 31° gevolgd door 9°, zoowel als door 7°, het direct plaatsen in 15° (resp. met 27 en 22 geslaagde bloemen) veel gunstiger dan dat men daarmee langer wacht; maar bij 3 w. 31° is dit juist andersom; bij 4 w. 31° gevolgd door 9° is alles vrij gunstig tot zeer gunstig, maar toch ook naar het schijnt iets gunstiger 3 weken te wachten; bij 4 w. 31° gevolgd door 7° is 3 weken wachten daarentegen weer ongunstiger. Er is hieruit geen conclusie te trekken. Want het is toch niet aan te nemen dat na 2 w. 31° en na 3 w. 31° dit effect juist tegengesteld zou zijn. Overzien we de proeven met 28° en 23° C. voorbehandeld, die in 27 van de 30 proeven 20 tot 30 geslaagde bloemen opleveren, dan is een voorkeur voor direct of later overbrengen in 15° nog moeilijker vast te stellen. Wij hebben daarom ook alle uitkomsten van 28° en 23° C. samengesteld, die direct en die welke na 10 en na 21 dagen werden overgebracht.

Dan zien we:

300 planten *direct* uit de koude in 15° C. gebracht gaven 232 bloemen
300 „ 10 d. later „ „ „ „ „ „ „ 237 „
300 „ 21 d. later „ „ „ „ „ „ „ 255 „

Indien men dus alles door elkaar neemt zou men zeggen, dat 21 dagen

TABEL 4. Vroege bloei van Imperator 1939—1940.

Voorbehandeling	Geplant in	6 cm loof	direct in	na 10 d. in	na 21 d. in	1e bloem open	Geslaagde bloemen v. d. 30
2 weken 31°	9°	30 Oct.	15°	—	—	2 Febr.	27
2 weken 31°	9°	„	—	15°	—	7 Febr.	15
2 weken 31°	9°	„	—	—	15°	12 Febr.	12
2 weken 31° *in zand*	9°	3 Nov.	—	—	15°	7 Febr.	26
3 weken 31°	9°	7 Nov.	15°	—	—	21 Febr.	6
3 weken 31°	9°	„	—	15°	—	21 Febr.	16
3 weken 31°	9°	„	—	—	15°	26 Febr.	20
4 weken 31°	9°	13 Nov.	15°	—	—	27 Febr.	20
4 weken 31°	9°	„	—	15°	—	27 Febr.	24
4 weken 31°	9°	„	—	—	15°	28 Febr.	27
2 weken 31°	7°	14 Nov.	15°	—	—	9 Febr.	22
2 weken 31°	7°	„	—	15°	—	11 Febr.	16
2 weken 31°	7°	„	—	—	15°	18 Febr.	10
4 weken 31°	7°	28 Nov.	15°	—	—	24 Febr.	29
4 weken 31°	7°	„	—	15°	—	26 Febr.	30
4 weken 31°	7°	„	—	—	15°	3 Mrt.	21

TABEL 5. Vroege bloei van Imperator 1939—1940.

Voorbehandeling	Geplant in	6 cm loof	direct in	na 10 d. in	na 21 d. in	1e bloem open	Geslaagde bloemen v.d. 30
3 weken 28°	9°	7 Nov.	15°	—	—	14 Febr.	24
3 weken 28°	9°	„	—	15°	—	17 Febr.	25
3 weken 28°	9°	„	—	—	15°	20 Febr.	20
4 weken 28°	9°	13 Nov.	15°	—	—	19 Febr.	22
4 weken 28°	9°	„	—	15°	—	21 Febr.	26
4 weken 28°	9°	„	—	—	15°	21 Febr.	24
5 weken 28°	9°	18 Nov.	15°	—	—	25 Febr.	22
5 weken 28°	9°	„	—	15°	—	28 Febr.	18
5 weken 28°	9°	„	—	—	15°	2 Mrt.	27
3 weken 28°	7°	18 Nov.	15°	—	—	16 Febr.	17
3 weken 28°	7°	„	—	15°	—	16 Febr.	24
3 weken 28°	7°	„	—	—	15°	16 Febr.	26
5 weken 28°	7°	2 Dec.	15°	—	—	4 Mrt.	28
5 weken 28°	7°	„	—	15°	—	4 Mrt.	27
5 weken 28°	7°	„	—	—	15°	12 Mrt.	27

later overbrengen gemiddeld iets meer goede bloemen geeft (23 op 300 planten = bijna 8 % of 2,3 per 30 stuks).

Dit is echter een grove sommeering en na de zeer verschillende voorafgaande behandelingen kan dit na de eene behandeling wel gelden, maar

TABEL 6. Vroege bloei van Imperator 1939—1940.

Voorbehandeling	Geplant in	6 cm loof	direct in	na 10 d. in	na 21 d. in	1e bloem open	Geslaagde bloemen v. d. 30
3 weken 23°	9°	7 Nov.	15°	—	—	14 Febr.	23
3 weken 23°	9°	„	—	15°	—	17 Febr.	21
3 weken 23°	9°	„	—	—	15°	23 Febr.	24
4 weken 23°	9°	13 Nov.	15°	—	—	19 Febr.	25
4 weken 23°	9°	„	—	15°	—	23 Febr.	22
4 weken 23°	9°	„	—	—	15°	26 Febr.	30
5 weken 23°	9°	18 Nov.	15°	—	—	25 Febr.	29
5 weken 23°	9°	„	—	15°	—	1 Mrt.	25
5 weken 23°	9°	„	—	—	15°	1 Mrt.	28
3 weken 23°	7°	20 Nov.	15°	—	—	15 Febr.	17
3 weken 23°	7°	„	—	15°	—	20 Febr.	20
3 weken 23°	7°	„	—	—	15°	19 Febr.	26
5 weken 23°	7°	2 Dec.	15°	—	—	26 Febr.	25
5 weken 23°	7°	„	—	15°	—	28 Febr.	29
5 weken 23°	7°	„	—	—	15°	3 Mrt.	23

na de andere wellicht niet. Geeft men bijv. slechts 3 weken 23° of 28° C. gevolgd door 7° dan leveren alléén deze 2 proeven reeds 18 bloemen meer op bij 21 d. later dan bij direct overbrengen. Maar men ziet, dat na 2 weken en 4 weken 31°, gevolgd door 7°, direct-overbrengen naar 15° juist 20 bloemen méér oplevert dan 21 dagen later overbrengen, en zoo heft dit de conclusie over 3 w. 23° en 28°, gevolgd door 7°, weer geheel op.

Wij kunnen alles samengenomen dus zeggen, dat *na de meer gunstige behandelingen van 3 tot 5 weken 23° en 28° C. er tot dusver geen duidelijk voordeel is aan te toonen, als de bloemaanleg in 9° nog 3 weken langer plaats vindt of direct in 15° wordt voortgezet.*

Bij de behandelingen met 2 en 3 w. 31°, waar duidelijker verschillen op den voorgrond traden, spreken de uitkomsten elkaar naar het schijnt te veel tegen om hier een conclusie te durven trekken. Wij zullen deze proeven nog eens herhalen om te zien of bijv. de tegenstelling na 2 en na 3 w. 31° C. wel bevestigd wordt. Daar waar slechts 2 of 3 weken vóórwarmte gegeven wordt is de kans op het gelukken der bloemen reeds riskanter en daardoor ontstaat hier meer grilligheid in het aantal geslaagde bloemen per groep.

B. Wat is het effect van de kortere of langere voorwarmte met 23°, 28° en 31° C.?

Daar de resultaten over het algemeen in dit jaar zeer gunstig waren zijn de tegenstellingen — afgezien van 2, 3 en 4 weken 31° C. — niet frappant.

Wij kunnen er allereerst op wijzen dat deze hooge temperaturen, die voor het *welslagen* van de bloemen van zooveel belang zijn, den bloei zelf sterk verschuiven. D.w.z. in deze hooge temperaturen is de groei uiterst gering, terwijl evenmin een versnellende nawerking optreedt. Zoodoende wordt allereerst het bereiken van de looflengte van 6 cm *door elke week hooge temperatuur 5 tot 7 dagen verlaat.* Het begin van den bloei weerspiegelt eveneens deze verlating, maar met veel sterker schommelende cijfers. *Een week langer warmte verlaat in den regel 4 tot 7 dagen,* zelden korter of langer; alleen bij 2 en 3 w. 31° is de verlating ± 2 weken. *Twee weken langer warmte, gevolgd door 7° C., geeft in den regel 13 tot 17 dagen verschuiving* van den bloei; gevolgd door 9° C. is de verschuiving meestal ± 11 dagen, maar na 31° C. voor-temperatuur is de verlating omstreeks drie weken.

Om van het slagen van de bloemen een indruk te krijgen, sommeeren we de 3 groepen van 2 weken, van 3 weken enz.

2 w. 31°—9°	54	3 w. 28°—9°	69	3 w. 23°—9°	68
3 w. 31°—9°	42	4 w. 28°—9°	72	4 w. 23°—9°	77
4 w. 31°—9°	71	5 w. 28°—9°	67	5 w. 23°—9°	82
2 w. 31°—7°	48	3 w. 28°—7°	67	3 w. 23°—7°	63
4 w. 31°—7°	80	5 w. 28°—7°	82	5 w. 23°—7°	77

Deze getallen geven dus het aantal geslaagde bloemen op telkens 90 planten. Wij moeten er op wijzen dat 3 w. 31°—9° (zie ook tabel 4) stellig door een of andere, misschien toevallige oorzaak een veel *te* laag aantal bloeiers opleverde. Na 4 en 5 weken voorwarmte leveren 7 van de 8 groepen 70 tot ruim 80 bloemen op telkens 90 planten. Na 3 weken is dit bedrag in 4 groepen 63 tot 69.

Vier tot vijf weken voorwarmte levert dus een veilige waarborg voor het slagen van een groot aantal bloeiers (80 tot 90 %). Telt men alle 8 groepen met *4 en 5 weken voorwarmte* samen, dan leverden deze 720 planten met zeer uiteenloopende verdere behandeling 608 goede bloeiers op, dat is 84 %. Zooals boven gezegd is, beteekent een langere tijd voorwarmte in 't algemeen ongeveer evenveel verlating van den bloei. *Door 3 weken 23° of 28° C. te geven gevolgd door 9° kan men rekenen op ruim 65 % bloeiers* en een bloei die ± midden Februari begint.

Om in de eerste helft van Februari bloemen te hebben moet men slechts 2 weken voorwarmte geven, waarschijnlijk het best met 31°. De uitkomst kan soms heel goed zijn, zie bijv. 2 w. 31°—9° en —7° direct in 15° gebracht. In vorige jaren hebben wij ook met 1 week 31° wel goede resultaten gehad. Het is duidelijk dat de uitkomst bij korter durende voorwarmte steeds riskanter wordt. Toch is het wel degelijk aan te bevelen een niet groote hoeveelheid *met 1 en 2 weken 31° C. te behandelen, omdat men dan einde Januari en begin Februari* Imperator *reeds voor een belangrijk percentage, bijv. 50 % tot soms 75 % in bloei kan hebben.* Tegenover de grootere risico staat ook de hoogere waarde in dien tijd.

C. Bestaat er een voorkeur tusschen 7° of 9° als prepareer-koude? Zooals in de vroegere mededeeling reeds is beschreven en hier herhaald is, wordt in 7° de looflengte 6 cm ongeveer 14 dagen later bereikt dan in 9°, maar de bloemaanleg is dan intusschen verder voortgeschreden dan toen 9° de 6 cm lengte bereikt had (zie tabel 3). Overigens is de tijd van bloeien en het aantal bloeiers over het algemeen genomen hetzelfde als na 9° C. Hoogstens zou men kunnen zeggen dat bij een korte voorwarmte (2 en 3 weken) het iets veiliger is 9° te gebruiken, niet 7°, terwijl na 4 en 5 weken voorwarmte de uitkomst met 7° C. ook zeer gunstig is. Dit komt overeen met de vroeger vermelde ervaringen.

Daar wij anderzijds echter geen duidelijke voordeelen van 7° boven 9° kunnen aantoonen, is het voor de praktijk eenvoudiger zich aan 9° te houden, daar deze temperatuur veel meer gangbaar is. Echter kunnen we zeggen dat het geheele gebied van 7° tot 9° goed is voor het prepareeren van Imperator.

D. De gunstige uitkomst van een zoo groot aantal groepen in dit jaar 1939—'40 hebben wij eerst toegeschreven aan den voedzamen grond die, gedraineerd met fijne gravel, dit jaar de trekkistjes vulde. De bollen waren dit jaar gemiddeld 2 gram zwaarder dan andere jaren, zoodat die factor wellicht ook reeds de uitkomsten gunstig heeft beïnvloed. In tabel 4 wordt ook één proef vermeld, die ter vergelijking in duinzand geplant werd evenals in vorige jaren. Hoewel deze slechts 2 weken 31° C. ontving was de uitkomst zeer goed (26 bloemen). In zoover is het dus weer de vraag of het wel de voedzame grond is die de uitkomsten zoo begunstigde. In verband met de resultaten van 1937—'38 is echter *een goede voedzame grond in elk geval aan te bevelen.*

Daar de bollen dit jaar ongeveer 2 gram zwaarder waren dan in vorige jaren, zou het ook mogelijk zijn, dat hierin alleen reeds de oorzaak lag van den gunstigen bloei in dit jaar. Veiligheidshalve gebruike men steeds de grootst mogelijke bollen bij het trekken, inzonderheid de grootste als men zeer vroeg forceeren wil, dus met 1 of 2 weken 31° C.

Om enkele twijfelachtige details zullen eenige proeven nog herhaald worden. In hoofdzaak echter is de bloei van Imperator als een der late en mooiste Hollandsche Irissen thans voor een ieder bereikbaar in de geheele maand Februari door de aanwijzingen te volgen, die in deze publicatie en reeds in Mededeeling N⁰. 57 gegeven zijn.

Zeer kort samengevat kunnen wij zeggen, dat een goede bloei in de 2e helft van Februari gewaarborgd is, als men den bollen 4 tot 5 weken voorwarmte geeft, die van 23° tot 28° C. mag varieeren; daarna planten in kistjes met voedzamen goed doorlatenden grond bij 9° of iets lager; — als het loof minstens 6 cm uit den bol is kan naar een kas van 15° C. worden overgebracht.

Door de voorwarmte te wijzigen in 1 of 2 weken 31° C. kan de bloei vervroegd worden tot de eerste helft van Februari of einde Januari, soms nog vroeger. De bloei kan ook dan nog gunstig uitvallen, maar de uitkomst is meer wisselvallig.

De voor Imperator gevonden behandeling werd dit jaar ook eens beproefd op een nieuw gewonnen blauwe hybride H. C. van Vliet van de N.V. Vereen. Bloembollenculturen te Noordwijk. Deze hybride is verwant met Imperator, maar bloeit iets vroeger.

De 160 bollen, die voor deze proeven beschikbaar waren, werden op 6 wijzen behandeld. Behandeling en uitkomst worden in tabel 7 weergegeven.

TABEL 7. Vroege bloei van H. C. VAN VLIET. 1939—1940.

Voorbehandeling	Geplant in	6 cm loof	direct in	na 21 d. in	1e bloem open	Aantal bollen	Geslaagde bloeiers	Percentage bloeiers
2 weken 31°	9°	3 Nov.	15°		23 Jan.	30	30	100%
2 weken 31°	9°	3 Nov.		15°	31 Jan.	30	29	96%
5 weken 23°	9°	22 Nov.	15°		14 Febr.	30	29	96%
5 weken 23°	9°	20 Nov.		15°	21 Febr.	30	29	96%
5 weken 23°	7°	3 Dec.	15°		20 Febr.	20	17	85%
5 weken 23°	7°	2 Dec.		15°	24 Febr.	20	18	90%

De bloei van H. C. v. Vliet begint in de zes proeven na prepareeren in 9° resp. 10 — 12 — 11 — 9 dagen vroeger dan bij Imperator en 6—8 dagen vroeger na prepareering in 7°. De uitkomsten zijn, vooral na gebruik van 9°, bijzonder gunstig en, wat inzonderheid van belang is, na slechts 2 weken 31° even goed als na 5 weken 23°. Aldus is deze variëteit zeer geschikt voor vroegen trek, allereerst doordat ze reeds uit haar aard ruim 10 dagen vroeger bloeit, maar ook doordat ze aan een kortdurende voorwarmte blijkbaar reeds voldoende heeft om zeer goed te bloeien. Bij 6 cm loof kan direct in 15° C. worden overgebracht; later overbrengen geeft enkel verlating. Deze variëteit kan dus in de 2e helft van Januari vlot bloeien, mits men flinke bollen uitkiest en in de kistjes voedzamen doorlatenden tuingrond gebruikt.

Hiermede is dus tevens bewezen, dat de voor Imperator gevonden gegevens ook voor naverwante variëteiten een betrouwbare basis kunnen zijn. Voor een variëteit als Wedgwood raadplege men Meded. N^0. 64 (Proc. Kon. Ned. Akad. v. Wet. p. 879).

Wageningen, Juli 1940.

SUMMARY.

Early-flowering Dutch Irises ("Imperator") III.

Up to now (Comm. Nr 48 in 1936 and Nr 57 in 1938) it had been found that the earliest flowering was obtained by giving only 1 week

31°, followed by 9°; that this quick forcing is rather risky, but that adequate nourishment beforehand or also during the experiment much improves the result; moreover, that a longer previous heating of 5 weeks at 23° to 28°, followed by 9° or 7°, gives with great certainty a very satisfactory bloom, bulbs from sandy soil being in this respect not inferior to those from clay soil.

At 9° and when the foliage is 6 cms long, flower-formation is exactly in its very first stage. The question was now, whether for this flower-formation it was necessary to remain first for another 2 to 3 weeks at 9° (which gives retardation), or whether the flower-origination proceeded equally well when transferring directly to 15°. Table 1 (1938) shows the course of the flower-formation at 9°—13° and at 15°; table 2 and 3 (1939) at 9° and 15° after preparing at 9° and at 7°.

The outcome is that the *flower-formation* proceeds very well at 15°, after the foliage has reached a length of 6 cms, and more quickly than at 9°. Also with the further *getting into bloom* it appeared that, when the foliage is 6 cms long, the flower-formation may further take place at 15°.

So the full flower-formation occurs when the forcing in the greenhouse has already started, which rarely happens with forced plants.

Tables 4, 5 and 6 give the result of 45 ways of treatment; the last two columns contain the dates at which flowering begins and the number of successful flowers from 30 bulbs.

The results fully confirm those formerly obtained. Four to five weeks of previous heat are a safe guarantee for a large number of successful bloomers (80—90 %). The longer previous heat is given, the later flowering sets in.

Three weeks at 23° or 28°, followed by 9°, gave over 65 % of bloomers, flowering beginning about the middle of February.

One or two weeks at 31° is more risky, but in this way it is possible to obtain 50 %, sometimes as many as 75 % of bloomers, which then begin to flower towards the end of January and early in February. The risk is greater, but on the other hand the value is enhanced.

A nourishing permeable soil in the boxes is recommended.

Summed up very briefly the conclusion is:

Good flowering in the second half of February is warranted if the bulbs are given 4 to 5 weeks of previous heat, which may range from 23° to 28° C., then they are planted in boxes with nourishing soil at 9° or a little less; when the foliage is at least 6 cms out of the bulbs, they may be transferred to a greenhouse of 15° C.

By modifying the previous heating to 1 or 2 weeks at 31°, flowering may be accelerated to the first half of February or the end of January, sometimes to a still earlier date, but the result is more uncertain then.

A new slightly earlier hybrid H. C. van Vliet, akin to Imperator, (see table 7) gave excellent results when treated in these ways; by its own nature it is already some 10 days earlier and after only 2 weeks of previous heating at 31° 96 to 100 % of the bulbs flowered.

Chemistry. — *On the mutual influence of potential determining and non-potential determining ions in sols.* By H. R. KRUYT and MARGA KLOMPÉ.

(Communicated at the meeting of September 28, 1940.)

The facts discussed in this communication became manifest in an elaborate investigation on the connection between flocculation value and sol concentration. We hope in course of time to make detailed communications on this subject. The facts mentioned in this publication form an additional complication of the main theme, but are significant for their own sake and on account of other, related, phenomena, for which reason we wish here to consider them separately.

In our investigation we started from a negatively charged AgI-sol. In order, however, to avoid certain complications we made use of a so-called repeptisation sol. The preparation is as follows: under ice-cooling the sol is made in the usual way from KI and $AgNO_3$, and without dialysis at once precipitated with an excession of $AgNO_3$. Immediately after this the precipitate is decanted 10 times with ice water, after which, under rapid stirring and ice-cooling, it is repeptised with 40 m. Mol KI final concentration. Then follows electrodialysis which subjects the sol to intense purification [1]). The rapid mode of working and the strong cooling prevent excessive recrystallisation of the AgI-particles. Thus the surface remains active, which makes repeptisation to 90 % possible. The sol is then brought to the pI required. The control of the pI was done in the way described in H. DE BRUYN's thesis [2]). We propose later to give a detailed discussion of the other properties of repeptisation sols and their difference from the normal ones. We know that such a sol consists of AgI-particles, having I-ions as charging ions, with H^+ ions as counter ions. We worked with a sol of pI 4, one of pI 5 and one of pI 6. For the present we shall restrict ourselves to the results obtained with the sol of about pI 5.

Various dilutions were made of that sol, but by adding KI we made sure that all of them had the same pI as the original sol.

Of such a series of dilutions we next determined flocculation concentrations for non-potential determining ions, for KNO_3, $Ba(NO_3)_2$ and $La(NO_3)_3$. We here restrict ourselves to one salt, viz. $Ba(NO_3)_2$. As it soon appeared that the pI of the sol was changed by the addition of an indifferent electrolyte, we carefully checked the dependence of pI on the $Ba(NO_3)_2$ concentration. The subjoined figure and table represent the results.

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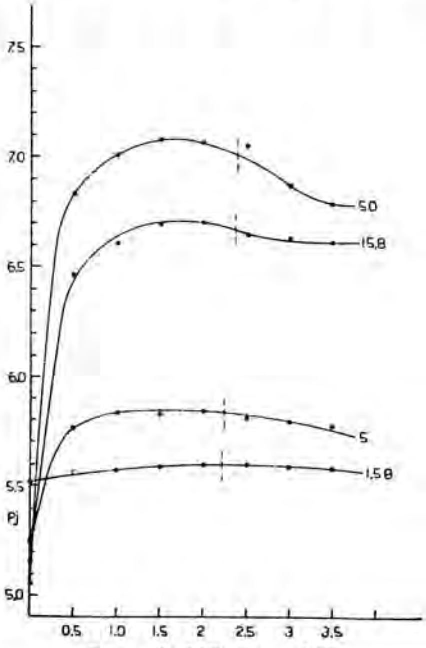
[1]) H. DE BRUYN and S. A. TROELSTRA, Koll. Z. 84. 192 (1938).

[2]) H. DE BRUYN, Diss. Utrecht 1938.

The pI values are set out as ordinates, the barium salt concentrations as abscissae. Each of the curves applies to a sol of the AgI concentration indicated near the line, namely in m Mols/L (final concentration, i.e. after the addition of the electrolyte). The pI is here increased by two factors, namely the dilution with electrolyte and the influence of the salt. In order to eliminate in the figure the first factor, the initial points of the curves do not represent the pI of the pure sol, but the pI after dilution with water 1 : 1. We see that this influence of the dilution is strongest as the sol is more diluted. The concentrated sol, with the greatest surface can more easily yield an aftermath of I-ions, so there the pI will change least. See our subsequent considerations.



Conc. $Ba(NO_3)_2$ in mM/L

So for the undiluted sol with AgI 50 m Mol/L we find that the pI has increased by two units. In the more diluted sols we find the same phenomenon, but less marked, as the sol is more diluted.

Final conc. $Ba(NO_3)_2$ in m Mols/L	pI sol R I 50 m Mols AgI/L	pI sol R II 5.8 m Mols AgI/L	pI sol R III 5 m Mols AgI/L	pI sol R IV 1.58 m Mols AgI/L
pure sol	4.98	4.93	4.96	5.10
sol + H_2O 1 : 1	5.05	5.16	5.24	5.41
0.5	6.84	6.46	5.77	5.56
1	7.01	6.61	5.83^5	5.57
1.5	7.07	6.69	5.83	5.59
2	7.06	6.70	5.84	5.60
2.5	7.05	6.64	5.81^5	5.60
3	6.86^5	6.63	5.80	5.59
3.5	6.79	6.60	5.78	5.58
flocculation value in m Mols/L	2.35	2.32	2.26	2.20

Experiences and considerations of the last few years in this laboratory [1] account for this phenomenon, which is so astonishing at first sight. We shall consider the condition of the sol before the addition of the indifferent electrolyte. There is at the periphery of the AgI-particle an electric double layer, the potential leap $\Delta\varepsilon$ of which is given in the relation of Nernst:

$$\Delta\varepsilon = -\mathrm{k} \log [\mathrm{I}^-] = \mathrm{k} \times \mathrm{pI}.$$

When an indifferent electrolyte is added, the double layer is compressed according to the theory of Gouy-Smoluchowski-Stern. The capacity of the double layer consequently decreases, and in the case of the charge remaining unchanged, the potential difference also decreases, but according to the relation of Nernst mentioned above, it is determined entirely by the pI. In order to maintain the connection the charge of the surface must be increased at the expense of I-ions from the intermicellary liquid. Now theoretically there are two possibilities:

1. There are so many I-ions in the liquid that the needed increase of the charge in the wall scarcely affects the pI. In this case potential difference and pI before and after the addition of the electrolyte are the same.

2. There are not enough I-ions in the intermicellary liquid, so that the removal effects a considerable change in the pI. In this case a number of I-ions will move to the surface — and hence from the liquid — until a new pI value has been formed and a corresponding $\Delta\varepsilon$, according to the formula of Nernst. The first case will be promoted by two circumstances: many I-ions in the intermicellary liquid, i.e. a sol of low pI, or little colloid surface, i.e. a low sol concentration.

In a sol of 50 m Mol/L and pI 5 neither the one, nor the other is the case. So the addition of the indifferent electrolyte, and with it the change in the capacity of the double layer, will cause a considerable increase of the pI of the sol. As the sol concentration decreases the phenomenon will become less manifest, just as is shown in our figure. The second conclusion too was confirmed by further investigation. When one starts from a sol of pI 4 the phenomenon is less evident.

This shows the difficulty of treating the connection between sol concentration and flocculation concentration. For even when the initial pI values are the same in all dilutions, the final pI values will differ after the addition of the flocculating electrolytes.

For the rest it is also interesting to note the localisation in our figure of the flocculation concentration. They are marked by a dotted line on each curve. It is seen that the pI shift is practically completed long before the flocculation value is reached.

After these results had been found with the negative sol, Dr. I. KUBAL

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[1] Especially H. DE BRUYN, Diss. l.c. Cf. H. R. KRUYT and H. DE BRUYN, Z. physik. Chem. **A 186**, 282 (1940).

((Prague) in the previous academic year made similar investigations in this laboratory with the positive sol. These too will be fully discussed later. We will now only state that essentially he obtained the same results, although circumstances are much more unfavourable in that case: it should be borne in mind that the solubility product of AgI is 10^{-16} so the neutral point is at pI 8 (pAg 8) and the point of zero charge at pI 10 (pAg 6). A sol of pAg 4 still proves insufficiently stable, so one has to work with pAg 3.8. Even in the highest attainable sol concentration the pAg shift on the addition of an indifferent electrolyte is comparatively slight, though it considerably passed the experimental error, so that maxima were obtained which were lower but evidently similar to those in our figure.

These investigations draw the attention to the fact that one should be extremely careful in interpreting the distinction — perfectly justified in itself — between potential determining and non-potential determining electrolytes (called indifferent in the above). For the latter do indeed (though indirectly) change the potential leap, especially in a sol with a low concentration of potential determining electrolyte (high pI in negative, high pAg in positive sols respectively) and a large development of colloid surface (high sol concentration).

Physics. — *The mechanism of emulsion formation in turbulent flow.* II *). Theoretical Part and Discussion. By P. H. CLAY. (Laboratorium voor Technische Physica van de Technische Hoogeschool te Delft.) (Communicated by Prof. J. M. BURGERS.)

(Communicated at the meeting of September 28, 1940.)

Zusammenfassung.

Die Eigenschaften des turbulenten Strömungsfeldes, die bei der mechanischen Bereitung grober Emulsionen eine Rolle spielen dürften, werden besprochen. Auf Grund davon werden das Zusammenfliessen und das Zerbrechen der Tröpfchen einer Emulsion in turbulenter Bewegung behandelt. Bei dem Zusammenfliessen lassen sich unterscheiden das direkte Zusammenfliessen durch die Wirkung der dynamischen Kräfte beim Zusammenstoss von zwei Tröpfchen, und das verzögerte Zusammenfliessen durch die Wirkung der Kohäsionskräfte. Das Ergebnis der Messungen (Teil I) deutet auf das wirkliche Vorkommen beider Prozesse.

Die möglichen Ursachen des Zerbrechens werden untersucht. Mit Hinsicht auf das Messresultat stellt sich heraus, dass das turbulente Zerbrechen der Tröpfchen Aehnlichkeit zeigt mit dem Zerstäubungsprozess im Dieselmotor und als eine Wirkung der Schwankungen des statischen Druckes betrachtet werden kann. Das Zerbrechen stellt der Grösse der Tröpfchen einer Emulsion eine obere Grenze, die in grober Annäherung quantitativ gedeutet werden kann.

7. *The turbulent field.*

In every point of a turbulent field of flow we distinguish the mean velocity (components U, V and W) and the velocity fluctuation (q, with components u, v and w). Here we always take $V = W = 0$, unless another assumption is made. Simultaneously with the velocity fluctuations there are fluctuations in the (static) pressure (p), in the velocity gradients ($\partial u/\partial y$ etc.) and in the pressure gradients ($\partial p/\partial x$ etc.). A consequence is the high diffusive power of the turbulent field.

Henceforth we denote the root mean square values of the various fluctuations by means of an index s.

For an elaborate theoretical and experimental treatment of turbulence we refer to the compilatory work of S. GOLDSTEIN [1]). Many more recent data can be found in the *Proceedings of the Vth International Congress for Applied Mechanics* [2]). Here we make use of the notation which

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*) Part I appeared in these Proceedings **43**, 852 (1940).

GOLDSTEIN applies for the treatment of the statistical theory of turbulence, as it has been developed by TAYLOR, VON KARMAN and others.

It has been established [3, 4]) experimentally that the frequency distributions of the components of the velocity fluctuations fulfill "GAUSS' *normal error law*". The probability that in a given point and at a given time u will lie between u and $u + du$ is:

$$w(u)\,du = \frac{1}{u_s \sqrt{2\pi}}\, e^{-\frac{u^2}{2u_s^2}}\, du .$$

Writing $|u|/u_s = f$, the chance that $f \geqslant f_1$ will be:

$$W(f_1) = \sqrt{\frac{2}{\pi}} \int_{f_1}^{\infty} e^{-\frac{f^2}{2}}\, df \quad . \quad . \quad . \quad . \quad . \quad . \quad (7,1)$$

For instance, for $f_1 = 2$ we get: $W(f_1) = 5 \,.\, 10^{-2}$; and for $f_1 = 3$: $W(f_1) = 3 \,.\, 10^{-3}$.

As yet there are no experimental or theoretical data concerning the corresponding distributions of probabilities for the fluctuations of the velocity gradients, of the pressure and of the pressure gradients. It may be supposed, however, that in the main these probabilities will have the same character as (7, 1).

In connection with our apparatus the following special data are of interest. First we consider the mean velocity fluctuations in straight and sufficiently long tubes and channels. In the case of flow through a channel of width $2b$, the distance of a given point from the central plane will be called y. In the case of flow through a tube with radius R, the distance of a given point from the axis of the tube will be called r. Following the custom we take v in the channel along $-y$ and in the tube along $-r$.

It has been proved that in these cases in every point u_s^2, v_s^2 and w_s^2 practically are proportional to the local mean shearing stress, $\tau_{xy} = \tau_{xy}(y)$ in channels, $\tau_{xy} = \tau_{xy}(r)$ in tubes. In the following we denote the shearing stress at the wall by τ_0 and we write $\tau_0 = \varrho U_\tau^2$.

Moving from the wall to the axis of the tube or to the central plane of the channel three regions are to be distinguished: a laminar boundary layer, a turbulent boundary layer and a turbulent central region (compare publ. 2, p. 350 and 367). The laminar boundary layer has a thickness of about $3\eta/\varrho U_\tau$ [6, 7]). Even in this boundary layer there still are velocity fluctuations [5, 8]).

The mean fluctuations in the turbulent boundary layer for the greater part depend on the shearing stress at the wall τ_0; at a distance d from the wall u_s/U_τ has a maximum value, which in first approximation is independent of Re: this maximum value is 2,3 in tubes and 2,6 in channels. The distance d approximately amounts to $20\,\eta/\varrho U_\tau$ in channels and to $40\,\eta/\varrho U_\tau$

in tubes. At the same place w_s/U_τ reaches a maximum value, which seems to increase a little with increasing Re, but which is always about as large as u_s/U_τ 6)); v_s/U_τ always is smaller and shows a lower maximum value, which varies somewhat with Re, at a greater distance from the wall also depending on Re.

According to FAGE's data 6) the maximum of q_s/U_τ in a tube increases from 2,9 to 3,5 when Re increases from 8000 to 20000. The influence of the turbulent boundary layer reaches unto a distance from the wall of $250\,\eta/\varrho U_\tau$.

In the turbulent central region the mean velocity fluctuations are connected with the local mean shearing stress as mentioned above, but towards the axis or towards the central plane they approach to constant values ($q_s/U_\tau = 1{,}44$ in a tube 6); $q_s/U_\tau = 1{,}34$ in a channel 2)).

TAYLOR 9) introduced a linear scale λ, which he called the "size of the smallest eddies", for the treatment of the mean velocity gradient $(\partial u/\partial y)_s$. He assumed:

$$(\partial u/\partial y)_s^2 = 2\, u_s^2/\lambda^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7,2)$$

In isotropic turbulence at a sufficiently high REYNOLDS' number λ has to fulfill the relation:

$$\lambda/L = A \sqrt{\eta/\varrho u_s L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7,3)$$

as TAYLOR showed by means of energetic considerations. Here L is a measure for the "scale" of the turbulence and generally it will be determined by the geometry of the apparatus. In (7, 3) the constant $A \approx 2$.

According to data of PRANDTL and REICHARDT which are mentioned by TAYLOR 11), (7, 3) also holds for turbulent flow through a channel in the turbulent central region and in the turbulent boundary layer up to the place where u_s/U_τ reaches its maximum value. In the turbulent boundary layer within a certain (small) distance from the wall $(dU/dy)^2$ will obtain a value comparable to $\overline{(\partial u/\partial y)^2}$, however.

No experimental data are known for the root mean square values of the pressure fluctuation p_s. TAYLOR supposed:

$$p_s = {}^1/_2\, k\, \varrho\, q_s^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7,4)$$

and calculated k for a mathematical model of turbulence, which gave an approximate description of isotropic turbulence. GREEN 13) extended this work and obtained 0,7 as the most probable value for k; in a tube this would give 3 to 4 for the maximum value of p_s/τ_0, corresponding with the maximum value of q_s/U_τ, when Re increases from 8000 to 20000. In a channel this will be about the same.

The field of flow in an arrangement 1, 15) similar to our model arrangement, but with very long cylinders, in first approximation is another case of parallel shearing in layers. Therefore the general properties of the field of velocity fluctuations in this case will correspond with those of the fields discussed above and three regions can be distinguished in the same way. The rotation and the fact that shearing stresses exist everywhere, however, bring about the following important deviations: 1) the existence of the annular vortices, which form part of the mean flow (see No. 3); 2) a more intense mixing and a smaller contrast between the turbulent boundary layer and the turbulent central region; 3) the annular vortices also transport momentum and will contribute to the resistance couple, i.e. to the shearing stresses (τ_1 at the outer cylinder and $\tau_2 = \tau_1 R_1/R_2$ at the inner cylinder). This contribution will become relatively smaller with increasing Re (see Fig. 6). Therefore the field of the mean components of the velocity fluctuations is to a less extent determined by the distribution of the mean shearing stress; the maximum values of q_s/U_τ and of p_s/τ_2 (in the turbulent boundary layer at the inner cylinder) will increase with Re, but they will remain smaller than in the tube or in the channel.

8. *The process of coalescence.*

We now will discuss the emulsion formation in our apparatus using the picture of the turbulent field given in No. 7. We suppose that the differences between the turbulent fields of flow described in No. 7, and the actual turbulent fields of flow in our apparatus filled with emulsions of low volume concentrations, have no preponderant influence on the emulsifying properties. (We restricted ourselves to volume concentrations of 12 % in the circuit and of 5 % in the model.)

First we consider the elementary processes and we begin with the coalescence of a droplet 1 whose radius is a_1 with a droplet 2 whose radius lies between a_2 and $a_2 + da_2$ and of which there are $n(a_2)\, da_2$ per cm^3 emulsion. We distinguish two problems:

a. The chance of collision between these droplets in the turbulent field. This is a hydrodynamic problem. We can derive approximate relations for the collision frequency $f(a_1, a_2)\, n(a_2)\, da_2$, if we suppose that the droplets follow the undisturbed velocity fluctuations at their centres. We denote by b the mean relative velocity of droplet 2 with regard to droplet 1 when their centres would pass each other at a distance l. If we suppose that both droplets remain spherical the required relation becomes:

$$f(a_1, a_2) = \int_0^{a_1+a_2} 2\pi b\, l\, dl.$$

In this equation the relation between b and l will be determined by the properties of the field of turbulent velocities.

The droplets, however, will not follow the local velocity fluctuations completely; certainly they will not do so when they approach each other very closely. When there is a difference between the densities of the liquids the mean relative velocity b will be greater for droplets of different sizes, especially if one of the droplets is rather large. In this case we expect an increased collision frequency therefore.

b. The chance of coalescence of the droplets at a collision. There are two possibilities:

b1. When the forces occurring during the collision break the coherence of the surface of the droplets, they may coalesce directly. This process has been investigated experimentally by GORBATSCHEW [16, 17]) for drops of water colliding in air ($a_1 = a_2 \approx 0{,}05$ cm). Direct coalescence appears to occur only when the relative velocity b before the collision lies in a certain region, which depends on a_1 and a_2, on the collision angle etc.

b2. After the collision the droplets may cling together by cohesive forces. If they remain in contact long enough, they will coalesce. The problem in this case is whether the cohesive force will be able to withstand the hydrodynamic forces which are trying to separate the droplets again. The cohesive force will depend on the difference γ_1 between the surface energy of the free droplets and the surface energy of the droplets in the common area of contact, and on the forms of the droplet surfaces around the area of contact; γ_1 will be connected with the physico-chemical properties of the surface layers of the droplets. This cohesive force cannot be calculated generally therefore. DERJAGUIN [18]) and BRADLEY [19]) calculated the maximum cohesive force K between plastic or elastic spheres by means of simplifying assumptions and obtained:

$$K = 4\pi\gamma_1 a_1 a_2/(a_1 + a_2).$$

We may qualitatively expect a similar relation for liquid spheres. For $a_1 < a_2$ the separating forces may be supposed to be proportional to the second or to a higher power of a_1. If γ_1 has a constant value, the chance of this type of coalescence will be greatest for small droplets with large ones. On the other hand we expect that the chance of direct coalescence will be lower for the smaller droplets in an emulsion. In special cases a part of the droplets may have electrical charges, which may influence the coalescence.

Process *b1* is of short duration. It can not be identified on the photographs. Process *b2* is slow and the photographs actually show it (No. 6, remark 3*a*). It corresponds with the coagulation process in emulsions, colloids etc.

The last columns of the tables *A* and *B* qualitatively give the rapidity of coalescence in the various emulsions. The expected influence of a density difference actually occurs in the model measurements (compare series *A*—*K* ($\varrho' = \varrho$) with *O*—*R* ($\varrho' > \varrho$)). Table *A* shows that emulsions with similar physical properties may have a different rapidity of coalescence. This points to an influence on the coalescence of the physico-chemical properties of the surface layers between the two liquids which are difficult to control sufficiently. The emulsion of series N, which contained a little bit of an appropriate emulsifier, did not show any coalescence indeed.

The process of coalescence will occur everywhere in the turbulent field and there will be a great variety in the hydrodynamic circumstances at the collisions. Therefore we do not expect that the coalescence will favor droplets of special radii in the emulsion. The action of coalescence in an emulsion expresses itself in a certain rate of variation of the size distribution of the droplets, which will depend on the phenomena discussed and on the size distribution itself.

9. *The bursting process.*

A droplet bursts if it cannot endure the forces acting on its surface in the turbulent field. It is not known which property in the turbulent field is responsible for the process of bursting. The coherence of the surface of our droplets in first approximation is given by $2\gamma/a$. The influence of the other physical properties as viscosities and densities will depend on the nature of the process of bursting, but it will be of secondary importance. Therefore we suppose that this bursting property will have the same dimension as $2\gamma/a$ (dyne/cm^2). With some probability two possibilities occur:

a. Bursting as in a laminar field of flow. TAYLOR [20, 21]) has investigated this phenomenon theoretically as well as experimentally; TOMOTIKA [22]) has extended TAYLOR's theoretical work. From him we take the supposition that the droplet will burst if it dwells in a region where the velocity gradient α exceeds a certain value α_1, determined by:

$$C_\alpha \eta \alpha_1 = 2\gamma/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9,1)$$

C_α depends on η'/η; according to TAYLOR it varies from 7 at $\eta'/\eta \gg 1$ to 0,8 at $\eta'/\eta \ll 1$. The region where $|\alpha| \gtrless \alpha_1$ must be larger than the droplet; the whole droplet will deform considerably before it bursts. This process will take some time. TAYLOR expected that this way of bursting exists in turbulent flow as well.

b. In the jet of an atomizer in an oil engine bursting of the fuel droplets is caused by other phenomena of which little is known in detail. See e.g. a paper by O. KLÜSENER [23]). We suppose that the droplet will burst, if the pressure at its surface locally falls more than $-p_1$ below the mean value. We can imagine that this will cause a local deformation of the droplet surface in the form of a tiny protuberance, what may lead to the separation of a small droplet. The following simple experiment may illustrate our picture of this way of bursting. We hang a droplet of kerosine in a little ring of thin wire in water and we let a fine jet of water skim the droplet surface at a small distance. In this way we produce a local region of lower pressure at the droplet surface; if the velocity of the water in the jet exceeds a certain value a bursting process of the kind mentioned will start. Here we assume:

$$-C_p p_1 = 2\gamma/a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9,2)$$

In this case the droplet will burst very rapidly and it will be deformed only locally. Therefore a higher viscosity of the droplet η' will counteract this way of bursting and will lower C_p.

Bursting will occur preferentially in the turbulent boundary layer or in the laminar one. Process *b*, e.g., will have the greatest chance in the regions where p_s is largest, that is in the turbulent boundary layer. Process a may be caused by the turbulent fluctuations in the velocity gradient in the turbulent boundary layer (see No. 7). The mean velocity gradient dU/dy may have influence on the occurrence of process *a* as well. It may be preponderant in the laminar boundary layer and in the adjacent part of the turbulent boundary layer.

According to No. 7 the maximum value of p_s occurs at a distance *d* from the wall; in the model apparatus at the inner cylinder *d* varies from 0,5 cm (4—6, 2) to 0,04 cm (5—6, 4); in the curved tubes of the circuit *d* varies from 0,1 cm (45—7, 9) to 0,001 cm (21—25, 7). In some of the photographs made with the model arrangement the droplets in the boundary layer still are sufficiently sharp, and these photographs make it probable that we have to do with process *b* only, since wholly deformed droplets are not to be seen (compare No. 6 and see Fig. 7).

Entering more into details we may ask ourselves how the processes of bursting in the turbulent field discussed may really occur. First we suppose that we have to do with process *b*. The chance that a droplet with radius *a* will burst is the chance that at its surface a region exists with $p \leqslant p_1$, provided relation (9, 2) holds between *a* and p_1. This chance will be connected with the chance $W(f_1)$ that in a certain point $|p| \geqslant |p_1|$, which is considered in No. 7. We expect that the chance of bursting per unit of time will show a very steep decrease for higher values of f_1, just as does the function $W(f_1)$ (see 7, 1)). In combination with (9, 2) this gives a qualitative picture of the relation between the chance of bursting and *a*: above a certain value of *a* it will be rather large; below this *a* it will decrease very rapidly with decreasing *a*. The influence of the process of bursting on the size distribution of the droplets in an emulsion will depend also on the way in which the droplets are broken up; it will show this characteristic dependency on *a* even stronger.

In a stationary state of dispersion the two elementary processes, as we have treated them now, must have led to an equilibrium. The process of bursting practically will have put a limit to the size of the droplets then, since, due to the dependency of the chance of bursting on *a*, even a much more rapid coalescence could keep up droplets which are only slightly larger.

We denote the largest droplet radius by a_m. In a stationary emulsion relation (9, 2) then will connect with a_m a maximum value of the pressure fluctuation, which we will call $-p_{1m}$.

If we had to do with process *a* in the turbulent boundary layer the reasoning would be analogous and a_m would be connected with a maximum value α_{1w} of the velocity gradient by relation (9, 1). If dU/dy were the main cause of bursting the reasoning would be simplified but the result would be the same.

The size distributions found show a steep decrease at larger values of *a*, which can be explained by means of this conception of the process of bursting (compare No. 6, remark 1*b*). The characteristic radius a_{95} which we calculated from the size distributions obtained for the emulsions investigated, can be identified with this a_m in the cases of stationary emulsions.

10. *Analysis of the measurements.*

The measurements made with the model apparatus supply sufficient data to enable us to check the conclusions of No. 9. In the laminar boundary layer or in the turbulent one at the inner cylinder bursting of the droplets will be most probable. We suppose in connection with No. 7 for the highest values of the fluctuations in these regions and for λ (see (7, 3)):

$$u_s \backsim v_s \backsim w_s \backsim \omega R_1 \text{ and } \lambda \backsim (R_1-R_2)/\sqrt{Re} \quad . \quad . \quad (10, 1)$$

Next we assume that the following relations will hold for the maximum values of the fluctuating properties which we introduced in No. 9:

$$-p_{1m} \backsim p_s \text{ and } \alpha_{1m} \backsim (\partial u/\partial y)_s \backsim u_s/\lambda \text{ (see 7, 2).} \quad . \quad . \quad (10, 2)$$

In stationary emulsions this gives the following relations for the possibilities of bursting which we considered in No. 9:

a. For the "turbulent" process *a* (see (7, 3) and (9, 1)):

$$2\gamma/a_{95} \backsim C_\alpha \eta\, u_s/\lambda \backsim C \alpha \varrho\, \omega^2 R_1^2/\sqrt{Re} \quad . \quad . \quad . \quad (10, 3)$$

b. For process *b* (see (7, 4) and (9, 2)):

$$2\gamma/a_{95} \backsim C_p\, p_s \backsim C_p \varrho\, \omega^2 R_1^2 \quad . \quad . \quad . \quad . \quad . \quad (10, 4)$$

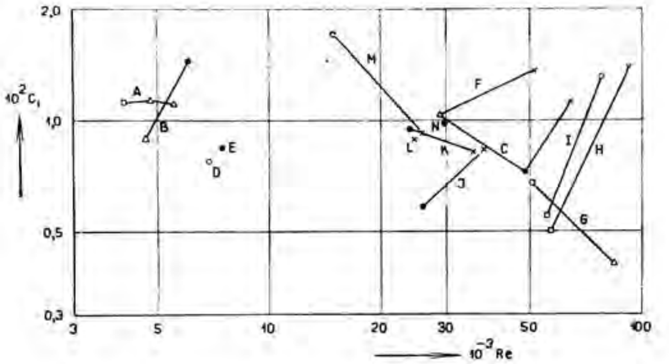
c. For the "laminar" process *a*, when the action of dU/dy is preponderant (in the region which is very near to the inner cylinder, where $\eta(dU/dy) = \tau_2$) (see (9, 1)):

$$2\gamma/a_{95} \approx C_\alpha \tau_2.$$

Table *A* gives $2\gamma/a_{95}\, \varrho\omega^2 R_1^2 = C_1$ and $2\gamma/a_{95}\, \tau_2 = C_2$ for all model measurements made on stationary emulsions.

For C_1 we should find a value proportional to $Re^{-1/2}$ in case *a* and a value independent of *Re* in case *b*. In case *c* we should find $C_2 \approx C_\alpha$. If we compare the values found for C_2 with TAYLOR's data for C_α as given in No. 9, it appears that, for the greater part, the values of C_2 are much higher, whereas the influence of the viscosity η' of the dispersed phase on C_2 is opposite to that on C_α. Therefore the process of case c does certainly not occur.

Fig. 13 gives C_1 as a function of REYNOLDS' number for the slower coalescing emulsions (series *A*—*M*). According to Fig. 13 the variation of C_1 with *Re* in no one of the emulsions corresponds with our relations (10, 3) and (10, 4) for the cases *a* and *b*; apart from a certain scattering in the values of C_1, however, the general course of C_1 with *Re* certainly points to (10, 4) and the influence of the viscosity η' appears to be small. Even series *N*, where coalescence is absent, gives an appropriate value of C_1. Fig. 14 gives C_1 as a function of REYNOLDS' number for some emulsions with more intense coalescence (series *O*—*R*). The mean value of C_1 in Fig. 14 is only a little bit lower than in Fig. 13. These results are in agreement with the picture of the bursting process given above and the conclusion is again that process *b* is the main cause of bursting.



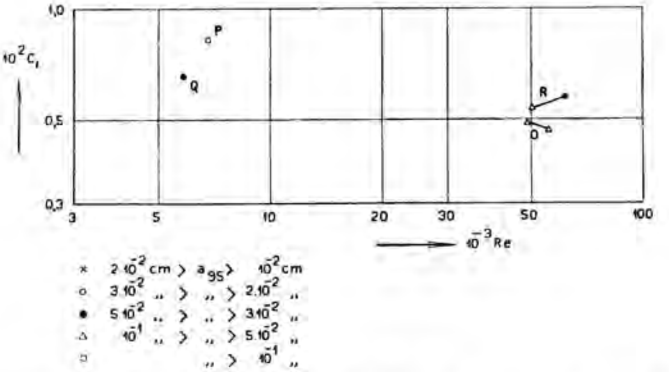


Fig. 13 (above). $C_1 = 2\gamma / a_{95}\, \varrho \omega^2 R_1^2$, for the stationary emulsions of series *A*—*M* made in the model arrangement, as a function of REYNOLDS' number.

Fig. 14 (below). The same for the stationary emulsions of series *O*—*R* made in the model arrangement.

C_1 still appears to be subjected to the following influences:

1. A slight decrease of C_1 when the coalescence is much faster has already been mentioned above; the influence of the volume concentration of the dispersed phase seems to be negligible.

2. In a given emulsion there is an influence of a_{95}. The variation of C_1 with a_{95} shows some similarity in different emulsions.

3. In the series D, G and H the ratio η'/η was high. If anything this seems to have produced somewhat lower values of C_1, as was expected for case b (see No. 9).

4. Figs. 13 and 14 indicate that there is still a slight decrease of C_1 with increasing REYNOLDS' number. This may point to some defect in the supposition (10, 1).

Quantitative experimental data which could determine the ratio between p_s and $\varrho\omega^2R_1^2$ in the model are lacking. In order to be able to judge the results quantitatively nevertheless, we introduce the supposition that, for the highest values of Re which were applied in the model apparatus, the maximum value of p_s/τ_2 is the same as the maximum value of p_s/τ_0 for flow through channels and through tubes at large values of Re which was mentioned in No. 7. Hence we assume that $p_s/\tau_2 = 4$ at the highest values of Re applied; then our measurements give, with a mean value of 10^{-2} for $C_1 : 2\gamma/a_{95}p_s \approx 6$. This result seems to be reasonable. A similar reasoning for case a shows that also in this respect case a is very improbable.

The investigations in the circuit contain a number of measurements too small to allow a similar analysis. We can compare these measurements with those in the model arrangement, however, on the quantitative basis. The region of the circuit where bursting is most intense will be the boundary layer region in the curved tubes (see No. 2). For these measurements table B gives $2\gamma/a_{95}\overline{\tau_{ok}} = C_k$, where $\overline{\tau_{ok}}$ is the mean shearing stress at the wall in the curved tubes. If we accept a value of 4 for $p_s/\overline{\tau_{ok}}$, then in the stationary emulsions we find for $2\gamma/a_{95}\,p_s$ values ranging from 6,7 to 3,4 in the rapidly coalescing emulsions of the series S and T, and values ranging from 9,5 to 5,5 in the scarcely coalescing emulsion of series U, while C_k always decreases with a_{95} in one series. Consequently there is agreement in the measurements made in both apparatus to some extend.

From the consistency in the results of this analysis we conclude that it is highly probable indeed that disturbances of the turbulent fields in our apparatus as caused by the pump in the circuit and by the slits in the model had a negligible influence on the emulsion formation, as we already assumed above.

The greatest handicap for more accurate investigations is a lack of knowledge about turbulence in general and about the turbulence in our apparatus in particular. The influence of a_{95} on C_1 and that on C_k generally seem to be in opposite directions, but the regions of a_{95} investigated are different in the two arrangements.

We draw attention to the fact that our measurements include a very large region in Re. Therefore it seems probable that the way of bursting which we found sets in as soon as turbulence occurs. This conclusion can be checked with the aid of the values of $2\gamma/a_{95}p_s$ found and the data given in No. 7 and No. 9.

We remark that in our case b the regions where $p \lessapprox p_1$ always are smaller than a_{95}. With increasing Re in a given emulsion a_{95} will decrease more rapidly than the size of these regions will do, however, and in this way the pressure gradients may begin to play a part and cause lower values of C_1 and of C_k. A calculation, which we shall not give in detail, shows that the relative influence of this phenomenon will increase with increasing values of $\eta U_m \sqrt{Re}/\gamma$. Here U_m is a characteristic mean velocity.

11. *The genesis of the size distributions. Final observations.*

Our picture of the emulsion formation leads to the conclusion that, if suddenly a lower Re is established, the bursting process for a short period will practically disappear. By comparing the size distribution of the emulsions of series O and Q in the model (see Fig. 11) and of the emulsion of series T in the circuit (see Fig. 12, curves 3 and 4), we can separately observe the influence of coalescence in these emulsions. In Fig. 12 this reminds us of the usual coagulation in an emulsion or colloidal solution and points again to the process of coalescence by cohesion. In the model arrangement sometimes approximately the same behaviour is observed. In Fig. 11 and in the emulsions of the series J and K we get a different impression, which seems to point to the process of direct coalescence. The accuracy of the measurements still is too low to be sure about this point, however. Always the new a_{95} is established slowly, but often a few large droplets make their appearance rather soon.

During a temporary excess of bursting the coalescence remains active as well; the new equilibrium value of a_{95} is always reached in some minutes as can be seen from the tables. An excess of bursting does not generally cause an excess in the number of the small droplets as one might expect.

Of course the new state of dispersion is established the sooner as there is more coalescence.

Our investigations do not go so far as to give a general explanation of the size distributions existing in the emulsions. They seem to be caused chiefly by an interaction between the smallest and the largest droplets in the emulsion, except in the case of direct coalescence.

In general we found that the homogeneity of our stationary emulsions — which means the occurrence of a relatively low number of smaller droplets — was promoted by:

a. a higher volume concentration of the dispersed phase (see e.g. Fig. 12, series U and V),

b. a higher REYNOLDS' number (see Figs. 10, 11 and 12).

In this connection we remark that in a stationary emulsion a more rapid coalescence causes a more frequent bursting of the droplets as well, and consequently does not promote homogeneity in general.

As for the technical applications mentioned in the introduction, we remark that the intensity of the interactions between the droplets in an emulsion may be as important as the existing size distribution.

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Acoustics. — *The residue and the mechanism of hearing*. By J. F. SCHOUTEN. (Natuurkundig Laboratorium der N.V. Philips' Gloeilampenfabrieken, Eindhoven-Holland.) (Communicated by Prof. G. HOLST.)

(Communicated at the meeting of September 28, 1940.)

§ 1. *Introduction.*

Experiments on subjective sound analysis revealed ¹) that the ear but partly analyses a complex sound into pure tones corresponding to the individual frequencies in the Fourier spectrum of that sound. OHM's law of subjective sound analysis applies to the lower harmonics only. In a periodic sound the higher harmonics cannot be perceived separately as pure tones but are perceived collectively ²) as *one* component of sharp tone quality: the *residue*. The pitch of the residue is equal to that of the fundamental tone.

In terms of this residue some paradoxical phenomena, in particular those related with the problem of the missing fundamental, found a ready explanation.

We shall now turn to the theoretical aspects of the radical change in OHM's law necessitated by the existence of the residue. The validity both of OHM's law and of HELMHOLTZ' rule ³) always has been held as a supreme test of the *place theory* of hearing. This theory is based upon the assumption that different frequencies excite different restricted areas of the basilar membrane. Such mechanism, indeed, might enable the ear to perform a Fourier analysis of the sound (OHM's law), without its being able to take note of the phase relations between the various sinusoidal components (HELMHOLTZ' rule).

Now what are the consequences of the existence of the residue as to our conceptions of the mechanism of the inner ear? For many sounds, especially those rich in harmonics, one is almost nearer the truth in saying that the ear performs no Fourier analysis at all, then in saying that it does, the residue being by far the most prominent component and only a small portion of the harmonics being separately perceptible.

Is the existence of the residue in conflict with the principles of a *place theory* at all, or with its customary interpretation only? We hope to show that, as to the *analysing* mechanism, there is no conflict, once the limited

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¹) J. F. SCHOUTEN, The perception of subjective tones, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 1086—1093 (1938).

²) J. F. SCHOUTEN, The residue, a new component in subjective sound analysis Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 356—365 (1940).

³) H. v. HELMHOLTZ, Lehre von den Tonempfindungen.

resolving power of that mechanism is taken rigorously into account. The actual existence of the residue may then be explained by a particular assumption as regards the functioning of the *transmitting* mechanism.

§ 2. *The Perception of Pitch.*

The place theory proper is a theory regarding the analysing mechanism only. An additional assumption regarding the transmitting mechanism is necessary to account for the capacity of the ear to distinguish different frequencies as tones different in pitch. There are two possibilities. Either it is performed by obtaining knowledge of the *place* of the stimulated receptors on the basilar membrane or by the different *response* of those receptors. In the first case the nerve fibers need only transmit a quantitative message determining the measure of excitation. Whatsoever the stimulus by which a receptor is excited, the pitch ascribed to it will always be that of that particular receptor and fiber. In the second case the nerve fibers also have to transmit a qualitative message, depending upon the quality of excitation of the particular receptor, which will determine the ultimate pitch perception. In that case the perception of a particular pitch is not restricted to excitation of a particular receptor and nerve fiber.

The first case may well be compared with a totally colourblind eye looking into a spectroscope. Different frequencies will be seen as lines different in place but alike in colour. The frequencies can be told from the *place* of each line. The second case corresponds to a normal eye looking into a spectroscope. This eye is able to tell the frequency of the lines, irrespective of their particular place, by virtue of their *colour*.

The first theory (the place-place theory) is the one most generally held, it is often tacitly considered as a necessary extension of the place theory proper. A difficulty arose from the fact that the resolving power of the inner ear is comparatively poor. The excitation curves are of the order of a full tone wide, while for very low frequencies probably even no local excitation exists at all. This led GRAY [4]) to pronounce the "principle of maximal stimulation". He assumed that, although one frequency excites a certain region of the basilar membrane, the point of maximal stimulation will be the point determining the pitch. There are serious objections to be made against that principle, but it affords a very elegant explanation of the phenomena of pitch-shift of weak tones under influence of strong tones of neighbouring frequency.

Returning now to our problem of the residue we find ourselves confronted with the extremely important experimental fact that in a complex sound *two* components may be present which have almost the same pitch, although the excited regions on the basilar membrane are far apart. All harmonics, from the tenth upwards, thus ranging in our case from about 2000 to 4000 cycles/sec. are heard together as one component of pitch 200.

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[4]) A. A. GRAY, Journ. of Anat. and Physiol., **34**, 324 (1900).

This proves, assuming the place theory proper to be correct, that excitation of a given area of the basilar membrane may give rise to perceptions of widely different pitch.

Thus, returning to the analogy with the eye, in which the basilar membrane is compared with the retina, we find that the ear should be considered as an eye equipped with colour sense looking into a spectroscope and deriving the wavelength of the lines *not* from their place in the spectrum but from their colour. Just the same as the eye, in the case of a grating spectroscope, will recognize a red line in a region of otherwise blue lines, so the ear will recognize a component of low pitch in a region which, for pure tones, would give rise to perceptions of very much higher pitch.

The distinctness of pitch, one of the outstanding problems in the first extension of the place theory proper, is, in the second extension, not determined by the narrowness of the stimulated area or by the distinctness of a maximum of stimulation but by the *equality of "colour" of the whole stimulated area*, be it wide or narrow.

Hence even those low tones to which the whole basilar membrane responds uniformly may have a definite pitch by virtue of the "colour" of excitation. The function of the analysing mechanism thus consists merely in dissecting a sound into different components on the basilar membrane which, then, may be separately perceived. Pitch, however, is determined not by the place but by the quality ("colour") of excitation. It is not excluded, and even probable, that the place of excitation (number of receptor or nerve fiber) has some other effect on the ultimate tone quality e.g. a variable amount of sharpness. Thus the pitch of the residue would be determined by its colour, its sharpness by the region of excitation.

§ 3. *The Mechanism of the Residue.*

We now turn to the question whether, on the basis of the place theory proper, the analysing mechanism reveals any property which might give rise to a collective perception of higher harmonics with a pitch equal to that of the fundamental tone.

This property is present indeed. Any pure tone will excite a certain area of the basilar membrane expressed by its excitation curve or, to put it the other way round, any particular receptor will respond to a degree expressed by its response curve [5]), to a certain band of frequencies. As yet, the

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[5]) The different wording "response curve" and "excitation curve" is not immaterial. As *response curve* we define the response of one receptor to different frequencies. As *excitation curve* we define the excitation of different receptors by one frequency. Thus if the receptors were to respond uniformly up to various frequency limits, the response curves would extend *below*, the excitation curves *above* those limits. In our case, where narrow curves and a continuous set of receptors are considered, the difference is often negligible.

width of this band is unknown, since different methods lead to estimations of widely different order. We may, however, assume that the half-value width is at least half a tone.

The lower harmonics, being an octave, a fifth, a fourth etc. apart, will chiefly excite narrow bands of receptors divided by regions in which the response is comparatively small. Conversely the receptors in those regions will respond almost exclusively to particular harmonics. For higher harmonics, however, which, in relative measure, come closer together, the receptors will be excited in comparable measure by several harmonics at the time. Hence, those receptors will not perform a sinusoïdal oscillation, but will respond with a complex wave form *revealing the periodicity of the total wave form*. The overlapping of the excitation curves of various harmonics necessarily leads to the introduction of the fundamental *period* in the response of the receptors.

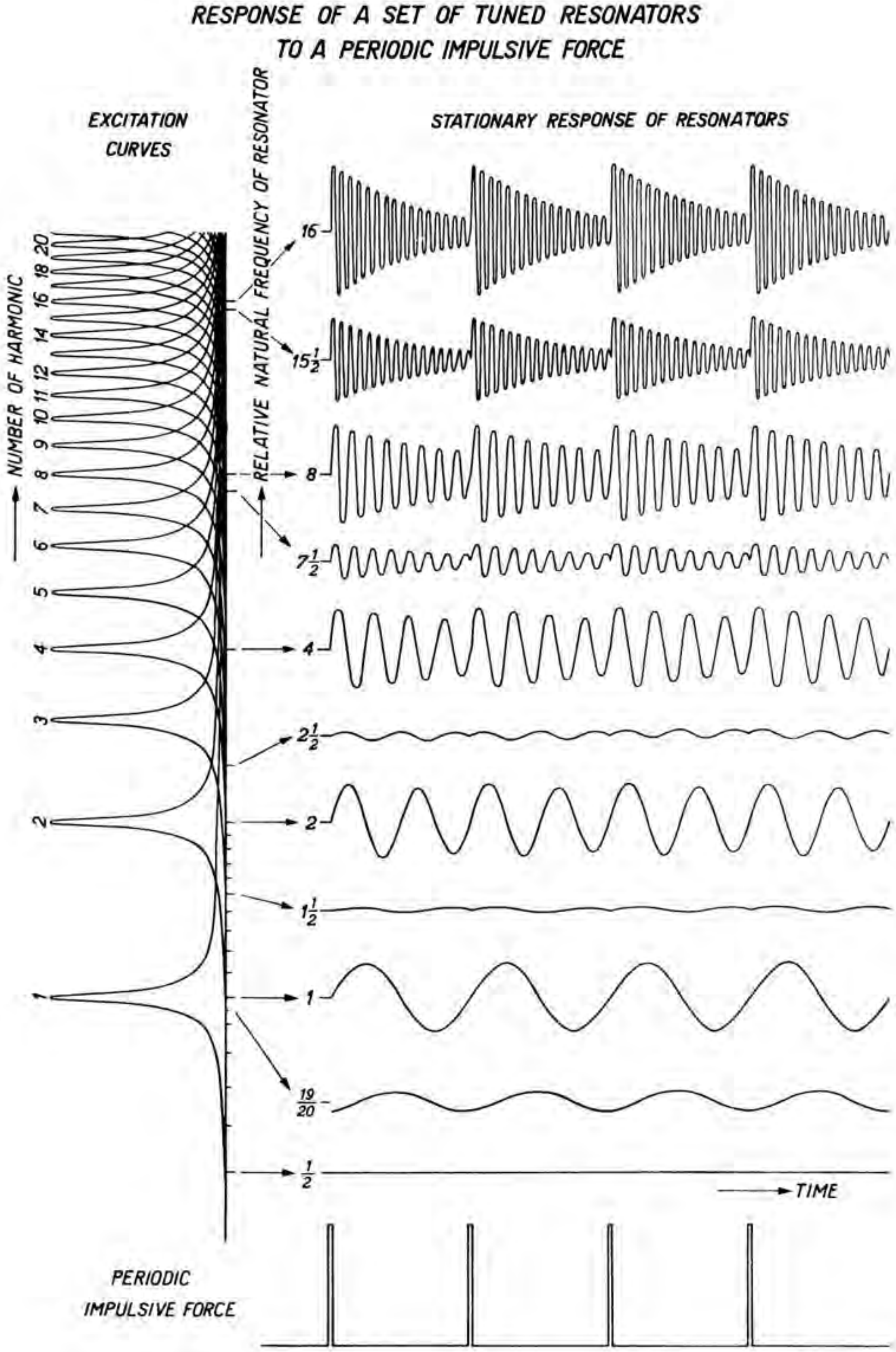
If thus the ear were to perceive pitch by virtue of the periodicity of the excitation of the receptors, the existence of the residue would be explained.

It should be remembered that the only reason why it was originally assumed that pitch is determined by the place of the excited receptor is for the sake of simplicity, so as not to make a complicate assumption if a simpler one serves as well. The simpler assumption now being proved untenable, it is necessary to make further assumptions as to the nature of the transmitting mechanism. In the "colour" theory of pitch perception it is assumed that from each receptor not only a quantitative but also a qualitative message is transmitted towards higher centres.

This may take place in two ways which we shall call the one-fiber and the multiple-fiber principle. In the one-fiber principle the clue as to the frequency has to be transmitted by the quality of discharge of the nerve fiber (telephone-theories), e.g. by the number of discharges per second. Another possibility, however, is given by the multiple-fiber principle which is generally assumed to account for the colour sensitivity of the eye. It is supposed, there, that the clue as to the colour is transmitted by the relative amount in which several fibers are excited by the same receptor. Two fibers would thus account for dichromatic vision, three fibers or two pairs of fibers for trichromatic vision.

We mention these possibilities merely to show that the "telephone" principle is by no means the only principle accounting for a colour theory of pitch perception and that, in view of our knowledge of the eye, the multiple-fiber principle is well worth considering too.

In fig. 1 an illustration is given of the response of a set of mechanical resonators to a driving force given by a "periodic impulse". Although the spatial analysis in the inner ear may very well be accounted for by other principles than that of mechanical resonance, the mechanical model, yet, in its essential features, displays the properties to be expected from all systems having a limited resolving power.


RESPONSE OF A SET OF TUNED RESONATORS
TO A PERIODIC IMPULSIVE FORCE
EXCITATION
CURVES
RELATIVE NATURAL FREQUENCY OF RESONATOR
STATIONARY RESPONSE OF RESONATORS
NUMBER OF HARMONIC
1
2
3
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18
20
16
15½
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7½
4
2½
2
1½
1
19/20
1/2
TIME
PERIODIC
IMPULSIVE FORCE

Fig. 1.

Graph showing the response of a set of tuned mechanical resonators to a periodic impulsive force.

The graph gives the amplitude x of the system:

$$m\ddot{x} + r\dot{x} + sx = F(t)$$

in which the constants are so adjusted that the sensitivity of the resonators and the relative width of the excitation curves are independent of the natural frequency. This is performed by taking s constant, $r \sim \frac{1}{\nu}$ and $m \sim \frac{1}{\nu^2}$ if ν is the natural frequency of the resonator $\left(\frac{r}{2m} = \frac{\nu}{10}\right)$.

The graph shows that in the low frequency region the resonators almost exclusively respond to one harmonic (1, 2, 3 etc.). The regions of very strong excitation (1, 2, 3 etc.) are separated by regions (½, 1½, 2½) in which the excitation is very small. In the high frequency region the response of the resonators increasingly shows a periodicity equal to that of the fundamental tone. Spatial analysis becomes poorer and poorer (compare 7½ with 8, 15½ with 16).

§ 4. *On beats.*

It is curious to consider that our present interpretation of the residue amounts to the supposition that this residue is *the perception as a separate component of the beats of the higher harmonics.*

That brings us back to a controversy even older than that between SEEBECK and OHM. In 1800 THOMAS YOUNG 6) supposed that the beats of two pure tones might be heard as a separate tone: "The greater the "difference in the pitch of two sounds, the more rapid the beats, till at last, "like the distinct puffs of air in the experiments already related, they "communicate the idea of a continued sound; and this is the fundamental "harmonic described by TARTINI."

This interpretation was strongly contested by HELMHOLTZ and was since abandoned, at least in physical and technical circles, the argument being, that beats have no physical existence as a separate tone and hence will never excite the corresponding receptor. If, none the less, a tone of corresponding pitch is heard, it *must* be due (according to the old extension of the place theory proper) to the presence of a pure tone of that frequency and, hence, to non-linear distortion.

Thorough studies of the properties of beats and beat-notes were carried out by KÖNIG, LORD KELVIN and TER KUILE 7). HERMANN 8) went in detail

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6) THOMAS YOUNG, Outline of Experiments and Inquiries respecting Sound and Light. Phil. Trans. Roy. Soc. London, I. 106—150 (1800).

7) R. KÖNIG, Ueber den Zusammenhang zweier Töne, Pogg. Ann. Physik, **157**, 181—237 (1876).

SIR W. THOMSON, On Beats of Imperfect Harmonics, Proc. Roy. Soc., **9**, 602—612 (1878).

TH. E. TER KUILE, Over phasen, zwevingen en klankaard. Ac. Thesis Amsterdam (1904).

8) L. HERMANN, Zur Theorie der Combinationstöne, Pfl. Arch. f. Physiol., **49**, 499—518 (1891).

into the theoretical consequences of these phenomena. The essence of these studies is the following. Slow beats will be heard not only if the two frequencies are almost identical but also on approximation to each of the harmonies 1 : 2, 1 : 3, 2 : 3, 3 : 4, 3 : 5, 4 : 5 etc. This was confirmed lately by LOTTERMOSER [9]). If the frequency of the beats increases the beats will change into beat-notes and thus into additional components of definite pitch. HELMHOLTZ' view, consisting in the supposition that the beats should be considered as beats between harmonics of almost common frequency and that the beat-notes should be considered as difference tones, is contested. According to KÖNIG (l.c. p. 235): "the origin of the beat-notes "is simply the periodic coincidence of similar maxima of the two wave "forms". He does not consider this to be in conflict with the place theory, although his arguments are not quite clear. To HERMANN we owe a very thorough investigation of the applicability of non-linear distortion to the explanation of beats and beat-notes. He finally, almost reluctantly, finds himself compelled (l.c. p. 517) "to drop HELMHOLTZ' hypothesis of "resonators in the ear, elegant as it might be" and (p. 514) "to return "to the old, so simple and natural, derivation of difference tones from beats, "viz. to ascribe to the ear the property to respond to any periodicity, within "certain limits of frequency, with a sensation of tone".

Once it is realized that HELMHOLTZ' theory consists of *two* assumptions: the place theory of the analysing mechanism (place theory proper) and the place theory of the transmitting mechanism, it is seen that HERMANN's criticism applies to the latter theory only. If the latter is replaced by the periodicity theory, the place theory proper even gains in probability since beat-notes, of which the residue is the most striking example, are most pronounced in those sounds where, according to the place theory proper, the different excitation curves show the highest grade of overlapping.

§ 5. *Consequences.*

An interesting consequence may be drawn with regard to persons having so-called "tonal lacunae" or "islands of deafness". If a tone to which such person is normally deaf is made loud enough the unsevered regions above and below the deaf region will respond to it. According to our theory such a person should still correctly perceive the pitch of the tone, whereas, according to the principle of maximal stimulation, he should perceive two pitches. According to TROLAND [10]) such persons indeed perceive the pitch correctly.

A paradoxical consequence may be drawn with regard to *high tone deafness.* In a periodic sound all high components are perceived together

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[9]) W. LOTTERMOSER, Bemerkung zu den subjektiven harmonischen Teiltönen. Ak. Zeitschr., 2, 148—149 (1937).

[10]) L. T. TROLAND, The Psychophysiology of Auditory Qualities and Attributes. J. Gen. Psych., 2, 28—58 (1929). In this paper a very clear distinction is made between theories of the analysing and the transmitting mechanism.

as a residue. Hence *high* tone deafness should chiefly affect one of the *lowest* components (the residue) of a complex sound.

A third consequency must be drawn with regard to the lower frequency limit of sound perception. One must distinguish between a pure tone of low pitch and a residue of that pitch. The loudness of the former will be determined by the sensitivity of the receptors to that frequency but not the loudness of the latter. Many contradictory results as regards this lower limit may be due to a residue having been listened to instead of a pure tone.

As a fourth consequence we must realize that the determination of periodicity is by no means always unambiguous. In those cases the pitch should show the same ambiguity.

§ 6. *Summary.*

The inner ear mechanism can be divided into the analysing mechanism within the cochlea and the transmitting mechanism from the cochlea towards higher centres.

The analysing mechanism consists of a set of receptors. Any one frequency will excite a number of receptors to a various degree expressed by the excitation curve. This leads to a spatial analysis on the basilar membrance of a complex sound into its sinusoïdal components. The analysis is limited by the width of the excitation curves. If the distance of two or more sinusoïdal components is small compared to the width of the curves, these will materially overlap and hence the receptors in that region will respond to several frequencies at the time.

The transmitting mechanism consists of the nerve fibers. The nerve fibers transmit both the quantity and the quality (periodicity) of excitation of the receptors. The former will determine loudness, the latter pitch, similar to the perception of brightness and colour in the eye. The locus of stimulation (determining pitch in the place-place theory) will most probably affect the tone quality (sharpness).

To account for the distinctness of the pitch of a pure tone the "principle of maximal stimulation" is no longer necessary, since this distinctness will, independent of the width of the stimulated area, be determined by the distinctness and the equality of the periodicity of the response of the receptors. Both will be ideal in the case of pure tones.

For complex sounds the number of pure tones perceptible in it will be determined by the efficiency of the spatial analysis and, hence, by the resolving power of the analysing mechanism. As soon as the excitation curves overlap beats or even (with increasing frequency) beat-notes (residues) will arise at the cost of the constituting pure tones. The pitch of those components is determined by the *periodicity* of the response of the receptors. The most striking example is found with the high harmonics of a periodic sound. All receptors in that region will respond to several harmonics at the time, thus revealing in their response a periodicity equal to that of the fundamental tone. Hence, those high harmonics, which are not separately perceptible, will be heard together as a separate component with a pitch equal to that of the fundamental tone. This component, the "residue", is distinguishable from the fundamental tone by its sharpness.

It thus remains right to say that the ear performs a Fourier analysis of the sound, but for the imperfections due to the limited resolving power of the analysing mechanism. At the same time it is also right to say, with the old investigators, that any periodicity may be heard as a separate note, provided that, as mostly will be the case, the analysing mechanism at some region responds with that periodicity.

For determining the mere components of a complex sound it is adequate to look at the *Fourier spectrum* of the sound. For determining the pitch of each separate component it is adequate to look at the *wave form* of the response of the receptors. That is the answer to the final query in our first paper (l.c. p. 1092) and the reconciliation of the "spectrum" and "periodicity" theories of hearing.

Eindhoven, July 1940.

Astrophysics. — *The Stark-Effect of the Heliumlines 4471.6 and 4470.2 in Spectra of Class B. Stars.* By S. VERWEIJ. (Communicated by Prof. A. PANNEKOEK.)

(Communicated at the meeting of September 28, 1940.)

1. In 1929 (Astrophys. J. 69, 70) STRUVE and ELVEY investigated in photographs of the spectra of *B*-type stars the differences in the helium lines between giants and the denser dwarfs. In dwarfs as contrasted with giants some *He* lines are broadened and show a very small displacement, and moreover forbidden helium lines appear. Especially the behaviour of λ 4471.6 appeared to be a good indicator for the dwarf character. In dwarfs this line is broadened and shows besides to the red side a component λ 4469.8. STRUVE identified this component with the strongest forbidden *He* line ($2p$—$4f$) which makes its appearance in the presence of electric fields.

Since then the profile of *He* 4472—4470 has been measured more exactly. In Observatory 61, 1938, p. 53 STRUVE pointed out that the ordinary Stark effect shifts the $2p$—$4d$ line (λ 4471.6) towards the red so that the stellar broadening should be one-sided. The forbidden line ($2p$—$4f$) is shifted only towards the violet. Consequently the space between the two lines should approach the background of the spectrum, whereas in reality a considerable darkening is observed there; the maximum observed intensity between these two lines is about 0.90. STRUVE tried to find the explanation in a broadening by collisional damping. He concludes: "The rough computations given in this note are, of course, not sufficient for a complete theory of the Stark-effect in *He* I. But they show that such a theory is required".

In *MN* 99 of the R.A.S., 1939, p. 150 FOSTER and DOUGLAS try to demonstrate in a rather cursory way that STRUVE's interpretation must be incorrect. They find that *He*, $2p$—$4f$, is strongly displaced to violet by electric fields. At its zero field position the f-line has intensity zero and at its 10 kv/cm position an intensity only one-eighth of the d-line for the same field. A wide weak absorption from λ 4470 to λ 4466 can thus be explained, but not the relatively strong line at 4471—4469 which mostly has been identified with it.

They suggest a strengthening of the d-absorption in the position of the f-line owing to resonance between $4d$ and $4f$ levels and a predominantly one-way energy cycle.

It is the purpose of the present paper to investigate the profile of *He* I by $2p$—$4d$ and $2p$—$4f$ absorption under various circumstances by means of an exact computation based as much as possible on constants theoretically known.

2. The theoretical determination of a line profile in the spectrum of a star with given effective temperature T_1 and surface gravity g proceeds in a number of consecutive steps. Firstly we want to know the variation of T and the electron pressure P with the depth in the stellar atmosphere, after which for a sufficient number of atmospheric layers in different depths the spreading of the diffusion coefficient s can be derived. Then at various values of the wave length λ we must determine the variation of s/k, the ratio of diffusion and absorption, with the depth in the atmosphere. Finally for each wave length in the realm of the line the energy radiated at the surface of the star can be computed.

The increase of T with the depth is determined by the relation $T^4 = \frac{1}{2} T_1{}^4 (1 + {}^3/_2\, \tau)$. The variation of P is given by $\triangle P = (dP/dp) \times \times\, g/k \times \triangle t$. Here τ and t are optical depths respectively determined by the mean absorption-coefficient $\bar{k}$ and the continuous absorption-coefficient k. The absorption-coefficients were taken from Publications Amsterdam N⁰. 4, Addendum.

The diffusion-coefficient is spread by Stark effect, Doppler effect and damping. In a constant homogeneous electric field a *He* line is split up by Stark effect in a number of components. STUART FOSTER 1) calculated with the aid of quantum mechanics for the Stark components of *He* $2p$—$4d$, and $2p$—$4f$ the theoretical displacements from the original d-line at 10, 20, 30, 40, 60, 80 and 100 kv/cm field strength. At 10, 40 and 100 kv/cm the intensities of these components were also calculated as fraction of the intensity of the diffuse line without external field.

At a point within the mol-electric field various fieldstrengths prevail each with an own probability. Every Stark component is spread out according to the probability function $W(\beta)$ of Amsterdam Publication 5, table 3. The probability $W(\lambda)$ that a component is displaced over a distance $\triangle\lambda$ is equal to $W(\beta)$ where the relation between $\triangle\lambda$ and β is given by $F = 46.8\, \beta\, (2P/T)^{2/3} = 46.8\, \beta\, \varrho^{2/3}$. Here F is the field strength which gives a displacement $\triangle\lambda$ to this component. By multiplying each ordinate, after making $\int W(\lambda) d\lambda = 1.00$, with the corresponding intensity I summation over all components gives a function $S(\lambda) = \Sigma I W(\lambda)$. To the σ components double weight was given. This function $S(\lambda)$ determines the intensity distribution in this helium line as it is spread by the Stark effect in a gas with a given density value $\varrho = 2P/T$.

In the computation of $S(\lambda)$ the displacements and intensities of the components were determined by graphical interpolation from the data given by STUART FOSTER. For $F < 2$ kv/cm the intensities of all $2p$—$4f$ components were taken zero. Table I and fig. 1 give the values of $S(\lambda)$

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1) Proc. R. S., A **117**, 137 (1928).

TABLE I. 1000 $S(\lambda)$ — Starkspreading. $\Delta\lambda$ is the distance from 4471 56 in angstrom units.

$\Delta\lambda$ / $\varrho \times 10^3$	0.001	.002	.003	.004	0.005	0.01	.02	.025	0.05	0.1	.15	.2	.25	.3	.4	.5	.7	1.0	1.2	1.5	2.0	2.5	3	4	5	7
2	403000	249000	84700	—	30800	6960	2420	1610	568	268	172	115	87	69	44	28	15	8	5							
5	103500	147400	119700	—	66200	16200	5040	3310	1166	517	324	203	159	125	78	54	29	15	10	6	4					
10	—	62400	72200	69000	64500	26800	9000	5660	1810	815	485	305	227	186	120	85	42	22	16	10.5	5.4	3.2	2.0			
20	—	—	23200	28400	30800	27300	14500	10170	3287	1315	761	477	353	270	180	118	57	30	22	14	6.9	4.8	2.8	1.4		
50						9410	10450	10060	5467	2647	1519	952	654	505	304	201	97	50	34	21	11	6.7	4.1	2.0	1.3	
100									4518	3258	2245	1497	1144	865	540	321	155	78	51	29	16	9.4	5.5	2.8	1.6	.71
150									2730	2608	2209	1715	1348	1061	693	448	207	105	67	37	20	12	7.2	3.4	1.8	.84
200									1715	1943	1852	1645	1421	1183	826	550	263	130	84	48	24	14	8.5	3.9	2.2	.98
300									848	1142	1252	1262	1224	1156	934	698	376	192	127	68	34	20	9.7	5.5	2.9	1.3
600										360	—	535	—	614	634	620	512	340	246	146	75	43	24	11	5.6	2.4

8.4	6.4	5.4	4.4	3.9	3.4	2.9	2.6	2.4	2.1	1.9	1.8	1.7	1.65	1.6	1.55	1.5	1.45	$\Delta\lambda$ / $\varrho \times 10^3$
									1.2	1.9	2.5	3.1	3.4	3.7	4	4.5	5	2
					0.6	0.9	1.3	1.7	2.9	4.1	5.7	7.8	8.8	9.8	11	12	15	5
				0.8	1.2	1.9	2.4	2.9	5.4	7.4	11	14	17	20	22	25	36	10
	.35	.55	0.9	1.2	1.9	2.8	4.0	5.0	8.6	13	19	25	34	39	48	47	44	20
.38	.76	1.1	2.0	2.7	3.9	6.3	9.0	11	20	31	51	73	89	101	90	54	20	50
.64	1.2	2.1	3.7	4.8	7.8	12	18	23	45	70	108	133	145	130	74	25	8.8	100
.88	1.9	2.9	5.4	7.5	11	19	28	37	72	106	149	153	140	106	43	14	3.0	150
1.2	2.5	4.2	7.4	9.7	15	25	40	51	93	131	161	141	116	78	34	8	1.8	200
1.6	3.5	6.1	11	15	24	41	62	83	130	149	143	102	74	44	20	3.7	0.7	300
3.4	7.2	12	23	33	53	86	112	121	124	96	68	38	—	13	–	1.0	—	600

for some values of ϱ. Here $\int S(\lambda)d\lambda = 1.00$. The very rapid ascent of the S-curve from the undisturbed position of the d- and f-line and the relatively small distances of the two maxima from these positions (λ 4471.56, λ 4470.16) are worth special attention.

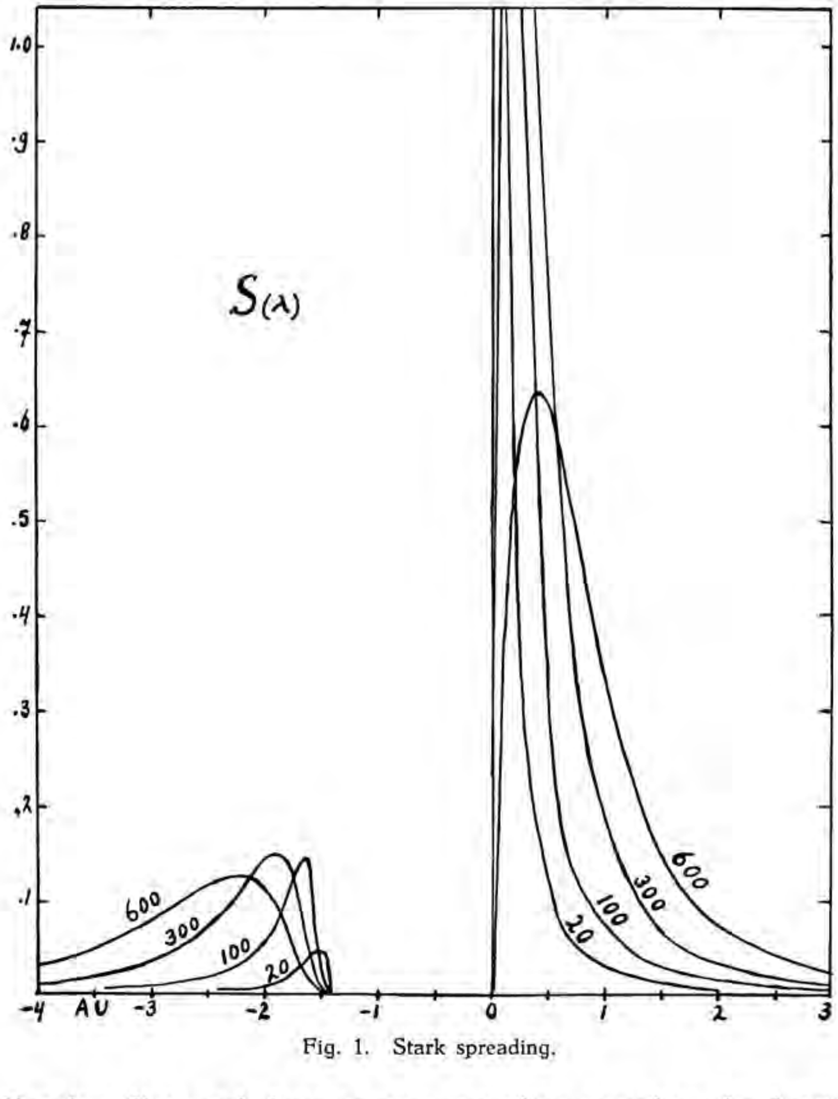

Fig. 1. Stark spreading.

Doppler effect and damping give spreading according the function:

$$\frac{s}{n}(v)=\frac{s}{n}(0)\times\frac{a}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-y^2}\,dy}{(a/2)^2+(v-y)^2}\qquad\frac{s}{n}(0)=\frac{\sqrt{\pi}\,.\,e^2}{m(H)\,.\,mc^2}\cdot\frac{\lambda_0^2}{\triangle\lambda(D)}$$

$$v=\frac{\lambda_0-\lambda}{\triangle\lambda(D)}\qquad a=\frac{\triangle\lambda(N)}{\triangle\lambda(D)}\qquad\triangle\lambda(N)=\frac{\lambda^2}{2\pi c}\cdot\gamma\qquad\triangle\lambda(D)=\frac{\lambda}{c}\sqrt{\frac{2RT}{\mu}}.$$

UNSÖLD [2]) gives the solution of this integral for the cases that $a \ll 1$. In the considered atmospheres the influence of damping is important, so a is mostly larger. Here the solutions for various values of a and v as given by MITCHELL and ZEMANSKY [3]) were used.

Firstly we must determine the value of the damping constant γ which consists of two parts: collisional damping γ_1, and radiation damping γ_2. The theory of collisional damping was first developed by UNSÖLD for stellar atmospheres. During the time that a charged particle passes a radiating atom the frequency of the latter is alterated by Stark effect and the total phase difference is $\int \triangle\omega\, dt$. A broadening collision is defined by the condition $\int \triangle\omega\, dt \geqslant 1$.

If the Stark effect is quadratic ($\triangle\lambda = c_1 \times F^2$) we have

$$\gamma_1 = 33.7\, C^{2/3} (RT)^{1/6} (\mu_1^{-1} + \mu_2^{-1})^{1/6} N \qquad C = 0.0615\, c_1\, \lambda^{-2}.$$

If the Stark effect is linear ($\triangle\lambda = c_2 \times F$) we have

$$\gamma_1 = 1534\, C^2 (RT)^{-1/2} (\mu_1^{-1} + \mu_2^{-1})^{-1/2} N \qquad C = 4.30 \times 10^8 \times c_2\, \lambda^{-2}$$

N is the number of charged particles per cm^3 = $^1/kT$ × partial pressure; μ_1 and μ_2 are the atomic weights of charged particle and radiating atom; λ is given in Angstrom units and F in kv/cm. So in case of quadratic effect $\gamma_1 \sim PT^{-5/6}$ and in case of linear effect $\gamma_1 \sim PT^{-3/2}$.

The Stark effect in *He* I 4471.6 is linear between 10 and 100 kv/cm, for weaker fields it seems to be quadratic. So neither of the two solutions can directly be used. By computing γ_1 for various values of c_1 and c_2 a sufficient accurate value of γ_1 can be approximated.

Fig. 2 where the abscis denotes the displacement $\triangle\lambda$ in Angstrom units and the ordinate the field strength in kv/cm gives an impression of the amount of collisional damping. The curves represent $\triangle\lambda = c_1 F^2$ or $\triangle\lambda = c_2 F$. Near these curves are given the amounts of γ_1 for a layer at the optical depth $t = 0.3$ in a stellar atmosphere with $T_1 = 20.000°$ and $\log\, g = 4.4$. In computing γ_1 the ion pressure was taken equal to the electron pressure and the atomic weight of the disturbing ion was taken one. The dashed curves give the real displacement $\triangle\lambda$ of the centre of all π and σ components with F. The classical dampingconstant γ_2 is

$$\frac{8\pi^2}{3} \cdot \frac{e^2}{me} \cdot \frac{1}{\lambda_0^2} = 1.11 \times 10^8.$$

Besides modern theory gives a correctionfactor which for $2p$—$4d$ is 0.33, as computed by GOLDBERG [4]). We see that the collisional dampingconstant

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[2]) UNSÖLD, Physik der Sternatmosphären, p. 163.

[3]) MITCHELL and ZEMANSKY, Resonance radiation and excited atoms. F. HJERTING gives this integral for values of a up to 1.0 (Aph. J. **88**, 508; 1938).

[4]) Astrophys. J. **90**, 423 (1939).

is many times larger than the classical one. Interpolation gives for $2p$—$4d$ γ_1 about 1700 times the classical γ_2; for $2p$—$4f$ this factor is about 1.8 times larger. In the depth $t = 0.01$ γ_1 is for $2p$—$4d$ still 350 times the classical γ_2.

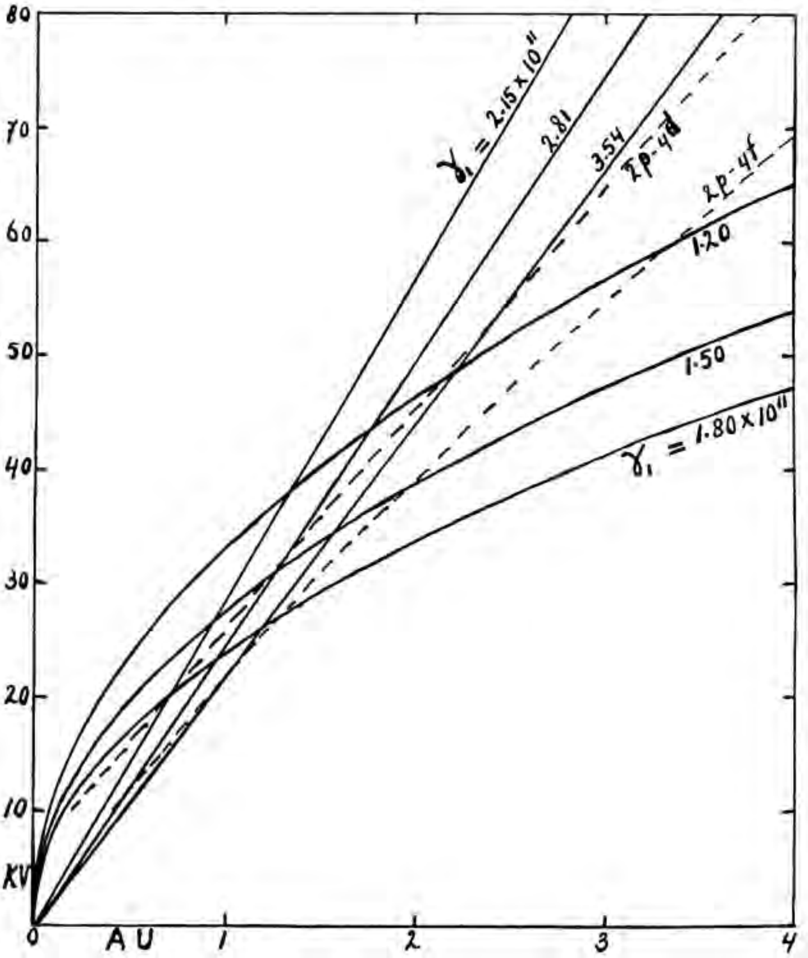

Fig. 2.

Table II and fig. 3 give for various values of ϱ the spreading of the diffusioncoefficient s/n by the combined action of Stark effect, Doppler effect and damping. For the derivation of this spreading the intensity in each point of the S-curve was spread according to the Doppler-damping curve. Finally the total absorption $\int s/n \; d\lambda$ was brought equal to $\frac{\pi e^2 \lambda^2}{m\, m(H) c^2} = 10.6 \times 10^{10}$. In this table is also given the temperature used for computing the Doppler-damping spreading. If this spreading is determined for a same value of $\varrho = 2P/T$ but with an other T the result will differ a little from the table value. As appears from the dependence of

TABLE II. $10^{-10} \times \frac{s}{n}$. Spreading by Starkeffect, Dopplereffect and damping.

$\triangle\lambda$ / $\varrho \times 10^3$	0.0	.01	.02	.025	.03	.04	.05	0.1	.15	.2	.3	.4	.5	.6	.7	1.0	1.2	1.5	2.0	2.5	3	4	5	T
2	46.0	46.1	45 5	—	43.8	—	40.4	26.6	—	5.26	1.09	—	.36	—	.17	.10	.06							16840
5	42.1	42.4	42.1	—	41.5	40.1	38.3	26.8	—	7.08	2.01	—	.71	—	.37	.19	.12	.07						16840
10	38 3	38.7	39.1	—	38.9	37.9	36.6	26 9	—	9.15	3.11	—	1.17	—	.57	.27	.18	.10						16840
20	31.7	—	—	33.7	—	—	33.1	27.5	—	11.9	4.80	—	1.66	—	.82	.38	.26	.14						16840
50	19.4						22.4	22.2	19.1	15.0	8.35	—	3.12	—	1.61	.78	.52	.31	.16					16960
100	11.6						14.0	15.2	15.1	14 0	10.4	—	5.05	—	2.68	1.30	.87	.51	.25	.12	.06			17560
150	8.30						9.79	10.8	11.1	11.1	9.71	—	5.89	—	3.39	1.73	1.18	.70	.37	.21	.12	.05		18350
200	6.30							8.10		8.88	8.55	—	6.25	—	3.98	2.11	1.49	.93	.47	.28	.16	.07		18950
300	4.04							4.95		5 56	5.93	5.82	5.39	—	4.22	2.70	2.00	1.29	.68	.42	.26	.14	.08	17000
600	2.15							2.39		2.61	2.81	2.93	3.01	3.05	3.01	2.67	2.36	1.89	1.25	.82	.51	.28	.17	17500

6	5	4	3.5	3	2.5	2.2	2.0	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.0	.9	.8	.7	.6	.5	.4	.3	.2	0.1	0.0	$\triangle\lambda$ / $\varrho \times 10^3$
						.01	.01	.02	.02	.03	.04	.04	.03	.01	.01	.01	.01	.01	.01	.01	.02	.03	.05	.23	3.30	23.0	46.0	2
					.02	.03	.04	.05	.06	.08	.11	.11	.06	.02	.01	.01	.01	.02	.02	.03	.04	.06	.11	.35	3.40	20.4	42.1	5
					.03	.05	.07	.09	.12	.16	.20	.21	.13	.05	.03	.03	.03	03	.04	.05	.07	.11	.24	.52	3.90	17.9	38.3	10
					.05	.08	.12	.16	.22	.29	.35	.33	.20	.08	.04	.04	.05	.06	.09	.12	.16	.23	.37	.72	3.47	14.3	31.7	20
				.07	.12	.21	.32	.40	.52	.62	.62	.49	.32	.21	.16	.15	.15	.17	.21	.29	.39	.51	.73	1.37	3 22	9.45	19.4	50
			.09	.14	.26	.42	.59	.70	.76	.79	.70	.57	.43	.34	.29	.28	.30	.33	.37	.46	.60	.80	1.12	1.78	3.22	6.46	11.6	100
		.09	.14	.23	.42	.61	.77	.83	.86	.84	.73	.61	.49	.42	.37	.36	.37	.42	.47	.59	.77	.99	1.32	2.01	3.19	5.27	8.30	150
		.13	.20	.32	.57	.78	.89	.93	.92	.86	.75	.64	.56	.50	.45	.45	.47	.51	.59	.70	.83	1.06	1.38	1.95	2.86	4.36	6.30	200
	.14	.25	.35	.51	.72	.81	.83	.82	.79	.76	.72	.67	.64	.62	.61	.62	.64	.68	.75	.85	.99	1.19	1.43	1.82	2.35	3.13	4 04	300
.18	.28	.43	.53	.62	.67	.67	.67	.66	.66	.66	.66	.67	.68	.70	.73	.76	.80	.85	.91	.99	1.10	1.21	1.35	1.52	1.72	1.93	2.15	600

$\triangle\lambda(N)$ and $\triangle\lambda(D)$ on T and P this difference will be very small in the temperature range of this work. For this reason this table II was also used for other temperatures, so much work was saved without injuring the final result.

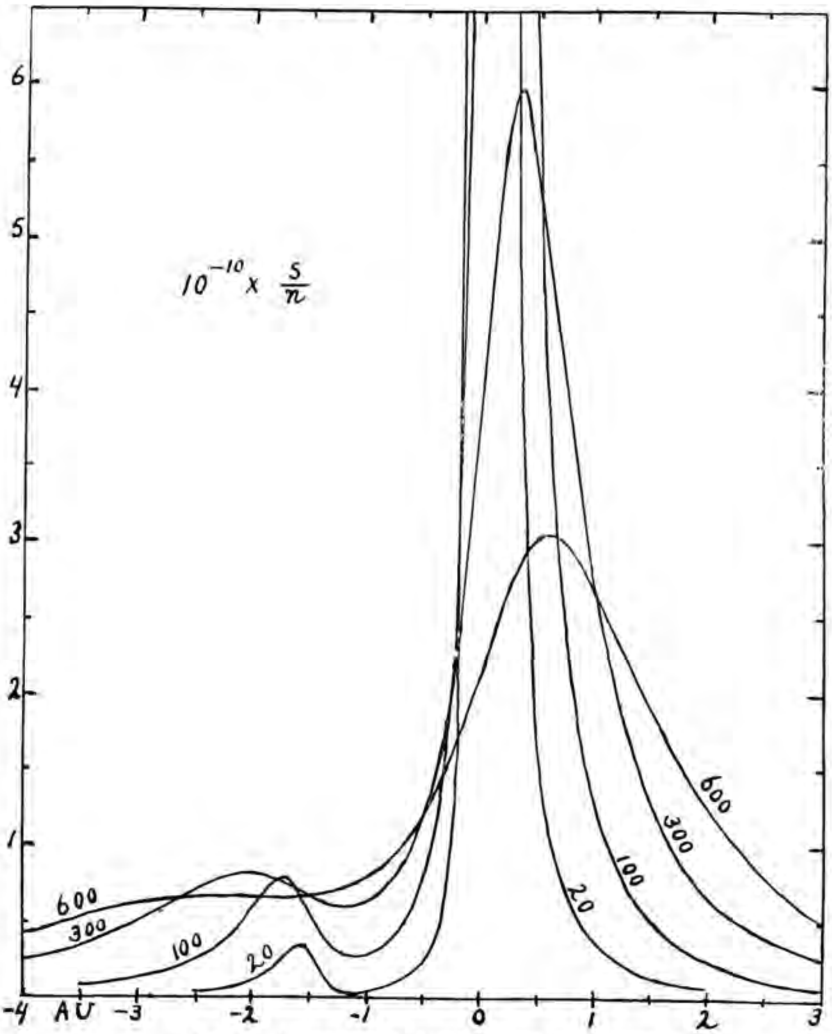

Fig. 3. Spreading of the diffusion coefficient.

The concentration n of the He atoms producing the line is equal to $f \times \varepsilon \times ij \times x_0$. Of the helium present in an atmospheric layer a fraction x_0 is neutral, which can be computed with.

$$\log\ n_1/n_0 = -\log P - [5.09]/T + 2.5 \log T + 0.12.$$

Of the neutral atoms a fraction ij given by $\log ij = -[5.02]/T + 0.95$ is in the $2p$ level. The fraction f of these atoms produces the considered $2p$—$4d$ line. ε is the ratio of the amount of helium to hydrogen in a stellar atmosphere. We must take $f \times \varepsilon = 0.0005$ for getting theoretical profiles of the same strength as the observed ones. This means, as the theoretical value [5]) of $f = 0.129$, that $\varepsilon = 1 : 250$.

For a given stellar atmosphere the ratio $s/k = n/k \times s/n$ now can be derived for some wave length $\triangle\lambda$ outside the centre of the line for various optical depths. Then the variation of s/k with the optical depth t for this $\triangle\lambda$ is known. The residual intensity $r = H_0/H_0\ (s = 0)$ at this $\triangle\lambda$ can now be found by determining the value H_0 of the radiated energy at the surface of the star by solving numerically the differential equations

$$\frac{dJ}{dt} = 3\,(1 + s/k)\,H \qquad \frac{dH}{dt} = J - E_0\,(1 + Ct) \qquad C = {}^3/_2\,.\,{}^1/_4\,h\nu/kt\,.\,\bar{k}/k.$$

Table III gives the results. Here, as in the former tables, the last figure is not reliable. The profiles of the spectral lines determined by these values for various T_1 and *log g* are given in fig. 4.

3. As appears from fig. 4 the profiles computed according to the theory described above have in the main a good agreement with the observed profiles [6]). They show the strong line at λ 4471.6 and next to it the weaker line near λ 4470. The *p*-line is asymmetrical with a very small shift of the maximum to red. At variance with the interpretation of FOSTER and DOUGLAS the *f*-line is rather strong and is only a little shifted to violet. The distance of the two maxima is for normal stars with $log\ g = 2$ and 4 respectively about 1.5 and 1.7 Angstrom units. The space between the two lines shows an important absorption. However, this absorption for g not too large is smaller than measured in observed profiles. Here, then, seems to remain a discrepancy between observation and this theory, for which other causes have to be sought.

This theoretical determination of *He* profiles enables us to survey the variation of these profiles with temperature and gravity and to get a theoretical foundation for conclusions about these parameters from the profile of this *He* line.

With increasing temperature the intensity of the line increases from about 12000° till a flat maximum is reached at 16000—20000°. For still higher temperatures the intensity decreases slowly. The maximum is for small g reached at lower temperature than for larger g. Fig. 5 shows for $\log\ g = 4$ and 2 the variation with T_1 of the width of the red wing measured from the position of the maximum of the *d*-line to the point with 5 and 10 % absorption. The dashed curves give the variation of this width with the spectral class as determined by ELVEY [7]) from observations, making use of the temperature scale derived by KUIPER [8]).

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[5]) Astrophys. J. **90**, 428 (1939).

[6]) Observatory **61**, 54 (1938). —MN of R.A.S. **99**, 156 (1939).

[7]) Astrophys. J. **70**, 153 (1929).

[8]) Astrophys. J. **88**, 446 (1938).

TABLE III. 1000 r. Residual intensities.

$\Delta\lambda$ *log g.*	6	5	4	3.5	3	2.5	2.2	2.0	1.9	1.8	1.7	1.6	1.5	1.4	1.3	1.2	1.1	1.0	.9	.7	.5	.3	.2	0.1	0.0	0.05	0.1	0.2	.3	.5	.7	1.0	1.5	2.0	2.5	3	4	5
$T_1 = 12.000°$																																						
2																						983	887	755	677	691	723	838	918	960	981							
3																						990	950	867	823	818	828	881	936	972	987							
4																						989	981	957	923	919	914	931	950	980	987							
5																						—	992	986	983	—	980	980	980	984	989							
6																						—	—	—	—	—	—	—	—	—	—							
$T_1 = 14.000°$																																						
2							990	988	985	980	976	970	970	980	994	995	995	995	994	990	981	920	735	462	360	378	433	652	801	885	925	958	984					
3							988	982	976	970	959	955	959	974	988	992	992	992	990	984	973	936	830	641	518	523	574	707	805	893	926	957	981					
4							986	979	973	971	968	970	975	981	986	989	989	989	989	984	971	944	911	831	755	735	742	781	834	903	935	962	977					
5						990	988	987	986	986	987	988	989	991	992	992	992	992	992	988	983	977	969	952	937	—	921	920	926	942	953	969	986					
6						—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	992	991	989	988	—	987	987	986	986	986	987	988					
$T_1 = 16.000°$																																						
2							988	983	980	974	964	956	956	964	990	993	993	993	993	985	975	898	573	280	201	214	258	488	695	830	893	936	973					
3						986	978	971	962	953	939	928	933	958	978	986	986	984	982	972	954	886	650	390	280	286	326	499	669	805	873	929	969	982				
4				991	985	981	970	960	952	946	935	938	948	959	972	974	974	974	973	962	930	876	790	605	471	455	476	561	664	788	856	920	964	977				
5			989	984	980	976	967	965	963	963	964	964	970	973	978	978	978	978	976	968	959	930	902	853	770	—	738	760	788	837	890	930	959	978	985	990		
6			993	990	988	987	987	987	987	987	987	987	986	986	985	985	985	984	983	981	978	976	973	969	961	—	951	947	949	951	954	961	971	976	981	986		
$T_1 = 18.000°$																																						
2							988	982	977	970	960	953	953	972	990	993	993	993	991	985	973	887	564	278	202	214	256	465	683	824	879	935	973					
3						983	972	962	951	940	923	909	909	948	972	983	983	982	978	960	940	842	568	310	220	227	265	416	586	745	835	900	956	980				
4				987	977	963	945	918	903	888	872	880	896	926	950	959	959	959	955	930	886	799	636	425	280	272	298	407	516	685	767	857	924	956	980			
5		985	976	968	956	932	915	900	896	896	900	910	919	927	938	941	941	938	935	918	882	815	740	630	506	—	460	495	556	665	758	830	903	940	964	974	985	
6		982	974	969	963	960	959	959	959	959	959	960	960	960	958	956	954	952	950	946	935	923	910	872	836	—	805	794	801	827	847	873	902	932	949	967	980	
$T_1 = 20.000°$																																						
2							988	982	978	971	961	955	955	970	988	993	993	993	991	985	974	893	592	305	224	238	276	490	694	826	890	935	974					
3						981	971	960	952	938	918	902	906	942	971	984	984	984	978	961	940	851	585	331	239	245	282	432	586	739	834	897	956	980				
4				981	971	954	929	902	884	865	850	853	877	909	930	944	945	945	939	910	868	765	613	404	276	269	296	386	491	647	741	823	909	949	975			
5		977	960	948	930	895	874	853	846	845	845	856	874	893	901	906	906	902	895	869	829	739	650	510	418	—	373	403	457	570	656	752	845	903	937	956	977	
6	968	956	940	930	917	910	907	905	905	905	906	908	908	908	907	905	902	898	895	879	848	820	777	733	673	—	643	619	632	658	698	748	813	856	897	926	959	973
$T_1 = 25.000°$																																						
2							993	990	985	980	977	970	970	980	993	995	995	995	994	991	980	922	659	362	272	290	338	559	747	870	920	957	983					
3						987	980	974	965	952	940	933	938	960	982	990	990	990	984	976	952	883	650	393	289	295	336	493	642	790	870	921	967	980				
4				988	979	965	950	922	908	890	880	889	905	928	948	956	958	956	952	932	894	807	673	461	336	322	342	434	533	689	781	857	930	963	980			
5		980	968	953	934	905	870	866	860	860	864	878	893	903	910	918	918	918	909	886	840	757	686	566	456	—	419	436	486	590	687	769	855	913	944	964	980	
6	962	945	924	915	900	888	884	884	884	887	890	891	891	893	890	886	881	878	873	856	830	786	750	711	667	—	646	627	623	625	656	711	775	827	870	903	946	964

S. VERWEIJ: THE STARK-EFFECT OF THE HELIUMLINES 4471.6 AND 4470.2 IN SPECTRA OF CLASS B. STARS.

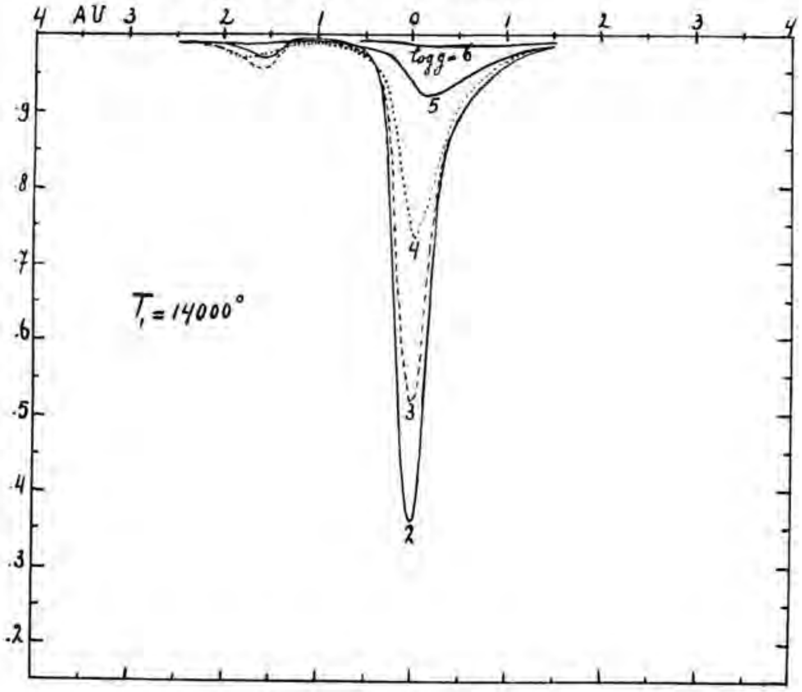

Fig. 4a.

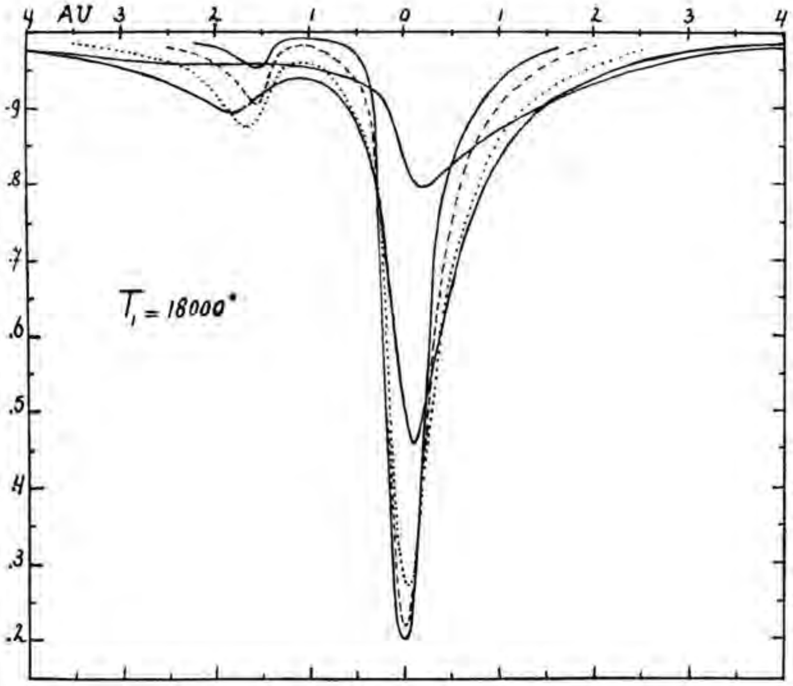

Fig. 4b.

Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

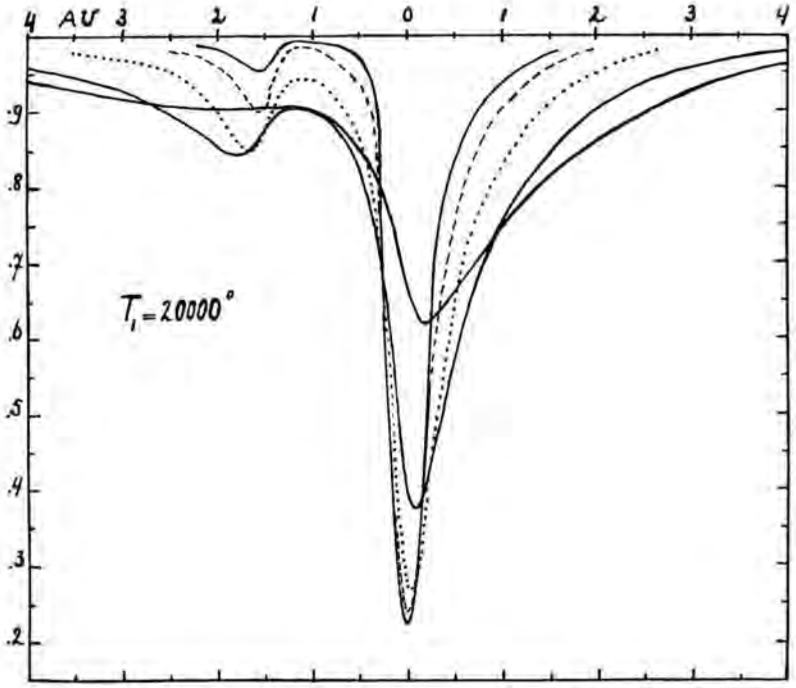

Fig. 4c.

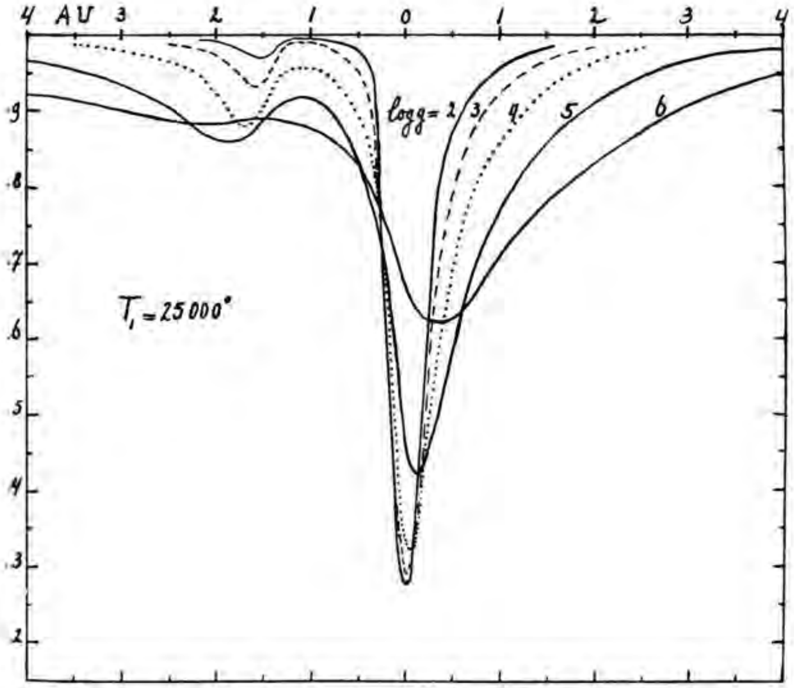

Fig. 4d.

Fig. 4. Line profiles.

The influence of the surface gravity appears in the displacement of the maximum of the *d*-line to red, in the width of the red wing and in the ratio *i* of the total *f* absorption to the total *d*-absorption.

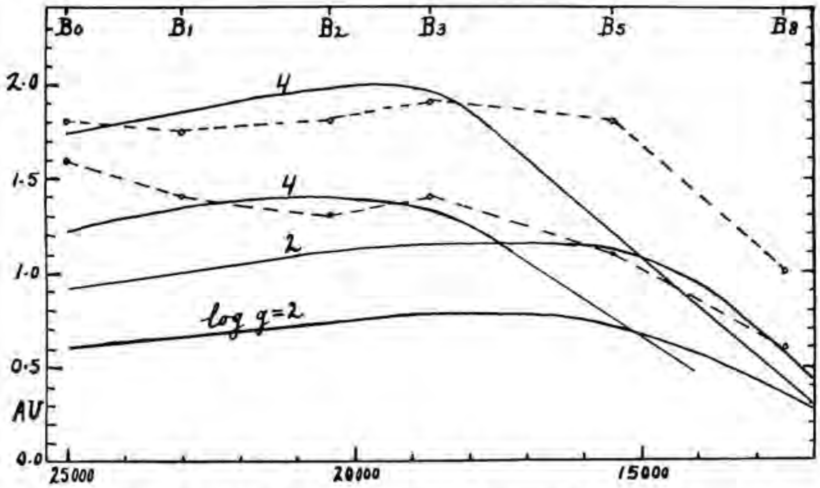

Fig. 5. Variation of width of red wing with effective temperature.

The displacement of the maximum of the *d*-line is small; observations have given the same result. In spite of the large displacement of the separate Stark components the shift of the core of the *d*-line to red is for $log\ g = 2$ and 4 only 0.01 and 0.03 A. This is caused by the rapid ascent of the *S*-curve and the small value of ϱ in the atmospheric layers where this core originates. For $log\ g = 5$ the shift is a little more than 0.1 A and for $log\ g = 6$ it is 0.2—0.3 A. But then the line becomes very flat and weak.

The width of the red wing increases with the gravity. Fig. 6 gives the variation of this width at 10 % absorption with *log g*. It is proportional

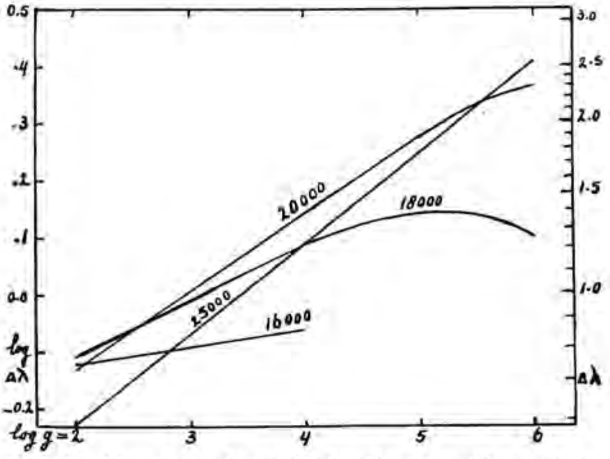

Fig. 6. Variation of width of red wing with gravity.

to a power of g with an exponent of 0.16, 0.13, 0.10 and 0.03 respectively for an effective temperature of 25000, 20000, 18000 and 16000°. In spectra of white dwarfs with $\log g > 6$ this line fails or will be weak and very much broadened. This corresponds with investigations of Kuiper in observed spectra of white dwarfs.

The value of i increases with the gravity. For the determination of i the overlapping wings of *d*- and *f*-line were separately drawn up to the continuous background, in accordance with former computations. From these two profiles the separate surfaces were determined. Fig. 7 gives i for various values of T_1 and $\log g$. The value of i varies more rapidly with g than the width of the red wing. So the ratio i is a very good indicator for the value of the surface gravity. Elvey 9) gives for three stars ζ *Cas*, ι *Her* (B_3) and γ *Peg* (B_2) as a mean $i = 0.29$ (uncertainty of the estimate is 30 %). Struve 10) gives for γ *Peg* (B_2) $i = 0.19$ and for 67 Ophiuchi ($B5p$) $i = 0.05$. The theoretical values of fig. 7 show a good agreement with i from observations.

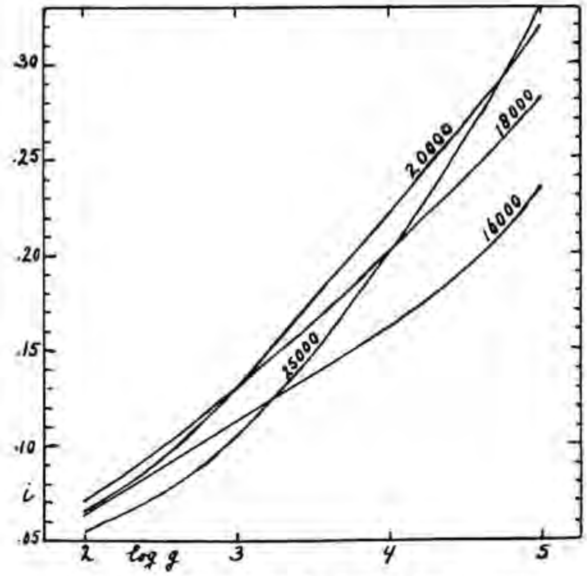

Fig. 7. Variation of the ratio i with gravity.

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9) Astrophys. J. **69**, 237 (1929).

10) Astrophys. J. **70**, 240 (1929).

Mathematics. — *Absolute Invarianten von fünf Geraden in R_4.* Von O. BOTTEMA. (Communicated by Prof. R. WEITZENBÖCK).

(Communicated at the meeting of September 28, 1940.)

1. In einer der ersten einer Reihe von Arbeiten zur projektiven Liniengeometrie im vierdimensionalen Raum hat R. WEITZENBÖCK für die Figur von fünf Geraden l_1, l_2, l_3, l_4, l_5 Invarianten der Gestalt $A_{i,jk,rs}$ angegeben. [1]) Eine solche Invariante, z.B. $A_{1,23,45}$ wird dabei so bestimmt, dasz ihr Verschwinden die Bedingung gibt, dasz die Schnittebene der Verbindungsräume S'_{23} (durch l_2 und l_3) und S'_{45} (durch l_4 und l_5) die Gerade l_1 trifft. Man hat also

$$A_{1,23,45} = \sum_{ik} \varphi_{ik} (S'_{23} S'_{45})_{ik}$$

wo φ_{ik} die Linienkoordinaten der Geraden l_1, $(S'_{23} S'_{45})_{ik}$ die Ebenenkoordinaten der genannten Schnittebene sind. Die Gleichung des Raumes S'_{23} lautet bekanntlich $\Sigma x_1 a_{23} \alpha_{45} = 0$ wo a_{ij} bzw. α_{ij} die Linienkoordinaten von l_2 bzw. l_3 sind. Die Invarianten A sind also Funktionen der Linienkoordinaten der fünf Geraden. Man kann in dieser Weise auch Invarianten für vier Geraden definieren, indem man z.B. l_5 mit l_2 zusammenfallen läszt.

WEITZENBÖCK hat den Satz bewiesen, dasz bei vier und bei fünf Geraden jede projektive Invariante sich durch Invarianten A ausdrücken läszt; überdies bilden diese Invarianten bei willkürlich vielen Geraden auch eine Rationalbasis für alle projektiven Invarianten.

2. Wir betrachten im Folgenden die Figur von fünf Geraden und die Invarianten $A_{i,jk,rs}$ der Eigenschaft, dasz *die Indizen i, j, k, r und s sämmtlich verschieden* sind. Die Invarianten sind dann linear in den Koordinaten jeder der fünf Geraden. Da überdies die Invarianten A je dasselbe Gewicht haben, ist das Verhältnis zweier dieser Invarianten offenbar *eine absolute Invariante* der Figur. (Vier Geraden haben keine absolute Invariante). Wir wollen eine *geometrische Deutung* dieser absoluten Invarianten herleiten. Es wird sich zeigen, dasz sie zum Teil gewöhnliche Punktdoppelverhältnisse sind.

3. Die drei Geraden l_1, l_2 und l_3 haben im Allgemeinen eine einzige Transversale m_{45}; die Gerade m_{45}, l_4 und l_5 haben eine Transversale n_{45}.

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[1]) R. WEITZENBÖCK, Die projektiven Invarianten von vier und fünf Geraden in R_4, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 245—252, (1939).

Die Gerade m_{45} wird also von den Geraden l_1, l_2, l_3 und n_{45} geschnitten; sind A, B, C, P die Schnittpunkte, so ist das Doppelverhältnis dieser vier Punkte eine absolute Invariante der Figur l_1, l_2, l_3, l_4, l_5. Wir zeigen dasz dieses Doppelverhältnis dem Quotient zweier Invarianten A gleich ist.

Wir geben der durch m_{45} und n_{45} bestimmten Ebene die Gleichungen $x_1 = x_2 = 0$ und den kollinearen Punkten A, B und C bzw. die Koordinaten (0 0 1 0 0), (0 0 0 1 0) und (0 0 1 1 0). Die Schnittpunkte von l_4 und l_5 mit der Geraden n_{45} seien D und E; ihre Koordinaten bzw. (0 0 0 0 1) und $(0\,0\,p\,q\,r)$; die Koordinaten des Punktes P sind dann $(0\,0\,p\,q\,0)$. Wir können für unser Ziel auf eine weitere Bestimmung des Koordinatensystems verzichten. A', B', C', D' und E' seien Punkte von bzw. l_1, l_2, l_3, l_4 und l_5, welche bzw. von A, B, C, D und E verschieden sind; die Koordinaten von A', B', C', D' und E' seien bzw. a_i, b_i, c_i, d_i, e_i, $(i = 1, 2, 3, 4, 5)$.

Die Bestimmung der Invarianten A musz mit einiger Sorgfalt geschehen, da wir gewisse Vorzeichenregeln zu beachten haben, welche WEITZENBÖCK angegeben hat (S. 246). Für die Koordinaten der Gerade l_1 wählen wir $p_{31} = a_1$ u. s. w.; für diejenigen von $l_2 : p_{41} = b_1$ u. s. w.; für $l_3 : p_{31} = c_1$ u. s. w.; für $l_4 : p_{51} = d_1$; für $l_5 : p_{31} = e_1$, u. s. w. S'_{23} ist der durch l_2 und l_3, also durch B, C, B' und C' bestimmte Raum; er erhält die Gleichung

$$\begin{vmatrix} b_2 & b_5 \\ c_2 & c_5 \end{vmatrix} . x_1 + \begin{vmatrix} b_5 & b_1 \\ c_5 & c_1 \end{vmatrix} . x_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} . x_5 = 0\,;$$

S'_{45} ist der Raum durch l_4 und l_5, also durch D, E, D' und E'; seine Gleichung wird

$$\ldots . + q . \begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix} . x_3 - p . \begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix} . x_4 = 0.$$

Für die Invariante $A_{1,23,45}$ haben wir also:

$$A_{1,23,45} = - \begin{vmatrix} a_1 & a_2 & a_5 \\ b_1 & b_2 & b_5 \\ c_1 & c_2 & c_5 \end{vmatrix} . \begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix} . q.$$

Um $A_{2,31,45}$ zu bestimmen, brauchen wir noch den Raum S'_{31}; die bezügliche Gleichung ist

$$\begin{vmatrix} c_2 & c_5 \\ a_2 & a_5 \end{vmatrix} . x_1 + \begin{vmatrix} c_5 & c_1 \\ a_5 & a_1 \end{vmatrix} . x_2 + \begin{vmatrix} c_1 & c_2 \\ a_1 & a_2 \end{vmatrix} . x_5 = 0.$$

Hieraus ergibt sich:

$$A_{2,31,45} = \begin{vmatrix} b_1 & b_2 & b_5 \\ c_1 & c_2 & c_5 \\ a_1 & a_2 & a_5 \end{vmatrix} \cdot \begin{vmatrix} d_1 & d_2 \\ e_1 & e_2 \end{vmatrix} \cdot p.$$

Wir haben also:

$$\frac{A_{2,31,45}}{A_{1,23,45}} = -\frac{p}{q} = -(A\,B\,C\,P),$$

wo $(A\,B\,C\,P)$ das Doppelverhältnis der vier Punkte A, B, C, P bedeutet. Allgemein haben wir also den folgenden Satz:

Ist i, j, k, r, s *eine Permutation der Zahlen* 1, 2, 3, 4, 5: m_{rs} *die Transversale der Geraden* l_i, l_j, l_k; n_{rs} *die Transversale über* l_r, l_s, m_{rs}, *sind* $P_{i,rs}$, $P_{j,rs}$, $P_{k,rs}$ *und* P_{rs} *die Schnittpunkte von* l_i, l_j, l_k *und* n_{rs} *mit* m_{rs}, *so ist*

$$\frac{A_{i,jk,rs}}{A_{k,ij,rs}} = -(P_{k,rs}\,P_{i,rs}\,P_{j,rs}\,P_{rs}).$$

Für das Verhältnis von zwei Invarianten A, welche in einem Indizespaar übereinstimmen, finden wir also eine einfache geometrische Bedeutung.

Die invariante $A_{i,jk,rs}$ ist symmetrisch in den Paaren j, k und r, s, schiefsymmetrisch in j, k und in r, s:

$$A_{i,jk,rs} = A_{i,kj,rs}$$
$$A_{i,jk,rs} = -A_{i,rs,jk}.$$

Es gibt also 15 wesentlich verschiedene Invarianten. Mit Hilfe der Invariante $A_{i,jk,rs}$ wird sowohl das Doppelverhältnis auf der Geraden m_{jk} wie auf m_{rs} ausgedrückt, d.h. eine bestimmte A gehört zu 2 der 10 Geraden m. Umgekehrt gehören zu einer dieser Geraden 3 Invarianten A, so zur Geraden m_{45} die Invarianten $A_{1,23,45}$, $A_{2,31,45}$ und $A_{3,12,45}$. Die 6 Verhältnisse von je 2 dieser Zahlen sind — mit dem Minuszeichen versehen — die 6 Werte, welcher das Doppelverhältnis der 4 bezüglichen Punkte auf m_{45} fähig ist. Dasz wir es hier mit den 6 assozierten Werten eines einzigen Doppelverhältnisses zu tun haben, wird durch die von WEITZENBÖCK herrührende Gleichung

$$A_{1,23,45} + A_{2,31,45} + A_{3,12,45} = 0$$

gewährleistet.

Während die Gleichung $A_{i,jk,rs} = 0$ ausdrückt, dasz die auf m_{jk} und auf m_{rs} liegenden Punktquadrupeln nicht aus vier *verschiedenen* Punkten bestehen, bedeutet offenbar die Bedingung $A_{i,jk,rs} = A_{j,ik,rs}$ dasz der auf m_{rs} gelegene Wurf *harmonisch* ist u.s.w.

4. Das Doppelverhältnis $(P_{k,rs}\, P_{i,rs}\, P_{j,rs}\, P_{rs}) = -\dfrac{A_{i,jk,rs}}{A_{k,ij,rs}}$ wollen wir mit D_{kij} bezeichnen. Es gibt offenbar 60 solcher Zahlen; die 6 Zahlen D welche die Indizen i, j und k haben, sind die 6 assozierten Werte des Doppelverhältnisses auf der Geraden m_{rs}.

Betrachten wir jetzt das Verhältnis zweier Invarianten A, welche *nicht* in einem Indizenpaar übereinstimmen, so müssen wir zwei Fälle unterscheiden, welche bzw. durch $\dfrac{A_{i,jk,rs}}{A_{j,rk,is}}$ und $\dfrac{A_{i,kj,rs}}{A_{i,rj,ks}}$ vertreten werden.

Wir haben

$$\frac{A_{i,jk,rs}}{A_{j,rk,is}} = \frac{A_{i,jk,rs}}{A_{j,rk,is}} \cdot \frac{A_{r,jk,is}}{A_{r,jk,is}} = \frac{A_{i,rs,jk}}{A_{r,is,jk}} \cdot \frac{A_{r,jk,is}}{A_{j,rk,is}} = D_{ris} \,.\, D_{jrk}.$$

Und

$$\frac{A_{i,kj,rs}}{A_{i,rj,ks}} = \frac{A_{i,kj,rs}}{A_{i,rj,ks}} \cdot \frac{A_{s,ik,jr}}{A_{j,ik,rs}} \cdot \frac{A_{j,ik,rs}}{A_{s,ik,jr}} = -\frac{A_{i,kj,rs}}{A_{j,ik,rs}} \cdot \frac{A_{s,ik,jr}}{A_{i,ks,jr}} \cdot \frac{A_{j,rs,ik}}{A_{s,jr,ik}} =$$
$$= D_{jik} \,.\, D_{isk} \,.\, D_{sjr}.$$

Hieraus geht hervor: *Die absoluten Invarianten, welche man erhält indem man das Verhältnis zweier Invarianten A bestimmt, die nicht in einem Indizenpaar übereinstimmen, sind dem Produkte von zwei oder von drei Doppelverhältnissen auf den Geraden m gleich.*

5. Die doppelverhältnisse D auf den verschiedenen Geraden m sind nicht von einander unabhängig. Man hat

$$\frac{A_{i,jk,rs}}{A_{k,ij,rs}} \cdot \frac{A_{j,kr,si}}{A_{r,kj,si}} \cdot \frac{A_{k,rs,ij}}{A_{s,kr,ij}} \cdot \frac{A_{r,si,jk}}{A_{i,rs,jk}} \cdot \frac{A_{s,ij,kr}}{A_{j,si,kr}} = -1$$

also

$$D_{kij} \,.\, D_{rkj} \,.\, D_{skr} . D_{irs} \,.\, D_{jsi} = 1$$

d.h. eine Bedingung für die Doppelverhältnisse auf den fünf Geraden m_{ij}, m_{jk}, m_{kr}, m_{rs} und m_{si}. Bei jeder zyklischen Permutation der Indizen i, j, k, r, s gibt es eine solche Bedingung. WEITZENBÖCK hat gezeigt, dasz von den 15 Invarianten A 6 linear-unabhängig sind. Daraus geht hervor, dasz sämmtliche Doppelverhältnisse D auf den 10 Geraden m durch 5 von ihnen bestimmt sind. Beispielsweise hat WEITZENBÖCK die folgenden Invarianten als linear-unabhängig angewiesen:

$J_1 = A_{1,23,45}$, $J_2 = A_{1,24,35}$, $J_3 = A_{1,25,34}$, $J_4 = A_{2,13,45}$, $J_5 = A_{2,14,35}$, $J_6 = A_{3,14,25}$.

Sie bestimmen 5 Doppelverhältnisse:

$$D_{123}=-\frac{J_4}{J_1},\ D_{214}=-\frac{J_2}{J_5},\ D_{314}=\frac{J_3}{J_6},\ D_{325}=-\frac{J_5}{J_6},\ D_{154}=\frac{A_{5,41,23}}{J_1}=-\frac{J_5+J_6}{J_1}.$$

Umgekehrt sind die Verhältnisse der 6 Invarianten J und also diejenigen der 15 Invarianten A durch die 5 Zahlen D bestimmt. Wir haben also:

Die Doppelverhältnisse auf den 10 Geraden m_{ij} können aus 5 von ihnen bestimmt werden, z.B. aus denjenigen auf den Geraden m_{14}, m_{45}, m_{52}, m_{23} und m_{35}.

Mathematics. — *Théorème sur l'itération d'une représentation conforme.* Par Prof. J. WOLFF. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of September 28, 1940.)

Soit $z_1(z)$ une fonction représentant le demi-plan $D(Rz > 0)$ conformément sur un domaine D_1 intérieur à D et dont la frontière f est une courbe de JORDAN intérieure à D, s'étendant à l'infini dans les deux directions $\arg z \rightarrow -\frac{\pi}{2}$ et $\arg z \rightarrow \frac{\pi}{2}$ tel que $|z|$ finit par croître. Soit $z_1(1) = 1$, $z_1(\infty) = \infty$.

Appelons λ la dérivée angulaire à l'infini de $z_1(z)$. Alors, grâce aux hypothèses,

$$\lim_{z \rightarrow \infty} \frac{z_1(z)}{z} = \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

quelle que soit la manière dont z tend vers l'infini dans D[1]).

Indiquons par D_2 le domaine obtenu en appliquant la transformation $z_1 = z_1(z)$ à tous les points z de D_1; par D_3 le domaine obtenu en appliquant la même transformation à tous les points de D_2, et ainsi de suite. D_n contient D_{n+1}, $n = 1, 2, \ldots$. Soit N l'ensemble commun à tous les D_n.

Dans une note présentée à la fin d'Avril 1940 aux *Comptes rendus* de l'Ac. des Sc., Paris, j'ai démontré que

si $\lambda > 0$, *alors* N *contient un domaine dont l'ouverture à l'infini égale* π.

Démontrons maintenant que

si N *contient un domaine* $\triangle$ *contenant un point* $\alpha \neq 1$ *et son image* $\alpha_1 = z_1(\alpha)$, *alors* $\lambda > 0$, par conséquent N contient un domaine dont l'ouverture à l'infini égale π.

Démonstration. $\triangle$ faisant partie de N, tout point z de $\triangle$ possède dans D des antécédents z_{-n} de tous les ordres, définis par $z_0 = z$, $z_1(z_{-n}) = z_{-(n-1)}$, $n = 1, 2, \ldots$. Les fonctions harmoniques $u_n(z) = \arg z_{-n}$ dans $\triangle$ ont leurs valeurs entre $-\frac{\pi}{2}$ et $\frac{\pi}{2}$ et $u_n(z)$ est une conjugúee de $\log |z_{-n}|$.

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[1]) J. WOLFF. *Sur la représentation des bandes.* Compositio mathematica I, p. 217, (1934).

Traçons dans $\triangle$ un polygone Γ d'extrémités α_1 et α. Alors

$$\log |a_{-n}| - \log |a_{-(n-1)}| = \int_{\alpha_1}^{\alpha} \frac{\partial u_n}{\partial \nu} \, ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Γ étant le chemin d'intégration et la normale ν ayant un sens convenable. Or, sur Γ les nombres $\partial u_n / \partial \nu$ sont bornés dans leur ensemble, par suite il existe une constante M telle que

$$\left| \frac{a_{-n}}{a_{-(n-1)}} \right| < M, \qquad n = 1, 2, \ldots . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Je dis que a_{-n} tend vers l'infini avec n. En effet, dans le cas contraire il existerait une constante R et une suite (p) d'entiers p croissant à l'infini tels que $|a_{-p}| < R$. Or, d'après l'hypothèse $z_1(1) = 1$,

$$\lim_{p \to \infty} z_p = 1 \text{ uniformément pour } |z| < R, \ z \text{ dans } D,$$

d'où la contradiction $\alpha = 1$.

De (1) et (3) résulte maintenant que

$$\lambda \geqq M^{-1} > 0 \qquad C. Q. F. D.$$

Remarque 1. $\triangle$ ne peut pas contenir le point $z = 1$, parceque ce point est sur la frontière de N et que N contient $\triangle$.

Remarque 2. Remplaçons l'hypothèse, que $|z|$ finit par croître sur f, par l'hypothèse moins restrictive:
il existe deux nombres m et ϱ tels que sur tout arc fini de f, dont les extrémités z' et z'' satisfont à $|z'| = |z''| > \varrho$, l'oscillation de $\log |z|$ ne surpasse pas le nombre m. Alors le théorème reste exact. En effet, la relation (1) est à remplacer par

$$\limsup_{z \to \infty} \left| \frac{z_1(z)}{z} \right| \leqq \lambda e^{m}$$

$$\liminf_{z \to \infty} \left| \frac{z_1(z)}{z} \right| \geqq \lambda e^{-m}$$

ce qui résulte de l'article cité[1]). Or, la relation (3) exige que

$$\frac{1}{M} \leqq \lambda e^{m}$$

donc λ est encore > 0.

Mathematics. — *Sur les fonctions holomorphes dont l'ensemble des valeurs est soumis à certaines restrictions.* Par Prof. J. WOLFF. (Communicated by Prof. J. G. VAN DER CORPUT).

(Communicated at the meeting of September 28, 1940.)

M. H. UNKELBACH a montré que les fonctions holomorphes dans le cercle-unité qui satisfont aux prémisses du lemme de SCHWARZ et qui, en outre, ne prennent pas une certaine valeur donnée, remplissent des inégalités intéressantes. [1]).

Dans la présente communication nous établirons des inégalités d'un caractère analogue, valables pour une classe de fonctions plus étendue que celle considérée par M. UNKELBACH. Pour plus de facilité le cercle-unité sera remplacé par un demi-plan dans les énoncés.

Théorème I. *Soit* $w(z)$ *holomorphe et à partie réelle positive dans le demi-plan* $D\,(\Re z = x > 0)$, *supposons que les limites angulaires de* $w(z)$ *aux points* 0 *et* ∞ *soient* 0 *et* ∞ *respectivement. Si l'ensemble des valeurs de* $w(z)$ *ne contient pas entièrement la demi-droite* $\left(|w| > 0,\ \frac{\pi}{2} > \arg w = \frac{\pi}{2} - \psi \geqq 0\right)$, *alors la dérivée angulaire* $w'(0)$ *au point* 0 *et la dérivée angulaire* λ *de* $w(z)$ *à l'infini remplissent l'inégalité*

$$w'(0) \geqq \frac{\psi^2}{\sin^2 \psi}\,\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

L'inégalité ne peut pas être améliorée: la fonction $w(z)$ *définie par*

$$\frac{w+\bar{a}}{w-a} = e^{\frac{1}{Az+B}},\ A = \frac{\lambda}{a+\bar{a}},\ B = \frac{1}{2i\psi} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

appartient à la classe considérée, elle ne prend pas la valeur a *d'argument* $\frac{\pi}{2} - \psi$ *et pour elle* (1) *devient l'égalité.*

Démonstration. Soit $a = |a|\,e^{\left(\frac{\pi}{2}-\psi\right)i}$, $0 < \psi \leqq \frac{\pi}{2}$, une valeur telle que $w(z) \neq a$ dans D. La fonction uniforme dans D

$$\Phi(z) = \left(\log \frac{w+\bar{a}}{w-a}\right)^{-1}.$$

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[1] H. UNKELBACH. *Ueber beschränkte Funktionen, deren Wertevorrat gewisse Lücken aufweist.* Mathem. Annaler **115**, p. 205.

le logarithme étant fixé tel que la limite angulaire de $\Phi(z)$ pour $z \rightarrow \infty$ est infinie, a sa partie réelle positive. Sa limite angulaire pour $z \rightarrow 0$ est égale à $\{2i(k\pi+\psi)\}^{-1}$ ou $\{2i(k\pi-\psi)\}^{-1}$, k entier. En indiquant par λ_Φ la dérivée angulaire de Φ à l'infini et par $\Phi'(0)$ celle au point $z=0$, on a, d'après un théorème connu,

$$\Phi'(0) \geqq \lambda_\Phi. \qquad (3)$$

Or

$$\lambda_\Phi = \frac{\lambda}{\alpha+\alpha} \quad \text{et} \quad \Phi'(0) = \frac{w'(0)\, sin\, \psi}{2\,|\alpha|\,.\,(k\pi+\psi)^2} \quad \text{ou} \quad \frac{w'(0)\, sin\, \psi}{2\,|\alpha|\,.\,(k\pi-\psi)^2}.$$

Dans tous les cas (3) donne l'inégalité (1).

La fonction (2) donne la représentation conforme de D sur une surface de RIEMANN à une infinité de feuillets couvrant D, à l'exception du point de ramification α. Faisons quelques remarques.

1. Si D contient une courbe continue γ_z s'étendant de $z=0$ à $z=\infty$, sur laquelle $|arg\, z|$, aux voisinages de 0 et ∞, est plus petit q'un angle fixe $< \frac{\pi}{2}$, et si l'image γ_w de γ_z inclut avec la partie supérieure de l'axe imaginaire un domaine où α est extérieur, alors dans (1) on peut remplacer ψ par $\pi-\psi$.

2. Supposons $w(z)$ univalente, $w'(0) < \infty$, $\lambda > 0$.

Soit α un point de la frontière du domaine $\triangle$, transformé de D par $w(z)$, alors $w(z) \neq \alpha$ dans D. De (1) et de la remarque faite résulte l'énoncé suivant:

η étant la racine entre 0 et π de l'équation

$$\frac{sin\, \eta}{\pi-\eta} = \left(\frac{\lambda}{w'(0)}\right)^{\frac{1}{2}},$$

l'image de la demi-droite $arg\, z = \frac{\pi}{2}$ est entièrement dans l'angle $-\frac{\pi}{2}+\eta \leqq arg\, w \leqq \frac{\pi}{2}$ et l'image de la demi-droite $arg\, z = -\frac{\pi}{2}$ est entièrement dans l'angle $\frac{\pi}{2}-\eta \geqq arg\, w \geqq -\frac{\pi}{2}$.

3. Une homographie conduit à l'énoncé:

Soit $w(z)$ holomorphe et $|w(z)| < 1$ dans le cercle $D(|z|<1)$, et soit $pq = c$ un arc circulaire dans D, dont les extrémités p et q sont sur la circonférence $C(|z|=1)$. Si $w(z)$ ne prend pas toutes les valeurs situées sur l'arc ouvert c, tandis que les limites angulaires de $w(z)$ sont p et q aux points p et q respectivement, alors les dérivées angulaires $w'(p)$ et $w'(q)$ remplissent l'inégalité

$$w'(p)\, w'(q) \geqq \frac{\psi^2}{sin^2\, \psi} \qquad (4)$$

ψ étant l'angle $\leqq \frac{\pi}{2}$ entre C et c.

Les remarques analogues aux remarques 1 et 2 sont évidentes.

Théorème II. *Soit $w(z)$ holomorphe et à partie réelle positive dans le demi-plan D, et supposons que D contienne un point $\alpha = \mathrm{a} + i\mathrm{a}'$ tel que l'équation, $w(z) = \alpha$ ait une seule racine simple $\beta = \mathrm{b} + i\mathrm{b}'$ dans D. Alors la dérivée λ de $w(z)$ à l'infini remplit l'inégalité*

$$\lambda b \leqq 2(\mathrm{a} - \lambda b) \,.\, \mathfrak{R}\left(\log \frac{\mathrm{a}}{b\, w'(\beta)}\right)^{-1}, \quad \ldots \ldots \quad (5)$$

sauf pour les fonctions $w(z) = \dfrac{\mathrm{a}\, z}{b} + \mathrm{c}$, $\mathfrak{R}\mathrm{c} \geqq 0$. Pour celles-ci le second membre de (5) n'a pas de sens.

L'inégalité (5) *ne peut pas être améliorée: la fonction $w(z)$ définie par*

$$\frac{w + \bar{\alpha}}{w - \alpha} = \frac{z + \bar{\beta}}{z - \beta}\, e^{\frac{2(\mathrm{a} - \lambda b)}{\lambda z}}, \quad \beta > 0 \,.\, \ldots \ldots \quad (6)$$

appartient à la classe considérée, elle prend une seule fois la valeur α au point unique β, et pour elle (5) *devient l'égalité.*

Démonstration. $w(z)$ n'étant pas de la forme $\dfrac{\mathrm{a} z}{b} + \mathrm{c}$, on sait que $\mathrm{a} - \lambda b > 0$. De plus $0 < |w'(\beta)| < \dfrac{\mathrm{a}}{b}$. Donc pour démontrer (5) nous pouvons nous borner au cas $\lambda > 0$.

La fonction uniforme dans D

$$\Phi(z) = \left(\log \frac{(w + \bar{\alpha})(z - \beta)}{(w - \alpha)(z + \bar{\beta})}\right)^{-1},$$

le logarithme étant fixé tel que la limite angulaire de $\Phi(z)$ pour $z \to \infty$ est infinie, a sa partie réelle positive. La dérivée angulaire λ_{Φ} de $\Phi(z)$ à l'infini est égale à $2^{-1}(\mathrm{a} - \lambda b)^{-1} \,.\, \lambda$, et $\Phi(\beta) = \left(\log \dfrac{\mathrm{a}}{b\, w'(\beta)}\right)^{-1}$. D'après un théorème connu on a

$$\lambda_{\Phi} \leqq \frac{\mathfrak{R}\Phi(\beta)}{b},$$

ce qui conduit à (5).

La fonction (6) donne la représentation conforme de D sur une surface de Riemann S à une infinité de feuillets couvrant D. Le point α, image du point β de D, appartient à S comme point simple, le point de ramification d'ordre infini α, limite angulaire de l'image de z pour $z \to 0$, n'appartient pas à S.

Signalons quelques conséquences du théorème II.

Supposons $\lambda > 0$ et $w(z)$ univalente. En vertu de (5) on a, en posant $\Re z = x$, $\Re w = u$, en tout point de D

$$\lambda x \leqq 2(u - \lambda x)\, \Re \left(\log \frac{u}{x\, w'(z)} \right)^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

On tire de (7) les inégalités suivantes valables dans D:

$$|\arg w'(z)| \leqq \frac{u}{\lambda x} - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$|w'(z)| \geqq e^{-2\left(\frac{u}{\lambda x} - 1\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Toute fonction $w(z)$ holomorphe et à partie réelle positive dans D jouit, comme on sait, des propriétés suivantes: $|w'(z)| \leqq \frac{u}{x}$ et la limite angulaire de $w'(z)$ à l'infini, comme celle de $\frac{u}{x}$, est égale à λ. Pour les fonctions $w(z)$ holomorphes, à partie réelle positive, et univalentes dans D on a d'après (8) et (9):

1⁰ *$\arg w'(z_n)$ tend vers zéro pour $n \to \infty$ sur toute suite $z_n \to \infty$, angulaire ou non, pourvu que $\frac{u(z_n)}{x_n} \to \lambda$.*

2⁰ $\frac{u}{x} \geqq |w'(z)| \geqq e^{-2\left(\frac{u}{\lambda x} - 1\right)}$ *partout dans D.*

Théorème III. *Toute fonction $w(z)$ satisfaisant aux prémisses du théorème II remplit l'inégalité*

$$\left| \frac{w''(\beta)}{w'(\beta)} + \frac{1}{b} - \frac{w'(\beta)}{a} \right| \leqq \frac{2}{b} \log \frac{a}{b\,|w'(\beta)|} \quad . \quad . \quad . \quad . \quad (10)$$

et pour la fonction (6) *l'égalité se produit, pourvu que $\beta > 0$, $w'(\beta) > 0$.*

La démonstration se fait en appliquant l'inégalité $|\Phi'(\beta)| \leqq \frac{\Re\, \Phi(\beta)}{b}$, remplie par la fonction $\Phi(z)$ qui a servi à démontrer le théorème II.

En particulier, si $a = \beta = 1$, $w'(1) > 0$:

$$\left| \frac{w''(1)}{w'(1)} \right| \leqq -2 \log w'(1) + 1 - w'(1).$$

Si $w(z)$ est univalente, l'inégalité

$$\left| \frac{w''}{w'} + \frac{1}{x} - \frac{w'}{u} \right| \leqq \frac{2}{x} \log \frac{u}{x\,|w'|} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

existe partout dans D.

Puisque $\frac{u}{x|w'|} \geqq 1$ on a $log\, \frac{u}{x|w'|} \leqq \frac{u}{x|w'|} - 1$ et $\left|\frac{1}{x} - \frac{w'}{u}\right| \geqq \frac{1}{x} - \frac{|w'|}{u}$. Donc l'inégalité

$$|w''| \leqq \frac{2u}{x^2} - \frac{|w'|}{x} - \frac{|w'|^2}{u} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

existe partout dans D.

D'une part l'inégalité (12) est plus faible que (11), d'autre part elle dit plus que l'inégalité $|w''| \leqq \frac{2u}{x^2}$ remplie dans D par toute fonction holomorphe et à partie réelle positive.

Mathematics. — *Ueber ein System von drei Strahlenkongruenzen.* Von PH. DWINGER. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of September 28, 1940.)

Einführung.

In dieser Note untersuchen wir ein System von drei Strahlenkongruenzen, deren Strahlen einander involutorisch zugeordnet sind, so dasz gilt:

1. Die Strahlenkongruenzen haben dieselbe Mittelfläche.
2. Zu jedem Strahle eines Strahlensystemes gehören zwei Hauptnormalen; diese Hauptnormalen sollen die zwei zugeordnete Strahlen sein.

§ 1.

Wir wählen eine Darstellung der Strahlensysteme die auf STUDY zurückgeht[1]).

Eine Gerade wird dann dargestellt durch einen dualen Einheitsvektor

$$\mathfrak{A} = \mathfrak{a} + \varepsilon \bar{\mathfrak{a}}$$

$\mathfrak{a}$ ist der Richtungsvektor ($\mathfrak{a}^2 = 1$) und $\mathfrak{a}$ ist das vektorielle Moment der in der Gerade wirkenden Einheitskraft in Bezug auf den Ursprung.

$a + \varepsilon b$ sind sogenannte duale Zahlen, wofür gilt $\varepsilon^2 = 0$.

Eine Kongruenz wird dann dargestellt durch

$$\mathfrak{A}(u, v) = \mathfrak{a}(u, v) + \varepsilon \bar{\mathfrak{a}}(u, v).$$

(u und v sind Parameter)

Man erhält die quadratische duale Differentialform

$$d\mathfrak{A}^2 = \left(\frac{\partial \mathfrak{A}}{\partial u} \frac{\partial \mathfrak{A}}{\partial v}\right) du\, dv$$

oder

$$d\mathfrak{A}^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2$$

$$E = e + \varepsilon \bar{e},\ F = f + \varepsilon \bar{f},\ G = g + \varepsilon \bar{g}.$$

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[1]) E. STUDY. Geometrie der Dynamen. Leipzig (1903). S. 195.

PH. DWINGER. Differentiaalmeetkundige beschouwingen over lijnenstelsels. Diss. Leiden (1938). S. 5.

Die Wurzeln der Gleichung

$$4\,(e\,g - f^2)\,k^2 - 2\,(\bar{e}\,g - 2\,f\,\bar{f} + \bar{e}\,g)\,k + (\bar{e}\,\bar{g} - \bar{f}^2) = 0 \quad . \quad . \quad . \quad (1)$$

sind die Hauptverteilungsparameter k^1 und k^2 der Kongruenz.

Die Gleichung

$$\begin{vmatrix} e\,du + f\,dv & \bar{e}\,du + \bar{f}\,dv \\ f\,du + g\,dv & \bar{f}\,du + \bar{g}\,dv \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

bestimmt zwei Systeme Gröszen, wofür wir bzw. (n^1, n^2) und $\bar{n}^1, \bar{n}^2)$ setzen.

Diese Gröszen werden in folgender Weise normiert:

$$e\,n^{1^2} + 2\,f\,n^1\,n^2 + g\,n^{2^2} = 1$$
$$e\,\bar{n}^{1^2} + 2\,f\,\bar{n}^1\,\bar{n}^2 + g\,\bar{n}^{2^2} = 1.$$

Ist nun $S\,(u,v)$ eine Ortsfunktion, so definieren wir die invarianten Ableitungsgleichungen

$$S_1 = n^1\,S_u + n^2\,S_v$$
$$S_2 = \bar{n}^1\,S_u + \bar{n}^2\,S_v.$$

Die Integrabilitätsbedingung wird

$$S_{12} + q\,S_1 = S_{21} + \bar{q}\,S_2$$

(q und $\bar{q}$ sind Invarianten und Funktionen $n^1, n^2, \bar{n}^1$ und $\bar{n}^2$).

Zu jedem Strahle gehören nun zwei Geraden $\mathfrak{B}$ und $\mathfrak{G}$, sodasz

$$\mathfrak{B} = \mathfrak{A}_1\,(1 - \varepsilon\,k^1)$$
$$\mathfrak{G} = \mathfrak{A}_2\,(1 - \varepsilon\,k^2).$$

Wir nennen $\mathfrak{B}$ und $\mathfrak{G}$ die zwei zu $\mathfrak{A}$ gehörenden Hauptnormalen (d.h. Kehlpunktsnormalen zu den Hauptflächen durch diesen Strahl).

$\mathfrak{B}$ und $\mathfrak{G}$ bilden mit dem Strahle $\mathfrak{A}$ ein invariantes orthogonales Achsenkreuz. Der Punkt durch dieses Achsenkreuz bestimmt ist der Mittelpunkt der Gerade und beschreibt die Mittelfläche der Kongruenz.

Das System invariante Ableitungsgleichungen [2]) ist nun

$$\left.\begin{array}{llll}
\mathfrak{A}_1 = & & (1 + \varepsilon\,k^1)\,\mathfrak{B} & \\
\mathfrak{A}_2 = & & & (1 + \varepsilon\,k^2)\,\mathfrak{G} \\
\mathfrak{B}_1 = & -(1 + \varepsilon\,k^1)\,\mathfrak{A} & & + A\,\mathfrak{G} \\
\mathfrak{B}_2 = & & & - B\,\mathfrak{G} \\
\mathfrak{G}_1 = & & - A\,\mathfrak{B} & \\
\mathfrak{G}_2 = & -(1 + \varepsilon\,k^2)\,\mathfrak{A} & + B\,\mathfrak{B} &
\end{array}\right\} \quad . \quad . \quad . \quad (3)$$

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[2]) PH. DWINGER, loc. cit. S. 45.

wobei

$$A = a + \varepsilon \bar{a} = - \{q + \varepsilon (k_2^1 + q\, k^1 - q\, k^2)\}$$

$$B = b + \varepsilon \bar{b} = - \{\bar{q} + \varepsilon (k_1^2 + \bar{q}\, k^2 - \bar{q}\, k^1)\}$$

A und B sind duale Invarianten. Die Integrabilitätsbedingung wird

$$q A + A_2 + B_1 + \bar{q} B - 1 - \varepsilon (k^1 + k^2) = 0 \quad . \quad . \quad . \quad . \quad (4)$$

Die Hauptrichtungen und damit die Hauptflächen durch jeden Strahl werden durch

$$\frac{du}{dv} = \frac{n^2}{n^1} \quad \text{und} \quad \frac{du}{dv} = \frac{\bar{n}^2}{\bar{n}^1}$$

bestimmt.

Die Kehlpunkte der Hauptflächen fallen zusammen im Mittelpunkt, und die Flächenormalen im Kehlpunkt sind bzw. $\mathfrak{B}$ und $\mathfrak{G}$.

Eine Regelfläche durch $\mathfrak{A}$ wird durch

$$\dot{\mathfrak{A}} = \alpha\, \mathfrak{A}_1 + \beta\, \mathfrak{A}_2$$

bestimmt, wobei

$$\alpha = n_1 \dot{u} + n_2 \dot{v}$$

$$\beta = \bar{n}_1 \dot{u} + \bar{n}_2 \dot{v}$$

(u und v sind Funktionen des Regelflächeparameters). (n_1, n_2) und $(\bar{n}_1, \bar{n}_2)$ sind kovariante Vektoren, die bestimmt werden aus

$$\sum_{i=1}^{2} n^i n_i = 1 \qquad \sum_{i=1}^{2} \bar{n}^i \bar{n}_i = 1$$

$$\sum_{i=1}^{2} n^i \bar{n}_i = 0 \qquad \sum_{i=1}^{2} \bar{n}^i n_i = 0.$$

Durch $\alpha = 1$, $\beta = 0$ und $\alpha = 0$, $\beta = 1$ sind also die Hauptflächen bestimmt.

§ 2.

Gegeben seien die Strahlenkongruenzen $\mathfrak{A}(u, v)$, $\mathfrak{B}(u, v)$ und $\mathfrak{G}(u, v)$.

$\mathfrak{A}$, $\mathfrak{B}$ und $\mathfrak{G}$ sind duale Einheitsvektoren, die Funktionen sind der Parameter u und v. Die Strahlen dieser drei Kongruenzen sind einander zugeordnet, so dasz zu einem Strahle der Kongruenz $\mathfrak{A}$, ein Strahl der Kongruenz $\mathfrak{B}$ und einer der Kongruenz $\mathfrak{G}$ gehören, beide mit denselben Bestimmungszahlen u und v wie $\mathfrak{A}$. Auf derselben Weise sind zu einem Strahle des Systemes $\mathfrak{B}$ ein Strahl der Kongruenz $\mathfrak{G}$ und einer der Kongruenz $\mathfrak{A}$ zugeordnet und schliesslich zu einem Strahle der Kongruenz $\mathfrak{G}$ ein Strahl der Kongruenz $\mathfrak{A}$ und einer der Kongruenz $\mathfrak{B}$.

Wir betrachten nun das Strahlensystem $\mathfrak{A}(u, v)$. Zunächts müssen wegen der involutorischen Zuordnung (siehe „Einführung") $\mathfrak{B}$ und $\mathfrak{G}$ die beiden Hauptnormalen sein eines Strahles $\mathfrak{A}$ (also die $\mathfrak{B}$ und $\mathfrak{G}$ mit derselbe u und v).

Betrachten wir nun das System $\mathfrak{B}$ und stellen wir die Bedingung dafür auf, dasz die Hauptnormalen eines Strahles des Systemes $\mathfrak{B}$ die zugeordneten Strahlen von $\mathfrak{A}$ und $\mathfrak{G}$ sind.

Es gilt dann

$$\dot{\mathfrak{B}} = \alpha \mathfrak{B}_1 + \beta \mathfrak{B}_2.$$

Wir bestimmen nun die beiden Hauptrichtungen und bestimmen dazu den Real- und Dualteil der in α und β quadratischen Differentialform

$$\dot{\mathfrak{B}}^2 = \alpha^2 \mathfrak{B}_1^2 + 2 \alpha \beta \mathfrak{B}_1 \mathfrak{B}_2 + \beta^2 \mathfrak{B}_2^2 .$$

Aus (3) folgt

$$R(\dot{\mathfrak{B}}^2) = \alpha^2 (1 + \mathbf{a}^2) - 2 \alpha \beta \mathbf{a} b + \beta^2 b^2 \quad . \quad . \quad . \quad . \quad (5^1)$$

$$D(\dot{\mathfrak{B}}^2) = \alpha^2 (2 k^1 + 2 \mathbf{a} \bar{\mathbf{a}}) - 2 \alpha \beta (\mathbf{a} \bar{b} + \bar{\mathbf{a}} b) + \beta^2 . 2 b \bar{b} \quad . \quad . \quad (5^2)$$

Die Gleichung für die beiden Hauptrichtungen findet man durch (2). Dieser Gleichung musz jedenfalls eine Richtung (α_1, β_1) genügen so dasz

$$\varrho \dot{\mathfrak{B}} = \alpha_1 \mathfrak{B}_1 + \beta_1 \mathfrak{B}_2 = -\alpha_1 (1 + \varepsilon k^1) \mathfrak{A} + (\alpha_1 A - \beta_1 B) \mathfrak{G} = \mathfrak{G}.$$

(ϱ ist ein dualer Proportionalitätsfaktor).

Also gilt $\alpha_1 = 0$.

Der Gleichung (2) musz also genügen $\frac{\alpha}{\beta} = 0$. Der Koeffizient von β^2 in der Gleichung ist also null. Dieser Koeffizient ist

$$-\mathbf{a} \bar{b} . 2 b \bar{b} + (\mathbf{a} \bar{b} + \bar{\mathbf{a}} b) b^2 = 0,$$

oder

$$b^2 (\bar{\mathbf{a}} b - \mathbf{a} \bar{b}) = 0.$$

Nun ist $b^2 \neq 0$, denn sonst würde die Kongruenz $\mathfrak{A}$ zylindrisch sein, also ist

$$\bar{\mathbf{a}} b - \mathbf{a} \bar{b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

oder

$$\frac{\bar{\mathbf{a}}}{\mathbf{a}} = \frac{\bar{b}}{b}.$$

Wir setzen diesen Bruch gleich k^3.

Also ist

$$\left.\begin{aligned}\bar{\mathrm{a}} &= \mathrm{a}\,k^5\\ \bar{b} &= b\,k^3\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (7)$$

Wir behaupten, dasz die Bedingung

$$\bar{\mathrm{a}}\,b - \mathrm{a}\,\bar{b} = 0 \quad \ldots\ldots\ldots\ldots \quad (7)$$

nicht nur notwendig, sondern auch hinreichend ist.

Zuerst wollen wir zeigen dasz $\mathfrak{A}$ die andere Hauptnormale ist. Dazu beweisen wir, dasz ein Paar (α_2, β_2) der Gleichung (2) genügt so dasz gilt

$$\varrho\,\dot{\mathfrak{B}} = \alpha_2\,\mathfrak{B}_1 + \beta_2\,\mathfrak{B}_2 = \mathfrak{A}.$$

Also wegen (3)

$$\alpha_2\,A - \beta_2\,B = 0 \text{ oder } \frac{\alpha_2}{\beta_2} = \frac{B}{A}.$$

Man findet fast mühelos, dasz auch diese Bedingung lautet $\bar{\mathrm{a}}\,b - \mathrm{a}\,\bar{b} = 0$.

Betrachten wir nun das Strahlensystem $\mathfrak{G}$ und fordern wir dasz $\mathfrak{A}$ und $\mathfrak{B}$ die Hauptnormalen von jedem Strahle sind, so finden wir in ähnlicher Weise wie oben dasz dafür die Bedingung (6) genügend ist.

Wir haben also ein System von drei Strahlenkongruenzen gefunden, deren Strahlen einander so zugeordnet sind, dasz die zu jedem Strahle einer Kongruenz gehörenden Strahlen der anderen Kongruenzen, die Hauptnormalen des ersten Strahles sind. Demzufolge haben auch die Strahlensysteme dieselbe Mittelfläche. Wir können diese Systeme bestimmen durch die Differentialgleichung

$$\mathrm{a}\,\bar{b} - \bar{\mathrm{a}}\,b = 0,$$

oder: (siehe (4))

$$q\,(k_1^2 + \bar{q}\,k^2 - \bar{q}\,k^1) - \bar{q}\,(k_2^1 + q\,k^1 - q\,k^2) = 0.$$

§ 3.

Wir untersuchen nun dieser Systeme weiter.

Wir betrachten die Kongruenz $\mathfrak{B}$

$$\dot{\mathfrak{B}} = \alpha\,\mathfrak{B}_1 + \beta\,\mathfrak{B}_2$$

oder

$$\dot{\mathfrak{B}} = -\alpha\,(1 + \varepsilon\,k^1)\,\mathfrak{A} + [\alpha\,\mathrm{a} - \beta\,b + \varepsilon\,k^3\,(\alpha\,\mathrm{a} - \beta\,b)]\,\mathfrak{G}.$$

wegen (2) und (7).

Durch eine projektive Transformation setzen wir:

$$\left.\begin{aligned} \alpha^* &= \alpha a - \beta b \\ \beta^* &= -\alpha \end{aligned}\right\} (\triangle \neq 0)$$

also

$$\dot{\mathfrak{B}}^2 = \alpha^{*2} + \beta^{*2} + \varepsilon\,(2\,k^3\,\alpha^{*2} + 2\,k^1\,\beta^{*2}) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hieraus ergibt sich dasz die Hauptverteilungsparameter k^3 und k^1 sind und weiter noch, dasz die Hauptrichtungen bestimmt sind durch

$$\left.\begin{aligned} &\text{I.} \quad \alpha^* = 1, \beta^* = 0 \quad \text{oder} \quad \alpha = 0, \beta = 1 \\ &\text{II.} \quad \alpha^* = 0, \beta^* = 1 \quad \text{oder} \quad \alpha = b, \beta = a \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Wir betrachten nun die Kongruenz $\mathfrak{G}$

$$\dot{\mathfrak{G}} = \alpha\,\mathfrak{G}_1 + \beta\,\mathfrak{G}_2 = -\beta\,(1 + \varepsilon\,k^2)\,\mathfrak{A} + [-\alpha a + \beta b + \varepsilon\,k^3\,(-\alpha a + \beta b)]\,\mathfrak{B}.$$

Wir setzen weiter

$$\begin{aligned} \alpha^{**} &= \qquad -\beta \\ \beta^{**} &= -\alpha a + \beta b \end{aligned}$$

und finden dann

$$\dot{\mathfrak{G}}^2 = \alpha^{**2} + \beta^{**2} + \varepsilon\,(2\,k^2\,\alpha^{**2} + 2\,k^3\,\beta^{**2}) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Hieraus folgt wieder dasz die Hauptverteilungsparameter k^2 und k^3 sind und dasz die Hauptrichtungen durch

$$\left.\begin{aligned} \alpha^{**} = 1,\ \beta^{**} = 0 \ \text{oder}\ \alpha = b,\ \beta = a \\ \alpha^{**} = 0,\ \beta^{**} = 1 \ \text{oder}\ \alpha = 1,\ \beta = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

bestimmt werden.

Es gilt also der *Satz:*

Die Hauptverteilungsparameter von $\mathfrak{A}$ *sind* k^1 *und* k^2, *von* $\mathfrak{B}$ k^3 *und* k^1 *und von* $\mathfrak{G}$, k^2 *und* k^3.

Es gilt auch noch der *Satz:*

Von jedem Strahle sind die Hauptflächen die komplementären Regelflächen[3] *einer der beiden Hauptflächen eines jeden der zugeordneten Strahlen.*

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[3] Die komplementäre Regelfläche wird gebildet von den Tangenten der Regelfläche in dem Kehlpunkt senkrecht auf der Regel.

Beweis (für $\mathfrak{B}$):

Die erste Hauptfläche von $\mathfrak{B}$ wird durch $\alpha=0$, $\beta=1$ bestimmt (siehe (9)), und diese Richtung bestimmt auch die zweite Hauptrichtung von $\mathfrak{A}$, also ist die erste Hauptfläche von $\mathfrak{B}$ zugleich komplementäre Regelfläche der zweiten Hauptfläche von $\mathfrak{A}$.

Die zweite Hauptfläche von $\mathfrak{B}$ wird bestimmt durch $\alpha=b$, $\beta=a$ ebenso wie die erste Hauptrichtung von $\mathfrak{G}$. Die Strahlen $\mathfrak{B}$ stehen senkrecht im Kehlpunkt der Strahlen $\mathfrak{G}$ und bilden also die komplementäre Regelfläche der ersten Hauptfläche von $\mathfrak{G}$.

§ 4.

In diesem Abschnitt stellen wir die invarianten Ableitungsgleichungen (also analog an (3)) auf für $\mathfrak{B}$ und $\mathfrak{G}$. Offenbar stellen die Indizen $_1$ und $_2$ in (3) die Ableitungen nach der Bogenlänge des sphärischen Bildes der Hauptflächen von $\mathfrak{A}$ dar.

Wir betrachten zuerst die Kongruenz $\mathfrak{B}$. Wir finden die invarianten Ableitungsgleichungen für diese Kongruenzen wenn wir in (3) die folgenden Verwechslungen vornehmen

$$\begin{array}{l}
\mathfrak{A} \rightarrow \mathfrak{B}\\
\mathfrak{B} \rightarrow \mathfrak{G}\\
\mathfrak{G} \rightarrow \mathfrak{A}\\
k^1 \rightarrow k^3\\
k^2 \rightarrow k^1\\
{}_1 \rightarrow {}_{1*}\\
{}_2 \rightarrow {}_{2*}
\end{array}$$

Die Indizen $_{1*}$ und $_{2*}$ stellen die Ableitungen nach der Bogenlänge des sphärischen Bildes von den Hauptflächen von $\mathfrak{B}$ dar.

Es gilt also

$$\left.\begin{array}{llll}
\mathfrak{B}_{1*} = & & (1+\varepsilon k^3)\,\mathfrak{G} & \\
\mathfrak{B}_{2*} = & & & (1+\varepsilon k^1)\,\mathfrak{A}\\
\mathfrak{G}_{1*} = & -(1+\varepsilon k^3)\,\mathfrak{B} & & +A^*\,\mathfrak{A}\\
\mathfrak{G}_{2*} = & & & -B^*\,\mathfrak{A}\\
\mathfrak{A}_{1*} = & & -A^*\,\mathfrak{G} & \\
\mathfrak{A}_{2*} = & -(1+\varepsilon k^1)\,\mathfrak{B} & +B^*\,\mathfrak{G} &
\end{array}\right\} \quad . \quad . \quad (12)$$

Wir berechnen die dualen Invarianten A^* und B^*.

Dazu berechnen wir $\mathfrak{A}_{1*}$

$$\mathfrak{A}_{1*} = \alpha\,\mathfrak{A}_1 + \beta\,\mathfrak{A}_2,$$

die α und β folgen aus

$$\alpha^* = \alpha\, a - \beta\, b$$
$$\beta^* = -\alpha,$$

wobei $\alpha^{*2} + \beta^{*2} = 1$ und $\alpha^* = 1$ und $\beta^* = 0$, also $\alpha = 0$, $\beta = -\frac{1}{b}$.

Wir haben also

$$\mathfrak{A}_{1*} = -\frac{1}{b}\,\mathfrak{A}_2 = -\frac{1}{b}\,(1 + \varepsilon\, k^2)\,\mathfrak{G}$$

$$A^* = \frac{1}{b}\,(1 + \varepsilon\, k^2)\; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

B^* berechnen wir aus

$$\mathfrak{G}_{2*} = \alpha\,\mathfrak{G}_1 + \beta\,\mathfrak{G}_2 .$$

Wegen $\alpha^{*2} + \beta^{*2} = 1$ und (9) gilt $\alpha = 1$, $\beta = -\frac{a}{b}$.

Also

$$\mathfrak{G}_{2*} = -\mathfrak{G}_1 - \frac{a}{b}\,\mathfrak{G}_2 = \frac{a}{b}\,(1 + \varepsilon\, k^2)\,\mathfrak{A}$$

also

$$B^* = -\frac{a}{b}\,(1 + \varepsilon\, k^2)\, . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

Die invarianten Ableitungsgleichungen für $\mathfrak{G}$ lauten

$$\left.\begin{array}{llll}
\mathfrak{G}_{1**} = & & (1 + \varepsilon\, k^2)\,\mathfrak{A} & \\
\mathfrak{G}_{2**} = & & & (1 + \varepsilon\, k^3)\,\mathfrak{G} \\
\mathfrak{A}_{1**} = & -(1 + \varepsilon\, k^2)\,\mathfrak{G} & & +A^{**}\,\mathfrak{B} \\
\mathfrak{A}_{2**} = & & & -B^{**}\,\mathfrak{B} \\
\mathfrak{B}_{1**} = & & -A^{**}\,\mathfrak{A} & \\
\mathfrak{B}_{2**} = & -(1 + \varepsilon\, k^3)\,\mathfrak{G} & +B^{**}\,\mathfrak{A} &
\end{array}\right\} \; . \; . \; . \quad (15)$$

wobei

$$\left.\begin{array}{l}
A^{**} = -\frac{b}{a}\,(1 + \varepsilon\, k^1) \text{ und} \\[2ex]
B^{**} = \frac{1}{a}\,(1 + \varepsilon\, k^1)
\end{array}\right\} \; . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

Nun sind bei einer Kongruenz $\frac{a}{a}$ und $\frac{\bar{b}}{b}$ die Verteilungsparameter der komplementären Regelflächen der Hauptflächen.

Wegen (13), (14) und (16) gilt also der *Satz:*

Die zwei komplementären Regelflächen der Hauptflächen der Kongruenzen $\mathfrak{A}$, $\mathfrak{B}$ *und* $\mathfrak{G}$ *haben dieselbe Verteilungsparameter. Sie sind bzw.* k^3, k^2 *und* k^1.

Schliesslich gelten noch die folgenden Beziehungen

$$a\, a^*\, a^{**} = b\, b^*\, b^{**} = -1$$

und

$$\bar{a}\, \bar{a}^*\, \bar{a}^{**} = \bar{b}\, \bar{b}^*\, \bar{b}^{**} = -\, k^1\, k^2\, k^3.$$

Chemistry. — *Die Analogie zwischen dem molekularen Bau und dem mechanischen Verhalten von Cellulose und Kautschuk.* Von P. H. HERMANS. (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of September 28, 1940.)

1. *Einleitung.*

Die theoretischen Betrachtungen über die Bildung, das deformatorische Verhalten und die mechanischen Eigenschaften der künstlichen Fasern aus regenerierter Cellulose standen bis jetzt fast ausnahmslos auf dem Boden der sogen. „Micellentheorie". Diese geht bekanntlich auf VON NÄGELI zurück und wurde durch die Arbeiten von AMBRONN, sowie durch die Ergebnisse der seit etwa dem Jahre 1920 auf diesem Gebiet angewandten röntgenspektrographischen Untersuchungsmethode neu belebt. Obgleich in den letzten Jahren die extreme Auffassung der kristallinen „Micelle" als individuelle Elementarbausteine der Faser für den Fall der natürlich gewachsenen Objekte verlassen worden ist [1]) und der Vorstellung eines statistischen Charakters des gittermässig geordneten Anteiles der Fasern Platz gemacht hat, spielen die Micelle in den Betrachtungen über die Cellulose-Lösungen und die regenerierte Cellulose noch immer eine wesentliche Rolle, allerdings in der modifizierten Vorstellung der „Fransenmicelle" [2]).

Ueber den mizellaren oder molekularen Lösungszustand der Cellulose in den bekannten Spinnlösungen gehen die Meinungen noch immer auseinander. Bekanntlich hat STAUDINGER seit Jahren die Auffassung der molekularen Dispersion verfochten. Nach ihm sind also die in den regenerierten Fasern nachweisbaren mikrokristallinen Anteile („Micelle") sekundäre, aus den Kettenmolekülen durch einen Rekristallisationsvorgang entstandene Gebilde.

In dieser Abhandlung soll gezeigt werden, dass diese Auffassung weitgehend zutrifft und soll darauf hingewiesen werden, dass bei der Deformation der regenerierten Cellulose makromolekulare Vorgänge die mass-

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[1]) F. T. PEIRCE, D. MILES, S. M. NEALE, W. T. ASTBURY: The colloid Aspects of Textile Materials, Trans Faraday Soc. 1933; H. MARK und O. KRATKY: Z. physik. Chem. B. **36**, 129 (1937) und insbesondere die vorzügliche Darstellung des Gegenstandes bei A. FREY-WYSSLING, Submikroskopische Morphologie des Protoplasma's und seiner Derivate, Berlin 1938 und weiter R. HOSEMANN (Bunsen-Tagung) Z. angew. Chem. **53**, 332 (1940).

[2]) Für eine neuzeitliche Darstellung und Literaturübersicht dieses ganzen Gebietes sei verwiesen auf einen Aufsatz von O. KRATKY, „Der micellare Aufbau der Cellulose und ihre Derivate", Z. angew. Chem. **53**, 153 (1940).

gebende Rolle spielen und dass die Deutung ihres Feinbaues daher weiterhin in erster Linie an Hand molekularer Vorstellungen vor zu nehmen ist. Weitere Fortschritte auf diesem Gebiete werden erst möglich sein, wenn man die Cellulose ihrer bisherigen Sonderstellung als Substanz, deren Verhalten unter allen Umständen vorzugsweise auf Grund von micellaren Vorgängen zu deuten sei, beraubt und sie nicht mehr mit anderen Augen betrachtet als die übrigen Linearpolymeren, wie die kautschukähnlichen Substanzen und die fibrillären Eiweisskörper, für die der molekulare Charakter ihres Feinbaues und Deformationsmechanismus schon zu den festgewonnenen Erkenntnissen gezählt werden kann.

2. *Das röntgenspektrographische und optische Verhalten frischer regenerierter Cellulosefäden.*

Durch langsames Ausfliessenlassen von Viskose aus einer Glaskapillare in eine konzentrierte Salzlösung erhält man unter geeigneten Bedingungen völlig isotrope hochgequollene Xanthogenatfäden, die sich in ebenfalls noch hochgequollene isotrope Cellulosefäden umsetzen lassen [3]). In diesen Fäden sind die dispergierten Celluloseteilchen ohne irgendwelche Vorzugsorientierung, so wie sie sich in der kolloiden Lösung befanden, in irgend einer Weise zu einem zusammenhängenden Gelgerüst zusammengetreten. Die Eigenschaften und das Verhalten dieser Fäden gestatten wertvolle Rückschlüsse auf die Natur der konstituierenden Teilchen.

Das Röntgendiagramm dieser gequollenen Fäden gleicht dem einer Flüssigkeit und verrät keinerlei gittermässig geordnete Bestandteile. Dass dies nicht auf eine Ueberstrahlung eventueller Kristallinterferenzen durch das diffuse Wasserspektrum zurückzuführen ist, geht daraus hervor, dass beim Dehnen dieser Fäden allmählich ein Faserdiagramm erscheint, dass neben dem Wasserspektrum die „kristallinen" Interferenzen der Hydratcellulose zu erkennen gibt. Mit zunehmendem Dehnungsgrad wird das Diagramm klarer und zeigt eine höhere axiale Orientierung des kristallinen Anteiles an. Es errinnert dies an das Verhalten des amorphen Kautschuks und anderer polymerer Substanzen, die ebenfalls erst beim Dehnen mikrokristallin werden. Werden die gequollenen isotropen Fäden getrocknet, so erscheinen ebenfalls Kristallinterferenzen, nun aber natürlich in der Form eines Debye-Scherrer-Diagrammes ohne Vorzugsorientierung der Kristallite. Dieser Vorgang ist der Kristallisation des amorphen isotropen Kautschuks beim Abkühlen zu vergleichen, der dann ebenfalls ein D.S.-Diagramm ergibt.

Beim Wiederaufquellen der getrockneten Fäden in Wasser [4]) bleibt nunmehr das kristalline D.S.-Diagramm in aller Schärfe erhalten: Der

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[3]) P. H. HERMANS und A. J. DE LEEUW, Kolloid-Z. **81**, 321 (1937); P. H. HERMANS: Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 798 (1939).

[4]) Die frischen isotropen Cellulosefäden hatten bei diesen Versuchen einen Quellungsgrad von 5—6, die Wiedergequollenen einen von 2.1—2.3.

sekundäre Quellungsvorgang der Cellulose verläuft bekanntlich *„inter-micellar"*.

Beim Trocknen der gedehnten primären Gelfäden nimmt das Faserdiagramm an Schärfe und Klarheit zu, was auf eine weitere Zunahme des kristallinen Anteilen hinweist; beim Wiederaufquellen bleibt es dann ebenfalls erhalten. Durch eine eingehende Analyse des polarisationsoptischen Verhaltens derselben Objekte, bei der von der WIENER-AMBRONN'schen Imbibitionsmethode Gebrauch gemacht wurde zur Unterscheidung zwischen Form- und Eigendoppelbrechung, konnte auf unabhängigem Wege ebenfalls dargetan werden, dass sowohl beim Dehnen der isotropen, als auch beim Entquellen der gedehnten primären Gelfäden eine sehr beträchtliche Zunahme des kristallinen Faseranteiles stattfindet [5]. Tabelle I gibt ein Beispiel von der beobachteten Zunahme der mit dem kristallinen Anteil verknüpften „reinen Eigendoppelbrechung" (für den Orientierungsfaktor korrigierten E-do) in Funktion des Dehnungsgrades v.

TABELLE I.

Zunahme der „reinen" Eigendoppelbrechung mit der Dehnung als Mass der Menge des kristallinen Anteiles.

v	1.06	1.29	1.47	1.60	1.72	1.84	1.94
$(n_a - n_o) \cdot 10^3$	7	13	16	20	24	23	27

Die regenerierte Cellulose wird also erst beim Dehnen, bezw. beim Entquellen kristallin.

Für die Deutung dieser Erscheinungen liegt es auf der Hand anzunehmen, dass die isotropen hochgequollenen primären Gelfäden ein lockeres wasserreiches Gerüst darstellen, das aus individuellen Fadenmolekülen aufgebaut ist, die in irgend einer noch näher aufzuklärenden Weise an bestimmten Stellen, die wir nach einem Vorschlag von FREY-WYSSLING mit der neutralen Bezeichnung „Haftpunkte" andeuten werden [6], mit einander zusammenhängen. Die Konfiguration, welche die Ketten dabei annehmen, ob gerade oder geknäuelt, ist erst später zu diskutieren.

Beim Entquellen schrumpft das Gerüst zusammen, nähern sich die Ketten und finden sich manche Kettenabschnitte an statistisch verteilten Stellen in genügendem Masse in paralleler Anordnung zur Bildung gittermässig geordneter Bereiche zusammen [7]. Bei der Dehnung des primären

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[5] P. PLATZEK, Diss. Utrecht 1939; P. H. HERMANS und P. PLATZEK, Z. physik. Chem. A. **185**, 269 (1939).

[6] A. FREY-WYSSLING, loc. cit. S. 75. Die Bezeichnung „Haftpunkte" nimmt über die Art der Bindungsweise nichts vorweg.

[7] Vergl. die Betrachtungen über das KRATKY'sche Prinzip der „Ordnung in kleinsten Bereichen": O. KRATKY, Kolloid-Z. **68**, 347 (1934). P. H. HERMANS, ebenda **83**, 71 (1938).

Gerüstes findet dieser Vorgang ebenfalls in gewissem Masse statt und wird das Einschnappen bestimmter Kettenabschnitte in gegenseitigem Gitterverband durch die bei der Dehnung erfolgende Parallelrichtung der Ketten erleichtert. Bei nachfolgender Entquellung kann der nämliche Vorgang noch weiter fortschreiten und nimmt der kristalline Anteil weiter zu.

3. *Der molekulare Charakter des Deformationsmechanismus.*

Hat man es in den primären Gelfäden tatsächlich mit einem aus Fadenmolekülen bestehenden Gerüste zu tun, so erhebt sich die Frage, wie dieses Gerüst wohl gebaut sein mag und welche Konfiguration und Gestalt die Kettenmoleküle dabei annehmen. Bei den Versuchen zur Lösung dieses Problems begegnet man einer Reihe von grossen Schwierigkeiten und diese Lösung ist denn auch noch keineswegs vollständig gelungen. Doch können durch eine aufmerksame Betrachtung der Erscheinungen, die sich bei der Deformation von isotropen Fäden bei verschiedenen Quellungsgraden beobachten lassen, wertvolle Hinweise gewonnen werden. Es ergibt sich, dass man zur Deutung dieser Erscheinungen erfolgreich Gebrauch machen kann von Vorstellungen, die bei der Deformation anderer linearpolymer Substanzen, namentlich der des Kautschuks, schon geläufig sind.

Da wir zunächst über das Wesen der Quellung nicht orientiert sind, wollen wir zuerst einmal den ungequollenen (an der Luft getrockneten) isotropen Faden betrachten und uns auf den Standpunkt stellen, dass dessen Feinbau mit dem eines isotropen Kautschukfadens vergleichbar ist.

Die hohe Dehnbarkeit der kautschukähnlichen Körper hängt bekanntlich mit dem Vorhandensein langer Hauptvalenzketten mit erheblicher innerer Beweglichkeit („Biegsamkeit" infolge „freier" Drehbarkeit der Kettenglieder gegeneinander) zusammen. Im ungedehnten amorphen Kautschuk streben diese Ketten nach einer geknäuelten, statistisch wahrscheinlichsten Gestalt [8]). Beim Dehnen findet durch einen noch nicht in allen Einzelheiten ergründeten Vorgang, ein Geradeziehen und Gleichrichten, also eine Orientierung der Ketten in die Zugrichtung statt. Die hohe Dehnbarkeit in der Grössenordnung von 400—1000 Proz. hängt eng mit der Entfaltung der geknäuelten Ketten zusammen. Man kann theoretisch über ihren maximalen Wert keine begrenzende Aussagen machen; sie ist schlechthin „gross".

Die Reversibilität der Dehnung (elastizität) findet darin ihren Grund, dass beim Aufheben der angelegten Spannung die Ketten, durch die innere Wärmebewegung getrieben, spontan nach den entropiereicheren geknäuelten, ungeordneten Zustand zurückstreben. Dieser Mechanismus ist ausser mit der inneren Beweglichkeit der Ketten auch mit einer grossen, fast flüssigkeitartigen Verschiebbarkeit benachbarter Kettenabschnitte gegeneinander verknüpft; sie entspricht der Mikro-Brownschen Bewegung nach KUHN [9]). Das heisst aber nicht, dass nun auch, wie in einer normalen Flüssigkeit, die ganzen Makromoleküle leicht gegeneinander beweglich sind. In dem kautschukartigen Zustand ist der Platzaustausch der Makromoleküle, die Makro-Brownsche Bewegung nach KUHN, stark

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[8]) Ueber das vermutliche Mass der Knäuelung vgl. insbesondere H. MARK, Der feste Körper, Leipzig 1938, S. 103.

[9]) W. KUHN, Z. angew. Chem. **49**, 858 (1936); **51**, 640 (1938); **52**, 289 (1939); Kolloid-Z. **87**, 3 (1939).

erschwert, bezw. verhindert [10]). Wäre dies nicht der Fall, so würde im gedehnten Zustand ein aneinander Abgleiten der gerichteten Moleküle, ein rasches Fliessen der Substanz eintreten, wodurch die hervorgerufene innere Spannung innerhalb einer gewissen Relaxationszeit vernichtet und die elastische Rückfederung aufgehoben würde. Die gestreckten Ketten als Ganzes müssen in irgend einer Weise im Inneren der Substanz in ihrer gegenseitigen Lage festgehalten werden. (Es ist dies nach W. KUHN [11]) zur Zeit eines der Zentralprobleme auf dem Gebiete der Hochelastizität.) Bei unvulkanisierten Kautschukpräparaten findet tatsächlich solch ein Fliessen in gewissem von der Temperatur abhängigem Masse statt, mit als Folge eine unvollkommene Umkehrbarkeit der Dehnung. Bei vulkanisiertem Kautschuk nimmt man bekanntlich Querbrücken zwischen den Ketten an, welche diese an einzelnen Stellen fest untereinander verknüpfen und so das Fliessen verhindern. In gestrecktem Zustande befänden sich dann zwischen den eingeklemmten Enden des Präparates lückenlose Reihen von durch Hauptvalenzen miteinander verbundenen Atomen. Nur solche Kautschukpräparate, die also durchgehende Netze von festverbundenen Ketten enthalten, können eine völlig reversible Dehnbarkeit aufweisen [12]).

Wollen wir nun untersuchen, ob die lufttrockene isotrope Cellulose in ihrem Feinbau dem isotropen Kautschuk ähnlich ist, dann ist zunächst noch zu berücksichtigen, dass die hohe reversible Dehnbarkeit des letzteren auf bestimmte Temperaturen beschränkt ist. Bei abnehmender Temperatur nimmt die Kohäsion zwischen benachbarten Ketten zu, die innere Beweglichkeit „friert ein" und Dehnbarkeit sowie Elastizität sinken stark ab, um schliesslich zu verschwinden.

Bei der Cellulose, mit ihren vielen polaren Gruppen, sind die Kohäsionskräfte zwischen den Kohlehydratketten von vornherein sehr viel grösser, als die zwischen den Kohlenwasserstoffketten des Kautschuks [13]); infolgedessen ist eine Erschwerung der inneren Beweglichkeit und damit eine beschränkte Dehnbarkeit zu erwarten. Der Orientierungsvorgang bricht, ebenso wie beim gefrorenen Kautschuk, durch einen anderen störenden Vorgang, den Bruch des Präparates, ab, bevor vollständige Orientierung der Ketten erreicht ist. Auch die Fähigkeit zur spontanen Rückfederung wird verschwinden, indem die Moleküle einander in den neu erzwungenen Lagen festhalten, genau so wie dies beim starken Abkühlen des bei Zimmertemperatur gedehnten Kautschuks der Fall ist, der dann ebenfalls im orientierten Zustand beharrt.

Bei der Cellulose darf man nun im gequollenen Zustande wieder eine vermehrte Beweglichkeit erwarten, da Quellung die intermolekularen Kohäsionskräfte schwächt. Bei der Cellulose wird der Quellungsgrad somit eine analoge Rolle spielen wie die Temperatur beim Kautschuk [14]).

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[10] Zustand der „fixierten Flüssigkeiten" nach K. UEBERREITER, Z. physik. Chem. B. **45**, 361 (1940); Z. angew. Chem. **53**, 247 (1940).

[11] Privatmitteilung des Herrn Prof. W. KUHN vom 2.12.1939.

[12] Siehe darüber besonders J. M. BURGERS, First Report on Viscosity and Plasticity. Amsterdam 1935, S. 98. (Begriff der „vollkommenen Elastizität".)

[13] Vgl. z.B. den Schmelzpunkt und Siedepunkt der Glucose mit dem des Isoprens.

[14] Die Temperatur wobei auch die Cellulose kautschukähnlich werden würde liegt so hoch, dass die Cellulose dabei zerstört werden würde; vgl. Fussnote 13.

Der isotrope Cellulosefaden entspricht diesen Voraussetzungen voll und ganz. Nach dem Trocknen über Phosphorpentoxyd ist er völlig undehnbar [15]), wie Kautschuk bei der Temperatur der flüssigen Luft. Lässt man den Faden an der Luft etwa 10 Proz. Feuchtigkeit absorbieren, dann ist seine Dehnbarkeit auf etwa 120—130 Proz. gestiegen und diese lässt sich weiter auf 200 Proz. erhöhen, wenn man den Faden in einer warmen feuchten Atmosphäre (Temp. 60°, Wassergehalt der Cellulose etwa 40 Proz.) dehnt. Es ist dabei zu betonen, dass die Orientierung bei dieser hohen Dehnung noch keineswegs eine vollständige ist (vgl. weiter unten).

Eigentümlicherweise gelingt es nun nicht, durch noch weitere Erhöhung des Quellungsgrades die Dehnbarkeit weiter zu steigern. Beim Einlegen in Wasser wird der Quellungsgrad etwa $q = 2.2$ [16]) und die maximale Dehnbarkeit ist dann ebenfalls etwa 120—130 Proz. Auch die oben beschriebenen noch höher gequollenen primären Gelfäden weisen alle ungefähr diese Dehnbarkeit auf. Diese Unstimmigkeit ist jedoch nur eine scheinbare, wie jetzt zu zeigen ist. Sie hängt mit einer bisher übersehenen, interessanten Eigentümlichkeit des Mechanismus der Quellung zusammen.

4. *Verlauf der Orientierung in Funktion der Dehnung.*

In früheren Arbeiten haben wir, z.T. in Zusammenarbeit mit O. KRATKY, experimentelle Methoden entwickelt, die es gestatten, den Verlauf des Orientierungsvorganges in Funktion der Dehnung quantitativ zu verfolgen [17]). Es wurden dazu ausser der neuen Methode der Quellungsanisotropie auch polarisationsoptische Messungen und die Röntgenmethode herangezogen. Die Ergebnisse der drei Methoden stimmen innerhalb der theoretisch zu erwartenden Abweichungen gut miteinander überein und wir stehen infolge dessen mit diesen Messungen auf sicherem Boden. Im folgenden werden wir uns einfachkeitshalber nur auf Ergebnisse der Methode der Quellungsanisotropie beziehen. Diese Methode hat den Vorzug, bei allen Quellungsgraden gut anwendbar zu sein und sie ist unempfindlich für das Verhältnis zwischen kristalliner und amorpher Substanz. Die Quellungsanisotropie ist 1 für den isotropen und ∞ für den völlig orientierten Zustand. Sie verläuft somit der Orientierung symbat.

Fig. 1 zeigt den Verlauf der Quellungsanisotropie in Funktion des experimentellen Dehnungsgrades v [18]). Die Orientierung steigt umso rascher an, je nachdem der Quellungsgrad der Objekte höher ist. In Tabelle II sind die jeweils bei 100 Proz. Dehnung ($v = 2$) erreichten Werte der

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[15]) In diesem Zustande lässt sich die Cellulose sogar im Mörser zu einem Pulver zerstössen.

[16]) Unter Quellungsgrad verstehen wir das Verhältnis des Volums des gequollenen zu dem des völlig trockenen Fäden.

[17]) Kolloid-Z. **81**, 143, 300 (1937); **82**, 58 (1938); **86**, 245; **87**, 296; **88**, 68, 172 (1939). Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 798 (1939). Z. physik. Chem. A **185**, 260 (1939) und z.T. noch unveröffentliche Untersuchungen.

[18]) Verhältnis der Länge des gedehnten zu der des isotropen Fadens.

Quellungsanisotropie Q und die daraus errechneten mittleren Orientierungswinkel α_m [19]) der Ketten zusammengestellt.

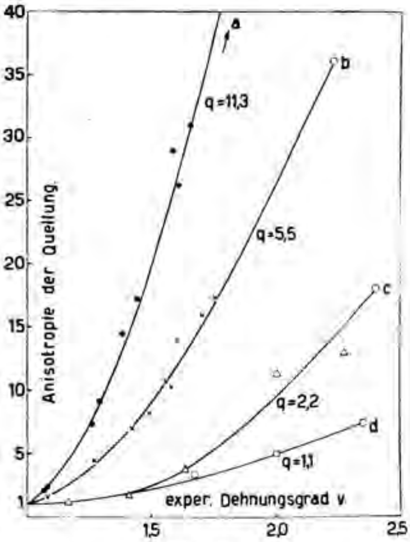

Fig. 1.

Quellungsanisotropie in Funktion des exper. Dehnungsgrades v bei verschiedenen Quellungsgraden q. O = Bruchpunkt.

TABELLE II.

Quellungsanisotropie Q und mittlerer Orientierungswinkel α_m verschieden gequollener Fäden nach Dehnung um 100%.

Quellungsgrad	Q	α_m
11.3	± 48	7°.5
5.5	25.3	11°.2
2.2	10.2	13°.5
1.1	5.0	25°.3

Bei gleicher prozentualer Dehnung ist die Orientierung umso vollständiger, je nachdem der Quellungsgrad des gedehnten Fadens höher liegt! Genau das gleiche Resultat ergeben die optischen- [20]) und Röntgenversuche [21]). Dieses bemerkenswerte Ergebnis wird noch interessanter und aufschlussreicher, wenn folgendes berücksichtigt wird.

Aus dem gesamten Versuchsmaterial darf geschlossen werden, dass beim Trocknen eines orientierten, gequollenen Fadens sein Orientierungsgrad nicht (oder höchstens nur in ganz geringem Masse) verändert.

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[19]) P. H. HERMANS, Kolloid-Z. **88**, 68 (1938).

[20]) P. H. HERMANS und P. PLATZEK, 1939 loc. cit.

[21]) Bisher unveröffentlichte Arbeit, zusammen mit O. KRATKY.

Trocknen wir nun einen primären stark gequollenen Gelfaden ($q = 11.1$) einmal im isotropen und andermal im stark gedehnten Zustand (siehe Figur 2), so ergibt sich folgendes. Der isotrope Gelfaden 1 von der Länge 224 mm und mit dem Quellungsgrad $q = 11.1$ schrumpft isotrop zu dem isotropen trockenen Faden 3 von der Länge $224 \times 11.1^{-1/3}$ mm und wird daher 100 mm lang. Der anisotrope Gelfaden 2, von der Länge 464 mm, der durch eine Dehnung $v = 2.07$ aus 1 entstanden ist, schrumpft dagegen anisotrop (viel mehr in der Dicke als in der Länge) und wird 390 mm lang (Faden 4). Vergleichen wir nun die beiden trockenen Fäden 3 und 4, so ist 4 um 3,9 mal länger geworden als 3. Dem experimentellen Dehnungsgrad $v = 2.07$ entspricht daher ein „auf den trockenen Zustand bezogener Dehnungsgrad" $v_t = 3{,}9$. Es ergibt sich nun bei allen Versuchen, dass für sämtliche Quellungsgrade der Orientierungsgrad immer die gleiche Funktion von v_t ist, d.h. dass die Kurven der Fig. 1 zu einer einzigen, für den Verlauf der Orientierung charakteristischen Kurve zusammenfallen, wenn man v durch v_t ersetzt. (Siehe Fig. 3.) Wir sehen daraus, dass die verschiedenen Objekte beim Dehnen zwar umso eher den Bruchpunkt erreichen, je nachdem der Quellungsgrad niedriger liegt, aber einen identischen Verlauf der Orien-

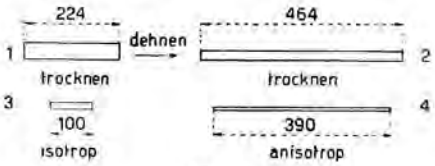

Fig. 2.

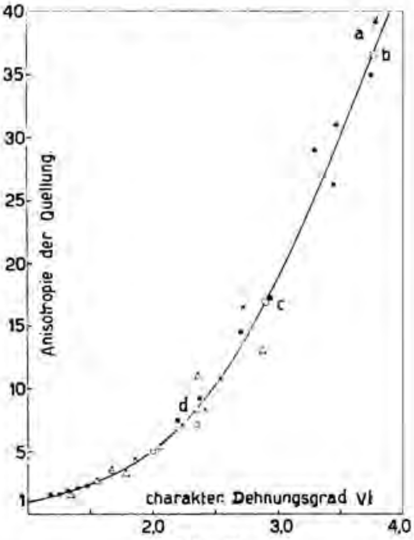

Fig. 3.

Quellungsanisotropie in Funktion des charakteristischen Dehnungsgrades v_t für die gleichen Objekte des Fig. 1. O = Bruchpunkt.

tierung in Funktion von v_t aufweisen[22]. Die Bruchpunkte haben dabei keine genau definierten, sondern von Versuch zu Versuch ziemlich schwankende Lagen, die sich ausserdem, wie ein oben schon angeführtes Beispiel zeigte, bisweilen durch Kunstgriffe verschieben lassen und u.a. auch von der Zuggeschwindigkeit abhängen. *Die charakterische Orientierungskurve bleibt dabei jedoch immer dieselbe.* Das „Brechen" ist eben ein sekundäres, den Orientierungsvorgang störendes Erreignis, das eine Sonderbetrachtung erfordert und an dem wir hier vorübergehen.

Man darf aus alldem schliessen, dass dem lufttrockenen isotropen Faden eine viel höhere „potentielle Dehnbarkeit" zugeschrieben werden muss als sich (wegen mangelnder „innerer Beweglichkeit") experimentell erreichen lässt. Es ist ein wichtiges, mit dem Feinbau und dem Quellungsmechanismus eng verbundenes Problem für sich, wieso man — bei Erhöhung der inneren Beweglichkeit durch Quellung — zwar tatsächlich viel höhere Orientierungsgrade erzielen kann, aber dazu mit zunehmender Quellung immer kleinere Dehnungen anzuwenden braucht. Wir gehen an dieser, für die Theorie der Gele zweifelsohne wichtigen Frage einstweilen vorüber und hoffen in einer anderen Arbeit darauf zurückzukommen.

Dehnt man also in hochgequollenem Zustand, aber betrachtet das Ergebnis der Dehnung nach Ueberführung der Objekte in den trockenen Zustand, dann lassen sich auch bei der Cellulose ausserordentlich hohe Dehnungsgrade experimentell realisieren. Im Beispiel der Fig. 2 war dieser Dehnungsgrad 3.9 (290 Proz.). Wir haben aber Dehnungsgrade v_t bis zu 7 (600 Proz.) beobachtet. Diese hohe Dehnbarkeit der Cellulose erfordert, zumal sie — wie noch unten zu zeigen ist — prinzipiell umkehrbarer Natur ist, für den entquollenen Zustand die Annahme geknäuelter Kettenmoleküle. Es lässt sich zeigen, dass Modelle mit starren Stäbchen einer Dehnbarkeit von weit über 100 Proz. nie gerecht werden können.

Die Analogie zum Kautschuk ergibt sich ebenfalls, wenn man „Kraft-Dehnungsdiagramme" gequollener Fäden in Funktion von v darstellt. Fig. 4 zeigt davon ein charakteristisches Beispiel[23].

5. *Umkehrbarkeit des Dehnungsvorganges bei der Cellulose.*

Im vorangehenden haben wir das weitgehend analoge Verhalten zwischen Cellulose und Kautschuk in folgender Beziehung dargetan:

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[22] Den Dehnungsgrad v_t haben wir daher den „charakteristischen" Dehnungsgrad genannt. In früheren Arbeiten (Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **42**, 798 (1939); Kolloid-Z. **89**, 344, 349 (1939)) haben wir schon gezeigt, dass auch der merkwürdige Verlauf der Kraft-Dehnungsdiagramme dieser Fäden überhaupt erst verständlich wird, wenn man auch dort v_t statt v als Dehnungsmass einführt. Auch hieraus ergibt sich in überzeugender Weise, dass v_t nicht etwa eine willkürlich gewählte, sondern eine mit dem Feinbau und dem Quellungsmechanismus eng verbundene Grösse darstellt.

[23] Die Kurven der kautschukartigen Substanzen sind dem Buche von R. HOUWINK: Elasticity, Plasticity and Structure of Matter, Cambridge 1938, S. 226 entnommen.

1. Ursprünglich amorphes Röntgendiagramm übergehend in ein „kristallines" Debye-Scherrer-Diagramm beim Entquellen bezw. Abkühlen und ein „kristallines" Faserdiagramm beim Dehnen, das sich dann beim Entquellen bezw. Abkühlen weiter verstärkt 24).

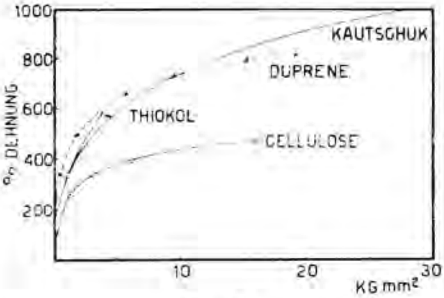

Fig. 4.
Zugspannung-Dehnungsdiagramme von vulkanisierten Kautschuk, Dupren, Thiokol und gequollener primärer Cellulosefäden in gleichem Masstab. (Spannung auf wirklichen Querschnitt bezogen).

2. Sehr hohe Dehnbarkeit mit uniaxialer Orientierung der Kettenmoleküle.
3. Bis auf fast Null zurücklaufende Dehnbarkeit beim Entquellen bezw. Abkühlen.
4. Aehnlicher Verlauf der Kraftdehnungskurven.

Wir haben uns nun noch mit der Reversibilität des Dehnungsvorganges auseinanderzusetzen. Haben wir mit unseren Betrachtungen das richtige getroffen, so müsste es auch bei der Cellulose prinzipiell möglich sein den Orientierungsvorgang rückläufig zu machen und zwar unter Verkürzung der Präparate bis auf ihre ursprüngliche Länge, falls ein ununterbrochenes molekulares Netz mit festen Haftpunkten vorliegt, oder bis auf eine grössere Länge wenn auch ein Fliessen stattgefunden hat (vgl. vulkan. und unvulkan. Kautschuk). Dass bei der Cellulose der Rückweg nicht spontan eingeschlagen wird 25), ist daraus zu verstehen, dass die neuen Berührungsstellen, die auf dem Hinweg zwischen den gleichgerichteten Ketten entstehen, infolge der hohen Kohäsions-Kräfte, die der Kohlenhydratkette innewohnen, fixiert bleiben, genau so wie beim gefrorenen orientierten Kautschuk. Sowie man diesen durch Temperaturerhöhung zur spontanen Rückfederung bringen kann, gelingt dasselbe bei der Cellulose durch starkes Aufquellenlassen. Diese Erscheinung wurde von uns schon

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24) Für das einschlägige Verhalten des Kautschuks, siehe P. A. THIESSEN und W. WITTSTADT: Z. physik. Chem. B **41**, 33 (1938).

25) Eine partielle spontane Rückfederung wird allerdings bekanntlich beobachtet. Sie nimmt mit dem Quellungsgrad der Präparate stark zu und wurde schon früher von uns als kautschukähnlich gedeutet. Siehe P. H. HERMANS und A. J. DE LEEUW, Kolloid-Z. **81**, 300 (1937); **82**, 58 (1938). K. H. MEYER hat schon früher diese Möglichkeit erwogen, Kolloid-Z. **59**, 214 (1932).

früher unter der Bezeichnung Aufquellungsretraktion beobachtet und beschrieben [26]). Es wurden dabei zwar beträchtliche, aber keine vollkommenen Rückfederungen erhalten. Neuerdings ist es uns jedoch gelungen, bei geeigneten Objekten einen Rückgang der Deformation bis nahe zum isotropen Zustand zu erzielen [27]). Untersucht man den Orientierungsgrad, so ergibt sich, dass dieser auf dem Rückweg wieder genau derselben Funktion vom charakteristischen Dehnungsgrad v_t entspricht, wie auf dem Hinweg. Es findet daher wirklich eine Umkehrung des Orientierungsvorganges statt und man bemerkt keinerlei Andeutung dafür, dass auf dem Hinweg ein Fliessen im oben erwähntem Sinne stattgefunden hat. Auch bei der regenerierten Cellulose müssen daher durchgehende Netze von durch feste Haftpunkte mit einander verbundenen Hauptvalenzketten angenommen werden. Welcher Art diese Haftpunkte sind, sei dahingestellt [28]). Die Hydratcellulose ist offenbar im Wesen ein nahezu vollkommen elastischer Körper [12]), aber einer mit sehr grossem innerem Widerstand.

Die Aufquellungsretraktion der Cellulose ist molekular-mechanisch den analogen, bei Eiweisstoffen beobachteten, Erscheinungen weitgehend vergleichbar, wie z.B. der Kontraktion des Kollagens [29]), der Rückfederung gestreckter Fibrinfäden beim Aufquellen [30]) und den ähnlichen Vorgängen beim Muskeleiweiss und beim Keratin des Haares. Ebenso wie die polare Cellulose, können auch die fibrillären Eiweisskörper ihre hochelastischen Eigenschaften erst im gequollenen Zustande entfalten. Wir wollen noch darauf hinweisen, dass KARGIN und MICHAILOV [31]) auf unabhängigem Wege schon vor einem Jahre zu ganz ähnlichen Auffassungen über den Feinbau der Cellulose gekommen sind.

6. *Schlusswort.*

Wir halten es für erwiesen, dass beim Entstehen und bei der Deformation künstlicher Fasern molekulare Vorgänge die entscheidende Rolle spielen. Die Micellen sind durch Rekristallisation entstandene sekundäre Gebilde, die zwar als Bestandteile der fertigen Faser zu berücksichtigen sind, aber deren Charakter erst in ergänzendem Sinne bestimmen. Die Streitfrage über die molekulare oder micellare Natur der Spinnlösungen dürfte zugleich damit zugunsten der ersteren entschieden sein;

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[26]) P. H. HERMANS und A. J. DE LEEUW, loc. cit.

[27]) Unveröffentlichte Versuche.

[28]) Wahrscheinlich kommen dafür Wasserstoffbindungen in bestimmter Konfiguration in Frage; vgl. die jüngsten Ansichten von K. FREUDENBERG über den Bau der Kohlenhydrate, Ber. **73**, 609 (1940).

[29]) Siehe z.B. A. KÜNTZEL und F. PRAKKE, Biochem. Z. **266**, 243 (1933).

[30]) U. EBBECKE, Kolloid-Z. **91**, 134 (1940).

[31]) V. A. KARGIN and M. V. MICHAILOV, Acta Physicochim. URSS **11**, 343 (1939). Vgl. P. H. HERMANS, Die Naturwiss. **28**, 264 (1940).

diese Lösungen enthalten keine gittermässig geordneten Teilchen. Damit ist natürlich noch gar nichts gesagt über die Möglichkeit des Vorkommens kinetisch einheitlicher Assoziationen von Fadenmolekülen in der Lösung.

Der Autor ist dem HOOGEWERFF-Fonds und Herrn Prof. Dr. H. R. KRUYT, Utrecht, verpflichtet für die Unterstützung, bezw. Förderung der Untersuchungen; Herrn Dr. P. PLATZEK dankt er für seine wertvolle Mitarbeit.

Breda-Ginneken, im August 1940.

Botany. — *The possible occurrence of carbonic anhydrase in green leaves.* By W. F. H. M. MOMMAERTS. (Communicated by Prof. L. G. M. BAAS BECKING.)

(Communicated at the meeting of September 28, 1940.)

§ 1. *Introduction.* The green plant takes up CO_2 from the atmosphere as a starting material for the processes of photosynthesis. According to some of the schemes for the mechanism of photosynthesis, which have been proposed, CO_2 enters the reactions as such, whereas in other schemes it does so in the hydratated form. An intermediate standpoint has been taken by BAAS BECKING & HANSON (1) who consider the hydratation of CO_2 as the BLACKMAN-reaction of photosynthesis.

According to the latter assumption the enzyme carbonic anhydrase would be of importance in photosynthesis, being the "enzymefactor" responsible for a number of photosynthetic characteristics. But also if CO_2 enters photosynthesis in the hydratated form, the presence of the carbonic anhydrase would be needed, because, according to BURR's (2) calculation, photosynthesis proceeds some hundreds to 2000 times as fast as hydratation of CO_2 could proceed under comparable conditions.

BURR himself reports negative results of his attempts to demonstrate carbonic anhydrase in plant tissues. The question therefore could be taken as definitely settled, if not contradictory results had been published by NEISH (3). Therefore it may be useful to describe the investigations about this question which were carried out in 1938 in the Botanical Laboratory in Leiden.

§ 2. *Methodical.* For the determination of carbonic anhydrase there are three groups of methods (4): gasometric and colorimetric methods, and one which makes use of the HARTRIDGE & ROUGHTON rapid reaction apparatus. The first and last groups are especially suited for kinetic experiments. The colorimetric methods are simple, and are considered to be more sensitive (5). Only the colorimetric method was available. The principle is as follows:

In a U-shaped tube (fig. 1), the following two stock-solutions are introduced, separated by mercury:

A: $NaHCO_3$, 0,02 N, with some drops alcoholic phenolred solution.

B: CO_2, 0,005 N.

The tube is cooled in ice for 10 minutes, and thereupon the fluids are mixed by suitable shaking. The p_H of the mixture becomes lower than that of A, causing a colour-change of the indicator to yellow, but the speed of this transmutation is limited by the reaction $CO_2 + H_2O \rightarrow H_2CO_3$.

Therefore, when no catalyses is present, the complete colour-change of the indicator takes some time, e.g. 1—2 minutes. The acceleration of the phenomenon is an indication of the presence of a catalyst.

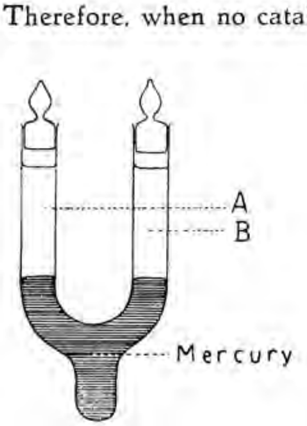

Fig. 1.

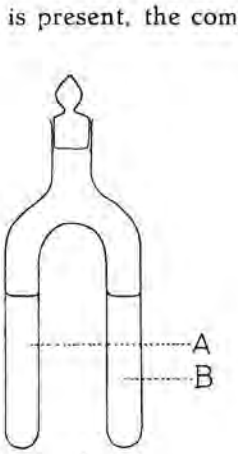

Fig. 2.

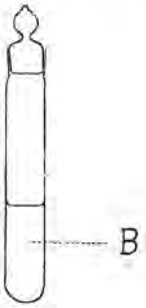

Fig. 3.

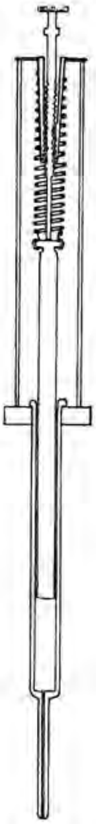

Fig. 4.

The method in this technical form has some disadvantages, especially the necessity of continuous cleaning of the mercury. One has tried to avoid this drawback by the use of a tube, as illustrated in fig. 2 but its use is to be excluded since the solutions exchange CO_2 during cooling; often a colour-change already occurs before mixing.

The following procedure has been devised:

About a dozen glass-stoppered tubes (fig. 3) of Jena-glass are filled for about 2/5 with solution B by means of a micro-pipette of KROGH—VAN DAM, as is used for micro-oxygendeterminations in water (6) (fig. 4). They are cooled in ice for 10 minutes, together with a little flask, containing solution A. Thereupon, with the same pipet-syringe, which has been thoroughly cleaned in the meantime, equal volumes of liquid A are mixed with the B already present in each tube. Mixing occurs practically instantaneous, as has been controled by neutralising a diluted acid, containing phenolphtalein, with alcaly. The enzyme-containing material may be admixed to A, or taken in the needle of the pipet, after the latter is filled with B, by a turn of the screw. The time is measured, in which the colour of the indicator becomes equal to that of a standard-tube, prepared at forehand in the same way. After some practice, the manipulations can be carried out with great uniformity. The results of a number of equal experiments may be treated with a statistical method. Control experiments with blood gave reaction-times of about 1 second.

With a photoelectric measurement of the colour-change it may also be possible to use the method for kinetic studies.

§ 3. *Experimental results.* Determinations of the carbonic anhydrase activity have been carried out with leaves of Spinacia oleracea L., Heracleum Sphondylium L., Lathyrus odoratus L. and Elodea canadensis Michx., using different modes of preparation, as leaf extracts with pure water, or with the addition of some $CaCO_3$, and with chloroplast-substance in crude preparations and in a purified form, as has been described in a previous publication (7) as preparations of chloroplast-grana.

Recently MENKE (8) has critisized this designation; MENKE found in preparations of intact chloroplasts a higher chlorophyllcontent as in the "grana". I must concede that his objections are sound, and that also my conclusions about the protein-chlorophyll compound are unproved. Apparently these preparations are contaminated with protoplasmic substance; their origin from grana has been proved by direct microscopic examination.

In all these cases, both in crude leaf-extracts, and in these preparations of contaminated grana, the results of the enzyme-determinations were negative, in accordance with the findings of BURR.

Table of some experimental results:

In all cases: CO_2, 0.005 N
$NaHCO_3$, 0.02 N
indicator: phenolred
leaf-extract: one turn of the screw of the pipette, filled with $NaHCO_3$ and indicator.
temperature: 0° C.

"Grana suspension" of Heracleum Sphondylium L. in aqua dest.
time of colour-change without extract:

36, 37, 30, 38 sec.

id., with suspension:

32, 42, 42, 34, 30 sec.

id., suspension heated:

30, 34, 24, 40, 36, 39 sec.

"Grana suspension" of Lathyrus odoratus L., var. Pinkii; precipitated with acetone (40°), suspensed in aqua dest.
time of colour-change without suspension:

89, 92, 90, 90 sec.

id., with suspension:

86, 90, 93, 94 sec.

id., suspension boiled:

± 90 sec.

"Grana suspension" of Lathyrus odoratus L. var. Pinkiï in aqua dest.
time of colour-change without suspension:

43, 50, 49, 47 sec.

id., with suspension:

47, 58, 30, 31, 34 sec.

id., suspension boiled:

42, 47, 37, 69, 56 sec.

Idem:
time of colour-change with suspension:

71, 53, 64, 68, 44, 60 sec.

id., suspension boiled:

57, 90, 85, 62, 64 sec.

Leaf extract of Spinacia oleracea L. (with addition of some $CaCO_3$) during the mincing of the leaves:
time of colour-change with suspension:

56, 61, 53, 51 sec.

id., suspension boiled:

69, 74, 61, 79, 70 sec.

In this case a very feeble acceleration is detectable. It is however not real: heating of the $CaCO_3$ drives of some CO_2, and makes the p_H more alcaline!

For comparison:
time of colour-change without addition:

76, 79, 81, 77 sec.

id., with blood:

nearly instantaneous (6 times).

§ 4. *Discussion.* The absence of carbonic anhydrase leads to the conclusion, that photosynthesis begins with CO_2 *as such*, as has been explained in § 1. On the other hand, the work of ARENS (9) clearly shows, that aquatic plants as Elodea, Vallisneria and others, take up bicarbonates from the water, and excrete carbonates or hydroxides, or, in other words, take up CO_2 in the hydratated form (HCO_3^-). The question in how far this phenomenon only applies to whole leaf, or may also be applied to the individual cells is of minor importance for the present considerations, but at some place in the plant the hydratated form must be converted in the CO_2-form. The question arises, whether the absence of carbonic anhydrase offers difficulties for the explanation of the photosynthetic rates of aquatic plants.

For the carbon dioxide-carbonic acid equilibrium the following equations hold (10):

$$CO_2 + H_2O \leftrightarrows H_2CO_3 \leftrightarrows H^+ + HCO_3^- \quad . \ . \ . \ . \ . \ (1)$$

hydratation reaction:

$$-\frac{d[CO_2]}{dt} = -k_{CO_2}[CO_2] + k_{H_2CO_3}[H_2CO_3] \quad . \ . \ . \ . \ (2)$$

this equation only holds good at $p_H < 8$. At $p_H > 8$ also:

$$= \frac{d[CO_2]}{dt} = -k_{CO_2 .OH}[CO_2][OH^-] + k_{HCO_3}[HCO_3^-] \quad . \ . \ (2a)$$

since here also the reaction $CO_2 \pm OH^- \leftrightarrows HCO_3^-$ goes on. At p_H 10.0, this equation predominates.

For the dehydratation the reversed equations may be derived. When, to simplifie the relations, in a case as photosynthesis, the concentration of the CO_2 or H_2CO_3 is kept zero, the intensities of resp. hydratation and dehydratation are represented by the equations:

$$-\frac{d[CO_2]}{dt} = -k_{CO_2}[CO_2] \quad . \ . \ . \ . \ . \ . \ . \ . \ (3)$$

$$-\frac{d[H_2CO_3]}{dt} = -k_{H_2CO_3}[H_2CO_3] \quad . \ . \ . \ . \ . \ . \ (4)$$

The last case (equation 4) is now under consideration. Assuming the intensities of photosynthesis of aquatic plants to be of the same order of magnitude as those of e.g. Sinapis and Helianthus (cited from BURR, loc. cit.), in which cases, (for equation 3) an acceleration of 100—250 times by the carbonic anhydrase was required, we may safely conclude, that in the case of aquatic plants the uncatalysed dehydratation reaction procedes with sufficient speed for the production of CO_2, because, at 18° C, the value of $k_{H_2CO_3}$ is about 500 times that of k_{CO_2} (10). The higher concentration involved, and the more alkaline p_H, at which also the reaction $HCO_3^- \rightarrow CO_2 \pm OH^-$ proceeds, corroborate this assumption in special cases.

Therefore it may be concluded that CO_2 as such takes part in photosynthesis, as it has been proposed in several schemes as f.i. by V. NIEL (11), GAFFRON and WOHL (12), ORNSTEIN c.s. (13) and WOHL (14).

Summary.

1°) The possible significance of a carbonic anhydrase for the mechanism of photosynthesis has been mentioned.

2°) A technical improvement of the colorimetric determination of carbonic anhydrase has been described.

3°) In green leaves no carbonic anhydrase is detectable, which leads to the conclusion that the process of photosynthesis starts with CO_2.

4°) This conclusion is not in disagreement with the findings of ARENS, concerning the fact, that aquatic plants take up CO_2 in the form of HCO_3^--ions, because the dehydratation of this HCO_3^--ions can proceed with sufficient speed without catalyses.

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Botany. — *Report on Fervorization of Plant Nutrient-Substrata.* By A. RADERMACHER and Z. KLAS, with the collaboration of Prof. V. VOUK. (Laboratory for Plant-physiological Research of the University of Zagreb, Yugo Slavia, Director Prof. V. VOUK.) (Communicated by Prof. G. VAN ITERSON Jr.)

(Communicated at the meeting of September 28, 1940.)

In this communication we give a short account of the behaviour of some plants in nutrient-substrata which have been heated to a high temperature. This heating of nutrient-substrata has been applied for a long time in plant-physiological, and especially in micro-biological, investigations. Generally this treatment has been called sterilization. For reasons which shall be given later, we suggest for this treatment the term *fervorization* [1]). Hereinafter, therefore, by fervorization shall be understood the heating of plant nutrient-substrata to a high temperature.

In the sense of this definition, soils, nutrient-solutions, water, and salts were fervorized and then used for the cultivation of plants. We reserve for ourselves the right to publish a more detailed report concerning these experiments in another place.

The incentive to our research was V. VOUK's (1) work on sterilization of cultures with brown coal. By virtue of his experiments, which took place at the laboratory for plant-physiological research of the University of Zagreb, Prof. VOUK stated, with special emphasis, that sterilization of the soil highly increases the yield.

I. *General remarks on the method of investigation.*

From a study of earlier work on the influence of heat on nutrient media it follows that an analysis was generally made in three directions, the aim being to explain the effect of heat sterilization by chemical, physical, or micro-biological changes in the soil.

The chief defect of this research work hitherto would seem to lie in the circumstance that all the factors possibly being active have not been examined separately. Only in one case time- and temperature-factors have been separately examined (without, however, yielding a result). It seemed important to us to examine the influence of heat on the nutrient salt components. This required the use of the classic water-cultures. Hereby one of the components which is regarded as being an essential factor in the influence of heat, the micro-biological one, is excluded. This makes it easier to interpret the results of the experiments.

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[1]) Fervor, latin: siedende Hitze, i.e. heating to the boiling temperature; ferve facere: to heat to the boiling temperature.

This method of physiological analysis with the aid of nutrient-solutions was the right one, as the first experiments already showed. By heating the v. d. Crone's nutrient-solution in the autoclave, and charging such a nutrient-solution with young *Sinapis alba* plants, we at once obtained the same increase of the yield, as had been formerly observed after the heating of soils. This striking result led us to a new interpretation of the influence of heat. As in our water-cultures we had almost entirely excluded the micro-biological influence of heat sterilization the result in our case was evidently *an effect of the high temperature and not of the absence of living germs.*

That the treatment of soils by high temperatures not merely leads to a micro-biological change, but also effects a change in other qualities of the soil, had already been observed by CZERMAK [1]. In his publication he therefore qualifies sterilization as "lediglich die Wirkung der Hitze auf die physiologischen Bodeneigenschaften".

For the heating of nutrient media, which generally leads to changes of a chemical, a physical and, also, a micro-biological nature, we should like to introduce the term "fervorization". The new state of the media is indicated collectively by the term *fervor-state*. This fervor-state induces an effect on plants which we will designate *fervor-effect*. This effect is manifested in different ways.

Further consideration of the experiments with water-cultures logically led us to divide the influence of heat on the nutrient-solution into the influence on the nutrient-salts and that on the water. Here, too, the first experiment showed that our way of reasoning was the right one. From this experiment and from other ones it followed that by mixing previously *fervorized water* with the unfervorized salts of the v. d. Crone-solution, the same fervor-effect was obtained, as was shown by the fervorized nutrient-solutions and by the fervorized soils.

The technic of our experiments was generally similar to that of heat sterilization, i.e. the substrata were heated in the autoclave to 137° C with a pressure of 2½ atm. during a certain time. Difficulties were experienced when the substratum had to be heated to 70° C and 100° C. The temperature of 100° C was reached in the autoclave without pressure, and that of 70° C by making a water bath in the autoclave.

In the following pages some of the experiments on the influence of fervorization on the development of plants will be briefly communicated.

II. *Experiments on the influence of fervorization of soils.*

In the literature there is already mentioned that the so-called sterilization increases the crop yield. In our experiments we wished to investigate the causes of this effect more thoroughly. Further, we endeavoured to answer some questions related to the germination, the growth of shoots, and the

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[1] CZERMAK, Landw. Versuchs. Stat. **76**, p. 91.

crop at various life periods of the plants, former investigations of different writers having, as a rule, been restricted to the first phase of development only. Furthermore, we wished to examine more exactly the effects of different temperatures in the presence of moisture, whereby the duration of the fervorization was to be reckoned with. We mainly wished to determine the duration of heating, and the temperature giving the most favourable effect.

In this communication we deal with experiments which have only been carried out with a soil which has been qualified as field-soil. This is a loamy, clayey, heavy soil, which had been taken from the experimental fields of the Botanical Gardens, the pH of which was 8.22. For the cultivation of plants we mixed 5 parts of this soil with 1 part of sand. The experiments were carried out with MITSCHERLICH's vessels.

a. *The influence on germination.* Seeds of *Sinapis alba* were bought, from which specimens of similar size were selected before they were put into the ground. 30—50 seeds were sown in rows at the same depth. The soil was fervorized twice, each time during 1 hour, at 137° C in the autoclave. For the purpose of comparison, seeds were also sown in unfervorized soil. Normal germination was equal to 70—80 %.

The results are given in Table I, in which the germination in unfervorized soil is put equal to 100 %. The results refer to 4 experiments.

TABLE I. Germination in 2 × 137° C fervorized field-soil after 4, 8 and 14 days, expressed in percentages. Number of germinated seeds in unfervorized field-soil = 100 %.

Number and date of experiment	4 days	8 days	14 days
1. (10.XII.38. — 10.VI.39)	37.50	101.67	
2. (31.XII.38. — 10.VI.39)	3.12	128.41	
6. (1.VI.39. — 5.VII.39)	47.06	163.63	156.25
7. (20.VI.39. — 4.VII.39)	32.20	113.92	113.92

If we examine the Table, we are first of all struck by the fact that on the fourth day of germination, a distinct retardation is seen, i.e. in fervorized soil on the whole there germinated far less than 50 % of the seeds which germinated in the unfervorized soil. After 8 days there was a considerable change. The percentage of germination in fervorized soil not only reached that in unfervorized soil, but even more or less surpassed it. After a further period of cultivation the results showed no change. Thus, the higher figure for the germination of *Sinapis alba* in fervorized soils may always be expected at the end of the experiment.

The *fervor-effect on germination, therefore, consists in a retardation of the germination in the beginning, and later on in an increase of the* *germination.* To arrive at a general conclusion with regard to the fervor-effect on germination these experiments will have to be made with the greatest variety of seeds.

Another interesting fact was observed in these germination experiments. In the first retardative phase of the fervor-effect the seedlings, especially the cotyledons of the germinated seeds, were generally smaller, though greener (better production of chlorophyll), and they produced more anthocyan at the underside. In addition it was noticed that the roots formed in this retardative phase were of a heterogeneous character. The root system in the fervorized soil showed a more or less rich ramification, more root-hairs and less growth of the tap-root. The tap-roots of the plants in normal soil, on the contrary, were long and thin, with few lateral roots and they produced many root-hairs. Similar facts had already been observed by RUSSEL and PETHERBRIDGE [1]). This difference in growth of the roots is a remarkable phenomenon, but, nevertheless, the roots of the seedlings in fervorized soil (Fig. 1 on Plate I) looked healthy, as might be concluded from the strong development of the root-hairs. Perhaps the stronger growth of plants in fervorized soil in the later stages of the development is partly due to the strong formation of the root system. Indeed the further discussion of the fervor-effect, especially that in nutrient-solutions, will show that, as a rule, the fervor-effect expresses itself more strongly in the development of the root system than in that of the shoot.

b. *The influence on the development of shoots.* Tables II and III show the fervor-effect on the development of shoots by heating at 137° C during 2 × one hour. Observations were made during the whole development of the plant till the flowering. In Table II the fervor-effect refers to the height

TABLE II. Fervor-effect on the height of shoots in 2 × 137° C fervorized field-soil, expressed in percentages. Height of shoots in unfervorized field-soil = 100 %.

Number and dates of experiments	20 d.	40 d.	62 d.	102 d.	113 d. Final height
8. (20.VI.39 — 13.X.39)	85.71	103.54	102.08	121.21	127.06

TABLE III. Fervor-effect on the dry weight of the shoots in 2 × 137° C fervorized field-soil, expressed in percentages. Dry weight in the unfervorized field-soil = 100 %.

Number and dates of experiments	20 d.	40 d.	62 d.	102 d.	113 d.
8. (20.VI.39 — 13.X.39)	—	134.40	146.54	259.34	345.47

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[1]) E. J. RUSSEL and PETHERBRIDGE, Journ. of Agric., Vol. **19**, No. 10, p. 909.

of shoots and in Table III to the dry weight of shoots. In both Tables it can be seen that the fervor-effect is visible after 40 days. Generally it may be said that, during the early development of the shoot, there is but little difference between plants in fervorized and in unfervorized soil to the advantage of the latter, and many negative experimental results of former investigators (e.g. MERKENSCHLAGER) 1) may be associated with this fact. Only in five weeks old cultures there are distinct differences to be seen in the height of plants in favour of those in fervorized soil. Moreover the effect expresses itself not only in a greater height but also in a generally strong, vegetative growth, and especially in the greater proliferance of the leaves, as is shown by the dry weight. As can be seen in Table III the dry weight of the crop increases from the 40th till the 113th day from 134.40 % to 345.47 %. The fervor-effect, therefore, is extraordinarily high. The crop increases to over 2½ times the normal. The same result was obtained in some experiments concerning the crop of seeds, on which we shall report in another place.

c. *The influence of the duration of fervorization.* Table IV shows the experimental results of the effect of the duration of fervorization at 137° C. These values refer to the height of shoots, to the fresh weight and to the dry weight. If, first of all we compare the rows of the end-effect (printed in bold type) after 113 days, the general result appears to be that the fervorization effect expresses itself more in the increase of the plant-mass than in the height. As for the influence of the duration of fervorization on the height of shoots, the greatest effect was observed at 8 × 137° C, and with regard to fresh and dry weight there seems to be an optimum between 3 × 137° C and 4 × 137° C (Figures on Plate II).

TABLE IV. Fervor-effect in 1 × 137° C, 2 × 137° C, 3 × 137° C, 4 × 137° C and 8 × 137° C fervorized field-soil after 51, 62, 102 and 113 days, expressed in percentages. Height and weight in the unfervorized field soil = 100 %.

	Field-soil experiment 8 (20.VI.39. — 13.X.39)											
	Height of shoots				Fresh weight of shoots				Dry weight of shoots			
	51 d.	62 d.	102 d.	113 d.	51 d.	62 d.	102 d.	113 d.	51 d.	62 d.	102 d.	113 d.
1 × 137° C	130.77	133.33	122.71	**136.76**	192.63	165.11	130 85	**220.07**	283.90	165.81	131.27	**240.95**
2 × 137° C	105.28	102.08	121.21	**126.47**	156.81	126.53	323.11	**352.10**	147.82	146.51	259.31	**345.47**
3 × 137° C	100.00	110.42	146.97	**135.29**	157.27	124.87	321.79	**368.98**	160.87	141.86	293.07	**362.03**
4 × 137° C	97.43	122.91	151.51	**136.76**	117.13	123.85	401.14	**483.12**	106.52	124.65	302.65	**417.99**
8 × 137° C	130.77	118.75	157.57	**144.11**	186.38	131.02	277.43	**364.01**	186.95	138.23	230.35	**337.52**

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1) MERKENSCHLAGER, Sinapis alba, München 1925.

The photo shows us one series of these experiments, and proves that the optimum has already been reached more or less after fervorizing three times during 1 hour at 137° C. The difference is most apparent if one compares the result with that of the 1 and 2 × fervorization. Anyhow the optimum was not yet reached when the soils were fervorized 2 ×, as we generally did in our experiments. This holds good only for the plant here examined (*Sinapis alba*), and for the soil which was used.

d. *The influence of the temperature of fervorization.* We observed the fervor-effect for the first time after heating in the autoclave at 137° C, and therefore we were very much interested to know, whether it would be possible to obtain the effect at a lower temperature, especially at temperatures below 100° C, i.e. temperatures also to be met with under natural circumstances.

The results of our experiments are combined in Tables V and VI, and if we compare the bold typed end-effects for the height of shoots, the fresh and the dry weight, we cannot fail to observe that the effect of fervorization at 70° C is already perceptible, though it is not nearly so strong as that of fervorization at 137° C.

TABLE V. Fervor-effect in 1 × 70° C, 1 × 100° C, 1 × 137° C fervorized field-soil after 62 and 102 days, expressed in percentages. The values for the unfervorized field-soil = 100 %.

Field-soil experiment 8 (20. VI. — 13. X. 39)						
	Height of shoots		Fresh weight		Dry weight	
	62 d.	102 d.	62 d.	102 d.	62 d.	102 d.
1 × 70° C	116.66	**125.75**	148.03	**111.77**	160.46	**114.58**
1 × 100° C	122.90	**124.24**	136.24	**139.32**	154.18	**132.81**
1 × 137° C	133.33	**122.71**	165.11	**130.85**	165.81	**131.27**

TABLE VI. Fervor-effect in 2 × 70° C, 2 × 100° C and 2 × 137° C fervorized field-soil after 62 and 102 days, expressed in percentages. The values for the unfervorized field-soil = 100 %.

Field-soil experiment 8 (20. VI. — 13. X. 39)						
	Height of shoots		Fresh weight		Dry weight	
	62 d.	102 d.	62 d.	102 d.	62 d.	102 d.
2 × 70° C	116.66	**118.18**	74.32	**161.62**	92.10	**169.28**
2 × 100° C	150.00	**122.72**	74.53	**156.42**	222.79	**125.79**
2 × 137° C	102.08	**121.21**	126.53	**323.11**	146.51	**259.31**

It is clear that in both series of experiments 2 × fervorization appears to be considerably more effective than 1 × fervorization. We may conclude to the possibility that fervorization takes place under natural circumstances, i.e. when the heating of the soil by the sun lasts a long time. The effect is the resultant of a time- and a temperature-component. (Figures on Plate III.)

The results of the experiments on fervorization of soils may be summarized as follows:

1. The fervor-effect on germination appears in two phases. In the beginning there is a retarded germination, lasting only for a few days and, after about 7—8 days, changing into a phase of stimulated growth. Further the retarding phase expresses itself in a delay in the development of the shoot, in an increase in the formation of chlorophyll and anthocyan, and especially in the ramification of roots and the formation of root-hairs.
2. The fervor-effect on the development of the shoot is expressed not so much in the development of height as in the generally stronger vegetative growth of all parts of the plant above the ground, particularly in the stronger development of the leaves. The fervor-effect is also noticeable in the development of flowers and seeds, as 2—2½ times greater crops are obtained.
3. The intensity of the fervor-effect is dependent on the duration of fervorization and also on the temperature. Maximal fervorization applied to *Sinapis alba* in the case of the soil used here (loamy, clayey, heavy soil) at 137° C is about 3—4 times fervorization during 1 hour. The fervor-effect is seen, to a lesser degree, after heating at a temperature of 70° C.

All these results refer to *Sinapis alba* only.

III. *Experiments on the influence of fervorization of nutrient-solutions.*

In this communication we deal with the fervorization of v. d. Crone's nutrient-solution. This is one of the best known nutrient-solutions, often succesfully applied in physiological research of the higher plants (BENECKE) [1]. For these researches it received special consideration because it contains an important undissolved residue. We also made some experiments with other nutrient-solutions, to which we shall refer in a more detailed publication.

Zea Mais (pure line isolated by Prof. Dr. TAVČAR, Zagreb–Maksimir), *Vicia Faba*, *Solanum lycopersicum*, *Sinapsis alba*, *Fagopyrum esculentum* and *Tagetes erecta* were our test plants. Some of the chief results of these researches will be communicated in the following tables. The values obtained with the normal unfervorized v. d. Crone's nutrient-solution are put equal to 100 %. Here we shall merely note that there is no great difference between the value of pH of the fervorized (varying from 5.08

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[1] BENECKE, Zeitschrift f. Bot. **1**, 235 (1909).

to 6.80) and that of the unfervorized v. d. Crone's nutrient-solution (pH 6.48).

According to the values in table VII all the test plants, with regard to the increase in height and in crop show a distinct fervor-effect. This was most marked in the height of shoots and in the total dry weight. Furthermore, we draw the attention to the high dry weight of the roots. Admittedly the test plants did not all react uniformly. For the height of shoots, *Vicia Faba, Solanum lycopersicum* and *Tagetes erecta* showed about 3—4 times higher values in the fervorized solution than in the unfervorized solution, and for the total dry weight more than the double values. The abnormal height of *Vicia Faba* is especially remarkable. *Sinapis alba* generally is slightly less responsive to fervorization. The fervor-effect of *Zea Mais* was the smallest; this may have been so owing to the small quantity of nutrient-solution.

TABLE VII. Results of the cultivations in 2 × 137° C fervorized v. d. Crone's nutrient-solution, expressed in percentages. The values for the unfervorized v. d. Crone's nutrient-solution = 100 %.

	Height of shoots	Length of roots	Fresh weight of shoots	Dry weight of shoots	Dry weight of roots	Total dry w.
Zea Mais (21.IV. — 17.VII.1939)	106.59	152.17	103.99	104.38	182.83	114.16
Vicia Faba (28.IV. — 27.VI.1939)	421.05	238.20	290.38	214.55	225.00	216.06
Sinapis alba II (29.IV. — 12.VI.1939)	208.79	—	114.85	123.80	225.92	131.05
Solanum lycopersicum (30.IV. — 28.VI.1939)	342.04	101.61	188.34	200.00	306.45	220.25
Tagetes erecta II (25.VIII. — 6.IX.1939)	293.29	137.03	228.13	211.70	404.71	231.42

Tables VIII and IX show the results of fervorization at different temperatures and of different duration. The values refer, as in the preceeding Tables, to the height and to the crop. If these values in Table VIII are compared with each other, it is evident that when fervorizing once at temperatures of 70° C, 100° C and 137° C, the fervor-effect clearly appeared at 137° C. At 70° C it is hardly perceptible and expresses itself distinctly only in the dry weight of the roots. (Figures on Plate II.)

We obtained about the same result with fervorization at 100° C. The fervor-effect was strongest with *Sinapis alba* and weakest — probably for the above mentioned reason — with *Zea Mais*.

As regards Table IX, here we see a similar result as in Table VIII, i.e. the fervor-effect appeared strongest at 2 × 137° C. At 2 × 70° C and at 2 × 100° C the values fluctuate. But, if we compare the results of 1 × 137° C and 2 × 137° C, the increase of the fervor-effect at 2 × 137° C is striking.

TABLE VIII. Results of the cultivations in 1 × 70° C, 1 × 100° C and 1 × 137° C fervorized v. d. Crone's nutrient-solution, expressed in percentages. The values for the unfervorized v. d. Crone's nutrient-solution = 100 %.

	Height of shoots			Length of roots			Fresh weight of shoots			Dry weight of shoots			Dry weight of roots			Total dry weight		
	70°	100°	137°	70°	100°	137°	70°	100°	137°	70°	100°	137°	70°	100°	137°	70°	100°	137°
Zea Mais (21.IV. – 17.VII.1939)	71.43	109.89	79.12	86.96	121.74	117.39	94.99	81.82	65.95	85.77	67.58	70.17	107.57	100.50	120.71	88.48	71.68	76.46
Vicia Faba (28.VI. – 27.VI.1939)	96.48	83.16	249.12	114.65	114.61	175.28	109.62	102.35	194.39	97.18	83.10	158.69	122.22	150.00	172.22	100.80	92.77	160.65
Sinapis alba II (29.IV. – 12.VI.1939)	125.94	160.67	252.30	—	—	—	118.59	131.25	125.96	124.50	137.96	111.61	203.70	203.70	150.00	130.13	142.63	114.34
Solanum lycopersicum (30.IV. – 28.VII.1939)	109.09	103.41	256.82	104.84	106.45	108.06	91.75	94.85	196.21	98.48	95.45	205.30	109.68	116.13	232.26	100.61	99.39	210.43

TABLE IX. Results of the cultivations in 2 × 70° C, 2 × 100° C and 2 × 137° C fervorized v. d. Crone's nutrient-solution, expressed in percentages. The values for the unfervorized v. d. Crone's nutrient-solution = 100 %.

	Height of shoots			Length of roots			Fresh weight of shoots			Dry weight of shoots			Dry weight of roots			Total dry weight		
	70°	100°	137°	70°	100°	137°	70°	100°	137°	70°	100°	137°	70°	100°	137°	70°	100°	137°
Zea Mais (21.IV. – 17.VII.1939)	72.53	72.53	106.59	132.61	123.91	152.17	96.51	64.29	103.99	72.54	66.93	104.38	117.17	80.81	182.83	78.10	68.66	114.16
Vicia Faba (28.IV. – 27.VI.1939)	78.95	143.86	421.05	82.02	132.58	238.20	72.20	201.97	290.38	68.54	138.03	214.55	63.89	225.00	225.00	108.03	150.60	216.06
Sinapis alba II (29.IV. – 12.VI.1939)	137.66	185.36	208.79	—	—	—	137.62	112.42	114.85	130.03	119.41	123.80	155.55	159.26	225.92	131.84	122.24	131.05
Solanum lycopersicum (30.IV. – 28.VII.1939)	104.55	129.55	342.04	133.87	119.35	101.61	90.61	137.32	188.34	103.79	144.70	200.00	96.77	138.71	306.45	102.45	143.56	220 25

The experiments show that in fervorizing nutrient-solution, time- and temperature-components are of equally great significance as in the fervorization of soils. As further variations of time and temperature components were not made, it is impossible to say anything about the optimum of the influences of these components.

It was established that:

1. By fervorization of the v. d. Crone's nutrient-solution at 137° C, distinct fervor-effects were obtained with *Vicia Faba*, *Solanum lycopersicum*, *Tagetes erecta* and *Sinapis alba*.
2. *The fervor-effect of cultures in the v. d. Crone's nutrient-solution is the same as in soil-cultures.* It appeared in the height of the plants and in the weight of the crop, but more especially in the dry weight of the roots.
3. It appears that also in this case duration and temperature of the fervorization are of importance. Hitherto the strongest fervor-effect was obtained by fervorizing twice during 1 hour at 137° C (Figures on Plate III).

IV. *Experiments on the influence of fervorization of the water.*

In continuation of the physiological analysis of the fervorization, experiments were made in which the two components of the v. d. Crone's nutrient-solution, i.e. the salts and the water were fervorized separately. The distilled water was fervorized twice during one hour at 137° C and, after cooling down to roomtemperature, the salts were added. These experiments, carried out with *Sinapis alba*, *Fagopyrum esculentum* and *Tagetes erecta*, had very remarkable results (Table X).

It appeared *that fervorization of the water was sufficient to produce the fervor-effect.* From the Table, as well as from the photos (Figures on Plate IV), it follows that the fervor-effect of the fervorized water was more or less equal to that of the fervorized v. d. Crone's nutrient-solution.

TABLE X. Fervor-effect in 2 × 137° C fervorized water with unfervorized v. d. Crone's nutrient-salts, expressed in percentages. The values for the unfervorized v. d. Crone's nutrient-solution = 100 %.

	Height of shoots		Length of roots		Fresh weight of shoots		Dry weight of shoots		Dry weight of roots		Total dry weight	
	H_2O, 2×137° + Cr. S.	Cr., 2×137°	H_2O, 2×137° + Cr. S.	Cr., 2×137°	H_2O, 2×137° + Cr. S.	Cr., 2×137°	H_2O, 2×137° + Cr. S.	Cr., 2×137°	H_2O, 2×137° + Cr. S.	Cr., 2×137°	H_2O, 2×137° + Cr. S.	Cr., 2×137°
Sinapis alba (14. V. — 4. VII. 1939)	249.41	279.68	—	—	77.49	110.56	106.51	139.96	90.99	92.79	104.94	134.88
Fagopyrum esculentum (24. V. — 17. VII. 1939)	179.20	217.11	77.01	121.84	138.02	156.20	150.69	160.64	336.11	347.22	160.38	170.39
Tagetes erecta II (25. VII. — 6. XI. 1939)	276.47	293.29	122.22	137.03	238.12	228.13	180.05	211.70	364.70	404.71	198.81	231.42

It is expressed in the height as well as in the crop of the plants. *Sinapis alba* showed noticeably smaller crops than *Fagopyrum esculentum* and *Tagetes erecta*. Both of the last-mentioned plants exhibit the fervor-effect especially in the dry weight of root, which was strikingly large in contrast to that of *Sinapis alba*.

With regard to fervorization of the salts, so far, one experiment has been carried out, but we do not consider the results sufficiently definitive.

It was established that *previous fervorization of the distilled water at 137° C to which the v. d. Crone's nutrient-salts are added is sufficient to produce the fervor-effect.*

General summary. If, at the end of our communication, we cast a restrospective glance, it seems to us that a new line of investigation has been established. As a matter of fact there exists already an extensive literature on the benefit of heating of soils — the so-called soil sterilization (RICHTER, KÖNIG, DIETRICH, SCHULZE, CZERMAK, RUSSEL and PETHERBRIDGE, BEWERLEY, a.o.) — and the results of those investigations have already been put into practice. We observe, however, that the cause of the amelioration of the soil was not clear hitherto. It was shown by our experiments with nutrient-solutions that this amelioration is not merely the result of sterilization, but also of general changes in the state of the nutrient substratum, caused by heating. To bring this effect to the fore we proposed the term "fervorization".

Heat sterilization is neither equivalent to chemical sterilization nor to sterilization by freezing. The results of the micro-biological cultures in moist soil sterilized by heat must be judged with caution, for they include not only the normal growth of the cultures but also the growth influenced by fervorization. Therefore it seems important to compare cultures on substrata, sterilized by chemicals, by freezing, and by heat.

We are quite alive to the fact that the investigations which we have described must be supplemented in many ways. Nutrient substrata and salts will have to be examined on their behaviour in connection with a far greater variety of plants.

Above all it will be necessary to throw more light upon the changes caused by the fervorization of water. We consider the investigation of this problem a task for chemists and physicists, whereas for physiologists it is of interest to examine whether fervorized water exercises influence on other phenomena of life.

Finally, it should be remembered that fervorization possibly takes place under natural circumstances. The great fertility of the otherwise poor soil in the tropics may be partly due to the fervorization of the soil caused by the radiation of heat rays by the sun. There may be similar conditions in other climates.

A. RADERMACHER AND Z. KLAS: REPORT ON FERVORIZATION OF PLANT NUTRIENT-SUBSTRATA.

PLATE I.

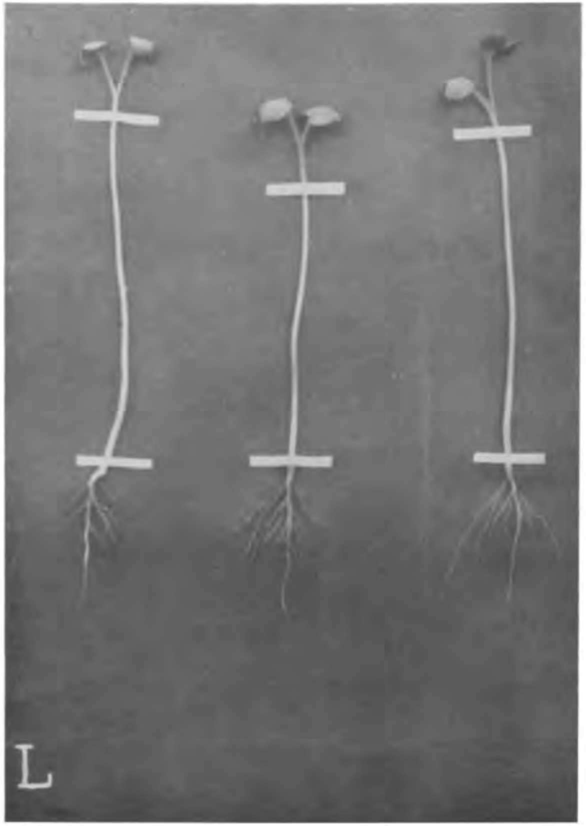


Fig. 1. 3 young seedlings of *Sinapis alba* in 2 × 137° C fervorized garden-mould.

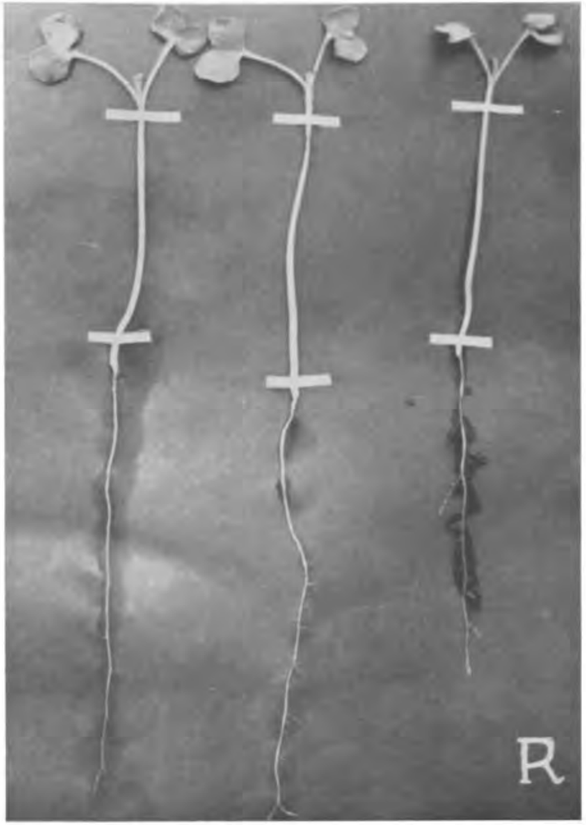
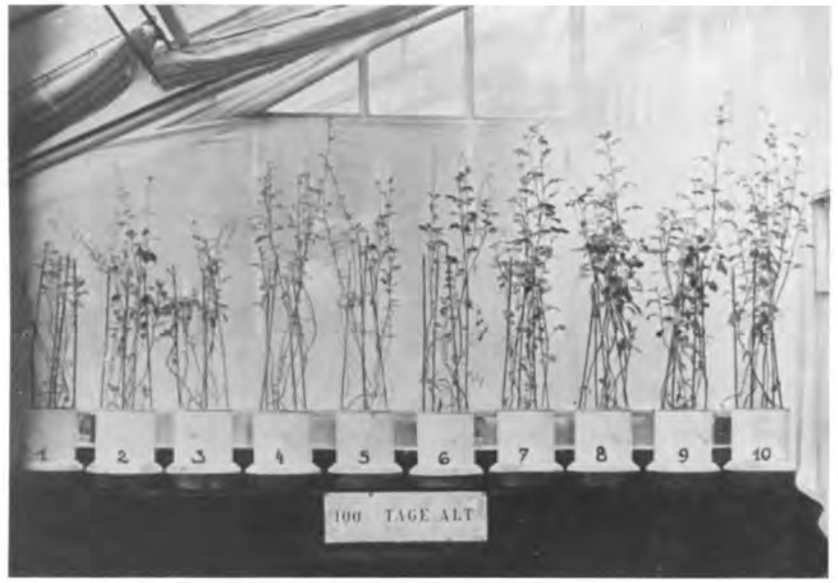


Fig. 2. 3 young seedlings of *Sinapis alba* in unfervorized garden-mould.

A. RADERMACHER AND Z. KLAS: REPORT ON FERVORIZATION OF PLANT NUTRIENT-SUBSTRATRATA.

PLATE II.

Sinapis alba in fieldsoil. 77 days old. From l. to r.: 1 = normal, 2 = 1 × 137° C, 3 = 2 × 137° C, 4 = 3 × 137° C, 5 = 4 × 137° C and 6 = 8 × 137° C.



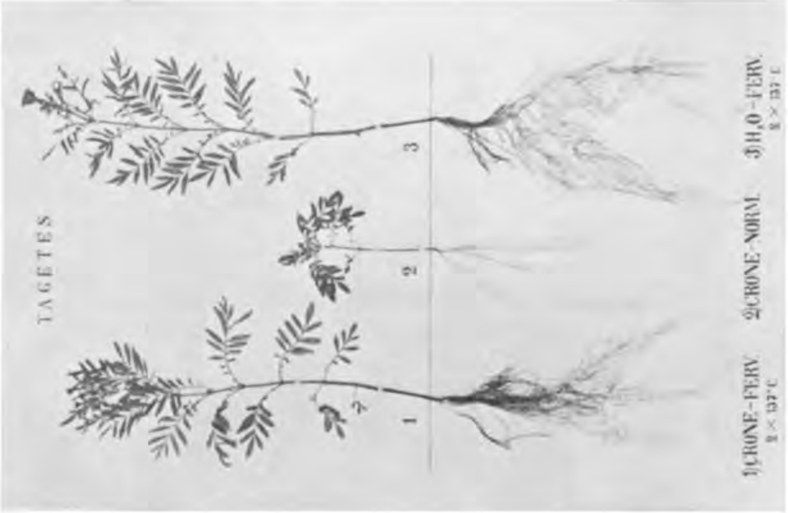
The whole row of the same culture after 100 days. 1 = normal, 2 = 1 × 70 C, 3 = 2 × 70° C, 4 = 1 × 100° C, 5 = 2 × 100° C, 6 = 1 × 137° C, 7 = 2 × 137° C, 8 = 3 × 137° C, 9 = 4 × 137° C and 10 = 8 × 137° C.

PLATE III.

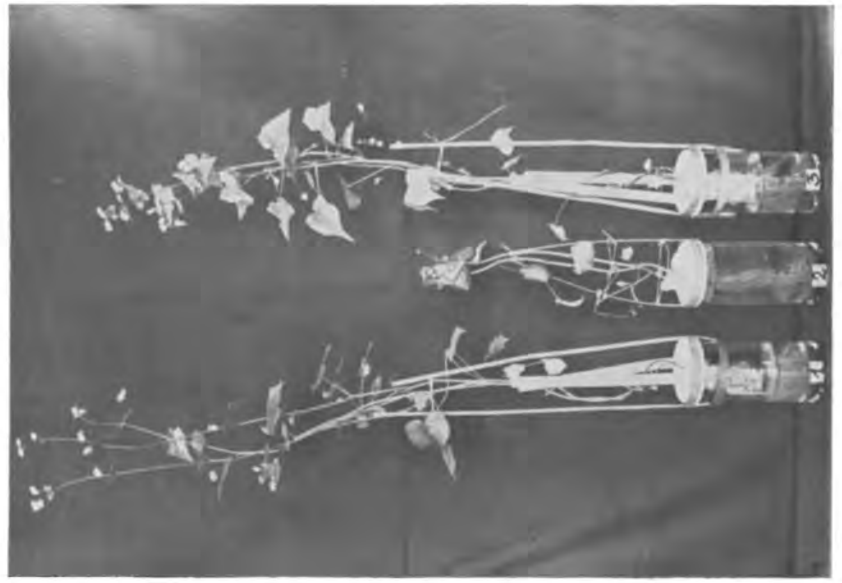
Sinapis alba in v. d. Crone's nutrient solution. 39 days old. From l. to r.: 1 = normal, 2 = 1 × 70° C, 3 = 2 × 70° C, 4 = 1 × 100° C, 5 = 2 × 100° C, 6 = 1 × 137° C, 7 = 2 × 137° C.

Vicia Faba in v. d. Crone's nutrient solution. 40 days old. From l. to r.: 1 = normal, 2 = 1 × 70° C, 3 = 2 × 70° C, 4 = 1 × 100° C, 5 = 2 × 100° C, 6 = 1 × 137° C, 7 = 2 × 137° C.

PLATE IV.



Tagetes erecta in v. d. Crone's nutrient solution.
Final effect after 74 days.



Fagopyrum esculentum in v. d. Crone's nutrient solution.
From l. to r.: 1 = 2 × 137° C, 2 = normal, 3 = H_2O,
2 × 137° C + v. d. Cr. salts (unfervorized).

Geology. — *On devonian limestones with Clathrodictyon cf. spatiosum and Heliolites porosus from Eastern Borneo.* By M. G. RUTTEN. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of September 28, 1940.)

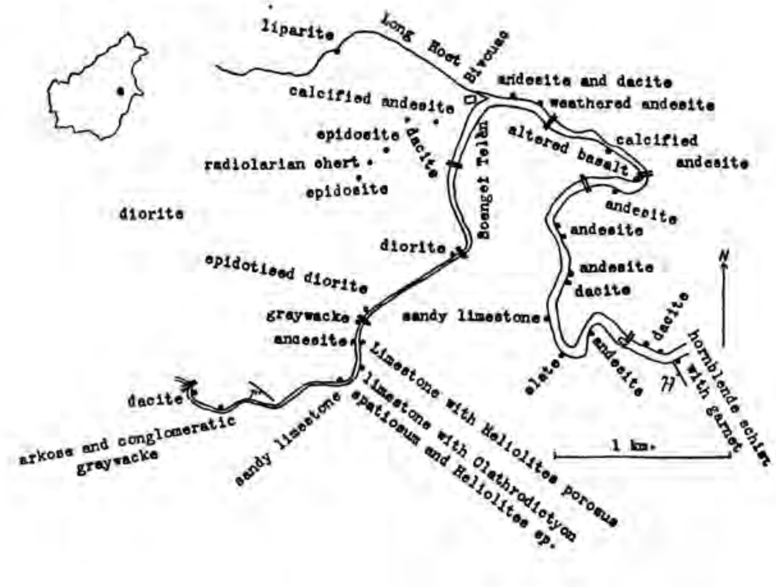
The fossils here described originate from the middle part of Eastern Borneo, where they were collected by Mr. H. WITKAMP, during the so-called "Midden Oost Borneo Expeditie" in 1925. They are stored at the "Mineralogisch Geologisch Instituut" at Utrecht.

The fossils are found in a dark, somewhat recrystallised limestone, which forms part of the formation of "Oude Leien" or Old Slates. This formation, which is built up of slates, quartzitic sandstones, graywackes and cherts which locally carry Radiolaria, is characterised by a regional metamorphism and belongs to the higher parts of the epizone. The Old Slates very seldom carry fossils, while up to the present, it has been impossible to trace distinct zones by the lithology of the components of the series. It is known to contain fossils of both Triassic and Lower Tertiary age in Western Borneo, and by the fossils here described, the range is extended downwards towards the Devonian. In fact the "Oude Leien" do not belong to a given formation, but in this group are included most of the lightly metamorphosed sediments from Borneo, the "Oude Leien" being distinguished by their facies only.

The fossils were found in the Soengei Telèn, a tributary of the Soengei Mahakam, just above the confluence of the Long Hoet with the Soengei Telèn (Soengei = Long = River). The exact locality is shown in the accompanying map, where also the rocks found in the neighbourhood are indicated.

The age of the Old Slates in this particular part of Borneo must be Lower or Middle Devonian, and most probably the series belongs to the Lower Devonian. The reasons are as follows. *Heliolites porosus* is known from the Lower and Middle Devonian of Europe (lit. 4, p. 58) and from the Middle Devonian of the Altai Region of Western Siberia (lit. 2, p. 27). *Clathrodictyon* almost exclusively occurs in the Silurian, and the species *Clathrodictyon spatiosum*, with which our fossil seems best to tally, is known only from Silurian erratic boulders from Northeastern Germany. We may now ask, if, in Borneo, *Clathrodictyon* has extended its range upwards, or *Heliolites porosus* downwards. According to KÜHN (lit. 3, p. 106) the Stromatoporoids tend to appear at a younger date, the farther East the fossils are found. It thus seems most probable, that in Borneo

Clathrodictyon lived during the Devonian also. We may assume that the fossils found define the Devonian, and, in accordance with the Silurian habit of the genus *Clathrodictyon*, it seems most probable that they belong to the Lower Devonian.



Map of a part of the Soengei Telèn. Borneo. After H. Witkamp.

Fig. 1.

My thanks are due to Prof. H. GERTH of Amsterdam, who helped me with advice and literature.

Clathrodictyon cf. spatiosum BOENHKE. (Pl. I, figs. 3, 4).

Cl. spatiosum BOENHKE, lit. 1, pp. 172, 173, figs. 19, 20.

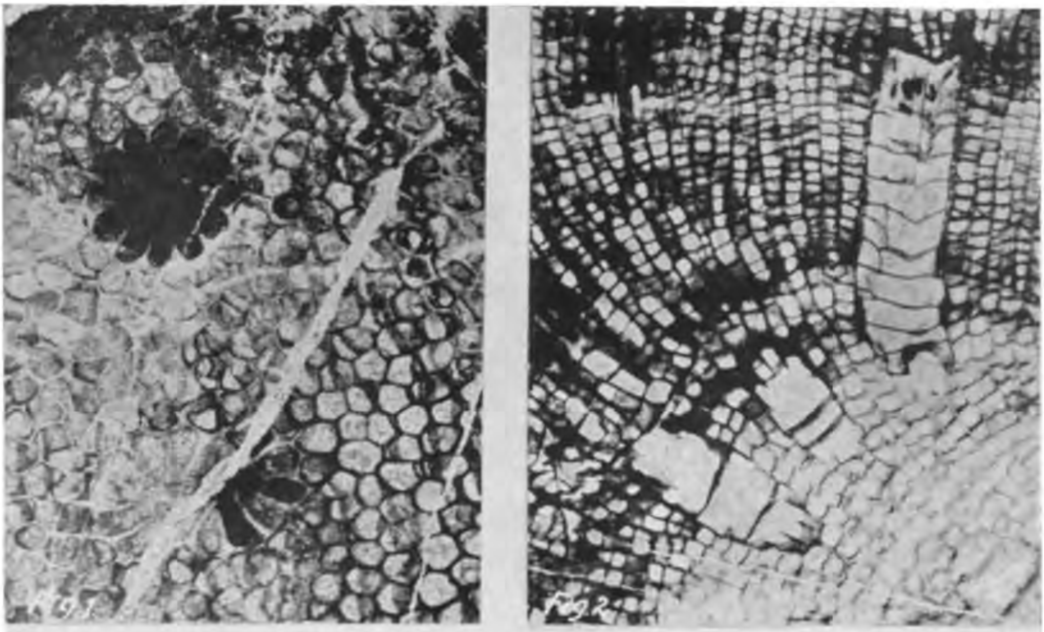
Coenosteum massive, large. I have only one specimen, which is roughly ovoid in shape, and measures 19 cm. by 10 cm., but it shows two fracture-planes, and must have formed a much larger fossil, when complete. The surface is eroded, and clearly shows the slightly flexuous laminae. The mode of attachment is unknown. There are no mamelons or astrorhizae to be seen, and zooïdal tubes are absent.

The internal structure could be studied from polished surfaces, and from two thin sections, from which the illustrations were taken. In vertical section the laminae appear as regular, continuous, rather thick lines, which are only slightly undulated, and show no signs of minute crumpling. The radial pillars are irregular, and do not reach beyond one interlaminar space. There are about two laminae to 1 mm., but the distance between the laminae is rather variable. In tangentional section we see the small round pillar-heads, with very few definite trabeculae.

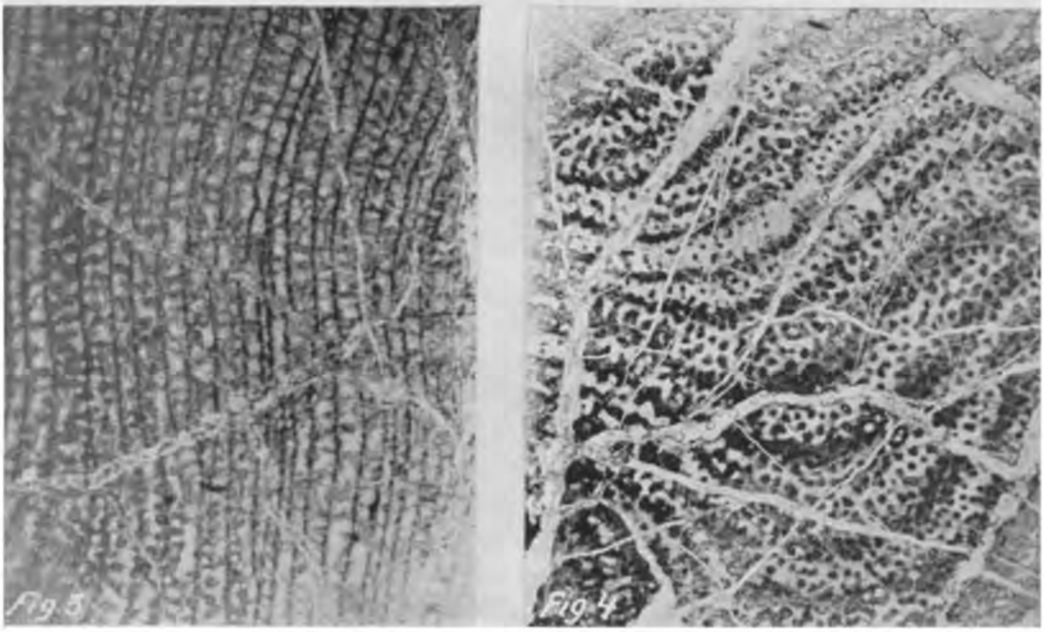
The general structure is very reminiscent of *Clathrodictyon spatiosum*,

M. G. RUTTEN: ON DEVONIAN LIMESTONE WITH CLATHRODICTYON CF. SPATIOSUM AND HELIOLITES POROSUS FROM EASTERN BORNEO.

PLATE I.



Figs. 1, 2. Two sections through a colony of *Heliolites porosus*. Fig. 1; horizontal section, × 10. Showing a calice in the upper left hand corner, and a second lower down, transversed by a calcite vein. Fig. 2; vertical section × 6, showing three calices.



Figs. 3, 4. Sections through *Clathrodictyon* cf. *spatiosum*. × 4. Fig. 3; vertical section. The horizontal laminae are placed vertically in this figure, the bottom of the fossil is to the left. Fig. 4; slightly oblique horizontal section, showing the round pillar-heads, with only rare trabeculae, between the slightly flaxuous laminae.

the differences are that in *Cl. spatiosum* the laminae are about 1 mm. apart, so that the internal structure is wider; and that the fossil from Borneo does not show dark median lines in the laminae and only rare trabeculae at the pillarheads. By these differences the specimen shows a transition towards the species *Clathrodictyon linnarsoni* (ROSEN) (lit. 5, p. 155, tab. XVIII, figs. 8—11*a*) or to *Clathrodictyon regulare* NICHOLSON (lit. 6, p. 5, tab. I, figs. 7, 8). These two species however are much smaller, whilst their internal structure is much more finely cut, so that our specimen tallies best with *Cl. spatiosum* BOENHKE.

Heliolites porosus GOLDFUSS. (Textfig. 2 and Pl. I, figs. 1, 2).

Heliolites porosus GOLDFUSS, in LINDSTRÖM, lit. 4, pp. 53—58, pl. II, figs. 29—37, pl. III, figs. 3—7.

Several specimens, embedded in a dark, somewhat recrystallised limestone. The corals can be studied only in thin-sections and on polished surfaces. The calcite of the corals is recrystallised to some extent also, tending to obscure their internal structure. Locally, however, the limestone is impregnated with black matter, and here the structure clearly stands out.

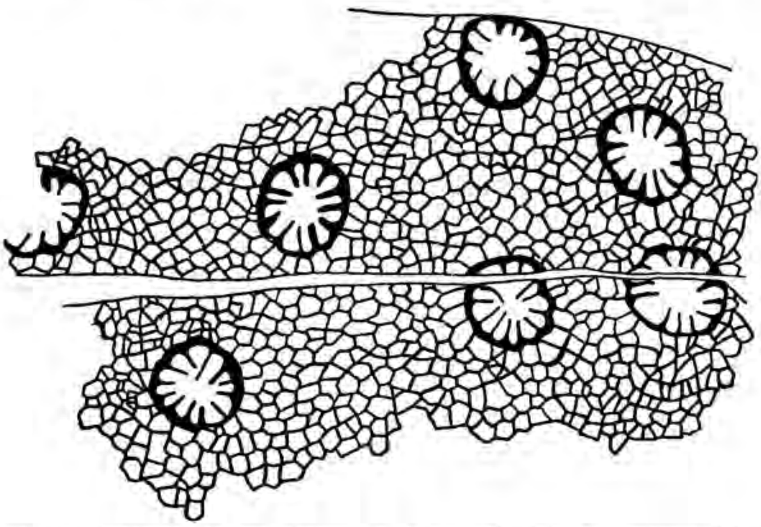


Fig. 2. Horizontal section through a colony of *Heliolites porosus*, × 5. Drawn by camera lucida.

The colonies attain about 5 cm. in diameter, the calices about 2 mm. Number of septa twelve. The length of the septa is variable, in some calices they reach halfway inside, in others they are short, so that the form is then reminiscent of the *Heliolites decipiens variety*. The distance between the calices is variable, but always larger than the diameter of the calices itself. The structure, which can be best studied on the accompanying figures, is identical with that of *Heliolites porosus*.

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Physiologie. — *Untersuchungen über die Erregungsleitung in Gastropodennerven* [1]). (Aus dem Institut f. vergl. Physiologie der Reichsuniversität Utrecht). Von B. J. KRIJGSMAN. (Communicated by Prof. H. J. JORDAN).

(Communicated at the meeting of September 28, 1940.)

Die Physiologie des Nerven- und Muskelsystems der Gastropoden bietet dem Forscher in verschiedener Hinsicht interessante Probleme. Ich erinnere bloss an die Kontraktilitäts- und Viskositätsphänomene der Muskeln dieser Tiere und an die dekrementielle Ausbreitung der Muskelkontraktion (neuere Uebersicht JORDAN 9). Um einen tieferen Einblick in die Leistungsfähigkeit des Nervensystems zu erhalten, haben JORDAN und LULLIES (10) die Reaktion des Fussnerven von Aplysia auf elektrische Reizung untersucht (Fussmuskelsystem als Indikator). Sie konnten bei Reizung mit einzelnen Induktionsschlägen keine absolute refraktäre Periode finden und bei Doppelreizung mit gleichgerichteten Stromstössen ohne Intervall fanden sie immer eine bedeutende Summation, welche bei zunehmendem Intervall (bis 2-3 msec) geringer wurde und darauf wieder langsam anstieg. Mit Narkoseversuchen stellten sie fest, dass eine allmähliche Abnahme der Leitfähigkeit auftritt. Aus diesen Befunden wurde geschlossen, dass die Erregung im Aplysianerven mit Dekrement geleitet wird.

Diese Resultate veranlassten mich eine Beantwortung der folgenden Frage zu suchen: *Welche elektrophysiologische Phänomene treten bei Reizung im Gastropodennerven auf?*

Methodik.

Für die Versuche wurde der N. intestinalis von Helix pomatia benutzt. Der Nerv wurde frei präpariert von der ihn umgebenden Muskelscheide (vgl. KRIJGSMAN 14-18) und in einer feuchten Kammer aufgestellt. Unpolarisierbare Zn-$ZnSO_4$-Elektroden (Kaolinspitzen mit Schneckenblut hergestellt) dienten als Ableitungselektroden; zur Reizung wurden Platinelektroden angewandt. Als Reizapparatur wurden grosse Induktorien verwendet mit 4 V. im primären Kreis. Die Registrierung fand statt mit dem Einthovenschen Saitengalvanometer; bei den langsam verlaufenden elektrischen Erscheinungen im Helixnerven ist Anwendung einer schlaffen Saite möglich. Die Ausschläge wurden mit einer Filmcamera aufgenommen; Zeitmarkierung mit einer Stimmgabel von 100 Hz. Zimmertemp. 18-21°. Die Ableitung war monophasisch.

Der N. intestinalis ist sehr schnell ermüdet und in dem Stück des

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[1]) Mit Unterstützung der Niederl. Regierung aus den Einkünften der „Zomerpostzegels".

Nerven, das sich zwischen den Reizelektroden befindet, werden manchmal sekundäre Kathoden gebildet und tritt leicht Blockierung des Aktionsstroms auf. Ich habe daher immer frisch isolierte Nerven verwendet und diese nur so lange für die Versuche benutzt, als Kontrollreize von konstanter Grösse ein konstantes Resultat ergaben. Ueberdies wurde nur mit absteigenden Reizen gearbeitet bei kleinem interpolarem Abstand.

Ruhe- und Aktionsstrom.

Das bei Ableitung zwischen einem abgetöteten Teil und der intakten Aussenseite des frischen N. intestinalis erhaltene Ruhepotential beträgt maximal 12 mV.

Reizt man den Nerven mit absteigenden Oeffnungsschlägen von zunehmender Stärke, so findet man einen zweigipfeligen Aktionsstrom. Dieser besteht aus den Aktionspotentialen von zwei Fasergruppen (vgl. Abb. 1): Eine Fasergruppe mit niedriger Schwelle, die sich bei 5-7 mal dem Schwellenwert der empfindlichsten Fasern voll entwickelt hat (Spannung 12 mV, Dauer 35 msec.) und eine zweite Gruppe, die sich von 12 bis 37 mal dem niedrigsten Schwellenwert entwickelt. Fortgesetzte Reizverstärkung ruft keine Aenderung dieses zweigipfeligen Aktionsstroms hervor; *der Nerv gibt, wie der Vertebratennerv, bei einer gewissen Reizstärke seine momentane Höchstleistung, die nicht überschritten werden kann.*

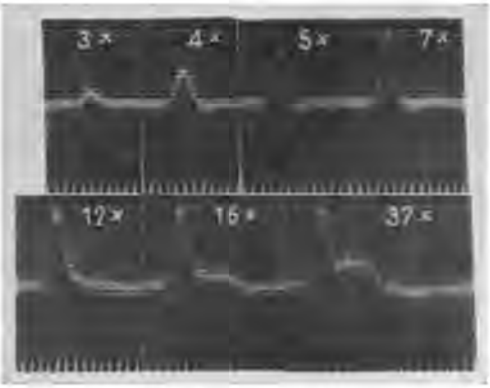

Abb. 1. Der Aktionsstrom im N. intestinalis von Helix pomatia bei zunehmender Reizstärke (Induktionsöffnungsschläge). Zuerst (bei einer Reizstärke bis zu 7 mal dem Schwellenwert) tritt eine schnelle Fasergruppe mit niedriger Schwelle in Aktion, darauf (bei 12—37 mal dem Schwellenwert) eine zweite, langsamere Gruppe mit höherer Schwelle. Die Reizstärke von 37 mal dem Schwellenwert ist maximal für alle Fasern des Nerven. Zeitmarkierung = 10 msec.

Fortpflanzungsgeschwindigkeit.

Die Geschwindigkeit der Aktionswellen im N. intestinalis von Helix habe ich berechnet durch Messung der Zeit zwischen Reizmoment und

Aktionswelle im Elektrogramm, wobei der Reizeinbruch zur Markierung des Reizmomentes diente. Abb. 2 gibt das Resultat von 54 Versuchen mit maximalen Reizen, bei verschiedenem Ableitungsabstand. Die Zahlen der Geschwindigkeit der schnellsten Fasern (P, vergleiche das in der Abb. eingezeichnete Schema des Aktionsstroms) bei verschiedenem Ableitungsabstand lassen sich mit geringer Streuung zu einer geraden Linie anordnen (Linie P). In 10 msec. werden rund 2.5 mm. zurückgelegt und in 25 msec. rund 9 mm. Daraus kann berechnet werden, dass die Leitungsgeschwindigkeit der schnellsten Fasern 40-48 cm/sec. beträgt. Für die mittlere Geschwindigkeit der schnellen Gruppe (Linie Q) fand ich rund 32 cm/sec., für die langsamsten Fasern der schnellen Gruppe (Linie R) 15-26 cm/sec., für die mittlere Geschwindigkeit der langsamen Fasern (Linie S) 7.7-8 cm/sec. und für die langsamsten Fasern (Linie T) 5.1 cm/sec. Die geringe Leitungsgeschwindigkeit hat zur Folge, dass schon nach dem Zurücklegen einer kleinen Strecke der Aktionsstrom eine grosse Dispersion aufweist (vgl. Abb. 7).

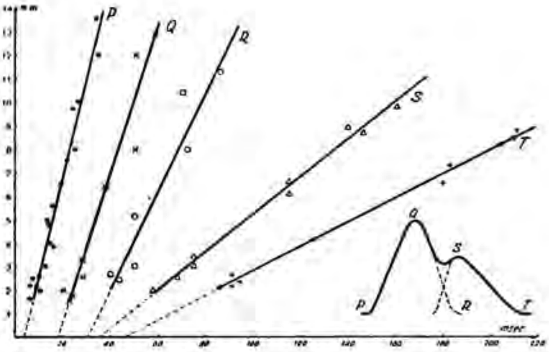

Abb. 2. Die Leitungsgeschwindigkeit von 5 verschiedenen Fasergruppen des N. intestinalis. Horizontal: Zeit in msec. Vertikal: zurückgelegte Strecke in mm. Eingezeichnet: Schema des Aktionsstroms.

Der Abb. 2. kann man weiterhin einen ungefähren Massstab für die Latenzzeiten des Aktionsstroms entnehmen. Zieht man die Geschwindigkeitslinien weiter nach unten durch (gestrichelt) bis zur Abzisse, so schneidet z.B. die Linie P die Abzisse bei 4 msec., d.h. die Latenz dieser schnellen Fasern beträgt 4 msec. Aehnlich zeigen die Fasern Q eine Latenz von 18 msec., die Fasern R von 30 msec., die Fasern S von 34 msec. und die Fasern T von 48 msec. *Die Latenzzeit ist somit desto grösser je höher die Schwelle liegt und je langsamer der Aktionsstrom geleitet wird.*

Die dicksten Fasern im N. intestinalis von Helix haben einen Durchmesser von 10-15 μ. Die grösste Leitungsgeschwindigkeit (40-48 cm/sec.) ist somit wahrscheinlich diejenige dieser dicken Fasern.

Die refraktäre Periode.

Die refr. Per. der Invertebratennerven und im allgemeinen der markarmen Nerven ist bedeutend grösser als bei myelinhaltigen Nervenfasern. Eine abs. refr. Per. von 5 msec. ist bei markarmen Nerven keine Seltenheit (z.B. LULLIES 20).

Im N. intestinalis von Helix habe ich die refr. Per. bestimmt durch Applikation von zwei gerade maximalen Einzelreizen, welche in verschiedenen Zeitintervallen dem Nerven durch dasselbe Elektrodenpaar zugeführt wurden. Zur Feststellung der Reaktionsgrösse wurde die Oberfläche der Aktionsstromkurve berechnet. Die Oberflächen der Kurven, die von bedingendem und Prüfreiz hervorgerufen worden waren, wurden zusammengezählt. Diese Summe wurde 100 % genannt und die Oberflächen der Kurven, die der Reaktion auf Doppelreize entsprachen, als Prozente dieses Maximalwertes berechnet. In der Abb. 3 sind die Resultate von Versuchen an 42 Nerven zusammengestellt.

Man sieht, wie bei einem Zeitintervall von 0 bis 7 msec. die Reaktion 50 % beträgt; der Nerv ist somit während dieser Zeit absolut refraktär. Bei etwa 7 msec. ist die abs. refr. Per. zu Ende und die Reaktionsgrösse steigt allmählich an. Bei 34 msec. ist die rel. refr. Per. noch nicht ganz abgelaufen. Die sich am schnellsten erholenden Fasern besitzen somit eine abs. refr. Per. von rund 7 msec. und die sich am langsamsten erholenden Fasern haben eine rel. refr. Per. von mehr als 34 msec.

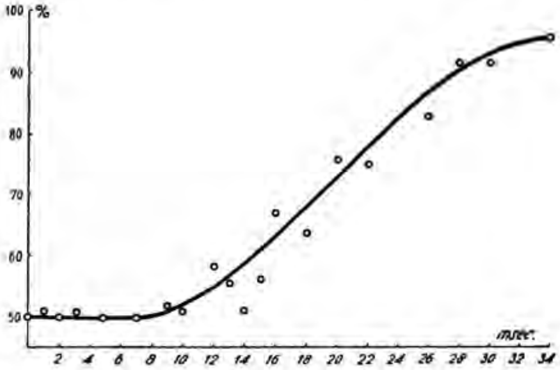

Abb. 3. Die refraktäre Periode des N. intestinalis, festgestellt durch Anwendung von zwei maximalen Induktionsschlägen mit verschiedenem Zeitintervall. Horizontal: Zeit in msec. Vertikal: Oberfläche der Aktionsstromkurve in % (die Oberflächen der Reaktionen auf den bedingenden und den Prüfreiz wurden zusammen gleich 100 % gesetzt). Die abs. refr. Per. (Aktionsstromoberfläche = 50 %, da nur auf den bedingenden Reiz reagiert wird) dauert 7—8 msec., die rel. refr. Per. ist nach 34 msec. noch nicht ganz zu Ende.

Ist der Einzelreiz gerade maximal für die schnelle Fasergruppe und der Doppelreiz bei Intervall von 0 msec. gerade noch subliminal für die langsame Fasergruppe, so kann die refr. Per. der schnellen Fasergruppe absonderlich untersucht werden, da die langsamen Fasern gar nicht erregt werden und das Auftreten der zweiten Reaktion am Ende der abs. refr. Per. somit einwandfrei beobachtet werden kann. Abb. 4 gibt einige Stadien eines solchen Versuches wieder.

Abb. 4. Die refraktäre Periode der schnellen Fasergruppe, festgestellt mit zwei Induktionsschlägen, die für diese Fasergruppe gerade maximal waren.
1 = Reaktion auf den bedingenden Reiz. 2 = Reaktion auf den
Doppelreiz mit Zeitintervall von 5 msec. 3 = „ „ „
„ „ „ „ 10 „ . 4 = „ „ „
„ „ „ „ 15 „ . 5 = „ „ „
„ „ „ „ 20 „ .
Zeitmarkierung = 10 msec.

Man sieht, wie sich bei einem Reizintervall von 10 msec. eine zweite Reaktion entwickelt, die bei zunehmendem Intervall grösser wird. Aus diesen Versuchen konnte ich berechnen, dass die abs. refr. Per. von 7 msec. diesen schnellsten Fasern zugehört und dass ihre ganze Refraktärzeit rund 25 msec. dauert. *Die umgekehrte Beziehung zwischen Leitungsgeschwindigkeit und Dauer der refr. Per.*, welche bei Vertebratenaxonen festgestellt wurde, *ist somit auch für den Helixnerven gültig.*

Die Fasern des N. Intestinalis stimmen in verschiedenen Eigenschaften überein mit den C-fasern von Vertebratennerven: Die Leitungsgeschwindigkeit, die Dauer der refr. Per. und die Dauer der latenten Addition (s.S. 1071) sind von der gleichen Ordnung.

Asynchrone Entladungen.

Die Untersuchungen der letzten Jahren ergaben, dass in Nerven verschiedener Tierarten unter bestimmten Umständen eine repetierende Aktion auftreten kann nach starken Einzelreizen und bei Durchgang eines konstanten Reizstroms. Im Helixnerven habe ich festgestellt, dass bei Reizung mit einzelnen Induktionsschlägen in der Nähe der maximalen Reizstärke asynchrone Nachentladungen auftreten können, welche einige Sekunden anhalten. Je stärker der Reiz, desto heftiger und länger diese repetierende Aktion wird. Registriert man diese Nervenaktion mit einem ballistischen Galvanometer, so erzielt man kein Reaktionsmaximum, da dieses träge Galvanometer synchrone und asynchrone Entladungen summiert. Die Tabelle I bringt davon ein Beispiel. Man sieht wie zwischen Reizstärken von 26 und 47 mal dem Schwellenwert (erste Reihe) die Oberfläche der Aktionsstromkurve ihr Maximum erreicht (zweite Reihe). Bei weiterer Reizverstärkung nimmt sie nicht mehr zu, es tritt aber nach dem synchronen Aktionsstrom eine repetierende Nervenaktion auf, welche bei zunehmender Reizstärke immer heftiger wird. Das ballistische Galvanometer (dritte Reihe) summiert dagegen synchrone Nervenreaktion und Nachentladungen, denn seine Ausschläge werden mit zunehmender Reizstärke fortwährend grösser.

TABELLE I.

Reizstärke in Schwellenwerten	Saitengalvanometer. Oberfläche des synchronen Aktionsstroms in mm²	Ballist. Galvo. Ausschlag in cm
10	62	2.3
16	99	5.2
26	127	6.9
47	141	8.4
106	140 + schwache Repetition	10.9
150	139 + mässige „	11.7
230	142 + starke „	12.7
390	141 + sehr starke „	13.1

Diese Resultate bringen eine Erklärung für die Befunde von Jordan und Lullies, die bei zunehmender Reizstärke eine immer höhere Muskelkontraktion beobachteten. Aehnlich wie das ballistische Galvanometer wird nämlich der träge Gastropodenmuskel synchronen Aktionsstrom und asynchrone Nachentladungen zu einer einzigen mechanischen Reaktion summieren. Bei Tieren, deren Nerven zu Repetition und deren Muskeln zu Summation der nervösen Impulse neigen, *eignen sich Muskel oder ballistisches Galvanometer demnach nicht zur Bestimmung der refr. Per. des Nerven* .

Lokale Erregung.

Die Erscheinung, dass ein subliminaler Reiz lokal auf dem Nerven für kurze Zeit „etwas" hinterlässt, das sich zu einem zweiten Reiz addieren kann, ist schon längst bekannt. Es steht zur Zeit fest, dass dieses „Etwas" eine elektrische Ladung ist, welche durch den unterschwelligen Reiz auf den Nerven deponiert wird (Lokalpotential, Elektrotonus). Nach den neueren Untersuchungen (z.B. Eichler (2—6); Katz (12, 13); Hodgkin (7—8)) ist diese Ladung nicht die einzige Ursache der Additionsfähigkeit.

Die lokale Aufladung löst nämlich eine lokale Reaktion des Nerven aus, die sogen. lokale Antwort, indem die Grenzmembran unter dem Einfluss der Reizladung schon etwas aufgelockert wird und die negative Plasmaladung demnach teilweise nach aussen entweicht. Diese lokale Antwort, die ja einen physiologischen Prozess darstellt, ist dadurch gekennzeichnet, dass sie einige Zeit zu ihrer Entwicklung braucht und daher nach dem Abklingen des Reizes noch weiter anwachsen kann. Misst man somit die elektrische Komponente der unterschwelligen Prozesse an der Reizkathode, so findet man ein Potential (Resultante des Lokalpotentials und der lokalen Antwort), welches nach dem Ende des Reizes nicht in Form einer Exponentialkurve absinkt, sondern eine nach oben konvexe Kurve bildet.

Zur Untersuchung der unterschwelligen Prozesse im N. intestinalis von Helix pomatia habe ich den Einfluss eines subliminalen bedingenden Reizes auf den Effekt eines schwach submaximalen Prüfreizes untersucht. Abb. 5 gibt davon ein Beispiel. Aufnahme *a* gibt das Resultat des einzelnen Prüfreizes wieder. In Aufnahme *b* wurden bedingender und Prüfreiz zur gleichen Zeit gegeben. Die Reaktion ist jetzt grösser, indem der Prüfreiz bei gleichzeitiger Applikation des bedingenden Reizes mehr Fasern über die Schwelle heben kann. Auch in der Aufnahme *c* (Reizintervall 7 msec.) ist eine grössere Reaktion aufgetreten als in Aufnahme *a*, zum Zeichen, dass der Einfluss des bedingenden Reizes nach 7 msec. noch nicht zu Ende ist. Merkwürdigerweise ist dieser Einfluss nicht nur noch nicht zu Ende, sondern sogar grösser als bei 0 msec. Bei 23 msec. (Aufnahme *d*) ist noch eine ganz schwache latente Addition vorhanden.

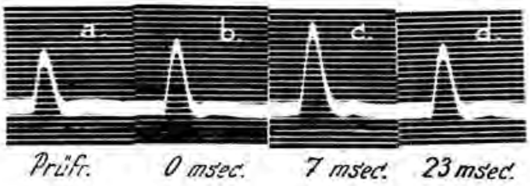

Abb. 5. Latente Addition in der schnellen Fasergruppe des N. intestinalis. Bedingender Reiz subliminal, Prüfreiz submaximal. *a* = Reaktion auf den Prüfreiz einzeln. *b* = Doppelreiz mit Intervall 0 msec. *c* = Doppelreiz mit Intervall 7 msec. *d* = Doppelreiz mit Intervall 23 msec.

Aus diesem Beispiel ist ersichtlich, dass auch beim Helixnerven eine latente Addition existiert, indem ein unterschwelliger Reiz „etwas" auf dem Nerven hinterlässt, das nach 23 msec. noch nicht ganz abgeklungen ist. Dieses „Etwas" ist jedoch nicht am grössten bei einem Intervall von 0 msec., sondern wächst noch eine Zeit lang weiter an um erst später abzunehmen.

Zur quantitativen Beurteilung der unterschwelligen Prozesse habe ich die Resultate mehrerer Versuche in einer Kurve zusammengefasst (Abb. 6). Aus dieser Kurve ist ersichtlich, dass bei Reizintervall 0 msec. eine gewisse latente Addition existiert, da der Effekt grösser ist als 50 %. Bei steigendem

Reizintervall zeigt diese Addition zuerst einen erheblichen Anstieg, gipfelt bei rund 8 msec., sinkt dann herunter und ist erst nach 28 msec. verschwunden. Das fortgesetzte Wachstum der Additionsfähigkeit nach dem Abklingen des bedingenden Reizes zeigt an, dass hier keine rein physikalische Aufladung durch den Reizstrom stattgefunden hat, denn es ist nicht ersichtlich, wie eine physikalische Aufladung noch weiter fortschreiten kann, nachdem die Ladungsquelle (der Reizstrom) schon beseitigt ist. Es muss neben der physikalischen Aufladung ein lokaler Erregungsprozess hervorgerufen worden sein, welcher Zeit braucht zu seiner Entwicklung. Diese lokale Erregung müssen wir, im Anschluss an die Resultate anderer Forscher, als einen physiologischen Prozess betrachten. Obgleich andere Forscher bei ihren Objekten bis jetzt nur eine konvexe Abweichung der Reizbarkeitskurve und keine wirkliche Steigerung beobachteten, ist es nicht notwendig, dass eine solche Steigerung bei allen Nerven fehlt. Das hängt nämlich ganz von dem Verhältnis zwischen Elektrotonus und lokaler Antwort ab. Ist die lokale Antwort gross im Verhältnis zur physikalischen Aufladung, so wird die Resultante einen Anstieg zeigen können. Das ist beim Helixnerven, wo die elektrotonischen Vorgänge gering und die lokalen physiologischen Prozesse beträchtlich sind, offenbar der Fall.

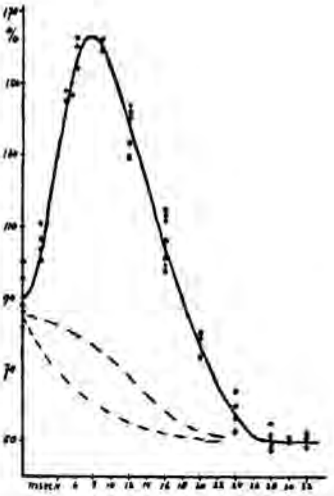

Abb. 6. Latente Addition im N. intestinalis. Bedingender Reiz subliminal, Prüfreiz submaximal. Horizontal: Zeitintervall zwischen bedingendem und Prüfreiz in msec. Vertikal: Oberfläche der Aktionsstromkurve in % (Reaktion auf den Prüfreiz = 50 %). Die gestrichelten Kurven geben den Verlauf der latenten Addition nach HILL (Exponentialkurve) und nach KATZ (konvexe Kurve) wieder.

Impulsleitung mit Dekrement.

Es steht heutzutage fest, dass die Impulsleitung in den myelinhaltigen Vertebraten- und Invertebratenaxonen nach dem Alles- oder Nichtsgesetz vor sich geht. Für die dünnen markarmen Axone von Invertebraten dagegen steht die Alles- oder Nichtsleitung nicht unbedingt fest (z.B. JORDAN und LULLIES (10), KAHN und KUSNEZOV (11)).

Bei dem Helixnerven habe ich diese Frage in einer grossen Anzahl von Versuchen untersucht. An den ganz frisch isolierten Nerven, der keine Aeste hat, wurden sechs Platinelektroden gelegt, in einem Abstand von ungefähr 3 mm von einander. Mit Hilfe von Schaltern konnte jede dieser Elektroden Reizkathode oder -Anode werden. Auf diese Weise konnte der Nerv an verschiedenen Stellen gereizt werden durch einfache Umschaltung; da die Ableitungselektroden nicht verlegt wurden, registrierte ich die Aktionsströme nach Zurücklegung verschieden langer Strecken. Abb. 7 gibt ein Bild eines solchen Versuches, wobei mit absteigenden gerade maximalen Einzelreizen gearbeitet wurde. Erstens tritt hierbei die grosse Dispersion der Aktionswellen bei grösserem Ableitungsabstand zutage (vgl. S. 1067). In Aufnahme I ist dieser Abstand 2.5 mm; man sieht die schnelle und langsame Aktionswelle, die gesamte Aktion dauert rund 100 msec. In Aufnahme II, wo der Ableitungsabstand 5.5 mm beträgt, dauert die Aktion schon viel länger (150 msec.), die beiden Hauptwellen sind weiter auseinandergerückt und jede Fasergruppe fängt an, sich in mehrere kleine Gruppen aufzuteilen. In Aufnahme III beträgt der Ableitungsabstand 8.5 mm; die beiden Hauptgruppen sind nur noch undeutlich zu unterscheiden und die ganze Aktion gibt durch die grosse Dispersion ein sehr gedehntes und wenig scharfes Bild (Dauer 230 msec.).

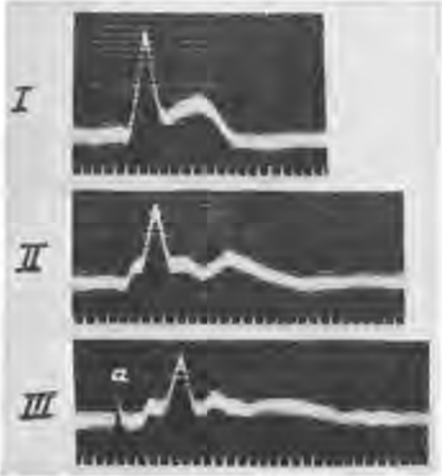

Abb. 7. Dispersion und Abnahme der Oberfläche des Aktionsstroms bei verschiedenem Ableitungsabstand. In Aufnahme I hat der Aktionsstrom 2.5 mm, in Aufnahme II 5.5 mm und in Aufnahme III 8.5 mm zurückgelegt. a = Reizeinbruch. Zeitmarkierung = 10 msec.

In diesen Aufnahmen ist weiter schon schätzungsweise zu beobachten, dass die Oberfläche der Aktionsstromkurve bei grösserem Ableitungsabstand kleiner wird. Quantitativ ist dies in der Abb. 8 ersichtlich, wo diese Oberflächen gemessen sind und in % abgetragen wurden (100 % = Aktionsstromoberfläche bei dem kleinsten Ableitungsabstand). Es ist ersichtlich, dass die Aktion bei zunehmendem Ableitungsabstand regelmässig an Grösse abnimmt; 8 mm von der Reizstelle ist sie ungefähr um 10 % kleiner geworden, auf 13—14 mm 20 % und auf 20 mm rund

25 %. Daraus darf m. E. geschlossen werden, *dass die dekrementielle Leitung eine normale Funktion dieses Nerven ist.*

Ist dieser Schluss richtig, so wird der Aktionsstrom, der unterwegs einen progressiven Verlust erleidet, nicht in jedem Punkt des Nerven die ganze vorhandene Energiemenge ausnutzen *und somit keine abs. refr. Per. hinterlassen.* Ist der Reiz maximal, so wird nur an der Reizkathode eine abs. refr. Per. entstehen, da der Aktionsstrom unterwegs ständig kleiner wird und nach Zurücklegen einer bestimmten Strecke sogar erlöschen muss. Wird der Reiz kleiner und kleiner, so wird auch die Nervenstrecke, welche die Aktionswelle zurücklegt bevor sie erlischt, selbstverständlich immer kürzer. Der Impuls kommt somit weiter, je stärker der Reiz ist und man wird demnach für weiter von der Reizstelle entfernte Punkte eine höhere Schwelle finden müssen, als in der Nähe der Kathode. Das ist tatsächlich nachweisbar. Bestimmt man nämlich auf verschiedenem Abstand der Ableitungsstelle den Schwellenwert, so findet man den Verlauf der Abb. 9. Man sieht wie mit zunehmendem Abstand die „Schwelle" ganz regelmässig ansteigt. D.h. also: Der Reiz muss stärker sein, soll der Aktionsstrom eine entferntere Stelle erreichen können. *Die Nichtgültigkeit des Alles oder Nichtsgesetzes ist damit bewiesen.*

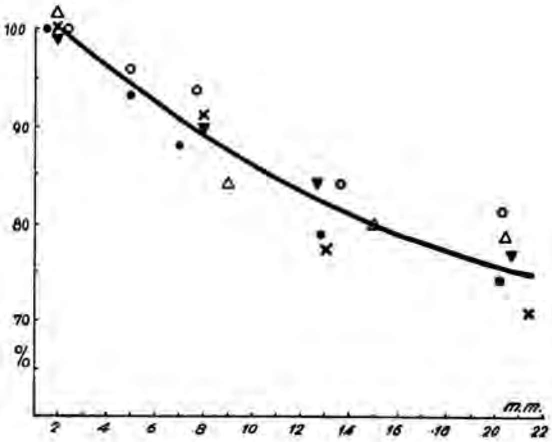

Abb. 8. Abnahme der Aktionsstromoberfläche bei verschiedenem Ableitungsabstand. Horizontal: zurückgelegte Strecke in mm. Vertikal: Aktionsstromoberfläche in % (die Oberfläche des Aktionsstroms beim kleinsten Ableitungsabstand wurde gleich 100 % gesetzt).

Nerven mit dekrementieller Leitung besitzen, wie aus dem Vorstehenden geschlossen werden kann, *keine Propagationsschwelle* (vgl. auch EICHLER), *da man durch Aenderung des Ableitungsabstandes jeden beliebigen „Schwellenwert" erhalten kann* [1]). Eine „Schwelle" existiert hier nur an

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[1]) Der Ausdruck „Schwelle" in den vorigen Abschnitten besitzt also nur relativen Wert.

der Reizstelle selbst; sie entspricht dem Reizpotential, das imstande ist eine minimale Aktion mit minimaler Ausbreitung auszulösen. Damit fällt aber der Unterschied zwischen dieser Schwelle und der von anderen Autoren (KATZ (12), LEDINGHAM und SCOTT (19), ARVANITAKI (1), usw.) beschriebenen Schwelle der lokalen Antwort fort.

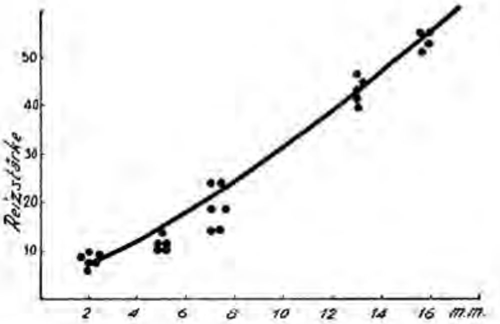

Abb. 9. Zunahme des Schwellenwertes bei grösserem Ableitungsabstand. Horizontal: Ableitungsabstand in mm. Vertikal: Schwellenwert in relativen Elektrizitätsmengen.

Es wird nun sehr schwierig, noch einen Unterschied zu finden zwischen der lokalen Antwort des Alles- oder Nichtsnerven und der dekrementiellen Aktion des Helixnerven. Beide stimmen nämlich in allen Eigenschaften überein. Sie verlaufen nicht nach dem Alles- oder Nichtsgesetz, sondern sind in Höhe und Ausbreitung vollkommen abhängig von der Reizstärke; beide haben eine niedrige, nicht genau bestimmbare Schwelle, welche nicht mit der Propagationsschwelle identisch ist. Bei dem Alles- oder Nichtsnerven wird bei Verstärkung des Reizes die Propagationsschwelle erreicht. Durch Aenderung der Ionenkonzentration (HODGKIN) oder Ermüdung (PUMPHREY, SCHMITT und YOUNG (21)) gelingt es daselbst aber, die Propagationsschwelle so weit nach oben zu verschieben, dass sie auch durch die stärksten Reize nicht mehr überschritten wird. Man bekommt dann bei Reizverstärkung eine immer grössere lokale Antwort, die an der Reizstelle so hoch werden kann wie sonst der Aktionsstrom, sich viele cm weit ausbreitet und diphasisch sein kann, kurzum genau so aussieht wie die Aktionswelle des Helixnerven. Man kann den Nerven mit dekrementieller Leitung somit auffassen als einen *Nerven, in dem die Propagationsschwelle normalerweise so hoch liegt, dass sie auch durch die stärksten Reizen nicht erreicht wird und in dem somit immer nur lokale (beschränkte, dekrementielle) Aktionen erzeugt werden können.* Man kann sich die Aktion im Gastropodennerven demnach folgendermassen vorstellen: Ein ganz schwacher Reiz erzeugt in den empfindlichsten Fasern eine kleine Aktion, ein stärkerer Reiz vergrössert diese Aktion und verursacht auch schon messbare Aktionen in weniger empfindlichen Fasern, usw. Jeder Axon zeigt demnach graduelle Aktionen in Abhängigkeit von der Reizstärke und von seiner Empfindlichkeit.

Diese dekrementielle „Scheinleitung" hat bei den Gastropoden zweifelsohne eine grosse Bedeutung, da bei diesen Tieren die Zentren und die synaptialen Verbindungen nicht genügend entwickelt sind um die Regulierung der Reflexe völlig beherrschen zu können. *Die dekrementielle Aktion hat hier die Aufgabe, die Erregung zu übermitteln und je nach der Reizstärke abzustufen.* Bei Tieren, wo die Zentren die Reflexregulierung gänzlich beherrschen, ist diese dekrementielle Aktion des peripheren Nerven reduziert auf einen kleinen Relikt (die „lokale Antwort") und es hat sich daselbst, ermöglicht durch die Entwicklung der Markscheide, ein sekundäres Phänomen entwickelt, nämlich der Aktionsstrom mit dem Alles- oder Nichtsgesetz. Durch dieses Gesetz wir der Axon degradiert zu einem einfachen Telegraphenkabel, der die Impulse während der Uebermittlung nicht abstufen kann (KRIJGSMAN (17)), sondern sie unverändert weiter gibt. *Die lokale Erregung hat hier nur noch Bedeutung als Schrittmacher für den Aktionsstrom.*

LITERATUR.

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Psychology. — *Die Sprache.* Von G. RÉVÉSZ. (Communicated by Prof. A. P. H. A. DE KLEYN.)

(Communicated at the meeting of September 28, 1940.)

Die Spracharten; — die Aeusserungen der Sprachfunktion; — das sinnliche Material und die autonomen Aufbauprinzipien der Spracharten; — die Sprache als symbolische Form; — die Grundfunktionen der Sprache; — die Definition der Sprache; — die artikulatorische Struktur und die grammatische Form der Sprache; — die gemeinsame Basis der Sprachen; — der Ursprung der Sprache; — die teleologische Bedeutung der Sprache; — die Sprachsituation; — die Sprache als Werkzeug des Denkens und als geistiger Kontakt; — die Sprache als Bildnerin der Hand und Grundbedingung der Arbeit; — die Sprache als Gestalterin des Weltbildes und Trägerin der Erkenntnisse; — die Sprache als Entwicklungsprodukt; — die erbbiologische Grundlage der Sprache; — die personale, soziologische, massen- und sozialpsychologische und allgemeine kulturelle Bedeutung der Sprache.

LEITSATZ.

Die Sprache ist mit dem Denken die höchste und umfassendste Funktion des menschlichen Geistes und zugleich das Wesensmerkmal des Menschen.

GRUNDTATSACHEN UND HAUPTGESICHTSPUNKTE.

1. *Die Spracharten.*

Es gibt vier voneinander unterscheidbare Spracharten, nämlich die Laut-, Gebärden-, Schrift- und Bildsprache.

Erläuterung: Natürliche und autochthone Spracharten sind die *Laut-* und die ebenso spontan entstandene und entwickelte natürliche *Gebärdensprache.*

Die Sprache par excellence ist die *Wortlautsprache*. Alle Grundfunktionen der Sprache sind an die Wortlautsprache gebunden. Die *Gebärdensprache* ist in dieser Richtung viel beschränkter, aber nicht weniger ausdrucksvoll. Beide Spracharten sind Sprachen sui generis, die unmittelbar aus der Sprachfunktion, ohne nachweisbare Mitbeteiligung der anderen entstanden sind. Die einzig biologisch wie psychologisch unanfechtbare Annahme ist, Laut und Gebärde gemeinsam als *die* gestaltenden Kräfte der Sprache anzusehen, da beide gleichwertige Mittel des Ausdruckes des inneren Lebens darstellen.

Es besteht kein Grund die hypostasierte Ursprache anders zu denken als Lautsprache. Die Auffassung, die Gebärdensprache als die ursprüngliche Sprachart anzusehen, woraus sich die Lautsprache allmählich entwickelte, lässt sich weder durch logische noch durch historische Argumente begründen. Es ist eine Konstruktion, die ihre Ueberzeugungskraft der schrankenlosen Anwendung des Entwicklungsgedankens zu verdanken hat.

Die *Schriftsprache*, wie auch die *Fingersprache* und die *rationell ausgebildete Gebärdensprache* (Gebärdenwortsprache) der Taubstummen, gewisser Indianerstämme und einiger Menschengruppen (z.B. Napolitaner, Zistensermönche) sind aus den beiden autochthonen Spracharten, vornehmlich aber aus der Wortlautsprache entstanden. Die Schrift- und Fingersprache stellt eine reine, eindeutige Uebertragung der Phoneme in optische bezw. Bewegungszeichen dar (durchgängige Anwendung des Alphabets), die grammatisch ausgebildete Gebärdensprache indessen setzt die Lautworte (Wortverbindungen) in eindeutige Bewegungsbilder (Gebärdenworte) um. Diese Spracharten besitzen also keine Autonomie, lehnen sich den natürlichen Spracharten an und entfalten sich *mit* ihnen und *durch* sie.

Die *Bildsprache* (Piktographie und Gebärdenzeichensprache) stellt im Gegensatz zu den obigen sekundären Spracharten die Wortlautsprache nicht in ihrer morphologischen und grammatischen Struktur dar, sondern bildet sie nach ihrem *Sinne* nach, d.h. *bildhaft*, in anschaulicher, womöglich allgemein verständlicher Ausdrucksform [1]). Die zwischen Bild- und Lautsprache bestehende enge Beziehung drückt sich u.a. in dem Umstand aus, dass bildhaft nur dasjenige dargestellt wird, was zugleich auch durch die Lautsprache verständlich gemacht werden kann, ferner, dass in den meisten Bildsprachen (wie in der chinesischen, aegyptischen, altkretischen) sich die Bildzeichen allmählich zu Wortzeichen, sogar zu Lautzeichen umbildeten. Die ursprüngliche Form der Bildsprache lässt sich auch in den Schriftzeichen noch nachweisen.

Aus diesen Erörterungen geht hervor, dass alle Spielarten der Sprache, mit Ausnahme der Gebärdensprache, auf die Wortlautsprache zurückgehen oder wenigstens von ihr im weitesten Masse mitbestimmt sind.

2. *Die Aeusserungen der Sprachfunktion.*

Die Veranlagung zur Sprache, der sog. innere Sprachsinn, äussert sich der Hauptsache nach im Sprachverständnis und im aktiven Sprechen.

Erläuterung: Sprachtätigkeit und Sprachverständnis ergänzen sich zur sinnvollen Einheit, zur Sprache, was nicht bedeutet, dass beide Funktionen im Einzelnen immer auch auf der gleichen Höhe stehen müssen. Als erstes Zeichen der Sprachfähigkeit dürfen wir das Sprach- bezw. Wortverständnis ansehen. Noch *vor* der Sprachtätigkeit ist das kleine Kind imstande einige Worte und Sätze als solche zu verstehen. Der Sprechakt verzögert sich bei ihm infolge des noch unentwickelten Sprachmechanismus und der gehemmten Spontaneität. Wörter, worauf kleine Kinder (oder Tiere)

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[1]) Z.B. Darstellung eines Hauses durch blosses Zeichnen des Schemas in der Luft mit der Hand, oder Darstellung der Lüge durch Hand- und Zungenbewegungen, die auf die „Zweizüngigkeit" deuten.

zweckentsprechend reagieren, ohne ihren Sinn verstanden zu haben, sind bloss Laute, die durch Dressur oder Erfahrung mit gewissen Handlungen und Unterlassungen fest assoziiert sind.

3. *Das sinnliche Material und die autonomen Aufbauprinzipien der Spracharten.*

Jede Sprachart hat ihre eigene sinnlich-anschauliche Ausdrucksform und innerhalb dieser ihre eigene Struktur.

Erläuterung: Die Sprache bedient sich akustischer, optischer und kinaesthetischer Gebilde. Die Lautsprache wird von phonetischen, die Gebärdensprache von motorischen, die Bild- und Schriftsprache von optischen Zeichen dargestellt und phonetischen, motorischen bezw. optischen Gestalt- und Strukturgesetzen unterworfen.

Das phonetische System der Sprachen ist strengen Lautgesetzen unterworfen; die natürliche Gebärdensprache umfasst im wesentlichen allgemein verständliche Gebärden (daher können sich fremdsprachige Menschen und Taubstummen ohne weiteres mit einander durch Gebärden verständigen); die künstlichen Gebärdensprachen sind zusammengesetzt teils aus natürlichen Gebärden, teils aus bildlichen (piktographischen) Darstellungen und aus willkürlich festgelegten Bewegungssymbolen. Die Bildsprache verwendet in der Regel allgemeinverständliche bildliche Darstellungen, die infolge der Schematisierung und der Zufügung von Symbolzeichen ihre Allgemeinverständlichkeit leicht einbüssen. Das Mittel der Schriftsprache besteht aus symbolischen Schriftzeichen, die zu den darzustellenden Dingen und Ereignissen keine unmittelbare Beziehung haben.

4. *Die Sprache als symbolische Form.*

Die Sprache besteht im Wesentlichen aus einem System von Laut-, Schrift- und Gebärdensymbolen.

Erläuterung: Eine innere Beziehung zwischen Wortlaut und Bedeutung gehört zu den Ausnahmen. Das gilt umsomehr, da die affektiven lautlichen Aeusserungen nicht zu der Sprache gehören; sie sind keine Symbole, sondern unmittelbare Reaktionen auf gewisse innere Zustände des Subjekts. Die Interjektionen, die ursprünglich mit den inneren Erregungen zwangsmässig zusammenhängen, nehmen vielfach symbolische Bedeutung an, sobald sie in die Sprache Eingang finden (ach, oh, eh, weh, bah, pfui etz.). Auch die onomatopoetischen Bildungen, die zu den äusseren Ereignissen in Beziehung stehen, werden allmählich ein Teil des nicht-interjektionellen Wortbestandes, sie verlieren dadurch ihre ursprüngliche Beziehungen zu den Eindrücken, durch die sie hervorgerufen sind.

Der Zusammenhang zwischen Zeichen und Bedeutung lässt sich noch am ehesten bei der Gebärdensprache feststellen. Viele Gebärden in den natürlichen und künstlichen Gebärdensprachen weisen ausdrücklich auf ihre ursprüngliche Bedeutung. Das steht nicht im Widerspruch zu der

Erfahrung, dass die natürlichen und konventionellen Gebärdensprachen die Tendenz haben, die Gebärden zu einem System von Symbolen zu entwickeln.

5. *Die Grundfunktionen der Sprache.*

Als Grundfunktionen der Sprache können die Benennung (Bezeichnung), die Darstellung und Mitteilung, mithin auch die begriffliche Formulierung gelten.

Erläuterung: Nicht alle Spracharten vermögen alle diese Funktionen zum Ausdruck zu bringen. Bei der Wortlautsprache und ihren Derivaten (Schrift- und Fingersprache) sind alle angeführten Funktionen manifest. Die Gebärdensprache besitzt die Nennfunktion nicht, und auch die logisch-grammatikalische Funktion lässt sich nur durch eine künstliche, sich an die Wortlautsprache anlehnende Gebärdensprache andeuten. Die Bildsprache eignet sich bloss für Darstellung und Mitteilung.

6. *Die Definition der Sprache.*

Unter Sprache verstehen wir *die Funktion, durch die wir mit Hilfe einer Anzahl von gegliederten (artikulierten) und in verschiedenen Sinnverbindungen auftretenden Laut- bezw. Bewegungs- oder Zeichengebilden unsere Wahrnehmungen, Urteile, Wünsche, etz. darzustellen und in der Absicht gegenseitiger Verständigung anderen mitzuteilen imstande sind.*

Erläuterung: Von dieser Definition ausgehend können wir die Ausdrucksbewegungen und die damit im engen Verband stehenden Gebärden und mimischen Bewegungen als solche nicht als Spracharten betrachten; sie stellen nur unmittelbare Reaktionen innerer Zustände dar, die die Sprache begleiten und ergänzen.

Noch weniger dürfen die Laut- und Bewegungsäusserungen der Tiere mit dem Namen „Sprache" bezeichnet werden. Sie entbehren aller wesentlichen Merkmale der Sprache, geben nur dem jeweiligen vital-affektiven Zustand des Tieres Ausdruck. Die Tierlaute stehen, wie ausdrucksvoll sie auch sein mögen, zu der menschlichen Sprache in keiner Beziehung. Sie sind unartikulierte, phonetisch gestaltlose Geräuschtöne, ohne Sinn- und anschaulichen Bedeutungsgehalt. Der Begriff „Tiersprache" ist irreführend und wissenschaftlich nicht berechtigt.

7. *Die artikulatorische Struktur der Sprache.*

Alle Spracharten sind artikuliert.

Erläuterung: Die Artikulation bezieht sich bei der Sprache nicht allein auf die physikalisch-phonetische bezw. motorische Gliederung, sondern auch auf die Funktion der lautlichen bezw. kinaesthetischen Gebilde als Ausdruck. Die Lautkomplexe müssen innerhalb der Gliederung eine besondere Betonung, die Gebärden einen besonderen Nachdruck haben und diese Art der Artikulation, des dynamischen Gefüges, hängt von dem Sinn ab [1]).

Am ausdrücklichsten kommt die Artikulation mittels Gliederung, Betonung, Rhythmus und Sprachmelodie in der Lautsprache zum Ausdruck. Am schwersten lässt sich die Artikulation bei der Bildsprache aufzeigen, da bei ihr das Zeitmoment nicht unmittelbar, sondern nur infolge der sinnvollen Erfassung des Dargestellten manifest wird.

8. *Die grammatische Form der Sprache.*

Die Sprache besitzt eine grammatikalische Struktur.

Erläuterung: Es gibt keine Sprache ohne grammatikalische Beziehungen und Kategorien. Selbst die primitivsten Sprachen entbehren die grammatikalische Struktur nicht. So wissen wir, dass ein auf der Halbinsel Malakka lebendes, kulturell tief stehendes Volk, das Sakai-Volk, eine Sprache in 3 Dialekten von äusserst komplizierter grammatikalischer Struktur spricht. Dasselbe wird berichtet von der primitiven Sprachen der Pygmäen Zentral-Afrikas. Ob bei den sog. Wurzelsprachen (aus isolierten, einsilbigen Wörtern bestehenden Sprachen ohne grammatikalische Kategorien) die grammatische Beziehungen völlig fehlen, ist noch fraglich.

Auch bei der natürlichen Gebärdensprache finden wir deutliche Ansätze zu grammatikalischen Formen. Unterschiede von Subjekt, Objekt, Attribut, Verbum und Adverb werden angedeutet. Demgegenüber ist die Bildsprache an sich jeder Grammatik bar; da sie aber infolge ihrer engen Beziehung zu der Lautsprache grammatikalisch erfasst wird, kann sie nicht als a-grammatikalisch betrachtet werden.

9. *Die gemeinsame Basis der Sprachen.*

Das gemeinsame Fundament der Sprachen wird durch die auf dem inneren Sprachsinn beruhenden symbolischen und grammatikalischen Funktionen gebildet.

Erläuterung: Die einheitliche Basis macht es verständlich, dass jedem Sprachsystem sich beliebig viele andere Sprachen eindeutig zuordnen lassen und dass jede Sprache in eine andere übersetzbar ist.

10. *Der Ursprung der Sprache.*

Der Ursprung der Sprache ist ein unlösbares Problem.

Erläuterung: Die Untersuchungen hinsichtlich des Ursprungs der Sprache haben zu der Aufstellung verschiedener Hypothesen geführt. Eine kritische Prüfung dieser Annahmen, sowie die Tatsache, dass für die Erkenntnis des Ursprungs der Sprache uns jede sprachgeschichtliche Grundlage fehlt, zeigt die Fruchtlosigkeit der darauf verwendeten, an sich bemerkenswerten Arbeit.

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[1]) Die Artikulation ist ein geistiges Moment, daher sind die Lautäusserungen der Tiere nicht artikuliert, selbst die Lautimitation der Papageien entbehrt jeder Artikulation.

Wenn man unter Vorstufe der Sprache solche Aeusserungen versteht, die gewisse die Sprache konstituierenden Elemente in sich schliessen, worunter sich die Mitteilungsabsicht und ein System von konstanten Zeichen befinden müssen, dann liegt die Sprache in dieser vermeintlichen Vorstufe bereits vor. Fehlen indessen wesentliche Elemente bezw. Funktionen, wie dies z.B. bei den Ausdrucksbewegungen oder Interjektionen der Fall ist, so sind wir nicht berechtigt, diese Aeusserungen als Vorstufe der Sprache zu betrachten, unabhängig davon, ob diese Tätigkeiten phylogenetisch oder ontogenetisch zeitlich aufeinander folgen oder nicht.

11. *Die teleologische Bedeutung der Sprache.*

Der Sinn der Sprache liegt in der gegenseitigen Einwirkung der Menschen aufeinander.

Erläuterung: A spricht B an, in der Absicht, verstanden zu werden. Bestünde diese Absicht nicht, dann hätte keine Sprache entstehen können. Auch der Entfaltung, Differenzierung und Bereicherung der Sprache liegt der Zweck als Prinzip zugrunde.

12. *Die Sprachsituation.*

Die ursprüngliche und allgemeine Form der sprachlichen Beziehung ist der Dialog.

Erläuterung: Die ursprüngliche Sprachsituation ist das Zwiegespräch. Auch ein zwischen drei Personen geführtes Gespräch lässt sich auf den Dialog zurückführen. Beim Sprechen handelt es sich um eine gegenseitige Beziehung, um Frage und Antwort, bezw. um sprachliche Aktion und sprachliche Reaktion. Durch welche Mittel das Ansprechen und das Reagieren geschieht, ist hierbei gleichgültig.

Eine Modifikation des Dialoges ist das einseitige Sprechen bei doppelseitiger Beziehung: der eine spricht in der Absicht verstanden zu werden, der andere ist nur bestrebt, das Gesagte zu begreifen, ohne zu antworten.

Gelegentlich hört die doppel- bezw. wechselseitige Beziehung ganz auf oder büsst ihre Bedeutung ein, wenn z.B. der Erlebnis- oder Denkinhalt aus innerem Drang, gewissermassen unabsichtlich, sprachlich formuliert zum Ausdruck kommt. Man denke hierbei an die dichterische Sprache, an die Sprache der Intuition, der Offenbarung, des Gebets, schliesslich an das „sprechende" Denken. Die Sprache richtet sich in diesen Fällen an die Gemeinschaft oder an ein vorgestelltes Wesen oder an uns selbst.

13. *Die Sprache als Werkzeug des Denkens.*

Die Sprache ist das adäquateste Werkzeug des Denkens und die den Verstand steuernde Kraft.

Erläuterung: Die Sprache ist nicht bloss das Mittel des logischen, formulierenden Denkens, sondern des Denkens überhaupt. Folglich gilt der obige Satz auch dann, wenn ein hypologisches, emotionelles, sprachloses und intuitives Denken angenommen wird, denn keine dieser Arten des Denkens ist ohne Voraussetzung und ohne Mitwirkung der Sprache vorstellbar. Denn auch der nicht ausgesprochene, nicht formulierte Gedanke entfaltet sich in der Sprachsphäre. Besonders das logische und formulierende Denken ist mit der Sprachfunktion so verflochten, dass der Anteil beider Funktionen an dem manifesten Gedanken meistens nicht zu scheiden ist, zumal jeder Gedanke nur durch sprachliche Formulierung seinen endgültigen Inhalt und sein Beziehungsreichtum erhält. Taucht ein ursprünglicher Gedanke plötzlich im Bewusstsein auf, so tritt er entweder unmittelbar in sprachlicher Form auf oder der ihn vorbereitende Vorgang geht (im Bewusstsein oder im Unbewussten) unter Mitwirkung der Sprache oder mindestens unter Miterregung der Sprachsphäre vor sich.

Begriffe können nur von sprachveranlagten Wesen gebildet werden. Nur Wesen, die die Fähigkeit besitzen, Sprache zu verstehen und sich mittels Sprache zu verständigen, sind in der Lage, Begriffe zu bilden und zu fixieren.

Der Satz, dass *alles* Denken die Sprache notwendig voraussetzt, lässt sich nicht halten. Um diesem Satz eine allgemeine Geltung zu verleihen, müsste man einerseits das Gebiet der Denkfunktion einschränken, andererseits die Phantasie und das anschauliche Erfassen von Zusammenhängen dem Denken absprechen.

Am meisten tritt die Sprachfunktion bei künstlerischer Tätigkeit zurück, ohne aber auch hier ihre Bedeutung gänzlich einzubüssen. Sie wird in den darstellenden Künsten, insbesondere dort, wo konstruktive Aufgaben zu lösen sind, wie in der Architektur und in allen Zweigen der dekorativen Künste, eine wichtige Rolle zu spielen haben. Nur in der musikalischen Gestaltung dürfte die Sprachfunktion keinen entscheidenden Einfluss ausüben. Daran ist nicht zu zweifeln, dass auch die musikalische Gestaltung in die Sphäre des Denkens hineinragt; die Frage ist nur, ob die schöpferische und darstellende Tätigkeit in der Musik spezifische und sprachlich geformte Denkinhalte voraussetzt. Das kann jedenfalls zugegeben werden, dass die aus stetem Zusammenwirken von Inspiration und Technik hervorgehende musikalische Schöpfung abstrakt-formale Bestimmungen enthält, die in die Sphäre der Sprache hineinragen und vielfach von einem schematischen Plan und gedanklicher Vorarbeit ausgehen. Auch der Umstand, dass Musiker, vor allem Komponisten über ihren schöpferischen Wachstumsprozes aufschlussreiche und sprachlich gut formulierte Aussagen bieten, kann für diese Auffassung geltend gemacht werden.

14. *Die Sprache als geistiger Kontakt.*

Die Sprache ist das adäquateste und reichhaltigste Mittel der Kommunikation, des gegenseitigen Verständnisses und des geistigen Kontaktes überhaupt.

Erläuterung: Alle Grundfunktionen der Sprache stehen im Dienste der geistigen Verständigung. Geistiger Kontakt entsteht in erster Reihe durch die *Sprache*, in zweiter Reihe durch *sprachbezogene* Manifestationen anderer Art. Als Verständigungsmittel können daher auch einzelne Gebärden und mimische Bewegungen, ferner bildliche Darstellungen verschiedener Art in Betracht kommen, sofern sie sprachbezogen, folglich in eine Sprache übersetzbar sind [1]). Dasselbe gilt für Bilder, Skulpturen, soziale Manifestationen, religiöse Objekte und Gebräuche, ferner für laut- und bewegungslose Signale.

Es ist vielleicht nicht überflüssig darauf hinzuweisen, dass Laute und Gebärden nur dann als Mittel der geistigen Verständigung gelten, wenn sie begriffen, von *beiden* Seiten aus sprachbezogen aufgefasst werden. Ist das nicht der Fall, dann handelt es sich nicht mehr um gegenseitige geistige Verständigung, sondern höchstens um Reaktionen auslösende Verhaltungsweisen infolge von Gewohnheitsbildung und Dressur.

15. *Die Sprache als Bildnerin der Hand.*

Die Sprache als Manifestation des Geistes wirkt bei der Entfaltung der menschlichen Hand in entscheidender Weise mit.

Erläuterung: Die Hand des Menschen, die weisende, „sprechende" und arbeitende Funktion der Hand ist das Erzeugnis des Geistes. Die Hand des Menschen ist darum fähig mittels Gebärden Erlebnisse und Gedanken zum Ausdruck zu bringen und freie Arbeit zu leisten, weil sie ein Organ des denkenden und sprechenden Menschen ist. Die Hand, die biologisch gesehen ein Greiforgan ist, musste in ihrer Form und Funktion durch den menschlichen Geist erst zum Ausdrucks- und Arbeitsorgan umgeschaffen werden.

16. *Die Sprache als Grundbedingung der Arbeit.*

Absichtliche und planmässige Arbeit, rationelle Arbeitsmethoden und der Arbeit angepasste Arbeitsgeräte und Werkzeuge bedürfen der Sprachfunktion und werden durch die Sprache in weitem Masse bestimmt.

Erläuterung: Menschen, die im obigen Sinne arbeiten, sprechen; solche, denen die Sprache gänzlich fehlt (schwere Idiotie, verwahrloste Taubblindheit), arbeiten nicht. Wo die Sprachfunktion stark gestört ist, wie bei hochgradiger Aphasie, wird auch die Arbeitsfähigkeit schwer geschädigt. Höhere Sprachentwicklung und mannigfaltigere Entwicklung der Arbeit hängen eng miteinander zusammen.

Arbeitsgeräte zu schaffen und sie sinnvoll und mannigfaltig zu verwenden, ist nur der Mensch imstande. Das „Werkzeug" des Tieres ist stets ein Organ des Tieres selbst. Gewisse Manipulationen der Affen als Werkzeuggebrauch und Werkzeugerschaffung zu bezeichnen, beruht auf einer unberechtigten Uebertragung dieser Begriffe auf tierische Betäti-

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[1]) Z.B. Gebärde der Geringschätzung: teilweises Schliessen der Augenlider, Wegwerfen der Augen, des Kopfes oder des ganzen Körpers. — Gebärde für den Befehl sofort hinauszugehen: der Arm wird je nach dem Grade der Heftigkeit rasch hochemporgeschleudert.

gungen, die bei einer gründlichen Analyse sich als von der menschlichen Arbeit grundsätzlich verschieden erweisen.

17. *Die Sprache als Gestalterin des Weltbildes.*

Die Sprachfähigkeit ist die notwendige Voraussetzung des Schaffens eines gegliederten und sinnvollen Weltbildes.

Erläuterung: Die durch die Sprache entstandenen Bezeichnungen, Namen, Begriffe (und die durch sie bedingten repräsentativen Vorstellungen und anschaulichen wie abstrakten Schemata) stellen die Grundbedingungen für die Erkenntnis der Dinge dar. Nur der sprechende bezw. der mit Sprachfunktion begabte Mensch kann den Sinn und die Bedeutung eines Gegenstandes oder Ereignisses erkennen, die Einzelwahrnehmungen zu einer anschaulich-sinnvollen Einheit vereinigen und vermöge dieser geistigen Tätigkeit sich ein gegliedertes und sinnvolles Weltbild schaffen. Die konkreten Wahrnehmungsinhalte können sich nur dann zu einem einheitlichen Bild verdichten, wenn sie sich zueinander in Beziehung setzen; sonst fehlt das Ferment für die erforderliche Synthese. Zu dieser Synthese gelangt man ausser durch eine lebendige Vorstellungstätigkeit nur mit Hilfe von Wörtern und Begriffen, folglich mittels Denken und Sprache. Fehlen diese Funktionen oder werden sie verhindert in Wirksamkeit zu treten, so wird die Erkenntnis der Aussenwelt nicht möglich, was auch auf die phänomenale Beschaffenheit der wahrgenommenen Welt entscheidenden Einfluss ausübt.

18. *Die Sprache als Träger der Erkenntnisse.*

Einen unübersehbaren Reichtum von Kenntnissen, die die Menschheit allmählich erworben hat, bewahrt die Sprache.

Erläuterung: Dieser Satz bezieht sich in erster Linie auf die sprachlich formulierten Erfahrungen, Erlebnisse, Ideen, Strebungen, die von Generation auf Generation überliefert und durch die Sprache unabsichtlich übertragen werden. Dasselbe gilt auch für den blossen Wortschatz. In den Wörtern und Begriffen, in den Synonyma, Heteronyma, in der Lautsymbolik, den analogischen Bildungen, ferner in der Vieldeutigkeit der Wörter und in den Urschöpfungen sind Kenntnisse, Erfahrungen, völkerpsychologische und charakterologische Merkmale verborgen, die noch immer ein ziemlich unerforschtes Gebiet der Sprachphilosophie und vergleichenden Sprachwissenschaft bilden.

19. *Die Sprache als Entwicklungsprodukt.*

Die Sprache ist ein Entwicklungsprodukt sowohl vom ontogenetischen wie vom phylogenetischen Standpunkte aus.

Erläuterung: Sowohl die Sprache jedes Individuums wie die Sprachen der Völker sind nach gewissen Gesetzen verlaufenden Entwicklungsprozessen unterworfen. Selbst die künstlichen Sprachen entfalten sich notgedrungen, weil sie eben Sprachen sind.

20. *Die erbbiologische Grundlage der Sprache.*

Die Sprache ist erworben und nicht vererbt. Vererbt ist die Disposition zum Sprechen.

Erläuterung: Jedes Kind muss die Sprache erwerben, erlernen, um dann allmählich eine mehr oder weniger charakteristische individuelle Sprachgewohnheit auszubilden. Die Disposition zum Sprechen schliesst die latente Fähigkeit des Erlernens und Gebrauchens *aller* Sprachen in sich. Der Umstand, dass die affektiven Laut- und Bewegungsäusserungen der Menschen vererbt sind, weist darauf hin, dass sie nicht zu dem Grundstock der Sprache gehören.

21. *Die personale Bedeutung der Sprache.*

Ausser ihrer grossen Bedeutung für das Denken und für die Phantasietätigkeit *bildet die Sprache vor allem die Voraussetzung für die intellektuelle Entfaltung und die Existenz der psychisch-realen, moralischen und geschichtlichen Persönlichkeit des Menschen.*

Erläuterung: Alle Störungen in der sprachlichen Entwicklung wirken unheilvoll auf die Verstandestätigkeit. Schwere Fälle der Idiotie sind gänzlich sprachunfähig; die Sprachtätigkeit der Imbezilen ist unbeholfen, ihr Sprachschatz ausserordentlich dürftig. Die Mehrzahl der Taubstummen erreicht trotz langes Sprachunterrichtes das geistige Niveau der Normalen nicht. Aphasie wirkt auf die geistige Tätigkeit regressiv. Sprachhemmungen beeinflussen die geistige Tätigkeit ungünstig.

Ohne Sprache besteht kein Ich-Bewusstsein und ohne Ich-Bewusstsein lässt sich keine moralische Persönlichkeit denken. Gewissen, Verantwortungs- und Pflichtgefühl sind nur denkbar, wenn auf Grund von Sittenregeln und des allgemeinen Sittengesetzes ethische Entscheidungen getroffen werden können. Diese Urteile wie alle anderen sind aber unabänderlich an die Sprache gebunden.

Die „historische" Persönlichkeit entsteht dadurch, dass der Mensch die Etappen und die wichtigsten Ereignisse seines eigenen Lebens der chronologischen Ordnung entsprechend in sprachlicher Form sich bewusst macht und mit dem realen, erlebten Ich in Verbindung setzt.

22. *Die soziologische Bedeutung der Sprache.*

Die Sprache gehört zu den wichtigsten Entstehungsbedingungen einer auf geistiger Grundlage aufgebauten Gemeinschaft.

Erläuterung: Ohne Sprache als Mittel der Verständigung und der begrifflichen Formulierung gibt es keine geistige Gemeinschaft und Gesellschaft. Jede auf Regeln, Gesetzen, Anschauungen beruhende Organisation bedarf der bezeichnenden, darstellenden bezw. formulierenden Funktion der Sprache. Willkürliche, rationelle Aenderungen in der Organisation der Geimeinschaft setzen die Sprache voraus.

23. *Die massen- und sozialpsychologische Bedeutung der Sprache.*

Die Sprache gehört zu den mächtigsten Faktoren der (geistigen) Beeinflussungsmöglichkeit des Menschen.

Erläuterung: Moralische und soziale Gesetze, religiöse, ethische und gesellschaftliche Forderungen und Traditionen bedürfen der sprachlichen Formulierung und bestimmen so nachhaltig die Verhaltungsweise der menschlichen Gruppen. Die lebendige Sprache hat im Leben der Massen und Gruppen den allergrössten Einfluss. Worte, Gesten, sprachbezogene Symbole (Fahne, Wappen, Kreuz etz.) haben grossen Assimilationswert und üben die stärkste suggestive Wirkung in positivem und negativem Sinne aus.

24. *Die allgemeine kulturelle Bedeutung der Sprache.*

Die Sprache ist die Grundbedingung aller Kultur und Zivilisation.

Erläuterung: Kultur und Zivilisation hängen von der sprachlichen Entwicklung der Menschheit ab. Hohe Kultur bei primitivem Zustand der Sprache lässt sich ebenso wenig vorstellen, wie primitive Kultur neben hoch entwickelter Sprache. Es lassen sich kaum solche Kulturgüter und zivilisatorische Errungenschaften finden, die zu der Sprache nicht in engster Beziehung stünden. Die Sprache beeinflusst demnach alles, was der menschliche Geist und das menschliche Gemüt im Bewussten und Unbewussten hervorbringt 1).

„Die Sprache ist gleichsam die äussere Erscheinung des Geistes der Völker; ihre Sprache ist ihr Geist und ihr Geist ihre Sprache" (W. v. HUMBOLDT).

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1) Eine Anzahl der hier aufgestellten Sätze, insbesondere die über die symbolische Form, die Definition, den gemeinsamen Basis, den Ursprung der Sprache, ferner die Auffassung betreffs der Sprachsituation und der Rolle der Sprache beim geistigen Kontakt, bei der Bildung der Hand, des Weltbildes und der Persönlichkeit, bedürfen einer weiteren Begründung, die ich in einer zusammenfassenden Studie hoffe zu veröffentlichen, wobei auch die einschlägige Literatur berücksichtigt wird.

Anatomy. — *On the equal growth of the neocortex during the ontogenesis of the sheep* [1]). By R. BRUMMELKAMP. (Communicated by Prof. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of September 28, 1940.)

The question whether or not the neo-cortex develops equally is important for our insight into the distribution of the nervous nucleus volume. According to BRUMMELKAMP and VAN VEEN [2]) this distribution is such that in cortex columns of the same base, penetrating the cortex perpendicularly from pia to marrow, the sum of the individual nucleus volumes, [K], is constant. Furthermore do the individual nucleus volumes K lie at such minimal distances from each other, that the areas of the surfaces, O_T, of territories T (of the same shape and touching each other on all sides) enveloping the nuclei are proportionate to the enveloped nucleus volumes: K (:) O_T. Finely it could be made true that the nuclei are arranged in layers whose number is constant.

If these rules also apply to the growing cortex, and if the cortex develops through general cell divisions, comprising all cells, they imply a proportionate enlargement of the various cortex areas, c.q. they make it extremely probable.

From the above it can be easily deduced that the total nucleus volume which can find a place per layer in one area unit, must be constant. For the sake of convenience we substitute the enveloping territories by globes, touching each other, and neglect the dead spaces between the globes. We note that generally the area (S) of the largest section of a globe bears ratio to the area (O) of the globe surface, so that we can also say: K (:) S. Let us suppose globes of equal size to be packed as closely as possible in one layer on the area unit (minimal packing). The number N of globes which in this manner can be placed on the area unit will bear inverse ratio to the area S, i.e. N (:) 1/S. From this it follows that the total nucleus volume (N.K.), which can be placed per layer on the area unit within the enveloping globes, is constant.

When the nuclei divide, growing again to their original size, the area taken up by the surrounding globes will also have doubled. This doubling of the area may be effected by the globes finding a place under (infra-position) or by the side of each other (juxta-position), or by a combination of the two.

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1) From the laboratory of developmental pathology, Amsterdam.

2) R. BRUMMELKAMP and P. VAN VEEN, The distribution of the nervous nucleus volume in the neo-cortex, Verh. der Kon. Nederl. Akad. v. Wetensch. afd. Nat. IIde sectie, Deel XXXIX, No. 1, 1940.

In the case of complete or partial infra-position, the number of superposed layers will increase. According to the above this is possible only until a certain number is reached, which will be the same in all the cortex areas, after that only juxta-position is possible.

When the growing process is such that first the required number of layers is reached through infra-position then the subsequent growth of the cortex will only be possible through juxta-position (i.e. through area-enlargement).

If, during the growth, there are general cell divisions comprising all the nuclei, this implies that during the process of juxta-position the cortex area increases everywhere by the same enlargement factor (proportionate area growth of the cortex fields).

It is clear that this is also the case when infra-position and juxta-position are combined, provided the two processes are brought about in the same proportion.

On the ground of literature study and measurements taken by themselves, BRUMMELKAMP and OFFRINGA [1]) have shown the constancy of the area proportion of the frontal brain to that of the rest of the neocortex during the *human ontogenesis* (foetus of 180 mm, 252 mm, and 360 mm; adult and microcephalian) and *phylogenesis* (troglodytes niger, simia satyrus, adult human), hence that the enlargement of these areas is proportionate.

It will now be seen that, in the *ontogenesis of the sheep*, from which it is easy to get a lot of embryonic samples, there is also a proportionate enlargement in size of the neocortex areas.

In early stages of development already, there are visible in the sheep a number of grooves, which enable us to divide the cortex area into clearly marked fields.

These grooves are laterally the fissura rhinalis and the fissura limitans superior fossae Sylvii; on the convexity the fissurae coronalis, ansata and supra-sylvia; medially the fissura splenialis. The fields bounded by these grooves are: the fronto-medial (1), the fronto-lateral (2), the caudo-lateral (3), caudo-intercalair (4), and medial (5) fields. For the demarcation of these fields we refer to table I; see also figures 1—7.

As soon as the material is sufficiently hardened after treatment with Bouin's fluid and alcohol 70 %, the cortex is dissected perpendicularly to the surface, according to the demarcations of table I, so as to divide it into 5 separate parts or fields 1—5.

The cortex area of each field is now determined by covering it with silk paper, following BRODMANN and HENNEBERG. This method was also

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[1]) R. BRUMMELKAMP and J. OFFRINGA, The relative growth of the frontal brain during human onto- and phylogenesis, Acta neerl. morph. Vol. III, No. 2 ,1940.

TABLE I.
(Demarcation of the fields)

Field 1 (frontal-medial)	frontal	frontal pole;
	medial	corpus callosum (genu);
	lateral	fissura coronalis;
	caudal	fissura ansata (medial branch) and perpendicular from end medial branch of the fissura ansata to corpus callosum.
Field 2 (fronto lateral)	frontal	frontal pole;
	medial	fissura coronalis, lateral branch of the fissura ansata, and anterior branch of the fissura suprasylvia;
	lateral	anterior branch of the fissura lim. sup. fossae Sylvii and horizontal branch of the fissura praesylvia;
	caudal	processus acuminus of the fissura lim. sup. fossae Sylvii.
Field 3 (caudo-lateral)	frontal	processus acuminus of the fissura lim. sup. fossae Sylvii;
	medial	arcus and posterior branch of the fissura suprasylvia;
	lateral	posterior branch of the fissura lim. sup. fossae Sylvii and caudal part of the fissura rhinalis lateralis;
	caudal	caudal pole.
Field 4 (caudo-intercalairial)	frontal	fissura ansata;
	medial	fissura splenialis;
	lateral	fissura suprasylvia (arcus and posterior branch);
	caudal	caudal pole.
Field 5 (caudo-medial)	frontal	perpendicular from end medial branch of the fissura ansata to corpus callosum;
	medial	corpus callosum (corpus and splenium);
	lateral	fissura splenialis;
	caudal	fissura splenialis.

applied and found reliable by BRUMMELKAMP and OFFRINGA in determining the cortex area of the human foetus.

Before covering the surface of the cortex fields with narrow strips of silk paper, these strips were moistened with alcohol 96 %, then they were pressed against the brain surface with a small brush and cut into the right shape. The thin silk paper adjusts itself beautifully to the surface. On applying the strips we used binocular lenses, Zeiss make. After the part to be measured was covered, it was submerged in alcohol, which made the paper easily detachable. The strips could then be dried and weighed on milligram scales. In order to diminish the experimental error we photographed them with an enlargement apparatus, taking care always to use the same scale and then cut out the prints obtained.

Setting to work in this way we determined the areas of field 1—5 in the brains of 5 foetal, one new-born and one adult sheep; the brains used are shown in fig. 1—7. The following percentual values of the areas of fields 1 to 5 are found:

TABLE II.
(Percentual division of field areas)

Brains of figure	Fronto-medial (1)	Fronto-lateral (2)	Caudo-lateral (3)	Caudo-interc. (4)	Caudo-medial (5)
1	15.79	20.92	17.44	33.00	12.83
2	15.61	23.16	16.22	33.91	11.06
3	13.43	23.54	18.83	33.07	11.30
4	16.70	22.41	16.98	31.37	12.08
5	16.85	21.94	17.01	33.81	10.91
6	15.77	20.37	18.10	33.12	12.01
7	13.92	22.11	19.19	33.18	11.58

It is apparent from the percentual division of the field areas that, not regarding the divergence in absolute size, the relative size has remained practically constant in the various stages of development.

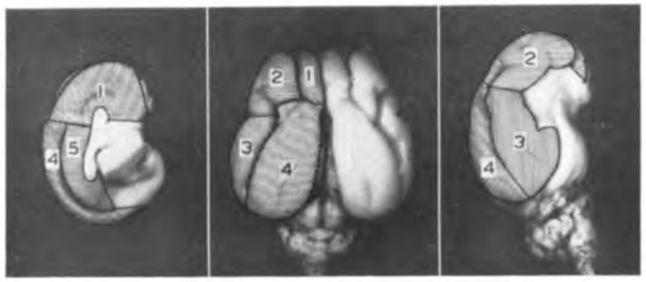
The mutual agreement of the percentages in these greatly different stages of development is striking and can hardly be interpreted in more than one way. It shows that during these stages of ontogenesis the surface areas of field 1—5 increase by the same enlargement factor, that there is, therefore *equal growth of the cortex fields.*

The way in which the cortex grows is intimately connected with this proportionate increase in area. Local processes of growth seem to be out of the question in the case of equal development. If one would adhere to the theory of local growing processes, they must be conceived as regulated in such a way that the equal enlargement of the fields is retained. It is hard to see the reason of such regulation. The proportionate enlargement of the cortex fields therefore must be accounted for by assuming that *the neo-cortex develops equally in all its parts,* a result which is in full accordance with the supposition applied for above.

R. BRUMMELKAMP: ON THE EQUAL GROWTH OF THE NEOCORTEX DURING THE ONTOGENESIS OF THE SHEEP.

1.

2.



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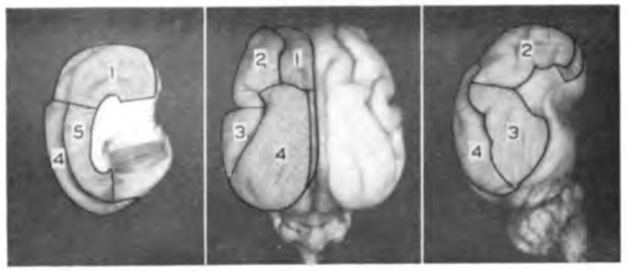

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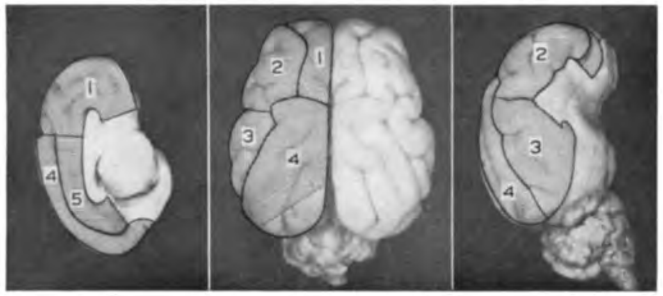
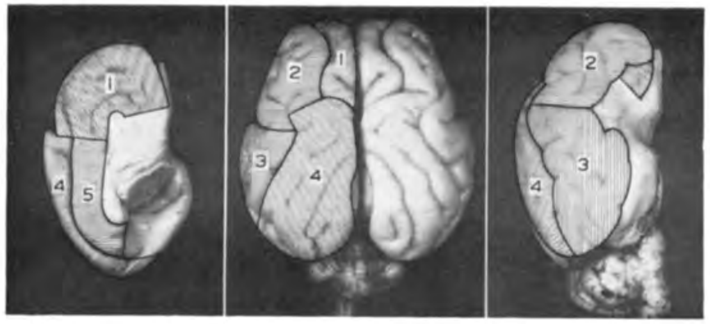


Fields: 1. fronto-medial; 2. fronto-lateral; 3. caudo-lateral; 4. caudo-intercalarial; 5. caudo-medial.

5.



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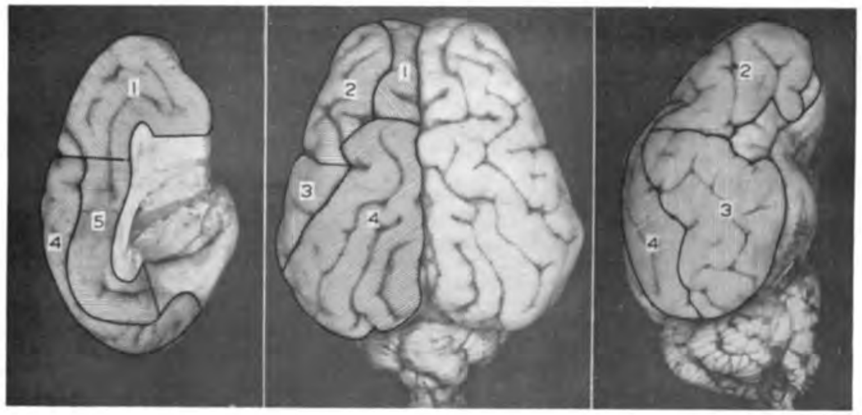

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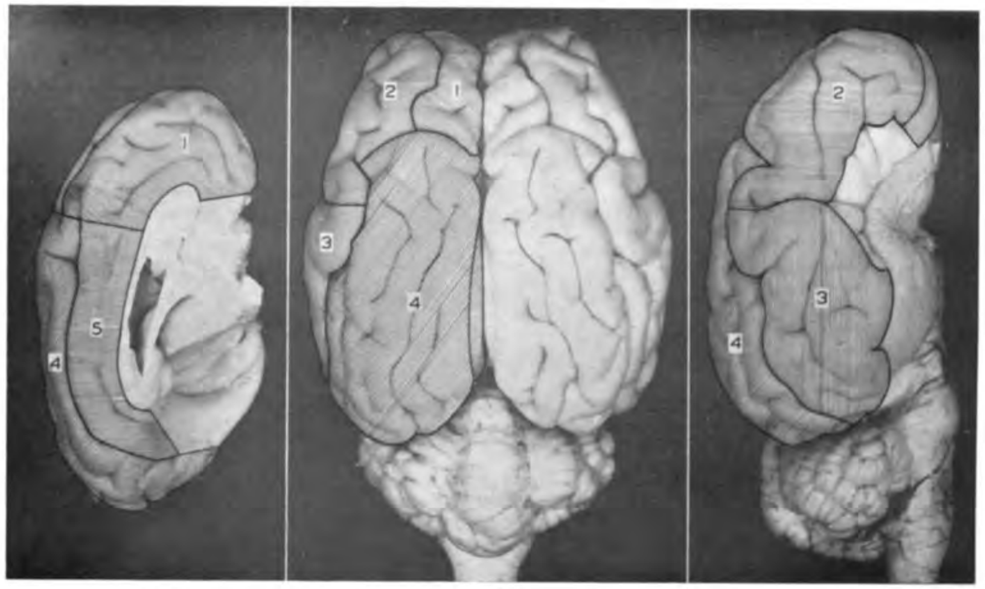

Fields: 1. fronto-medial; 2. fronto-lateral; 3. caudo-lateral; 4. caudo-intercalarial; 5. caudo-medial.

Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

Anatomy. — *The nucleus of* BELLONCI *in birds.* By J. L. ADDENS. (From the Dutch Central Institute for Brain Research, Amsterdam.) (Communicated by Prof. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of September 28, 1940.)

In a paper published in these Proceedings (Vol. **41**, 1938) I described in the rabbit and aardvark a nucleus, lying in the common stem of the lateral components of the stria medullaris, which I homologized with the nucleus ovalis of reptiles. Before me, in mammals, the same nucleus had been noticed by Miss GILBERT (1935) [1] in the human embryo, who already interpreted it as a nucleus ovalis, and by YOUNG (1936) in the guinea pig, who, however, regarded it as but an encapsulated portion of the interstitial nucleus of the stria terminalis.

The nucleus ovalis is a small ellipsoidal cell group in the foremost part of the thalamus, first described by HUBER and CROSBY (1926) in the American alligator [2] and subsequently also demonstrated for the other reptilian orders, with the exception, hitherto, of snakes. Most recently it was reported by STRÖER (1939) for Lacerta vivipara. Looking for homologues, in the Anamnia, of this cell group, I came to the conclusion in the paper cited at the outset that it corresponded with a nucleus described by BELLONCI (1888) in the frog as the "vorderer oberer Nucleus des Corpus geniculatum thalamicum", a homology, already suggested by CAIRNEY (1926) and PAPEZ (1935). HERRICK (1925), who studied this nucleus in great detail in amphibians, and found that it is a correlation centre for olfactory, optic and somesthetic impulses, called it nucleus of BELLONCI. This term I also applied to the homologous nuclei of reptiles and mammals, since HERRICK's term (1925) had the priority over HUBER and CROSBY's (1926). Moreover, the term ovalis may give rise to confusion with similar ones, used for other thalamic nuclei, viz., ovoidalis and ovoideus.

The nucleus of BELLONCI, first noticed in amphibians, thus having been demonstrated for reptiles and mammals, it was but natural, as a first step to augment our knowledge of the occurrence of this structure, to try to find it also in the remaining class of amniotes, viz., birds.

I started this search by the aid of a set of series of the sparrow (Passer

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[1] See for titles of papers already cited in the previous communication, the list of literature given there.

[2] This is not quite correct. As set forth in the historical survey at the end of the paper, RENDAHL (1924) already before HUBER and CROSBY (1926) described this nucleus in Varanus salvator and Alligator mississippiensis, calling it nucleus fasciculi septi.

domesticus), viz., a transverse and a sagittal WEIGERT–PAL-paracarmine series, a horizontal WEIGERT–PAL-alum-carmine series, and a transverse series stained for cells by HUBER's toluidin blue method. Having found the nucleus sought for, it was also studied in a transverse WEIGERT–PAL series, counterstained with various sorts of carmine, of the dove (Columba domestica) and in a transverse HUBER series of the same animal. This bird, due to the stronger development of the olfactory system, proved better suited for the study of the connections.

An inspection, in the sparrow, of the foremost part of the thalamus, the region where the nucleus of BELLONCI is found in reptiles and mammals, immediately revealed the presence of a clearly circumscribed nucleus, which by its general appearance made the impression to be the sought for structure. As in the cases, so far known, it was a small ellipsoidal body, whose neuropil stained strongly with paracarmine. In the dove about the same relations were found. As was to be expected, this clearly circumscribed and rather conspicuous nucleus had been described before. It appeared to be the nucleus lateralis anterior of EDINGER and WALLENBERG (1899), a cell group about which, despite its distinctness, a great confusion has arisen, which I shall try to disentangle at the end of the paper.

I shall start to put forward my findings in the sparrow, and already now shall speak of a nucleus of BELLONCI, leaving the argumentation for later.

In the sparrow the nucleus of BELLONCI is an ellipsoidal body of dense neuropil with numerous, mostly small cells in it, lying immediately dorsal to the margin of the foremost part of the chiasma, separated, however, from it by the tractus isthmo-opticus (figs. 1, 2, 3, 4). The major axis of this ellipsoidal body extends in a transverse plane from ventromedial to dorsolateral, and is but little longer than the longer of the minor axes, which is directed sagittally. The third axis is considerably shorter than the other two.

The nucleus of BELLONCI lies dorsolateral to the voluminous cell group, mostly designated without specification as lateral geniculate, but which I agree with KUHLENBECK (1937) in regarding as the ventral lateral geniculate (figs. 1, 3). This nucleus begins a little more caudally than the nucleus of BELLONCI. Just in front of the latter the nucleus supraopticus and nucleus of the diagonal band of BROCA are situated, the former ventrally and applied to the optic tract, the latter more dorsally (fig. 4). The nucleus of BELLONCI is immediately succeeded by the nucleus rotundus (figs. 3, 4).

The ramus dorsalis of the tractus septo-mesencephalicus courses dorsal to the nucleus of BELLONCI, so that it is wedged in between this tract and the tr. isthmo-opticus (figs. 1, 2, 3, 4). Though closely applied to these tracts, so that it may synapse with them, there is no conclusive evidence, in our WEIGERT–PAL preparations, of such a supply, no more than by the optic tract. As far, however, as it is possible to judge from WEIGERT–PAL preparations, the chief supply of the nucleus of BELLONCI is provided

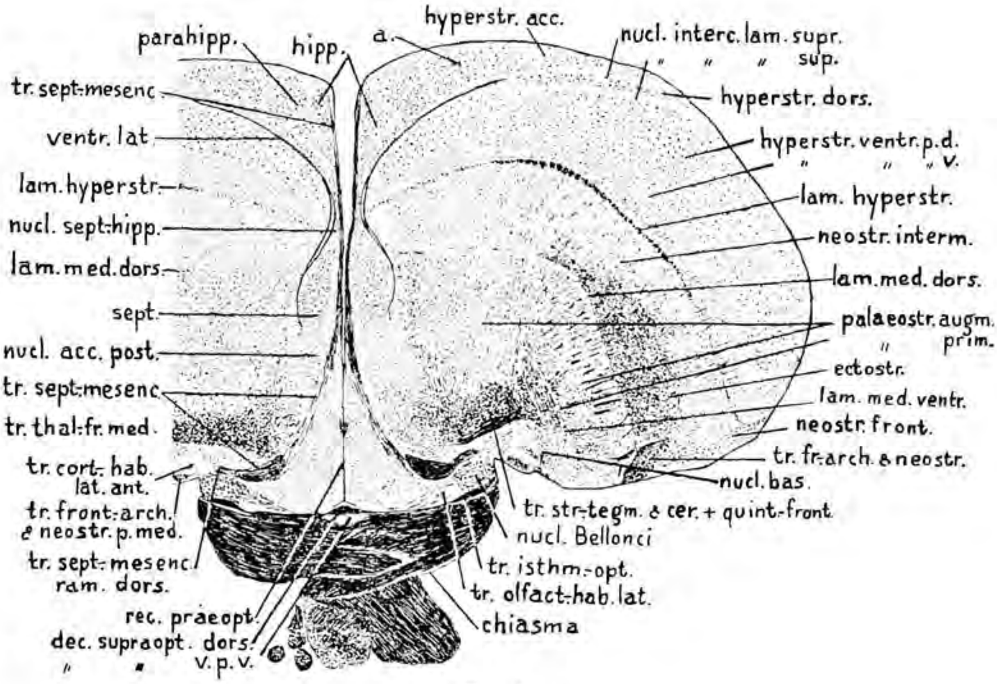

Fig. 1.

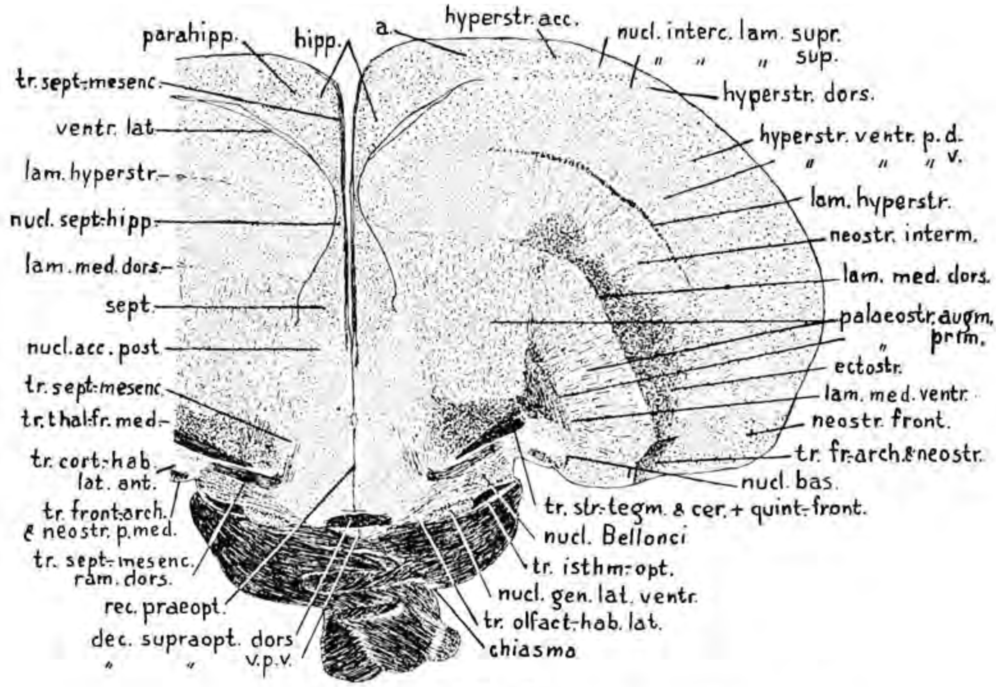

Fig. 2.

Figs. 1 and 2. SPARROW. Cross-sections through foremost part and middle of nucleus of BELLONCI, resp., the former just in front of the ventral lateral geniculate, the latter a little after beginning of this nucleus. WEIGERT–PAL-paracarmine. × 10.

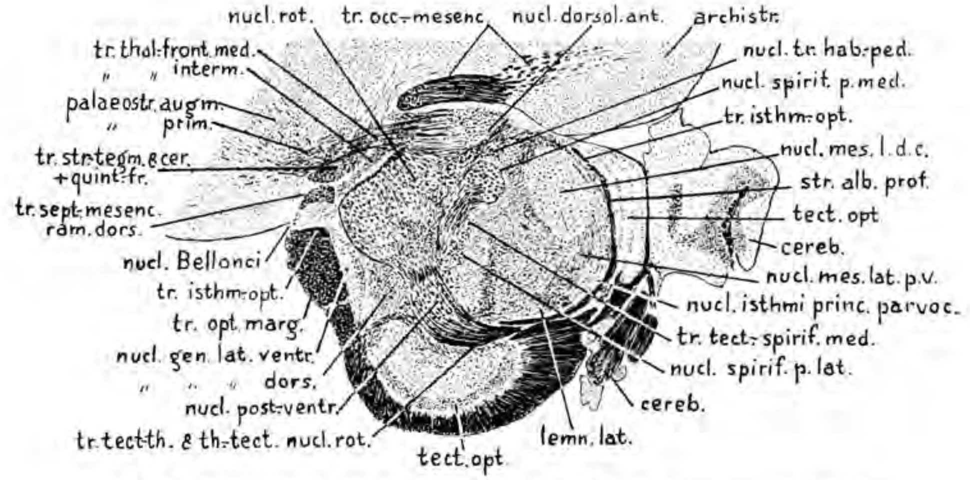

Fig. 3. SPARROW. Sagittal section through middle (in transverse direction) of nucleus of BELLONCI. WEIGERT–PAL-paracarmine. × 10.

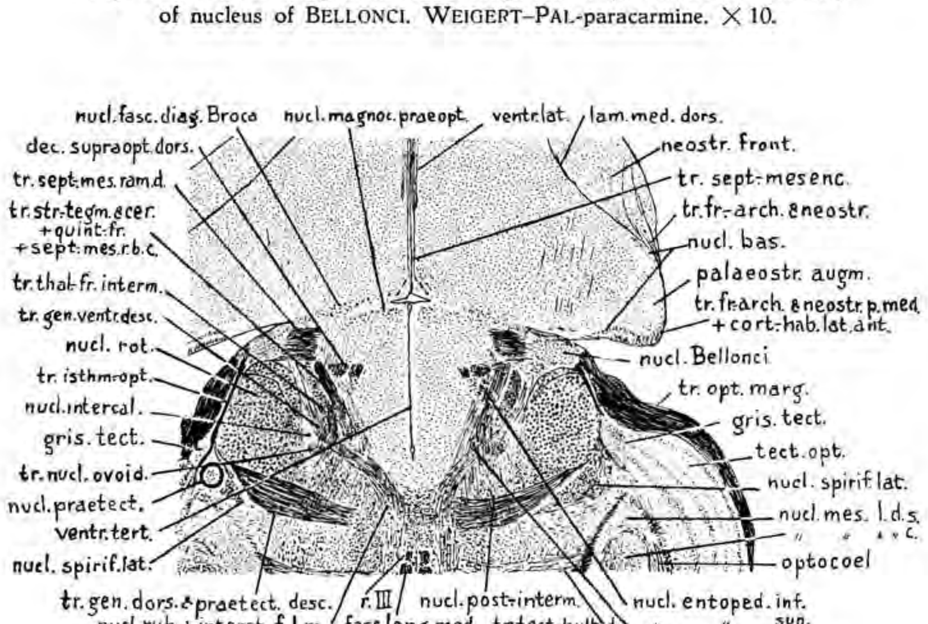

Fig. 4. SPARROW. Horizontal section through middle (in vertical direction) of nucleus of BELLONCI. WEIGERT–PAL-alum-carmine. × 10.

EXPLANATION OF SOME OF THE ABBREVIATIONS USED IN THE FIGURES.

a, differentiated portion of hyperstriatum accessorium (see HUBER and CROSBY, 1929).

comp. sept.-mes. str. m., component from tractus septo-mesencephalicus to stria medullaris.

gris. tect., griseum tecti, tectal gray of HUBER and CROSBY (1929).

nucl. acc. post., nucleus accumbens pars posterior.

nucl. bas., nucleus basalis.

nucl. interc. lam. sup., nucleus intercalatus laminae superioris (see KAPPERS, HUBER and CROSBY, 1936).

nucl. interc. lam. supr., nucleus intercalatus laminae supremae (see KAPPERS, HUBER and CROSBY, 1936).

nucl. mes. l. d. c., nucleus mesencephalicus lateralis pars dorsalis centralis.
nucl. mes. l. d. s., nucleus mesencephalicus lateralis pars dorsalis superficialis.
nucl. periv. m. acc., nucleus periventricularis magnocellularis accessorius (see KUROTSU, 1935).
nucl. periv. m. pr., nucleus periventricularis magnocellularis principalis (see KUROTSU, 1935).
nucl. post.-interm., nucleus postero-intermedius.
nucl. post.-ventr., nucleus postero-ventralis.
nucl. sept.-hipp., nucleus septo-hippocampalis.
tr. gen.-front. ventr., tractus geniculo-frontalis ventralis (see FREY, 1937).
tr. sept.-mes. r. b. c., tractus septo-mesencephalicus ramus basalis caudalis.

by a mass of loosely arranged fine myelinated fibres that seem to come from the dorsal supraoptic decussation (figs. 1, 2). These fibres diverge in the direction of the nucleus, embrace and traverse it, but at its lateral side again collect. Farther they could not be followed with certainty in the sparrow. In the dove, where they are much better studied, we shall revert to them.

Also in this bird the nucleus of BELLONCI is an ellipsoid (figs. 5, 6), but here the major axis, in contradistinction with the sparrow, where it lies in a transverse plane, is directed sagittally, while the longer of the minor axes, which is but little shorter than the major axis, extends almost vertically.

In the dove the nucleus of BELLONCI shows about the same relationships to the neighbouring nuclei as in the sparrow, but those to the neighbouring and possibly synapsing tracts are somewhat different, in that the ramus dorsalis of the tractus septo-mesencephalicus is only in contact with the oral extremity of the nucleus. Very soon this tract, leaving the nucleus, turns dorsomedially, and runs caudalward along the surface of the brain (figs. 5, 6).

No more than in the sparrow the origin of the fibres coming from ventromedial could be ascertained in the dove, though they are far more numerous here. All that I can say is that they seem to come from the dorsal supraoptic decussation. Part of them swing medially across the dorsal peduncle of the lateral forebrain bundle and join the stria medullaris, another part do so with the ramus dorsalis of the tractus septo-mesencephalicus. The latter, however, is connected by a compact strand of fibres, coursing along the surface of the brain, with the stria medullaris (fig. 6). These connecting fibres, in part at least, may be the same as that portion of the fibres coming from ventromedial which joined the tractus septo-mesencephalicus.

Evidently the loose fibre mass coming from ventromedial represents the tractus olfacto-habenularis lateralis, which originates in the preoptic and hypothalamic regions. This fibre mass has not been seen by HUBER and CROSBY (1929), who state that they could not identify a tractus olfacto-habenularis lateralis in birds. The strand of fibres, however, connecting the septo-mesencephalic tract with the stria medullaris, has been observed

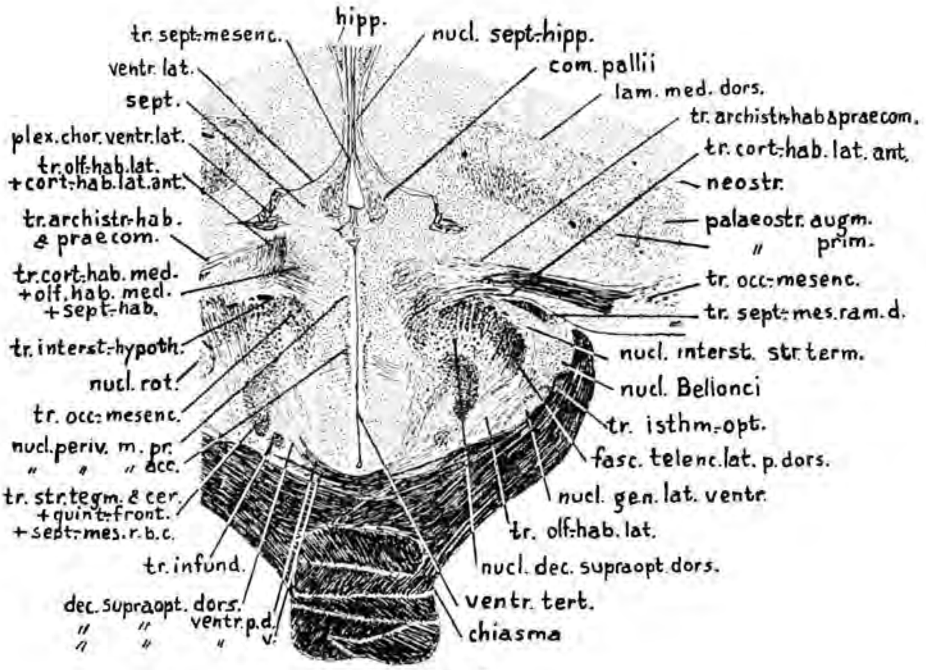

Fig. 5.

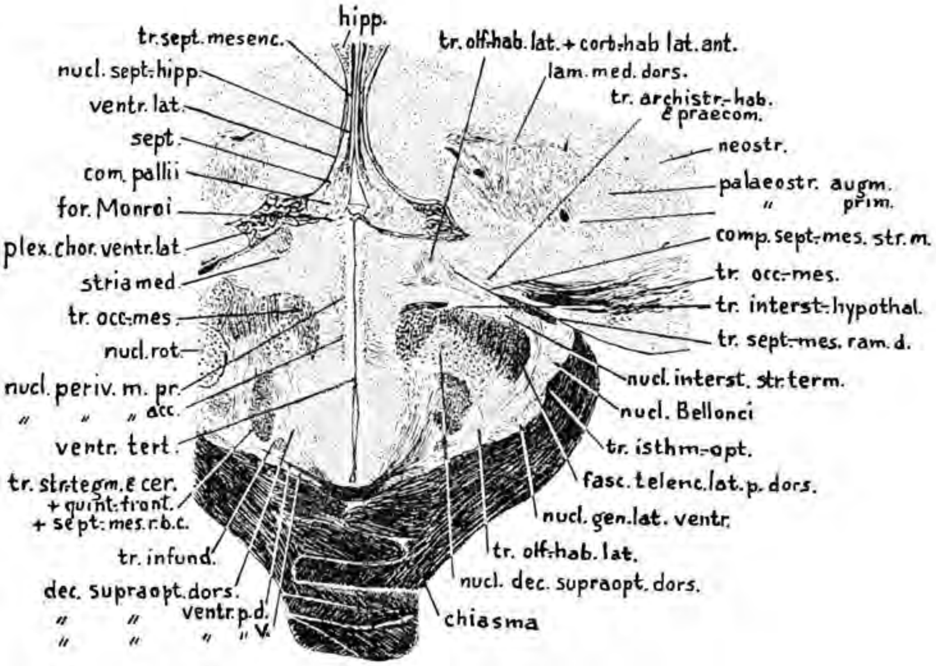

Fig. 6.

Figs. 5 and 6. DOVE. Cross-sections through hindmost part of nucleus of BELLONCI and beginning of stria medullaris, fig. 6 some sections behind fig. 5. WEIGERT–PAL-paracarmine, × 8.

by HUBER and CROSBY (their component from septo-mesencephalic system to habenula).

The tractus olfacto-habenularis lateralis is not reported here for the first time for birds, it was described in them long ago as tractus strio-habenularis by MÜNZER and WIENER (1898) and EDINGER and WALLENBERG (1899). Also KAPPERS and THEUNISSEN (1908) mention this bundle.

Medial to the nucleus of BELLONCI and coextensive with its caudal half lies a cell mass, which is closely applied to the dorsal peduncle of the lateral forebrain bundle (figs. 5, 6). At its beginning it only covers the lateral aspect of the lateral forebrain bundle, but gradually it expands medially so as to cover this bundle dorsolaterally too. Also in the sparrow this nucleus is present, but here it lies farther backward (fig. 7). No doubt this is the crusta peripeduncularis of EDINGER and WALLENBERG (1899), which cell mass is only referred to of later authors by GROEBBELS (1924), who in the dove and chick is inclined to identify it with his geniculatum laterale tertium anterius [1]) and by RENDAHL (1924, p. 315), who could not find it in the chick. It may be the same as the bed nucleus of the tractus thalamo-frontalis medialis of HUBER and CROSBY (1929, p. 89). I suppose this cell group to be the interstitial nucleus of the stria terminalis [2]), on the ground that it occupies exactly the same position relative to the lateral forebrain bundle and nucleus of BELLONCI as the latter does in crocodilians (compare figs. 5 and 6 of the dove with fig. 2d. of Crocodilus porosus in the previous paper).

We shall now put forward our arguments for regarding the nucleus lateralis anterior of birds as the homologue of the nucleus of BELLONCI of amphibians, reptiles, and mammals.

In the first place the position of the two nuclei is quite similar, both being situated in the foremost part of the thalamus, dorsal to the oral extremity of the ventral lateral geniculate, with the restriction, however, that in anures the nucleus of BELLONCI has shifted rather caudad along the dorsal side of the lateral geniculate (ADDENS, 1938, fig. 1). In mammals the nucleus of BELLONCI occupies its usual rostral position, but here the lateral geniculate has shifted caudad, so that there is a gap between these nuclei.

As to the development of the nuclei, we are comparing, the nucleus of BELLONCI is known to arise from the ventral thalamus (see ADDENS, 1938). About the development of the nucleus lateralis anterior of birds RENDAHL (1924) and KUHLENBECK (1937) unfortunately are at variance. According to RENDAHL (1924), in the chick this nucleus splits off from the foremost

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[1]) This is not correct. Both in the dove and chick the crusta peripeduncularis lies farther forward and more medially than GROEBBELS' geniculatum laterale tertium anterius. In my opinion the latter is the foremost dorsal part of the tectal gray of HUBER and CROSBY (1929).

[2]) In the previous paper, following CAIRNEY (1926) for Sphenodon, I labelled this nucleus as the interstitial nucleus of the tractus amygdalo-praeopticus.

upper part of a cell plate, which for the rest gives rise to the ventral lateral geniculate ¹), and thus would belong to the ventral thalamus. According to KUHLENBECK (1937), working also with the chick, the nucleus lateralis anterior, however, is a rostral differentiation of the primordium of the dorsal lateral geniculate, which belongs to the dorsal thalamus.

In amphibians as well as in reptiles and mammals the nucleus of BELLONCI is an ellipsoid, whose major axis surpasses the longer of the minor axes two times or more in length. In the first class the major axis of the ellipsoid is directed sagittally, whereas in reptiles and mammals it is situated in a transverse plane. In the previous paper I explained this difference by a turning of the nucleus through 90°. It may as well be, however, that it has been brought about by a compression in sagittal direction. Like the nucleus of BELLONCI also the nucleus lateralis anterior is an ellipsoid, with, however, but a slight difference in length between the major axis and the longer of the minor axes.

As to the histological structure of the nucleus of BELLONCI, in amphibians this nucleus is a dense mass of neuropil, with but a few cells within it, almost all of these being situated outside the neuropil, viz., on its medial side. In reptiles and mammals, however, the cells lie within the neuropil. The avian nucleus lateralis anterior is likewise a dense mass of neuropil, with the cells within it.

A comparison of the connections of the two nuclei also strongly speaks in favour of their homology. In the previous paper it was set forth that the chief supply to the nucleus of BELLONCI (in amphibians and crocodilians at least) was by the tractus olfacto-habenularis lateralis ²), and that a contribution from the optic might be deemed certain in amphibians, probable in reptiles, and not excluded in mammals. The connection with the optic was confirmed of late by STRÖER (1939), who found that in specimens of Triturus taeniatus where in early stages the primordium of one eye was extirpated, the neuropil of BELLONCI on the heterolateral side was smaller than on the other.

As to the connections of the nucleus lateralis anterior, when describing this nucleus in the sparrow and dove, I already expressed as my opinion

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¹) As set forth below, the nucleus lateralis anterior is identical with RENDAHL's nucleus fasciculi septi (labelled ä by him), while the ventral lateral geniculate of KUHLENBECK and us corresponds with RENDAHL's corpus geniculatum thalamicum.

²) In the previous paper I distinguished in crocodilians a tractus olfacto-habenularis lateralis anterior and posterior, identifying the former with the tractus olfacto-habenularis lateralis of Miss CROSBY (1917) and the latter with her tractus olfacto-habenularis posterior. As I now perceive, the latter identification is not correct. The tractus olfacto-habenularis posterior of Miss CROSBY arises near the posterior end of the hemisphere from the nucleus of the lateral olfactory tract and the ventro-medial nucleus. The fibres, however, described by me as tractus olfacto-habenularis lateralis posterior, from an unknown source ascend through the forebrain bundle to the stria medullaris. They have already been seen by UNGER (1911) (his tractus transversalis taeniae) in crocodilians and by CAIRNEY (1926) in Sphenodon. Thus the two terms proposed by me have to be dropped.

that its chief supply is by fibres coming from ventromedial and ascending to the stria medullaris, which I interpreted as the tractus olfacto-habenularis lateralis. I could not decide, however, in my WEIGERT–PAL preparations whether the optic tract, the tractus isthmo-opticus and the ramus dorsalis of the tractus septo-mesencephalicus, all of which are closely applied to the nucleus lateralis anterior, actually synapse with it. Regarding this, however, the Japanese SHIINA (1932), in his monograph on the nucleus lateralis anterior, gives some, though not very convincing, information. Partly on the basis of normal preparations (WEIGERT–PAL), partly of MARCHI and GUDDEN experiments, he came to the conclusion that fibres of the ramus dorsalis of the tractus septo-mesencephalicus pass only through the nucleus, but that optic fibres actually end here.

Contrary to HUBER and CROSBY (1929), who did not see the tractus olfacto-habenularis lateralis, SHIINA described these fibres in all the birds, investigated by him, but regarded them as arising in the nucleus lateralis anterior and running in medial direction, partly to join the supraoptic decussations. Doubtless, however, the fibres in question ascend to the stria medullaris, as described and figured here for the dove (figs. 5, 6).

Finally it may be mentioned that the only connection HUBER and CROSBY (1929) name for the nucleus lateralis anterior is their tractus thalamo-frontalis intermedialis (better intermedius). SHIINA confirms this both in normal and MARCHI preparations and, moreover, on the same basis, mentions a connection with the tractus thalamo-frontalis lateralis. In my WEIGERT–PAL preparations I can confirm neither of these statements.

From this discussion of the connections of the nucleus lateralis anterior of birds thus much seems to be sure that the tractus olfacto-habenularis lateralis provides the chief supply, and that a connection with the optic is very probable, just the same conclusions we arrived at for the nucleus of BELLONCI of amphibians and reptiles. Thus, with the possible exception of embryological development, all criteria for the homology of nuclei available: position in the adult, connections, histological structure, and shape are in favour of the homology advocated by us.

The only other interpretation so far offered of the nucleus lateralis anterior of birds, is by KUHLENBECK (1937). As mentioned above, according to him, in the chick this nucleus is a rostral differentiation of the same cell plate which for the rest gives rise to the dorsal lateral geniculate. If, moreover, it could be proved that the nucleus lateralis anterior receives optic fibres, this nucleus should simply be called pars rostralis corporis geniculati dorsalis. The anterior part of the optic tract, however, which, according to KUHLENBECK, possibly supplies the nucleus lateralis anterior in all probability is nothing but our tractus olfacto-habenularis lateralis.

While, apart from KUHLENBECK's, no attempts have been made to find homologues for the nucleus lateralis anterior of birds, either lower or higher in the scale of vertebrates, it has been tried several times to homologize the nucleus of BELLONCI of amphibians or ovalis of reptiles with centres of higher forms in a different manner from ours. If these attempts had been successful, they would, of course, invalidate one or more of the conclusions arrived at by us, and so we have to discuss them. For want of space, however, we can do so only with the two most important among these hypotheses, viz., those of HUBER and CROSBY (1929), which is the same as that of KAPPERS (1938), and of LE GROS CLARK (1932).

HUBER and CROSBY (1929) are inclined to homologize their avian nucleus superficialis parvocellularis with the nucleus ovalis of reptiles. This nucleus superficialis parvocellularis is the same as the nucleus of the septo-mesencephalic tract of EDINGER and WALLENBERG (1899). The name was first used by RENDAHL (1924), who, however, under this term included the nucleus lateralis of EDINGER and WALLENBERG. The nucleus superficialis parvocellularis is a band of gray matter along the lateral aspect of the dorsal thalamus throughout almost the whole of its extent, in which the ramus dorsalis of the septo-mesencephalic tract on its way to the midbrain splits up for a great part (fig. 7).

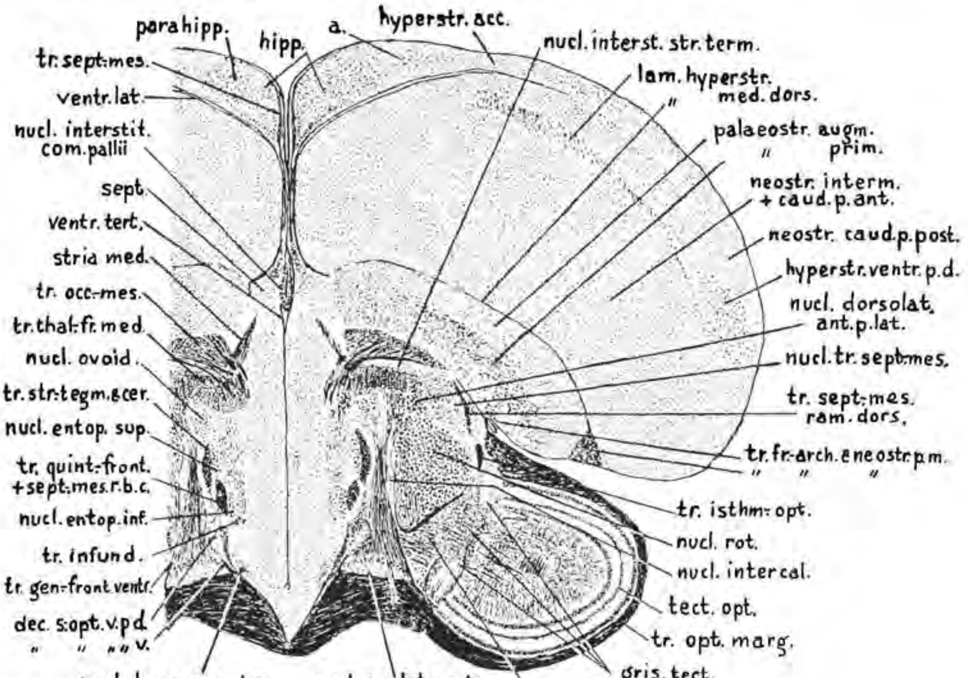

Fig. 7. SPARROW. Cross-section approximately through middle of thalamus. WEIGERT–PAL-paracarmine. × 10.

The first argument of HUBER and CROSBY in homologizing the nucleus superficialis parvocellularis of birds with the nucleus ovalis of reptiles is that the position of the two (dorsal to the lateral forebrain bundle and lateral to the nucleus dorsolateralis anterior) is approximately the same. From our side, however, it may be stated that the position of the nucleus lateralis anterior relative to this bundle and nucleus, in the dove (figs. 5, 6) at least, is not so very much different.

The second argument of HUBER and CROSBY in favour of the homology of the nucleus superficialis parvocellularis and nucleus ovalis' is derived from the connections. Further study of their alligator material suggested that there are fibres from the medial forebrain bundle (or associated with it) which swing around the ventral side of the lateral forebrain bundle and into the nucleus ovalis. Such fibres would be homologous with the ramus dorsalis of the septo-mesencephalic tract and the nucleus with the nucleus of that tract or nucleus superficialis parvocellularis. In the text-book of KAPPERS, HUBER and CROSBY (1936) this statement, however, is not repeated.

Contributory evidence to a homology of the nucleus superficialis parvocellularis and nucleus ovalis, according to HUBER and CROSBY, is found in the distribution of optic fibres to both these nuclei. For the nucleus ovalis, as set forth above, the presence of optic connections may hold good, but neither for the sparrow, nor the dove I can confirm HUBER and CROSBY's findings of optic fibres going to the nucleus superficialis parvocellularis.

Our chief objection against the homology advocated by HUBER and CROSBY, is that a connection of the nucleus superficialis parvocellularis with the tractus olfacto-habenularis lateralis is entirely lacking. Moreover, and this also is an important point, this nucleus is voluminous and of considerable length, extending from about the beginning of the thalamus to the posterior commissure, whereas the nucleus ovalis is small and confined to the foremost part of the thalamus. And, as the latter is chiefly an olfactory centre, it is extremely improbable that in the microsmatic birds it has developed into a structure several times its size.

The second attempt, in which we are concerned, at homologizing certain thalamic centres is LE GROS CLARK's (1932) regarding the nucleus of BELLONCI of amphibians as the forerunner of the dorsal lateral geniculate of mammals. Already in reptiles, according to LE GROS CLARK, a dorsal lateral geniculate may be recognized (the homologue consequently, in his opinion, of the nucleus of BELLONCI of amphibians). In the American alligator, according to LE GROS CLARK, this centre is represented by a differentiated lateral portion of the nucleus dorsolateralis anterior, described by HUBER and CROSBY (1926), which is said by these authors to receive some optic fibres. In Sphenodon these superficial cells even form a fairly circumscribed nucleus, which has been termed the dorsal nucleus of the lateral geniculate body by CAIRNEY (1926) and DURWARD (1930).

Thus LE GROS CLARK's view about the fate of the nucleus of BELLONCI of amphibians in amniotes (represented diagrammatically in fig. 3 of his paper, 1932) is entirely different from ours.

It must be objected to LE GROS CLARK that it is very doubtful if the differentiated lateral portion of the nucleus dorsolateralis anterior of the

American alligator, leaving alone if it is a constant feature, really receives optic fibres. In the transverse WEIGERT–PAL-paracarmine series of this animal at my disposal the staining of the cells was not sufficient to decide whether this differentiated portion was present, but I greatly doubt if optic fibres end in the region where it must be situated. Moreover, this cell group, if present at all, does not correspond with the dorsal lateral geniculate, described by CAIRNEY (1926) and DURWARD (1930) in Sphenodon. The latter, in my opinion, really is the homologue of the mammalian dorsal lateral geniculate. But also in the crocodilians there are a ventral and a dorsal lateral geniculate, the corpus geniculatum laterale of the alligator of HUBER and CROSBY, according to my interpretation, being the dorsal lateral geniculate, while their nucleus tractus tecto-thalamici cruciati is the ventral one (see fig. 3 of the previous paper).

As stated at the outset, despite its distinctness, a great confusion prevails about the avian nucleus I interpret as the nucleus of BELLONCI. In order to try to make an end to this confusion a short historical survey about the nuclei involved in it, may be appended here.

The first authors to notice our nucleus were EDINGER und WALLENBERG (1899) in their classical paper on the bird's brain, who, however, do not mention it in the text, but only indicate it in their figures.

KAPPERS (1921) in the first edition of his text-book in two figures relating to Pratincola rubicola (Vol. II, figs. 468, 469; p. 875, 876) labelled a nucleus in the dorsolateral part of the thalamus as nucleus lateralis anterior. It appears, however, from the text that the nucleus lateralis of EDINGER and WALLENBERG is meant.

Not before the lapse of the quarter of a century after EDINGER and WALLENBERG's (1899) paper, the nucleus lateralis anterior was seen again, by RENDAHL (1924), in the embryo and adult chick. He, however, erroneously identified it with the nucleus of the septo-mesencephalic tract of EDINGER and WALLENBERG, calling it nucleus fasciculi septi, and thus gave rise to the subsequent confusion. To the nucleus of the septo-mesencephalic tract RENDAHL gave a new name, viz., nucleus superficialis parvocellularis, including with this cell group the nucleus lateralis of EDINGER and WALLENBERG. RENDAHL found his nucleus fasciculi septi also in reptiles (Varanus salvator and Alligator mississippiensis). From his description it is clear that this is no other than the nucleus of BELLONCI (alias ovalis) of these animals. Thus RENDAHL (1924) already before HUBER and CROSBY (1926) observed the nucleus ovalis, and moreover, already recognized its homology with the nucleus lateralis anterior of birds.

CRAIGIE (1928) has followed RENDAHL in his error, the nucleus he describes in humming birds as nucleus tractus septi, being the nucleus lateralis anterior of EDINGER and WALLENBERG.

It was HUBER and CROSBY (1929), who in their comprehensive studies on the avian diencephalon for the first time correctly identified the nucleus lateralis anterior. They failed, however, to see that this nucleus was mistaken by RENDAHL for the nucleus of the septo-mesencephalic tract.

In his second paper on the avian brain (kiwi) CRAIGIE (1930) still designates the nucleus lateralis anterior as nucleus tractus septi. Moreover, he now described another cell group immediately dorsomedial to the latter as nucleus lateralis anterior, whereby the prevailing confusion was still enhanced. In his last paper, dealing with this subject, CRAIGIE (1931), however, realized that his nucleus tractus septi was identical with the nucleus lateralis anterior of EDINGER and WALLENBERG, and that, consequently, the nucleus of the kiwi, lying dorsomedial to the latter and called nucleus lateralis anterior by him, was something else. In my opinion this nucleus is the crusta peripeduncularis of EDINGER and WALLENBERG, our interstitial nucleus of the stria terminalis (figs. 5, 6, 7).

Finally it may be mentioned, that, like HUBER and CROSBY, to whom belongs the credit to have done so for the first time, also SHIINA (1932) and KUHLENBECK (1937) correctly identified the nucleus lateralis anterior.

SUMMARY.

It is argued that the nucleus lateralis anterior of birds, first described by EDINGER and WALLENBERG (1899), is the homologue of the nucleus of BELLONCI of amphibians, reptiles and mammals.

The crusta peripeduncularis of EDINGER and WALLENBERG is homologized with the interstitial nucleus of the stria medullaris.

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Anatomy. — *Verdubbeling van den hoektand.* Door TH. E. DE JONGE-COHEN. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of September 28, 1940.)

Wanneer wij in de oudere literatuur de vraag tegenkomen, of óók bij den hoektand overschrijding van het normale aantal mogelijk is, dan vinden wij deze vraag gemeenlijk in dier voege beantwoord, dat deze mogelijkheid ontkend dan wel betwijfeld wordt — in het gunstigst geval wordt de manifestatie van een *overtolligen* cuspidatus als hooge uitzondering erkend.

Nu moge dit antwoord in het licht onzer huidige opvattingen weinig bevredigend — immers onvolledig — heeten, men verlieze niet uit het oog, dat de oudere onderzoekers, die niet of nauwelijks bekend waren met het begrip tandverdubbeling, slechts een gebrekkig inzicht hadden in het geheele probleem der hyperodontie.

Thans weten wij, dat hyperodontie nòch bij de incisivi nòch bij de postcanine gebitselementen tot die marquante anomalieën gerekend mag worden, wier teratologische merkwaardigheid alleen reeds beschrijving genoegzaam motiveert. In scherpe tegenstelling daarmede vormt overschrijding van het normale aantal der cuspidati een zóó uitermate zeldzaam phænomeen, dat de waarneming van een geval van hoektandverdubbeling in melkgebit en permanente serie ons de gereede aanleiding biedt, om in aansluiting op onze beide vorige publicaties ten deze (1) opnieuw de aandacht voor dit onderwerp te vragen.

Reeds bij oppervlakkige beschouwing blijkt, hoe ongewoon spaarzaam de gegevens zijn, welke de literatuur ons omtrent de doorbraak van meerdere cuspidati verstrekt — om het even, of daarbij van hoektandverdubbeling sprake is dan wel van overtollige hoektanden. Nu behoeft dit ons niet al te zeer te verrassen; morphotisch toch vertegenwoordigt de caninus ongetwijfeld het meest gefixeerde element in 's menschen gebit: zijne vormanomalieën zijn nòg veel zeldzamer dan bij den medialen snijtand of eersten molaris. Zoo ligt het derhalve voor de hand, dat óók zijne numerieke afwijkingen zich door even uitzonderlijke zeldzaamheid kenmerken.

Is het onder deze belichting derhalve begrijpelijk, dat eerst in 1924 de mededeeling van EGGER de mogelijkheid van agenesie der cuspidati boven elken twijfel verhief, niet zóólang behoefden wij op de eerste gevallenbeschrijving van hoektandverdubbeling te wachten: wel ontbeerde BAŠTÝŘ [1889 (2)] den steun van röntgenologie en photographie, zijn pen en teekenstift daarentegen getuigen op meesterlijke wijze van 's schrijvers feilloozen clinischen blik! Zijne uitnemende beschrijving vormt stellig de eerste volledig gedocumenteerde observatie van hoektandverdubbeling (afb. 1 en 2) [1]. Daarnaast vinden wij nog een aantal andere gevallen vermeld — zoo b.v. door DEPENDORFF, ESCHLER, GOTTARDI, MATHIS, SCHEFF, STAFNE en TAVIANI [2] (3—9) — doch stellig kunnen deze niet alle den toets der critiek doorstaan: zoo is ons b.v. uit persoonlijke waarneming gebleken, dat het door ESCHLER beschreven specimen in werkelijkheid een geval van verdubbeling van den lateralen incisivus voorstelt! Trouwens: ook de overige gevallen zouden wij niet gaarne alle zonder eenige restrictie voor onze rekening durven nemen.

Immers dienen wij er terdege rekening mede te houden, dat zich zoowel mediaal als distaal van den caninus overtollige elementen manifesteeren kunnen, veelal niet meer dan aplastische tandrudimenten, welke genetisch niettemin in geenerlei relatie tot dezen behoeven te staan. Bovendien echter kan de laterale incisivus in de configuratie zijner kroon dermate aan zijnen distalen synergeet herinneren — MÜHLREITER vestigde daar al lang terug de aandacht op — dat wij reeds bij vroeger gelegenheid (10) betoogden, dat het wel eens zoude kunnen blijken „dat de z.g. overtollige hoektanden niets anders zijn dan al dan niet overtollige gecaniniseerde laterale snijtanden: en daarop is, meenen wij, tot nog toe nooit de aandacht gevestigd [3])!"

* * *

Niettemin: vast staat, dat óók bij den hoektand het normale getal overschreden kan worden. En dan dringt zich als vanzelf de vraag op den voorgrond, wat dan wel de genetische beteekenis daarvan zijn moge. Bij de beantwoording dezer vraag willen wij geen poging wagen, de causale genese te achterhalen — wij weten het immers bij voorbaat reeds: ignorabimus!

Anders staan wij tegenover de formale genese. Deze heeft reeds bij vroegere onderzoekers de bijzondere aandacht getrokken en terecht hebben zij er den nadruk op gelegd, dat de phylogenese ten eenenmale in gebreke blijft, ons eene aannemelijke verklaring te schenken van de manifestatie van *overtollige* hoektanden.

In het licht der tegenwoordige opvattingen is het probleem aldus echter verkeerd gesteld. Immers, aan den hierboven geschetsten gedachtengang ligt de onjuiste præmisse ten grondslag, dat overschrijding van het normale aantal tanden *uitsluitend* op atavisme zoude kunnen berusten. Welnu:

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[1] Dat ook deze auteur zijn geval als overtolligen tand determineert, is in het licht onzer boven gegeven beschouwingen alleszins begrijpelijk.

[2] De door dezen laatste beschreven casus is daarom zoo opmerkelijk, wijl ook hier de anomalie zich in melkdentitie en blijvende serie kenbaar maakt. Of wij in zijn geval echter met verdubbeling dan wel met de manifestatie van een overtolligen cuspidatus te doen hebben, laat zich nòch uit zijne beschrijving nòch uit de bijgevoegde afbeeldingen met zekerheid vaststellen.

[3] Op. cit. pag. 720.

zonder nu nog de extreme opvattingen van HERBST (11) onder dit opzicht te deelen, mogen wij thans wel als vaststaand aannemen, dat — met name in het gebied der fronttanden — deze vermeerdering óók door verdubbeling, *schizodontie*, tot stand komen kan. Voor nadere bijzonderheden naar BOLK's bekende onderzoekingen ten deze verwijzende (12), brengen wij opnieuw in herinnering, dat in volkomen overeenstemming met het morphologisch uiteenloopend karakter van beide snijtanden de splitsingstendenz bij den lateralen gemeenlijk véél geprononceerder blijkt dan bij den medialen: reeds eerder wezen wij erop, dat bij dezen laatsten volledige splitsing veel minder frequent voorkomt.

Evenzoo mogen wij in dezen zelfden gedachtengang verdubbeling van den hoektand als eene difformiteit van waarlijk unieke zeldzaamheid beschouwen — *en dat wel zéér in het bijzonder, wanneer deze zich, gelijk in ons geval, in beide dentities kenbaar maakt* (afb. 8 e.v.).

Duidelijk is echter, dat de genese der tweelingstanden — beide immers derivaat van éénzelfden aanleg — het begrip „overtollige tanden" geen ruimte laat. Overgangsvormen daarentegen, tusschenvormen, zullen wij bij beide snijtanden in tal van gradaties aantreffen.

Hoe is het nu met den hoektand in dit opzicht gesteld? Het ligt voor de hand, dat óók *overgangsvormen* bij dezen tot de allergrootste uitzonderingen zullen behooren. Als zoodanig beschouwen wij het in afb. 3 en 4 weergegeven præparaat, welks vormdifferentiatie onmiddellijk de gedachte wakker roept aan de schizogene varianten der incisivi: het atrophisch karakter van zijn middelste randtuberculum vormt een scherp contrast met de ontwikkeling zijner laterale kroonsegmenten, welke zich beide als het ware reeds tot zelfstandige elementen trachten te emancipeeren.

Nòg progediënter ontwikkelingsstadium heeft de door BUSCH (13) waargenomen casus bereikt, uit wiens beschrijving wij navolgenden passus citeeren (zie ook afb. 5): „Ein gewaltiger Eckzahn endet in zwei getrennten Schmelzspitzen, zwischen denen noch eine kleinere Schmelzspitze eingeschaltet ist[4])."

Echter ook het melkgebit levert enkele fraaie voorbeelden. Zoo ontleenen wij aan ons eigen materiaal een geval, welks bilateraal-symmetrische splitsingstendenz alléén op den wortel betrekking heeft: door de bijzondere tijdsomstandigheden kunnen wij slechts de RÖNTGENogrammen weergeven (afb. 6). Maar met nadruk brengen wij het voor enkele jaren door HERBST[5]) (11) beschreven object in herinnering, omdat het adspect zijner kroon (vd. afb. 7) volkomen beantwoordt aan het beeld der schizodontie.

Ons eigen geval (afb. 8—13a) betreft een negenjarigen knaap; terwijl in zijn familie geenerlei verdubbelingsverschijnselen voorkwamen — ook in andere organen van serialen aanleg niet — waren in diens bovenkaak rechts twee normale melkhoektanden tot doorbraak gekomen, terwijl

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[4]) L.c. pag. 482—483, ib. fig. 10.

[5]) L.c. pag. 371 en afb. 33.

RÖNTGENoscopisch bovendien de aanleg van twee blijvende cuspidati vastgesteld kon worden. Voor nadere bijzonderheden naar de desbetreffende afbeeldingen verwijzende, onderstrepen wij nochtans, dat deze laatste — van welke de laterale na zijne eruptie verwijderd moest worden — eveneens beide van normale structuur bleken.

⁂

Besluiten wij met eene korte opmerking over de phylogenetische beteekenis der besproken dysmorphie. In morphologischen zin progessief — substraat immers eener orgaanverdubbeling — vormt zij eene neutrale variatie, welke de grenzen eener normale variabiliteit weliswaar verre overschrijdt, voor de historische ontwikkeling van ons dentaal systeem nochtans van geenerlei belang is.

SAMENVATTING.

Dat de oudere onderzoekers overschrijding van het normale aantal der gebitselementen gemeenlijk tot de ontwikkeling van overtollige tanden terugbrachten, is begrijpelijk: immers, eerst BOLK stelde daarnaast duidelijk de mogelijkheid van tandverdubbeling door kiemsplijting (*schizodontie*) in het licht.

Weliswaar hield BOLK zich daarbij in eerste instantie met de snijtanden bezig, doch, ons óók ten aanzien van den hoektand bij den door hem gegeven verklaringsmodus aansluitend, beschrijven wij een geval, hetwelk wij als uitzonderlijk zeldzaam voorbeeld van hoektandverdubbeling in lacteale en permanente serie interpreteeren.

ZUSAMMENFASSUNG.

Dasz die älteren Forscher Ueberschreitung der normalen Anzahl der Gebisselemente fast ausnahmslos auf die Entwicklung überzähliger Zähne zurückführten, ist verständlich: denn erst BOLK betonte ja daneben die Möglichkeit der Zahnverdopplung durch Keimspaltung (*schizodontie*).

Zwar beschäftigte sich BOLK dabei in erster Linie mit den Schneidezähnen, doch — uns auch hinsichtlich des Eckzahnes dem von ihm gegebenen Erklärungsmodus anschliessend — beschrieben wir einen Fall, den wir als ausserordentlich seltenes Beispiel von Eckzahnverdopplung in lactealer und permanenter Serie interpretierten.

SUMMARY.

It stands to reason, that older investigators usually ascribed the surpassing of the normal number of dental elements to the development

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Afb. 1.

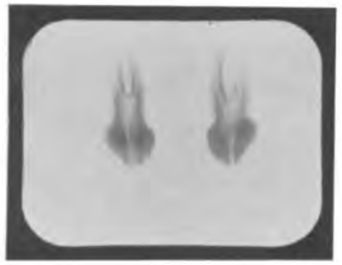

Afb. 2.


Afb. 3.


Afb. 4.


Oberer Eckzahn, welcher in drei Kronenspitzen ausläuft. Ansicht von der facialen Seite. Natürl. Grösse.

Afb. 5.


Afb. 6.

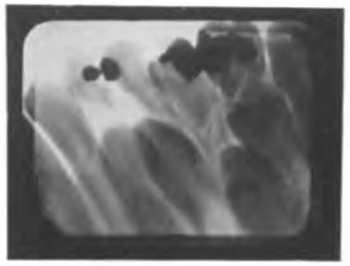

Afb. 7.

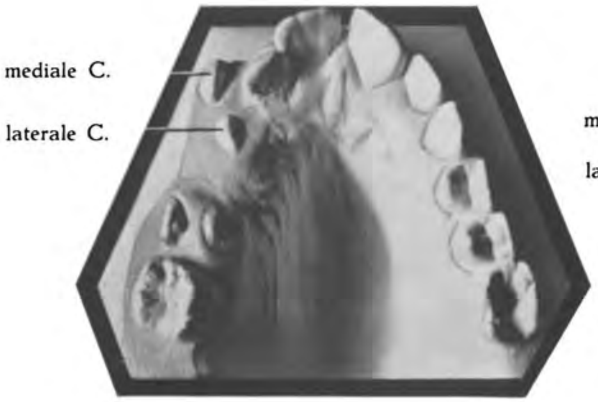
Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

Afb. 8.

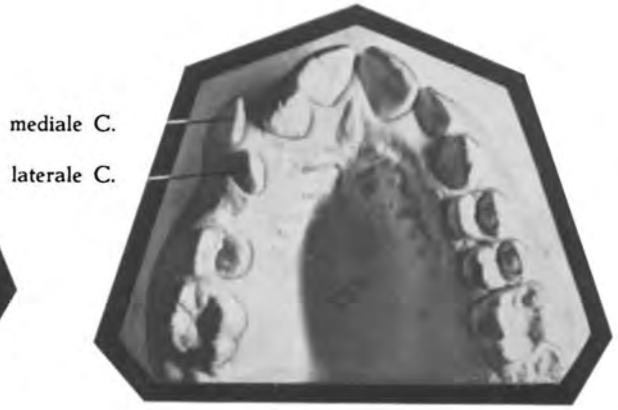

Afb. 8*a*.


Afb. 9.


Afb. 10.



Afb. 11.



Afb. 13.


Afb. 12.


Afb. 13*a*.

of superfluous teeth: for it was BOLK, who first clearly advanced the possibility of toothdoubling through germdivision (*schizodontia*).

It is true, that in the first instance BOLK occupied himself with incisors, but in agreement with his mode of explanation applied to the canine, we describe a case, which we interpret as an exceptionally rare exemple of doubling of the canine in the lacteal and in the permanent series.

RÉSUMÉ.

On s'explique, que les savants aient autrefois ramené le dépassement du nombre normal des éléments de la dentition au développement de dents supplémentaires, puisque c'est BOLK, qui a été le premier à mettre clairement en évidence la possibilité du dédoublement des dents par suite de la scission des germes dentaires (*schizodontie*).

BOLK s'est occupé en premier lieu, en traitant ce sujet, des incisives, il est vrai, mais en adoptant pour la canine également le mode d'explication, qu'il a proposé, nous décrivons un cas, que nous envisageons comme un exemple exceptionnellement rare du dédoublement d'une canine, tant dans la série des dents de lait que dans celle des dents permanentes.

Anatomisch-embryologisch Laboratorium
der Universiteit van Amsterdam.

TOELICHTING OP DE AFBEELDINGEN.

Verdubbeling in het blijvende gebit.

Afb. 1 en 2	: ontwikkeling van twee hoektanden (geval-BASTÝR).	
Afb. 3 en 4	: overgangsvorm (schizogene variatie), van faciaal en oraal gezien.	
Afb. 5	: u.s. (geval-BUSCH).	

Verdubbeling in het melkgebit.

Afb. 6	: geval van bilateraal-symmetrische wortelsplitsing en	schizogene variatie-BOLK.
Afb. 7	: splitsingstendenz van de kroon (geval-HERBST)	

Verdubbeling in beide dentities.

Afb. 8, 9*a* en *b*:	toestand op negenjarigen leeftijd — lacteale resp. blijvende elementen in situ.
Afb. 10	: beide rechter melkcuspidati in originali.
Afb. 11—13	: toestand nà wisseling der cuspidati.
Afb. 13*a*	: laterale blijvende cuspidatus in originali.

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Medicine. — *Paradoxical Effect of Oestrone in Male Animals.* V. *Further Arguments in Favour of the Hormonal Etiology of the Hypertrophy of the Prostatic Gland.* (From the department of Pharmacology, University of Leiden.) By J. H. GAARENSTROOM, S. E. DE JONGH and D. J. KOK. (Communicated by Prof. J. v. D. HOEVE.)

(Communicated at the meeting of September 28, 1940.)

Since a couple of years a definite tendency exists to see a causative relation between the prostatic hypertrophy and a dysfunction of the testicles, the latter resulting in a change of the hormonal equilibrium (male hormone-oestrone or a comparable compound) in favour of the latter.

We will not repeat here all our arguments, but under the reference to earlier papers (1) only remind of the so-called "paradoxical" enlargement, in the castrated male animal, of the derivatives of the Wolffian duct, obtainable with oestrone injections, an enlargement, which has typical places of predilection in each species, e.g. the prostatic gland in the dog.

These not repeated arguments are based upon a great number of analogies between the enlarged prostata of the castrate, treated with oestrone and of the patient, suffering from spontaneous "prostatic hypertrophy". The imposing similitude, however, is seriously disturbed in one respect, the histological structure. The aim of this communication is to denervate the objections against our theory, based upon this difference.

In short the latter, striking as it is, can be described, e.g. for the dog, as follows:

Whereas the increase in size mainly depends upon growth of muscle and connective tissue in both cases, important difference exists in the behaviour of the epithelium. The pathological findings in the clinical picture of the hypertrophy of the prostate gland should be interpreted, after REISCHAUER (2) and POLLAK (3) as a penetration of the new-built lobes of muscular and connective tissue by the glandular tubes. The tubes are *wide*, with a well developed *cylindrical epithelium* in one layer, which has a typically male apparition. It is classified as "male", because the same strongly arborised gland-complexes, covered with a high epithelium (though surrounded with less muscle and connective tissue) can be obtained in a castrate by injection of testosterone. However, in the "artificially enlarged" prostate, solitary small glandular tubes lay here and there in the fibro-muscular tissue; they *hardly* have *any lumen*, (in young castrated animals it is rather the picture of epithelial cords), and a *low cubic epithelium*, which, if high doses of oestrone are used, even converts into the *stratified form with cornification*.

On critical study these differences yield two questions to be solved: A: What is the cause of the "male" apparition of the epithelium in the prostate of the patients, and B: why does a metaplasia into a stratified epithelium in our experimental animals exist?

The second point gave us the lesser trouble, so herewith will be done first. From earlier experiments we knew already that this is to be contributed to overdosage with oestrone. Trying to obtain a growth within some weeks for which nature takes a much longer time, we are bound to too high dosages, so that in the excretory tubes or even the glandular tubes themselves a stratified epithelium is produced. We could (1) show already that this is not a necessary detail in the syndrome of the "artificial" prostatic hypertrophy: we could double the prostata weight with low dosages of oestrone in some weeks without the least metaplasia. We are customed to use dosages, of middling height, with which an augmentation of the prostata to a manyfold of the initial weight can be obtained and with which only the biggest excretory tubes show metaplasia of the epithelium. This may remain an aesthetic shortcoming, for technical reasons the production of a considerable enlargement in not too long a period is favourable for the experiments whereas, after the above deliberations, it can be done without objections against the theory.

The first point yields more difficulties and these don't lay in the "insufficiently male" aspect of the artificial prostatic hypertrophy, but rather in the male (often excessively male) one of the spontaneous form, since we wish to relate the latter with a *shortage* of male hormone!

When indeed during the period of development of the spontaneous hypertrophy the oestronelike element relatively prevails, it must be postulated that during this time the "male" effect of testosterone on the epithelium is more easily obtained than its counteraction of the prostata growth, promoted by oestrone.

One might suppose that a higher level of testosterone in the body would cause a still more "male looking" prostatic gland, what, however, conflicts with the histological picture of this gland during the period of optimal function of the testicles. Thus, one is guided to the supposition that *the epithelial effect of testosterone is enhanced by the relatively higher level of oestrone in the animal.* For androsterone this is known since long already ("pacemaker effect" FREUD (4)). With testosterone KUN & PECZENIK (5) and VAN DER WOERD (6) saw signs of it in the rat.

When this supposition is correct, no improbability or contradiction remains: Testosterone, because of insufficient quantities, is not able to prevent the growth, caused by oestrone, but (even grace to that oestrone) it can make the epithelium keep its normal aspect. Our experiments, in order to test the above considerations, were performed with four litters of young dogs, including a total of 13 male animals.

They were castrated at an age of ca 6 weeks. A short time afterwards they underwent several treatments, that will be described later on. After the autopsy a series of organs, under which the prostata bears the main importance, were weighed. The prostata was sectioned for histological investigation.

The treatment, body- and organweight of each dog are entabulated. The relative prostata weights refer to the body weight of the corresponding animal at autopsy, the relative dosages to the average of the body weights of the whole litter at the beginning of the experiment and at autopsy.

Litter I (Dog H.I.J.). All dogs received oestrone. Two of them moreover different quantities of testosterone.

The dosage of oestrone proved to be too low: the prostata is enlarged (cf. relative weights dogs M, P, Q and S) but not much. The quantities of testosterone, high as they were, could easily bring about a male (resp. supermale) structure in the prostatic gland. Since this was accompanied by distinct (normal!) growth, the experiment does not deliver arguments for or against our supposition.

Litter II (Dog K.L.M.).

Dosage of oestrone is doubled, testosterone far-going reduced. Third animal: control, treated with oil. Though the oestrone effect is definitely more distinct here than in dog H (as appears from weight and histological picture of the prostata (K)) testosterone easily prevails over the oestrone activity (L).

Again, however, a demonstrable, though small enlargement exists. We approach the optimal ratio between the two components, which is reached in

Litter III (Dog N.O.P., viz. fig. 1—3).

Dog O received in absolute sense somewhat less oestrone, relatively, however, somewhat more than in the foregoing experiment. In dog N this was combined with very little testosterone. Dog P received testosterone only.

In this dog P no demonstrable effect upon the prostata could be shown: no gain of weight (c.f. dog M and Q), and histologically no masculinization. In dog N the weight of the prostata is definitely lower than in dog O (absolutely as well as relatively). Nevertheless the gland had a more "male" apparition than those of dogs O and P: *instead of the epithelial cords in dogs O and P real tubes proved to have developed.*

Thus, in the presence of oestrone, the prostata-enlarging effect of which was hindered in the mean time, this quantity of testosterone revealed a masculinizing influence, to which it was not capable when acting alone.

Litter IV (dog Q, R, S, T, viz. fig. 4—7). A fourth pup in the litter produced us the advantage of an untreated control animal. For the rest it was corroborative for exp. III with a still smaller dosage of oestrone. Nevertheless the prostatic weight of the animal (T), treated with oestrone and testosterone comes again absolutely and relatively below that of the animal exclusively treated with oestrone (R); this quantity of testosterone, which alone caused no growth nor development of the prostata worth mentioning (S) induced an epithelial development of the gland to a certain degree in dog T, that received besides testosterone oestrone.

Dog	Daily dose in γ of		Same pro av. Kg dog		Bodyweight dog in Kg			Weight Prostata in gr	Same pro Kg dog at autopsy
	Test. pr.	Oestr.	Test. pr.	Oestr.	Beginning of the experim.	End of the experim.			
H	—	30	—	12.6	1.6	3.3	2.4	0.80	0.24
I	500	30	208	12.6	1.6	3.2		1.22	0.38
J	2000	30	832	12.6	1.2	3.3		4.29	1.30
K	—	200	—	22.4	6.7	13.4	8.9	4.93	0.37
L	700	200	79	22.4	5.7	11.6		5.40	0.47
M*	—	—	—	—	5.3	10.8		1.05	0.09
N	160	133	35	29	4.0	5.3	4.6	2.43	0.46
O	—	133	—	29	3.7	5.2		3.92	0.75
P	160	—	35	—	4.5	5.1		0.53	0.10
Q*	—	—	—	—	2.7	6.0	4.4	0.45	0.08
R	—	80	—	18	2.6	5.5		1.82	0.33
S	160	—	36	—	3.1	6.5		0.60	0.09
T	160	80	36	18	2.8	6.0		1.59	0.27

* Injected with oil as a control solution.

The differences between the "artificial prostatic hypertrophy" of the castrate and the spontaneous form of the non-castrate are no doubt not entirely taken away with the above experiments, but still we are very content already, to have been able to show, that with testosterone, in certain, carefully chosen dosages, a decrease in size of the prostatic gland, without preventing the hypertrophy as a whole, can be obtained together with a glandular development in "male" sense.

It is probable, that a somewhat higher dosage of testosterone, administered over a longer period (e.g. half a year) together with oestrone, should give rise in the hypertrophied prostata to an almost complete development of the glandular tissue. This sort of experiments, in which the average relative dosage partially remains an uncertainty unto the finish because of the growth of the dog, sothat the success is technically beyond the control of the experimentator, is not very inviting and in our opinion superfluous at the progress of our experiences.

We see the etiology of the hypertrophy of the prostate as follows:

The level of testosterone-like substances in the body decreases gradually,

J. H. GAARENSTROOM, S. E. DE JONGH and D. J. KOK: PARADOXICAL EFFECT OF OESTRONE IN MALE ANIMALS. V. FURTHER ARGUMENTS IN FAVOUR OF THE HORMONAL ETIOLOGY OF THE HYPERTROPHY OF THE PROSTATIC GLAND.

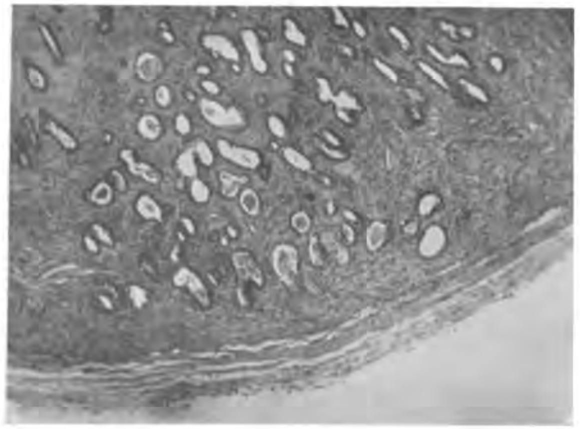

Fig. 1. N. Oestrone + testosterone prop.

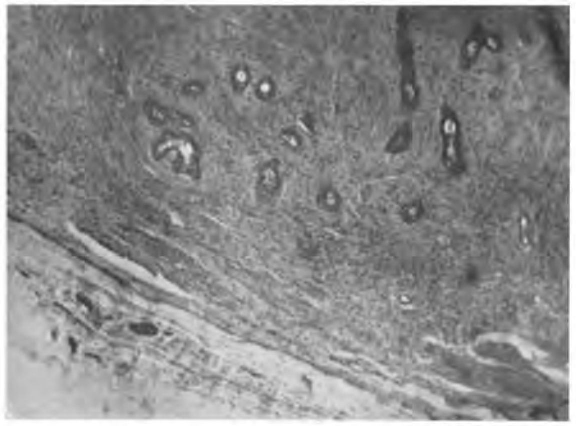

Fig. 2. O. Oestrone.

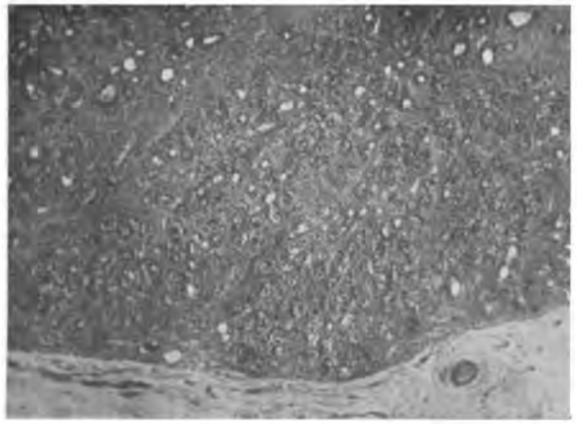

Fig. 3. P. Testosterone prop.

Fig. 1—3. Glandular part of prostatae of dogs N—P.

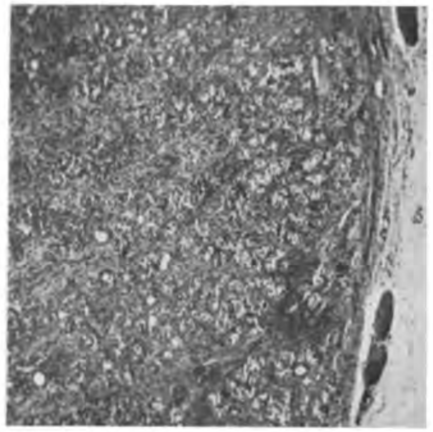

Fig. 4. Q. Oil.

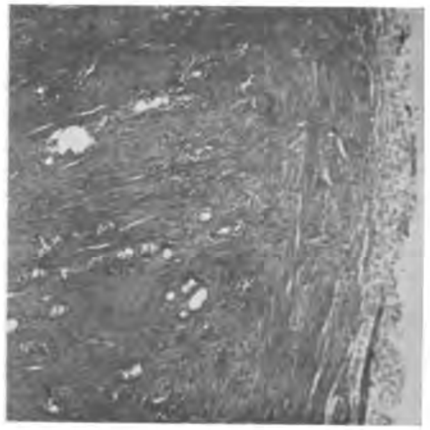

Fig. 5. R. Oestrone.

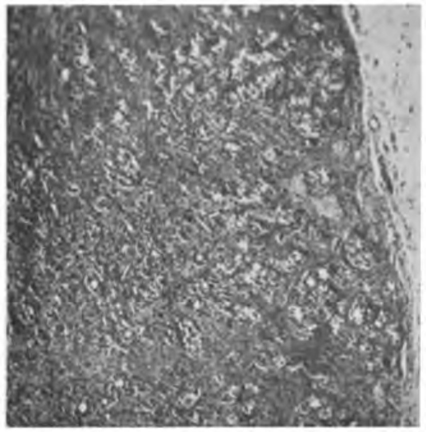

Fig. 6. S. Testosterone prop.

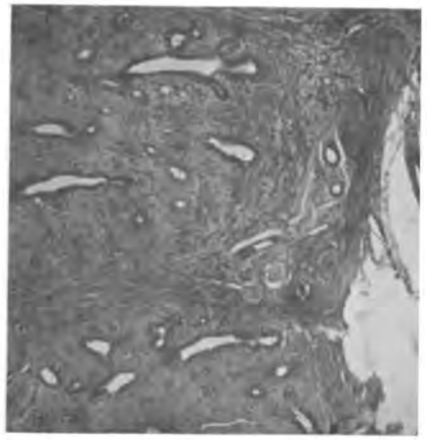

Fig. 7. T. Oestrone + testosterone prop.

Fig. 4—7. Glandular parts of dogs Q—T.

Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

sothat the oestrone-like component increases, relatively at least. When the former level comes so low, that the oestrone effect can not be fully hindered, the prostatic hypertrophy begins by growth of muscle and connective tissue. The small quantity of testosterone is on the other hand capable, with the aid of oestrone, to make the glandular system grow, also into the new developed parts of the organ, and to procure (remain procuring) a "male" aspect to the whole gland. When in exceptional cases the testosterone level would sink still lower, the male aspect could not be maintained and the gland would obtain the appearance of the uncomplicated "oestrone-prostate". Such an aspect actually was found by ZUCKERMAN and GROOME (7) in one of nine dogs suffering from spontaneous hypertrophy of the prostate. If the testosterone level decreases only slightly the ability to temper the oestrone-hypertrophy is mainly or totally preserved and a hypertrophy of the prostate does hardly or not occur.

Summary.

In a series of 13 male dogs, in which the influence of varying quantities of oestrone, testosterone and their combinations upon the weight and the histological picture of the prostatic gland was investigated the following was found:

1. A certain quantity of testosterone, together with oestrone, gives rise to a beginning differentiation of the epithelium which fails in case of each of the substances being given alone in the same dose.
2. This quantity of testosterone alone does not cause any growth of the prostata. It inhibits, but only very partially, the rank growth induced by oestrone, so that still a pathological hypertrophy remains.

Herewith the main difference between the spontaneous prostatic hypertrophy and the oestrone hypertrophy of the castrate is taken away and a new corroboration is furnished for the view that the hypertrophy of the prostatic gland is due to a "dysorchidy", consisting of a relative change of the equilibrium male hormone-,,oestrone-like substance", in favour of the latter.

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Medicine. — *The action of the so-called Restropic Factor.* By P. RUITINGA Jr.[1]. From the department of Pharmacology of the University of Leiden (Holland). Director: Prof. Dr. S. E. DE JONGH). (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of September 28, 1940.)

WETZLER–LIGETI and WIESNER came to the supposition of a pituitary substance, the so-called "restropic-factor", stimulating the reticulo-endothelial system (R. E. S.), since they could state, that congo-red, when intravenously administered to rabbits, proved to disappear more readily from the blood, when the animals were treated with certain extracts from the anterior pituitary lobe.

In an earlier investigation we could show, that this substance is not identical with some other substances of the anterior lobe, also related perhaps with the R. E. S. or with parts of it (viz. the substance, promoting blood demolition, the lienotropic factor and the thymotropic factor). This, however, does not needs argue against an activation of the R. E. S. by the restropic factor. But it is, on the other hand, questionable whether the rapidity, with which the congo-red disappears, runs parallel with the R.E.S. activity, and this being the only argument on which W. L. and W. based their hypothesis, we preferred to follow this more closely, before accepting their conclusions.

In literature one finds friends (ADLER and REIMANN, STERN and WILHEIM) as well as adversaries (LETTERER, NINOMIYA, RICHARDSON) of the idea, that the R. E. S. activity may be measured on the congo-red level in the blood. One of the vulnerable spots in this test is, that no one could show the storage of the congo-red in the body (at least, when the congo-red was given in the usual quantities). LETTERER claims (justly in our view), that when the disappearance of the congo-red is taken as a measure for the accumulating power, the latter must be showable histologically.

In order to establish, whether the hypothesis of WIESNER c.s., of a pituitary substance, stimulating the R. E. S., was right, experiments were done with congo-red and lithium-carmine (of the latter substance the accumulation is known to be readily showable!).

I. *Experiments with Congo-red.*

Since the storage of congo-red in the R. E. S. has never been exactly shown histologically, we also followed the *excretion* after the administration

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[1] Further details and literature, see Diss. Leiden 1939.

of WIESNER extracts, thus trying to find a correlation between the congo-red index [2]) and the *excretion*, for the unfortunately not too improbable case, that such a correlation between the congo-red index and the *storage* should not be showable.

As we also wished to investigate the effect of an impoverishment of WIESNER substance in the body, we were bound to use the rat, because of technical details (hypophysectomy!). So the congo-red test had to be transported from the rabbit on this animal. Hence the technique had to be changed a bit, since we now were compelled to work with so much smaller amounts of blood, but principally we followed the method, described by ADLER and REIMANN and later changed by STERN and WILHEIM, in rats as well as in rabbits.

Rats of 150—200 g. body weight received 0.8 ccm of a fresh 1 % congo-red solution in the femoral vein. After 6 and 60 minutes 0.1 ccm blood was tapped from the tail and brought in 1.4 ccm of a 3.8 % sodium citrate solution. After centrifuging 1 ccm of both fluids was colorimetered with a standard after addition of 0.2 ccm 10 % HCl. All experiments with the rats were done in duplo, like we did with our rabbits.

Eventual changes in the c.-r. index could only have any value for us, when in the normal untreated rat this index is individually constant (as it was shown for the rabbit). By measuring the index, twice at least, in some rats, we could confirm this and thus dared ascribe eventual changes in the c.-r. index to the concerning operative interference or treatment.

We then followed the blood congo-red level, to see whether also in the rat congo-red disappears more quickly after the injection of WIESNER extracts:

Two normal rats received during 4 days, twice daily, 0.5 ccm WIESNER extract intraperitoneally. In both animals the c.-r. index increased after the treatment with the extract and returned to the initial value after the last injection, just like it was known for the rabbit.

Then the influence of hypophysectomy upon the c.-r. index was followed, by measuring the latter first (twice) before the operation and then on the 10th day afterwards.

Table I shows, that the index was considerably reduced after hypophysectomy in all cases.

This means that with hypophysectomy just the opposite result was obtained as with the injection of WIESNER extract. Since, after hypophysectomy, the c.-r. index came to so low a value as does not occur in normal animals, we now could try to find out whether this is due to a difference in the congo-red *storage* between normal and hypophysectomized animals. Because of the much slower disappearance of the congo-red in

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[2]) With congo-red index is meant the number, indicating the decrease of the blood congo-red level during the 6th—60th minute after the intravenous injection, counted in procents of the concentration after 6 minutes. Thus a high index means a quick disappearance of the congo-red.

TABLE I.

	C.-r. index before hypophysectomy	C.-r. index 10 days after hypophysectomy
Rat B 4785	26 23	13
Rat B 5633	28 32	15
Rat B 5635	28 28	14
Rat B 5636	31 39	20

the latter, we might expect, following WIESNER, a smaller accumulation than in the former. We therefore tried to show congo-red histologically in liver and spleen one hour after an intravenous injection.

Neither we, however, succeeded. So, we had to look for other causes in view of the varying rapidity with which congo-red disappears from the blood under different circumstances. We believed them to lay in the excretion.

The bile is so much considered to be the main way of excretion (OKA), — NINOMIYA even based a functional liver test upon it —!, that we immediately and exclusively directed our investigations in that way.

In normal rats a method was worked out and then, with the thus obtained experience, the experiments were done in the following way:

In rats of 150—200 g. body weight the duodenum was ligated in the pyloric area and ca. 10 cm further on; the abdomen was closed and 0.8 ccm of a 10 % congo-red solution intravenously injected. One hour after this injection the animal was killed and the ligated duodenal section taken away. We measured the quantity of congo-red in this section in % of the total amount of injected congo-red, by comparing it colorimetrically with a standard, obtained by adding a known quantity of congo-red to the contents of a duodenal section, similarly ligated in a controlrat. The congo-red excretion was estimated during one hour, since at the measuring of the c.-r. index the disappearance of the substance was also defined over the period of one hour after the injection. In the other parts of the intestine, no congo-red could be found back, so that the substance obviously cannot reach the lumen of the gut by way of the intestinal glands, but only via the liver with the bile.

Control of the excretion before and on the 10th day after operation in the same rat (in order to see whether hypophysectomy influences the congo-red excretion), was impossible, since the above gauge of the congo-red level means the death of the animal.

We therefore took a group of normal and one of hypophysectomized rats, operated upon 10 days before, and defined in both groups the excretion by the liver as well as the c.-r. index in order to obtain an impression about the difference between the decrease of the congo-red level in the blood and the excretion by the liver, in normal and hypophysectomized rats.

As is shown in table II the percentage of the total amount of congo-red, that appeared in the duodenal section, is much smaller in hypophysectomized rats than in normal ones.

TABLE II.

		C.-r. index	% Congo-red in the intestine from the injected quantity
normal rats	B 5879	32	16
	5880	39	18
	5877	33	14
	5883	38	7
	5885	34	7
	5889	34	17
	5891	35	8.5

		C.-r. index before the hypophysect.	C.-r. index after the hypophysect.	% c.-r. in the intestine from the injected quantity
hypophysectomized rats	B 5633	30	17	0.25
	5592	—	23	2.5
	5759	—	—	0.25
	5760	—	8	0.5
	5886	41	16	3
	5882	38	6	0.25
	5890	38	26	3

We further investigated, whether the restropic factor is able to keep the liver function normal and to prevent the decrease of the c.-r. index after this operation:

A WIESNER extract, that made the c.-r. index increase in normal rats and rabbits was administered to two groups of hypophysectomized rats, during 7 days (from the 4th to the 11th day after the operation).

In one group only the c.-r. index was defined, in order to see, whether the extract could prevent its decrease: this was indeed the fact, as shows table III and a considerable decrease followed after the last injection of the extract (cf. table I, dealing with untreated, hypophysectomized animals).

In the 2nd group the congo-red excretion by the liver was estimated on the 11th day after hypophysectomy and so the influence of the extract hereupon investigated (vid. table IV).

TABLE III.

	C.-r. index before the hypophysect.	C.-r. index after hypophysectomy (injection during 11 days)		
		8 days	11 days	16 days
Rat B 6201	37	40	45	20
Rat B 6202	39	30	35	23
Rat B 6203	34	—	33	16

TABLE IV.

	Congo-red excretion in $^0/_0$ from the injected quantity 11 days after hypophysectomy
Rat B 6204	14
Rat B 6205	22
Rat B 6208	22

The percentage congo-red of the total amount, excreted by the liver, proves to be much higher in these animals than in untreated, hypophysectomized ones and partially even higher than in normal rats (cf. table II).

The excretion with the bile did not interest us exclusively as such, but especially its relative value in view of the also defined c.-r. index. In normal rats we found an average index of 34, after hypophysectomy 16 and after injection of pituitary extract in hypophysectomized animals 38, so that a far-going substitution of the pituitary activity in this respect proves to be possible. The average excretion with the bile was respectively 13, 2 and 19 %.

Now the question is: how far reaches the correlation between c.-r. index and the excretion with the bile?

Of course the above numbers are not precise and furthermore there are arguments that the first group must be corrected to lower values and the second to higher ones.

Since 6 minutes after the i.v. injection part of the congo-red must have escaped from the blood flow already, the excretion after one hour is deducted from a too low initial value, so that the index is found *too high*. On the other hand on each moment (and therefore also at the end of the experiment) a certain quantity has left the liver cells without having reached the intestine already, so that the values for the excretion with the bile must needs be somewhat *too low*.

In every case, however, the decrease of the c.-r. index of the untreated hypophysectomized animals goes together with an almost as important decrease in the excretion, so that the influence of the pituitary gland, resp.

of a WIESNER extract on the c.-r. index can be almost completely reduced to an increased excretion with the bile.

Conclusion.

A WIESNER extract, that made the congo-red index distinctly increase in the rabbit as well as in the normal rat, prevented in rats the decrease of the c.-r. index and the diminishment of the excretion by the liver, that normally occur after hypophysectomy. The restropic factor promotes the liver function, at least as far as the excretion of congo-red is concerned. It is questionable, if it influences the R. E. S., but it is shown that an eventual influence hereupon must be of little importance.

II. *Experiments with Lithium-carmine.*

By reducing the c.-r. index to a question of excretion, WIESNER's deduction (influence of the active principle of certain pituitary extracts on the R. E. S.) is bereft of all arguments. **Thus the name "restropic factor" seems to be mistaken at all.** Before accepting this far-going conclusion, however, we wished to follow the influence of the hypophysis on the accumulation of a dye substance in a more direct way. We therefore used lithium-carmine and investigated, whether a quantitative difference exists in the storage of this substance between normal and hypophysectomized animals: 8 rats of precisely the same body weight (210 gr) were selected: 4 of them underwent hypophysectomy; 10 days later all animals received 1 cc of a 5 % lithium-carmine solution intravenously. Four hours afterwards all rats were killed; liver and spleen were extirpated and prepared for histological investigation. Furthermore, as always happened, the sella turcica of the operated animals was examined microscopically, to confirm the effectiveness of the operation. In none of the cases a rest was found. From the liver preparations results that the asteroid cells ("Kupfercells") contain rather much carmine. There is no distinct difference between the carmine accumulation of the livers of normal and hypophysectomized animals. It seems, however, that the latter contain somewhat more! The spleen contains, in the normal as well as in the hypophysectomized rat, only little carmine.

It now had to be researched whether notwithstanding all this, lithium-carmine too disappears more slowly from the blood in the hypophysectomized than in the normal rat. We therefore introduced the term carmine index, the definition of which is analogous to that of the c.-r. index.

The technique was the same as in the congo-red experiment; we only administered this time 0.8 ccm of a 5 % lithium-carmine solution intravenously. A serious disadvantage of this method, however, is the toxicity of the carmine; most of the animals came in a temporal shock and some even died immediately after the injection. The general status often became so bad, that we could hardly obtain some blood from the tail Heart punction often became necessary, so that the mortality further increased.

We first established that the carmine index was individually constant in the normal, untreated rat and then controlled the influence of hypophysectomy hereupon.

As is shown in table V, the carmine index decreases considerably after hypophysectomy, similar to the behaviour of the c.-r. index. Once being established that the lessened rapidity, with which the carmine disappeared from the blood was not a consequence of a decreased storage in the R.E.S., the question arose what then might be the cause; a question, which we again tried to solve on the base of the excretion.

TABLE V.

	Li. Carmine-index before hypophysectomy	Li. Carmine-index 10 days after hypophysectomy
Rat B 5763	41	13
Rat B 5764	41	22
Rat B 5958	40	33
Rat B 5960	34	25
Rat B 6015	43	22

We further tried to establish, more or less in passing, whether carmine, as was shown for congo-red, disappears more readily from the blood, after the administration of a Wiesner extract and whether the latter is able to prevent the decrease of the carmine index (like that of the c.-r. index) after hypophysectomy. The experiments failed, however, because once the extract proved to influence neither the c.-r. index and all rats died after the carmine injections in 2 other cases.

The opinion of OKA, that carmine, unlike congo-red, is not excreted by the liver but by the kidneys we could confirm, so that it now became necessary to follow the renal carmine excretion before and after hypophysectomy and the influence of the restropic factor hereupon.

In order to study the influence of the kidneys upon the lowering of the blood carmine level with relation to the carmine-index, the percentage of the injected amount of carmine found back in the urine after one hour ought to have been defined. At that time, however, often no urine was produced, nor could be obtained even with the catheter. Therefore the urine was gathered over a longer period. After some experiments we decided to define the excretion over a period of 12 hours.

Once being determined, that the carmine excretion by way of the urine was constant in normal, untreated rats (table VI) the excretion was defined in a group of rats before and the 10th day after hypophysectomy (table VII). In the hypophysectomized rat the quantity of carmine, excreted with the urine, is much smaller than in the normal animal. In normal animals carmine excretion is independent of the quantity of produced urine (table VI) since these animals seem to be able to concentrate the carmine to a high degree; hypophysectomized animals, however, seem to have partially lost this capacity.

TABLE VI.

Normal rats		cc urine in 12 hours	$^0/_0$ Carmine from the injected quantity
Rat 6254	18.7.1939 0,5 cc Carmine solution	6.5	21
	22.7.1939 „ „ „ „	9.5	22
Rat 6255	18.7.1939 „ „ „ „	6.0	23
	22 7.1939 „ „ „ „	10.0	23

TABLE VII.

	Before hypophysectomy		10 d. after hypophysectomy	
	cc urine in 12 hours	$^0/_0$ Carmine from the injected quantity	cc urine in 12 hours	$^0/_0$ Carmine from the injected quantity
Rat B 6079	—	27	0.5	0.5
Rat B 6072	—	19	2	4
Rat B 6215	5	20	2	3

The experiment, in which we tried to define the influence of the restropic factor on the carmine excretion gave no results, because of the high mortality of the rats or the ineffectiveness of the extracts; in one rat only the experiment succeeded: there an active Wiesner extract failed to prevent the diminishment of the carmine excretion with the urine after hypophysectomy, this by contrast with the congo-red excretion with the bile. We do not wish to attach great value to this result.

Conclusion.

The decrease of the blood carmine level, after an intravenous injection of this substance, takes place more slowly after hypophysectomy than before this operation. This is not caused by a decreased accumulation by the reticulo-endothelial system, but (just like applies for congo-red), partially, at least, without doubt by a decreased function of the organs, which play the main part in the elimination of those substances from the body; *in casu* the kidneys.

Discussion.

In this paper an effort was made to analyse the importance of the pituitary gland, considering the fate of intravenously injected dye substances. It was found that the hypophysis promotes (be it by different ways) the excretion of a not accumulated substance (congo-red) as well as of an accumulated one (carmine). This elucidates the more rapid disappearance from the blood in normal animals (or those treated with pituitary extracts) than in hypophysectomized ones. The factors of the pituitary extracts, which must be responsible therefore in both cases, are perhaps not identical. In the case of the congo-red it is the so-called restropic factor; for the carmine the substance is still unknown. Hypophysectomy not diminishing the accumulation, each relation between pituitary gland and accumulation capacity must be left uncertain in the corresponding case. The name restropic factor for a substance, that makes congo-red disappear more rapidly from the blood stream (i.e. promotes its excretion with the bile) is in our view based upon a wrong postulate and ought to disappear from literature.

Once knowing that the so-called restropic factor has most probably nothing to do with the R. E. S., we need not be astonished, that in our earlier investigations this factor proved to be not identical with the substance, promoting blood demolition, nor with the lienotropic neither with the thymotropic factor, since now every logical relation has proved to be absent.

Summary.

1. No relation can be shown between the restropic factor and the reticulo-endothelial system.

2. The excretion of intravenously injected congo-red depends on a liver function, which decreases after hypophysectomy and which is influenced by the restropic factor.

3. Hypophysectomy does not inhibit the accumulation capacity of the R. E. S. for lithium carmine.

4. The excretion of lithium carmine, when i.v. administered, is determined by a renal function, which too decreases after hypophysectomy.

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Medicine. — *Pregnanediol estimations in the urine of laboratory animals.* From the department of Pharmacology, University of Leiden. By A. TH. KNOPPERS. (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of September 28, 1940.)

Introduction.

The substance pregnanediol 3 α—20 α, for which we propose to use the name pregnanediol (mainly in clinical literature is often spoken of pregnandiol) has drawn much attention in recent time as a decay and partially as an excretion product of progesterone in man; this opened a field of investigation of physiological as well as of diagnostical (1, 2, 3, 4) importance.

Attempts to make VENNING and BROWNE's estimation in the urine of man (5) suitable for laboratory animals, failed (6). With BEALL's (11) method no such attempts were made as yet.

MARKER c.s. (7, 8, 9, 10), however, who performed a large series of sterole investigations upon cows, could show pregnanediol in several cases. In the horse an excretion product of a somewhat different structural formula (allo-pregnanediol 3 β—20 α) proved to play the main rôle. The urine of the pregnant hog proved to contain no pregnanediols but pregnanolones, which are less advanced reduction products of progesterone.

The aim of our experiments, described below, was to find a method, eventually with appropriate modifications, adequate for the investigation of the progesterone metabolism in the laboratory animal and in case of failure, to detect the causes thereof.

Experimental Part.

Methodical.

For the determination of pregnanediol excretion we first applied the method, elaborated by VENNING & BROWNE. Epitomizing, this method involves the following steps:

Starting-material ca 1 L. urine. Ample butylalcoholic extraction; evaporation in vacuo; solution in sodium-hydroxyde (0,1 N); shaking with butylalcohol; evaporation in vacuo to dryness; solution in water. Pregnanediol glucuronide is precipitated by acetone. The weight is defined after purification. As a criterium for the purity the melting-point is taken (268°—270° for the pure substance). For the identification the mixture melting-point with pure sodium-pregnanediol-glucuronide is used.

Since laboratory animals void only little urine, we first tested the method with *smaller* quantities of human material, but further in the usual way.

Three portions of 80 cc of the urine of a woman in the 7th month of pregnancy were employed. The yields were 2.4 mgs (melting-point 257°), 3.5 mgs (m.p. 257°) and 1.4 mgs (m.p. 250°). The last result may not be counted with, because of the too low melting-point. The wide differences between the two yields with the same melting-point is striking; both quantities lay at the lower limit of what might be expected (normal $>$ 40 mgs pro L. urine or $>$ 3.2 mgs pro 80 cc).

In 80 cc urine, voided during the 2nd half of the menstrual cycle, no sodium pregnanediol glucuronide could be shown (normal 3—6 mgs pro L.).

These results are in good accordance with those of VENNING & BROWNE's, who credit their method with only qualitative value in case the yields don't surpass 4 mgs.

In view of the relatively more serious losses, when small portions are examined, it seemed us to be advantageous to use larger quantities of urine, e.g. $>$ 200 cc.

This is *a fortiori* the case, when the urines of pregnant rabbits are examined, which often show undefinable, disturbing precipitations. This no doubt pertains to the fact, that the butylalcoholic extract had a dark reddish-brown colour, which resisted all the following treatments. It becomes still more intense, if the urine has been kept for a longer time; this cannot be prevented by keeping the urine in the refrigerator, until the wanted quantity is pooled, neither by adding toluene as a conservative. This explains, that from 4.8 mgs sodium p.gl., solved in 80 cc rabbit's urine nothing was found back. The negative results of BUXTON & WESTPHAL (6) have lost much of their value herewith.

In order to eliminate impurities, the following experiments were done:

I. 200 cc of the urine of a woman in the 7th month of pregnancy was divided in two equal portions. The first portion was elaborated following VENNING & BROWNE. Yield 2.14 mgs (m.p. 257°). The other one was shaken with norit. The filtrate was clear and treated as above. Yield nothing.

II. 600 cc of the urine of a woman in the 8th month was divided in 3 portions of 200 cc. The first portion was elaborated following V. & B. Yield 10.7 mgs (m.p. 257°). The second was shaken with norit and further treated as the first portion. Yield nothing. The third portion was extracted with butylalcohol as usual. The extract was shaken with norit (1.25 g pro 100 cc). The filtrate was colourless and further treated following V. & B. The melting-point of the mixture of the thus obtained sodium pregnanediol-glucuronide with *pure* sodium p.gl. showed no decrease (260°).

The gravimetrical determination failed.

III. A sample of the urine of a woman in the 7th month was divided in 2 equal parts. One of these was elaborated following V. & B. Yield: 4 mgs (m.p. 257°); the butylalcoholic extract of the second was shaken with norit and the filtrate treated following V. & B. Yield: 2.7 mgs (m.p. 260°).

IV. 400 cc of the urine of a woman in the 8th month was divided in 2 parts of 200 cc. The first portion was elaborated following V. & B. Yield: 13.1 mgs (m.p. 260°), the butylalcoholic extract of the second was shaken with norit. Yield: 7.9 mgs (m.p. 265°).

These experiments indicate that the insertion of a purification of the butylalcoholic extract with norit (1.25 g/100 cc) in VENNING & BROWNE's method causes a constant loss of ca 35 %.

On the other hand a purer final product was obtained, as is shown with the estimation of the melting-points.

V. 14.5 mgs sodium pregnanediol glucuronide was added to 400 cc normal rabbit's urine. The mixture was neutralized and divided in 2 equal parts.

The first portion (n.b.: 200 cc!), elaborated following V. & B. yielded 4.0 mgs (m.p. 256°). The 2nd portion (V. & B. treatment with norit-purification) yielded 4.8 mgs (m.p. 260°). By simple calculation (a loss of 35 % with the norit purification and 15 % with V. & B.'s method) the added sodium p.gl. could be found back *quantitatively* with the method of VENNING & BROWNE improved by purifying with norit.

With the thus completed method the eventual presence of sodium pregnanediol glucuronide in the urine of pregnant or treated rabbits could be verified.

We further employed the method, described by BEALL (11).

Since this author used considerable amounts of urine (ca 200 L.) we had to modificate his method for smaller amounts (viz. ca 500 cc).

Methodical: (vid. Biochem. Journal **31**, 35 (1937)).

11 g sodium benzoate is added to 500 cc urine. With a 5 % solution of HCl the pH is brought to 2, after which a precipitation (mainly benzoic acid) appears. Shaking during 2 hours. The precipitation is stirred in ample acetone (ca 120 cc). Evaporation. The residual is dissolved in a slight excess of NaOH (10 %). Shaking at 70° C with 60 cc toluene in a Dewar bottle. Divided with a separatory funnel. The aqueous part is kept up.

a. Toluene is washed out with NaOH, 2N, 20 cc; four fold washing, each time with 15 cc water; toluene shaken with Na_2SO_4 exsicc.; toluene evaporated in vacuo; dry residual solved in boiling light petroleum; precipitation; crystallization from hot acetone (m.p. 230°—232°); recrystallization with hot ethylalcohol; precipitation; defining of the melting-point. (Free-) pregnanediol: m.p. 238°—240°.

b. Aqueous part: Brought to pH 2; precipitation is boiled with 105 cc HCl 0,25 N during 2 hours (Hydrolysis of conjugated pregnanediol);

after cooling rendered alkaline with NaOH 10 % (in slightly excessive dosage). Further treatment as aforementioned. Final result: Pregnanediol, originally conjugated, but set free by hydrolysis.

> Technical difficulties were not spared to us at the application of this 2nd method. Consecutively paraffinum and a sulphur containing, nitrogen-free organic substance were demonstrated. These difficulties were put aside. In view of these findings, it is necessary to perform the toluene extraction with cork or glass stopples.

Further the method, when applied for urine from animals, proved to be only apt for qualitative estimation, the yields being too small to allow quantitative conclusions. This is mainly due to losses, occurring in consequence of the recrystallizations, what is not astonishing with so small yields.

Investigation.

In our experiments rabbits and dogs were used. Investigated were a series of urines of pregnant animals and of normal beasts, treated with ample doses of progesterone (3 × 10 mgs daily). In the latter case all the urine, voided *until 48 hours after the last injection* was examined. The results are entabulated: (See table following page.)

These results show, that neither in the rabbit nor in the dog, during pregnancy or after the injection of large quantities of progesterone, the excretion of sodium pregnanediol glucuronide can be demonstrated.

Once only a very small quantity of pregnanediol could be shown in the urine of a pregnant rabbit with the aid of BEALL's method. This could *not* be confirmed in the urine of another pregnant rabbit.

Principially a three-fold cause can be indicated for the fact, that generally in the urine of rabbits and dogs no pregnanediol is to be found under circumstances, favourable thereto in other respects:

1. Our method is inadequate. *Added* pregnanediol, however, could regularly be detected.
2. The animals were able (like the horse and the dog are) to convert progesterone into some other form of pregnanediol or a less reduced substance, not demonstrable with the employed method. In one pregnant rabbit, however, pregnanediol could be shown to a certainty. The above supposition should imply the improbability, that this rabbit had a metabolism, different from that of the others.
3. The convertion of progesterone in pregnanediol (in man too, obviously far from quantitative!) has a maximum, more dependent upon the capacity (i.e. the bulk) of the body, than upon the quantity of progesterone, the latter contains. The mentioned capacity will be small in the usual experimental animals, so small even that the limit of determinability is reached or surpassed.

Arguments in favour of the last possibility are given with observations,

Species	Condition	Quantity of urine	Method	Yield of pregnanediol
Rabbit No. I ♀	Pregnant 12th—15th day	155 cc	VENNING en BROWNE	0 mgs
Id	Id 17th day	80 cc	Id	0 mgs
Id	Id 18th—22nd day	560 cc	Id	0 mgs
Rabbit No. II ♀	Id 15th—20th day	460 cc	Id. with norit purification	0 mgs
Id	Id 20th—26th day	700 cc	Id Id	0 mgs
Dog No. I ♀	Id 6th—7th week	830 cc	Id Id	0 mgs
Rabbit No. III ♀	Id 12th—14th day	700 cc	BEALL	0.75 mgs after precipitation with acetone m.p. 228° — 230°. M.p. of the mixt. with pure pregnanediol: no decrease!
Rabbit No. IV ♀	Id 13th—14th day	600 cc	BEALL	0 mgs
Dog No. I ♀	Id 6th—7th week	600 cc	BEALL	0 mgs
Rabbit No. V ♀	1 progest. inj. Urinalysis unto 72 h. afterwards	780 cc	V. en B. with norit purif.	0 mgs
Rabbit No. VI ♀	One day 30 mgs progesterone. Urinalysis: to 48 h. afterwards	480 cc	Id Id	0 mgs
Id	Two foll. days 30 mgs progest. Urinalysis: to 48 h. afterwards	800 cc	Id Id	0 mgs
Rabbit No. V ♀	Two foll. days 30 mgs progest. Urinalysis: to 48 h. afterwards.	690 cc	BEALL	0 mgs

quoted in literature, made after the administration of uncommonly high dosages of progesterone to human females.

It is in good accordance herewith, that in the one pregnant rabbit, in which it was certainly shown, pregnanediol was present in only minimal quantities.

Conclusions:

1. After inducing in VENNING & BROWNE's method a purification of the butylalcohol extract with norit, added sodium pregnanediol glucuronide could be found back to a satisfying degree.

2. With the aid of this combined method (V. & B. plus norit purification) we were able to show, that neither the pregnant rabbit nor the pregnant dog excretes demonstrable quantities of pregnanediol.

3. With BEALL's method we succeeded once only in the demonstration of a trace of pregnanediol (0.75 mgs) in the urine of a pregnant rabbit.

4. It is argumented, that rabbits at least are capable to produce pregnanediol, but that under the present experimental circumstances animals of a certain size are required for a successful determination.

5. Thus, the usual experimental animals are not adequate to investigate upon, in order to gain a more detailed insight into the metabolism and the decay of progesterone.

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Sérologie. — *Sur un nouveau phénomène observé dans la réaction de fixation.* (Premier mémoire.) Par ONG SIAN GWAN. (Communicated by Prof. E. GORTER.)

(Communicated at the meeting of September 28, 1940.)

1. Etant donné un sérum humain quelconque, normal ou pathologique, il se peut qu'on obtienne le résultat suivant: Les réactions de fixation faites séparément avec deux antigènes différents A et B donnent des résultats négatives. Mais si l'on réalise la même réaction avec un mélange de A et de B, la réaction peut être positive et même fortement positive.

2. *Antigènes employés.* Ils sont trois: l'antigène spermatozoïde, le nouvel antigène tuberculeux de M. BESREDKA et l'antigène syphilitique.

L'antigène spermatozoïde est constitué par un mélange d'une partie de sperme frais humain et de deux parties d'eau physiologique phéniquée à 0,75 p. 100 [1]). Le mélange introduit dans des ampoules scellés se conservent indéfiniment. On l'agite de temps à autre et on utilise le liquide surnageant obtenu soit par centrifugation, soit par décantation. L'antigène limpide est d'une couleur jaunatre et on l'emploie tel qu'il est.

L'antigène syphilitique (extrait alcoolique de coeur de veau) est préparé d'après la méthode de BORDET–RUELENS légèrement modifiée.

3. *Technique de la réaction.* La technique adoptée est celle de M. BESREDKA. On fait la lecture immédiatement après, une deuxième deux heures et souvent une troisième 15 à 20 heures plus tard. Le résultat est exprimé en nombre d'unités d'alexine (une unité d'alexine = 0,1 cc d'alexine dilué au $^1/_{15}$) déviées par 1 cc de sérum [2]). Si la réaction est très prononcée, on la répète avec le sérum dilué et on tient compte de cette dilution.

Notons que les tubes témoins contenant le mélange d'antigènes (0,3 cc d'antigène spermatozoïde + 0,3 cc d'antigène tuberculeux) ne fixent pas d'alexine en absence de sérum. Dans certains cas on a utilisé la moitié du mélange (0,3 cc) et l'on a obtenu le même résultat.

4. Nous avons montré dans une note précédente [1]), la présence de sensibilisatrices antispermatozoïdes dans le sang de l'homme et de la femme. Désignons par *spermoréaction*, la réaction de fixation effectuée avec l'antigène spermatozoïde.

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[1]) C. R. Acad. Sc. **202**, 1874 (1936).

[2]) Comme il s'agit ici d'une détermination quantitative, il est nécessaire d'utiliser un même échantillon d'alexine pour chaque série de réactions. Dans ce but l'alexine fut conservée dans l'air liquide (—190° C). Elle se conservait ainsi indéfiniment. Même diluée au 1/15, le titre ne semblait pas modifier au bout de plusieurs jours.

Ainsi, en faisant sur chaque sérum trois réactions de fixation respectivement avec l'antigène spermatozoïde, l'antigène tuberculeux et le mélange de ces antigènes à parties égales, on peut distinguer trois groupes de résultats.

I. Dans le premier groupe le sérum donne une spermoréaction et une réaction de tuberculose négatives, mais il donne une réaction positive ou fortement positive avec un mélange de deux antigènes mentionnés.

II. Dans le deuxième groupe le sérum donne une réaction de fixation positive, soit avec l'antigène spermatozoïde seul, soit avec l'antigène tuberculeux seul, ou bien les deux réactions sont positives. Mais la réaction réalisée avec le mélange de deux antigènes est toujours plus fortement positive que la somme des réactions obtenues séparément avec les deux antigènes.

III. Enfin, le troisième groupe est constitué soit par des réactions négatives pour les trois antigènes employés, soit par une spermoréaction ou une réaction de tuberculose positive ou toutes les deux positives. La réaction faite avec le mélange de deux antigènes est égale ou même inférieure à la some des réactions obtenues séparément avec les deux antigènes.

On peut les résumer dans le tableau 1. On constate que le sérum du premier groupe donne le phénomène observé le plus net. Que le sérum du deuxième groupe donne le phénomène encore positif et qu'enfin dans le dernier groupe le phénomène ne se manifeste plus.

TABLEAU 1.

Groupes	Résultats des réactions		
I	$S = 0$	$T = 0$	$ST +$
II	$S +$	$T = 0$	$ST > S$
	$S = 0$	$T +$	$ST > T$
	$S +$	$T +$	$ST > S + T$
III	$S = 0$	$T = 0$	$ST = 0$
	$S +$	$T = 0$	$ST \leqslant S$
	$S = 0$	$T +$	$ST \leqslant T$
	$S +$	$T +$	$ST \leqslant S + T$

S = spermoréaction.
T = réaction de tuberculose.
ST = réaction réalisée avec un mélange d'antigènes spermatozoïde et tuberculeux.

5. Ainsi, sur 185 sérums d'homme et de femme examinés, 45 ou 24,3 p. 100 appartiennent au premier groupe, 86 ou 46,5 p. 100 au deuxième et le reste 54 ou 29,2 p. 100 au dernier groupe. Donc, plus que deux tiers (70,8 p. 100) des sérums examinés montrent le phénomène observé.

Les résultats des trois groupes de sérums examinés sont présentés dans les tableaux 2,3 et 4. Le tableau 2 (groupe I) met en évidence le nombre

TABLEAU 2. Groupe I.

Nombre d'unités d'alexine déviées par 1 cc de sérum	ST + fréquence	Nombre de réactions de B.W. et de floculations positives
5	2	1
10	14	1
15	8	—
20	6	1
25	3	2
30	8	2
35	2	1
75	2	1
Total	45	9
Nombre total d'unités d'alexine déviées	925	
Moyenne	20.6	

TABLEAU 3. Groupe II.

*	A			B			C			
	S	ST	**	T	ST	**	S	T	ST	**
5	—	—	—	—	—	—	11	8	—	—
10	9	—	—	8	—	—	33	17	—	—
15	4	7	1	4	1	—	3	10	2	—
20	2	—	—	4	2	—	1	3	1	—
25	—	3	1	—	4	—	—	—	7	3
30	—	1	—	5	5	3	—	2	17	3
35	—	2	1	—	3	1	—	—	3	2
40	1	1	—	—	—	—	—	2	—	—
45	—	—	—	—	2	—	—	3	3	—
55	—	—	—	—	1	1	—	—	1	—
60		1	—	—	1	—	—	2	6	2
65	—	—	—	—	2	—	—	—	—	—
75	—	—	—	—	—	—	—	—	1	1
85	—	—	—	—	—	—	—	—	1	—
90	—	—	—	—	—	—	—	—	4	1
100	—	1	—	—	—	—	—	—	—	—
150	—	—	—	1	—	—	—	1	—	—
165	—	—	—	—	1	—	—	—	—	—
225	—	—	—	—.	—	—	—	—	1	—
275	—	—	—	—	—	—	—	—	1	—
Total	16	16	3	22	22	5	48	48	48	12
***	230	480		520	910		450	965	2410	
Moyennes	14.4	30.0		23.6	41.4		9.4	20.1	50.2	

* Nombre d'unités d'alexine déviées par 1 cc de sérum.
** Nombre de réactions de B. W. et de floculations positives.
*** Nombre total d'unités d'alexine déviées.

d'unités d'alexine déviées par le sérum en présence d'un mélange de deux antigènes. La spermoréaction et la réaction de tuberculose sont ici toutes deux négatives. La moyenne d'unités d'alexine déviées par 1 cc de sérum est égale à 20,6 unités.

Le tableau 3 montre les résultats du groupe II. Il est divisé en trois sousgroupes différents: A, B et C. Dans le sousgroupe A, la spermoréaction (S) et la réaction réalisée avec un mélange d'antigènes spermatozoïde et tuberculeux (ST) sont positives, mais $ST > S$. En effet, la différence entre les moyennes arithmétiques de ST et de S est égale à 30—14,4 = 15,6 unités.

Par contre, dans le sousgroupe B la spermoréaction est négative, la réaction de tuberculose (T) et ST sont positives, mais $ST > T$. La différence entre les deux moyennes arithmétiques est égale à 41,4—23,6 = 17,8 unités.

Enfin dans le dernier sousgroupe C, toutes les trois réactions sont positives et $ST > (S + T)$. La moyenne arithmétique de ST est égale à 50,2 unités, la somme des moyennes arithmétiques de S et de T est égale à $9{,}4 + 20{,}1 = 29{,}5$ unités. La différence $\overline{ST} - (\overline{S} + \overline{T}) = 50{,}2 - 29{,}5 = 20{,}7$ unités.

La plupart des sérums examinés appartiennent à ce sousgroupe.

Le tableau 4 donne les résultats des sérums examinés, qui ne montrent plus le phénomène observé. Ce groupe III peut être divisé en quatre sous-

TABLEAU 4. Groupe III.

*	A		B			C			D			
	S T ST	**	S	ST	**	T	ST	**	S	T	ST	**
0	22	1	—	—	—	—	—	—	—	—	—	—
5	—	—	2	2	—	—	—	—	3	2	—	—
10	—	—	10	10	2	—	—	—	11	9	4	—
15	—	—	1	1	—	2	2	—	3	2	7	3
20	—	—	—	—	—	—	—	—	—	1	1	—
25	—	—	—	—	—	—	—	—	—	1	—	—
30	—	—	—	—	—	—	—	—	—	—	2	—
35	—	—	—	—	—	—	—	—	—	—	1	—
45	—	—	—	—	—	—	—	—	—	1	—	—
60	—	—	—	—	—	—	—	—	—	—	1	1
90	—	—	—	—	—	—	—	—	—	1	1	—
Total	22	1	13	13	2	2	2	0	17	17	17	4
***	0		125	125		30	30		170	310	410	
Moyennes			9.6	9.6		15.0	15.0		10.0	18.2	24.1	

* Nombre d'unités d'alexine déviées par 1 cc de sérum.

** Nombre de réactions de B. W. et de floculations positives.

*** Nombre total d'unités d'alexine déviées.

groupes A, B, C et D. Dans le sousgroupe A, toutes les réactions *S*, *T* et *ST* sont négatives. Dans le sousgroupe B, les deux réactions *S* et *ST* sont positives, la réaction *T* étant négative. Les moyennes de *S* et de *ST* sont égales à 9,6 unités; les deux réactions sont équivalentes.

Il est curieux de noter que, deux seulement sur 185 sérums examinés appartiennent au sousgroupe C, où *T* et *ST* sont positifs et *S* négatif. Les moyennes de *T* et de *ST* sont égales à 15 unités et les deux réactions sont donc équivalentes. Dans le dernier sousgroupe D, toutes les réactions *S*, *T* et *ST* sont positives, mais $ST \leqslant (S + T)$. En effet, la moyenne arithmétique de *ST* est égale 24,1 unités et celle de la somme des moyennes de *S* et de *T* est égale à 10 + 18,2 unités = 28,2 unités. La différence $\overline{ST} - (S + T) = 24{,}1 - 28{,}2 = -4{,}1$ unités.

Parmi les 17 sérums de ce sousgroupe il y en avait dix, dont deux syphilitiques, qui donnaient une différence négative.

On peut résumer les résultats des trois groupes de sérums examinés dans le tableau 5.

TABLEAU 5.

Groupes	Sous-groupes	Nombre de sérums	Moyennes d'unités d'alexine déviées			Différence $\overline{ST} - (\overline{S} + \overline{T})$
			$\overline{S}$	T	$\overline{ST}$	
I		45	0	0	20.6	+ 20.6
II	A	16	14.4	0	30.0	+ 15.6
	B	22	0	23.6	41.4	+ 17.8
	C	49	9.5	20.2	49.9	+ 20.2
III	A	22	0	0	0	0
	B	13	9.6	0	9.6	0
	C	2	0	15.0	15.0	0
	D	16	9.7	17.8	23.4	— 4.1
	Total	185				

Ces résultats mettent en évidence la différence $ST - (S + T)$, la plus grande parmi les sérums, dont la spermoréaction et la réaction de tuberculose sont négatives (groupe I), ou parmi ceux, dont les deux réactions sont positives (groupe II, C). Ils montrent que, la présence de sensibilisatrices antispermatozoïdes seules (groupe II, A), donne une différence moins grande que la présence de sensibilisatrices tuberculeuses seules (groupe II, B). Enfin, ils montrent que, la présence de deux sensibilisatrices simultanées peut donner un phénomène invers (différence négative, groupe III, D).

6. *Rôle de la syphilis dans la production du phénomène.* Il est probable que le sérum à réactions de BORDET–WASSERMANN (B. W.) et de

floculations positives donnent le phénomène plus marqué que le sérum non syphilitique.

Presque tous les sérums étudiés ont été examinés du point de vue de la syphilis par la réaction de BORDET–WASSERMANN et par les réactions de floculations d'après KAHN, MEINICKE (MKR) et SACHS–GEORGI. Les résultats de ces réactions ont été controlés par d'autres laboratoires spécialisés, à Paris: Hôpital St. Lazare et Hôpital Boucicaut, à Leyde: Institut d'hygiène et de bactériologie à l'Université et à Rotterdam: Laboratoire municipal à Coolsingel, aujourd'hui en ruine après le bombardement [3]).

Les réactions de BORDET–WASSERMANN et de floculations positives indiquées dans les tableaux 2,3 et 4 sont rapportées aux résultats de la réaction de fixation réalisée avec un mélange d'antigènes spermatozoïde et tuberculeux (*ST*). Les chiffres indiquent le nombre de sérums syphilitiques. On constate les résultats suivants:

Groupe I. 9 sérums syphilitiques sur 45 sérums examinés, soit 20 p. 100.
Groupe II. 20 sérums syphilitiques sur 86 sérums examinés, soit 23 p. 100.
Groupe III. 7 sérums syphilitiques sur 54 sérums examinés, soit 13 p. 100.

On en conclut que, parmi les sérums montrant le phénomène considéré (groupe I et II), le pourcentage de sérums syphilitiques est plus élevé que parmi ceux, où le phénomène ne se manifeste pas. (Groupe III). Le sérum syphilitique semble donc jouer un rôle dans la production du phénomène.

7. Voyons maintenant, si le sérum syphilitique donne le phénomène plus marqué que le sérum non syphilitique. Pour cela, nous allons comparer les moyennes d'unités d'alexine déviées par le sérum syphilitique et le sérum non syphilitique.

Pour savoir, si la différence entre deux moyennes $\bar{x}$ et $\bar{x}'$ est réelle on peut appliquer la distribution de t. Si la probabilité P pour que, t dépasse la valeur obtenue, prend une valeur très petite, par exemple inférieure à 5 p. 100, on en conclut que la différence est significative [4]).

En examinant les sérums du groupe I (tableau 2), on obtient:

Moyenne arithmétique d'unités d'alexine déviées par le sérum syphilitique $\bar{x} = 28{,}33$ unités. Moyenne arithmétique d'unités d'alexine déviées par le sérum non syphilitique $\bar{x}' = 18{,}61$ unités. On a $\bar{x} - x' = 9{,}72$, $s = 4{,}506$, $t = 5{,}788$, $n = 43$.

La probabilité cherchée pour que, $t > 5{,}788$ serait beaucoup plus petite que 0,001 et il n'est pas douteux que la différence soit réelle. Le sérum syphilitique peut donc fixer une quantité d'alexine plus importante que

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[3]) Nous remercions vivement M. L. E. DEN DOOREN DE JONG, directeur du laboratoire municipal à Rotterdam, qui a bien voulu nous fournir la plupart des sérums examinés.

[4]) R. A. FISHER, Statistical methods for research workers, London and Edinburgh, Oliver and Boyd, 1938.

R. A. FISHER and F. YATES, Statistical tables, London and Edinburgh, Oliver and Boyd, 1938.

le sérum non syphilitique. Ce résultat est en accord avec la conclusion obtenue plus haut en comparant les pourcentages de sérums syphilitiques dans les trois groupes différents de sérums examinés.

8. On peut également soumettre les résultats du groupe II au même calcul. Cépendant, il faudrait comparer dans ce cas la différence $ST - (S + T)$ du sérum syphilitique et du sérum non syphilitique. On a:

Moyenne arithmétique de différences $ST - (S + T)$ des sérums syphilitiques $x = 16{,}0$ unités. Moyenne arithmétique de différences des sérums non syphilitiques $x' = 19{,}9$ unités. On obtient $x - x' = -3{,}9$, $s = 25{,}356$, $t = -0{,}603$, $n = 84$.

On trouve dans le tableau de t, la probabilité $P \cong 0{,}55$ pour que, $t < -0{,}603$. Elle est donc beaucoup plus élevée que la précédente. Il est donc probable que dans ce cas la différence entre les moyennes n'est pas réelle.

9. *Réaction de fixation réalisée avec un mélange d'antigènes syphilitique et spermatozoïde. a. Sérum non syphilitique.* Sur 53 sérums examinés aucun n'a donné un phénomène positif, c'est-à-dire le sérum en présence d'antigène spermatozoïde ou d'un mélange d'antigènes spermatozoïde et syphilitique à parties égales, fixe exactement la même quantité d'alexine.

b. Serum syphilitique. Il est probable que la présence de réagines syphilitiques est indispensable pour produire le phénomène. Mais les réactions effectuées sur 22 sérums syphilitiques donnaient en général un phénomène invers, c'est-à-dire la somme des réactions réalisées avec les antigènes syphilitique et spermatozoïde séparément est plus grande que la réaction faite avec le mélange de deux antigènes mentionnés (tableau 6).

On peut savoir au moyen de la distribution de t si, la différence entre les réactions ΣS et $(\Sigma + S)$ est réelle.

Le tableau 6 donne les résultats suivants: $x = -5{,}8$, $s = 6{,}965$ $t = -3{,}906$, $n = 21$.

La probabilité cherchée pour que, $t < -3{,}906$ serait inférieure à 1 p. 1000. La différence entre les valeurs ΣS et $(\Sigma + S)$ est donc significative. On en conclut que l'antigène spermatozoïde ajouté à l'antigène syphilitique empêche la fixation de l'alexine, phénomène invers du précédent.

L'examen attentif du tableau 6 permet de faire les remarques suivantes: 1. Non seulement la moyenne $\Sigma S = 17{,}2$ est plus petite que la somme des moyennes $\Sigma + \bar{S} = 23{,}0$, mais elle paraît également inférieure à la moyenne $\bar{\Sigma} = 19{,}6$. La distribution de t montre cependant que la différence entre les valeurs ΣS et Σ n'est pas réelle. La probabilité cherchée serait égale à $P \cong 0{,}1$. Il semble donc que, si les réagines syphilitiques sont saturées par l'antigène syphilitique, le sérum n'est plus capable de fixer l'antigène spermatozoïde. 2. En comparant les différences $\Sigma S - \bar{S} = +13{,}8$ et $\Sigma S - \bar{\Sigma} = -2{,}4$ on peut en conclure que l'affinité de la réagine syphili-

TABLEAU 6. Sérums syphilitiques.

No des sérums	Dilution du sérum	Résultats des réactions de fixation			Différence $\Sigma S - (\Sigma + S)$
		Σ	S	ΣS	
115	1/7	10	0	10	0
116	non dilué	35	10	30	−15
117	1/3	40	5	33	−12
118	1/5	20	0	15	− 5
119	1/2	18	10	18	−10
120	1/5	10	0	10	0
121	1/2	5	10	10	− 5
141	1/2	25	0	20	− 5
142	1/2	30	10	25	−15
143	1/2	18	0	15	− 3
165	1/5	33	0	15	−18
186	1/5	10	0	5	− 5
187	1/15	30	0	18	−12
188	1/30	25	0	10	−15
199	1/5	10	5	13	− 2
200	non dilué	15	0	25	+10
201	non dilué	20	10	25	− 5
202	1/2	10	0	10	0
203	1/2	10	15	15	−10
204	1/2	13	0	13	0
205	1/2	19	0	23	+ 4
206	1/15	25	0	20	− 5
Moyennes		19.6	3.4	17.2	− 5.8

Σ = Réaction de BORDET–WASSERMANN.
ΣS = Réaction réalisée avec un mélange d'antigènes syphilitique et spermatozoïde.
1/5 = un volume de sérum dilué dans quatre volumes d'eau salée à 0,9 p. 100.

tique pour l'antigène syphilitique est plus prononcée que celle de la sensibilisatrice antispermatozoïde pour l'antigène spermatozoïde. Ceci est facile à comprendre puisque, la réagine syphilitique est un anticorps d'immunsérum, tandis que la sensibilisatrice antispermatozoïde est un anticorps normal.

Il est cependant possible dans certains cas d'obtenir un phénomène faiblement positif ($\Sigma S > \Sigma + S$) en faisant la réaction avec un sérum syphilitique convenablement dilué dans l'eau salée à 0,9 p. 100 ou dans le sérum normal. La condition exacte pour produire le phénomène n'est pas encore déterminée.

Il est à remarquer que, toutes choses égales, le sérum syphilitique dilué dans le sérum normal donnait une réaction de BORDET–WASSERMANN plus faible que celui dans l'eau salée à 0,9 p. 100. La dilution dans le sérum normal peut donner une réaction de BORDET–WASSERMANN négative, tandis que la même dilution dans l'eau salée à 0,9 p. 100 donne encore une réaction positive. L'action inhibitrice du sérum normal vis à vis de la bactériolyse était signalé en 1901 par CAMUS, GLEY et MÜLLER.

10. *Reaction de fixation réalisée avec un mélange d'antigènes syphilitique et tuberculeux.* Ce mélange fut essayé sur quatre sérums seulement, dont deux d'entre eux montrent un phénomène positif. Il semble donc que le phénomène se produit également avec ce mélange. L'antigène tuberculeux étant épuisé, nous regrettons de ne pouvoir continuer ces essais.

11. *Sérums d'autres espèces.* On a également cherché si, d'autres sérums que le sérum humain peuvent donner le phénomène. A cet effet on a examiné 7 sérums de lapin, 6 sérums de cheval, 2 sérums de mouton, 2 sérums de chat, un sérum de porc et un sérum de boeuf. En employant le mélange d'antigènes spermatozoïde et tuberculeux, on observe chez l'homme comme chez l'animal le même phénomène. Il est à remarquer que le sérum de lapin donne le phénomène très prononcé $[ST - (S + T) = + 35]$.

12. *Immunsérums.* Le sérum d'animal immunisé donne-t-il le même résultat que le sérum normal? Pour cela, on a examiné quatre sérums antitétaniques provenant de laboratoires différents, deux sérums antidiphtériques d'origine différente, un sérum antistreptococcique polyvalent de l'Institut Pasteur (I. P.) et un sérum anticolibacillaire polyvalent préparé par nous même ⁵). Le tableau 7 montre le résultat obtenu.

Les lapins 1, 2 et 3 ont reçu à 10 j. d'intervalle quatre injections intraveineuses de toxine tétanique et les lapins 4 et 6 ont reçu à 15 j. d'intervalle deux injections souscutanées de toxine tétanique incorporée dans la lanoline-vaseline et une injection souscutanée de la même toxine. Les chiffres entre parenthèses indiquent les résultats de la réaction avant l'immunisation. On voit que celle-ci n'a pas d'influence marqué sur le phénomène étudié. Il est à noter que, le sérum antitétanique R_1 donnait une forte réaction de fixation en présence d'antigène spermatozoïde. Le sérum en présence d'antigène dilué au $^1/_{20}$ était capable de fixer 10 unités d'alexine.

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⁵) C. R. Soc. Biol. **125**, 228 (1937).
Zeitschr. f. Immunitätsf. **93**, 248 et 278 (1938).

(à suivre.)

ERRATUM.

Palæontology. — *The fossil human remains discovered in Java by* Dr. G. H. R. VON KOENIGSWALD *and attributed by him to* Pithecanthropus erectus, *in reality remains of* Homo sapiens soloensis. *Continuation*. By Prof. EUG. DUBOIS.

Correction of Proceedings, Vol XLIII, No. 7, 1940.

p. 851, line 11 to the end of that alinea, read:

also based my researches on phylogenetic progress of the brain, especially the cerebrum, in Mammals, the principal result of which was, subsequently, likewise obtained in Birds, by LOUIS LAPICQUE.

NEDERL. AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS

VOLUME XLIII

No. 9

President: J. VAN DER HOEVE
Secretary: M. W. WOERDEMAN

CONTENTS

Physics. — *Critical speeds of rotating shafts.* By C. B. BIEZENO.

(Communicated at the meeting of October 26, 1940.)

1. *Introduction.* It is a well-known fact, that a rotating shaft, supported either in a statically determinate or a statically undeterminate way and bearing n centred and concentrated masses $m_1, m_2 \ldots m_n$ (in comparison of which the mass of the shaft itself be neglected), may whirl at most at n different angular speeds ω. However, up to now, no satisfactory proof has been given for the fact — tacitly assumed by all technicians —, that n, just now denoted as the upper limit of the number of unequal whirling speeds, in reality represents this number exactly. As will be seen in par. 2 the reciprocal squares of the critical angular speeds are roots of a so-called secular-equation of the n^{th} degree and therefore it only needs evidence that all these roots are *real, positive* and *unequal*. This communication deals with the unequality of the roots; only for completeness' sake the proof of their reality and positiveness, which can be found elsewhere, will be included [1]).

2. *The secular-equation having for roots the reciprocal squares of the critical angular speeds.* If at a whirling speed ω of the shaft the masses $m_i\,(i = 1, 2 \ldots n)$ have deflections $y_i\,(i = 1, 2 \ldots n)$, then it is evident that these deflections are maintained by the mass-reactions $m_i \omega^2 y_i\,(i = 1, 2, \ldots n)$. Therefore the following equations — where a_{ij} represents the ordinary MAXWELL-number with respect to the points i and j — hold:

$$\sum_{j=1}^{n} m_j\,\omega^2\,y_j\,a_{ij} = y_i \qquad (i = 1, 2, \ldots n). \quad . \quad . \quad . \quad . \quad (1)$$

They only admit a solution $y_i\,(i = 1, 2 \ldots n)$ different from zero if

$$\begin{vmatrix} m_1\,\omega^2\,a_{11} - 1 & m_2\,\omega^2\,a_{12} & m_3\,\omega^2\,a_{13} \ldots \ldots m_n\,\omega^2\,a_{1n} \\ m_1\,\omega^2\,a_{21} & m_2\,\omega^2\,a_{22} & m_3\,\omega^2\,a_{23} \ldots \ldots m_n\,\omega^2\,a_{2n} \\ \vdots & & \\ m_1\,\omega^2\,a_{n1} & m_2\,\omega^2\,a_{n2} & m_3\,\omega^2\,a_{n3} \ldots \ldots m_n\,\omega^2\,a_{nn} - 1 \end{vmatrix} = 0. \quad (2)$$

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[1]) See f.i. C. B. BIEZENO and R. GRAMMEL, Technische Dynamik, p. 807 (J. SPRINGER, Berlin).

If we put $\frac{1}{\omega^2} = z$ and $\sqrt{m_i m_j \alpha_{ij}} = a_{ij}$, this equation is equivalent with:

$$S_n \equiv \begin{vmatrix} a_{11}-z & a_{12} & a_{13} \dots a_{1n} \\ a_{21} & a_{22}-z & a_{23} \dots a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} & a_{n3} \dots a_{nn}-z \end{vmatrix} = 0 \; . \; . \; . \quad (3)$$

the determinant S_n being symmetrical with respect to its main diagonal as result of the MAXWELL-relation $\alpha_{ij} = \alpha_{ji}$. To prove that equation (3) has no *equal* roots, it will do to show that the two equations

$$T \equiv S_{n-1} \equiv \begin{vmatrix} a_{11}-z & a_{12} & \dots a_{1,n-1} \\ a_{21} & a_{22}-z & \dots a_{2,n-1} \\ \vdots & & \\ a_{n-1,1} & a_{n-1,2} & \dots a_{n-1,n-1}-z \end{vmatrix} = 0 \; . \; . \quad (4a)$$

and

$$U \equiv \begin{vmatrix} a_{12} & a_{13} \dots\dots\dots\dots a_{1n} \\ a_{22}-z & a_{23} \dots\dots\dots\dots a_{2n} \\ \vdots & \\ a_{n-1,2} & a_{n-1,3} \dots \; a_{n-1,n-1}-z \quad a_{n-1,n} \end{vmatrix} = 0 \; . \; . \quad (4b)$$

have no *common* roots. For, after a well-known theorem, a double root of equation (3) is a common root of all equations, which may be derived from (3) by putting its minors of the $(n-1)^{\text{th}}$ degree equal to zero. Therefore if only two of these equations, such as (4*a*) and (4*b*) are shown to have no common roots, it is impossible that (3) should have a double root. The non-existence of common roots of (4*a*) and (4*b*) will be proved in par. 5 by demonstrating that all roots of (4*a*) are real and positive and that all real roots of (4*b*) are negative.

3. *Some properties of the* MAXWELL *numbers* α_{ij}. We design by $\triangle_n$ the determinant of the n^{th} order

$$\triangle_n \equiv \begin{vmatrix} \alpha_{11} & \alpha_{12} \dots \alpha_{1n} \\ \alpha_{21} & \alpha_{22} \dots \alpha_{2n} \\ \vdots & \\ \alpha_{n1} & \alpha_{n2} \dots \alpha_{nn} \end{vmatrix} \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

by $\overset{i,j,k\ldots.}{\Delta_n^{p,q,r\ldots.}}$ the determinant composed of those elements of Δ_n, which are found in the rows $i, j, k\ldots$ and the columns p, q, r; more particular by $\overset{1,2\ldots n-1}{\Delta_n^{2,3\ldots n}}$ the determinant derived from Δ_n by suppressing the n^{th} row and the first column. Furthermore the symbol $a_{ij}^{u,v,w\ldots}$ will stand for the MAXWELL number a_{ij}, relating to the points (i, j) that appears if, in addition to the natural supports of the shaft, artificial ones are introduced in the points $u, v, w\ldots.$

Now we consider the shaft under the action of a unit-force $P=1$ at the point n and ask for the displacement $a_{n,n}^{1,2\ldots(n-1)}$ that occurs if in all points $1, 2\ldots(n-1)$ artificial supports are introduced. Let $X_1, X_2\ldots X_{n-1}$ be the unknown reactions in those points. Then we have

$$\left.\begin{aligned}
y_1 &= X_1\,a_{11} + X_2\,a_{12} + \ldots. + X_{n-1}\,a_{1,n-1} + a_{1,n} = 0\\
y_2 &= X_1\,a_{21} + X_2\,a_{22} + \ldots. + X_{n-1}\,a_{2,n-1} + a_{2,n} = 0\\
y_{n-1} &= X_1\,a_{n-1} + X_2\,a_{n-1,2} + \ldots + X_{n-1}\,a_{n-1,n-1} + a_{n-1,n} = 0\\
y_n &= X_1\,a_{n,1} + X_2\,a_{n,2} \qquad + \ldots + X_{n-1}\,a_{n,n-1} + a_{n,n} = a_{n,n}^{1,2\ldots(n-1)}
\end{aligned}\right\} \qquad (2)$$

and it follows at once — $X_1, X_2\ldots X_{n-1}$ and $a_{n,n}^{1,2\ldots(n-1)}$ being regarded as unknowns —

$$a_{n,n}^{1,2\ldots(n-1)} = \frac{\Delta_n}{\Delta_{n-1}} \quad \ldots\ldots\ldots\ldots \quad (3)$$

We learn from (3) that Δ_n and Δ_{n-1} have similar signs because $a_{n,n}^{1,2\ldots(n-1)}$ is essentially positive, and from this fact again it follows, that all determinants $\Delta_n, \Delta_{n-1}, \ldots \Delta_1$ have the same sign and that they all must be positive because the last one, $\Delta_1 \equiv a_{11}$, is essentially positive.

Once more we consider the shaft under the action of a unit force P at the point n but now ask for the deflection $a_{1,n}^{2,3\ldots k}$ in the point 1, which occurs if in the points $2, 3\ldots k$ artificial supports are introduced. If these supports exert the reaction $X_2, X_3\ldots X_k$, we have

$$\left.\begin{aligned}
y_1 &= X_2\,a_{12} + X_3\,a_{13} + \ldots + X_k\,a_{1k} + a_{1,n} = a_{1,n}^{2,3\ldots k}\\
y_2 &= X_2\,a_{22} + X_3\,a_{23} + \ldots + X_k\,a_{2k} + a_{2,n} = 0\\
y_3 &= X_2\,a_{32} + X_3\,a_{33} + \ldots + X_k\,a_{3k} + a_{3,n} = 0\\
&\;\vdots\\
y_k &= X_2\,a_{k2} + X_3\,a_{k3} + \ldots + X_k\,a_{kk} + a_{k,n} = 0
\end{aligned}\right\} \quad . \quad (4)$$

and it follows from these equations — $X_2, X_3 \ldots X_k$ and $a^{2,3\ldots k}_{1,n}$ being regarded as unknowns —

$$a^{2,3\ldots k}_{1,n} = \frac{\begin{vmatrix} a_{12} & a_{13} \ldots a_{1k} & a_{1n} \\ a_{22} & a_{23} \ldots a_{2k} & a_{2n} \\ \vdots & & \\ a_{k2} & a_{k3} \ldots a_{kk} & a_{kn} \end{vmatrix}}{(-1)^{k-1}\begin{vmatrix} a_{22} & a_{23} \ldots & a_{2k} \\ a_{32} & a_{33} \ldots & a_{3k} \\ \vdots & & \\ a_{k2} & a_{k3} \ldots & a_{kk} \end{vmatrix}} = \frac{(-1)^{k-1}\,\Delta_n^{\substack{1,2\ldots(k-1),k \\ 2,3\ldots k,\quad n}}}{\text{positive number}} \quad . \quad . \quad (5)$$

Therefore the sign of

$$\frac{\Delta_n^{\substack{1,2\ldots k-1,k \\ 2,3\ldots k,\quad n}}}{a_{1,n}} \text{ agrees with the sign of } (-1)^{k-1}\frac{a^{2,3\ldots k}_{1,n}}{a_{1,n}}. \quad . \quad . \quad (6)$$

In this proof it is unessential that the points $2, 3 \ldots k$ are consecutive; they may be as well a set of points arbitrarily chosen amongst the points $2, 3 \ldots (n-1)$.

4. *The sign of the quotient* $a^{2,3\ldots k}_{1,n} : a_1, n$. It will be shown in this paragraph that the sign of the quotient $a^{2,3\ldots k}_{1,n} : a_{1,n}$ is given by $(-1)^{k-1}$. To this end we prove the following theorem: If a beam, supported in any number of points $A_0, A_1 \ldots A_n$ of equal height, is submitted to the action of a single force P, the deflections, occurring in any span $A_i A_{i+1}$ have a uniform sign, whereas the deflections occurring in any two consecutive spans are opposite.

If we consider two consecutive segments $A_{i-1}A_i$ and $A_i A_{i+1}$ of the beam $A_o \ldots A_n$, each of them freely supported at its ends, but loaded in these points by the moments of transition M_{i-1}, M_i and M_{i+1} (see fig. 1) the MAXWELL numbers $\beta_{i-1,i} = \beta_{i,i-1}$, β^{*}_{ii}, β^{**}_{ii} and

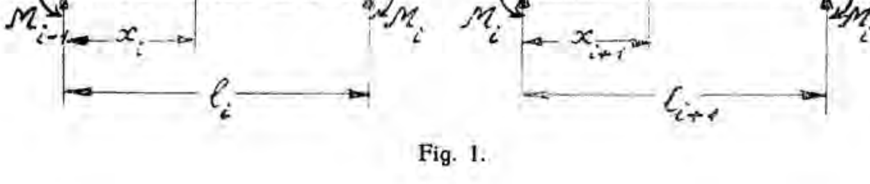

Fig. 1.

$\beta_{i+1,i} = \beta_{i+1,1}$ may be introduced. Every number represents the slope in the point indicated by the first index, caused by a unit-moment acting in the point indicated by the second index. The single and double asteriks of $\beta_{i,i}$ denote whether the number refers to the left or to the right span.

If furthermore $S_{i,i-1}$ defines the moment of the area of reduced moments of the span (i), with respect to its left support due to the forces acting on the freely supported beam $A_{i-1}A_i$ and similarly $S_{i+1,i+1}$ the moment of the area of reduced moments of the span $(i+1)$ with respect to its right support, due to the forces acting on the freely supported beam $A_i A_{i+1}$, the following well-known relation — named after CLAPEYRON — holds:

$$\beta_{i,i-1} M_{i-1} + (\beta^{*}_{i,i} + \beta^{**}_{i,i}) M_i + \beta_{i,i+1} M_{i+1} = \frac{S_{i,i-1}}{l_i} + \frac{S_{i+1,i+1}}{l_{i+1}} \quad \ldots \quad (1)$$

Expressed in the lengths l_i and l_{i+1} of the gaps, the (variable) moment of inertia I of the cross section and the modulus of elasticity E, the numbers $\beta_{i-1,i}$, $\beta^{*}_{i,i}$, $\beta^{**}_{i,i}$, $\beta_{i,i+1}$ have the values

$$\left.\begin{aligned} \beta_{i,i-1} = \frac{1}{l_i^2}\int_0^{l_i} \frac{x_i(l_i - x_i)}{EI}\,dx_i\,;\ \beta^{*}_{i,i} = \frac{1}{l_i^2}\int_0^{l_i}\frac{x_i^2\,dx_i}{EI}\,; \\ \beta^{**}_{i,i} = \frac{1}{l_{i+1}^2}\int_0^{l_{i+1}} \frac{(l_{i+1} - x_{i+1})^2\,dx_{i+1}}{EI}\,;\ \beta_{i,i+1} = \frac{1}{l_{i+1}^2}\int_0^{l_{i+1}} \frac{x_{i+1}(l_{i+1} - x_{i+1})}{EI}\,dx_{i+1} \end{aligned}\right\} . \quad (2)$$

They are essentially positive and it can be proved in a similar way as in par. 3 for the numbers a_{ij}, that

$$\beta^{**}_{i,i}\,\beta^{*}_{i+1,i+1} - \beta_{i+1,i}\,\beta_{i,1+i} > 0. \quad \ldots\ldots \quad (3)$$

Returning now to our continuous beam $A_0 \ldots A_n$ which may be either supported or clamped at A_0 and have a series of consecutive unloaded spans (1), (2) ... (i) for each of which the quantities $S_{i,i-1}$ and $S_{i+1,i+1}$ in eq. 1 are to be put equal to zero, we find by a repeated application of (1):

$$\left.\begin{aligned} &\beta^{**}_{0,0} M_0 + \beta_{0,1} M_1 = 0 \\ &\beta_{1,0} M_0 + (\beta^{*}_{1,1} + \beta^{**}_{1,1}) M_1 + \beta_{1,2} M_2 = 0 \\ &\beta_{2,1} M_1 + (\beta^{*}_{2,2} + \beta^{**}_{2,2}) M_2 + \beta_{2,3} M_3 = 0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ a. s. o.} \end{aligned}\right\} \quad \ldots \quad (4)$$

from which we deduce

$$M_0 = -\frac{\beta_{0,1}}{\beta^{**}_{0,0}} M_1\,; \quad M_1 = -\frac{\beta_{1,2} M_2}{\beta^{**}_{1,1} + \beta^{*}_{1,1} - \dfrac{\beta_{1,0}\beta_{0,1}}{\beta^{**}_{0,0}}}\,; \; M_2 = -\frac{\beta_{2,3} M_3}{\beta^{**}_{2,2} + \beta^{*}_{2,2} - \dfrac{\beta_{2,1}\beta_{1,2}}{\beta^{**}_{1,1} + \beta^{*}_{1,1} - \dfrac{\beta_{1,0}\beta_{0,1}}{\beta^{**}_{0,0}}}}$$

and in general

$$\frac{M_i}{M_{i+1}} =$$

$$= \underline{-\beta_{i,i+1}}\Big|\overline{\beta^{**}_{i,i} + \beta^{*}_{i,i} - \underline{\beta_{i,i-1}\beta_{i-1,i}}}\Big|\overline{\beta^{**}_{i-1,i-1} + \beta^{*}_{i-1,i-1} - \underline{\beta_{i-1,i-2}\beta_{i-2,i-1}}}\Big|\overline{\underline{\dots}}\Big|\overline{\beta^{**}_{1,1} + \beta^{*}_{1,1} - \frac{\beta_{1,0}\beta_{0,1}}{\beta^{**}_{0,0}}}$$

The fraction $M_i : M_{i+1}$ is negative and in absolute magnitude smaller than $\beta_{i,i+1} : \beta^{**}_{i,i}$. For if we denote by B_j the continual fraction

$$\beta_j = \beta^{*}_{j,j} - \underline{\beta_{j,j-1}\beta_{j-1,j}}\Big|\overline{\beta^{**}_{j-1,j-1} + \beta^{*}_{j-1,j-1} - \underline{\beta_{j-1,j-2}\beta_{j-2,j-1}}}\Big|\overline{\underline{\dots}}\Big|\overline{\beta^{**}_{1,1} + \beta^{*}_{1,1} - \frac{\beta_{1,0}\beta_{0,1}}{\beta^{**}_{0,0}}}$$

we know by (3) that

$$B_1 \equiv \frac{\beta^{*}_{1,1}\,\beta^{**}_{0,0} - \beta_{1,0}\,\beta_{0,1}}{\beta^{**}_{0,0}}$$

is essentially positive. Therefore

$$B_2 \equiv \beta^{*}_{2,2} - \frac{\beta_{2,1}\,\beta_{1,2}}{\beta^{**}_{1,1} + B_1} > \beta^{*}_{2,2} - \frac{\beta_{2,1}\,\beta_{1,2}}{\beta^{**}_{1,1}} = \frac{\beta^{*}_{2,2}\,\beta^{**}_{1,1} - \beta_{2,1}\,\beta_{1,2}}{\beta^{**}_{1,1}}.$$

The latter fraction again is positive so that $B_2 > 0$. Proceeding in this way we find at last

$$\frac{M_i}{M_{i+1}} = -\frac{\beta_{i,i+1}}{\beta^{**}_{i,i} + B_i}$$

where $\beta_{i,i+1}$, $\beta^{**}_{i,i}$ and B_i all represent positive numbers. Consequently

$$\frac{M_i}{M_{i+1}} < 0 \quad \text{and} \quad \left|\frac{M_i}{M_{i+1}}\right| < \frac{\beta_{i,i+1}}{\beta^{**}_{i,i}}.$$

The first of the two statements still leaves the possibility that the deflection of the span $A_i A_{i+1}$ should change its sign over the length l_i (obviously such change is excluded if M_i and M_{i+1} were both positive or negative). The second one however makes it certain that such change cannot occur. For if we try to find the moment M_i^* (M_{i+1} regarded as

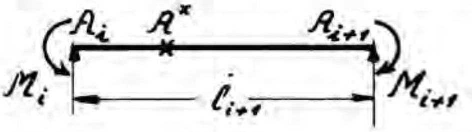

Fig. 2.

a given positive moment) that would neutralize the slope caused at A_i by M_{i+1}, (see fig. 2), the condition

$$M_{i+1}\,\beta_{i,i+1}+M_i^*\,\beta_{i,i}^{**}=0$$

must be fulfilled, and therefore

$$M_i^*=-\frac{\beta_{i,i+1}}{\beta_{i,i}^{**}}M_{i+1} \text{ or } \frac{M_i^*}{M_{i+1}}=-\frac{\beta_{i,i+1}}{\beta_{i,i}^{**}}.$$

If $\left|\dfrac{M_i}{M_{i+1}}\right|$ should be $>\dfrac{\beta_{i,i+1}}{\beta_{i,i}^{**}}$ a point A^* with zero deflection would arise at the right side of A_i. In our case however, where $\left|\dfrac{M_i}{M_{i+1}}\right|<\dfrac{\beta_{i,i+1}}{\beta_{i,i}^{**}}$ neither such point nor a zero-slope at the point A_i can appear.

Consequently the deflection of the beam changes its sign at every support and keeps it in every span. This being fixed the sign of the quotient $a_{1,n}^{2,3\ldots k}:a_{1n}$ can easily be found. For indeed the sign of a_{1n} corresponds with the sign of $(-1)^p$, p being the number of real shaft-supports between the points 1 and n. If, however, at the points $2,3\ldots k$ a set of $(k-1)$ supports is introduced the sign of $a_{1,n}^{2,3\ldots k}$ corresponds with the sign of $(-1)^{p+(k-1)}$. Therefore the sign of $a_{1,n}^{2,3\ldots k}:a_{1,n}$ is given by $(-1)^{k-1}$, and consequently the sign of the quotient: $\overset{1,2\ldots(k-1)\;k}{\Delta}{}_n^{2,3\ldots\;k\;\;n}:a_{1,n}$ which we knew already to agree with the sign of $(-1)^{k-1}a_{1,n}^{2,3\ldots k}:a_{1,n}$ (compare (6, 3) is positive for all values $2\leqq k\leqq(n-1)$.

5. *The unequality of the critical angular speeds.* With the notation introduced in the beginning of par. 3 the equations (2, 4*a*) and (2, 4*b*) can be written as follows:

$$\left.\begin{aligned}&z^{n-1}-\left[\sum_{i=1}^{n-1}m_i\,\overset{i}{\Delta}{}_n^{i}\right]z^{n-2}+\left[\sum_{\substack{i=1\\i<j}}^{n-1}\sum_{j=1}^{n-1}m_i\,m_j\,\overset{i,j}{\Delta}{}_n^{i,j}\right]z^{n-3}-\\&-\left[\sum_{\substack{i=1\\i<j<k}}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1}^{n-1}m_i\,m_j\,m_k\,\overset{i,j,k}{\Delta}{}_n^{i,j,k}\right]z^{n-4}+\ldots+(-1)^{n-2}m_1\,m_2\ldots m_{n-1}\Delta_{n-1}=0\end{aligned}\right\}\quad(1)$$

$$\left.\begin{array}{l} a_{1n}\, z^{n-2} + \left[\sum\limits_{i=2}^{n-1} m_i \,\triangle_n^{\substack{1,i\\ i,n}}\right] z^{n-3} + \left[\sum\limits_{\substack{i=2\\ i<j}}^{n-1} \sum\limits_{j=2}^{n-1} m_i\, m_j \,\triangle_n^{\substack{1,i,j\\ i,j,n}}\right] z^{n-4} + \\ + \left[\sum\limits_{\substack{i=2\\ i<j<k}}^{n-1} \sum\limits_{j=2}^{n-1} \sum\limits_{k=2}^{n-1} m_i\, m_j\, m_k \,\triangle_n^{\substack{1,i,j,k\\ i,j,k,n}}\right] z^{n-5} + \ldots + \\ \qquad\qquad\qquad + m_2\, m_3 \ldots m_{n-1} \,\triangle_n^{\substack{1,2\ldots(n-1)\\ 2,3\ldots n}} = 0 \end{array}\right\} \quad . \quad . \quad . \quad (2)$$

As to equation (1) — or (2, 4*a*) — it is well-known, that the set of functions

$$S_{n-1},\ S_{n-2},\ S_{n-3}, \ldots, S_1,\ +1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

constitutes a so-called set of STURM-functions. If therefore two arbitrary values $z = p$ and $z = q\ (p > q)$ are substituted in the functions (3), the loss of variations indicates the number of real roots between p and q. If for p and q the values ∞ and 0 are taken we find, with $z = \infty$ the following signs

$$(-1)^{n-1} \quad (-1)^{n-2} \ \ldots . (-1)^1 \quad +1$$

that is: $(n-1)$ variations.

On the contrary, if $z = 0$ is substituted, the values of $S_{n-1}, S_{n-2} \ldots (+1)$ are represented by the essential positive numbers

$$m_1\, m_2 \ldots m_{n-1} \triangle_{n-1},\ \ m_1\, m_2 \ldots m_{n-2} \triangle_{n-2},\ \ldots, m_1 \triangle_1,\ +1$$

which have no variations of sign at all.

Equation (1) therefore has $(n-1)$ *positive* roots. As to equation (2), if it be divided by the coefficient a_{1m} it follows from par. (4) that all coefficients of z^{n-2}, z^{n-3} a.s.o. are essentially positive and therefore this equation has *no* positive roots. Consequently common roots of (1) and (2) do not exist and this involves (comp. par. 2) the inequality of all critical speeds.

6. *Concluding remark.* It must be remarked that it is a fundamental feature of the foregoing proof, that all supports are rigid and that at these supports no zero-slope can occur if the shaft is loaded by a single force. Therefore supports — terminal ones eventually excepted — which should have a "clamping" effect, must be excluded. In fact it is seen at once that the shaft, of constant diameter, represented in fig. 3, for which

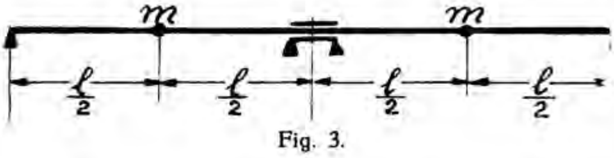

Fig. 3.

$a_{12}=0$, only has one single critical speed which is identical with the critical speed of the shaft represented in fig. 4.

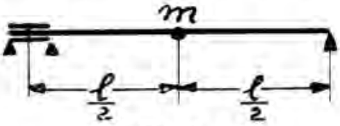

Fig. 4.

Elastic supports too must be excluded. For it is an easy matter — supposing that the middle support of the shaft represented in fig. 5 is

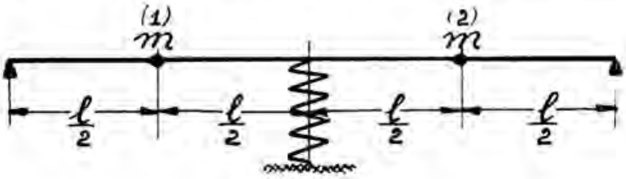

Fig. 5.

an elastic one with the elastic constant c — to calculate c in such a way that two equal critical speeds occur. Indeed, the condition for equal critical speeds

$$\left(\frac{a_{11}-a_{22}}{2}\right)^2+a_{12}^2=0$$

which follows from

$$\begin{vmatrix} m\,a_{11}-z & m\,a_{12} \\ m\,a_{21} & m\,a_{22}-z \end{vmatrix}=0$$

requires $a_{12}=0$. The constant c therefore must be chosen in such a way, that a load placed in (1) produces a zero-deflection in (2). If the shaft has a constant cross section we find $c=\frac{28}{3}\,\frac{EI}{l^3}$.

Therefore the theorem proved in this paper only holds for rigid, non-clamping supports. An exception has to be made for terminal supports, which must be rigid, but may be clamping.

Hydrodynamics. — *On the application of statistical mechanics to the theory of turbulent fluid motion. A hypothesis which can serve as a basis for a statistical treatment of some mathematical model systems.* II. By J. M. BURGERS. (Mededeeling N°. 39 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hoogeschool te Delft). *)

(Communicated at the meeting of October 26, 1940.)

9. *Discussion of the result obtained in Part I.* — The result which has been arrived at in Part I can be formulated as follows:

(*a*) The introduction of a regular point lattice with a finite spacing $\delta = \beta \nu$, instead of a continuous field, into the phase space for the variables ξ_n and η_n, has made it possible to derive formulae for the average statistical values of $\overline{\xi_n^2}$ and $\overline{\eta_n^2}$ which do not lead to a divergent expression for the total dissipation of energy. (*b*) These formulae present a similar course as the values of $\overline{\xi_n^2}$ and $\overline{\eta_n^2}$ which were estimated from an investigation of the solutions of the differential equations defining the mathematical model system. (*c*) The average values of $\overline{\xi_n^2}$ and $\overline{\eta_n^2}$ are directly connected with the magnitude of the "exterior force" P — compare eq. (28) —, in such a way that the "exterior force" practically is balanced by reactions deriving from the first few components of the secondary motion only. (*d*) It appeared possible to obtain a definite relation between P and the "velocity" U of the principal motion — compare eq. (30) — in which no other unknown constant appeared besides the factor β; the relation leads to a quadratic "resistance law".

The following observation still is of interest: From the calculations given in **8** it will be seen that we may write:

$$U = 6\,\nu\,N_s \quad \ldots\ldots\ldots\ldots \quad (31)$$

where N_s is the quantity defined by:

$$N_s = \frac{\Sigma\, n^2\,(\overline{\xi_n^2} + \overline{\eta_n^2})}{12\,P/\pi^2} \quad \ldots\ldots\ldots\ldots \quad (32)$$

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*) Continued from these Proceedings **43**, p. 945 (1940). — An error of print should be corrected at p. 945, line 9 from the top: the formula at the end of the line must be read: $N_1 \simeq 1.3 \sqrt{P\,\delta}$. The correct formula has been used in the subsequent calculations of that page, so that no further changes are necessary.

We may say that N_s measures the "*effective length*" of the spectrum of elementary components of the secondary motion; *i. e.* N_s is the number of components which should be counted in making up the total dissipation, if eq. (28) applied to all values of n until $n = N_s$, while beyond that value the spectrum would be suddenly cut off. The calculations given in 8 show that approximately: $N_s \simeq 2{,}7\, N_1$. (It would be important to find a more direct and more accurate way for calculating N_s than the procedure employed in 8).

It seems probable that equation (31) is representative of a relation of general character in systems of this kind. The fact that we have obtained a quadratic "resistance law" is a consequence of the circumstance that for the system considered N_s is proportional to $\sqrt{P}$.

The introduction of the point lattice into the phase space thus in a certain way has brought an answer to the problem stated in a previous communication on this subject [13]).

It should be understood that the introduction of this point lattice is no more than a mathematical artifice, applied in order to take notice of the peculiar properties of the system which are due to the simultaneous presence of non linear terms and of frictional terms in the equations. The value of the coefficient β, occurring in the expression for the lattice spacing δ, should be obtained from a more rigorous treatment of this subject; it is not to be considered as an undetermined quantity, which at the end of the calculations might be adjusted at will.

10. Apart from the introduction of the point lattice, another matter attracts notice, *viz.* the form of the dissipation equation which has been chosen as the basis for the statistical calculations. As will be seen from eq. (7) these calculations have been built upon the assumption that the average dissipation of energy in unit of time by the secondary motion should have a fixed value, given as PU. This assumption has the great advantage that it makes the calculations as simple and direct as possible. It differs, however, from the elimination procedures which had been applied in previous papers, where either the mean value of the velocity or that of the exterior force (the pressure gradient) was considered as being given.

It can be tried to apply a similar procedure to the system under consideration, but it is found that this leads to a rather unprobable result. We take the case where P is given, and consider U as a variable quantity, which may have the value U_m in the instantaneous state numbered m. When we write:

$$E_m = \tfrac{1}{2} \sum_n (\xi_{nm}^2 + \eta_{nm}^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

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[13]) Compare: On the application of statistical mechanics to the theory of turbulent fluid motion, Part. VII. these Proceedings **36**, 628 (1933).

then instead of eq. (7) the following two equations must be fulfilled by every sequence of instantaneous states [14]:

$$\sum_m f_{ms}(P-\nu U_m-E_m)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (34a)$$

$$\sum_m f_{ms}(PU_m-\nu U_m^2-\varepsilon_m)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (34b)$$

The method of the partition function cannot be directly applied in this case, as the expression $PU_m-\nu U_m^2-\varepsilon_m$ is not always of the same sign. However, the application of the methods for calculating the "most probable statistical state" gives:

$$\overline{f_m}=C\,e^{\lambda_1(P-\nu U_m-E_m)+\lambda_2(PU_m-\nu U_m^2-\varepsilon_m)} \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where λ_1 and λ_2 are two positive constants (λ_2 is related to $-\ln\sigma$ in sections 7—8).

As U_m may have all positive and negative values, it is found that the average value of U is equal to that value of U_m which makes $\overline{f_m}$ a maximum. Hence:

$$\overline{U}=(\lambda_2 P-\lambda_1\nu)/2\lambda_2\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The average values of ξ_n^2 and η_n^2 are given by a similar formula as (21), in which now:

$$\theta_n=e^{-(\lambda_1+\lambda_2\nu\pi^2 n^2)\xi^2/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

For not too large values of n this will lead to the following expression for $\overline{\xi_n^2}$:

$$\overline{\xi_n^2}\simeq\frac{1}{\lambda_1+\lambda_2\nu\pi^2 n^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

Now from (36) it will be seen that $\overline{U}$ will become small compared with P (*e.g.* will become of the order $\sqrt{P}$) only then, when $\lambda_1\nu\simeq\lambda_2 P$. In that case, however, λ_1 will greatly exceed the value of $\lambda_2\nu\pi^2 n^2$ for values of n which are smaller than an amount of the order $\sqrt{P/\pi\nu}$. This would lead to the result that $\overline{\xi_n^2}$ for such values of n should be independent of n, instead of being proportional to n^{-2}. I am inclined to consider such a result as unprobable, as most observations on turbulent motion give the impression that the components of a more coarse pattern (components of long wave length, large vortices) are preponderant in the balance of forces.

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[14] Compare the paper mentioned in footnote 1 (see p. 936 above), p. 44, eqs. (20.5) and (20.10).

11. The problem next presents itself whether the introduction of a point lattice into the phase space also may serve to bring to a more successful end the investigations on turbulence undertaken in some earlier papers [15]). It must be mentioned beforehand that these investigations had been based upon the assumption of a twodimensional field of flow, so that they cannot be considered as referring to the actual pattern of turbulent fluid motion in a tube or a channel, which pattern is three-dimensional in nature (it must be supposed that in this motion vortices with their axes parallel to the general direction of the flow play a preponderant role). Hence the twodimensional system should be considered as merely representing another model system.

We abandon the procedures for the elimination of the pressure gradient formerly employed; neither can we calculate the principal motion by equating $\mu dU/dy$ to the difference between the value of the total shearing force τ found from the pressure gradient and the value of $-\varrho\overline{uv}$ derived from the statistical calculations. As the difference between τ and $-\varrho\overline{uv}$ necessarily is small, it must be considered that this method, which was applied in some previous papers [16]), cannot guarantee a sufficiently accurate result. Hence a different procedure should be found for determining the principal motion, and it is this point which now brings the main difficulty into the problem. An attempt perhaps may be made along the following lines.

Provisionally we assume that the principal motion $U(y)$ is given. The secondary motion at any instant can be described by giving the stream function ψ, so that:

$$u = \partial\psi/\partial y, \qquad v = -\partial\psi/\partial x; \qquad \zeta = -\triangle\psi \quad . \quad . \quad . \quad (39)$$

We suppose that ψ can be resolved into a series of elementary components of the form:

$$\chi_{\text{I}}(y)\cos(\alpha x + \varepsilon) + \chi_{\text{II}}(y)\sin(\alpha x + \varepsilon),$$

in which the functions $\chi_{\text{I}}, \chi_{\text{II}}$ are chosen in such a way that these elementary components give a positive maximum value to the fraction:

$$\frac{1}{Vb}\,\frac{-\iint dx\,dy\,uv\,\dfrac{dU}{dy}}{\iint dx\,dy\,\zeta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

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[15]) On the application of statistical mechanics to the theory of turbulent fluid motion; these Proceedings **32**, 414, 634, 818 (1929); **36**, 276, 390, 487, 620 (1933).

[16]) Compare: On the resistance experienced by a fluid in turbulent motion, these Proceedings **26**, 582 (1923), and the papers mentioned in the preceding footnote. The method has been criticized already in Part VII of these papers, these Proceedings **36**, 620 (under II) (1933).

Here V represents the mean velocity of the principal motion over a cross section of the channel, while b is the breadth of the channel; these factors have been introduced in order that the fraction shall become a nondimensional quantity.

When we write: $\chi = \chi_{\rm I} + i\,\chi_{\rm II}$, it is found that χ must be a solution of the equation:

$$\chi^{\rm IV} - 2a^2\chi'' + a^4\chi - ia\frac{\Lambda}{Vb}\left(\frac{dU}{dy}\chi' + \frac{1}{2}\frac{d^2U}{dy^2}\chi\right) = 0. \quad . \quad (41)$$

As boundary conditions for χ we prescribe that both χ and χ' shall be zero at the walls of the channel; then for every value of a eq. (41) will admit solutions only when the parameter Λ is equal to a characteristic value of the equation; these characteristic values, which are nondimensional and which are dependent upon the magnitude of a, will be denoted by $\Lambda_{\alpha n}$. For any solution $\chi_{\alpha n}$ of (41) we have:

$$-\overline{(u_{\alpha n}\,v_{\alpha n})} = \tfrac{1}{2}\,a\,(\chi'_{{\rm I}\alpha n}\,\chi_{{\rm II}\alpha n} - \chi_{{\rm I}\alpha n}\,\chi'_{{\rm II}\alpha n})$$

$$\overline{(\zeta_{\alpha n})^2} = \tfrac{1}{2}\,(\chi''_{{\rm I}\alpha n} - a^2\,\chi_{{\rm I}\alpha n})^2 + \tfrac{1}{2}\,(\chi''_{{\rm II}\alpha n} - a^2\,\chi_{{\rm II}\alpha n})^2,$$

where the mean values are taken with respect to x.

We now prescribe as a normalizing condition for the functions $\chi_{{\rm I}\alpha n}$, $\chi_{{\rm II}\alpha n}$:

$$-\frac{1}{V}\int dy\,\overline{(u_{\alpha n}\,v_{\alpha n})}\,\frac{dU}{dy} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (42a)$$

It is then found that:

$$b\int dy\,\overline{(\zeta_{\alpha n})^2} = \Lambda_{\alpha n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42b)$$

It will be seen that the characteristic values $\Lambda_{\alpha n}$ necessarily are positive. — The normalizing condition (42*a*) ensures that into the development of the function ψ only such elementary components are admitted which can derive energy from the primary motion. A further consequence of (42*a*) is that the functions $\chi_{{\rm I}\alpha n}$, $\chi_{{\rm II}\alpha n}$ will have the dimensions of a length.

12. As a general expression for the stream function of the secondary motion we now write:

$$\psi = \sum_{\alpha n}\left[\xi_{\alpha n}\,(\chi_{{\rm I}\alpha n}\cos ax + \chi_{{\rm II}\alpha n}\sin ax) + \eta_{\alpha n}\,(-\chi_{{\rm I}\alpha n}\sin ax + \chi_{{\rm II}\alpha n}\cos ax)\right] \quad (43)$$

The coefficients $\xi_{\alpha n}$, $\eta_{\alpha n}$ have the dimensions of a velocity. In general they must be considered as functions of the time; a particular instantaneous state m of the secondary motion, however, can be specified by giving the values $\xi_{\alpha nm}$, $\eta_{\alpha nm}$ of the coefficients for that state. It is usually assumed that the parameter α may run through a series of equidistant values of the form $2\pi h/L$, where L is the length of the channel, while h is a positive integer.

In consequence of the choice adopted for the functions χ the dissipation integral (taken per unit of length of the channel) for the state of motion m can be expressed in the following form:

$$\varepsilon_m = \frac{\mu}{L} \iint dx\, dy\, (\zeta^2)_m = \frac{\mu}{b} \sum_{\alpha n} \Lambda_{\alpha n} (\xi^2_{\alpha n m} + \eta^2_{\alpha n m}) \quad . \quad . \quad (44)$$

The average statistical state of the system now may be calculated upon the assumption that the average dissipation of energy in unit time shall have a given value. In equation (7) we substitute the expression (44) for ε_m, and as before we suppose that the coefficients $\xi_{\alpha n m}$ and $\eta_{\alpha n m}$ only can take values which are integer multiples of a "threshold value" δ. This "threshold value" must have the dimensions of a velocity; judging from the equations of motion it probably will be of the order $\mu/\varrho b$.

Applying the methods described in sections 4—7 it is found that the statistical mean values of $\xi^2_{\alpha n}$ and $\eta^2_{\alpha n}$ again will be given by eq. (21), in which θ_n must be replaced by $\theta_{\alpha n} = \sigma^{\mu \Lambda_{\alpha n} \delta^2/b}$. It is convenient to write: $\sigma^{\mu \delta^2/b} = e^{-1/C}$, so that:

$$\theta_{\alpha n} = e^{-\Lambda_{\alpha n}/C} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Then for moderate values of α and n we may expect:

$$\overline{\xi^2_{\alpha n}} = \overline{\eta^2_{\alpha n}} \simeq \frac{\delta^2}{-2 \ln \theta_{\alpha n}} = \frac{\delta^2 C}{2 \Lambda_{\alpha n}} \quad . \quad . \quad . \quad . \quad . \quad (46)$$

whereas for large values of α and n:

$$\overline{\xi^2_{\alpha n}} = \overline{\eta^2_{\alpha n}} \simeq 2\, \delta^2 \theta_{\alpha n} = 2\, \delta^2 e^{-\Lambda_{\alpha n}/C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

(compare eqs. (24*b*), (24a) of Part I). The transition between the two types of approximation again will be determined by $\theta_{\alpha n} = 0{,}25$, *i.e.*:

$$\Lambda_{\alpha n} = C \ln 4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Equation (48) provides us with an indication concerning the limit of the spectrum of elementary components; in this respect it plays a similar role as the condition $(\nu \pi^2 N_1^2 \delta^2/2) \, . \ln \sigma = \ln 0{,}25$ applied in 8. The actual number N_s of components which should be counted in making up the total dissipation, assuming as the dissipation per component the value:

$$(\mu/b)\, \Lambda_{\alpha n} (\overline{\xi^2_{\alpha n}} + \overline{\eta^2_{\alpha n}}) = (\mu/b)\, \delta^2 C \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

still will be larger than the number of components which satisfy eq. (48), as can be judged from the relation between N_s and N_1 found for the original model system.

13. The formulae obtained in the preceding section lead to the following result: When the characteristic values $\Lambda_{\alpha n}$ of equation (41) have been calculated, and when values have been assumed for δ and for C, the average statistical values of $\xi^2_{\alpha n}$ and $\eta^2_{\alpha n}$ can be found from (46) and (47). The average statistical state of the secondary motion then is known, and it is possible to calculate the average value of $-\overline{uv}$ for this state as a function of y. Now this value practically must be equal to the value of the shearing force τ which can be found from the pressure gradient. When the calculations are started with an arbitrary expression for dU/dy in (41), it is not at all certain that this condition shall be fulfilled. It seems probable, therefore, that this condition can be satisfied only with a particular choice of dU/dy. If this should prove to be the case, then this particular function dU/dy might be considered as giving that distribution of the principal motion over the channel which apparently is in equilibrium with the average statistical state of the secondary motion.

The application of this condition to find the value of dU/dy will be a difficult matter, although it is not impossible that approximate calculations already may give much help.

Another point, however, should be noted. In eq. (41) we have to do with the quantity $\frac{1}{V}\frac{dU}{dy}$; hence the application of the condition cannot give the absolute values of the velocity, but only the relative distribution. The absolute value of V should be found from the dissipation condition, in a similar way as in the original model system the value of U was obtained from eq. (29) or from eq. (31). We must expect, therefore, that the value of V again will depend upon the number N_s of elementary components which must be counted in making up the total dissipation. Some rough estimates which were made in order to get a provisional idea pointed to a new difficulty, connected with the circumstance that the spectrum now is determined by two parameters, α and n: the value estimated for N_s was of a higher order than was expected. As it is no easy matter to obtain a reliable picture of the influence of the expression adopted for dU/dy upon the values of the characteristic numbers $\Lambda_{\alpha n}$, it has not been possible as yet to arrive at definite results.

Mathematics. — *Ueber algebraische Systeme von partiellen Differentialgleichungen erster Ordnung.* II. *Gleichungen mit einer Unbekannten.*

Beweis des Existenz theorems.

Von J. A. SCHOUTEN und W. VAN DER KULK.

(Communicated at the meeting of October 26, 1940.)

6. *Regularität der Gleichungen einer irreduzibelen algebraischen Mannigfaltigkeit in einem einzigen Punkt.*

Bekanntlich nennt man d die Dimension einer irreduzibelen algebraischen Mannigfaltigkeit $\mathfrak{M}$, wenn d der $n-1$ Verhältnisse der Koordinaten $w_1 \ldots, w_n$ eines allgemeinen Punktes algebraisch unabhängig sind i.b. auf den Körper der meromorphen Funktionen der $x^\varkappa$ und die übrigen $n-1-d$ sich algebraisch i.b. auf diesen Körper in diese d ausdrücken lassen [1]). Bilden die Tensoren $\underset{i}{P}^{\mu\lambda_1\ldots\lambda_{a_i}}$, $i=1,\ldots,\mathrm{N}$ eine Basis des zugehörigen Ideals von $\mathfrak{M}$, so sind

$$G_i(x^\varkappa, w_\lambda) \overset{\text{def}}{=} \underset{i}{P}^{\mu\lambda_1\ldots\lambda_{a_i}} w_\mu w_{\lambda_1} \ldots w_{\lambda_{a_i}} = 0;\ i=1,\ldots,\mathrm{N} \quad \varkappa, \lambda = 1,\ldots,n \quad (1)$$

die Gleichungen von $\mathfrak{M}$. Ist nun $w_\lambda = \overset{0}{w}_\lambda$ in $x^\varkappa = \underset{0}{x}^\varkappa$ ein zu $\mathfrak{M}$ gehöriger Vektor und schreiben wir

$$G_i^\varkappa(w_\lambda) \overset{\text{def}}{=} \left(\frac{\partial G_i}{\partial w_\varkappa}\right)_{x^\mu = \underset{0}{x}^\mu} \quad ; \quad i=1,\ldots,\mathrm{N}. \quad . \quad . \quad . \quad . \quad (2)$$

so bestimmen die Gleichungen

$$G_i^\varkappa(\overset{0}{w}_\lambda)\, w_\varkappa = 0 \quad ; \quad i=1,\ldots,\mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in der lokalen P_{n-1} von $\underset{0}{x}^\varkappa$ eine ebene Mannigfaltigkeit. Ist die Dimension derselben d, d.h. gibt es unter den N Hyperebenen (3) genau $n-1-d$ linear unabhängige, so nennen wir sie den *Tangentialraum* von $\mathfrak{M}$ in $\underset{0}{x}^\varkappa$, $\overset{0}{w}_\lambda$ [2]).

Wir beweisen jetzt den

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[1]) Vgl. E.a.G., S 112.

[2]) Vgl. E.a.G., S. 171.

Satz: Besitzt eine d-dimensionale irreduzibele algebraische Mannigfaltigkeit 𝔐 *in* $\underset{0}{x}^{\varkappa}$, $\overset{0}{w}_{\lambda}$ *einen Tangentialraum, so lassen sich* $n-1-d$ *der* $n-1$ *Verhältnisse der* w_{λ} *in der Umgebung von* $\underset{0}{x}^{\varkappa}$, $\overset{0}{w}_{\lambda}$ *schreiben als analytische Funktionen der* $x^{\varkappa}$ *und der d übrigen.*

Beweis: Da ein Tangentialraum existiert, hat die Matrix von $G_{l}^{\varkappa}(\overset{0}{w}_{\lambda})$ den Rang $n-1-d$. Durch Umnumerieren kann man also stets erreichen, dass die Determinante von $G_{j}^{\gamma}(\overset{0}{w}_{\lambda})$; $\gamma = d+2, \ldots, n$; $j = 1, \ldots, n-1-d$ nicht verschwindet. Aus der Homogenität der Gleichungen (1) in w_{λ} folgt

$$\overset{0}{w}_{\lambda}\, G_{j}^{\lambda}(\overset{0}{w}_{\mu}) = (a_{j}+1)\, G_{j}(\underset{0}{x}^{\varkappa}, \overset{0}{w}_{\lambda}) = 0 \quad . \; . \; . \; . \; . \; . \quad (4)$$

Wären nun $\overset{0}{w}_{1}, \ldots, \overset{0}{w}_{d+1}$ alle gleich Null, so würde aus (4) folgen

$$\overset{0}{w}_{\gamma}\, G_{j}^{\gamma}(\overset{0}{w}_{\lambda}) = 0 \;;\; \gamma = d+2, \ldots, n \;;\; j = 1, \ldots, n-1-d \; . \; . \quad (5)$$

und dies ist nicht möglich, da $G_{j}^{\gamma}(\overset{0}{w}_{\lambda})$ den Rang $n-1-d$ hat und $\overset{0}{w}_{d+2}, \ldots, \overset{0}{w}_{n}$ nicht alle zugleich Null sein können da sonst w_{λ} Null wäre. Durch Umnumerieren kann man also stets erreichen, dass *gleichzeitig* $\overset{0}{w}_{1} \neq 0$ ist und $\operatorname{Det}(G_{j}^{\gamma}(\overset{0}{w}_{\lambda})) \neq 0$ ist.

Wir schreiben jetzt $\omega_{2}, \ldots, \omega_{n}$ für die Verhältnisse $\frac{w_{2}}{w_{1}}, \ldots, \frac{w_{n}}{w_{1}}$ und

$$H_{j}(x^{\varkappa}, \omega_{\beta}) \overset{\text{def}}{=} G_{j}(x^{\varkappa}, 1, \omega_{\beta}) \;;\; \beta = 2, \ldots, n \;;\; j = 1, \ldots, n-1-d. \quad (6)$$

Dann ist

$$\frac{\partial H_{j}}{\partial \omega_{\beta}} = \frac{1}{w^{a_{j}}} \frac{\partial G_{j}}{\partial w_{\beta}} \quad ; \quad \beta = 2, \ldots, n \quad j = 1, \ldots, n-1-d. \quad . \quad (7)$$

Daraus folgt, dass in $\underset{0}{x}^{\varkappa}$, $\operatorname{Det}\left(\frac{\partial H_{j}}{\partial \omega_{\gamma}}\right)$ für $\omega_{\beta} = \frac{\overset{0}{w}_{\beta}}{\overset{0}{w}_{1}}$ nicht verschwindet.

Aus den Gleichungen

$$H_{j}(x^{\varkappa}, \omega_{\beta}) = 0 \;;\; \beta = 2, \ldots, n \;;\; j = 1, \ldots, n-1-d \quad . \; . \quad (8)$$

lassen sich also nach der Theorie der Funktionaldeterminanten $\omega_{d+2}, \ldots, \omega_{n}$ in einer Umgebung von $\underset{0}{x}^{\varkappa}$, $\overset{0}{\omega}_{\beta}$ als analytische Funktionen von $x^{\varkappa}, \omega_{2}, \ldots, \omega_{d+1}$ lösen

$$\left.\begin{array}{l} \omega_{d+2} = f_{d+2}\;(x^{\varkappa}, \omega_{\varepsilon}) \\ \;\vdots \qquad\quad \vdots \\ \;\vdots \qquad\quad \vdots \\ \omega_{n} \quad = f_{n} \quad (x^{\varkappa}, \omega_{\varepsilon}) \end{array}\right\} \left\{\begin{array}{l} \varepsilon = 2, \ldots, d+1 \\ \varkappa = 1, \ldots, n \end{array}\right\} \quad . \; . \; . \; . \quad (9)$$

Diese Gleichungen stellen nun einen allgemeinen Punkt einer irreduzibelen d-dimensionalen algebraischen Mannigfaltigkeit $\mathfrak{M}'$ dar [3]). Die Punkte von $\mathfrak{M}$ erfüllen aber (8), also auch (9), d.h. es ist $\mathfrak{M} \subseteq \mathfrak{M}'$. Da aber $\mathfrak{M}$ ebenfalls irreduzibel und d-dimensional ist, ist $\mathfrak{M} = \mathfrak{M}'$ [3a]). $\mathfrak{M}$ lässt sich also in der Umgebung von $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\beta$ durch ein System von Gleichungen der Form (9) mit in dieser Umgebung analytischen Funktionen der $x^{\varkappa}$ und ω_ε, $\varepsilon = 2, \dots, d+1$, darstellen, w.z.b.w.

In jedem Punkte $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\lambda$, wo ein Tangentialraum existiert, lässt sich aus $G_j^{\varkappa}(\overset{0}{w}_\lambda)$ (bis auf einen skalaren Faktor) ein einfacher kontravarianter $(n-1-d)$-Vektor konstruieren

$$u^{\varkappa_{d+2} \cdots \varkappa_n} = \{ G_1^{[\varkappa_{d+2}} \dots G_{n-1-d}^{\varkappa_n]} \}_{w_\lambda = \overset{0}{w}_\lambda}, \quad \dots \quad (10)$$

den wir den Tangential $(n-1-d)$-Vektor in $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\lambda$ nennen. Wir können das erhaltene Resultat also auch folgendermassen aussprechen:

Besitzt $\mathfrak{M}$ *in* $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\lambda$ *einen Tangentialraum, und ist daselbst die Bestimmungszahl* $u^{d+2 \dots n}$ *ungleich Null, so sind* $\overset{0}{w}_1, \dots, \overset{0}{w}_{d+1}$ *nicht alle zugleich Null. Ist z.B.* $w_1 \neq 0$, *so lässt sich* $\mathfrak{M}$ *in der Umgebung von* $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\lambda$ *durch* (9) *darstellen.*

Es ist noch zu bemerken, dass $\mathfrak{M}$ in der Umgebung von $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\lambda$ der vollständige Durchschnitt ist der $n-d-1$ algebraischen Hyperflächen (8), deren Tangentialräume in $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\lambda$ linear unabhängig sind, da $\dfrac{\partial H_j}{\partial w_\beta}$ den höchsten Rang $n-d-1$ hat.

7. *Das Existenztheorem.*

Wir beweisen jetzt das

Existenztheorem: Ist $\mathfrak{M}$ *eine d-dimensionale vollständige irreduzibele algebraische Mannigfaltigkeit,* $\underset{0}{x}^{\varkappa}$ *ein Punkt wo auch die lokale zu* $\mathfrak{M}$ *gehörige algebraische Mannigfaltigkeit irreduzibel ist und* $\overset{0}{w}_\lambda$ *ein zu* $\mathfrak{M}$ *gehöriger Vektor in* $\underset{0}{x}^{\varkappa}$, *während* $\mathfrak{M}$ *in* $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\lambda$ *einen Tangentialraum besitzt, so existiert mindestens ein in der Umgebung von* $\underset{0}{x}^{\varkappa}$ *analytisches zu* $\mathfrak{M}$ *gehöriges Gradientfeld* $\partial_\lambda s$, *dessen Feldwert in* $\underset{0}{x}^{\varkappa}$ *mit* $\overset{0}{w}_\lambda$ *zusammenfällt.*

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[3]) Vgl. E. a. G. § 29.

[3a]) Vgl. E. a. G. S. 112.

Beweis.

Das Koordinatensystem sei so gewäht, dass der Massvektor $\overset{n}{e}_\lambda$ nicht zu $\mathfrak{M}$ gehört, in $\underset{0}{x}^{\varkappa}$ weder mit $\overset{0}{w}_\lambda$ zusammenfällt noch in dem zu $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\lambda$ gehörigen Tangentialraum liegt, während für $d < n-2$ ausserdem die in $\underset{0}{x}^{\varkappa}$ durch $\overset{0}{w}_\lambda$ und $\overset{n}{e}_\lambda$ gelegte Verbindingsgerade ausser $\overset{0}{w}_\lambda$ keinen einzigen Punkt mit $\mathfrak{M}$ gemeinschaftlich hat. Für den Tangential-$(n-1-d)$-Vektor $u^{\varkappa_{d+2}\cdots\varkappa_n}$ in $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\lambda$ gilt also

$$u^{\varkappa_{d+2}\cdots\varkappa_n}\,\overset{n}{e}_{\varkappa_n} \neq 0 \quad \ldots\ldots\ldots\ldots \quad (11)$$

d.h. es ist wenigstens eine Bestimmungszahl mit einem Index n ungleich Null. Durch Umnumerieren kann man also erreichen dass $u^{d+2\ldots n}$ ungleich Null ist. Nach dem im vorigen Paragrafen erhaltenen Resultat kann man durch weiteres Umnumerieren erreichen dass ausserdem $\overset{0}{w}_1$ ungleich Null ist. Nach dem daselbst bewiesenen Satz lässt sich $\mathfrak{M}$ in der Umgebung von $x^{\varkappa}$, w_λ darstellen durch Gleichungen von der Form (9) oder auch

$$\left.\begin{array}{l} w_{d+2} = w_1\, f_{d+2}\left(x^{\varkappa}, \dfrac{w_\varepsilon}{w_1}\right) \overset{\text{def}}{=} \varphi_{d+2}\,(x^{\varkappa}, w_1, \ldots, w_{d+1}) \\ \vdots \\ w_n = w_1\, f_n\left(x^{\varkappa}, \dfrac{w_\varepsilon}{w_1}\right) \overset{\text{def}}{=} \varphi_n\,(x^{\varkappa}, w_1, \ldots, w_{d+1}) \end{array}\right\} \begin{array}{l} \varepsilon = 2, \ldots, d+1 \\ \varkappa = 1, \ldots, n \end{array} \Bigg\} \quad (12)$$

Da $\overset{0}{w}_1$ nicht verschwindet, sind auch die Funktionen $\varphi_{d+2}, \ldots, \varphi_n$ analytisch in der Umgebung von $\underset{0}{x}^{\varkappa}$, $\overset{0}{w}_\lambda$.

Das Problem ist also jetzt eine in der Umgebung von $\underset{0}{x}^{\varkappa}$ analytische Funktion $s(x^{\varkappa})$ zu finden, so dass

$$\left.\begin{array}{l} \partial_{d+2}\, s = \varphi_{d+2}\,(x^{\varkappa}, \partial_1 s, \ldots, \partial_{d+1} s) \\ \vdots \\ \partial_n\, s = \varphi_n\,(x^{\varkappa}, \partial_1 s, \ldots, \partial_{d+1} s) \end{array}\right\} \varkappa = 1, \ldots, n. \quad \ldots \quad (13)$$

ist, während ausserdem gelten soll

$$(\partial_\lambda\, s)_{x^{\varkappa} = \underset{0}{x}^{\varkappa}} = \overset{0}{w}_\lambda. \quad \ldots\ldots\ldots\ldots \quad (14)$$

Für $d = n - 2$ enthält das System (13) nur eine einzige Gleichung. Nach dem Existenztheorem von CAUCHY—KOWALEWSKI gibt es in diesem Falle eine einzige Lösung s, die für $x^n = \underset{0}{x}^n$ in eine beliebige in der Umgebung von $\underset{0}{x}^\varkappa$ analytische Funktion $\overset{0}{s}(x^1, \dots, x^{n-1})$ übergeht. Es gibt also unendlich viele Lösungen, die für $x^\varkappa = \underset{0}{x}^\varkappa$ nur den Bedingungen $\partial_1 s = \overset{0}{w}_1, \dots, \partial_{n-1} s = \overset{0}{w}_{n-1}$ genügen. Da aber $\overset{0}{w}_\lambda$ die Gleichung (12) erfüllt folgt aus (13), dass für jede dieser Lösungen auch $\partial_n s = \overset{0}{w}_n$ ist in $\underset{0}{x}^\varkappa$. Für $d = n - 2$ ist das Existenztheorem damit bewiesen.

Wir nehmen jetzt an der Beweis sei schon geliefert für eine d-dimensionale vollständige irreduzibele algebraische Mannigfaltigkeit in einer P_{n-2}, deren Grundkörper aus den meromorphen Funktionen in einem Gebiet einer X_{n-1} besteht (vgl. A.S. I [4]) S. 957). Den Induktionsbeweis teilen wir in drei Teile.

a. *Konstruktion der Hilfsfunktion* $\overset{0}{s}(x^1, \dots, x^{n-1})$.

Es sei $\mathfrak{M}_1$ die projizierende Mannigfaltigkeit von $\mathfrak{M}$ i.b. auf $\overset{n}{e}_\lambda$ (vgl. A.S. I S. 961). Diese wird dargestellt durch die Gleichungen

$$\left.\begin{array}{l} w_{d+2} = \varphi_{d+2}(x^\varkappa, w_1, \dots, w_{d+1}) \\ \vdots \\ w_{n-1} = \varphi_{n-1}(x^\varkappa, w_1, \dots, w_{d+1}) \end{array}\right\} \varkappa = 1, \dots, n \quad . \quad . \quad . \quad (15)$$

Nach A.S. I § 5 ist $\mathfrak{M}_1$ sowohl vollständig als irreduzibel, während aus (15) folgt, dass $\mathfrak{M}_1$ ausserdem $(d+1)$-dimensional ist. Man beweist leicht, dass $\mathfrak{M}_1$ für den uns allein noch interessierenden Fall dass $d < n - 2$ ist in $\underset{0}{x}^\varkappa, \overset{0}{w}_\lambda$ einen Tangentialraum besitzt[5]) und dass sich $n - 1 - (d + 1)$ kegelartige algebraische Hyperflächen bilden lassen, die $\overset{n}{e}_\lambda$ als Spitze haben, $\mathfrak{M}_1$ enthalten und in $\underset{0}{x}^\varkappa, \overset{0}{w}_\lambda$ Tangentialraüme besitzen, die linear unabhängig sind. $\mathfrak{M}_1$ ist in einer Umgebung von $\underset{0}{x}^\varkappa, \overset{0}{w}_\lambda$ der vollständige Durchschnitt dieser Hyperflächen

$$F_k(x^\varkappa, w_\beta) = F_k^{\beta_1 \dots \beta_{r_k}} w_{\beta_1} \dots w_{\beta_{r_k}} = 0; \; k = \mathrm{d} + 2, \dots, \mathrm{n} - 1 \quad \beta = 1, \dots, n - 1 \quad (15a)$$

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[4]) Mit A. S. I bezeichnen wir die erste Mitteilung, Proc. Ned. Akad. v. Wet. **43**, S. 955—963 (1940).

[5]) Wesentlich hierfür ist, dass die durch $\overset{0}{w}_\lambda$ und $\overset{n}{e}_\lambda$ gelegte Verbindungsgerade nur den Punkt $\overset{0}{w}_\lambda$ mit $\mathfrak{M}$ gemeinschaftlich hat.

sodass das System (15*a*) mit dem System (15) gleichbedeutend ist. Setzt man nun $x^n = \underset{0}{x}^n$ in (15) bzw. (16) so entsteht

$$\left.\begin{aligned} w_{d+2} &= \varphi_{d+2}(x^\alpha, \underset{0}{x}^n, w_1, \dots, w_{d+1}) \\ &\vdots \\ w_{n-1} &= \varphi_{n-1}(x^\alpha, \underset{0}{x}^n, w_1, \dots, w_{d+1}) \end{aligned}\right\} \alpha = 1, \dots, n-1 \quad . \quad (16)$$

bzw.

$$F_k(x^\alpha, \underset{0}{x}^n, w_\beta) = 0 \ ; \ \alpha, \beta = 1, \dots, n-1 \ ; \ k = \mathrm{d}+2, \dots, \mathrm{n}-1 \quad . \quad (16a)$$

und jedes dieser Systeme bestimmt in der P_{n-2}, die zu der X_{n-1} mit der Gleichung $x^n = \underset{0}{x}^n$ gehört, die selbe algebraische Mannigfaltigkeit $\mathfrak{M}_0$, von der man leicht beweist, dass sie vollständig ist (vgl. A. S. I, S. 962) sowie irreduzibel und *d*-dimensional. Ferner dass in $\underset{0}{x}^\alpha$, $\overset{0}{w}_\beta$ ein Tangentialraum existiert, und dass auch die lokale zu $\mathfrak{M}_0$ gehörige algebraische Mannigfaltigkeit in $\underset{0}{x}^\alpha$ irreduzibel ist. Nach der Voraussetzung existiert also eine Funktion $\overset{0}{s}(x^\alpha)$; $\alpha = 1, \dots, n-1$, die den Gleichungen

$$\left.\begin{aligned} \partial_{d+2}\overset{0}{s} &= \varphi_{d+2}(x^\alpha, \underset{0}{x}^n, \partial_1\overset{0}{s}, \dots, \partial_{d+1}\overset{0}{s}) \\ &\vdots \\ \partial_{n-1}\overset{0}{s} &= \varphi_{n-1}(x^\alpha, \underset{0}{x}^n, \partial_1\overset{0}{s}, \dots, \partial_{d+1}\overset{0}{s}) \end{aligned}\right\} \alpha = 1, \dots, n-1 \quad . \quad (17)$$

oder auch dem mit diesem System gleichwertigen System

$$F_k(x^\alpha, \underset{0}{x}^n, \partial_\beta\overset{0}{s}) = 0 \ ; \ k = \mathrm{d}+2, \dots, \mathrm{n}-1 \quad \alpha, \beta = 1, \dots, n-1 \quad . \quad (17a)$$

genügt und für welche in $\underset{0}{x}^\alpha$ ausserdem noch gilt

$$\partial_\beta\overset{0}{s} = \overset{0}{w}_\beta \quad ; \quad \beta = 1, \dots, n-1 \quad \dots \dots \quad (18)$$

b. Konstruktion der Funktion $s(x^\nu)$.

Nach dem Existenztheorem von CAUCHY—KOWALEWSKI existiert eine einzige Funktion $s(x^\nu)$, die der letzten der Gleichungen (13):

$$\partial_n s = \varphi_n(x^\nu, \partial_1 s, \dots, \partial_{d+1} s) \quad \dots \dots \quad (19)$$

genügt, während ausserdem gilt

$$s(x^\alpha, \underset{0}{x}^n) = \overset{0}{s}(x^\alpha) \quad ; \quad \alpha = 1, \dots, n-1. \quad \dots \quad (20)$$

Aus (18) folgt für $x^\varkappa = \underset{0}{x}{}^\varkappa$

$$\partial_\beta s = \overset{0}{w}_\beta \quad ; \quad \beta = 1, \ldots, n-1, \qquad (21)$$

und aus (19) folgt also, dass für $x^\varkappa = \underset{0}{x}{}^\varkappa$

$$\partial_n s = \varphi_n (\underset{0}{x}{}^\varkappa, \overset{0}{w}_1, \ldots, \overset{0}{w}_{d+1}) \qquad (22)$$

ist. Da aber $\underset{0}{x}{}^\varkappa, \overset{0}{w}_\lambda$ zu $\mathfrak{M}$ gehört und also der Gleichung (12) genügt, folgt in $x^\varkappa = \underset{0}{x}{}^\varkappa$

$$\partial_n s = \overset{0}{w}_n, \qquad (23)$$

sodass die Funktion s der Gleichung (14) genügt.

c. *Beweis dass s auch den ersten n—d—2 Gleichungen* (13) *genügt.*

Der Voraussetzung nach liegt $\overset{n}{e}_\lambda$ nicht in dem Tangentialraum zu $\underset{0}{x}{}^\varkappa, \overset{0}{w}_\lambda$. Es gibt also mindestens eine Hyperfläche mit einer Gleichung

$$F(x^\varkappa, w_\lambda) = F^{\lambda_1 \ldots \lambda_r} w_{\lambda_1} \ldots w_{\lambda_r} = 0, \qquad (24)$$

mit $F^{\lambda_1 \ldots \lambda_r}$ analytisch in einer Umgebung von $\underset{0}{x}{}^\varkappa$, die $\mathfrak{M}$ enthält und deren Tangentialraum zu $\underset{0}{x}{}^\varkappa, \overset{0}{w}_\lambda$ nicht durch $\overset{n}{e}_\lambda$ geht. Infolgedessen ist

$$\frac{\partial F}{\partial w_n} \neq 0 \qquad (25)$$

in $\underset{0}{x}{}^\varkappa, \overset{0}{w}_\lambda$. Setzt man nun in F für w_n ein $\varphi_n(x^\varkappa, w_\eta)$; $\eta = 1, \ldots, d+1$, so entsteht eine Funktion

$$\left. \begin{aligned} E(x^\varkappa, w_\beta) \overset{\text{def}}{=} F(x^\varkappa, w_\beta, \varphi_n(x^\varkappa, w_\eta)) \; , \; \varkappa = 1, \ldots, n \quad \beta = 1, \ldots, n-1 \\ \eta = 1, \ldots, d+1 \end{aligned} \right\} \quad (26)$$

die in der Umgebung von $\underset{0}{x}{}^\varkappa, \overset{0}{w}_\lambda$ analytisch ist und auf $\mathfrak{M}_1$ verschwindet. Denn ist $w_\beta(x^\varkappa)$ ein beliebiger Punkt von $\mathfrak{M}_1$, so genügt w_β den Gleichungen (15) und daraus folgt, dass $(w_\beta, \varphi_n(x^\varkappa, w_\eta))$ den Gleichungen (12) genügt und also ein Punkt von $\mathfrak{M}$ ist und infolgedessen $F(x^\varkappa, w_\lambda)$ zum Verschwinden bringt.

Das Verschwinden von E auf $\mathfrak{M}_1$ bringt mit sich, dass es eine Gleichung gibt von der Form

$$\left. \begin{aligned} E(x^\alpha, w_\beta) = J^k(x^\varkappa, w_\beta) F_k(x^\varkappa, w_\beta) \; ; \; k = \mathrm{d}+2, \ldots, \mathrm{n}-1, \\ \beta = 1, \ldots, n-1 \\ \varkappa = 1, \ldots, n \end{aligned} \right\} \qquad (27)$$

wo die rechts auftretenden Funktionen J^k (die nicht notwendig rational in den w_β zu sein brauchen) in der Umgebung von $\underset{0}{x}^{\varkappa}, \overset{0}{w}_\beta$ analytisch in $x^{\varkappa}, w_\beta$ sind[6]). Setzt man nun $w_\beta = \partial_\beta s$ in (27), so entsteht infolge (19)

$$\left.\begin{array}{l} F(x^{\varkappa}, \partial_\lambda s) = E(x^{\varkappa}, \partial_\beta s) = J^k(x^{\varkappa}, \partial_\beta s)\, F_k(x^{\varkappa}, \partial_\beta s); \\ \qquad\qquad\qquad\qquad k = \mathrm{d}+2, \ldots, \mathrm{n}-1 \\ \qquad\qquad\qquad\qquad \beta = 1, \ldots, n-1 \\ \qquad\qquad\qquad\qquad \varkappa, \lambda = 1, \ldots, n \end{array}\right\} \quad . \quad . \quad (28)$$

wo die $J^k(x^{\varkappa}, \partial_\beta s)$ jetzt in der Umgebung von $\underset{0}{x}^{\varkappa}$ analytisch sind in $x^{\varkappa}$. Es verschwinden nun sowohl $F(x^{\varkappa}, w_\lambda)$ als $F_k(x^{\varkappa}, w_\lambda)$, ($w_n$ tritt in den F_k nicht auf), auf $\mathfrak{M}$. Da $\mathfrak{M}$ vollständig ist, verschwinden daselbst also auch die Polynome

$$\left.\begin{array}{l} K_k(x^{\varkappa}, w_\lambda) \overset{\text{def}}{=} D(F_k; F)^{\mu\lambda_1\ldots\lambda_r k\varkappa_1\ldots\varkappa_r}\, w_\mu w_{\lambda_1} \ldots . w_{\varkappa_r} \\ \qquad\qquad\qquad\qquad k = \mathrm{d}+2, \ldots, \mathrm{n}-1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (29)$$

Für diese Polynome gelten also genau wie für F Gleichungen von der Form

$$K_k(x^{\varkappa}, \partial_\lambda s) = L_k^l(x^{\varkappa}, \partial_\beta s)\, F_l(x^{\varkappa}, \partial_\beta s), \quad k, l = \mathrm{d}+2, \ldots, \mathrm{n}-1 \quad (30)$$

wo die $L_k^l(x^{\varkappa}, \partial_\beta s)$ analytisch in $x^{\varkappa}$ sind in der Umgebung von $\underset{0}{x}^{\varkappa}$. Schreiben wir $F_k(x^{\varkappa}, \partial_\beta s) \overset{\text{def}}{=} \theta_k(x^{\varkappa})$, so ist (30) gleichbedeutend mit den Kongruenzen

$$\left.\begin{array}{l} K_k(x^{\varkappa}, \partial_\lambda s) = D(F_k; F)^{\mu\lambda_1\ldots\lambda_r k\varkappa_1\ldots\varkappa_r}\, (\partial_\mu s)(\partial_{\lambda_1} s)\ldots(\partial_{\varkappa_r} s) \equiv 0 \mathit{\ mod\ } (\theta_l) \\ \qquad\qquad\qquad\qquad\qquad\qquad k, l = \mathrm{d}+2, \ldots, \mathrm{n}-1 \end{array}\right\} . \ (31)$$

Auf diese Kongruenzen wenden wir jetzt die allgemeine Formel

$$\left.\begin{array}{l} D(X_a; X_b)^{\mu\lambda_1\ldots\lambda_{a+b}}\, w_\mu w_{\lambda_1} \ldots w_{\lambda_{a+b}} = 2(a+1)(b+1)\, X_a^{\mu\lambda_1\ldots\lambda_a} \\ \qquad\qquad\qquad\qquad X^{\mu\nu_1\ldots\nu_b}\, (\partial_{[\mu} w_{\lambda]})\, w_{\lambda_1} \ldots w_{\nu_b} + \\ + (a+1)\, X_a^{\mu\lambda_1\ldots\lambda_a}\, w_{\lambda_1} \ldots w_{\lambda_a}\, \partial_\mu (X_b^{\lambda\nu_1\ldots\nu_b}\, w_\lambda w_{\nu_1} \ldots w_{\nu_b}) - \\ - (b+1)\, X_b^{\mu\lambda_1\ldots\lambda_b}\, w_{\lambda_1} \ldots w_{\lambda_b}\, \partial_\mu (X_a^{\lambda\nu_1\ldots\nu_a}\, w_\lambda w_{\nu_1} \ldots w_{\nu_a}) \end{array}\right\} \quad . \quad (32)$$

an, die eine Verallgemeinerung ist der Formel (10) A. S. I, S. 958 und in diese Formel übergetzt für den dort berücksichtigten Fall, dass die

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[6]) Vergl. für den Beweis dieser Tatsache E. KÄHLER, Einführung in die Theorie der Systeme von Differentialgleichungen, S. 12.

eingeklammerten Ausdrücke in den beiden letzten Termen verschwinden. Da $\frac{\partial F_k}{\partial w_n}$ verschwindet, ergibt sich dann

$$\left.\begin{array}{r} \left\{\frac{\partial F_k}{\partial w_\beta}(x^\varkappa, \partial_\lambda s)\right\} \partial_\beta F(x^\varkappa, \partial_\lambda s) - \left\{\frac{\partial F}{\partial w^\mu}(x^\varkappa, \partial_\lambda s)\right\} \partial_\mu \theta_k \equiv 0 \; mod\,(\theta_l) \\ k, l = \mathrm{d}+2, \ldots, \mathrm{n}-1 \\ \beta = 1, \ldots, n-1 \\ \varkappa, \lambda, \mu = 1, \ldots, n \end{array}\right\}. \quad (33)$$

Da aber infolge (28)

$$\left.\begin{array}{r} \partial_\beta F(x^\varkappa, \partial_\lambda s) \equiv 0 \; mod\,(\theta_k, \partial_\beta \theta_k) \\ \beta = 1, \ldots, n-1 \\ \varkappa, \lambda = 1, \ldots, n \\ k = \mathrm{d}+2, \ldots, \mathrm{n}-1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

folgt

$$\left.\begin{array}{r} \left\{\frac{\partial F}{\partial w_n}(x^\varkappa, \partial_\lambda s)\right\} \partial_n \theta_k(x^\varkappa) \equiv 0 \; mod\,(\theta_l, \partial_\beta \theta_l) \\ \beta = 1, \ldots, n-1 \\ \varkappa, \lambda = 1, \ldots, n \\ k, l = \mathrm{d}+2, \ldots, \mathrm{n}-1 \end{array}\right\} \quad . \quad . \quad . \quad (35)$$

wobei zu bemerken ist, dass beim Ausschreiben des rechten Gliedes in $\theta_{\mathrm{d}+2}, \ldots, \partial_\beta \theta_{\mathrm{n}-1}$ alle Koeffizienten analytisch sind in der Umgebung von $\underset{0}{x}^\varkappa$. Da in diesem Punkte $\frac{\partial F}{\partial w_n}(x^\varkappa, \partial_\lambda s)$ übergeht in $\frac{\partial F}{\partial w_n}(\underset{0}{x}^\varkappa, \overset{0}{w}_\lambda)$ und dieser Ausdruck infolge (25) nicht verschwindet, kann man durch $\frac{\partial F}{\partial w_n}$ dividieren und es entsteht eine Gleichung von der Form

$$\left.\begin{array}{r} \partial_n \theta_k(x^\varkappa) = \varphi_k^l(x^\varkappa)\, \theta_l(x^\varkappa) + \varphi_k^{l\beta}(x^\varkappa)\, \partial_\beta \theta_l(x^\varkappa) \\ k, l = \mathrm{d}+2, \ldots, \mathrm{n}-1 \\ \beta = 1, \ldots, n-1 \\ \varkappa = 1, \ldots, n \end{array}\right\} \quad . \quad . \quad . \quad (36)$$

mit in der Umgebung von $\underset{0}{x}^\varkappa$ analytischen Koeffizienten im rechten Gliede. Nun ist ausserdem

$$\left.\begin{array}{r} \theta_k(x^\alpha, \underset{0}{x}^n) = F_k(x^\alpha, \underset{0}{x}^n, \partial_\beta s) = 0 \\ \alpha, \beta = 1, \ldots, n-1 \\ k = \mathrm{d}+2, \ldots, \mathrm{n}-1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

infolge (17*a*) und (20). Nach dem Existenztheorem von CAUCHY—KOWALEWSKI hat das System (36) mit den Anfangsbedingungen (37) eine einzige Lösung. Da aber $\theta_k(x^\nu) = 0$ offenbar eine Lösung ist, ist dies auch die einzige Lösung und damit ist gezeigt, dass $\partial_\lambda s$ auch die Gleichung (15*a*) und somit auch die mit dieser gleichwertige Gleichung (15) befriedigt, woraus hervorgeht dass s nicht nur der letzten der Gleichungen (13) sondern sämtlichen Gleichungen (13) genügt w. z. b. w.

8. *Schluss.*

Wir kehren jetzt zurück zu der in A. S. I § 2 gestellten Aufgabe die Lösungen zu bestimmen der Systeme

$$\underset{i}{P}^{\mu\lambda_1\ldots\lambda_{a_i}} (\partial_\mu s)(\partial_{\lambda_1} s)\ldots(\partial_{\lambda_{a_i}} s) = 0 \; ; \; i = 1, \ldots, N. \quad . \quad . \quad (38)$$

Zunächst bestimme man nach der in A. S. I § 3 angegebenen Methode die grösste vollständige Mannigfaltigkeit $\mathfrak{M}_\nu$, die in der Mannigfaltigkeit $\mathfrak{M}$, definiert durch die Gleichungen

$$\underset{i}{P}^{\mu\lambda_1\ldots\lambda_{a_i}} w_\mu w_{\lambda_1} \ldots w_{\lambda_{a_i}} = 0 \; ; \; i = 1, \ldots, N \quad . \quad . \quad . \quad (39)$$

enthalten ist. Sodann zerlege man $\mathfrak{M}_\nu$ in irreduzibele Mannigfaltigkeiten $\mathfrak{M}_{\nu 1}, \ldots, \mathfrak{M}_{\nu r}$, die alle vollständig sind. Sind dann z.B.

$$\underset{s}{Q}^{\mu\lambda_1\ldots\lambda_{a_s}} w_\mu w_{\lambda_1} \ldots w_{\lambda_{a_s}} = 0 \; ; \; s = 1, \ldots, M \quad . \quad . \quad . \quad (40)$$

die Gleichungen von $\mathfrak{M}_{\nu 1}$, so bestimme man in einem Punkt $\underset{0}{x}^\nu$ eine Lösung $\overset{0}{w}_\lambda$ dieser Gleichungen, so dass auch die lokale zu $\mathfrak{M}_{\nu 1}$ gehörige Mannigfaltigkeit irreduzibel ist und $\mathfrak{M}_{\nu 1}$ in $\underset{0}{x}^\nu, \overset{0}{w}_\lambda$ einen Tangentialraum hat. Es lässt sich dann eine Funktion s bilden, so dass $\partial_\lambda s$ (40) also auch (38) befriedigt und für welche ausserdem in $\underset{0}{x}^\nu$ gilt $\partial_\lambda s = \overset{0}{w}_\lambda$. In dieser Weise können alle Lösungen von (38) erhalten werden.

Weniger genau lässt sich das Existenztheorem auch folgendermassen formulieren:

Ist $\overset{0}{w}_\lambda$ *in* $\underset{0}{x}^\nu$ *ein zu einer vollständigen (nicht notwendig irreduzibelen) algebraischen Mannigfaltigkeit gehöriger Vektor nicht besonderer Lage, so lässt sich* $\overset{0}{w}_\lambda$ *zu einem zu* $\mathfrak{M}$ *gehörigen Gradientfeld fortsetzen*

oder auch

Die zu einer vollständigen algebraischen Mannigfaltigkeit gehörigen

Vektoren nicht besonderer Lage lassen sich zu Gradientfeldern aneinanderreihen.

Es sei noch bemerkt, dass das Existenztheorem folgende leicht beweisbare Umkehrung zulässt.

Lässt sich in jedem beliebigen Punkt $\underset{0}{x}^{\nu}$ *eines bestimmten Gebietes der* X_n *jeder zu einer algebraischen Mannigfaltigkeit* $\mathfrak{M}$ *gehörige Vektor* $\overset{0}{w}_{\lambda}$ *eines bestimmten Gebietes des* w_{λ}*-Raumes in diesem Punkt in einer Umgebung von* $\underset{0}{x}^{\nu}$ *zu einem zu* $\mathfrak{M}$ *gehörigen Gradientfeld fortsetzen, so ist* $\mathfrak{M}$ *vollständig.*

Chemistry. — *Thixotropy of suspensions in apolar liquids.* (Preliminary Communication.) By H. R. KRUYT and F. G. VAN SELMS.

(Communicated at the meeting of October 26, 1940.)

FREUNDLICH and RÖDER [1]) have made the interesting observation that suspensions of rice starch or quartz, which in water give systems that may be considered Newtonian liquids, occasion spontaneous thixotropy (so-called "false body") in CCl_4, cyclohexane respectively. One of us [2]) has attached theoretical consequences to this observation which affect not only the theory of thixotropy, but also the theory of gelatination generally.

In order to test and extend this course of thought an extensive investigation is being made, following a method different from RÖDER's, which has revealed facts that for the present throw a different light on the observations mentioned. It appears that thixotropy does, indeed, occur when starch is suspended in air dry condition (that is with a moisture percentage of ca. 12 %) in CCl_4, but that the phenomenon is *practically absent* when the starch is first dried in a vacuum dessiccator over chloride of calcium. In the same way suspensions of finely divided quartz *do not show thixotropy* in CCl_4 when this material has first been dehydrated during 1½ hours at ca. 360°.

Hence the explanation given so-far for the remarkable phenomenon of suspensions in an apolar medium is in need of revision. The presence of water in the phase boundary of, for instance quartz — CCl_4, apparently plays an all-important part. One is even inclined to think that it is not the apolarity of the organic liquid that causes the phenomenon but its "unmixing" with water.

Although the investigations are as yet only in the initial stage, and we hope to return to this problem more extensively at a later date, we considered these facts of such importance for the theory of thixotropy and gelatination, as to justify the immediate publication of this experience.

VAN 'T HOFF-*Laboratorium.*

Utrecht, October 1940.

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[1]) H. FREUNDLICH and H. L. RÖDER, Trans. Faraday Soc. **34**, 308 (1938).

[2]) H. R. KRUYT, Chimie et Industrie **42**, 587 (1939).

Mathematics. — *Un problème de la théorie des réseaux plans.* Par P. HAZEBROEK. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 26, 1940.)

1. Deux familles de courbes dans le plan constituent dans une région R un réseau plan, quand: 1° par chaque point de R passe une courbe de chaque famille, de telle sorte que ces deux courbes ne soient pas tangentes, 2° deux courbes appartenant à la même famille n'ont pas de point en commun.

Un réseau est exprimé analytiquement par trois équations $x = x(u, v)$, dans lesquelles le seul symbole x représente les trois coordonnées projectives planes d'un point x par rapport à un repère fixe. On obtient les deux familles de courbes en ne variant qu'*un* des paramètres u, v.

Par deux droites conjuguées en un point x d'un réseau nous entendons deux droites passant par x et harmoniques aux tangentes xx_u, xx_v. Par un réseau conjugué d'un réseau donné N nous entendons un réseau, dont toutes les paires de tangentes sont droites conjuguées par rapport à N.

Le sujet de notre article est la solution du problème suivant:

Y-a-t'il des réseaux, qui admettent une infinité de réseaux conjugués, dont les deux familles soient constituées exclusivement de droites?

Nous supposons qu'aucune des deux familles du réseau cherché ne soit formée uniquement de droites.

2. En traitant ce problème nous ferons usage de la méthode des repères mobiles de CARTAN, dont le lecteur est supposé au courant.

A chaque point de notre réseau nous associons un repère local aux sommets a_0, a_1, a_2, dont les coordonnées par rapport au repère fixe satisfont la relation

$$[a_0, a_1, a_2] = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Les coordonnées fixes d'un point sont des combinaisons linéaires des coordonnées de a_0, a_1 et a_2, de sorte qu'elles puissent être exprimées par l'équation

$$z = z_0\, a_0 + z_1\, a_1 + z_2\, a_2,$$

où z_0, z_1, z_2 sont les coordonnées locales de z.

De la même façon les coordonnées de da_0, da_1, da_2 sont exprimées par les équations

$$\left.\begin{aligned} d\,a_0 &= \omega_{00}\, a_0 + \omega_{01}\, a_1 + \omega_{02}\, a_2 \\ d\,a_1 &= \omega_{10}\, a_0 + \omega_{11}\, a_1 + \omega_{12}\, a_2 \\ d\,a_2 &= \omega_{20}\, a_0 + \omega_{21}\, a_1 + \omega_{22}\, a_2, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

dans lesquelles les ω_{ij} sont des expressions de PFAFF. En vertu de (1) nous avons

$$\omega_{00} + \omega_{11} + \omega_{22} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Les covariants bilinéaires des équations (2) doivent être 0, d'où il suit que

$$\omega'_{ij} = \sum_k [\omega_{ik}, \omega_{kj}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Le repère général dépend de 8 paramètres. Nous attacherons le repère au réseau de façon covariante, de sorte qu'il ne dépende que des deux paramètres essentiels u, v du réseau, les six paramètres inessentiels doivent être éliminés. Une différentiation qui ne fait varier que les paramètres inessentiels sera désignée par le symbole δ; en ce cas nous écrirons e_{ij} au lieu de ω_{ij}.

Nous considérons un point $x(u_0, v_0)$ de notre réseau et faisons coïncider a_0 et x, tandis que $a_0 a_1$ et $a_0 a_2$ sont des tangentes respectivement aux courbes $v = \text{const.}$ et $u = \text{const.}$ passant par x. En outre nous faisons coïncider a_1 et a_2 avec les transformés de LAPLACE correspondant respectivement aux directions de u et de v. Les points *géométriques* a_0, a_1 et a_2 sont alors définis complètement et déterminés analytiquement à un facteur près, avec la restriction résultant de l'équation (1). Il n'y a donc encore à éliminer que deux paramètres inessentiels. Nous avons encore

$$\delta a_i = e_{ii} a_i . \qquad (i = 1, 2, 3)$$

Tous les autres e_{ij} sont 0.

Par suite du choix de notre repère local nous avons $\omega_{01} = f(u, v) du$, $\omega_{02} = g(u, v) dv$, nous écrirons dorénavant $\omega_{01} = \omega_1$, $\omega_{02} = \omega_2$. Tous les ω_{ij}, pour lesquels $e_{ij} = 0$, sont des formes linéaires de du et dv, par conséquent aussi de ω_1 et ω_2. En particulier à cause du choix de a_1 et a_2 on a $\omega_{12} = b_1 \omega_1$, $\omega_{21} = b_2 \omega_2$. Puisque nous supposons qu'aucune des deux familles du réseau soit constituée uniquement de droites, b_1 et b_2 sont $\neq 0$.

De (4) il suit, en rappelant que $[\omega_{21}, \omega_2]$ et $[\omega_{12}, \omega_1]$ sont 0

$$\omega'_1 = [\omega_{00} - \omega_{11}, \omega_1], \quad \omega'_2 = [\omega_{00} - \omega_{22}, \omega_2]. \quad . \quad . \quad . \quad . \quad (5)$$

Donc

$$\delta \omega_1 = (e_{00} - e_{11})\, \omega_1, \quad \delta \omega_2 = (e_{00} - e_{22})\, \omega_2.$$

De même

$$\left.\begin{aligned} \omega'_{12} &= [\omega_{10}, \omega_2] + [\omega_{11} - \omega_{22}, \omega_{12}] \\ \omega'_{21} &= [\omega_{20}, \omega_1] + [\omega_{22} - \omega_{11}, \omega_{21}], \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

d'où il s'ensuit, en vertu de la formule $(k\omega)' = k\omega' + [dk, \omega]$,

$$\begin{aligned} \delta \omega_{12} &= b_1 (e_{11} - e_{22})\, \omega_1 = \omega_1\, \delta b_1 + b_1\, \delta\omega_1 \\ \delta \omega_{21} &= b_2 (e_{22} - e_{11})\, \omega_2 = \omega_2\, \delta b_2 + b_2\, \delta \omega_2 \end{aligned}$$

de sorte que, tenant compte de (3)

$$\delta b_1 = 3\, e_{11}\, b_1, \quad \delta b_2 = 3\, e_{22}\, b_2.$$

Il est donc permis de déterminer les deux paramètres inessentiels de telle façon que $b_1 = 1$ et $b_2 = 1$. Par là e_{11} et e_{22} deviennent 0 et notre repère est maintenant fixé aussi analytiquement.

De (4), (5) et (6) il découle:

$$\begin{aligned} &[3\,\omega_{11}, \omega_1] + [\omega_{10}, \ \omega_2] = 0 \\ &[\omega_{20}, \ \omega_1] + [3\omega_{22}, \omega_2] = 0. \end{aligned}$$

En vertu de ces relations nous posons

$$\left.\begin{aligned} &\omega_{00} = -(b+\beta)\,\omega_1 - (c+\gamma)\,\omega_2 \\ &\omega_{11} = b\,\omega_1 + \gamma\,\omega_2, \quad \omega_{10} = 3\,\gamma\,\omega_1 + \mu\,\omega_2, \\ &\omega_{22} = \beta\,\omega_1 + c\,\omega_2, \quad \omega_{20} = \nu\,\omega_1 + 3\,\beta\,\omega_2, \end{aligned}\right\} \quad \ldots\ldots \quad (7)$$

où $b, c, \beta, \gamma, \mu, \nu$ sont des invariants du réseau. Les équations de notre réseau sont donc

$$\left.\begin{aligned} &d\,a_0 = \{-(b+\beta)\,\omega_1 - (c+\gamma)\,\omega_2\}\,a_0 + \omega_1\,a_1 + \omega_2\,a_2 \\ &d\,a_1 = (3\,\gamma\,\omega_1 + \mu\,\omega_2)\,a_0 + (b\,\omega_1 + \gamma\,\omega_2)\,a_1 + \omega_1\,a_2 \\ &d\,a_2 = (\nu\,\omega_1 + 3\,\beta\,\omega_2)\,a_0 + \omega_2\,a_1 + (\beta\,\omega_1 + c\,\omega_2)\,a_2. \end{aligned}\right\} \quad \ldots \quad (8)$$

$$\left.\begin{aligned} &\omega_1' = (2\,\gamma + c)\,[\omega_1, \omega_2] \\ &\omega_2' = -(2\,\beta + b)\,[\omega_1, \omega_2]. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (9)$$

Pour une fonction arbitraire F nous posons

$$d\,F = F_1\,\omega_1 + F_2\,\omega_2.$$

On peut alors déduire des conditions d'intégrabilité (4) les relations suivantes

$$\left.\begin{aligned} &\gamma_1 - b_2 = 2\,\beta\,\gamma - b\,\gamma - b\,c - \mu + 1 \\ &\beta_2 - c_1 = 2\,\beta\,\gamma - c\,\beta - b\,c - \nu + 1, \end{aligned}\right\} \quad \ldots\ldots \quad (10)$$

$$\left.\begin{aligned} &\mu_1 - 3\,\gamma_2 = 3\,\mu\,(\beta + b) - 6\,\gamma\,(2\,\gamma + c) + 3\,\beta \\ &\nu_2 - 3\,\beta_1 = 3\,\nu\,(\gamma + c) - 6\,\beta\,(2\,\beta + b) + 3\,\gamma. \end{aligned}\right\} \quad \ldots \quad (11)$$

Nous examinons encore dans quelles conditions le réseau est asymptotique. Si nous entendons par d_1 et d_2 des symboles de différentiation dans les directions de u et de v, de manière que $d_1F = F_1\omega_1$, $d_2F = F_2\omega_2$, il est facile à trouver que le rapport anharmonique des points a_0, a_1, $a_0 + d_1a_0$, $a_1 + d_2a_1$ est égal à $\mu\omega_1\omega_2$. De même le rapport anharmonique des points a_0, a_2, $a_0 + d_2a_0$, $a_2 + d_1a_2$ est égal à $\nu\omega_1\omega_2$. Si ces deux rapports anharmoniques sont égaux entre eux, donc si $\mu = \nu$, le réseau est asymptotique ¹).

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¹) Selon la définition plus usuelle un réseau est asymptotique, s'il peut être considéré comme la projection des asymptotiques d'une surface. Pour faciliter les calculs nous nous sommes servis de la propriété employée dans le texte.

3. Soit $\omega_2^2 = k^2\omega_1^2$ $(k = k(u, v))$ l'équation differentielle d'un réseau conjugué de notre réseau. La tangente à la courbe $\omega_2 = k\omega_1$ en a_0 est $kz_1 - z_2 = 0$. La courbe $\omega_2 = k\omega_1$ sera une droite, si l'équation $kz_1 - z_2 = 0$ ne change pas en différentiant dans la direction $\omega_2 = k\omega_1$.

Pour qu'une droite l

$$h_0 z_0 + h_1 z_1 + h_2 z_2 = 0$$

reste fixe en différentiant dans une certaine direction, il faut que les coefficients de l'équation

$$(d h_0 - \Sigma\, \omega_{0i} h_i) z_0 + (d h_1 - \Sigma\, \omega_{1i} h_i) z_1 + (d h_2 - \Sigma\, \omega_{2i} h_i) z_2 = 0$$

soient proportionnels aux coefficients de l'équation de l. En se servant de cela pour la droite $kz_1 - z_2 = 0$, nous trouvons par différentiation dans la direction $\omega_2 = k\omega_1$ que k doit satisfaire à la condition

$$k_1 + k k_2 + 1 + k(\beta - b) - k^2(\gamma - c) - k^3 = 0. \quad . \quad . \quad . \quad (12)$$

Pour trouver la condition dans laquelle la courbe $\omega_2 = -k\omega_1$ est une droite, nous remplaçons k par $-k$ dans l'équation (12). Les conditions pour que ces deux équations existent à la fois, sont donc

$$k_1 = -k(\beta - b) + k^3, \quad k k_2 = -1 + k^2(\gamma - c). \quad . \quad . \quad . \quad (13)$$

Le fait que $dF = F_1\omega_1 + F_2\omega_2$ est une différentielle exacte, est exprimé en posant zéro le covariant bilinéaire; cela nous conduit en vertu de (9) à la relation

$$F_{12} - (2\gamma + c) F_1 = F_{21} - (2\beta + b) F_2. \quad . \quad . \quad . \quad . \quad (14)$$

En faisant usage de cette formule on déduit de (13), à l'aide de (10), l'équation pour k:

$$3ck^4 - (\mu + \nu - 6) k^2 + 3b = 0. \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Pour qu'il y ait une infinité de réseaux conjugués doublement réglés, il faut que les coefficients de l'équation (15) soient 0, donc

$$b = 0, \quad c = 0, \quad \mu + \nu = 6.$$

Nous posons

$$\mu = 3(1 + \vartheta), \quad \nu = 3(1 - \vartheta)$$

et commençons par supposer que $\vartheta \neq 0$, donc que le réseau n'est pas asymptotique.

4. Le cas $\vartheta \neq 0$. Les équations (9), (10) et (11) deviennent alors

$$\omega_1' = 2\gamma\,[\omega_1, \omega_2], \quad \omega_2' = -2\beta\,[\omega_1, \omega_2], \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\gamma_1 = 2\beta\gamma - 3\vartheta - 2, \quad \beta_2 = 2\beta\gamma + 3\vartheta - 2, \quad . \quad . \quad . \quad . \quad (17)$$

$$\vartheta_1 - \gamma_2 = 3\beta\vartheta - 4\gamma^2 + 4\beta, \quad \vartheta_2 + \beta_1 = 3\gamma\vartheta + 4\beta^2 - 4\gamma. \quad . \quad . \quad (18)$$

En vertu de ces deux dernières équations nous introduisons deux nouvelles fonctions ξ et η de telle sorte que

$$\gamma_2 = 4\gamma^2 - 4\beta + 3\xi\vartheta, \quad \beta_1 = 4\beta^2 - 4\gamma - 3\eta\vartheta, \quad . \quad . \quad . \quad (19)$$

$$\vartheta_1 = 3\vartheta(\beta + \xi), \quad \vartheta_2 = 3\vartheta(\gamma + \eta). \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Les covariants bilinéaires de $\gamma_1\omega_1 + \gamma_2\omega_2$ et $\beta_1\omega_1 + \beta_2\omega_2$ étant 0, on obtient

$$\left.\begin{aligned} \xi_1 &= \beta\xi - 3\xi^2 + 9\gamma - 7\eta \\ \eta_2 &= \gamma\eta - 3\eta^2 + 9\beta - 7\xi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

De même pour le covariant bilinéaire de $\vartheta_1\omega_1 + \vartheta_2\omega_2$

$$\xi_2 - \eta_1 = 2\gamma\xi - 2\beta\eta - 6\vartheta.$$

Nous pouvons donc introduire une fonction φ, de manière que

$$\left.\begin{aligned} \xi_2 &= 2\gamma\xi - 3\vartheta + 7\varphi + 5 \\ \eta_1 &= 2\beta\eta + 3\vartheta + 7\varphi + 5. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

De même façon que ci-dessus on trouve

$$\left.\begin{aligned} \varphi_1 &= 3\varphi(\beta - 2\xi) + 9\xi\vartheta + 3\xi - 12\beta + 3\eta\gamma + 3\eta^2 \\ \varphi_2 &= 3\varphi(\gamma - 2\eta) - 9\eta\vartheta + 3\eta - 12\gamma + 3\xi\beta + 3\xi^2 \end{aligned}\right\} \quad . \quad (23)$$

avec la condition d'intégrabilité

$$\varphi + \xi\eta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

En différentiant et faisant usage des relations (17)—(24) on déduit les nouvelles relations

$$\left.\begin{aligned} \xi^2\eta - 2\xi + 3\beta + \eta^2 - 3\eta\gamma - 3\xi\vartheta &= 0 \\ \xi\eta^2 - 2\eta + 3\gamma + \xi^2 - 3\xi\beta + 3\eta\vartheta &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (25)$$

Différentiant la première de ces équations dans la direction de $\omega_2 = 0$ et la seconde dans la direction de $\omega_1 = 0$ on trouve ensuite

$$\left.\begin{aligned} -3\gamma\vartheta - \xi^3\eta + 3\gamma\xi\eta - 3\xi\eta^2 + 3\xi\beta - 6\gamma + 4\eta &= 0 \\ 3\beta\vartheta - \xi\eta^3 + 3\beta\xi\eta - 3\xi^2\eta + 3\eta\gamma - 6\beta + 4\xi &= 0. \end{aligned}\right\} \quad . \quad (26)$$

Résolvant β et γ de (25) on trouve

$$3(\xi\eta - 1)\beta = (\eta^2 + 2\xi)(\xi\eta - 1) + 3(\eta^2 - \xi)\vartheta$$

$$3(\xi\eta - 1)\gamma = (\xi^2 + 2\eta)(\xi\eta - 1) - 3(\xi^2 - \eta)\vartheta.$$

Si $\xi\eta \neq 1$ on peut substituer β et γ dans (26) et on obtient alors

$$(\xi^2 - \eta)(\xi\eta - 1 - \tfrac{3}{4}\vartheta) = 0$$

$$(\xi - \eta^2)(\xi\eta - 1 + \tfrac{3}{4}\vartheta) = 0.$$

On peut alors distinguer, en supposant toujours que $\vartheta \neq 0$, les cas suivants:

1° $\xi \eta = 1, \quad \xi = \varepsilon, \quad \eta = \varepsilon^2 \, (\varepsilon^3 = 1)$.
2° $\xi = \eta = 0$.
3° $\xi = \eta^2, \quad \vartheta = \frac{4}{3} (\xi \eta - 1)$.

En comparant tous ces cas aux équations précédentes on trouve qu'ils mènent tous au cas $\vartheta = 0$. Les réductions sont tellement simples que nous les avons supprimées.

Nous trouvons donc comme premier résultat que *les réseaux cherchés sont, s'ils existent, asymptotiques.*

5. Le cas $\vartheta = 0$. Les équations (16)—(18) deviennent maintenant

$$\omega_1' = 2 \gamma [\omega_1, \omega_2], \quad \omega_2' = -2 \beta [\omega_1, \omega_2]; \quad \dots \dots \quad (27)$$

$$\beta_2 = \gamma_1 = 2 \beta \gamma - 2; \quad \dots \dots \dots \quad (28)$$

$$\beta_1 = 4 \beta^2 - 4 \gamma, \quad \gamma_2 = 4 \gamma^2 - 4 \beta. \quad \dots \dots \quad (29)$$

Il se trouve alors que toutes les conditions d'intégrabilité sont satisfaites.

Afin de déterminer les réseaux pour lesquels ces relations sont valables, nous posons

$$\omega_1 = f \, du, \quad \omega_2 = g \, dv.$$

On a maintenant pour une fonction arbitraire F

$$F_u = f F_1, \quad F_v = g F_2. \quad \dots \dots \dots \quad (30)$$

En outre il suit de (27)

$$\beta = -\frac{g_u}{2 f g}, \quad \gamma = -\frac{f_v}{2 f g}.$$

De $\beta_2 = \gamma_1$ et (30) il découle

$$(\log f/g)_{uv} = 0,$$

équation qui a la solution générale

$$\frac{f}{g} = \frac{U}{V}$$

où U et V sont des fonctions arbitraires respectivement de u et v. Par une transformation $u = u(\bar{u})$, $v = v(v)$ ω_1 et ω_2 se changent en $\omega_1 = \bar{f} d\bar{u}$, $\omega_2 = g \, dv$, f et g se transformant en $f = f \frac{du}{d\bar{u}}$, $g = g \frac{dv}{dv}$. On peut donc transformer les paramètres de telle manière que f devienne égal à g. Les paramètres sont alors déterminés à des constantes additives près.

Nous supprimons maintenant les barres et posons

$$\omega_1 = \frac{du}{\psi}, \quad \omega_2 = \frac{dv}{\psi};$$

Alors

$$\beta = \tfrac{1}{2}\,\psi_u, \quad \gamma = \tfrac{1}{2}\,\psi_v$$

et les équations (28) et (29) conduisent aux équations suivantes pour ψ

$$\begin{aligned}\psi\,\psi_{uu} &= 2\,\psi_u^2 - 4\,\psi_v,\\ \psi\,\psi_{vv} &= 2\,\psi_v^2 - 4\,\psi_u,\\ \psi\,\psi_{uv} &= \psi_u\,\psi_v - 4.\end{aligned}$$

Les trois solutions indépendantes de ces équations sont

$$\psi = 2\,(\varepsilon u + \varepsilon^2 v), \quad \varepsilon^3 = 1.$$

Considérons d'abord la solution $\psi = 2(u + v)$. Les invariants du réseau sont alors

$$\beta = \gamma = 1, \quad \mu = \nu = 3,$$

et les équations (8) deviennent

$$\begin{aligned}d a_0 &= -\left\{\frac{du}{2\,(u+v)} + \frac{dv}{2\,(u+v)}\right\} a_0 + \frac{du}{2\,(u+v)}\, a_1 + \frac{dv}{2\,(u+v)}\, a_2,\\ d a_1 &= 3\left\{\frac{du}{2\,(u+v)} + \frac{dv}{2\,(u+v)}\right\} a_0 + \frac{dv}{2\,(u+v)}\, a_1 + \frac{du}{2\,(u+v)}\, a_2,\\ d a_0 &= 3\left\{\frac{du}{2\,(u+v)} + \frac{dv}{2\,(u+v)}\right\} a_0 + \frac{dv}{2\,(u+v)}\, a_1 + \frac{du}{2\,(u+v)}\, a_2.\end{aligned}$$

Il se trouve que les deux autres solutions pour ψ mènent à ces mêmes équations si l'on remplace a_1 et a_2 par εa_2 et $\varepsilon^2 a_2$ et si l'on introduit εu et $\varepsilon^2 v$ comme nouvelles variables.

En éliminant a_1 et a_2 on trouve les équations différentielles suivantes pour le réseau

$$\begin{aligned}a_{0uu} &= \frac{1}{(u+v)^2}\, a_0 - \frac{3}{2\,(u+v)}\, a_{0u} + \frac{1}{2\,(u+v)}\, a_{0v},\\ a_{0vv} &= \frac{1}{(u+v)^2}\, a_0 + \frac{1}{2\,(u+v)}\, a_{0u} - \frac{3}{2\,(u+v)}\, a_{0v},\\ a_{0uv} &= \frac{1}{(u+v)^2}\, a_0 - \frac{1}{2\,(u+v)}\, a_{0u} - \frac{1}{2\,(u+v)}\, a_{0v}.\end{aligned}$$

6. Afin de résoudre ces équations nous posons $a_0 = x/(u+v)$ par quoi les équations deviennent

$$x_{uu} = x_{uv} = x_{vv} = \frac{1}{2\,(u+v)}\,(x_u + x_v).$$

La solution générale de ces équations est

$$x = C_1(u+v)^2 + C_2(u-v) + C_3$$

de sorte qu'en choisissant le repère fixe de façon convenable les équations de notre réseau soient

$$\begin{cases} \varrho x_1 = (u+v)^2 \\ \varrho x_2 = u-v \\ \varrho x_3 = 1. \end{cases}$$

Comme solution de notre problème nous trouvons donc un système de coniques qui ont entre elles une osculation de deuxième ordre en un point fixe P et qui ont comme tangente commune une droite fixe l (faisceau tangentiel osculateur). Par chaque point du plan (à l'exception de P et des points de l) passent deux coniques du système qui sont les deux courbes du réseau sortant de ce point. Du point de vue projectif il n'y a qu'un réseau satisfaisant les demandes de notre problème.

Le réseau trouvé est la projection des asymptotiques de la surface

$$\begin{cases} \sigma y_1 = \frac{1}{2}(u+v)^2 \log(u+v) - (u-v)^2 \\ \sigma y_2 = (u+v)^2 \\ \sigma y_3 = u-v \\ \sigma y_4 = 1. \end{cases}$$

Cette surface possède donc la propriété de contenir une infinité de réseaux conjugués dont les courbes sont situées dans des plans qui passent tous par un point fixe.

Anatomy. — *On the shape of froth chambers.* By S. T. BOK. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of October 26, 1940.)

In the tissues of the human body there is a great variety of froth structures and possibly the spatial networks formed in those tissues by protoplasmic and other rods are related to froth structures in so far that these rods may be found at the place of earlier froth edges. The quantitative investigation of these tissues is ever hampered by the lack of an adequate geometrical analysis of froth structures. We even find the most diverging communications in the literature about the shape of the froth chambers and hence a calculation of the most elementary properties of the shape of a froth — such as for instance the total edgelength in a given volume — is not even possible by approximation. Therefore we have investigated the shape of froth, the general results of which investigation are described in the following pages.

The properties of shape of froth are greatly different, according as the chambers are practically of the same size or not. When the difference in size is great the chambers are all more or less globular: in the spaces left between the larger globes (even when they nearly touch) the smaller globes can find sufficient room without materially affecting the shape of the larger ones. In froths the chambers of which are of the same size, on the other hand, the shape of the chambers approaches that of polyhedrons, i.e. bodies bounded by flat faces. For in that case the chambers are separated by thin, practically flat walls, meeting in approximately straight edges. Only froths with chambers of the same order of size will be discussed here.

To such a froth the two well known laws apply, viz.

1. three walls always meet in one edge (under approximately equal angles) and

2. four edges always meet in one point of junction (also under approximately equal angles).

From this it follows that the interfacial angle between two adjacent chamber walls is approximately 120° (i.e. about $^1/_3 \times 360^\circ$) and that the four edges meeting in one point will spatially behave approximately like the four lines connecting the centre of a regular tetrahedron with its four vertices, so that the angle between two intersecting edges will be approximately $109^\circ\, 28'\, 16,\ldots\ldots''$ (being the angle between the lines in the tetrahedron mentioned, that is an angle with $cos = -{}^1/_3$).

The two rules mentioned follow from the conditions for equilibrium in a froth. A froth consists of two substances which do not mix, either two liquids, or a liquid and a gas, of which the one filling the chambers is disperse, i.e. divided into a number of non-coherent parts, separated by the second, coherent, so continuous, substance, which forms the walls. When the disperse substance fills only a small part of the space we use the term emulsion, otherwise it is a froth. From this it follows that the spatial structure of an emulsion gradually passes into that of a froth, as the relative volume of the dispersed substance (emulsion globules or contents of the froth chambers) increases. The greater the relative volume of the froth chambers, the thinner the walls and the more apparent the froth character. Very thin walls, however, can only exist when the surface tension is great as compared with the specific gravities. The structure of a "fine" froth therefore mainly depends on the surface tension and this is approximately the same in all froth walls. For the surface tension depends on the nature of the two adjacent substances and of the curve radius of the surface. In a froth with a thin wall whose chambers are practically the same size, the greater part by far of the surface is very little curved, only a narrow strip along the edges showing a more marked curvature. So the potential energy, accumulated in the partition planes depends principally upon the nature of the two substances (which nature is the same everywhere), and on the total size of those partition planes. The attempt to establish the equilibrium, which therefore, is mainly an attempt to occupy the smallest possible area, results in a tangential tension in the froth walls, which is approximately the same everywhere. The three walls meeting in one edge exert approximately the same degree of tension, and the three powers (approximately equally great) will only establish an equilibrium if they form approximately equal angles, so that together the three walls must form approximately equal interfacial angles. And where a following froth plane intersects the edge, the interfacial angle between this plane and two of the former being again ca. 120°, the four intersecting lines, the edges, again intersect under approximately equal angles.

In the literature various theories have been put forward concerning certain polyhedrons, which are supposed approximately to have the shape of froth chambers and which, when a great number of them is piled up, are supposed more or less to fulfil the conditions for equilibrium mentioned.

BUFFIN's theory, that this is the rhombic dodecahedron, is the least felicitous, although many after him supported his theory, the last of his followers being SEIFRIZ in 1930. We may derive this rhombic dodecahedron from the situation, found in regularly arranged globes. The regular arrangement (one of the two so-called densest globe cumulations) is characterized by the fact that any 12 globes touching a "centre" globe (all globes having the same size), have their centres in the centres of the edges of a cube whose centre of gravity coincides with the centre of the central globe (see fig. 1, left top corner). If we imagine such regularly arranged globes equally increasing in volume while their centres remain in the same place, they will be flattened against each other in such a way that each plane of contact is found in the place of the tangent plane which, in the original configuration, may be imagined through the point of contact between the two globes. In the case of maximal growth, therefore, they become polyhedrons, which may also be construed in the regular arrangement by construing the mutual tangent plane of two globes through each point of contact. The lines of intersection of those tangent planes are the edges of the polyhedron obtained and it appears that this polyhedron is a rhombic dodecahedron (see fig. 1, right top corner). BUFFIN verified this construction by tying peas in a bag and swelling them in water. The flattened peas frequently had the shape of a rhombic dodecahedron. That this, however, does not correspond to the formation of a froth, is in my opinion

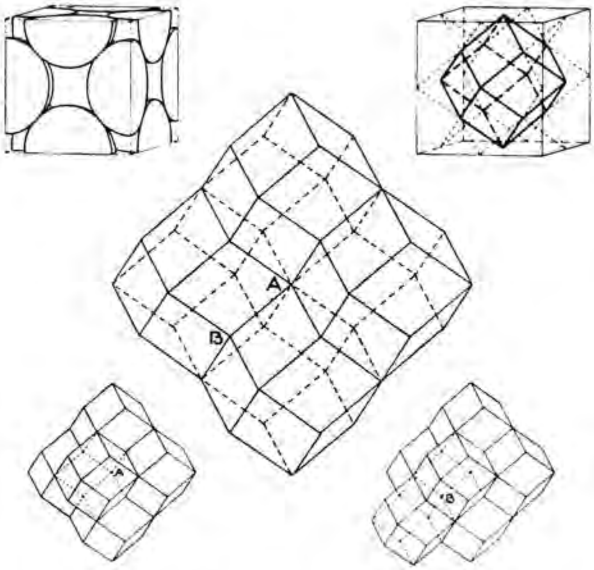

Fig. 1. Regularly piled up globes (see left top corner), when swollen form perfectly fitting rhombic dodecahedrons (see right top corner). In half the points of contact of the edges formed thus 8 edges meet (point *A* in the central figure), in the other points their number is 4 (point *B*). Continued piling up, namely, does not add edges to points *A* and *B*, as may be seen from the bottom figures.

clear from the fact that in this configuration of planes 4 edges meet in only half of all the points of junction. In the remaining points 8 edges meet (two of which always being in line, see A in fig. 1, centre). So this configuration does not fulfil the conditions of the second main rule of a froth (4 edges in every point of junction). And all the froths I examined fulfil this rule. Moreover, in froths I only found very few tetragons as walls and these resembled squares far more than the rhombs of a rhombic dodecahedron, which have angles of 109° 28′ 16,...″ and 70° 31′ 43,......″. Finally the size of one of these angles varies very much from the ideal value of 109° 28′ 16......″. So BUFFIN's thought construction and his experiment do not apply to froths, and this may be understood from the fact that the surface tension did not act as shaping power.

The same error is found in the starting point of Lord KELVIN (1887), who endeavoured to find a body which was to satisfy three requirements, 1. in cumulation the space should be entirely filled, 2. in every vertex 4 edges should meet and 3. the interfacial angles between the planes should be about 120°. The orthic tetrakaidecahedron mentioned by him fulfils these requirements. The two interfacial angles occurring in it are 109° 28′

16,......" and 125° 15′ 25,......". In this train of thought the surface tension does not play a part either and it follows that the result is not applicable to froth. This polyhedron is namely bounded by 8 regular hexagons and 6 squares, and in the froths some hexagons and squares do indeed occur, but they form a small minority. This solution does not fulfil the requirements either that the angles between the edges should be approximately 109° 28′ 16,......": in the orthic tetrakaidecahedron the angles are 90° and 120°.

Both (and similar) reasonings follow a faulty trend of thought: in searching for a polyhedron agreeing as much as possible with the rules of a froth, which on cumulation will completely fill the space. On nearer consideration it is this last requirement which appears to be contradictory to the fact that the shape of a froth depends on surface tensions.

The requirement of complete filling of space namely, implies the idea that the froth walls are flat. In such flat walls the surface tension must everywhere be exactly the same, the curvature being the same (see above) and in order to establish an equilibrium the interfacial angles between the planes must all be identical like the angles between the edges, which ought to be exactly 109° 28′ 16,......". But it is impossible for bodies bounded by flat faces to have only these angles, for the total of the angles of a polygon in a flat plane is always a complete multiple of 180°, and the angle mentioned of 109½° and that of 180° have no common divisor. *So a froth with purely flat walls cannot exist.* Only curving of the faces can solve the problem, on the one hand such a curving being attended with differences in the surface tension on account of which the interfacial angles and the angles between the edges may be somewhat unequal and on the other hand, owing to the curvature, the total of the angles of one face not needing to be exactly a full multiple of 180°. As a matter of fact I found the walls in all the froths examined to be slightly curved, as appeared from the division of light on the glistening froth walls. In figures 3 and 4, two photographs of froth of soapsuds, the edges are seen to be slightly curved. The shape of the froth chambers, therefore differs from a polyhedron with ideal angles, not only in that the angles between its edges as well as its interfacial angles deviate somewhat from the ideal values, but also in that the walls are slightly curved. And that means that the polyhedron does not completely fill the space on cumulation: the interfacial angles of this polyhedron need not be exactly 120° and the three polyhedrons meeting in one edge need not fill the space completely. The only requirements we may make of the approximate polyhedron are those of the two main rules and it is exclusively these requirements which are based on the conditions for equilibrium.

From the properties of a froth we can deduce yet another indication of this polyhedron, although this is not a conditio sine qua non. This indication follows from the consideration that all the chambers adjacent to one specific chamber will in principle have the same relation to that one chamber, there is at least no a priori reason why one should behave differently in principle from its neighbours. This suggests that in the froth structure there is a certain tendency to make the chamber shape approach to a regular polyhedron.

In my opinion there are four arguments in favour of this being the regular dodecahedron.

In the first place both the interfacial angles and the angles between the edges of the regular dodecahedron have values approaching pretty near to the values mentioned. In the other regular polyhedrons the difference of those angles and interfacial angles is considerably greater, as may be seen from the following table.

	Interfacial angle	Angle between the edges
regular tetrahedron	70° 31′ 44″	60°
,, hexahedron	90°	90°
,, octahedron	109° 28′ 16″	60°
,, dodecahedron	**116° 33′ 54″**	**108°**
,, icosahedron	138° 11′ 23″	60°
in froth approximately	**120°**	**109° 28′ 16″**

So the interfacial angle of the regular dodecahedron differs not quite $3\frac{1}{2}$° and the angle of the edges not quite $1\frac{1}{2}$° from the angles which in flat walls would establish the equilibrium. In all the other regular polyhedrons the differences are considerably greater.

The second argument is that it is only in accumulation of the regular dodecahedron (after modification in the above sense) that 3 planes meet in one edge and 4 edges in one vertex. When regular dodecahedrons are piled up not all the walls can coalesce, a.o. because the three interfacial angles (of over $116\frac{1}{2}$°) which can meet in one edge, leave an opening of ca. $360° - 3 \times 116\frac{1}{2}° = 10\frac{1}{2}°$. But round one regular dodecahedron 12 other nearly regular dodecahedrons can be placed, exactly fitting into the space if some of its interfacial angles are taken a few degrees greater. It appears then that along each edge 3 planes meet and 4 edges in each vertex (compare fig. 2), which corresponds with the figures in the froth structure.

A pile of regular hexahedrons (cubes) will fit perfectly without any modification. But when the vertices coalesce 4 planes will meet in each edge and 6 edges in each vertex, so we do not get the numbers 3 and 4 which invariably are found in froth structure.

When piling up regular tetrahedrons, narrow clefts are always left: the interfacial angle is just over $70\frac{1}{2}$° and so the five meeting in one edge, together fill 352½° which is 7½° short of the completion of 360°. When we imagine them slightly modified in shape (a.o. by enlarging the interfacial angles to averagely 72°), 5 walls will meet in each edge and 12 edges in each point of junction, which numbers deviate still more from the numbers of the froth structure.

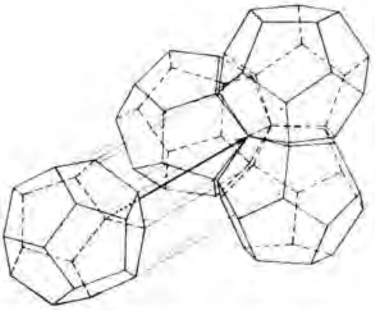

Fig. 2. Four regular dodecahedrons may be piled up nearly fitting round one point.

Regular octahedrons of equal size when piled up also leave clefts (interfacial angle 109½°, so too large for four and too small for five such octahedrons in one edge). Perfectly fitting piling may be obtained by reducing their interfacial angle from just over 109° to 90°. Six such bodies together form a rhombic dodecahedron [1]) and the space may be filled by piling up rhombic dodecahedrons. It is then seen that 4 walls meet along each edge and 8 edges in each vertex, which is again contradictory to the main rules of a froth.

Perfectly fitting accumulation of bodies approaching the shape of a regular icosahedron is not possible. Moreover, in a point of junction of a regular icosahedron 5 edges meet already, which is too great a number to fulfil the condition of the second main rule of froth structure. In accumulation this number would become greater still, and the discrepancy would be even more pronounced.

So the regular dodecahedron is the only regular polyhedron to which the shape of the froth chamber might approach under the condition that 3 planes must meet in one edge and 4 edges in one vertex.

The third argument is given by the consideration that on accumulation of globes of the same size 12 globes can always be in contact with one. On mutual flattening therefore bodies with 12 faces will be formed and if these bodies should approach the regular polyhedron this ought to be the regular polyhedron showing 12 planes, that is the regular dodecahedron.

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[1]) An octahedron has 6 vertices. Through 4 vertices a diagonal plane may be brought, bounded by four edges and having the shape of a square. The 6 octahedrons mentioned may now be placed so that a diagonal face of each coalesces with the wall of a cube. Of the planes within the cube 2 coalesce each time, of the planes outside the cube two are always in line. Together each two planes named secondly form one plane of the rhombic dodecahedron.

Fourth argument: by far the majority of the walls in a froth are pentagonal and the regular dodecahedron only consists of pentagons, the other regular polyhedrons are bounded by triangles (viz. the tetra-, octa- and icosahedron) or by squares (the hexahedron, the cube).

That most of the walls in a froth are pentagonal is a fact found in each of the froths investigated for this purpose (with chambers of about the same size). Meanwhile a superficial observation of such a froth may easily lead the investigator astray, as the surface chambers have a different shape. When, for instance, we examine a froth obtained by blowing air through a blowpipe into a layer of soapsuds in a cylinder glass (which froth entirely may fill the space of the glass above the liquid) we are at first struck by the lines along which the froth walls attach themselves to the glass wall and nearly all these lines form hexagons of a fairly regular shape. (The same is true of the lines connecting the walls between the upper chambers with the walls separating the chambers from the open air. If looked for, the same picture may be observed on the surface of the liquid at the bottom of the glass). In these surface chambers the conditions for equilibrium are different owing to the fact that on the outer surface of the froth there are only walls running in, not running out. This is the cause that walls between the surface chambers, especially in their most superficial parts, can place themselves pretty truly in such a position that all the interfacial angles are 120°, owing to which a section perpendicular to their edges must show a regular hexagon. And owing to the fact that these walls are perpendicular to the surface, this surface shows such a section.

The influence of the surface being less felt in those chambers that are situated farther away from the surface, we see the general froth structures discussed in this paper, only in the centre of the froth, at least when the cylinder glass is wide enough. In order to make them clearly visible I photographed a froth of soapsuds (made in a cylinder glass) by means of a lens with a comparatively small focus distance (5 cm) and a large opening (*F*: 3.5), owing to which only the interior gave a sharp picture. It is evident from the photographs that most of the walls are pentagonal (figures 3 and 4).

The preponderance of pentagons I found not only in all the froths consisting of a liquid and a gas, but also in froths of two liquids. Such a froth structure occurs in the yolk of an egg, for instance of a hen's egg. In it the yellow substance lies as a liquid in separate parts of about the same size, separated by walls of an uncouloured liquid. By boiling the yellow liquid coagulates: when carefully pushed or stirred with a small glass stick, a piece of hard boiled yolk, placed in a drop of water on an object glass will separate into a number of coagulated froth chambers. The photographs in fig. 5 demonstrate the fact that these bodies approach the shape of polyhedrons, mainly bounded by pentagons. (I am grateful to Prof. H. G. BUNGENBERG DE JONG for indicating this highly demonstrative example.)

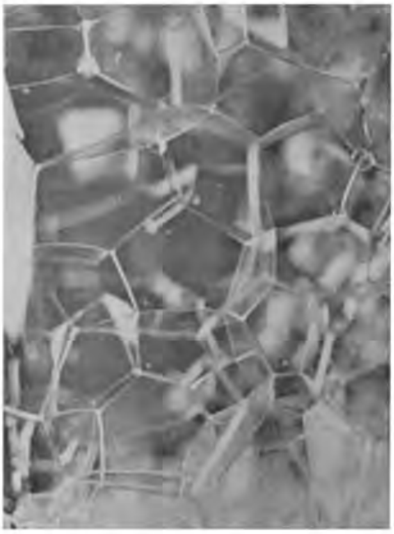

Fig. 3. The majority of walls in a froth of soapsuds are slightly curved pentagons.

Fig. 4. A few walls in a froth of soapsuds are hexagons, fewer still are tetragons (one of each is seen in the photograph).

Fig. 5. Coagulated yolk of egg with pentagonal faces.

Meanwhile, when bodies approaching in shape the regular dodecahedron are piled up, a phenomenon occurs, in consequence of which not all the walls can be pentagonal, but in which some hexagons and fewer tetragons must occur, which is also seen to be the case in the froth photographs.

Starting from a regular dodecahedron a rind of 12 other dodecahedrons may be built round the first, which slightly deviate from the regular dodecahedron in the above sense. By each time fixing new edges to each vertex, so that 4 edges meet in each vertex and the edges each time enclose pentagons, we can for an unlimited space of time continue building new rinds round the former ones, all consisting of pentagon dodecahedrons. But the shape of these dodecahedrons will deviate the more from the original, central dodecahedron as they are further removed from it. For it appears that their tangential dimensions (their dimensions in a direction perpendicular to the line connecting their centre with that of the central dodecahedron) ever become greater. The cause of this lies in the discrepancy between the growth of the tangential surface of the rind (proportionate to the square of its radius) and the increase in the number of dodecahedrons per rind. This number is determined by the number and the position of the uncovered pentagons of the previous rind, and it increases more slowly than the surface, so that one dodecahedron must cover a larger part of the rind as the distance from the central dodecahedron increases. If this construction is carried out (I did it with twisted copper-wire), the space can ever be taken up by additional pentagon dodecahedrons, but in doing so either the outer dodecahedrons must be larger in all directions, which leaves their shape approximately that of regular dodecahedrons but which increases their volume, or their volume may be left almost unchanged, but then their radial dimensions (along the line running through their centre and that of the central dodecahedron), should be taken increasingly smaller. Then they are more and more flattened, finally becoming broad, thin slices, which cannot exist as froth chambers. In no single way can the space over a large extension be occupied by bodies which are everywhere approximations of regular dodecahedrons of the same size.

This difficulty may be met by placing between the dodecahedrons some bodies with a greater number of planes, in which the interfacial angles need only be changed a little. Without entering into details it may be demonstrated from fig. 6, an arrangement of nearly fitting regular dodecahedrons, enclosing a hexagon. On either side of this hexagon a body is bounded, which on completion of the accumulation would be a tetrakaidecahedron, bounded by 12 pentagons and 2 hexagons. In the same way other configurations produce polyhedrons with a few tetragons. Indeed this occurrence of hexagons and tetragons among the great majority of pentagons is found in the froths. Fig. 4 shows a hexagon by the side of a tetragon. As a rule such a tetragon appears to be approximately a square.

Finally we can discuss briefly the relation between the rhombic dodecahedron, which was supposed to be the result of a regular cumulation of globes and the regular dodecahedron which approaches the actual proportion in the froth.

The twelve globes in a regular arrangement touching some central globe, are not arranged equally round that central globe. If of the cube on whose edge centres lie the centres of the peripheral globes, we select a vertex, we see that 3 globes which touch each other are arranged round it (fig. 7 left). Round the centre of a face of that cube, on the other hand, there are 4 globes, between which a greater space is left open than between the three globes mentioned first. Hence the peripheral globes can be divided in a natural way into 4 groups, each consisting of 3 globes touching each other. Owing to the fact that these three globes also touch the central globe, their three centres and the centre of the central globe lie in the vertices of a regular tetrahedron, being the most stable configuration of 4 globes. When we imagine lines drawn through the centre of the central globe and through each of the centres of these tetrahedrons, these lines also intersect 4 non-adjacent vertices of the cube (see fig. 7 right top corner), owing to which they form mutually equal angles of 109° 28′ 16, ...". They may therefore be conceived as the four trigonal polar axes of the regular system. The three peripheral globes of one group are equidistant from their axis and may be revolved round that axis. They then leave the regular arrangement, to recover it after a rotation of 120°.

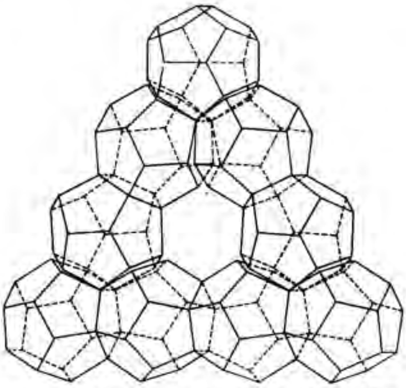

Fig. 6. Regular dodecahedrons can form nearly fitting hexagons.

In the regular arrangement it is seen that any peripheral globe always touches 4 other peripheral globes. The tangent plane between the peripheral globe selected and the central globe is therefore intersected by the four tangent planes between the central globe and the other peripheral globes mentioned, i.e. the tangent plane becomes a tetragon. The four peripheral globes touching the peripheral globe selected are not arranged regularly round it: they touch two by two, leaving two openings of the same size, situated diametrically opposite each other. The tetragon deducted above is therefore a rhomb. Hence the development of the rhombic dodecahedron. After the revolution round the polar axis described we find other proportions between the peripheral globes. It appears then that any peripheral globe has come to lie between 5 others: so in the case of growth in this configuration the globes will develop into dodecahedrons bounded by pentagons. These are no regular pentagons owing to the peripheral globes being dispersed irregularly round the selected one. But in this configuration the number of contacts between the peripheral globes is considerably smaller than in the regular one (viz. 12 instead of 36), for now only the three globes out of each group touch and there is no contact between globes of different groups. Therefore this configuration, which I propose to call tetratoid (see fig. 7 right bottom corner), leaves room enough for each peripheral globe to relinquish its contact with the other peripheral globes (a peripheral globe is no longer hemmed in by four other peripheral globes touching it). This opens the possibility that the 12 peripheral globes distribute themselves perfectly equally round the central globe (none of the peripheral globes touching another) and this configuration causes the regular dodecahedron. But the peripheral globes can also distribute themselves *approximately* regularly round the central one, and then the result is the polyhedron which is an *approximation* of the regular dodecahedron with which they have in common that they are both bounded by 12 pentagons.

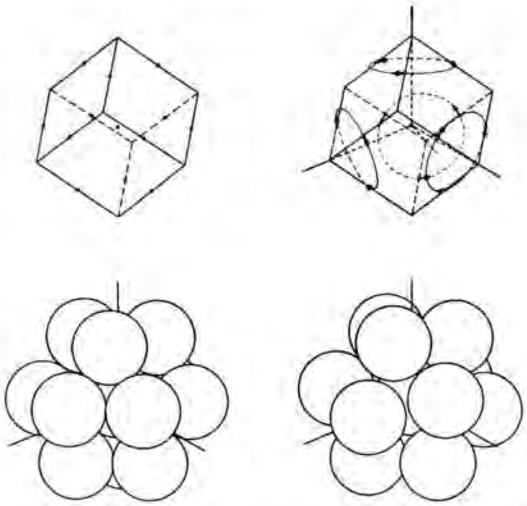

Fig. 7. The regular and the tetratoid arrangement of globes. The figures on the left demonstrate the regular, those on the right the tetratoid arrangement. The top figures show the position of the globe centres with regard to a cube. In the right figure they have revolved from the regular position round the four pictured trigonal polar axes by the length of the arrows (that is 15° in a positive sense).

From this systematic description it follows that the regular arrangement represents only one specific arrangement of the peripheral globes, and one with a maximum of mutual points of contact, which implies a minimum of freedom, and that from this, through rotation on the one hand, and through dispersion of the globes on the other, an infinite number of other positions may develop with fewer points of contact and hence with more freedom. The only other configuration with the maximal number of points of contact is the hexagon (different from the regular one in that out of the groups of three peripheral globes one isolated group has revolved 60° round its axis). Only in these two specific positions the 12 tangent planes are tetragons (in the regular one they are rhombs, in the hexagon they are rhombs and isosceles trapezia), in all the other positions they are pentagons.

The regular and the hexagonal arrangements can be constructed further than the first rind, while the structural plan remains exactly the same. This is not the case with the tetratoid and free arrangements. I have not succeeded in extending them with globes of the same size in such a way that the 12 peripheral globes surrounding any central globe are distributed so that each is surrounded by 5 others. This is in agreement with the fact mentioned above, that the extension with dodecahedrons, bounded by pentagons is not possible without changing the volume or the general shape of those dodecahedrons.

The proposition defended here, that the froth chambers are approximations of regular dodecahedrons has already been put forward by several investigators. Most of them only adduced the argument that among the regular polyhedrons it is only the regular dodecahedron which has interfacial angles and angles between the edges approximately agreeing to the angles which develop when 3 flat planes meet in one line under equal interfacial angles and when 4 straight lines meet in one point under equal angles. They did not account for the deviations from those angles and they did not refute the arguments of the authors who mentioned other polyhedrons as approximations of the shape of the froth chambers, so that the uninitiated reader had no arguments for a correct choice. I think I have provided such theoretical arguments in the above, and moreover to have shown that the regular dodecahedron is more in keeping with the actual proportions in a froth than other shapes mentioned in the literature. As for the latter, it was particularly shown that the majority of walls in a froth are *pentagons* (of the regular polyhedron only the dodecahedrons have pentagonal walls) and that among them there occur a few hexagons and tetragons, which must also occur now and then when approximately regular dodecahedrons are piled up.

Anatomy. — *Neuropilem and gliareticulum in the cortex of the cerebrum.*

By S. T. BOK. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of October 26, 1940.)

The investigation by R. C. RENES [1]) concerning the abundance of fibres in the cortex of the cerebrum has brought to light that everywhere between the different cells and vessels of the cortex there is the same total length of nerve fibre per spatial unit (except in those places where the myelin sheaths are numerous and so occupy an appreciable part of the space). In other words the neuropilem is equally dense everywhere in the cortex.

This makes it clear how it is that cortex areas, in which there occur mainly small nerve cells, are much richer in nerve cells than areas with larger nerve cells (by which we mean that in areas with small cells the number of nerve cells per spatial unit — the density of the nerve cells — is greater than in areas with large cells). For generally a larger nerve cell has a larger dendrite complex with a greater total length of dendrites than a smaller cell and because the richness in fibres is the same everywhere, this greater length of dendrites requires more space, so that the space between the nerve cells must be greater as those cells are larger.

Meanwhile RENES' investigation has not yet made it clear through what cause this distribution of fibres is the same everywhere in the cortex.

The constancy found in the length of the nerve fibres per spatial unit suggests that everywhere the fibres have reached their maximal development, that, as it were, the cortex is everywhere filled to the greatest possible extent with nerve fibres. But what is the factor, determining the maximum which brings to a close any further development of ever new branches between those existing already?

This is certainly not the mere fact that the space between the nerve fibres should have become too narrow to allow fibres of sufficient thickness to penetrate: in the BIELSCHOWSKY-preparation everywhere between the thin black-impregnated fibres there is a space which is averagely considerably more than ten times as great as the average thickness of those nerve fibres, so that certainly hundreds of times more fibres of the same size might find a place in the same space in the cortex.

RENES had investigated if, perhaps, the cause may be that there must always be a certain minimal distance between the nerve fibres. If such a minimal distance is indeed prescribed, the cortex space would be "filled up"

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[1]) R. C. RENES, Over de vezelrijkdom in de hersenschors. Thesis Leyden 1940.

much sooner (then the maximal length of nerve fibre per spatial unit would be much smaller) than if the fibres are allowed to touch. For this purpose he has measured the distance between neighbouring nerve fibres (= the length of their common perpendicular) and in one respect his result seemed to confirm this idea.

He has namely made these measurements in different architectonic fields and in each field separately in layers II—VI (layer I is not sufficiently impregnated in the BIELSCHOWSKY-method). In each field a small area was selected in each of the layers and in each of those areas the fibre distance was measured in a hundred cases. It was seen that in each area examined these distances vary greatly in size, but in all areas their size varies between practically the same minimum (0.4 μ) and about the same maximum (1.9 μ). So in each field he found a certain minimal fibre distance and this minimal distance appeared to be the same in all the areas examined (for the mutual variations between these minima practically remained below the value of the error of measurement: the average error of the individual measurements was determined at 0.027 μ and in the various areas examined the smallest of the distances found varied between 0.40 and 0.44 μ, once it was 0.46 μ). So it appeared that there is indeed a minimal fibre distance throughout the cortex, which was found everywhere, and within which no two nerve fibres cross each other.

But in two important points the results of his measurements are by no means in agreement with the theory tested.

In the first place the distribution of the various fibre distances was not in agreement with the idea that everywhere in the cortex there is the maximal quantity of nerve fibre which there might be if it was only limited by a minimal distance between the fibres. If this maximum should be reached it might be expected that in many places the fibres would cross at the minimal distance. It is true that greater distances would also occur owing to two fibres crossing at a certain place while they had already been fixed in place through neighbouring crossings, with minimal distance, but the greater distances would become less frequent as they became greater. A frequency curve of the different fibre distances found, would therefore coalesce with the zero line in values less than the minimal distance, at the minimal distance it would rise vertically to the top value after which it would slowly fall to the greater values.

The frequency curve made of the fibre distances found does not fulfil this expectation. After the minimal value of 0.4 μ it does, indeed, climb steeply, to fall slowly after reaching the maximum, but the top lies too far past the minimal value (namely at 0.56 μ) for this deviation to be attributable to errors in measuring. Only with an average error of measurement of 0.11 μ the vertical truncation of the curve in these circumstances would have been flattened to approximately the shape found in the actual measurements. But repeated measuring of the same distance of intersection proved the error to have been no more than 0.027 μ, so that we are justified in attributing a real significance to the gradual climbing of the frequency curve between 0.4 and 0.56 μ. So the distribution of the measurements found is not in keeping with the idea of a maximal filling limited by a minimal distance between the fibres prescribed.

In the second place the value of the minimal distance is contradictory to this idea: if the fibres could not come closer together than the minimal distance of 0.4 μ, the total fibre length per spatial unit could be many times greater than the length found. Neglecting the thickness of the fibre and supposing all the fibres to run parallel (in order to obtain maximal "filling") then a simple calculation shows that per 1000 μ^3 of space there would be 14450 μ of fibre, that is 72 times the length found (200 μ). Owing to the thickness of the fibres this value will probably be a little less, but not much, because the average fibre thickness may be estimated at less than ¼ the minimal distance between the fibres, and the total fibre length per spatial unit would be less than 4 % smaller. A greater influence has the fact that the fibres do not run parallel but in many different directions. Owing to this factor the possible length will be a few times smaller than calculated above, but it will certainly be dozens of times greater than the one found in the cortex.

So RENES' calculations show that the "fibre filling" is the same everywhere in the cortex, but they do not throw light on the factors which keep the filling constant. In particular they deny the supposition that there would be here a maximum of fibres, limited only because they must not or cannot approach any nearer than to a certain constant distance. So the constancy of the fibre filling must be owing to some other cause. What cause can that be?

The reconstruction of a small fibre area (measuring 15 × 15 × 15 μ), made by RENES — a stereoscopic picture of which is given in his thesis — showed that the fibres are not distributed equally about the space, but that comparatively large open spaces are found, between which the fibres are packed as in thin membranes or in rods. These open spaces are of about the same size, the centres of two membranes facing each other are about 5½ μ apart. So it is probable that the occurrence of these open spaces limits the quantity of nerve fibres and the question arises if the constancy in the fibre filling may be a consequence of a constancy of these open spaces.

About the nature of these open spaces we can state with certainty that they do not consist of cell bodies: they do not contain nuclei. Moreover they are far too numerous for that: from the measurement given it would follow that there ought to be about 9000 in an area of 100 × 100 × 100 μ, whereas there are only 80 to 120 cells (glia and nerve cells together).

The order in size of these open spaces corresponds to the meshes of the glia reticulum as drawn by SPIELMEYER, by HOLZER, and others. The supposition is obvious that they are vacuoli, between which protoplasm walls or protoplasm rods form the bedding for the nerve fibres.

Thus far it has not proved technically possible to obtain an insight through microscopic observation into the shapes of these vacuoli and protoplasm walls or rods. A purely theoretical deduction, however, reveals some remarkable circumstances, which suggest a very simple way in which the fibre quantity in the cortex would be limited to the measure found by these very vacuoli.

We may start from the provisional supposition that these vacuoli are non-coherent; then the protoplasm would lie between them in the shape of a froth, and then the above measurement of the froth chambers tells us, according to a calculation which will be given below, that the total edge length in this froth corresponds with the length of nerve fibres found ($200\,\mu$ per $1000\,\mu^3$). If the vacuoli should be more or less coherent this agreement would still hold, if only the protoplasm walls were penetrated, as in a sponge, the protoplasm rods being left in the place of the original edges of the froth structure. The fibre filling found would then be determined quantitatively by the simple rule: in one protoplasm rod one nerve fibre.

In the calculation indicated above the open spaces were conceived as chambers in a protoplasm froth and the reconstruction showed that these chambers are about equally large. In a communication about the shape of froth chambers [1]) I could determine that in a froth with about equally large chambers the latter approach the shape of a regular dodecahedron. For the calculation we therefore started from the conception that when we imagine flat planes between the open spaces these planes would be the walls of regular dodecahedrons. This conception only approaches the actual proportions, as has been explained in the communication mentioned, but for our purpose this approximation proves amply sufficient.

In the reconstruction the centres of neighbouring open spaces lie about 15 cm apart. As the enlargement is 27500, they are about $150000 : 27500 = 5.46\,\mu$ apart in the preparation, so that the radius of the inscribed globe of the dodecahedrons may be calculated as $\frac{1}{2} \times 5.46 = 2.73\,\mu$, from which follows the edge length as $2.46\,\mu$ [2]) and the contents $114\,\mu^3$ [3]).

Each regular dodecahedron is enclosed by twelve pentagons, meeting in $12 \times 5 : 2 = 30$ edges. In the cumulation 3 dodecahedrons meet in one edge, so that per dodecahedron there are $30 : 3 = 10$ edge lengths or $10 \times 2.46 = 24.6\,\mu$ of edge length. Per $1000\,\mu^3$ there are $1000 : 114 = 8.75$ dodecahedrons, giving a total of $8.75 \times 24.6 = 215\,\mu$ edge length. In the same volume RENES found $200\,\mu$ of fibre length.

Considering that the conception of a froth as an accumulation of regular dodecahedrons is only an approximation, this agreement is so great as to warrant the conclusion that practically there is as much fibre length in the

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[1]) See these Proceedings, p. 1180—1190.

[2]) From the radius $r = 1/20\, a \sqrt{250 + 110\sqrt{5}} = 1.11\, a$ or
$a = 0.9\, r = 0.9 \times 2.73 = 2.46\,\mu$

[3]) From the contents $c = 1/4\, a^3 (15 + 7\sqrt{5}) = 7.663\, a^3$
$= 7.663 \times 2.46^3 = 114\,\mu^3$.

neuropilem as the edgelength would be, if the cortex was a froth with a chamber size agreeing with the data of the reconstruction.

As the distance between the nerve fibres has been measured, it is attractive to use the froth structure with averagely 8.75 chambers per 1000 μ^3 as a provisional model, and to ask ourselves what would be the aspect in that model of the crossing of the fibres, of which there is each time one in an edge.

In a froth four edges always meet in one point of junction. A nerve fibre which we imagine in one of those edges will lie in another of those four edges after passing that point: so it occupies two out of the four edges of that one point of junction. Hence in the two remaining edges there will be a second fibre, which passes the first in this point of junction. According to this model there can only pass two fibres through the same junction, otherwise there would be more than one nerve fibre in one rod.

It is clear that the two fibres would intersect each other in this point of junction, if the edges as well as the fibres were mathematical lines without thickness. But in our model both have thickness. The mathematically conceived planes between the adjacent chambers discussed above served only to calculate the edgelengths, in our model these planes lie in a protoplasm wall of a certain thickness and thus the abstractly conceived mathematical edges also lie in protoplasm rods with a certain thickness, while the junctions where four rods meet are also protoplasm areas of a certain spatial extension. If this extension is large enough as regards the thickness of the fibre the two fibres can cross here and leave a certain distance between them. Then they lie no longer accurately at the site of the mathematical edges, but a little by the side of them, at least in these junctions where they cross one another.

The four edges meeting in one point of junction in a froth, together form averagely equal angles, that is to say, they run from that point like the lines connecting the centre of a regular tetrahedron with its four vertices. If a nerve fibre was situated exactly in the mathematical edge that fibre would make an angle at each point of junction it passes (viz. one of averagely just over 109°). If we set out to draw in our model the shortest fibre between two far removed points, which fibre was everywhere to coincide exactly with the mathematical edge, then this would not be a straight line but a broken one, making the angle mentioned at regular intervals, but otherwise approaching the straight line mentioned. It is then seen that this broken line is wound screw-wise round a nearly straight axis (apart from the fact that it consists of a number of straight parts, not being curved, therefore, like a mathematical screw line).

Because the nerve fibres in our model — at least in the points of junction of the edges — should not coincide exactly with the mathematical edges, the possibility arises to truncate and round off the angle which each fibre would otherwise have to make at each point of junction. A fibre thus rounding off every angle, will by this fact be a little shorter than the total of the mathematical edges it follows and the screw-like line mentioned will approach to a straight line. Maximal shortening would be obtained by such a fibre which would entirely flatten all its angles: such a fibre would be situated in the axis of the screw-wisely broken line of the corresponding mathematical edges. The question may be asked, what would be the distance between two such maximally straightened out fibres.

A purely mathematical treatment of this problem is not possible, as the froth chambers have not the exact shape of a regular dodecahedron. The property discussed, that regular dodecahedrons cannot be piled up exactly fitting, is the cause that the froth chambers must differ a little from regular dodecahedrons, and in the article mentioned on the shape of froth chambers it has been explained that on account of the surface tensions in these shapes, these deviations must a.o. consist in slight curvature of the walls and of the edges, which curvings differ from edge to edge and from chamber to chamber. Meanwhile it is possible in models of regular dodecahedrons to indicate, approximately at least, where these lines must be situated, and it is then seen that the distance of intersection between these axis lines in each point of junction is approximately 0.75 of the edge length. In our model, where the edge length represents 2.46 μ, this gives a distance of intersection of 1.85 μ and this is the maximal distance which these fibres can have with complete straightening out of their spiral course. This theoretically derived maximal distance of intersection of 1.85 μ comes fairly near to the maximal fibre distance of 1.9 μ, prevailing in the measurements, so that our model appears readily to account for the maximal fibre distance found by RENES.

Yet another consequence to be derived from the model agrees with a peculiarity of the fibres in the preparations, qualitatively as well as quantitatively.

If the maximal fibre distance was really determined by the above straightening out of a screw-like course to a straight one, then only those few fibres in which the maximal mutual distance of 1.9 μ had been found would entirely have given up their screw-like course, while all the other fibres had not quite straightened out to a straight line. So the many fibres should have a screw-like course with a pitch prescribed by our model. When we follow those mathematical rods which wind round a straight line, it is seen that the axis line is again on the same side of those rods if along that axis line a distance has been covered which is a little smaller than the distance between two adjacent dodecahedron centres. According to the model the pitch should therefore be a little less than $2r$ or $2 \times 2.73\ \mu$, so a little less than 5.46 μ.

As a matter of fact most fibres in a BIELSCHOWSKY preparation do not run straight but with fairly regular undulations. In a fibre which lies about in the direction of the optic axis of the microscope, and which therefore, when the micrometer-screw stands still is seen as a black point (small spot), we see in most cases that this point moves in an approximately circular direction when the micrometer-screw is turned: the undulation proves to be a consequence of the screw-like course. The pitch of this screw can be easily measured in those fibres which run parallel to the section plane. In the drawings RENES made to determine the length of the fibres, their undulation is plainly visible (see fig. 5 of his thesis) and from it the average pitch of 5 μ was measured. This measurement also lies in the same order of size as the measure of a little less than 5.46 μ theoretically derived above.

In order to get the nerve fibres in the junctions of the protoplasm rods to cross at the mutual distances found, these junctions must have special dimensions and the question arises what shape the protoplasm froth should have in connection with this requirement. When calculating this we are up against the uncertainty whether the nerve fibres can lie quite peripherally in their protoplasm layer, or if they have always to be surrounded on all sides by a layer of protoplasm of a prescribed thickness. On the ground of the minimal distance of intersection of 0.4 μ, RENES already pronounced the supposition that every nerve fibre must be surrounded by a protoplasm sheath 0.2 μ thick. Starting from this supposition, a calculation, which will be published in detail elsewhere, shows that in order to obtain the maximal distance towards the neighbouring one of 1.9 μ the junction must rather exactly have the shape that would arise if the adjacent froth chambers were globular and just touching. Where, on the other hand, the chambers are more or less flattened by contact, the junctions would be such that the distances of intersection would have to be smaller than the maximal value of 1.9 μ. The fibres crossing in these smaller junctions could not have a perfectly straightened out course. The variation of the distances between the fibres might therefore — in part at least — be based on a variation in the thickness of the froth walls (the protoplasm rods respectively). In this connection it is remarkable that already with very slight flattening of the froth walls (a very small penetration respectively) the protoplasm rods between the junctions would become too thin to contain more than one nerve fibre with its sheath 0.2 μ thick.

So here the froth structure also acts as a factor determining that per protoplasm rod there is as a rule only one nerve fibre.

The supposition that the protoplasm of the cortex is constructed like a froth (or spongelike as a network of rods, taking the place of the edges in that froth), the chamber size of which corresponds to the dimension of the open spaces in RENES' fibre construction, produces consequences which qualitatively and quantitatively correspond to the following properties, which had come to light through the measurements taken of the nerve fibres:

1. The total length of protoplasm rods (froth edges) is practically equal to the total length of nerve fibre, viz. 215 μ and 200 μ respectively per cell-free cortex area of 1000 μ^3;
2. The distance between two perfectly straightened out fibres, crossing

each other in the junction of the protoplasm rods (1.85 μ) practically agrees with the maximal value of the distance between neighbouring nerve fibres in the cortex (1.9 μ);

3. According to the supposed froth or reticulum structure of the protoplasm, the fibres which are not entirely straightened out (the great majority) ought to describe screw-like lines with a pitch of not quite 5.46 μ, while most of the nerve fibres in the BIELSCHOWSKY preparation do indeed follow screw lines, with a pitch of averagely 5 μ;

4. According to a calculation not included here for lack of space, the complete straightening out of the fibres with its corresponding fibre distance of 1.9 μ could occur only in such places where the chambers are not flattened by contact with each other. If the chambers are a little larger (everything calculated with a constant mutual distance between their centres) the flattening (or penetration) of the frothwalls must be attended with screw-like fibres and smaller distances between the fibres. The protoplasm rods between their junctions will then also become so thin that they can contain only one nerve fibre with its layer of protoplasm 0.2 μ thick. As we saw above, this is indeed the rule.

It stands to reason that the protoplasm froth or reticulum may be identified with the so-called glia reticulum, the reticulum of protoplasm rods, into which the extreme ends of the differentiated gliacells finally pass and which — according to the drawings occurring in the literature — agrees in order of size with the open spaces in the reconstruction of RENES.

According to the theory discussed in the above, the relation between the unmyelinated nerve fibres and the protoplasm in the cerebral cortex is the same as in the peripheral nerve system.

Anatomy. — *Some topographic relations of the orbits in Man and Anthropoids during ontogenesis, especially bearing on the ontogenetic development of the "rostrum orbitale"*. By J. ARIËNS KAPPERS. (Communicated by Prof. H. M. DE BURLET.)

(Communicated at the meeting of October 26, 1940.)

When examining the base of an adult human or anthropoidal skull we see how the two roofs of the orbits protrude more or less into that part of the skull cavity in which the frontal lobes of the brain are situated. Between these roofs a more or less deep space exists at the bottom of which the lamina cribrosa is found. Therefore the anterior part of the base is not flat in a transverse sense either in Anthropoids or in Man. That part of the frontal brain lobes, including the lobus olfactorius, which lies between the orbital roofs is commonly called rostrum orbitale. It will be our aim to discuss in Man and Anthropoids the form and as far as possible the genesis of the space between the orbital roofs in which the rostrum lies.

Looking at the well-known model of the primordial cranium of a human embryo of 8 cm crown-rump length, made by ZIEGLER after HERTWIG, we see how the ala orbitalis forms a nearly flat plate extending at right angles from the septum of the nose, so that the primitive roofs of the orbits practically do not rise above the level of the lamina cribrosa and the base of the skull is nearly flat in a transverse sense. The same is seen in the frontal section through the head of a human embryo of about two months old reproduced in figure 1. The section is not quite perpendicular to the aequatorial plane, so that the right side of the embryo, to the left in the figure, with which we will deal only, is cut more backward than the left side. The septum nasi protrudes above the level of the roof of the nose cavity and, what is not the same thing here, above the bottom of the fossa cranii, forming the crista galli at the anterior end of the skull base. On the left side of this crista in the figure the base of the skull, i.e. the roof of the orbit, slopes slightly laterally and upwards to the ascending wall of the brain cavity, being still nearly flat. Nothing is yet to be found of an elevation of the roof of the orbit protruding into the brain cavity.

No indication is seen of an *U*- or *V*-shaped space between the roofs of the orbits, in which in the adult stage the rostrum orbitale lies.

We now compare this section with figure 2, showing a frontal section through a senile human skull sawn perpendicularly to the lamina cribrosa, through the forepart of the crista galli. In this way the two sections to be compared are as homotopic as possible.

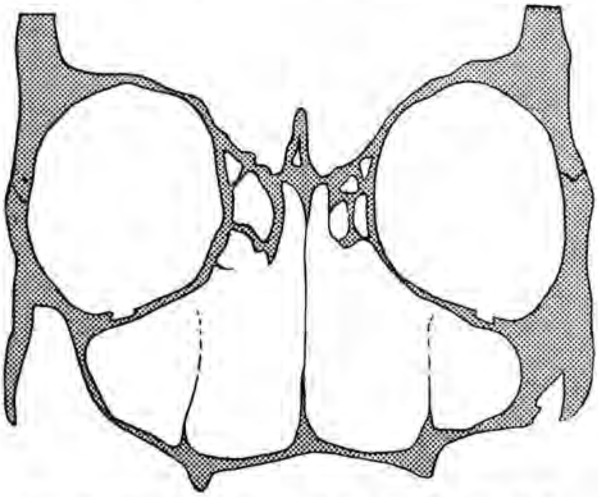

Fig. 2. Frontal section through senile human skull.

We chose this adult skull because in this specimen the pit beside the crista galli in which the olfactory bulbs are situated and the whole *U*-shaped space between the roofs of the orbits were so pronounced that one could speak of a real fossa interorbitalis encephalicus in which the rostrum orbitale lies.

Apart from some very similar relations in the two sections, we observe also a remarkable difference.

First we will point out the most important, quite similar relation between the position of the orbit as a whole towards the cavum nasi in the two sections. Connecting in thought the centres of the orbits, as done by BOLK, we see that this line in the two skulls not only runs beneath the upper border of the crista but even a little underneath the roof of the cavum nasi. BOLK, using a frontal section through the skull of a neonatus, laid great stress on this point, saying that also in this respect the adult human skull maintains its fetal features. He used for comparison a new-born and an adult skull. That we find the same state of affairs in a so much younger skull corroborates his point of view.

Passing to the differences which exist between the two sections, we may first call attention to figure 3, showing a frontal section through the skull of a human fetus of about the 8th month perpendicularly to the lamina cribrosa. A remarkable difference with regard to the young embryonic as well as to the adult skull, appears in the vertical height of the orbit relatively to that of the cavum nasi. In comparison with the two months old embryo, the vertical height of the orbit is increased very much relatively to that of the nose cavity. The line, however, connecting the centres of the two orbits would have the same position with regard to the tectum nasi.

This implies that the roof of the orbits has grown upwards, while the bottom has grown downwards. In this way the roofs are beginning to protrude into the cavum cranii, forming between them the *U*-shaped valley in which the rostrum orbitale lies instead of the flat plate existing in the younger embryo. Later on in the ontogeny the nose cavity grows downwards, as is well known, and beside it and underneath the orbits are then formed the maxillary sinuses. That is why in the adult skull the relation between the height of the orbit and that of the cavum nasi on this level nearly reaches again the early embryonic value. Thus the protrusion of the orbital roofs in adult Man which does not exist in early embryonic development, is primarily due to a vertical enlargement of the orbit relatively to the nose in a fetal stage of development. However, the relative position of the centres of the orbits to the tectum nasi remains the same during development and therefore the position of the orbits in adult Man, being quite lateral to the nose cavity, is an ontogenetically conservative one.

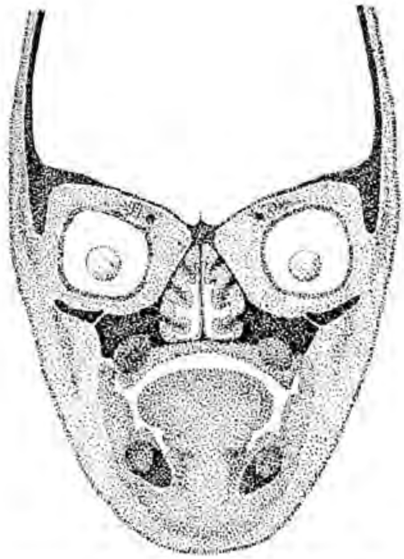

Fig. 3. Frontal section through head of human fetus aged 8 months.

Passing now to the Anthropoids it is well known that in the adult stage the *U*- or *V*-shaped "valley" between the orbits gives the impression of being much deeper here than in Man.

To get some information about the position of the orbits relatively to the nose in very young, fetal stages of development in Anthropoids is very difficult owing to the lack of material. Descriptions of anthropoid embryos are very rare and as far as we know sections through such embryos were never published. In 1885 DENIKER dissected a fetus of Gorilla and one of Gibbon and gave a very elaborate and thorough description of the whole and the parts. As far as the whole of its features were concerned, the Gorilla fetus was at the same stage of development as a human fetus of the 5th—6th month. The Gibbon fetus was older and probably in its last intra-uterine month. About the frontal bone of the young Gorilla DENIKER says:

> "La lamelle de l'os frontal, formant par sa face inférieure la voûte de l'orbite, est excessivement mince; elle n'est point encore bombée vers la cavité cranienne comme chez les gorilles même très jeunes. Par suite de cette disposition, la lame criblée de l'ethmoïde, encore cartilagineuse, se voit très nettement, tandis que, dans les crânes même les plus jeunes, elle est enfoncée profondément entre les frontaux et à peine visible."

This last statement, where he deals with the relation in Gorilla infants, may be slightly exaggerated, but what he says about the conditions in his Gorilla fetus is clearly visible in his figure of the skull base. The frontal part of the base, upon which the frontal part of the brain is situated, is very flat indeed and has not at all the shape which it has in postnatal stages. Here also the orbits are situated laterally to the nose cavity. Therefore nothing of a rostrum is seen on his figure taken from the lateral side of his endocranial cast of the Gorilla fetus.

Speaking of the skull of his Gibbon fetus, he says:

> "La lame criblée cartilagineuse a la forme ovoïde... les deux os frontaux se touchent presque en arrière et en avant de la lame criblée; chez le Gibbon adulte, ils vont se souder en cet endroit, et c'est à peine si l'on apercevra la lame criblée, enfoncée entre les deux os qui deviendront très bombés".

Looking at DENIKER's figure of the skull base of the Gibbon fetus, we see that the anterior part is somewhat more "bombé" than in the fetus of Gorilla and not quite so flat. Yet, there is still a very great difference in this regard in comparison with the adult stage. That it is still more flat in the Gorilla is probably due to the lesser age of the latter.

In figure 4, we show an X-ray photo of the skull of a Gorilla fetus, present in the collection of the museum of the Anatomy Department of the University of Amsterdam, the exterior of which was formerly described by BOLK. This specimen must have lived till the end of pregnancy. The orientation in the photogram is not far from the Frankfurt plane being perpendicular to the plate.

Here the orbital roofs are already vaulted, so that the formation of an orbital keel has begun, but it is curious to see how the line connecting the centres of the two orbital adituses would not run far above the brain base and the roof of the nose, quite contrary to the state of affairs in most adult Anthropoids, especially in Gorilla (see figure 6). This also resembles quite human conditions.

Descriptions of the primordial crania of Primates other than Anthropoids are rather rare. In 1902 FISCHER published and described a model of the primordial cranium of a Cercopithecus cynomolgus embryo of 25 mm crown-trunk length. His figure clearly shows that boths alae orbitales are remarkably flat, so that here also the orbits were lying towards the sides of the lateral nose walls and a rostrum orbitale must have been still absent.

FISCHER himself says about this region: "Die vordere Partie der Ala orbitalis (also damit auch die Cartilago spheno-ethmoïdalis) ist völlig eben, liegt auch etwa in einer Flucht mit der Lamina cribrosa, genau wie beim menschlichen Embryo, während bei den meisten erwachsenen Affen das Dach der Augenhöhle (Frontale) sich so stark gegen das Gehirn vorwölbt, dass die Siebplatte in eine tiefe Spalte zu liegen kommt." In a later article this author describes the model of the same specimen once more together with that of a Semnopithecus maurus embryo of 53 mm crown-trunk length. Speaking about the ala orbitalis he says here: "Sie ist als eine mächtige Platte entwickelt, die sich von der Lamina cribrosa und den Sphenoïdbalken aus als Dach der Augenhöhle seitlich ausspannt. In ihrer Form und Lage stimmen Affe und Mensch völlig überein, weichen aber von andern Säugern ab." The solum supraseptale of Reptiles, after GAUPP the homologon of the ala orbitalis, is a flat plate which rises in an oblique position lateraly to the median plane, whereas in Primates the position became nearly horizontal. Further on FISCHER says that in the Macacus cynomolgus embryo the upper sides of the alae lie in one plane with the lamina cribrosa, whereas in Semnopithecus the latter is already somewhat sunken between the orbital roofs. This also appears from the figures with which he illustrates his article. The difference between the two monkeys on this point, in our opinion, may be caused by the fact that the embryo of Semnopithecus was somewhat older, thus approaching the adult relations.

From these facts it appears that in the younger fetal stages of many, if not all, Primates the topographic position of the orbits relative to the nose cavity is the same: the plane through the orbital axes lies beneath the subcerebral part of the nose roof, in other words: the orbits are situated beside the nose cavity, and there is no rostrum orbitale, the skull base being nearly flat.

To get a more objective idea of the topographic position of the orbital roofs in our own postnatal material we studied quasi-frontal sections through the skull base.

To avoid a further destruction of our skulls of a great many Anthropoids and Men of different ages which for other purposes had been sawn along the median plane, we made frontal figures with the aid of MARTIN's diagraph of the inner wall of the skull cap and the skull base on the level of the posterior border of the lamina cribrosa of one half of the skull, the skull being oriented with the Frankfurt plane perpendicular to the drawing paper. After we had in this way got the relations on one side of the skull, the whole outline of the inside of the skull on this level could be easily reconstructed, supposing that the skull was symmetrically built. A so far occipital level was chosen, and not for instance the level of the frontal border of the lamina, because of the soon receding frontal bone in the Anthropoids. Starting our drawing in the mid-line on this level and in this orientation, we soon should reach the fronto-lateral side of the skull cap with the point of the diagraph without touching the orbital roofs at all.

The drawback of our method is that the point of greatest height of the orbital roof above the roof of the nose is not always fixed in the figure. This disadvantage was removed by projecting that point in the level of our drawing by means of the diagraph. It was not necessary to note also the median point of the nose roof on this more frontal level, because in this orientation the part of the roof between the two levels was almost perpendicular to the drawing-paper.

In figure 5 we give the drawing of an infantile and an adult specimen of all genera of Anthropoids, excepting Siamang. Of adult Man only we give two figures to show the possibility of individual variations in this region. It appears that the outline of the inside of the cavum cranii at this level in Man has a much greater extension than in the Anthropoids, the part of the frontal lobes above the orbital roofs being absolutely much

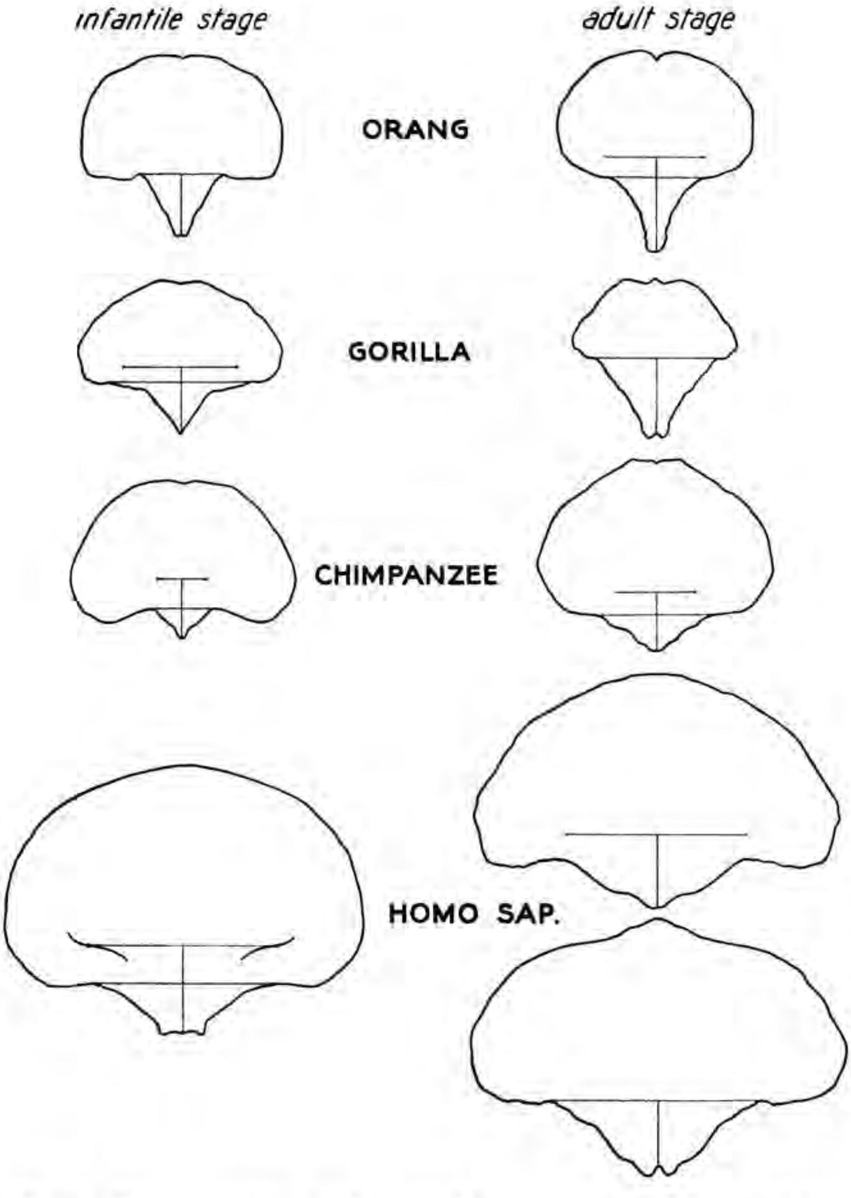

Fig. 5. Comparison of the development of the rostrum orbitale and the relative topography of the orbital roofs in young and adult Summoprimates. (For explanation see text). In the infantile Orang teeth shed was just beginning. The young Gorilla had a milk dentition which was not yet quite complete while in the Chimpanzee child a nearly complete milk dentition was present. The human infant was aged 1.5 years. All drawings ½ natural size.

broader and also higher than in the Anthropoids. The latter is partly due to the fact that in the large apes the frontal bone recedes very soon above the orbits or, in other words, the forehead is really receding, though not so much as is suggested by the big torus supra-orbitalis. A further relative difference at this level between the cavum cranii of Anthropoids and Man is the greater breadth relative to the height in the latter.

Passing to the fossa interorbitalis, containing the rostrum, we cannot demonstrate a clear postnatal ontogenetic difference in each genus separately, but we should remember that the infantile skulls used were not very young. An apparent fossa already occurs in each infantile skull. The impression of the depth of the fossa is determined by two factors, viz. the breadth and the real depth, i.e. the length of the perpendicular between the line connecting the highest points between the two roofs and the basis cranii in the mid-line, as demonstrated in the figures. If the distance between the highest points of the orbital roofs becomes greater, while the real depth of the fossa remains the same, we get the impression that the fossa is less deep than before.

It now appears that the absolute depth of the fossa measured between the roof of the nose and the line between the two highest points of the orbital roofs is not so much greater and at times even less in the Anthropoid skull than in that of Man. It is only the breadth of the fossa, measured between the two highest points, which is always much greater in Man and this is the principal cause why in the latter the fossa seems to be less deep. In fact it is only much broader than in the Anthropoids. The narrowness of the fossa in Apes makes it much more pointed dwonwards, thus justifying the name of rostrum orbitale or "bec encéphalique" for the part of the frontal brain situated in it. In Anthropoids half of the breadth and often even the whole breadth, is always less than the height, in Man on the contrary this breadth always exceeds the height.

Summarising we can say that the impression of the relatively greater depth of the cavity between the orbits in Anthropoids, in which the rostrum orbitale is situated — taken by neurologists when found in Man for a primitive feature —, in the main is rather due to the smaller distance between the orbits, i.e. the narrowness of the rostrum, than to an absolutely greater depth in comparison to the conditions as found in Man. Nevertheless it is true that the depth of the *U*- or *V*-shaped valley between the orbits in Anthropoids is greater relatively to the total base-vertex height on this level and this fact adds strongly to the impression of a greater rostral development in Anthropoids.

Discussion.

When tracing the ontogenesis of the *U*- or *V*-shaped valley between the orbital roofs in which the rostrum orbitale is situated, we saw that it is already present in young infantile stages of all Summoprimates and that its morphological alterations during postnatal development are only very small and not essential. Yet, this fact does not imply that the orbital roofs protrude during the whole of ontogenesis into the anterior part of the brain cavity. On the contrary, it appeared that in prenatal development of Primates there is a stage in which the roofs of the orbits in the primordial cranium, and still later on, lie nearly in one plane with the lamina cribrosa, the subcerebral roof of the nose. At this time nothing of an orbital keel is to be found, either in Apes or Man. Somewhat later, in Homo perhaps in the third month of pregnancy, the roofs of the orbits begin their vaulting and herewith their protruding into the anterior part of the brain cavity. This is a fact which hitherto has not been sufficiently realised and which has a certain value for the understanding of the phylogenetic laws and differences in this region between Man and Anthropoids.

Till now it was commonly thought that the rostrum in Man is absent or far less developed than in Anthropoids, owing to the greater development of the frontal part of the brain in Man. Thus BOULE and ANTHONY, speaking of this "bec encéphalique" say:

> "Chez l'homme normal au contraire (to the Apes), par le fait du développement des lobes frontaux, aussi bien dans le sens latéral que dans le sens vertical, la région cérébrale antérieure n'offre pas cet amincissement, et les parois supérieures des cavités orbitaires tendent à s'aplatir et à se placer sensiblement dans un même plan, le bec encéphalique est très reduit."

Consequently they try to explain the lack of a rostrum in Man in a simple mechanical way by the relatively greater development of the frontal brain lobes in Man, unconscious of the fact that also in the ontogenesis of the Apes there is a time when an orbital keel does not exist, though at that time the relative development of these lobes will in essentials not be different from the relations as found in the adult stage [1]. As we see, the development of the frontal brain becomes an insufficient explanation for the relative lack of a rostrum in Man. The view of BOULE and ANTHONY, however, is commonly accepted by others, as, for instance, MARTIN, GREGORY and TILNEY.

With the same right one could reverse the causes of the facts, if one could be content with such a simple causality in a living organism, saying: in Apes there is a big rostrum because the orbital roofs protrude far and high into the brain cavity, whereas in Man the orbital roofs protrude less so that the orbital keel is much less developed. Such a view, moreover, takes much better into account the ontogenetic development, in which, as we saw, there occurs a stage where the situation is probably equal in all Primates, no protruding orbital roofs existing at all. It is not

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[1] According to BRUMMELKAMP ('40), for instance, the relative size of different field areas of the neocortex of the sheep remains practically constant during ontogenesis.

very probable that from these stages on the relative development of the frontal brain will be very different between the genera [1]).

For these reasons it is incorrect to call a brain itself primitive, because it has a somewhat more than usually pronounced rostrum orbitale. For in saying so one would be inclined to consider the physiological possibilities of that brain as being primitive. In such cases it would be better to say that the morphological organisation of the organism, especially that of the facial part of the skull including the orbits, shows primitive or, rather, progressive features.

Another fact which is seldom taken into account by authors dealing with the rostrum orbitale and which is nevertheless of importance for the understanding of the shape of this formation, is the difference in position of the orbits towards each other in Man and Anthropoids. Already by the mere fact that in homotopic levels the absolute distance between the orbits (septum interorbitale of SCHWALBE) is greater in Man, the human orbital keel seems less high than it really is. This is to be seen in figure 5.

The process of rostral development may be described as follows. In Anthropoids as in Man in certain embryonic stages the orbits are situated quite laterally to the nose cavity, so that the orbital roofs and the tectum nasi are practically in one plane. Afterwards the roofs become vaulted primarily by a relatively greater increase of the orbital height, so that they begin to protrude into the frontal part of the brain cavity. In this way the space between the roofs in which the orbital keel of the frontal brain is situated develops. In all Primates this process will be the same. Then, in the group of Summoprimates, we see in non-hominids the beginning of another process: the turning upwards of the orbits as a whole. This implies that the line which connects the centres of the orbits, lying at first beneath the level of the lamina cribrosa, rises above this level. The later process is not equally pronounced in all Anthropoids. Most constant and most expressed it is in Gorilla, whose skull among all Summoprimates shows the greatest morphological differences between younger and older stages (see also Fig. 6). In the adult Gorilla the frontal part of the orbits lies beside the big sinus frontalis instead of beside the nose cavity as in Man. The ontogenetic ascent of the orbits has been already investigated by BOLK.

This ascent of the orbit as a whole cannot be observed in Man, where in this respect fetal conditions are preserved. More or less the same thing was already pointed out in 1885 by LISSAUER, who wrote that the splanchnocranium in Anthropoids turns upwards during ontogenesis, whereas that in Man turns downwards. KEITH, who also investigated these relations, says very pithily: "The face of the adult covers what was the forehead of the infant Anthropoid." This ontogenetic difference also distinctly appears

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[1]) BRUMMELKAMP ('38) found that the relation between the surface area of the frontal brain and that of the whole neocortex is constant in all Primates, Prosimiae excepted. Thus there is no elective development of the frontal lobes within the group of the higher Primates.

from the figures given by him in his article of 1910. Like BOLK, KEITH concludes that in many features "the infantile condition of Anthropoids becomes the permanent condition of adult Man."

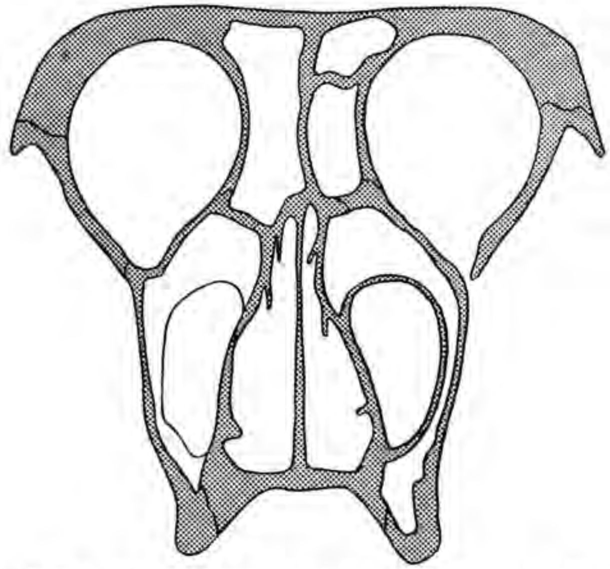


Fig. 6. Frontal section through skull of adult Gorilla. (After BOLK.)

When with these facts in mind we study the endocranial casts of Anthropoids, fossil and recent Men, given in figure 7, we see that they may play a part here also. In the photographs a constantly smaller rostrum is to be found from Ape to recent Man, which is, and not for the least part, also due to a constantly growing distance between the orbital cavities. Likewise a gradually decreasing penetration of the facial part of the skull into the neurocranium is to be deduced. In the cranial casts of lower palaeolithic Man such as the one from Ngandong, the rostrum orbitale is still much more pronounced than in recent Man. From lower palaeolithic Man onwards conditions are nearly the same as in his recent descendant. It is often forgotten that also in these relations a considerable individual variability exists, so that one has to be careful in giving a certain rostrum the epitheton primitive, especially since it is rarely expressed in measurements.

The fact, pointed out by C. U. ARIËNS KAPPERS, that in Anthropoids and also in some fossil Men the upper border of the rostrum, viewed from aside, runs upwards in relation to his lateral subcerebral horizontal line (see Gibraltar cast), whereas in recent Man this border remains beneath that horizontal, is without any doubt due to the fact that in Apes, and also in a much slighter degree in some fossil human skulls, the roof of the orbit, causing that border, runs more upwards in a frontal direction relative to his horizontal line than in recent Man. This again is due to the fact that in the more "primitive" cases the orbits and therefore also their superior walls are relatively more turned upwards in the skull. This agrees with our statement that in Anthropoids and also, though in a minor degree, in lower palaeolithic Man, as we could observe from some unpublished diagrams, the whole orbital pyramid points more upwards in a frontal direction than in recent human skulls.

The further forward we go in the skull of the large Apes, the more the orbits are situated above and beside the nose cavity, instead of lying only beside it. In a strict sense, therefore, it is impossible to make any direct deductions about the breadth of the inner nose in Anthropoids or about the development of the organ of smell with the aid of the interorbital breadth index, as SCHWALBE did, because the place of measurement of the breadth of the nose cavity on this level lies far above the roof of the latter.

This is very apparent from our figure 6, a frontal section of a Gorilla skull after BOLK, sawn just behind the third molar, so even somewhat caudally to the level, where the breadth of the "septum interorbitale", between the dacrya, must be measured. In Anthropoids there are no parts of the nose lying between the dacrya, as is the case in Man (see fig. 2), where not only the upper part of the nose cavity but also the sinus ethmoidales are situated between those two points. In the large Apes the sinus frontalis, if present, lies between them. Figure 6 again clearly demonstrates the high position which the orbits have relatively to the nose cavity in Anthropoids.

Concluding we can say that, simply speaking, the development of a rostrum orbitale is determined by the ontogenetic protruding of the orbital roofs into the frontal part of the brain cavity and that the extent of this protrusion is a different one among the group of Summoprimates, being greatest in Gorilla and least in Man. Therefore the development of a rostrum is primarily due to a more or less progressive development of the facial sphere. This being fetalised or conservatively developed in Man, the reduced rostrum may be explained as a secundary fetalised feature. The development of the frontal brain in the sense of BOULE and ANTHONY will scarcely play a part, if any, here.

Though we think we have demonstrated that the greater or lesser development of a rostrum is primarily due to a greater or lesser protruding of the orbital roofs into the brain cavity, we do not mean to say that these roofs exert any direct pressure on the frontal brain lobes, thus forming the rostrum. By no means this protruding is the simple direct causal factor in rostral development. As everywhere in an organism the diverse structures are adjusted to each other in a way which is of a far more complex nature. At best one could say that, comparing the facial sphere and the brain, the facial sphere in Anthropoids plays an increasingly important part during ontogenesis, whereas in Man the balance, as existing in young fetal times, is preserved.

Acknowledgements.

I am very much indebted to Prof. M. W. WOERDEMAN of the Anatomy Department, Amsterdam, for the use of skull material as well as for his permission to make X-ray photograms of the Gorilla fetus. Furthermore I am greatly obliged to Prof. VAN EBBENHORST TENGBERGEN for his assistance in making the röntgenograms of which one is published in this paper and to Prof. C. U. ARIËNS KAPPERS for the photograms of the endocranial casts of his collection.

Summary.

1. The topographic relations between the orbits and the nose cavity in Man and Anthropoids have been studied during ontogenesis. Special attention has been given to the relations between the orbital roofs and the cribriform plate, forming together the relief of the fossa cranii anterior.

2. It has been found that in young embryonic stages of all Summoprimates the anterior part of the skull base is almost flat. The orbits are situated beside the nose cavity. In consequence no *U*- or *V*-shaped valley between the orbital roofs, in which the rostrum orbitale of the frontal brain in adult stages lies, occurs as yet. Neither in embryonic Apes nor in embryonic Man is there such a rostrum at all.

3. Some factors are concerned with the formation of this valley in later ontogenesis. Primarily, and this is true of all Summoprimates, there is an increase of the vertical orbital height relative to that of the nose cavity. This causes a protrusion of the orbital roofs into the brain cavity. In the second place the orbits of Anthropoids turn upwards in the face so that their position, being at first quite lateral to the nose cavity, changes to one above and beside it. Therefore the orbital roofs will protrude relatively more in Anthropoids than in Man. These processes are almost completed at a young infantile stage.

4. The difference in form between the rostrum in Man and in the Apes is due to at least the following factors:

a. The absolutely lesser distance between the orbits in Anthropoids which makes the rostrum much narrower.

b. The greater height of the valley in which the rostrum lies in relation to the total height of the frontal brain in Anthropoids.

c. The turning upwards of the orbits in Anthropoids.

The greatest absolute height of the rostrum in Man is not very different from that in Anthropoids and therefore the whole difference, apart from the greater narrowness in Apes, is merely a relative one.

5. In pleistocene Man conditions are intermediate between recent Man and the Anthropoids, although they are much nearer to modern human conditions. Some more progressive, anthropoidal, developmental features, however, are apparent.

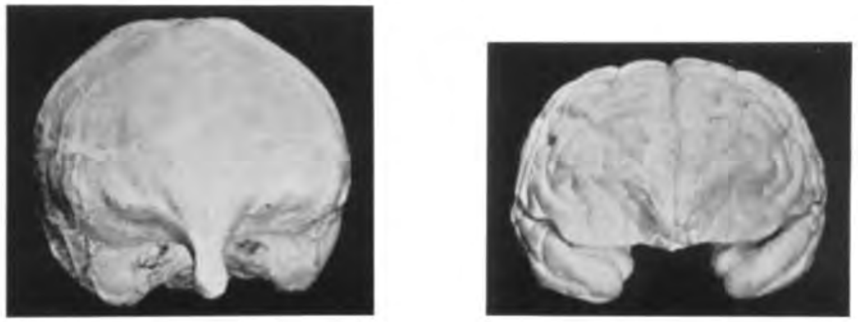
6. Certainly the relatively lesser development of the rostrum in Man is not simply due to a greater development of the frontal lobes, as was thought before. The cause may be sought in the more conservative development of the topographic orbital relations in Man in contrast to the more progressive one, especially that of the facial sphere, in Anthropoids. The relations in the orbital sphere of Man are fetalised in the sense of BOLK's theory.

J. ARIËNS KAPPERS: SOME TOPOGRAPHIC RELATIONS OF THE ORBITS IN MAN AND ANTHROPOIDS DURING ONTOGENESIS, ESPECIALLY BEARING ON THE ONTOGENETIC DEVELOPMENT OF THE "ROSTRUM ORBITALE".

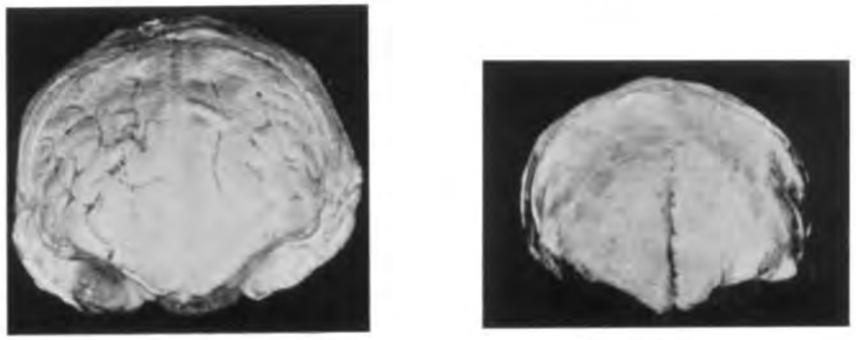
Fig. 1. Frontal section through head of human embryo aged 2 months.

Fig. 4. X-ray photogram through head of Gorilla fetus.

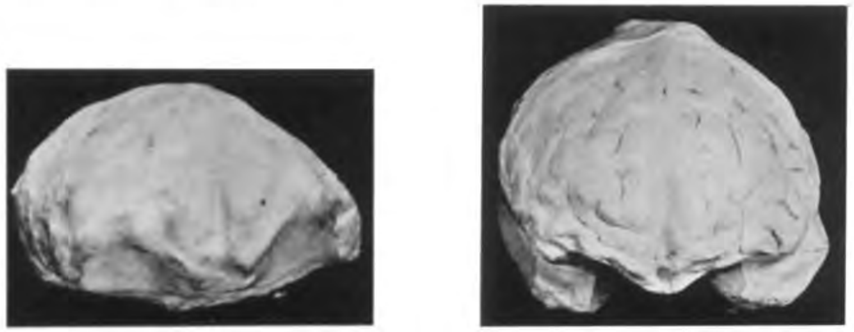
Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.



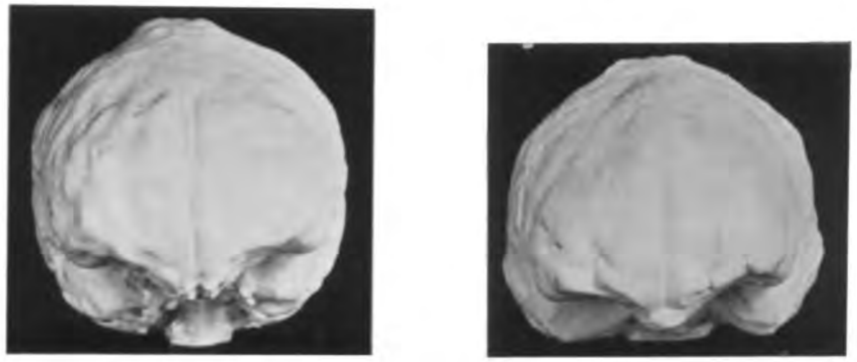
Gorilla. Chimpanzee.



Sinanthropus. Ngandong V.



Gibraltar (from aside). Predmost III.



Torres straits. Eskimo.

Fig. 7. Photograms of endocranial casts from Anthropoids, fossil and recent Man in different orientations.

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(Communicated at the meeting of October 26, 1940.)

First Part.

I. *The maternal oxygen- and carbon dioxide dissociation curves during pregnancy.*

In earlier work (1937) it had been established that the influence of pregnancy on the oxygen dissociation curve in the cow differs from that observed by BARCROFT et al. (1934) in the goat in this respect, that in the cow the curve remains within the limits of the field which we had fixed for normal non-pregnant animals, whereas in the goat the curve leaves the normal field at a gestation of 10 weeks, remaining outside during all the further time of pregnancy.

It was in the last two months only that in the cow the lower part of the curve, going until an oxygen pressure of 20 mm maximally, had been shown to become more inflected, so that the right limit of the field was slightly crossed by the curve.

It is true that this field is considerably wider than that in the goat, and also exceeds the dimensions determined for any other animal examined, man included.

The course of the oxygen dissociation curve had been watched in one and the same animal, using 3 cows and starting from the 4th month of pregnancy.

It seemed desirable to us to follow this course of the curves in a greater number of animals and using a different technique, because of the importance of that lower part of the curve for its shape and position, and for its steepness, which had been found to be increased during the last months. The position of the points for low oxygen pressure had not satisfied us and incited to new measurements. At the same time the carbon dioxide dissociation curves (carbon dioxide d.c.s.) had to be determined.

It was our intention to check the limits of the field for the curves of normal non-pregnant animals meanwhile.

Four healthy adult Frisian cows were used for these measurements. The blood was drawn from the jugular vein at various moments of pregnancy as is shown by table 1. (For table 1 see next page).

The blood was collected in a wide glass tube which contained a 5 % solution of ammonium oxalate with 2½ % sodium fluoride. The solution had been brought to a pH of 7.4. It had a volume of 1/50 of that of the blood with which it was thoroughly mixed. In some experiments the anticoagulant was used in crystalline form.

TABLE 1.

Number of animal	Duration of pregnancy in months
11	4
	6
	6½
	8
	9
12	2
	3
	6
	8
	8½
	9
13	6
	7
	7½
	8½
	9
14	5
	6
	9

However much BARCROFT's differential manometer had satisfied us, VAN SLYKE's constant volume apparatus was used in these experiments because of our interest in the position of the points for low oxygen pressure especially. The advantages offered by this apparatus, when a great number of determinations have to be carried out, was another reason to choose it.

Since a practically unlimited volume of blood was at our disposal, the determinations were made using 2 cc of blood.

The carbon dioxide curves were determined by measuring the carbon dioxide in 1 cc of true plasma, obtained from oxygenated blood, in the usual manner.

Results.

Before describing the oxygen d.c. during pregnancy, the limits for the field within which the curves for normal non-pregnant animals are found will be discussed. The shape, given in an earlier paper (1938), was verified by determining the curves in six normal non-pregnant animals. The new field, demonstrated in Fig. 1, agrees for far the most part with that given previously; its lower part has undergone a slight correction. In the same figure the field for the goat and for man has been drawn for comparison.

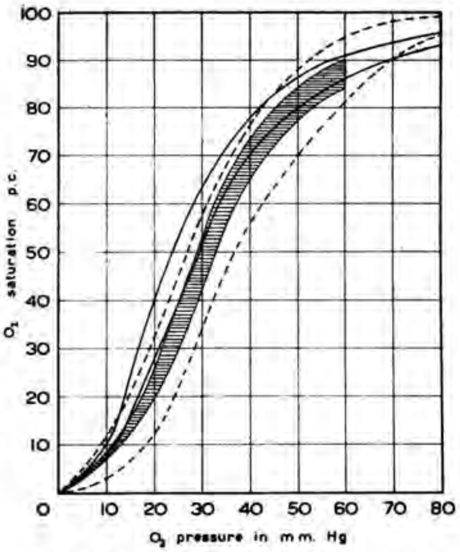

Fig. 1. The area of the O_2 dissociation curves of normal cows, not pregnant, and not having been so recently, is given by the dotted lines. CO_2 pressure 40 mm Hg., temp. 38° C. The field for the goat has been shaded, that for man has been drawn. The curves for the goat and for man according to BARCROFT.

The carbon dioxide d.c.s for true plasma of oxygenated blood, determined at the same time for the same animals, are enclosed by the limits given in Fig. 2. As the reader will find there the reserve alkali varies between 47.5 and 57.5.

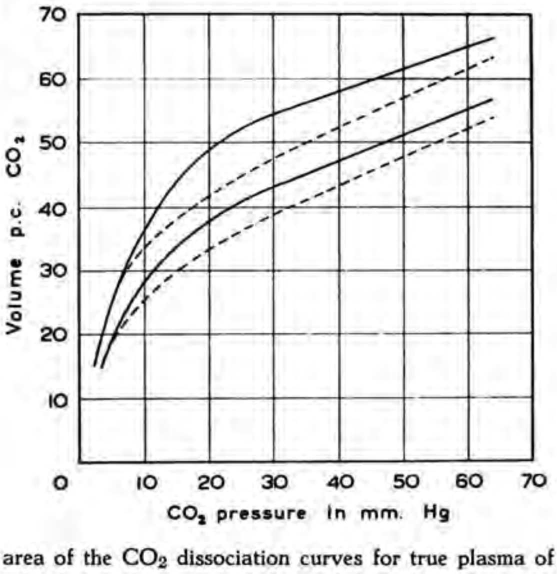

Fig. 2. The area of the CO_2 dissociation curves for true plasma of oxygenated blood of normal cows, not pregnant and not having been so recently, is given by the drawn lines. The dotted field is that for oxygenated blood, as had been determined in previous work. Temp. 38° C.

It is not sure that Fig. 2 gives a fully correct image of the difference between plasma and blood. The state of nutrition of the animals and the kind of nourishment may be supposed to have its influence here, since the true plasma curves were obtained from animals which had passed the summer in the meadow whereas the first set of curves, one excepted, had been determined in the beginning of spring, after the animals had been kept in the stable during the winter. In Fig. 3 a comparison is given of the carbon dioxide d.c.s of whole blood and true plasma from one and the same animal.

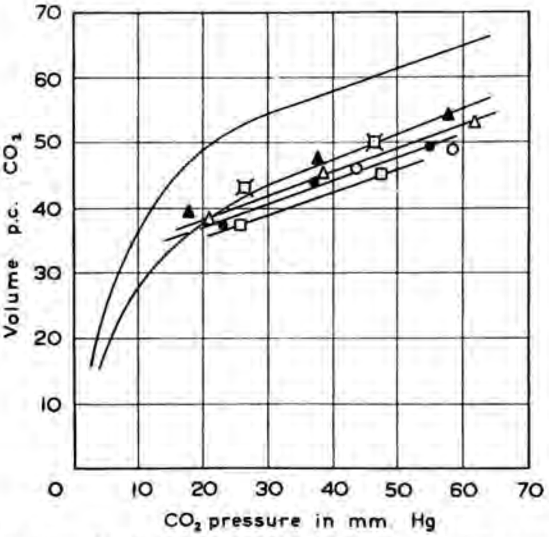

Fig. 3. Carbon dioxide dissociation curves of (*a*) maternal oxygenated blood (□) and oxygenated true plasma (¤). The animal was pregnant during 8 months. (*b*) maternal oxygenated blood (○) and oxygenated true plasma (△) of an animal, pregnant during 7 months, together with those of the foetal blood (● and ▲). Temp. 38° C.

Coming now to the oxygen d.c.s of the pregnant animal, it was established that the influence of gestation is perceivable in two respects. (1) The curves become steeper during pregnancy and (2) they become more inflected during the last 6 months. Speaking in general the position of the curves remains within the limits of the normal field. In some instances the right border line of the latter may be surpassed slightly by the lower end of the curve as a result of the increased inflection observed here during the 7th, 8th and 9th month.

The results, obtained in earlier experiments (1937) with respect to position and shape of the curves have been confirmed. The more steeple position however was found not to be restricted to the last months only, but it was detected to occur earlier, starting from about the 4th month of pregnancy.

In Fig. 4, 5, 6 and 7 a set of oxygen d.c.s is given, together with the carbon dioxide d.c.s, which may be considered as representative.

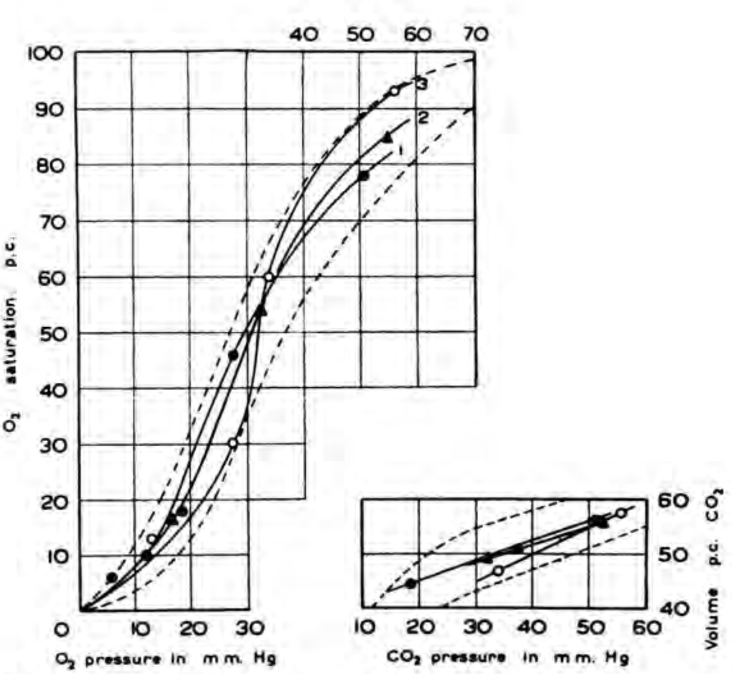

Fig. 4. O_2 dissociation curves and CO_2 dissociation curves of cow 12, pregnant about 2 months (1), 6 months (2) and 8 months (3). The dotted lines give the limits of the field for normal non-pregnant animals. For the O_2 d.c.s. CO_2 pressure 40—43 mm Hg; the CO_2 d.c.s. have been determined for true oxygenated plasma. Temp. for both 38° C.

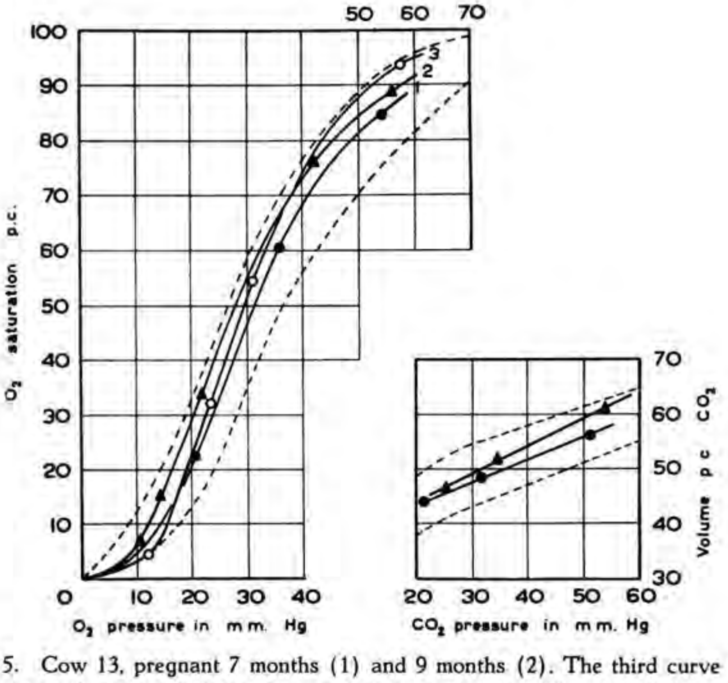

Fig. 5. Cow 13, pregnant 7 months (1) and 9 months (2). The third curve (3) was obtained one week later, 3 hours before parturition. The dotted lines give the limits of the field for normal non-pregnant animals. Rest as in Fig. 4.

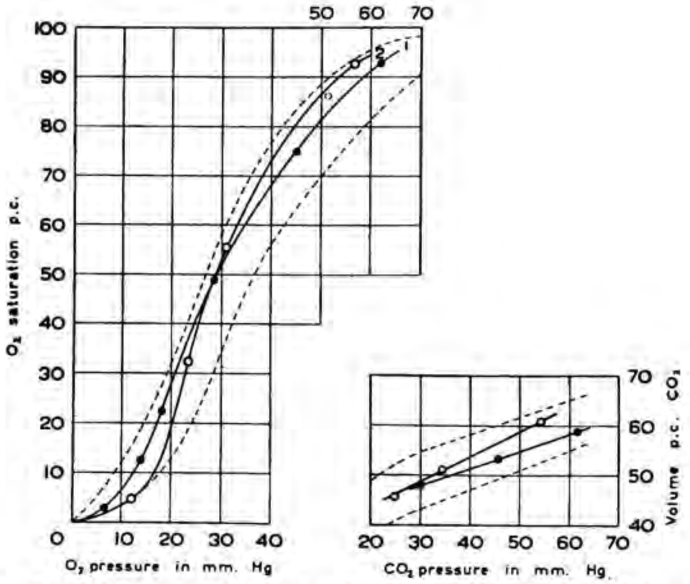

Fig. 6. Cow 13, pregnant 8½ months (1). Curve (2) was obtained from blood, taken 3 hours ante partum. The dotted lines give the limits of the field for normal non-pregnant animals. Rest as in Fig. 4.

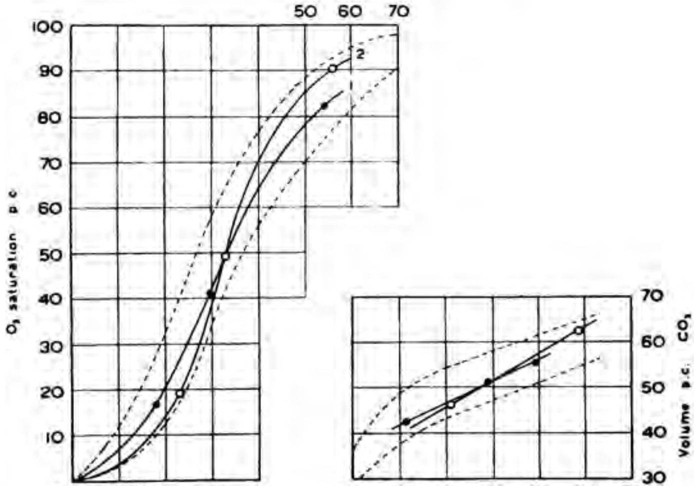

Fig. 7. Cow 14, pregnant 5 months (1) and 6 months (2). The dotted lines give the limits of the field for normal non-pregnant animals. Rest as in Fig. 4.

So it must be concluded that although the changes of the maternal oxygen d.c. during the period of pregnancy may not be called salient, they nevertheless are distinct when the bearing of the curves is watched in one and the same animal.

Concerning the causes by which these changes are produced the only conclusion that can be drawn with security is this that they are not found in changes of hydrogen-ion concentration.

Firstly it would be difficult to understand how both changes described could be produced by changes of the pH. Moreover the reserve alkali was rather constant during the greater part of the period of gestation, so that its influence was quite insufficient to produce the changes of position of the curves established.

The course of the reserve alkali in the pregnant animal is shown by Fig. 8. It shows that the carbon dioxide combining power of its blood is not lower than in the non-pregnant animal and that there is a noteworthy constancy of alkali reserve up to the ninth month. During that time it ranged between 50 and 55. In one animal (no. 13) it rose somewhat above this limit, reaching 58 in the midst of the 8th month. Then, at the end of the 9th month, a general rise is observed. In two instances in which the blood had been examined immediately before parturition and some few minutes after it, a fall of the reserve alkali could be stated to have taken place during this short time (about 1 hour). The values, found after parturition, have been plotted on the ordinate p.p.

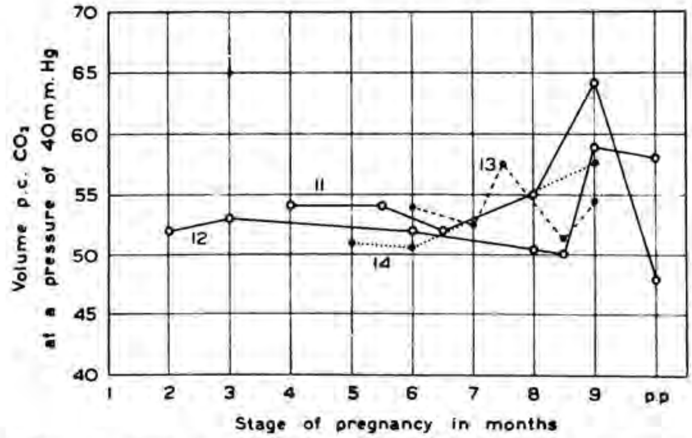

Fig. 8. Reserve alkali during the period of pregnancy in 4 pregnant cows. The figures at the curves give the number of the animals; p.p. = some few minutes post partum. The determinations were made in true plasma of oxygenated blood. Temp. 38° C.

The values of the reserve alkali and those for the hydrogen-ion exponent of true plasma at the various stages of gestation, at a carbon dioxide pressure of 40 mm Hg, are given in table 2. The latter were calculated by using the HENDERSON–HASSELBALCH formula.

In how far the changes of the oxygen d.c.s are due to alterations of the haemoglobin molecule on the one hand and to the effect of electrolytes on the other remains still an open question at this moment.

In none of the animals the maternal alkali reserve was lower than in normal non-pregnant animals. Most often it was found to correspond with the higher part of the normal field for non-pregnant animals.

TABLE 2.

Number of animal	Duration of pregnancy in months	Reserve alkali in true plasma (Vol. p.c. CO_2 bound at a CO_2 pressure of 40 mm Hg)	pH
11	4	54	7.38
	6	54	7.38
	6½	52	7.36
	8	55	7.38
	9	64	7.44
	10 min. p. p.	48	7.33
12	2	52	7.36
	3	53½	7.37
	6	52	7.36
	8	50½	7.35
	8½	50	7.35
	9	59	7.42
	5 min. p. p.	58	7.41
13	6	54	7.38
	7	52½	7.36
	7½	58½	7.41
	8½	51	7.35
	9	54½	7.38
14	5	51	7.35
	6	50½	7.35
	9	58	7.41

It is remarkable that so fundamental differences, as observed here, can be stated to exist at various points between ox- blood on one hand and the blood of man, and even the blood of the goat, on the other, both animals approaching each other so nearly in many respects. In the goat the maternal curve moves in the opposite direction and at the half of gestation it falls outside the normal field where it remains during the further time of pregnancy. Barcroft et al. (1934) found this displacement to be accounted for by the change in reaction: the increased hydrogen-ion concentration. In the ox there is no displacement to the right side at all and the change of hydrogen-ion concentration is quite different.

Also in man the results obtained in studying the maternal oxygen d.c. differ from those in the ox; Eastman, Geiling and de Lawder (1933) who examined the curve of the mother immediately after birth of the child, found it distinctly displaced to the right. A corresponding result has been obtained in 1936 by Leibson, Likhnitzky and Sax.

Consequently the gap between maternal and foetal oxygen d.c. which is found in the ox- blood as well as in the other species mentioned, is effected in the cow by removal of the foetal curve exclusively.

II. *The dissociation curves of the foetal and the maternal blood.*

Some years ago we examined the relation between foetal and maternal dissociation curve in the cow and published the curves, obtained from an animal, 8 months pregnant (1938). The opportunity to continue the work in a greater number of animals and at various stages of pregnancy was wanting at that moment.

In this chapter the results obtained in a greater number of experiments will be given.

Method.

Both oxygen- and carbon dioxide d.c.s were determined. The maternal blood was taken from a carotid artery, a sample of foetal blood being drawn from one of the umbilical vessels, after Caesarian section had been done under local anaesthesia; the foetus remained connected to its mother under conditions as physiological as possible. The blood was collected in the way as described in the foregoing pages.

The moment of pregnancy at which the curves were determined are clear from the following table.

Age of foetus in months	Sex	Number of cases
3	m	1
3½	m	1
4	f	1
5½	f	1
7	f	2
7	m	1
8	f	1
8	m	2
8½	—	1

Results.

In all cases irrespective of the age of the foetus, the oxygen dissociation curve of the foetal blood was found to the left of the maternal. The difference in position between both was considerable and the foetal curve was found to lie outside the field for normal non-pregnant animals in 9 out of 11 cases. In the two remaining instances the curves were found either for the greater part or completely within the limits of the normal field.

As was described in the first chapter of this paper the maternal curves remained within the normal field. In some instances the inflection in the last month of pregnancy had increased in such a way that the lower part of the curve crossed the right limit of the field but this was an exception. Of course the curves obtained at various ages of the foetus, can hardly be compared, since they were obtained from different animals.

In the figures 9—13 a set of curves is given for both mother and foetus on various points of foetal life.

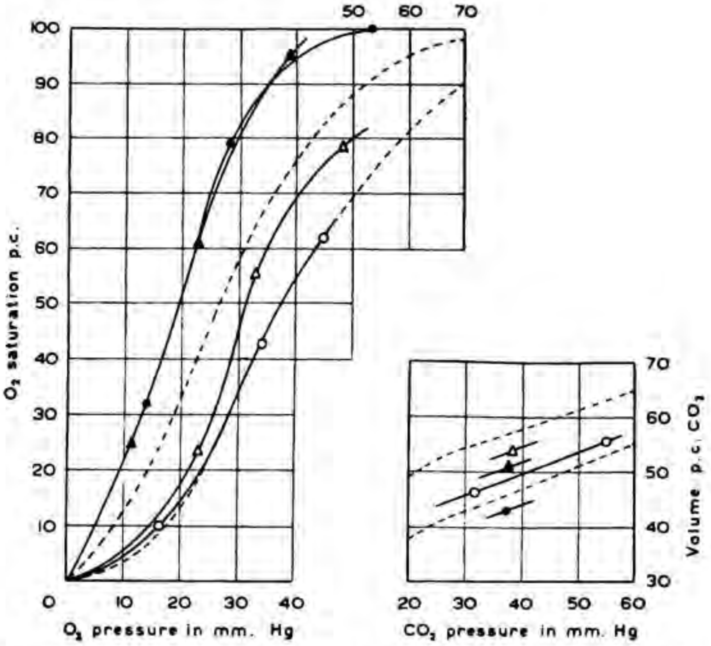

Fig. 9. O_2 dissociation curves of two foetus, old 3 (●) and 3½ months (▲) respectively, together with those of the mothers ((○) and (△)). The foetal curves coincide for a great part. CO_2 pressure 42—44 mm Hg., temp. 38° C. At the bottom the corresponding CO_2 dissociation curves for true plasma of oxygenated blood. The field for the normal non-pregnant animals has been dotted.

In all figures given here the foetal curve is found outside the normal adult field, as it was the rule. Two other examples are given by Fig. 14 in which the curve for a foetus, 4 months old, is found to have its position for the greater part within the normal area, whilst the dissociation curve of a foetus of 8½ months is found entirely within that field. If we consider the carbon dioxide d.c.s of Fig. 14 the explanation of these exceptions does not offer difficulties. In both foetus the reserve alkali was low and remained below the normal field. In the foetus of 8½ months it was even extremely low and less than 50 p.c. of the value which may be considered

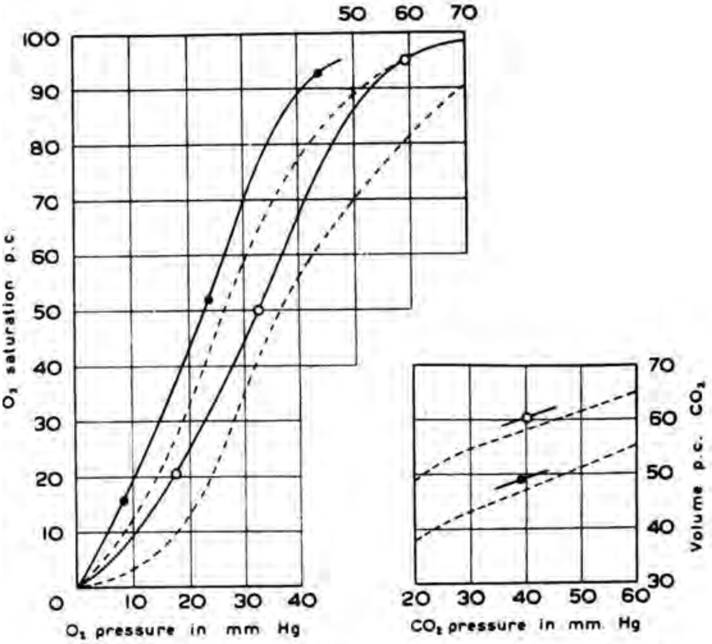

Fig. 10. Age of the foetus 5½ months. The rest as in Fig. 9.

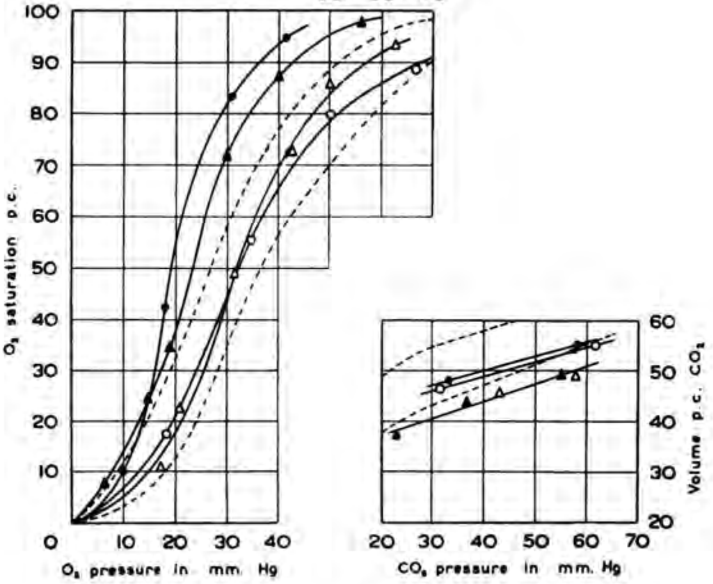

Fig. 11. Age of both foetus 7 months. The rest as in Fig. 9.

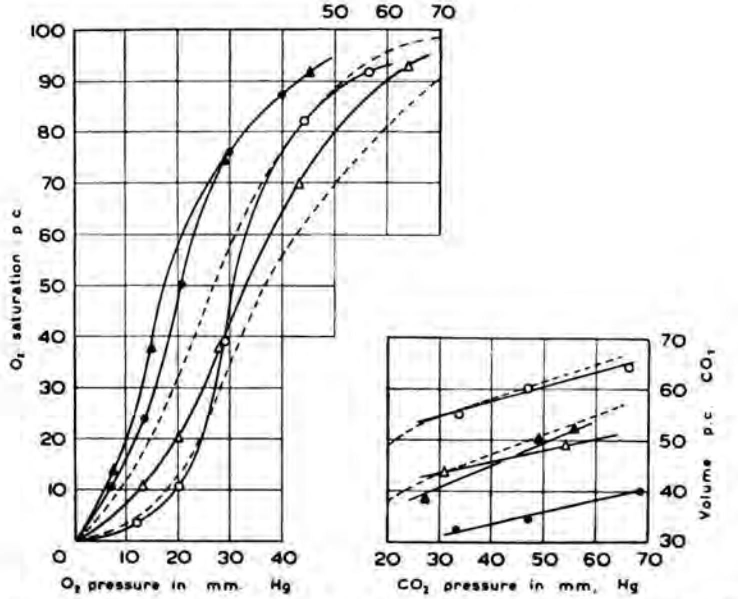

Fig. 12. Age of foetus 7 (●) and 8 (▲) months respectively. The rest as in Fig. 9.

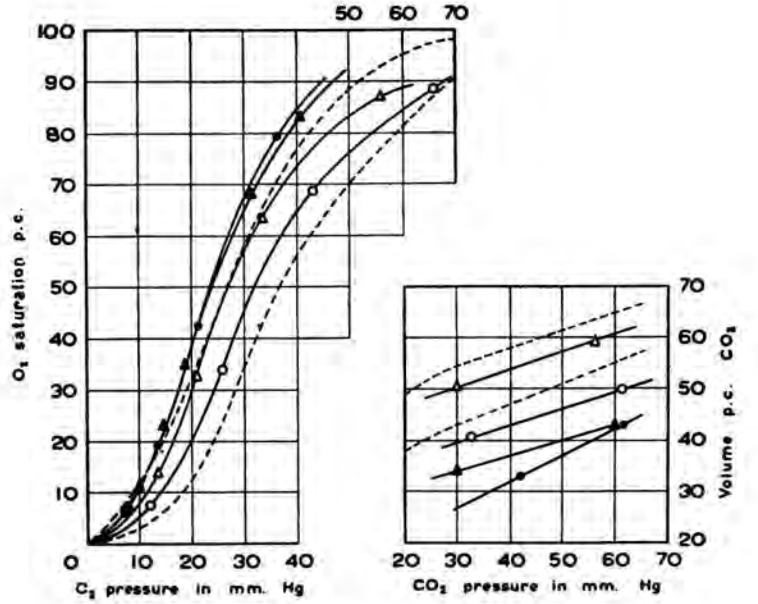

Fig. 13. Age of both foetus 8 months. The rest as in Fig. 9.

as being normal. It is the Bohr effect by which the position of the curves of Fig. 14 is brought about.

How is the course of the reserve alkali during foetal development?

During the first 6 months of intra-uterine life the foetal carbon dioxide curves fall within the field, established for normal non-pregnant animals, be it in its lower part. In the seventh months' foetus the curve was found a little below that field. As intra-uterine life procedes the gap between the two increases (See Fig. 12 and 13). This had also been found in the foetus of 8 months, described in an earlier paper (1938).

Fig. 14. Age of foetus 4 (●) and 8½ (▲) months respectively. The rest as in Fig. 9.

In the mother cows the reserve alkali was rather high also in this series. Up to about the 7th month of pregnancy the position of the curves correspond even to the higher half part of the normal field for non-pregnant animals. Later on a rather great dispersion of results was observed, which must probably be attributed to the condition of nutrition of the animals of this series. The pregnant animals described in chapter (I) were all fully sound and in an excellent state of nutrition. They belonged to the permanent stock of cattle of one of the Veterinary Institutes of the Faculty. The animals however which were sacrificed to the experiment together with their foetus had been bought for this purpose. Their state of nutrition was considerably less. So the lower reserve alkali, found in some of them will probably find its explanation in this circumstance. In 8 out of 11 experiments maternal reserve alkali was considerably higher than the foetal, in the remaining 3 cases there was scarcely any difference between the two.

A review of the reserve alkali, measured in both mother and foetus, is given in Fig. 15. Besides the 11 animals treated in this chapter, other measurements made in pregnant and in new-born animals, have been collected.

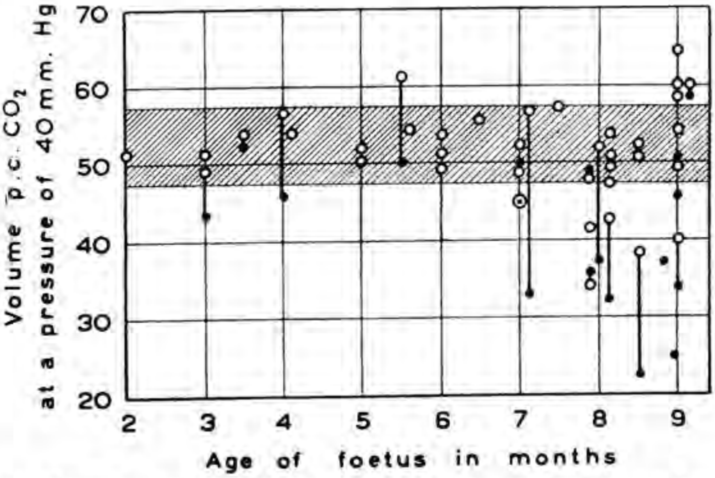

Fig. 15. Reserve alkali in maternal blood (○) and in blood of the foetus at various ages (●). The CO_2 dissociation field for normal non-pregnant adult animals has been shaded. The points indicate the volume p.c. CO_2, bound by true plasma at a gas pressure of 40 mm Hg.

The figure shows the relation between mother and foetus and moreover the fall of foetal reserve alkali in the last months.

Returning to the oxygen d.c. another point of difference between maternal and foetal curve has still to be stated, viz. concerning the shape of the curve. The foetal curve is less inflected than the maternal and during the first six months of prenatal life especially its lower part is so stretched that the hyperbolic form is approached. Until the 7th month this shape is conspicuous as may be seen in the figures 9, 10 and 14. So the moment of change in the foetal oxygen d.c. obviously coincides with the fall of alkali reserve.

Discussion.

This coincidence may be of importance for the foetus from a physiological point of view since both changes cooperate in facilitating giving off oxygen in the area of low oxygen pressure, as will be found in the foetal tissues. Especially the stretched oxygen d.c. may be called favourable. It is not only the oxygen binding power which, at a low oxygen pressure, is thus promoted but the volume of oxygen given off will be relative great at a slight fall of oxygen pressure. So the respiratory needs of foetal tissues during the first months may be considered as being contented in a very satisfactory way in consequence of this hyperbolic curve.

As the young embryo grows the oxygen needs of its tissues, the muscles especially, will increase considerably and the process of gas change will be performed at a higher level of the curve, i.e. at a higher oxygen pressure. The more inflected curve, exhibited by the foetal blood starting from the 7th month may be considered to meet this demand in a striking way. In the area of higher oxygen pressure oxygen will be given off now more readily than it was allowed by the hyperbolic curve.

A set of oxygen dissociation curves has been given for the goat by BARCROFT et al. (1934). These authors also found a gap between maternal and foetal curves but it was produced by displacement of both maternal and foetal curve. Whereas the foetal curve was found to the left of the normal, starting from about the 12th week, the maternal was drifting to the right and had already left the normal area at the 10th week of gestation.

The carbon dioxide balance between maternal and foetal blood in the goat was examined by KEYS (1934). His results differ considerably from ours, observed in the ox. Contrary to our findings he found a markedly higher alkali reserve in the foetal than in the respective maternal blood. The average foetal blood was able to bind 22 p.c. more carbon dioxide than the corresponding maternal at the same tension. The foetal reserve alkali was not much lower than what he considered as normal for the goat. The maternal one however was considerably below it. Unfortunately KEYS did not measure the reserve alkali in true plasma of either oxygenated nor reduced blood but in whole blood. If we try to make his figures comparable with ours the result is that the reserve alkali in true plasma of oxygenated blood was here between 35 and 40 in 4 cases and about 45 in the fourth.

It is difficult to follow his conclusions concerning the course of the reserve alkali during foetal life since they are based on determinations, partly made by others and the impression is given that they were made under conditions which were not very favourable.

Some figures for the reserve alkali in the pregnant woman are found in literature. The conclusion is that there is a definite drop of reserve alkali as pregnancy proceeds. WILLIAMSON (1923) found a minimum (33 in multiparas, 43½ in primiparas) at the end of gestation. After parturition it rose so that on the 10th day after delivery a normal level was reached again. In the child at term the carbon dioxide binding power of the blood was considerably higher than in the maternal blood.

So there are points of difference in many respects between the goat and the ox.

1. In the goat the first sign of pregnancy is exhibited by the maternal curve which, according to the authors as the result of increased hydrogen-ion concentration, leaves the normal field between the 7th and 10th week whilst any shift in the foetal curve is still lacking. In the cow there is no fall of reserve alkali in the mother, the maternal oxygen dissociation curve does not move to the right but even slightly to the left, maintaining however its position within the normal field.
2. In the cow the inflection of the curve increases.
3. The foetal curve however is much more mobile in the cow than in

the goat. As soon as in the 3rd month, i.e. in the first third part of intra-uterine life, it has taken its position far from the normal field (Fig. 9), whilst in the goat it is not before the 13th week (i.e. at the end of the second third part of gestation) that its removal becomes distinct and the curve partly falls outside the limits of the normal area.

4. In the goat the higher part of the curve remains inside those limits up to the 19th week.

5. Further the authors mentioned found the foetal and the maternal curves drawing towards one another again in the last week of foetal life of the goat, and one week later, after birth, the foetal curve had returned again in the adult field, both maternal and foetal curve having the normal degree of inflection. In the cow the foetal curve persists considerably longer in its foetal state concerning both position and shape. In the new-born calf the curve, although nearer to the maternal than it was in the foetal time, is still outside the normal field (1937).

So the divergences, exhibited by the foetus, are far more intensive in the cow than in the goat, and apparently the young animal is born in a condition, differing much more from the adult than the kid of the goat does.

6. It is interesting that in the goat the foetal blood, though not more alkaline than the normal, is more alkaline than the maternal. The same is the case in man. The reverse relation is found in the cow.

Although in the cow the difference between maternal and foetal curve is brought about by the properties of the foetal oxygen dissociation curve exclusively, that difference and its importance for foetal respiration is not

TABLE 3.

Oxygen pressure required in order to give an oxygen saturation of 50 p.c.

Age of foetus	Goat (BARCROFT)			Age of foetus	Cow		
	Foetus	Mother	Difference		Foetus	Mother	Difference
10 weeks	32	36	4	3 months	20	37	17
11 ,,	30	36	6	3½ ,, (3)	18	34	16
13½ ,,	24	34	10	4 ,,	30½	32½	2
14 ,,	30	38	8	5½ ,, (2)	23	33	10
15 ,,	24½	37	12½	6 ,,	10	30	20
16½ ,,	22	35	13	7 ,, (3)	20	32	12
18 ,,	25	40	15	8 ,, (4)	20	32	12
19½ ,,	21	35	14	8½ ,,	29	36¼	7¼
20½ ,,	24	34	10				
At birth	30	36	6	At birth (6)	22½	31½	9

The figures between brackets give the number of observations from which the average value has been taken.

less considerable. Also in the cow the avidity for oxygen is far greater in the foetal than in the maternal blood, and at a given oxygen pressure foetal blood will be saturated to a notably higher degree than the maternal. This will be evident from the figures, collected in table 3, in which the respective figures observed in the goat by BARCROFT, are given for comparison. The table shows the gas tensions required in order to give a 50 p.c. saturation to the blood. For both mother and foetus, the pressure required is lower in the cow than in the goat at a corresponding moment of pregnancy. The difference however between mother and foetus is generally greater in the cow than in the goat.

In table 4 the figures are given for the reserve alkali of true plasma for both mothers and foetus. The pH has been calculated.

TABLE 4.
Reserve alkali in maternal and foetal true plasma.

Age of foetus in months	Sex	Reserve alkali		Corresponding pH (calculated)	
		mother	foetus	mother	foetus
3	m	49½	44	7.34	7.28
3½	m	54½	52	7.39	7.36
4	f	56	46	7.40	7.30
5½	f	60½	50	7.43	7.34
7	f	49	49½	7.33	7.34
7	f	44	44	7.28	7.28
7	m	57½	33½	7.40	7.15
8	f	43	32	7.28	7.15
8	m	54½	37½	7.39	7.20
8	m	47	47	7.30	7.30
8½	—	39	23	7.22	6.98

In man, in which the oxygen dissociation curve for mother and child was examined immediately after birth of the baby (LEIBSON, LIKHNITZKY and SAX (1936)) the same events have been found as in the goat, in so far that the maternal curve is displaced considerably to the right, whilst the foetal curve corresponds nearly with that of the non-pregnant adult.

This displacement of the maternal curve in man and in the animals, except for the ox, deserves attention because of its physiological consequences. It shows a greater ability of the blood to give off its oxygen and foetal respiration will thus be improved, provided that maternal blood will be oxygenated in the lungs to the same degree in spite of the higher oxygen pressure in the blood. Is it reasonable to assume that in general alveolar oxygen pressure will be high enough to do so? No doubt it is an interesting point to investigate the composition of alveolar air in the course of pregnancy. It may be remembered here, that during pregnancy we found an increase of oxygen content of the alveolar air in the dog (1937). Starting from the 4th week of pregnancy, i.e. before the midst of foetal life, alveolar oxygen pressure increased with 6 to 12 p.c., whilst at the same time carbon dioxide pressure had fallen to under 90 p.c. of its original value.

In the description of our first observation, two years ago, it was already emphasized that differences of reserve alkali could not be responsible for the difference between maternal and foetal oxygen d.c. The new experiments affirm this conclusion since the pH of the foetal blood either agreed with that of the maternal or even remained below it. In spite of this fact the foetal oxygen curve was found at the left, which feature proves that the displacement to the right, promoted by the altered hydrogen-ion concentration, is counteracted by another influence by which it is inverted to a displacement in the opposite direction. Moreover it is not the position only but also the shape, by which both curves differ.

It is very probable that the signs observed are effected by the properties of foetal haemoglobin, which differ from those of the maternal. For the chemical work done in this field by many workers, the reader may be referred to a report given for the Sixteenth International Physiological Congress (1938). The question in how far electrolytes play a part here, cannot be settled.

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Psychology. — *Die menschlichen Kommunikationsformen und die sog. Tiersprache.* I. Von G. RÉVÉSZ. (Communicated by Prof. A. DE KLEYN.)

(Communicated at the meeting of October 26, 1940.)

1. *Die Kommunikation im Allgemeinen.*

Wollen wir den Begriff „Kommunikation" dem Sprachgebrauch entsprechend verwenden, so müssen wir die Bezeichnung „Kommunikation" auf solche Verbindungen beschränken, wo es sich um wechselseitige, von einander abhängige Veränderungen im Verhalten der kommunizierenden Individuen bezw. Gruppen handelt. Demzufolge können wir die *Kommunikation* als ein erbbiologisch fundiertes und zu den allgemeinen Lebensäusserungen und Lebensbedürfnissen gehörendes, auf Gegenseitigkeit beruhendes Verhalten zwischen artgleichen und artungleichen lebenden Wesen definieren. Die Kommunikation ist also mit der zwischen Individuen und Gruppen bestehenden gegenseitigen Verbindung nur dann eindeutig charakterisiert, wenn die Verbindung für die Art beförderlich und das Mittel der Verbindung, des Aufeinanderwirkens ausgebildet und der psychophysischen Beschaffenheit der miteinander in Kommunikation tretenden Arten bezw. sozialen Gruppen angepasst ist.

Aus dieser erklärenden Definition folgt, dass *die vitale Bedeutung, die Gegenseitigkeit, ferner die Fixierung und Angepasstheit der Mittel die wesentlichsten Merkmale jeder Form der Kommunikation bilden.* Wo diese Merkmale fehlen, wird man von Kommunikation nicht sprechen dürfen, sonst müssten wir alle Reaktionen lebender Wesen, insofern sie durch ein anderes Wesen ausgelöst werden, als kommunikative Aeusserungen ansehen. In diesem Falle müssten u.a. die an sich zweckmässigen, aber einseitigen Reaktionen, wie etwa die Flucht der Tiere beim Bröllen des Löwen, der Schreckruf der Hühner beim Anblick der Schlange, die Angriffsstellung der Katzen beim Riechen von Mäusen, das Weinen des Kindes beim Wahrnehmen fremder Gesichter als Folge einer kommunikativen Verbindung zwischen Beutetier und Löwe, Huhn und Schlange, Katze und Maus, Kind und Mensch betrachtet werden. Bei allen diesen Fällen fehlt das der Kommunikation zugrunde liegende oder erst durch sie zustande gebrachte Zusammenspiel der Partner.

Besonders muss man sich hüten, sich von der blossen Tatsache des Zusammenseins, der Vergesellschaftung irreführen zu lassen, und sie selbst schon als Ausdruck der Kommunikation zu betrachten. Nichts wäre unrichtiger als diese Auffassung. Die soziale Verbindung stellt noch kein zuverlässliches Zeichen für das Vorhandensein einer Kommunikation dar. Es gibt Tiere, die gelegentlich oder ständig zusammenleben, einander aufsuchen, Schlaf-, Schwarm- und Wandergesellschaften bilden, gemeinsam nach Nahrung suchen und Feinde angreifen, ohne dass zwischen ihnen Kommunikation in dem angegebenen Sinne bestehen würde. Dass Glieder einer solchen Gemeinschaft sich keines Kommunikationsmittels bedienen, verhindert das Zusammenleben, das Bestehen gemeinschaftlicher Interessen, selbst eine differenzierte Arbeitsteilung nicht. Wie vielseitig diese sozialen Tätigkeiten auch sein mögen, sie verdanken ihre Existenz unveränderlichen biologischen Gesetzen, die die Glieder der Gemeinschaft zu gewissen vorausbestimmten Verhaltungen geradezu zwingen.

Was die Ausdrucksform der Kommunikation anbelangt, so muss man wissen, dass nicht alle Laute, Bewegungen, Haltungen, die andere Wesen zu bestimmten zweckmässigen Reaktionen veranlassen, ohne Weiteres als Kommunikationsmittel zu gelten haben. Wenn der Schmerzlaut eines Tieres seine Artgenossen zur Flucht veranlasst, so folgt daraus noch nicht, dass diese Reaktion — die nicht einmal zweckmässig zu sein braucht — auf wechselseitiger Beziehung beruht. Gelegentlich kann ein beliebiger Reiz bei derselben Tierart dieselbe Reaktion auslösen. Auch der menschliche Schmerzlaut bildet an sich kein Kommunikationsmittel zwischen den Menschen, trotzdem kann er als Anzeichen eines bemitleideten Zustandes uns zur Hülfeleistung veranlassen. Allerdings kann der Mensch einen Schmerzlaut mit der Absicht ausstossen, um mit seinen Artgenossen in Kontakt zu treten. Wird dieser beabsichtigte Laut von den anderen als solcher verstanden, so ist die Lautäusserung zum Kommunikationsmittel ad hoc geworden. Seine Kommunikationsfunktion liegt aber nicht im Laut selbst, sondern in der Absicht.

Die gegebene Definition gibt genügende Anhaltspunkte, um zu entscheiden, ob einer reaktiven Handlung eine kommunikative Tendenz zu Grunde liegt oder nicht. Damit ist aber der Weg zur Erforschung des Kommunikationsproblems noch nicht gesichert. Wir müssen versuchen zuverlässige Gesichtspunkte zu finden, die uns in diesem äusserst komplizierten Gebiet zu leiten vermögen. Es scheint mir, dass diese Gesichtspunkte durch Feststellung der *Kommunikationsformen* zu gewinnen sind. Diese klassifikatorische Arbeit ist umso dringlicher, da sie zu Problemstellungen führt, die sowohl für die Entwicklungspsychologie, als auch für die Sprachpsychologie von grundlegender Bedeutung sein dürften.

a) Die erste Unterscheidung, die wir treffen müssen, ist die zwischen *absichtlicher* und *unabsichtlicher* Kommunikation. Absichtliche Kommunikation lässt sich nur bei Menschen antreffen. Sie ist mit der Willens- und Sprachfunktion in engster Beziehung [1]). Unabsichtliche, zwangsläufige Kommunikation zeigt sich vornehmlich bei Tieren, gelegentlich auch bei Menschen, insbesondere bei kleinen Kindern. Die Mittel der absichtlichen Kommunikation bilden die Gebärden- und Lautsprachen, Schrift- und Bild-

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[1]) Siehe darüber ausführlicher im zweiten Teil dieser Arbeit: Die Tiersprache. Proc. Ned. Akad. v. Wetensch., Amsterdam, **43**, N⁰. 10 (1940).

sprachen, schliesslich die konventionellen Signalsysteme. Die Mittel der unabsichtlichen Kommunikation bestehen aus Lauten, Bewegungen, Verhaltungen, die dem Triebmechanismus angehören.

b) Die Kommunikation kann entweder dem Interesse des „Senders" oder des „Empfangers" dienen oder zu Gunsten beider. Die ausschliesslich dem Lebensinteresse des Senders dienende Verhaltungsweise können wir *autistische*, die nur auf den Empfänger bezugnehmende *altruistische*, schliesslich die wechselseitige Interesse fördernde Verhaltungsweise *soziale* nennen. Ausschliesslich im Interesse des „angesprochenen" Partners geäusserte Zeichengabe kommt nur bei Menschen vor; sie ist immer von Absicht getragen. Meiner Kenntnis nach gibt es kein Tier, das Laut- oder Bewegungszeichen ausschliesslich zum Wohl anderer Tierindividuen hervorbringen würde; der Sender ist dabei immer beteiligt. Der Ruf der Henne zur Ernährung, Reinigung und Pflege ihrer Brut ist nicht ohne jeden Gewinn für die Mutter. In der Fürsorge liegt die Befriedigung wichtiger Lebensbedürfnisse der Mutter eingeschlossen: die Kleinen leisten aktiv zugleich als nehmende und als gebende Mitglieder des Gynopaediums der Mutter Dienste. Demgegenüber sind die autistischen und besonders die sozialen Kommunikationen in der menschlichen Gesellschaft wie im ganzen Tierreich weit verbreitet.

c) Die wichtigste Unterscheidung bezieht sich einerseits auf *gerichtete, adressierte*, andererseits auf *nicht-gerichtete, richtungslose* Kommunikationen. Diese Unterscheidung hat besonders vom entwicklungspsychologischen Standpunkt aus Bedeutung.

Die gerichtete Kommunikation setzt eine interindividuelle Beziehung voraus; es geht aus ihr die Tendenz hervor, den Empfänger des Rufes zu einer bestimmten Handlung zu veranlassen. Die ungerichtete Kommunikation unterscheidet sich von der gerichteten dadurch, dass bei ihr gerade diese *Tendenz* fehlt, durch gewisse Verhaltungen Einfluss auf bestimmte Individuen oder Gruppen auszuüben und zwischen Sender und Empfänger eine interindividuelle Beziehung, eine Art von Partnerschaft zu schaffen. Diese richtungslose Form der Verbindung kann dennoch rechtmässig als Kommunikation bezeichnet werden, da sie geeignet ist, durch der Art angepasste Mittel Tierindividuen bezw. Tiergruppen zu zweckmässigen und artfördernden Handlungen zu veranlassen.

Als Beispiele von gerichteten, adressierten Kommunikationen kommen in erster Linie die menschliche Sprache in ihren verschiedenen Erscheinungsformen (Laut-, Gebärden- und Zeichensprache), ferner einige zweckdienliche Laut- und Bewegungsäusserungen des kleinen Kindes in der vorsprachlichen Entwicklungsphase in Betracht ¹).

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¹) KARL BÜHLER hat in seiner „Sprachtheorie" (1934) die Sprache durch drei weitgehend unabhängige Variabeln charakterisiert, nämlich durch *Ausdruck*, *Appell* und *Darstellung*. (In seinen früheren sprachpsychologischen Arbeiten verwendete er die Termini: Kundgabe, Auslösung und Darstellung). Diese Merkmale sind besonders geeignet die Sprachsituation des Menschen darzustellen, dagegen nicht für eine Beschreibung der

Obgleich die an bestimmte Individuen bezw. Gruppen gerichteten kommunikativen Aeusserungen vornehmlich für die soziale Verbindung der Menschen bezeichnend sind, finden wir sie gelegentlich auch im Tierreich. Bestimmte Lautäusserungen und Körperhaltungen der Tiere zwecks Anlockung der Geschlechter, wozu die mannigfaltigen Stimmäusserungen und Balzstellungen der Vögel gehören, die Lockrufe des Muttertieres, ferner eine Anzahl von Lauten, Bewegungen und Körperhaltungen domestizierter Tiere und Affen dienen als Beispiele. Hierher gehören auch der durch Dressur und Gewohnheit entstandene Kontakt zwischen Mensch und Tier 1).

Allerdings ist es nicht immer leicht zu entscheiden, ob eine tierische Laut- oder Bewegungsäusserung gerichtet oder ungerichtet ist, ob eine tierische Reaktion als Folge einer gerichteten oder einer ungerichteten Aeusserung anzusehen ist. So wird sich z.B. in den meisten Fällen der sog. Warnungsruf, durch den eine ganze Herde zur Flucht veranlasst wird, als ungerichtete Kommunikationsform deuten lassen. In jedem konkreten Fall muss die Frage gestellt werden, ob die Lautäusserung wirklich einen Warmungsruf, ein Appell darstellt, um die Genossen von der Gefahr abzuwenden und sie zur Flucht zu drängen, oder geht es nur um einen nicht-gerichteten, nicht-adressierten Laut, etwa um einen Schrecklaut, der die Herde instinktiv zur Fluchtreaktion treibt. Meiner Ansicht nach hat die letztere Deutung eine grössere Wahrscheinlichkeit. Das Tier adressiert den Ruf nicht an seine Artgenossen, um ihnen gleichsam kundzugeben, dass eine Gefahr im Anzuge ist, sondern der Ruf stellt einen blossen Ausdruck seines *eigenen* Furchtzustandes dar, der bei den übrigen Mitgliedern der Herde triebhaft Schreck und demzufolge Fluchtreaktion auslöst. Welchen Eindruck auch die Verhaltungsweise des „signalierenden" Tieres und der Herde auf uns machen möge, es scheint kein zwingender Grund vorzuliegen, diese und ähnliche Fälle zu den adressierten Kommunikationen zu rechnen. Hat man die Gelegenheit einmal das ganze Schauspiel einer durch Schreckruf in Bewegung gesetzten Herde zu beobachten, so findet man es äusserst unwahrscheinlich, dass das lautgebende Tier

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Kommunikation im allgemeinen. Erstens schliesst der Begriff Appell bereits die Aufforderung zu einer Handlung in sich, was jedenfalls für die nicht-gerichteten Kommunikationsäusserungen nicht gilt, ausserdem hat dieses Wort einen zu imperativen Charakter (Befehl-Handlung). Zweitens bezieht sich das Moment der Darstellung nur auf die Sprache und nicht auf andere kommunikative Modalitäten. Schliesslich beschränkt sich das Schema BÜHLERs auf den Sender, die Reaktion des Empfängers wird dabei nicht berücksichtigt, wodurch die Gegenseitigkeit der kommunikativen Verbindung nicht recht zum Ausdruck kommt. Da BÜHLER sich nur die Aufgabe stellte, die Relationsfundamente des Sprachereignisses festzulegen, und nicht, wie ich, die Erscheinungsformen der Kommunikation im allgemeinen darzustellen, kann sein Schema für die Klassifikation der Kommunikationsformen nicht in Betracht kommen.

1) Die adressierte Kommunikation schliesst die Absichtlichkeit nicht notwendig in sich. Schon wegen des Fehlens der Sprachfunktion können wir bei Tieren nicht annehmen, dass die adressierten Laut- und Bewegungsäusserungen mit bewusster Zielvorstellung und vorstellungsmässiger Antizipation des Zieles verbunden sind.

die Absicht hätte, seine Genossen vor der Gefahr zu warnen. Sein ungestümes, rücksichtloses Benehmen während der Flucht weist nicht auf eine Partnerschaft, auf eine interindividuelle Beziehung.

Die sogenannte Fühlersprache der Ameisen, die vor allem dazu dient, den Speisebrei abzugeben, oder das Ausstossen eines kurzen Tones der wachehaltenden Bienen, der die Genossen des Stockes zum Ansammeln veranlasst, sind auch Fälle, die ich nicht zu den gerichteten Kommunikationsäusserungen rechnen möchte. Selbst der sog. Werbetanz der von einem ergiebigen Futterplatz zurückkehrenden Bienen, der die andern zum Ausflug nach dieser Stelle aufreizt, scheint nicht adressiert zu sein, obwohl die ausgezeichnete Beschreibung dieses Vorganges durch v. FRISCH leicht den Eindruck erweckt, als ob die „tanzenden" Bienen die Tendenz hätten, ihre Genossen von einem entdeckten Futterplatz zu verständigen [1]).

Es liegt nicht in meiner Absicht die Möglichkeit des Warnrufes bei Tieren in Abrede zu stellen. Ich stehe nur auf dem Standpunkt, dass solange keine unabweisbaren Argumente für adressierte, spezifisch gerichtete Laut- oder Bewegungsäusserungen vorliegen, man bei der Interpretation solcher Fällen die grösste Vorsicht walten lassen soll.

Die Frage, ob in einem konkreten Fall gerichtete Kommunikation vorliegt oder nicht, lässt sich am ehesten entscheiden, wenn man prüft, ob zwischen dem Sender und Empfänger ein *Kontakt* angenommen werden kann. Mit dem Wort „Kontakt" wollen wir die Wechselseitigkeit und die von *beiden* Teilen ausgehende Tendenz (Absicht) des Zusammenwirkens, des Aufeinander-Gerichtet-Seins, des Eingestellt-Seins bezeichnen. Mit Rücksicht auf die besonderen Entstehungsbedingungen und Auswirkungen der kommunikativen Aeusserungen wollen wir also vom Kontakt erst dann sprechen, wenn zwischen Individuen und Gruppen auf Grund einer *bereits vorhandenen, in mannigfaltiger Weise in Erscheinung tretenden wechselseitigen Beziehung mit Hilfe von zweckmässigen und auf die Art abgestimmten und von ihnen „verstandenen" Mitteln ein Zusammenspiel* zustande kommt.

Kontakt entsteht in erster Linie zwischen Individuen und Gemeinschaften von gleicher Art; aber auch zwischen artverschiedenen Wesen lässt sich gelegentlich diese Beziehung beobachten. Die Artungleichheit an sich schliesst das Entstehen des gegenseitigen Kontaktes keineswegs aus. Es ist nicht so selten, dass zwischen Haustieren, z.B. zwischen Hund und Katze kameradschaftliche Beziehung entsteht, die alle wesentlichen Merkmale des interindividuellen Kontaktes aufweist.

Die Frage, zwischen welchen gleichartigen oder ungleichartigen Individuen Kontakt zustande kommen kann, lässt sich nicht von vornherein beantworten. Besteht aber einmal ein Kontakt, so ist anzunehmen, dass dieser sich bei zahlreichen Lebenssituationen äussern wird. Weisen also Tiere nur eine einzige, bezw. eine sehr geringe Anzahl Kommunikationsäusserungen auf, dann müssen triftige Grüde vorliegen, um diese Kommu-

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[1]) K. v. FRISCH, Die Sprache der Bienen. 1923.

nikation trotzdem als gerichtet zu betrachten. Es ist nämlich sehr unwahrscheinlich, dass eine so tiefgreifende wechselseitige Beziehung, wie die des Kontaktes, in der Lebensweise der Tiere nur ganz spärlich in Wirkung treten sollte. Auch von diesem Standpunkte aus wird man z.B. die sog. Fühlersprache der Ameisen und die Lautäusserungen der Hunde nicht zu den adressierten Kommunikationen rechnen.

Der Kontakt wurzelt nicht immer im Biologischen. Beispiele liefern dafür die geistig fundierten Verständigungsformen der Menschen. Zuweilen lassen sich aus der Sphäre des rein-Biologischen herausragenden Kontaktbeziehungen auch bei Tieren feststellen. So ist z.B. der Kontakt zwischen gewissen Haustieren und Menschen nur zum Teil Resultat der Domestizierung, der Hauptsache nach Ergebnis der individuellen Gewohnheitsbildung und Erfahrung.

Ich lege ein ganz besonderes Gewicht auf das *Kontaktmoment*. Kommunikative Verbindungen mit und ohne Kontakt stellen die grössten Differenzen dar, die im Gebiet der Kommunikation überhaupt anzutreffen sind. Nur bei dieser Unterscheidung lassen sich die Formen der Kommunikation hinsichtlich ihrer Bedeutung für das Einzelindividuum, die ganze Verhaltungsweise der kommunizierenden Sozietätsmitglieder und die Möglichkeit einer Kommunikation zwischen Mensch und Tier verständlich machen.

2. *Die Sprache als spezifisch anthropologischer Begriff.*

Die angeführten Beispiele haben uns gelehrt, dass Kommunikationen zwischen artgleichen und artverschiedenen Individuen und Gruppen in verschiedener Weise entstehen können, nämlich gerichtet wie ungerichtet, absichtlich wie unabsichtlich, aktiv wie reaktiv, instinktiv wie geistig fundiert. Sie treten in verschiedenen Formen zum Ausdruck, als Bewegung, Körperhaltung, Lautäusserung, Gebärde und Sprache. Es ist deutlich geworden, das Kommunikation die Sprache grundsätzlich nicht voraussetzt, was sich schon daraus ergibt, dass es Kommunikationsformen gibt, die unabhängig von der Sprache entstehen und sich entwickeln. Die Sprachfunktion ist auch eine der Kommunikationsformen, selbst die höchstentwickelte, aber dann eine Kontaktform sui generis, mit besonderer Struktur und besonderen Funktionen und Ausdrucksmitteln ausgestattet.

Auf die morphologische und funktionelle Verschiedenheit der Sprache von den übrigen Kommunikationsformen muss besonders hingewiesen werden, da in der zoologischen und tierpsychologischen Literatur gelegentlich die Tendenz wahrzunehmen ist, die Sprache *als die allgemeine Kommunikationsform* zu betrachten, was zu Folge hat, dass man alle Laut- und Bewegungsäusserungen der Tiere, durch welche sie auf ihre Artgenossen und andere Tiere einwirken, unter dem Begriff der „Sprache" unterzubringen versucht [1]. Das kann so weit gehen, dass selbst die Verhaltungsvon der Sprache der *Tiere* spricht und diese von der der *Menschen* unter-

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[1] F. HEMPELMANN, Tierpsychologie, 1926, S. 529.

weisen bei der Paarung der Geschlechter als Formen der Sprache gedeutet werden.

Die Verwendung des Sprachbegriffes für Bezeichnung aller jener instinktiven Aktionen und Reaktionen, die den gegenseitigen Verkehr fördern und den gemeinsamen Interessen dienen, ist nicht nur unzweckmässig, sondern auch unstatthaft. Daran ändert nicht, wenn man nachdrücklich scheidet. Denn werden einmal die tierischen Kommunikationsformen unter den Oberbegriff „Sprache" gebracht, so besteht die Gefahr, zwischen tierischen und menschlichen Kommunikationsformen weitgehende Analogien zu statuieren und spezifische Funktionen der Sprache in die tierischen Kontaktformen hineinzuinterpretieren. Der Umstand also, dass ein Forscher ausdrücklich betont, dass die „Tiersprache" zur Unterscheidung der menschlichen Sprache den logischen Aufbau, die grammatikale Struktur, die bewusste Mitteilungsabsicht usf. entbehrt, genügt bei weitem nicht, um falschen Schlüsse vorzubeugen. Nicht selten begegnen wir Forschern, die an einer Stelle die menschliche Sprache von der sogenannten Tiersprache prinzipiell unterscheiden, an einer anderen Stelle ohne jede Zurückhaltung von einer „ausdrucksvollen und kräftigen Sprache des Männchens" und von einer „Laut- und Gebärdensprache bei Anlockung der Geschlechter" sprechen.

Die Anwendung spezifisch anthropologischer Begriffe überhaupt, wie z.B. die der Sprache, des Verstandes, des Willens in der Tierpsychologie ist nicht so unbedenklich, wie es uns auf dem ersten Blick erscheinen mag. Die Gefahren sind keineswegs dadurch aufgehoben, dass man sich dessen bewusst ist, dass der Begriff im übertragenen Sinne, metaphorisch verwendet wird. Man realisiert meistens nicht, dass man bei Anwendung von Lehnbegriffen leicht der Täuschung unterliegt, als ob zwischen dem Inhalt des ursprünglichen Begriffes und dem des Lehnbegriffes weitgehende Uebereinstimmungen bestünden. Die Macht der Wortidentität kann uns leicht zur Annahme der sachlichen Identität verleiten [1]).

Eine sprachpsychologische Grundtatsache ist es, dass jeder eindeutig definierte Begriff oder jedes im Sprachgebrauch eindeutig verwendete Wort jedem Versuch widerstrebt, seinen Umfang und Inhalt nach Bedarf zu modifizieren. Daraus folgt, dass ein Wort seine ursprüngliche Bedeutung auch dann durchsetzt, wenn es mit vollem Bewusstsein im bildlichen, im übertragenen Sinne verwendet wird. Trotz besten Vorhabens tritt die ursprüngliche Bedeutung in Bereitschaft, verschmilzt mit dem neuen Inhalt und beeinflusst unbemerkt unsere Ueberlegungen und Schlussfolgerungen.

Wie verderblich die Einführung des Begriffes der Sprache in die Tierpsychologie sein kann, lässt sich nicht besser demonstrieren als durch Hinweis auf Fragestellungen, die gerade dadurch entstanden sind, dass man aus dem Begriff „Tier*sprache*" solche Konsequenzen zog, die nur aus

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[1]) Vergl. dazu die Ausführungen in meiner Arbeit: Gibt es einen Hörraum? in den Acta Psychologica, 1937, Band III, S. 137.

dem ursprünglichen Begriff der Sprache folgen. Eine ganze Literatur entstand, die sich mit vollem Ernst zur Aufgabe stellte, die „Sprache" der Tiere zu untersuchen, mit dem Zwecke, diese zur menschlichen Sprache in Beziehung zu bringen. Dabei sind die Forscher nicht bei den autochthonen tierischen Kommunikationsformen, bei den spezifischen Laut- und Bewegungsäusserungen der Tiere stehen geblieben, — was wenigstens darauf hingewiesen hätte, dass sie das Wort „Sprache" tatsächlich in einer ganz anderen Bedeutung verwenden wie die Sprachwissenschaft. Sie haben sich durch die Macht des Wortes so sehr beeinflussen lassen, dass sie gewissen Tieren das Verständnis der menschlichen Lautworte zuerkannten, ja sogar die prinzipielle Möglichkeit des Anlernens der menschlichen Sprache — wenn auch in sehr beschränktem Masse — ernstlich erwogen. Immer wieder wurden Beobachtungen, Erfahrungen, experimentelle Ergebnisse mitgeteilt, die in mehr oder weniger verdeckter Form die Ansicht zum Ausdruck brachten, dass zwischen den tierischen und menschlichen Kommunikationsformen ansehnliche Uebereinstimmungen bestehen und dass der Unterschied zwischen Mensch und Tier in dieser Beziehung nicht so einschneidend ist, wie das von philosophischer, teilweise auch von psychologischer Seite aus dargelegt wird.

Meiner Ansicht nach sind auch die Hypothesen über die vermeintlichen Ansätze und Uebergangsformen der Sprache [1], die angeblich in den Warn- und Lockrufen, in Lautimitationen u.s.w. ihren Niederschlag finden, Folgen der unberechtigten Anwendung des Begriffes „Sprache". In dieser Weise werden gleichsam durch eine Zauberformel Tätigkeiten, die an sich mit dem Sprachakt nichts zu tun haben, mit der Sprache eng verbunden und geben zu irrtümlichen Lehren Anlass.

Nach diesen Auseinandersetzungen wird es niemanden überraschen, dass die Ansichten über die Kommunikationen in der Tierwelt äusserst schwankend und unklar sind. Bei der Uneinigkeit und Undeutlichkeit der Ansichten und wegen der dabei entstandenen Begriffsverwirrung schien es mir nicht überflüssig zu sein, auf diese Fragen einzugehen, und durch Klärung der Probleme das Fundament für eine befriedigende Lösung zu legen.

3. *Das Wesen der Sprache.*

Die erste und wichtigste Frage, über die wir uns in diesem Zusammenhang entscheiden müssen, ist des leitenden *Gesichtspunktes* und der *Methode*.

Zunächst müssen wir uns darüber klar werden, dass die Frage nach der sog. Tiersprache ausschliesslich auf Grund tierpsychologischen Tatsachen nicht gelöst werden kann. Jeder, der unvoreingenommen die seitens der Tier- und Entwicklungspsychologen aufgestellten Thesen und Theorien einer kritischen Betrachtung unterzieht, muss schliesslich zu der Ueber-

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[1] G. RÉVÉSZ, Het probleem van den oorsprong der taal. Nederl. Tijdschr. v. Psychologie, VIII, 1940.

zeugung kommen, dass die aufgeworfene Frage durch Aufzeigen der verschiedenen tierischen Kommunikationsformen und durch Hinweise auf gewisse Dressurleistungen, die ihrerseits die widersprechendsten Deutungen zulassen, mit logischer Gewissheit nicht zu beantworten ist. Man muss demnach trachten einen logisch rechtmässigen Ausgangspunkt zu finden, von dem die Erfahrungstatsachen eine natürliche und sinnvolle Deutung finden können. Dieser Ausgangspunkt ist in der *Begriffsbestimmung der Sprache* zu finden. Von da müssen wir ausgehen, selbst auf die Gefahr hin, dass unsere theoretischen Ueberlegungen vielleicht nicht durchwegs den Erfolg haben werden, die gegenteiligen Ansichten zu einer Revision zu zwingen. Nimmt man sich aber die Mühe die sog. Tiersprache vom sprachphilosophischen und sprachpsychologischen Standpunkte aus zu sehen, — meiner Ansicht nach die einzig berechtigte Stellungsnahme, — so wird man nicht bloss die Lehre von einer Tiersprache aufgeben, sondern zugleich die Widersinnigkeit der Problemstellung in ihrer heutigen Fassung in ihrem ganzen Umfang einsehen.

Der Begriff der Sprache ist dogmatisch nicht festgelegt. Die Entstehung einer Philosophie und Psychologie der Sprache hat eine notwendige und zugleich fruchtbare Erweiterung des Sprachbegriffes nach sich gezogen. Daraus erklärt sich, dass bezüglich der Definition der Sprache, trotz weitgehender Uebereinstimmung der Auffassungen, keine Einigkeit erzielt ist. Prinzipielle Argumente lassen sich gegen eine Erweiterung des Begriffes nicht anführen, solange dieses Streben nicht zu Ausschaltung unerlässlicher Merkmale der Sprache führt. Obgleich bei den verschiedenen Definitionen der Sprache das Gewicht einmal auf das eine, dann auf das andere Merkmal gelegt wird, ist es nicht zweifelhaft, dass es Eigenschaften gibt, die notwendig zu dem Inhalt einer jeden Definition der Sprache gehören müssen. Für unseren Zweck genügt, wenn wir von einer Begriffsbestimmung ausgehen, in der keine der konstitutiven Eigenschaften und Funktionen der Sprache fehlt, die aber auch keine unwesentliche in die Definition aufnimmt, folglich nicht mehr und nicht weniger umfasst als das, was der Sinn und die Bedeutung der Sprache notwendig fordert. Ich glaube, dass die von mir gegebene Definition diesen Forderungen entspricht.

Unter Sprache verstehen wir die Funktion, durch die wir mit Hilfe einer Anzahl von gegliederten (artikulierten) und in verschiedenen Sinnverbindungen auftretenden Laut- bezw. Bewegungs- oder Zeichengebilden (Wörter im erweiterten Sinne) unsere Wahrnehmungen, Gedanken, Urteile, Wünsche darzustellen und in der Absicht gegenseitiger Verständigung anderen mitzuteilen imstande sind [1]).

Diese Definition bezieht sich auf alle Arten der Sprache (Lautwort-, Gebärden-, Zeichen- und Schriftsprache), die jede ihre eigene sinnlich-anschauliche Ausdrucksform und ihre eigene Struktur hat. Auch umfasst sie

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[1]) G. RÉVÉSZ, Die Sprache. Proc. Ned. Akad. v. Wetensch., Amsterdam, **43**, N⁰. 8 (1940).

notwendig alle fundamentale Funktionen der Sprache, wie die benennende, kundgebende (oder mitteilende), signifische, grammatikalisch-logische Funktion. Demgegenüber schaltet sie alle Aeusserungsformen aus, die der in der Definition aufgenommenen Merkmale entbehren. Demzufolge kann eine Aeusserungsform, die keine gegenseitige Verständigung beabsichtigt, wie z.B. die natürlichen affektbetonten Ausdrucksbewegungen nicht als Sprache gelten. Auch die differenzierten Stimmäusserungen der Vögel und der Säugetiere, die keine Sinnverbindungen bilden, entsprechen den notwendigen Kriterien einer Sprache nicht. Dasselbe gilt von den Lauten und Bewegungen des kleinen Kindes in der praelingualen Periode, soweit sie unartikuliert sind und keine Darstellungsfunktion haben und nicht mit der Absicht gegenseitiger Verständigung erfolgen.

Man kann vier *Spracharten* unterscheiden: nämlich die Laut- und Gebärdensprache und die Schrift- und Bildsprache. Die natürlichen und autochthonen Spracharten sind die Laut- und natürliche Gebärdensprache, während die Schrift- und Bildsprache aus den beiden ersteren, vornehmlich aber aus der Wortlautsprache entstanden sind.

Das adäquateste und reichhaltigste Mittel der Ausdrucks und des geistigen Kontaktes stellt die *Lautsprache* dar; alle Grundfunktionen der Sprache sind an sie gebunden und die ganze geistige Tradition des Volkes ist in ihr eingeschlossen. — Die *Gebärdensprache* bildet eine Sprache sui generis, die unmittelbar aus der Sprachfunktion, ohne deutlich nachweisbare Mitbeteiligung der Lautsprache entstanden ist. Sie weist eigene Struktur- und Entwicklungsgesetze auf. — Die *Schriftsprache* ist indessen ein Derivat der Lautwortsprache, eine einfache Uebertragung dieser Sprachform in ein System von Zeichen, ganz ähnlich wie die *Fingersprache* in Bewegungen. Es liegt daher auf der Hand, dass die aus der Lautsprache transformierten Finger- und Schriftsprache ganz dieselbe grammatikalischen Kategorien wie die Lautsprache besitzen, und in ihrem Aufbau und ihrer Ausdrucksmöglichkeit grundsätzlich mit der Lautwortsprache übereinstimmen.

Auch die *konventionellen Gebärdensprachen*, wie die der nordamerikanischen Indianer [1]), der Napolitaner [2]) oder der Taubstummen müssen als Sprachen im eigentlichen Sinne des Wortes gelten. Sie sind zum Teil in Anlehnung an die gesprochene Sprache entstanden, zum Teil durch Uebereinkunft festgelegt und rationell weitergebildet. Sie werden im Sinne der Lautsprache und der durch Anschauung gewonnenen Vorstellungsschemata begriffen, Erwähnenswert ist, dass jede konventionelle Gebärdensprache, soweit sie ein in sich geschlossenes Sprachsystem repräsentiert, sich bis zu einem gewissen Grade aus eigener Kraft zu entwickeln vermag. Sie ist imstande gelegentlich solche Geschehnisse und Relationen zum Ausdruck zu bringen, die in der Lautsprache ihr Adäquates nicht oder noch

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[1]) G. MALLERY, Sign Linguage among North American Indians. 1881.

[2]) ANDREA DE JORIO, La mimica degli autichi investigata nel gestire napolitano. 1832.

nicht hat. (Beispiele liefert dafür die Gebärdensprache der Indianer und Araber [1]).

Eine besondere Stellung nimmt die *Bildsprache* (Piktographie) ein, da sie die lebendige Sprache nicht in ihrer morphologischen und grammatischen Struktur, sondern nur ihrem *Sinne nach* bildhaft darstellt. Die Bildsprache lehnt sich nicht, wie die Finger- und Schriftsprache der Lautsprache an, und ist in ihrer Struktur und Darstellungsweise von der gesprochenen Sprache bei weitem nicht so abhängig wie die natürliche Gebärdensprache. Die Bildsprache ist eine sprachliche Mitteilungsform eigener Art, die durch das Bildhafte unmittelbar Sinn erhält und verstanden wird. Sie wird vielfach in die lebendige Sprache übertragen, um verstanden zu werden; aber auch ohne Uebertragung lässt sie sich begreifen.

Demgegenüber stellen die *Signale*, die lautlichen Zeichengaben der Bergvölker, die Trommelsprachen der Naturvölker keine Sprachen dar. Aus dem Umstand, dass das Signal „rot" als „verbotener Zugang" oder ein bestimmter Trommelrhythmus als Aufforderung zum Kampf verstanden wird, folgt keineswegs, dass ein mehr oder weniger ausgebildetes Signalsystem der Sprache gleicht. Es handelt sich hier stets um willkürlich gewählte Zeichen, denen man nach Uebereinkunft diese oder jene Bedeutung erteilt, ganz ähnlich wie man etwa in der Physik mit σ 1/1000 Sec. oder mit π eine bestimmte Zahl bezeichnet. Der Unterschied zwischen Sprache und Signalsystem kommt besonders darin zum Ausdruck, dass die Signale dem Verlauf der in sprachliches Gewand umgesetzten Gedankengänge nicht folgen, jeglicher Struktur entbehren, starr sind und sich aus eigener Kraft nicht zu entfalten vermögen. Um verstanden zu werden, muss jedes Signal für sich bekannt sein: eine Ableitung des einen aus dem anderen ist nicht möglich. Ein Signalsystem tritt dadurch zu der Sprache in Beziehung, da es Sachverhalte verdeutlicht, die gewöhnlich in der Sprache ihren Ausdruck finden.

Eine besondere Rolle müssen wir im System der Verständigungsmittel den *Interjektionen* und *mimischen Bewegungen* zusprechen. Auch sie bilden keine Sprache, aber werden in die Sprache aufgenommen, insofern sie ausser Ausdruckswert noch symbolische Bedeutung haben. Es ist hinlänglich bekannt, dass diese Aeusserungen in bestimmten Situationen die Wörter vertreten können, aber man weiss auch, dass zu diesen Verständigungsmitteln nur derjenige greifen wird, der selber spricht, und diese Zeichen nur derjenige begreifen kann, der selber der Sprache fähig ist. Interjektionen und mimische Bewegungen sind demnach unselbständige Elemente der Laut- bezw. Gebärdensprache, die nur im Rahmen der Sprache Bedeutung haben.

Die Analyse der Sprachformen zeigt, dass nur eine einzige Sprachart existiert, die allen Anforderungen des gegenseitigen Verständnisses und

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[1] I. GOLDZIEHER, Ueber Gebärden- und Zeichensprache bei den Arabern. (Z. f. Völkerpsychol. u. Sprachwiss. Bd. 16).

Kontaktes gerecht wird, und das ist die Lautwortsprache. Ausser dieser müssen auch noch die Gebärden- und Bildsprachen als Sprachen von eigener Art und besonderer Ausdrucksfähigkeit betrachtet worden, da sie durch eine Anzahl von individuell gestalteten und in verschiedenen Sinnverbindungen wiederkehrenden „Zeichen" unsere Erlebnisse, Gedanken, Wünsche begrifflich ¹) oder bildhaft darzustellen und mitzuteilen befähigt sind.

Im Zusammenhang mit diesen Betrachtungen über die Sprache treten unvermeidlich Fragen auf, deren Lösung für unsere Auffassung betreffs der Stellung des Tieres im Kosmos und der Beziehung des Tieres zum Menschen von nicht zu unterschätzender Bedeutung ist.

Die erste dieser Fragen lautet: Gibt es Tierarten, die von Natur aus die Fähigkeit besitzen die menschliche Sprache zu verstehen und gar solche, die artikulierte Lautworte auszusprechen und in ihrer Bedeutung zu erfassen imstande sind? Anschliessend an diese Frage muss geprüft werden, ob Tiere durch Gebärden mit anderen Lebewesen einen Kontakt zustande bringen.

Ist der Nachweis erbracht, dass Tiere sowohl von der aktiven wie von der passiven Sprachtätigkeit ausgeschlossen sind, so entsteht die weitere Frage, ob sie über eigene, autochthone Verständiqungsformen verfügen, und zwar über solche, die verwandtschaftliche Züge mit der menschlichen Sprache aufweisen.

Gelingt es uns nicht, die Existenz solcher autochthonen Tiersprachen wahrscheinlich zu machen, so bliebe noch die Frage zu behandeln, in welcher Weise, durch welche Mittel Tiere zueinander und zu den Menschen in Kontakt treten können.

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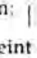
¹) Es ist möglich Gebärden oder andere anschauliche Zeichen an Stelle des Lautwortes zu setzen und ihnen einen begrifflichen Inhalt zu geben. Es kommt nicht auf das äussere Phänomen an, sondern auf den Inhalt und die Bedeutung. So kann ein konventionelles Zeichen oder ein Bild einen Begriff vertreten. Das Wort „Haus" ist ein Begriff, aber ebenfalls das Zeichen:  da es das Haus im allgemeinen und nicht exemplarisch darstellt. Allerdings scheint uns dieses Zeichen einen begrenzteren Umfang zu haben als das Wort „Haus", welches die Behausungen aller Art in sich fasst. (Vergl. dazu KANT's Lehre vom Schema in der Kritik der reinen Vernunft.)

Medicine. — *Biological Properties of Aethinyl-testosterone* 1). By L. A. M. STOLTE. (From the Department of Pharmacology, University of Leiden. Director Prof. Dr. S. E. DE JONGH.) (Communicated by Prof. J. V. D. HOEVE.)

(Communicated at the meeting of October 26, 1940.)

I. *Progesterone-like properties.*

In 1938 INHOFFEN and HOHLWEG (1) prepared a substance, that may be called aethinyl-testosterone, pregneninolone or anhydro-oxyprogesterone, according to the view one has upon the structural formula. The most remarkable property of the substance is its progesterone-like activity upon the endometrium of the immature rabbit not only after subcutaneous but *also* after *oral* administration. Progesterone being orally inactive, the new substance immediately gained importance for the gynaecological praxis. In the course of 1939—1940 2) we investigated, how far aethinyl-testosterone can be considered to be biologically identical with progesterone. We further compared the quantities required of both substances, in order to obtain a certain effect. Aethinyl-testosterone indeed proved to possess a number of properties that may be called progesterone-like. Besides, we could show (and in view of the chemical nature of the substance, this was not so very surprising), that it bears testosterone-like properties and finally it proved unexpectedly to exert certain oestrogenic activities.

Other publications have appeared already in literature, in agreement with our findings: RUZICKA c.s. (2) and EMMENS & PARKES (3) also found a progesterone-like effect upon the endometrium of the immature rabbit.

COURRIER & JOST (4) and EMMENS & PARKES described a testosterone-like influence upon the cock's comb and upon the sex accessories in the male rat.

The former investigators could maintain the pregnancy of castrated rabbits with subcutaneous injections of aeth.-test. in high dosages. The latter authors found that aeth.-test. after subcutaneous as well as after oral administration produces oestrus in castrated rats.

We intend to describe in 2 papers 1. the progesterone-like and 2. the testosterone-like and oestrone-like properties of aeth.-test.

A few times aeth.-test. was administered in oily solution (max. 1 mgm pro 3 ccm); mostentimes it was emulgated in 1 ccm water with 200 mgm gummi arabicum. For the oral administration a stomach tube was used.

a. CLAUBERG-*test.* Our research pertaining to the relation, as reported by INHOFFEN and HOHLWEG ($^{2}/_{3}$ mgm progesterone subcutaneously ∞ 2 mgm

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1) For further details and literature vid. diss. Leiden 1940.

2) Organon Ltd generously supplied us with aethinyl-testosterone.

aeth.-test. subc. ∞ 4 mgm aeth.-test. orally = 1 K E) was confined to only some experiments.

In as much as the small number of animals allowed any conclusions, we got the impression that in our experiments the relation was 1 : 4 : 8—10 instead of 1 : 3 : 6.

3 mgm aeth.-test. (subcut. in emulsion) caused a ++/+++ effect in one rabbit; for the same result 6—8 mgm aeth.-test. (orally in emulsion) were required. Conversely $^2/_3$ mgm progesterone subcut., already caused a +++ effect.

Administration in oily solution or in an alcohol-water suspension of aeth.-test. (as used by INHOFFEN c.s.) gave no better results.

In order to allow a comparison, the effect of 10 mgm test. propr. and of 20 mgm testosterone was investigated. The result was resp. +/++ and O/+. Herewith it had become improbable that the progesterone-like activity of aeth.-test. upon the endometrium is due to the testosterone-like properties (KLEIN and PARKES) (5) of this substance.

In our further experiments we based ourselves upon the following relation:

progesterone subc. : aeth.-test. subc. : aeth.-test. orally 1 : 4 : 8—10.

b. Formation of deciduomata.

The experiment was done as described by SHELESNYAK (6): 16 immature rats (ca 3 weeks of age) were first treated with 2.5 γ oestrone during 4 days daily in 2 injections of 0.1 cc oil each.

The animals then were divided over 4 groups and received resp. 0.2 mgm progesterone subcut., 1 mgm aethinyl-testosterone subcut., 2.5 mgm aeth.-test. orally and 0.25 mgm testosterone subcut., daily for 7 days in 2 dosages a day.

The 8th day after the beginning of the experiment a silk thread was drawn through one of the uterine horns in longitudinal direction. Autopsy the 12th day.

Macroscopically all hurted horns proved to be unevenly thickened over the total length; distinctly isolated tumors could not be shown.

Microscopically circumscript tumor-like structures, consisting in decidual cells, were found in the endometrium of the animals, treated with progesterone and in these only. Though in the other groups a scattered decidual reaction existed (but in the intact horns as well!), it could be shown by staining with Gieson that connective tissue was involved in the process, whereas in the real deciduomata in the progesterone animals no such tissue could be detected. Conclusively, it is possible to make deciduomata appear with the aid of crystalline progesterone in immature rats, that underwent a precursory treatment with oestrone.

Neither the 4-fold quantity of aethinyl-testosterone (subcut.), nor the 10 fold quantity of aeth.-test. (orally), nor finally the equal quantity of testosterone (subcut.) proved to be able to effect the same.

On the base of dates in literature, concerning oestrone and testosterone in combination with progesterone, this failure is possibly due to the oestrogenic properties of aethinyl-testosterone still to be described. (Oestrone + progesterone, vid. COURRIER (7), SELYE and MC KEOWN (8)). On the other hand a disturbing influence of the testosterone-like component must be considered improbable (testosterone + progesterone, BROOKSBY (9)).

c. *Influence upon the sensibility of the rabbit uterus to oxytocine.*

The influence upon the uterine motility was researched upon 3 castrated rabbits with abdominal window.

Rabbit I received 100 int. U oestrone daily during 12 days. With 0.01—0.001 Voegtlin U of Piton [1]) a distinct increase (when compared with "spontaneous") of the total duration of the uterine contractions could be obtained.

After the subcutaneous administration of daily 100 int. U of oestrone, combined with 4 mgm aeth.-test. orally, the threshold of the sensibility to Piton went up to 1—0.1 V U. A 9 days treatment with solely oestrone was followed by a decrease to 0.001 V U.

In rabbit II the threshold was 0.01—0.001 V U Piton after a treatment with 100 int. U oestrone daily during 10 days. After 100 int. U oestrone + 16 mgm aeth.-test. (subcut.) on 5 subsequent days it had become 10 V U; after a 11 days treatment with oestrone only, it went down as far as to 0.001—0.0001 V U.

The combination of the oestrone with 0.4 mgm progesterone made the threshold rise to 0.01—0.001 V U in 6 days.

In rabbit III a threshold of 0.001 V U was reached after 7 subsequent daily injections of 100 int. U oestrone. After 5 following days of 100 int. U oestrone + 10 mgm aeth.-test. (orally) the threshold increased to 1—0.1 V U. After 100 int. U oestrone during 9 days it had diminished to 0.1—0.01 V U; after 100 int. U oestrone, combined with 5 mgm aeth.-test. (subcut.) it ascended to 1—0.1 V U.

This showed, that aeth.-test., subcutaneously administered as well as orally, is able to lower the sensitiveness of the rabbit uterus to oxytocine. However, the uterine contractions make a very uncoordinate impression, when under the influence of aeth.-test. The uterus has a "spotty" appearance and the contractions don't proceed regularly over the surface of the organ, as is the case under the influence of progesterone. Furthermore it was observed that under the influence of aeth.-test. the spontaneous motility decreases only little or not at all. This too is contrarious to what has been reported in literature (VAN WOERKOM (10), REYNOLDS c.s. (11)) on progesterone.

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[1]) "Piton" (Organon) is an oxytocically active preparation manufactured from the posterior pituitary lobe.

d. *Maintenance of Pregnancy after castration.*

1. Pregnant *rats* were bilaterally castrated on the 5th, 10th or 14th day after sperm had been shown in the vaginal smear. To begin with the day before castration the animals were treated with resp. 1.25 mgm progesterone, 5 mgm aeth.-test. (subcut.) and 10 mgm aeth.-test. orally, all in 2 portions daily unto the 18th—20th day, whereupon autopsy was performed on the following day in order to establish wether the fetusses were present and if so to count the number of the living and the dead ones. A control group was solely treated with gummi arabicum.

TABLE I.

Day of castration	Daily dosage			
	G. arab.	1.25 mgm Prog.	5 mgm aeth. test. subc.	10 mgm aeth. test. orally
5th	4 rats —, —, —, —	3 rats +, +, —	3 rats —, —, —	3 rats —, —, —
10th	3 rats —, —, —	5 rats ++, ++, +, —	5 rats —, —, —, —, —,	—, —, —
14th	3 rats ++, +, +	3 rats ++, ++, ++	3 rats ++, +, —	++, +, +

++ Living fetusses.
\+ Dead fetusses or placent. remainders.
— absence of any remainder.

As the table (I) shows, the effect of aeth.-test. orally or subcutaneously administered, is inferior by far to that of progesterone in view of the "maintenance" of pregnancy, at least when the substances are given in the aforementioned quantitative relation of 8 : 4 : 1. At autopsy living fetusses had been found exclusively in those animals in which castration had taken place on the 14th day; the same finding, however, in the (castrated) controls, treated with nothing but gummi arabicum.

The evaluation of the results in the groups, castrated on the 5th day, is complicated by the fact, that the demonstration of sperm in the vaginal smear is not the *absolute* proof, that pregnancy has existed. The nidation of the ovum does not take place in the rat but on the 5th day, sothat at the time of castration the diagnosis of an eventual pregnancy cannot yet be delivered.

The "placental sign", i.e. the appearance of blood in the vaginal smear — what occasionally happens on the 8th day of pregnancy already — is another sign of pregnancy. (The absence hereof, however, does not exclude pregnancy; the latter may have come to an end before the 8th day). In 2 of the animals of each group, castrated on the 5th day this sign was present, so that at least in these cases pregnancy has certainly existed.

2. Six pregnant *rabbits* were bilaterally castrated on the 13th or 15th day. From the day before castration on, the animals received twice daily aethinyl-testosterone orally or subcut. during one week. A 7th rabbit, (also

pregnant and castrated) received 3.125 mgm *progesterone* daily. On the base of experiments reported in literature (COURRIER c.s. (12), ALLEN c.s. (13), PINCUS c.s. (14)) we expected this quantity to be amply sufficient to maintain the pregnancy of this animal. This appeared to be true: at autopsy, performed on the 7th day after castration, in both horns totally 5 living embryos were found, no dead ones, no placental remainders.

With aeth.-test., in the quantities as applied by us, we never succeeded in keeping an embryo alive. Neither the 4-fold of the active dose of progesterone, viz. daily 12.5 mgm aeth.-test. subcutaneously administered (2 rabbits), nor the 10-fold, viz. daily 31 mgm aeth.-test., orally administered (2 rabbits) proved to be able to procure the pregnancy of the castrated rabbit.

Even an increase of the dosage unto as much as 30 mgm subcut. (1 rabbit) and 80 mgm orally (1 rabbit) failed to yield any result, except placental remainders and, sometimes recognizable, fetusses.

These findings are not in disagreement with those of COURRIER's and JOST's. In their experiments 3 rabbits, castrated on the 4th day of pregnancy, received 5—10 mgm aeth.-test. orally daily; 2 other animals 20 mgm aeth.-test. daily by way of subcut. injection. Both dosages proved to be insufficient to maintain pregnancy.

A 2nd group of 4 rabbits received 40 mgm aeth.-test. subcut. daily and besides 10 mgm orally. Two of them were castrated on the 4th day and killed on the 10th or 12th. 4, resp. 8 nidated ova were found. The 2 other animals were castrated on the 12th day. One animal was killed 10 days later. Result: 1 living embryo and 5 on the way of resorption. The remaining animal was killed on the 17th day already: 9 normal fetusses were found. These experiments show that 40 mgm aeth.-test. subcut. and 10 mgm aeth.-test. orally, both daily administered, are able to make the nidation of the ovum possible in the castrate rabbit.

In one animal the authors could keep the fetus alive with this dosage during 5 days.

Our highest dosages were lower than the one with which COURRIER succeeded; the latter may probably be seen as a limit quantity.

This implies, that the relation between progesterone and aeth.-test. as found with the CLAUBERG test (1 : 4 : 8—10) does *not* pertain to the capacity to maintain pregnancy. COURRIER, when using aeth.-test. had to give the 15-fold of the active progesterone dose subcutaneously in order to keep the rabbit fetus alive. No details exist, concerning the required "oral" quantity.

e. *Counteraction of oestrone in male mice: suppression of paradoxical effect.*

It appeared from experiments by DE JONGH c.s. (15), that among the changes in the accessory reproductive organs of the spayed, adult mouse brought about under the influence of oestrone, one is specially counteracted by progesterone, namely the strong development of muco-reticular connective tissue around the ampulla ducti deferentis. With the aid of the qualitative and quantitative dates, resulting from these experiments, we investigated the influence of aeth.-test., orally administered, upon the aforementioned paradoxical oestrone effect. Spayed adult male mice were injected with 1 γ oestrone, twice daily, for 3 weeks. One-half of the animals received besides daily aeth.-test., emulgated in gummi arabicum.

The rank growth of connective tissue around the ampulla after the injection of 2 γ oestrone daily for 3 weeks can be counteracted with 200 γ progesterone, subcutaneously given. The quantities of Aeth.-test., used by us and administered by way of a stomach tube, were 6—10 times as high, viz. 1.2—2 mgm daily.

The daily application of the stomach tube (in preliminary experiments even twice daily!) implies a severe trauma and caused a high mortality among our mice, so that, from 3 separate experiments, only 9 animals remained, that had been treated with a combination of oestrone and aeth.-test. and that could be compared with an equal number, treated with oestrone only. From these experiments it appeared, that the growth of the periampullar connective tissue could be slightly inhibited with 1.2 mgm aeth.-test. *per os* daily and distinctly with 2 mgm.

In view of this counteraction, the relation of the effective doses progesterone — aeth.-test. *per os* corresponds rather well with the one, found with the CLAUBERG-test.

f. Suppression of the oestrone-oestrus in female mice.

DE FREMERY, KOBER and TAUSK (16) reported, that in castrate, adult mice the oestrus, produced with 7 int. U oestrone (min. dosage for *oral* oestrus effect), was suppressed if 3 rabb. U oestronefree, non crystalline corpus luteum hormone ("progestine") was simultaneously injected (1 U: insufficient; 2 U: not experimented with).

In similar experiments we now investigated the influence of aeth.-test., orally given, on the oestrus, produced with a limit dose of oestrone, subcut. injected.

Castrate, female mice received 0.1 γ or 0.12 γ oestrone, divided over 3 injections (in 0.1 cc oil). One half of the animals received besides also in 3 injections 12 or 20 mgm aeth.-test. *per os* (stomach-tube). Vaginal smear every day, 3rd and 4th day twice daily. After a week the experiment was repeated, with interchange of the groups, in view of the given substances.

With 12 mgm aeth.-test. neither the oestrus effect of 0.1 γ nor that of 0.12 γ oestrone could be counteracted. In 3 experiments with groups of 5 animals, 8 had reached a positive oestrus index ("*e, f. g.*") at the moment of their maximal reaction [1]), under the influence of 0.1 γ oestrone, while the combination with 12 mgm aeth.-test. *per os* yielded 9 positive signs. In the experiment in which 0.12 γ oestrone had been used, the results even were resp. 4 and 11! The significance of these findings will be dealt with more closely in our next paper on the (oestrogenic) activity of aeth.-test.

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[1]) Details about this index: S. E. DE JONGH, E. LAQUEUR, P. DE FREMERY, Biochem. Zschr. **250**, 1932.

With 20 mgm aeth.-test. orally a cross-test was done (2 series, each of 4 animals): with this quantity it proved to be possible to suppress the oestrus effect of 0.12 γ oestrone. The administration of solely 0.12 γ oestrone yielded 8 positive signs and only 2, when combined with 20 mgm aeth.-test *per os*. So 20 mgm aeth.-test. *per os* corresponds with 3 rabb. U progestine, (vid. the above research of DE FREMERY c.s.), as far as the inhibition of the artificial oestrus is concerned. When expressed in rabbit U (CLAUBERG test) aeth.-test. corresponds in our experiments with ca 2—3 R. U, so that a satisfying accordance has been found. However, besides a progesterone-like activity, aeth.-test. has testosterone-like properties and it is possible to counteract the oestrogenic effect of oestrone with testosterone also. For the discussion of the possibility, that the inhibition of the oestrone-oestrus with aeth.-test., rests upon a testosterone-like activity of aeth.-test., the reader is referred to our paper, dealing with this mechanism.

Summary.

The possibility of progesterone-like properties of aethinyl-testosterone (anhydro-oxyprogesterone, pregneninolone), subcutaneously or orally administered, has been studied. Mainly the following results were obtained:

1. In the CLAUBERG test (pregravid changes in the rabbit uterus) the effect of aeth.-test., subcutan. given, is ca ¼ that of progesterone and ca $^1/_8$—$^1/_{10}$ after oral administration.

2. It appeared to be impossible to provoke deciduomata in the immature rat and to maintain pregnancy in the rat and the rabbit after castration with aeth.-test. subcut. and aeth.-test. *per os*, resp. in dosages of 4 and 8—10 times the quantity, sufficient for progesterone.

3. Dates from literature (COURRIER) as well as our own experiments lead to the assumption that for the maintenance of pregnancy in the rabbit dosages of aeth.-test. are required, 15 times (subcut.) and 30—40 times (*per os*) the quantity sufficient for progesterone.

4. Aeth.-test., subcutaneously or orally administered, is capable of lowering the *sensitivity for oxytocine* of the rabbit uterus in the same quantitative relation, when compared with progesterone, as has been found with the CLAUBERG test.

5. Aeth.-test. *per os* is capable of inhibiting the effect of oestrone in castrated, male mice (paradoxical effect round ampulla ducti deferentis) and in castrated female mice (oestrus) in the same quantitative relation, when compared with progesterone, as has been found with the CLAUBERG test.

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Sérologie. — *Sur un nouveau phénomène observé dans la réaction de fixation.* (Deuxième mémoire.) Par ONG SIAN GWAN. (Communicated by Prof. E. GORTER.) *)

(Communicated at the meeting of September 28, 1940.)

13. *Mécanisme du phénomène étudié.* On pourrait l'expliquer comme suite. Il existe dans le sang normal ou pathologique des sensibilisatrices antispermatozoïdes 1) et tuberculeuses. Comme les anticorps sont des protéines et par conséquent des grosses molécules, il se peut q'une même molécule d'anticorps fixerait en même temps deux antigènes différents (fig. 1—C). Le complexe ainsi formé fixerait une quantité d'alexine plus élevée que celle obtenue par la somme des complexes anticorps-antigène spermatozoïde (fig. 1—A) et anticorps-antigène tuberculeux (fig. 1—B).

TABLEAU 7. Immunsérums.

Sérums	S	T	ST	ST − (S + T)
Sérum antitétanique, cheval R1	10 *	10	15 *	− 5
" " " I.P.	10	10	10	0
" " " 182	0	0	10	+ 10
" " " R3	25	25	150	+100
" " , lapin 1	0(0)	0(5)	0(35)	0
" " " 2	0	0	35	+ 35
" " " 3	0(0)	0(0)	0(30)	0
" " " 4	0(0)	0(0)	30(35)	+ 30
" " " 6	10(0)	10(0)	15(35)	− 5
" antidiphtérique, cheval 182	20	10	35	+ 5
" " " I.P.	0	0	15	+ 15
" antistreptococcique polyvalent, cheval I.P.	0	10	15	+ 5
Sérum anticolibacillaire polyvalent, lapin D7	20	20	35	− 5

Les chiffres entre parenthèses indiquent les résultats de réaction avant l'immunisation.
* Réactions effectuées avec l'antigène spermatozoïde dilué au 1/20.

Dans le cas où l'une des deux ou toutes les deux réactions spermatozoïde et tuberculeuse sont négatives, on pourrait supposer que les sensibilisatrices

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*) Premier mémoire, Proc. Ned. Akad. v. Wetensch., Amsterdam, **43**, 1133 (1940).

antispermatozoïde et tuberculeuse existent dans le sérum, si faibles soient ils. Que le complexe anticorps-antigène tuberculeux-antigène spermatozoïde (fig. 1—C) pourrait ainsi prendre naissance et qu'il serait capable de fixer même fortement l'alexine. De cette façon pourrait être expliqué les résultats de réactions des sérums du groupe I et du groupe II, A et B.

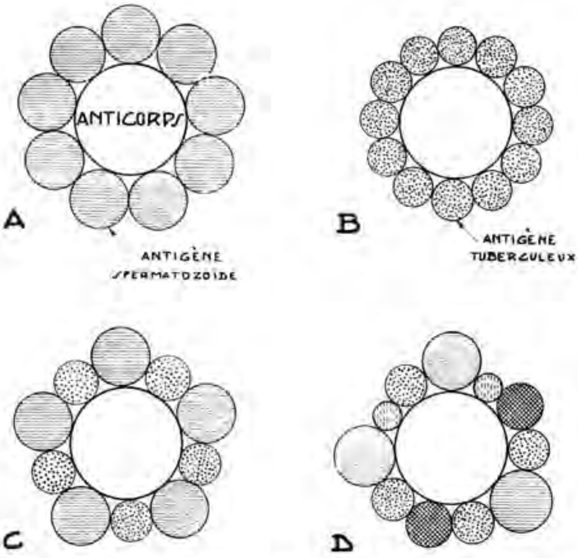

Fig. 1. A. Combinaison anticorps-antigène spermatozoïde. B. Combinaison anticorps-antigène tuberculeux. C. Combinaison simultanée entre anticorps et antigènes spermatozoïde et tuberculeux. D. Cimbinaison simultanée entre anticorps et plusieurs antigènes différents.

En généralisant l'hypothèse précédente, on pourrait supposer qu'un anticorps serait également capable de combiner simultanément plusieurs antigènes: $a_1, a_2, \ldots, a_n$ (fig. 1—D) et qu'un tel complexe fixerait une quantité d'alexine plus élevée que celle obtenue par la somme des réactions réalisées avec les antigènes: $a_1, a_2, \ldots, a_n$ séparément. Cette vue se rapproche de celle de M. KURT MEYER, qui a soustenu la conception d'anticorps doué de plusieurs affinités [6]).

Une deuxième hypothèse également probable est la suivante: On pourrait supposer que chaque molécule d'anticorps fixerait seulement l'antigène correspondant (fig. 1—A ou B) et qu'un mélange de complexes différents A et B fixerait une quantité d'alexine plus élevée que celle obtenue par la somme des complexes A et B séparément. Cette deuxième hypothèse est en désaccord avec les résultats obtenus sur le sérum syphilitique dilué dans un sérum normal, contenant des sensibilisatrices antispermatozoïdes. Ici le sérum normal ne contient pas de réagines syphilitiques et ne peut que former le complexe anticorps-antigène spermatozoïde. Par contre le sérum

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[6]) C. R. Soc. Biol. **122**, 26 (1936).

syphilitique dilué contient très peu ou pas de sensibilisatrices antispermatozoïdes et ne peut que former le complexe anticorps-antigène syphilitique. Et pourtant le mélange de deux sérums en présence d'un mélange d'antigènes syphilitique et spermatozoïde ne dévie pas une quantité d'alexine plus grande que celle obtenue par la somme des deux réactions séparément.

En supposant ces hypothèses vraies on devrait pouvoir diminuer et même supprimer le phénomène, si l'on parviendrait à empêcher dans un mélange, soit la fixation de l'un des deux antigènes par l'anticorps, soit la fixation d'alexine par l'un des deux complexes A ou B. A cet effet, des réactions de fixation ont été réalisées à des températures différentes: à 36° C dans l'étuve, à la température de laboratoire variant de 15 à 22° C et à 0° C (glace fondante). On sait que la fixation d'alexine diminue en même temps avec la température et qu'à zéro degré la fixation est minime ou nulle. Il est également admissible que la combinaison anticorps-antigène diminue en même temps avec la température.

Pour effectuer la réaction à 0° C, il était nécessaire de prendre les précautions suivantes: tous les réactifs et les tubes étaient plongés dans la glace fondante. Quand la température du liquide restait constante on faisait le mélange et on laissait la réaction s'effectuer pendant une heure dans la glace fondante. On ajoutait ensuite le système hémolytique et on portait les tubes à l'étuve à 36° C.

Le tableau 8 montre les résultats de réactions de fixation avec 20 sérums d'homme et un sérum antitétanique de cheval, effectuées à des températures différentes. Ils montrent que, la différence $ST - (S + T)$ diminue en même temps avec la température et qu'à 0° C elle est nulle ou négative (excepté le sérum 81). Le phénomène est donc supprimé. La diminution de réaction se manifeste surtout à 0° C dans la réaction de tuberculose; en effet la plupart des réactions sont devenues négatives.

On en conclut que, si l'un des deux antigènes n'est plus ou faiblement fixé par l'anticorps le phénomène ne se manifeste plus, ou bien que, si l'un des deux complexes A ou B n'est plus capable de capturer l'alexine le phénomène ne se montre pas non plus. Ces résultats ne montrent pas laquelle des deux hypothèses est vraie; ils prouvent seulement qu'on peut supprimer le phénomène, si l'on empêche l'une des deux réactions de se réaliser dans le mélange d'antigènes spermatozoïde et tuberculeux.

14. *Le point de vue chimique du phénomène étudié.* On admet en général que la réaction anticorps-antigène est un phénomène d'adsorption (BORDET). Cependant pour expliquer la spécificité plusieurs auteurs pensent qu'en dernier lieu il s'agirait d'une combinaison chimique. Pour savoir si une combinaison chimique prend naissance dans le phénomène considéré nous avons appliqué la détermination du pouvoir rotatoire. On sait que, lorsque deux dissolutions de substances actives sont mêlées, sans qu'il se produise une combinaison chimique, la déviation de polarisation est égale à la somme des pouvoirs rotatoires des substances dissoutes chacune dans le même volume de liquide. Au contraire, si une combinaison chimique

TABLEAU 8.

No des sérums	36° C				Température de laboratoire				0° C			
	S	T	ST	△	S	T	ST	△	S	T	ST	△
105	10	60	275	205	10	60	140	70	10	30	40	0
Sérum antitétanique R3	25	25	150	100	10	0	35	25	5	0	5	0
126	5	150	225	70	10	60	120	50	20	30	50	0
81	5	0	75	70	0	30	20	—10	5	0	10	5
77	5	5	55	45	5	0	5	0	5	0	0	— 5
114	10	45	90	35	10	40	70	20	10	30	40	0
35	20	30	85	35	30	15	60	15	15	0	15	0
36	10	15	60	35	10	10	20	0	10	0	10	0
110	0	30	55	25	0	30	50	20	0	15	15	0
63	0	5	30	25	0	0	15	15	0	0	0	0
73	5	0	30	25	0	0	10	10	0	0	0	0
82	0	5	30	25	10	20	30	0	5	0	5	0
103	0	0	25	25	5	10	20	5	0	0	0	0
111	0	0	20	20	0	0	20	20	10	0	10	0
104	0	30	45	15	0	20	55	35	10	0	10	0
127	10	40	60	10	10	20	35	5	10	0	5	— 5
106	10	10	30	10	0	5	15	10	5	0	5	0
107	10	10	30	10	0	0	15	15	0	0	0	0
72	5	15	30	10	0	15	20	5	0	0	0	0
52	5	10	25	10	10	10	25	5	0	0	0	0
124	10	15	30	5	10	20	40	10	25	10	25	—10

△ = différence ST—(S + T).
Les chiffres indiquent le nombre d'unités d'alexine déviées par 1 cc de sérum.

prend naissance, la déviation même l'accusera, et elle n'est plus égale à la somme des pouvoirs rotatoires de deux substances séparément.

Pour résoudre la question proposée nous avons déterminé le pouvoir rotatoire des mélanges suivants:

a. Sérum chauffé à étudier + mélange d'antigènes spermatozoïde et tuberculeux + alexine + eau physiologique Q. S. pour 4 cc.

b. Sérum chauffé à étudier + eau physiologique Q. S. pour 4 cc.

c. Mélange d'antigènes spermatozoïde et tuberculeux + eau physiologique Q. S. pour 4 cc.

d. Alexine + eau physiologique Q. S. pour 4 cc.

On a pris la précaution de mesurer exactement la même quantité de sérum, d'antigènes et d'alexine dans le tube réaction *a* et dans les tubes témoins *b*, *c* et *d*. Dans les expériences effectuées la quantité de réactifs utilisés était variable, de sorte qu'on pouvait mieux saisir le phénomène étudié. Les tubes étaient portés une à deux heures à l'étuve à 36° C et on procédait à la mesure de pouvoir rotatoire. Dans certains cas les tubes étaient gardés pendant trois heures à l'étuve. Notons que le pouvoir rotatoire des mélanges ne variait pas après être conservés pendant 24 h. en glacière (+5° C).

Toutes les mesures furent effectuées au moyen d'un polaristrobomètre de WILD, dont le principe est un polariscope de SAVART placé entre deux nicols. En faisant 16 lectures sur chaque mélange, on obtient une moyenne avec un écart type variant de 0°,02 à 0°,07. Toutes les mesures ont été faites avec un même tube de 20 cm de longueur et de 0,5 cm de diamètre. Quatre centrimètres cubes de liquide suffisent pour la mesure. La source de lumière était une lampe électrique à vapeur de sodium, fabriquée par Philips. ($\lambda \cong 5780$ Å).

Le tableau 9 représente les résultats de mesures avec 11 sérums, montrant une réaction de fixation très marquée.

TABLEAU 9. Sérums donnant des réactions de fixation très marquées.

No des sérums	Résultats des réactions de fixation			Réactions α_r en degrés	Témoins $\alpha_1+\alpha_2+\alpha_3$ en degrés	Différence $\alpha_r-(\alpha_1+\alpha_2+\alpha_3)$ en degrés
	S	T	ST			
75	10	20	30	—0.44	—0.30	—0.14
77	5	5	55	—0.59	—0.66	+0.07
80	10	15	30	—0.08	—0.27	+0.19
106	10	10	30	—0.27	—0.16	—0.11
122	10	15	30	—0 77	—0.92	+0.15
123	15	15	30	—1.44	—1.51	+0.07
129	10	40	60	+0.06	—0.11	+0.17
130	10	15	30	—0.95	—0.82	—0.13
131	0	10	30	—0.95	—0.99	+0.04
133	10	10	30	—1.34	—1.39	+0.05
135	10	10	30	—2.57	—2.66	+0.09
Moyennes	9.1	15.0	35.0	—0.85	—0.89	+0.04

Si dans le mélange *a* une combinaison chimique prend naissance, on devrait obtenir une différence nette entre le pouvoir rotatoire du tube réaction *a* et la somme des pouvoirs rotatoires des tubes témoins *b*, *c* et *d*. La moyenne arithmétique des pouvoirs rotatoires des tubes réactions *a* est égale à $x = -0°{,}85$ et celle de la somme des pouvoirs rotatoires des tubes témoins *b*, *c* et *d* est égale à $\bar{x}' = -0°{,}89$. Nous allons voir si la différence entre les deux moyennes x et $\bar{x}'$ est réelle.

On a: $\bar{x} - \bar{x}' = 0°{,}04$, $s = 0{,}751$, $t = 0{,}125$, $n = 20$.

Dans le tableau de *t*, on trouve pour $n = 20$, la probabilité $P \cong 0{,}9$ pour que, $t > 0{,}125$. La différence entre les deux moyennes n'est pas significative.

Puisque on a ici des observations parallèles, on pourrait aussi bien appliquer la deuxième méthode de la distribution de *t*. On trouve ainsi: $x = +0{,}04$, $s = 0{,}368$, $t = 0{,}361$, $n = 10$.

La probabilité pour que, $t > 0{,}361$ serait entre 0,7 et 0,8. On trouve donc le même résultat que ci-dessus: la moyenne des différences entre les valeurs observées n'est pas réelle.

Par comparaison avec les sérums donnant des réactions de fixation fortement positives, nous avons examiné 13 sérums, dont les réactions sont négatives ou faiblement positives. Le sérum 138 avait donné un mélange hémorragique, on était obligé de le mesurer à la lumière rouge. Le tableau 10 montre les résultats obtenus. Le calcul donne: $\bar{x} - \bar{x}' = +0{,}04$, $s = 0{,}695$, $t = 0{,}147$, $n = 24$.

TABLEAU 10.

Sérums donnant des réactions de fixation négatives ou faiblement positives.

No des sérums	Résultats des réactions de fixation			Réactions α_r en degrés	Témoins $\alpha_1+\alpha_2+\alpha_3$ en degrés	Différence $\alpha_r-(\alpha_1+\alpha_2+\alpha_3)$ en degrés
	S	T	ST			
100	0	0	0	−2.58	−2.27	−0.31
101	0	0	20	−2.47	−2.60	+0.13
102	0	0	10	−0.58	−0.49	−0.09
108	0	5	10	−0.88	−0.78	−0.10
109	0	0	10	−0.59	−0.68	+0.09
125	5	5	15	−0.72	−0.85	+0.13
128	0	0	15	−0.75	−0.89	+0.14
132	0	0	10	−1.17	−1.26	+0.09
134	0	0	15	−1.41	−1.78	+0.37
136	0	0	0	−2.18	−2.12	−0.06
137	0	5	20	−0.91	−1.23	+0.32
138	0	10	20	−1.38	−1.09	−0.29
140	0	0	0	−0.73	−0.85	+0.12
Moyennes	0	1.9	11.2	−1.26	−1.30	+0.04

Le tableau de t donne pour $n = 24$, la probabilité cherchée entre 0,8 et 0,9 pour que, $t > 0{,}147$. La différence entre les moyennes n'est pas significative.

La deuxième méthode de la distribution de t donne: $x = +0{,}04$, $s = 0{,}647$, $t = 0{,}223$, $n = 12$.

La probabilité cherchée serait entre 0,8 et 0,9 pour que, $t > 0{,}223$. Les deux méthodes donnent le même résultat; la moyenne des différences n'est pas réelle.

En résumé, les sérums montrant une réaction de fixation fortement ou faiblement positive donnent le même résultat. Il n'y a aucune différence entre le pouvoir rotatoire du mélange considéré comme réaction et la somme des pouvoirs rotatoires des composants employés. Dans la réaction de fixation et, surtout dans le phénomène étudié, une combinaison chimique ne semble pas prendre naissance. Si nous avions trouvé une différence réelle entre les moyennes des pouvoirs rotatoires, on est sûr qu'il s'agit d'une combinaison chimique. Mais malgré les résultats négatifs obtenus, il est prudent de ne pas écarter la possibilité d'une intervention chimique. Dans cette question si importante, touchant le mécanisme de la spécificité il est nécessaire de vérifier les résultats par une autre méthode physique, la spectroscopie.

15. *Importance du phénomène observé. a.* En premier lieu on pourrait probablement expliquer le mécanisme du phénomène de SANARELLI–SHWARTZMAN. On sait que ce phénomène, comme celui de l'anaphylaxie, consiste en deux étapes: une première injection préparante et une seconde injection déchaînante. On pourrait expliquer le mécanisme de ce phénomène comme suite: l'injection première et l'injection seconde donnent naissance in vivo en présence d'anticorps normaux, des combinaisons anticorps-antigènes du type *C* ou *D*. (fig. 1). Ces complexes devraient être pour l'animal plus toxique que les complexes du type *A* ou *B* séparément, engendrés par les filtrats microbiens et les anticorps normaux. Quand les injections préparante et déchaînante sont de la même substance, il faudrait imaginer l'intervention d'une deuxième substance, qui se trouverait dans l'organisme de l'animal.

b. L'allergie tuberculinique est probablement du même mécanisme. La tuberculine pourrait former avec l'anticorps ou la cellule tuberculeuse avec le concours d'une deuxième substance des combinaisons du type *C* ou *D*, qui sont responsables à l'allergie tuberculinique.

Les échecs de l'épreuve de PRAUSNITZ–KÜSTNER et de la transfusion en grosse quantité (8 à 10 cc) du sérum de cobaye tuberculeux et allergique à un cobaye sain, que nous avons tenté en collaboration avec M. KOURILSKY prouvent que, la deuxième substance ne se trouverait pas dans le sang du cobaye tuberculeux [7]).

c. L'observation suivante que nous avons fait autrefois pourrait également ainsi expliqué. Elle consiste en ceci: un sérum anticolibacillaire polyvalent agglutinant les souches employées à un titre variant de 1/300 à 1/32000, donne une agglutination maxima 1/32000 avec un mélange de douze souches de B. coli employées à parties égales [8]). On pourrait penser que les agglutinines colibacillaires sont fixées simultanément par les différentes souches de B. coli, formant ainsi des combinaisons du type *D*. Ces combinaisons provoqueraient une agglutination plus prononcée que les complexes du type *A* ou *B*, constitué par chacune des souches employées et les agglutinines colibacillaires.

d. Enfin, le phénomène observé est en rélation avec ce qu'on est convenu d'appeler la ,,synergie médicamenteuse". C'est-à-dire l'action d'un mélange de deux médicaments est dans certains cas plus prononcée que la somme des effets de deux médicaments séparément. En généralisant l'hypothèse précédente, on pourrait expliquer le mécanisme ainsi: deux molécules chimiques différentes, qui se combinent à une même cellule donneraient un

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[7]) Revue de la tuberculose **3**, 350 (1937).

[8]) Zeitschr. f. Immunitätsf. **93**, 282 (1938).

effet plus grand que la somme des effets, résultant de la combinaison de la cellule avec les deux molécules chimiques séparément.

Ce travail était commencé en 1935 dans le service du professeur E. SERGENT, au laboratoire de recherches de la clinique propédeutique à l'hôpital Broussais-La Charité, Paris. Il fut ensuite poursuivi au laboratoire KAMERLINGH ONNES, grâce aux professeurs W. J. DE HAAS et W. H. KEESOM, directeurs du laboratoire, qui m'ont accueilli avec la plus grande bienveillance et dont je les remercie sincèrement. Je dois également remercier M. C. A. CROMMELIN, directeur-adjoint du laboratoire, qui a bien voulu mettre à ma disposition les appareils nécessaires.

NEDERL. AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS

VOLUME XLIII

No. 10

President: J. VAN DER HOEVE
Secretary: M. W. WOERDEMAN

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Physics. — *The soft component of Cosmic Radiation.* By J. CLAY.

(Communicated at the meeting of November 30, 1940.)

In the cascade theory of electrons is calculated the number and energy distribution of electrons found under a certain layer of matter, produced by one electron of certain energy. But what will be the result of an electron radiation in total, will depend on the energyspectrum of the electrons.

For some years we have already tried to find the number of electrons and their energy but it was impossible to find reliable results on account of different disturbing factors. It seems to be necessary to measure the total number of incoming particles and to subtract the penetrating part, which is 80 % of the total number, to get the number of the soft particles only. To measure the decrease of these soft particles only, it is necessary to find the differences by applying thin layers of material and to find the decrease, in casu some percents of the total number. The constancy of the counters must be very high for this purpose and the numbers measured must be great in a short time, moreover it is necessary to do this in periods of very small barometer changes, — because the barometer variation is 5 % for 1 cm Hg. —; constancy of temperature is necessary and no magnetic-storms may occur. To meet these difficulties we used 5 countersystems. One countersystem controlled the number of the hard component (10 cm between the counters) and another controlled the hard and soft together by using no absorption material. For the determination of the soft component itself three other sets were used. Every set had 3 boxes in series and every box 3 counters parallel with an active surface of 820 cm^2. The opening cone was 24° and 50°. These counters were placed at distances of about 20 to 60 cm. The number of coincidences was from 120 to 60 per minute. We could measure 7200 to 3600 coincidences p.h. and within a period of 6 hours we may reckon with a natural fluctuation of 0.5 % to 0.7 %. For every layer we measured about 100000 coincidences. Mostly the value of coincidences for one layer of matter was enclosed between two measurements with another layer. We could be certain of the results for 1 % in the total number, that means 5 % in the number of the soft component alone in its full intensity.

With these sets the decrease was measured in Pb., Fe., Al., C., H_2O and paraffin.

We know the formulae of energyloss in different materials and for lead.

these formulae[1]) agree with the statistics of ANDERSON and NEDDERMEYER[2]) between 10^5 and 10^7 e-volt.

When we now take the energy for a certain layer and count the number of particles which can just penetrate this layer, we know the number of particles which have an energy above this limit. We find the integral spectrum $I = C f(E_{\min})$. From this function we find the differential spectrum. It might be necessary to take just the average energy between every two intervals and the numbers which are stopped in this interval, but when the distribution is of the same nature as those of the mesons, we can use an easier way and this seems to be real.

When the integral spectrum is of the form $I = C E^{-(S-1)}$, the differentialspectrum is $N(E) = \frac{dI}{dE} = -C(S-1)E^{-S}$.

If this function holds we may take $\log. I = -(S-1) \log. E$ and $S-1$ is found from the inclination of the line in log. I and log. E.

At first we took the relation between the energy and the range in lead for electrons given by HEITLER (fig. 1) and so we know the energy

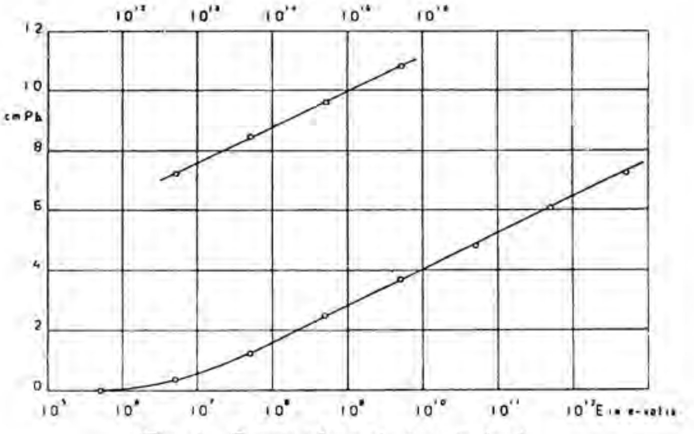

Fig. 1. Range of an electron in lead.

necessary for every layer. Then we put the log. of the remaining number of electrons below the different layers against log. E and we find the line given in fig. 2. In order to find the remaining number of the total soft rays it is necessary to separate the soft from the hard component. We know from many absorption experiments that the intensity of the total radiation consists in two parts, under more than 10 cm of Pb. the decrease is of another kind as under smaller layers and from an extrapolation of the line from 10 to more cm of lead to zero, we see that the increase of the hard component must be very small. Between 10 and 15 cm of lead the decrease is

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[1]) W. HEITLER, The Quantum Theory of Radiation (1936).

[2]) S. NEDDERMEYER and C. D. ANDERSON, Phys. Res., **51**, 884 (1937).

small on account of the effect of decay of mesons of lower energy (Fig. 3). So we decided to consider the difference between the number with no absorption layer and the number under a layer of 10 cm Pb. as the number of the soft rays. Another supposition is made by extrapolation of the HEITLER

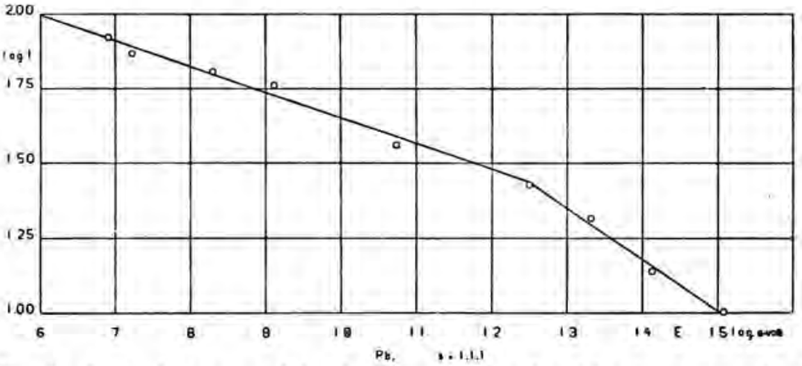

Fig. 2. Integral spectrum of the electron component found by absorption in lead.

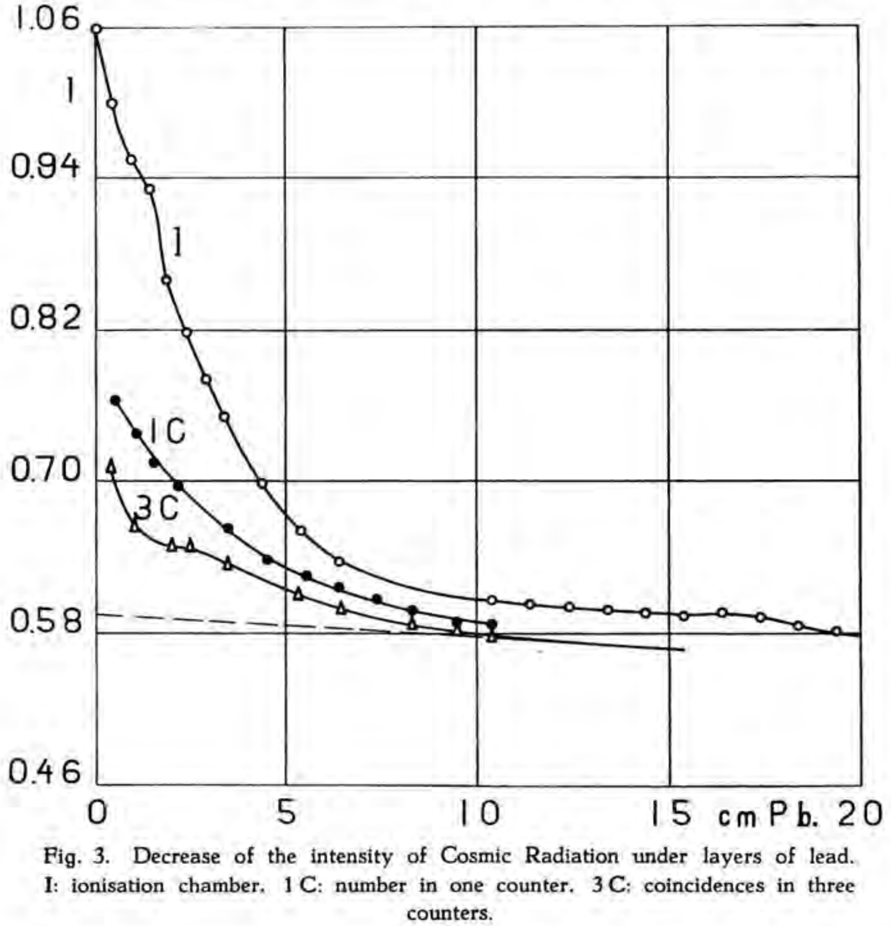

Fig. 3. Decrease of the intensity of Cosmic Radiation under layers of lead. I: ionisation chamber. 1 C: number in one counter. 3 C: coincidences in three counters.

formula above about 10^8 e-volt. We may fear that in this region the formula breaks down. For energies with a DE BROGLIE wavelength, small in relation to the dimensions of the electron, it will only be tentative. But at the moment we have no other way and for this reason we have made an extrapolation for higher energies. The value of the energy of an electron for a range of 10 cm Pb. is 10^{15} e-volt and the number with such energies and more will be very small. But from the phenomenon of the wide showers P. AUGER and his collaborators [3]) and L. JANOSSI & LOVELL [4]) there are indications that such energies may occur. From the graph, which gives the relation between log. I and log. E, we see that within the limits of uncertainty, the value of log. I is linear to log. E, until a value of 5.10^{12} e-volt, according to a range of 8 cm in Pb. is reached. $S = 1.11$. That means that the energy distribution is $N(E) = N_0\, E^{-1.11}$. And up to the energy of 5.10^{12} e-volt we have 80 % of the electrons. For the remaining part $S = 1.25$.

We can verify the value of S. because if we take another material we have to find the same value of S. For this reason we measured the decrease of the soft radiation in lighter materials as aluminium, paraffin, coal and water. But the difficulty is that we cannot bring enough material of this kind to stop the total soft radiation without making the distance between the counters too long and the cone in which we measure too small. In such a case the number of electrons would become too small to warrant a sufficient accuracy in a reasonable time. But there are more difficulties. At first we took paraffin as a suitable material to be placed between the counters and we could place a layer of 35 cm between the counters. We found that the absorption of paraffin is abnormally high in comparison with that of aluminium and later on the same was found in comparison with coal and with water, which were both normal in relation to aluminium and lead. But secondarily we found that coal gives an abnormally decrease for thicker layers, which can be explained by the production of secondary knockon electrons by mesons (fig. 4). It was found by SWANN and RAMSEY [5]) that the production of one-electron showers by mesons increases on decrease of density of matter, which process is unexplained at the moment. But it is possible to find a regular decrease in water down to a layer of 60 cm of water and when we make a graph of log. I against log. E, in which E means the energy, necessary for the ranges in water layers, we find again a linear relation within the limit of uncertainty of the measurements down to the value of 3.10^8 e-volt and with this value 40 % of the total number of electrons is stopped (fig. 5). When we calculated from this relation the value of $(S-1)$ we find 0.23 and for the differential spectrum $N(E) = N_0\, E^{-1.23}$.

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[3]) P. AUGER, R. MAZE, P. EHRENFEST and A. FRÉON, J. de Physique, **10**, 1 (1939).

[4]) L. JANOSSI and LOVELL, Nature, **142**, 716 (1938).

[5]) W. F. G. SWANN and W. E. RAMSEY, Phys. Res., **57**, 749 (1940).

We think that the difference with the distribution we found for Pb. is more than the limit of uncertainty. But we may be sure that the decrease of the number with energy is only a small fraction more than the inverse of the energy, which means that it is much smaller than is accepted generally, vide identical with that of the spectrum of the mesons which goes down with the third power of E.

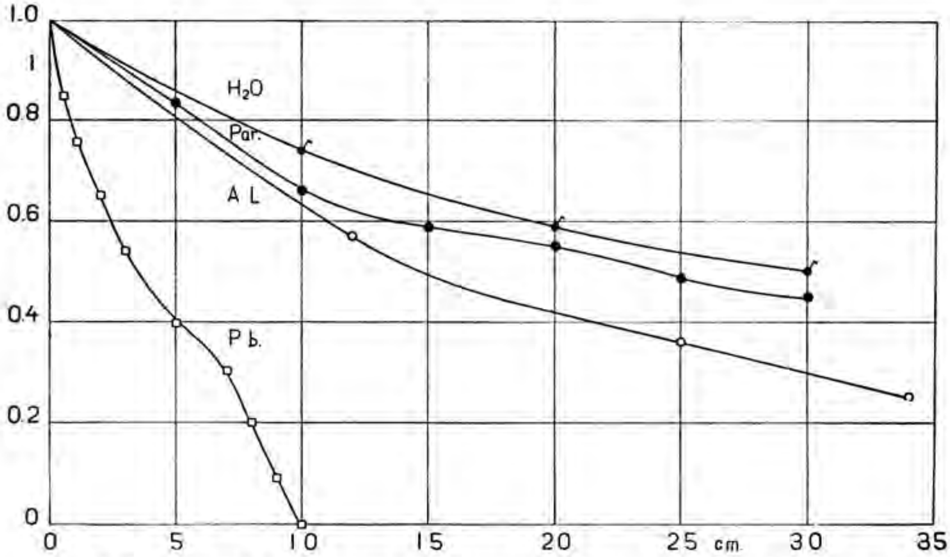
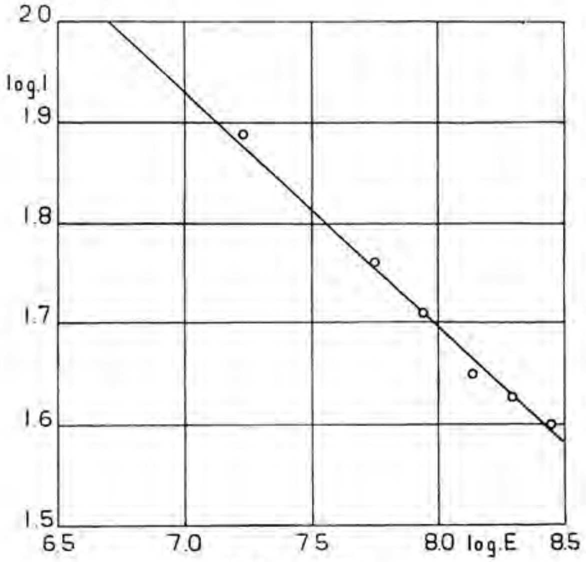


Fig. 4. Decrease of the electron component, produced in air, in H_2O, paraffin, Al and Pb.



H_2O $S = 1.23$

Fig. 5. Integral spectrum of the electron component found bij absorption in water.

What the reason is for the abnormally high absorption in paraffin is not clear, especially because it is not found in coal and not in water, which have about the same number of protons p.c.c.

It is often found that under a thick layer of matter there is no Rossi-maximum of showers. Only JANOSSI ⁶) reports he finds a soft shower-maximum under thick layers of material. In order to know the explanation the soft component was measured under thick layers of water (fig. 6),

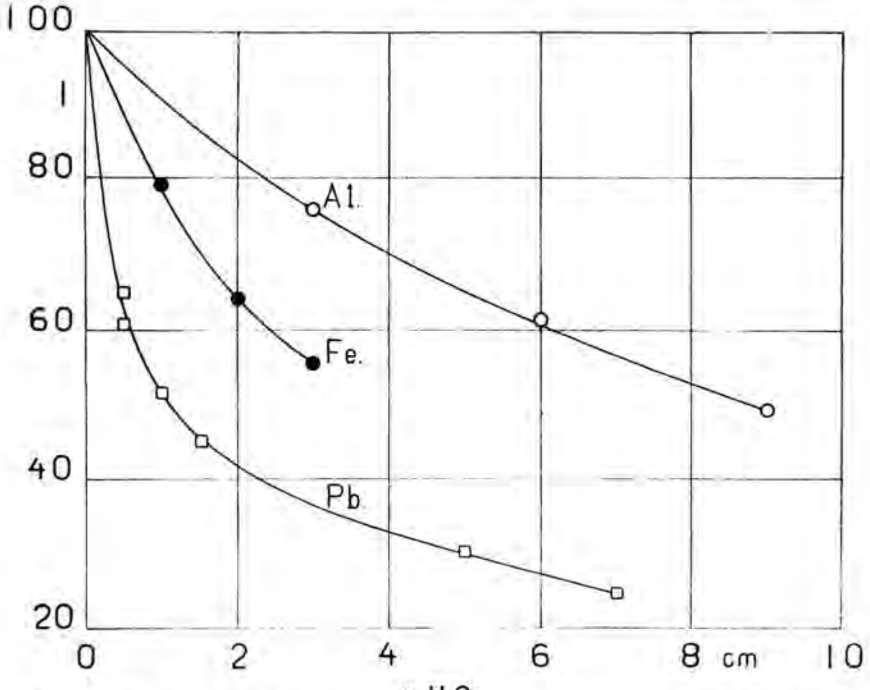


Fig. 6. Decrease of the electron component produced in water, in Al, Fe and Pb.

aluminium, iron and lead, and we found that the percentage of the soft radiation in relation to the hard component is under air 27 %, water 24 %, aluminium 20 %, iron 9.5 % and under lead 4.7 % (fig. 7), this is in agreement with the result of SWANN, mentioned above, that the number of soft secondaries in equilibrium with primaries in different material decreases with atomic number. We think that the real production in lead will not be smaller than it is in light material in a volume of the same number of electrons, but the loss of energy of the electrons necessary to escape from the volume element in which they are produced is larger, proportional to z^2, and this will decrease their number and their energy. The number of the secondaries in relation to the hard primaries is given in graph 8. And the energyspectrum, at least for the lower spectral part, can be found from the relation log. I to log. Ξ given in graph 9. We find that the

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⁶) L. JANOSSI, Proc. Ray. Soc., A. 167, 499 (1938).

decrease is faster as the atomic number of the material of the layer is higher.

From each of these radiations the decrease is measured in different materials and we find that the energy of the electrons is smaller also with increasing atomic-number, in agreement with earlier experiments, when the absorption of single-electrons was measured [7].

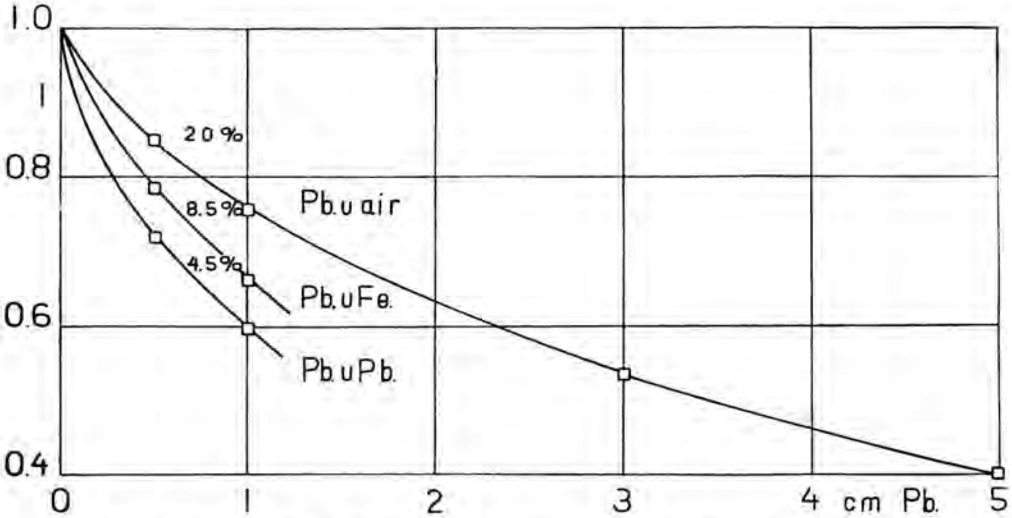

Fig. 7. Decrease in Pb of the electron component produced in air, Fe and Pb.

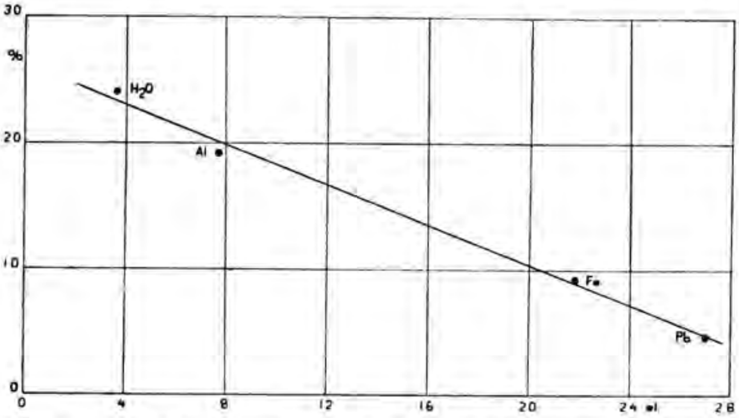

Fig. 8. Number of electrons found under H_2O, Al, Fe and Pb in relation to the number of mesons.

Graph 7 gives the result of this part of the experiments and their decrease was measured with smaller and wider opening-cones, 44° in one 100° in the other direction. One phenomenon is striking here: when we measure the number of electrons under lighter materials the production of electrons bij the mesons in the absorbing layer overcompensates the

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[7] J. CLAY, Physica, 3, 352 (1936).

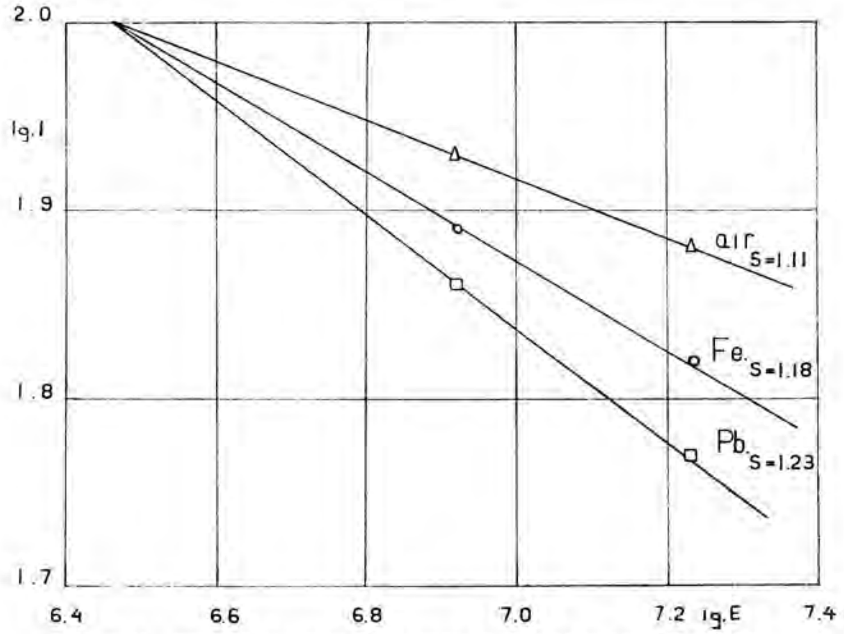

Fig. 9. Integral spectrum of electrons produced in air, Fe and Pb.

decrease. We see this in graph 10. The variation under absorbing paraffin, under Pb. and under Fe. and air. This now explains that the number of electrons under thick layers of light matter can be restored so far, that enough electrons are present again to give a maximum in the showercurve.

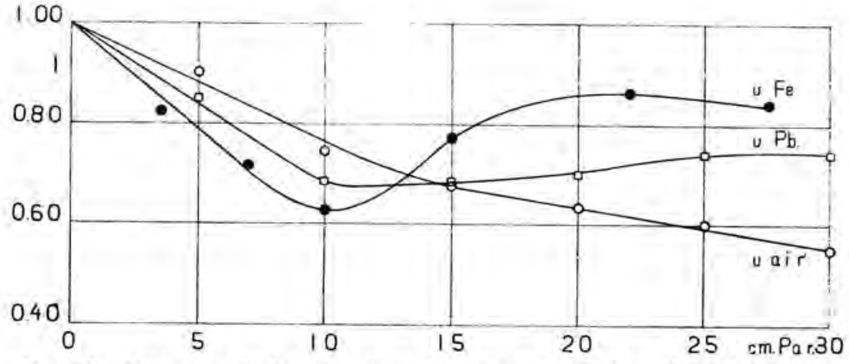

Fig. 10. Decrease of the electron component in paraffin. Produced in air Fe and Pb.

In order to have a full picture of soft radiation it is necessary to know the angular distribution of it. This is partly — from 0–60° — the same as that of the mesons [8]), $f(\theta) = 0.005 + 0.12 \cos^2 \theta$ p. cm^2 p. min., but for larger angles the percentage is larger and at a nearly horizontal direction it is 40 % of the total number instead of 20 %, as it is in the vertical.

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[8]) J. CLAY, J. T. WIERSMA and K. H. J. JONKER, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **41**, 706 (1938).

Palæontology. — *The fossil human remains discovered in Java by* Dr. G. H. R. VON KOENIGSWALD *and attributed by him to* Pithecanthropus erectus, *in reality remains of* Homo sapiens soloensis. *Conclusion*. By Prof. EUG. DUBOIS.

(Communicated at the meeting of November 30, 1940.)

As a matter of fact the morphologic similarity of the skull cap which I described in 1894, together with other fossil remains, as *Pithecanthropus erectus*, to the homologous part of a small *Hylobates* species is so striking that I never lost sight of this resemblance, nor feel at liberty to regard the "Java man" as a real, be it ever so primitive man. Evidence of that is to be found in a number of my publications, attesting, on the other hand, to distinctness of the *Pithecanthropus* from the *Hylobates* organism, and at the same time to the much greater distinctness from the chimpanzee.

Nevertheless VON KOENIGSWALD, recently, wrongly proclaimed that I changed my mind, a few years ago, from an originally and permanently hominide view to a, surprisingly, new Hylobate-one. HANS WEINERT, on his part, since 1928, wrongly imputed to me, originally, a preference of the chimpanzee for resemblance to *Pithecanthropus*.

To prove that these ways of putting things are wrong, I may reproduce, in the first place, some literal quotations from that monograph, the first scientific description of the fossil species under consideration [1]):

p. 1. „Classe. **Mammalia.**
„Ordnung. Primates.
„Neue Familie. *Pithecanthropidae*.

„Hirnschädel, absolut und im Verhältnis zur Körpergrösse viel geräumiger als bei den *Simiidae*, jedoch weniger geräumig als bei den *Hominidae*; Inhalt der Schädelhöhle ungefähr zwei Drittel vom durchschnittlichen Inhalte derjenigen des Menschen. Neigung der Nackenfläche des Hinterhauptbeines bedeutend stärker als bei den *Simiidae* [2]). Gebiss, obwohl in Rückbildung, noch vom Typus der *Simiidae* [3]). Femur in seinen Dimensionen dem menschlichen gleich und wie dieses für den Gang bei aufrechter Körperhaltung gebaut.

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[1]) EUG. DUBOIS, Pithecanthropus erectus. Eine menschenähnliche Uebergangsform aus Java. II und 40 Seiten in folio. Mit 2 Tafeln und drei Textfiguren. Batavia. Landesdruckerei. 1894. (Vorwort datiert: Tulung Agung (Java), Januar 1894.)

[2]) After complete preparation of the fossil skull cap, the nuchal occipital region appeared strikingly similar to that of the gibbons; in *Pithecanthropus* the inclination of the planum nuchale, particularly, was certainly not much stronger than in *Hylobates*.

[3]) The first lower premolar and the mandible then were not known.

„*Pithecanthropus erectus*, genus novum, species nova.

p. 3. „Die Oberfläche des Schädeldaches ist glatt, ... zeigt durch seine glatte Oberfläche und seine allgemeine Form grosse Aehnlichkeit mit dem Schädel von Anthropopithecus (vergl. Taf. I), jedoch noch mehr mit dem von *Hylobates* (Fig. 1)".

„Tafel I. Fig. 1. *Pithecanthropus erectus* n.g., n.sp. Schädeldach, von oben, nach Photographie.
„ 1a. „ „ n.g., n.sp. Schädeldach, von der linken Seite, nach Photographie.
„ 2. *Anthropopithecus troglodytes* Gmelin ♀ adult. Schädel von oben, nach Photographie.
„ 2a. „ „ Gmelin ♀ adult. Schädel von der linken Seite, nach Photographie.
Fig. 1 und 1a $^{1}/_{2}$ natürl. Grösse.
„ 2 und 2a $^{2}/_{3}$ „ „ (Reducirt zu ungefähr gleicher Grösse mit 1 und 1a.)

„Die Tafel zeigt die Verschiedenheit in der Bildung der occipitalregion und der Superciliarregion des Schädels, sowie die bedeutendere Wölbung bei *Pithecanthropus*. Wie aus den Textfiguren 1 und 2 1) zu ersehen ist, stimmt in diesen Beziehungen *Pithecanthropus* viel mehr mit *Hylobates* überein."

p. 12. „Aus der Beschreibung und aus den Vergleichen geht hervor, dass das fossile Schädeldach eine Art andeutet, die in ein anderes Genus als *Gorilla, Simia* und *Homo* eingereiht werden muss; diesem letzteren nähert es sich durch seine absolute Grösse und Wölbung, zeigt jedoch grosse Uebereinstimmung mit *Anthropopithecus* und, der Form nach noch mehr mit *Hylobates*."

p. 25. „Aus dem Vergleiche geht also mit Sicherheit hervor, dass zwischen dem Femur der fossilen Form und dem des Schimpanse bedeutende Unterschiede bestehen, hauptsächlich in der viel ansehnlicheren Schlankheit des fossilen Knochens und seiner relativ und absolut grösseren Länge, ferner in der Form des Schaftes und besonders des unteren Gelenkendes; ...

„Das Femur der Hylobatiden underscheidet sich von dem fossilen ausser durch seine absolute Grösse, hauptsächlich durch die Form der Condylen" ...

p. 26. „Dagegen stimmt dieses Genus in der relativen Dicke des Schaftes des Femur mit dem Menschen überein."

p. 27. „bei den Gibbons, die im Verhältnis zu ihrer Grösse keinen schwereren Oberkörper haben als der Mensch, finden wir auch ebenso schlanke Oberschenkelbeine wie beim Menschen" ...

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1) In Figure 2 the seeming inclination of the lower portion of the occipital much exceeds the real inclination, on account of damage of the fossil.

p. 28. „Man könnte nun annehmen, dass die grosse Länge der unteren Extremitäten wie bei den Gibbons in Correlation stand zu den als vervollkommnete Kletterorgane länger gewordene Arme. Diese Annahme erscheint aber unbegründet, wenn man bedenkt, um wie viel die fossile Form grösser ist, als die *Hylobates*-Arten. Schon beim Körpergewichte der Hylobatiden ist, wie es scheint, die erworbene Specialisirung an der Grenze des Erreichbaren angelangt. Der Siamang bewegt sich weniger schnell als die übrigen Arten derselben Gattung, und aus den Spuren von geheilten Knochenbrüchen, die so oft an den Skeletten von Hylobatiden zu sehen sind, geht hervor, dass die pfeilschnellen Bewegungen zwischen den federnden Aesten der breitkronigen indischen Waldbäume, wozu diese Tiere der bewunderenswürdig specialisirte Bau ihrer Extremitäten befähigt, nicht ohne Gefahr sind. Für den so grossen Körper der fossilen Form wäre eine solche, eher fliegen als klettern zu nennende, Bewegung ganz unmöglich. Andererseits ist es sehr merkwürdig, dass gerade die Gibbons, die bezüglich des Grossenverhältnisses zwischen Beinen und Oberkörper mit dem Menschen übereinstimmen, die einzigen Affen sind, die — wenn auch auf mangelhafte Weise — aufrecht gehen können, ohne dabei, wie Schimpanse, Gorilla und Orang-Utan thun, die Hände als Stütze zu gebrauchen. Der directe Beweis dafür, dass die javanische Form sich auf ganz andere Weise bewegt haben muss als die Hylobatiden dies für gewöhnlich thun, wird übrigens durch den gänzlich verschiedenen Bau des unteren Gelenkendes des Femur, insbesondere der Condylen, geliefert."

In my report „Pithecanthropus erectus, eine menschenähnliche Uebergangsform", read September 21, 1895 [1]), „habe ich die Profilcurven der zwei Spy-Schädel, ebenso wie die eines jüngst von CUNNINGHAM beschriebenen Schädels eines Microcephalen [2]) neben einander gezeichnet (Vergl. Figur 2). Dabei zeigt sich eine grosse Uebereinstimmung; es muss aber auffallen, dass der Scheitelteil des Trinilcraniums viel flacher ist als bei jenen. Wirklich ist dies eine Eigenschaft des fossilen Schädels, worauf ich jetzt hinweisen will, die, glaube ich, schwer an Menschenschädeln zu finden sein würde." [1]) The same figure, however, establishes conformity, in this respect, of *Pithecanthropus* with *Hylobates leuciscus*, *Semnopithecus maurus* and *Anthropopithecus troglodytes*, wanting what later I have called the vertex parietalis, a true human distinctive.

I may now quote, translated, from a paper of 1907 [3]):

p. 456. "I now regard *Pithecanthropus erectus* as nearly allied to the living Gibbons. The thigh-bone and the skull have, however, double the

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[1]) Compte-rendu des séances du Troisième Congrès international de Zoologie. Leyde. 1895, p. 265.

[2]) D. J. CUNNINGHAM, The brain of the microcephalic idiot. Scientific transactions of the Royal Dublin Society. May 1895. Plate 38, Fig. 5.

[3]) EUG. DUBOIS, Eenige van Nederlandschen kant verkregen uitkomsten met betrekking tot de Kendeng-fauna (fauna van Trinil). Tijdschrift van het Koninklijk Nederlandsch Aardrijkskundig Genootschap. 2e Ser., dl. 24, pp. 449—458. Mededeelingen 1907. *See pp. 456—457.

dimensions of the smaller species of these Long-armed Apes, for instance of the Wou Wou (*Hylobates javanicus*) from Java. The skull is ape-like rather than man-like in form, principally on account of the backward situation of the temporal constriction (not its depth, as RUDOLPH VIRCHOW supposed). The dentition was presumably still somewhat more human than that of the small Gibbons. The thigh-bone, adapted to the erect gait with stretched knee-joint, exhibits, however, yet vestiges of adaptation to climb. With the Gibbons, which distinguish themselves from all the other Apes and Monkeys by being in the habit of walking on their legs only, without the support of their hands — the relative length and the form of the thigh-bone vary in a high degree, and in this respect some individuals somewhat approach *Pithecanthropus*, though the knee-joint never is quite formed for the erect gait with stretched leg, as is certainly the case with *Pithecanthropus*.

"The feature, however, that distinguishes *Pithecanthropus* sharply from the Gibbons is, in the first place, exactly the resemblance of its cranial form to that of the small Gibbons. I cannot enough be pointed out, that the larger species of Gibbon, the Siamang (*Hylobates syndactylus*, possesses a much more truly monkey-like lowvaulted skull and a (somewhat) more bestial dentition than the smaller Gibbon species. Yet the difference in size of the animals is relatively small. And now, *Pithecanthropus*, with double the dimensions has retained the capacious skull of the smaller Gibbon species. The dentition is even less bestial.

"A definite part of the brain volume of *Pithecanthropus* can be measured accurately and compared with the homologous part of large and small Gibbons. What appears then? That the fossil form from Java, in the same progression as observed in the Gibbons, had twice as much brain as an (imaginary) Gibbon of double the size of the Wou Wou would have. Consequently, *Pithecanthropus* cannot be a Gibbon. Opposed to this is also the fact of its erect gait with stretched knee-joint.

"When now inspecting the endocranial cast, which gives us an idea of the form of the convolutions of the cerebrum, this appears to be of the human fundamental type, to represent, as it were, only the principal lines of the human forms. The inferior frontal convolution, in Man the seat of the centre of (articular) speech, is of the most simple, although typically human form. But we can also conceive the cerebral convolutions of *Pithecanthropus* developed from the still more simple ones of the Gibbons. Now it is a fact of general observation that with related animals the cerebral convolutions become more complicated according to the cerebrum increasing in size.

"In short, I consider *Pithecanthropus* to be a descendant of less spezialized (less long-armed) ancestors of the Gibbons (*Prohylobatides*), a descendant which had assumed the erect posture. Through this the arms and the hands, were, at least to a great extent liberated from locomotor functions, enabled to such tactile and prehensile faculties as we find reflected in the very much increased brain volume and the exceedingly human morphology of the cerebral surface."

The best information about the real morphologic, and taxonomic, character of the *Pithecanthropus* fossils, however, would be given by the publication of very good photographic and plaster cast representations. I regret, that I have not been enabled to effectuate such publications, until in the years 1924 to 1926, the Koninklijke Akademie van Wetenschappen te Amsterdam much obliged those who took a particular interest in this important matter of science, and much indebted me, by rendering assistance in effectuating both modes of publication; the exact figures, accompanied by a very concise text [1]), appeared at the same time that accurate casts became available.

From the figures and the text the hylobatoid similarity, or at least resemblance in a number of points, is again evidenced, together with not a few points of difference.

I may quote some of the principal items from the first paper:

p. 267. "The fronto-biorbital index (Schwalbe) *i.e.* its ratio to the external orbital facial breadth, which latter I estimate at 115 mm as a minimum, is at most 79. In a cranium of a Hylobates agilis, in many respects resembling that of Pithecanthropus, this index is 78.4. The post-orbital length index (Schwalbe) is 25.5, the distance between the orbital constriction and the bregma-transversal being 47 mm. This index is 25 at the same cranium of Hylobates agilis. The whole pre-cerebral part of the frontal bone is hylobatoid, like the rest.

"It is seen that to the sagittal arc-length of the cranial vault the frontal bone contributes 100 mm, the parietal bone 90 mm. and the upper part of the tabular portion of the occipital bone 45 mm. This is an entirely different ratio between the two first divisions of the vault from that in Homo sapiens and Homo neandertalensis, where the parietal arc is longer than, or equally long as, the frontal arc. In the Hylobatidae, on the other hand, the parietal bone is much shorter, in comparison with the frontal arc, than in Pithecanthropus. The latter's fronto-parietal index is 90, that of the large (p. 268) gibbon genus, Symphalangus, 53 on an average, and that of the small gibbon species has an average value of 42. The relative length of the frontal squama diminishes with increasing size of the body, in consequence of the diminution that this entails of the ratio between the volumes of the

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[1]) EUG. DUBOIS, On the principal characters of the cranium and the brain, the mandible and the teeth of Pithecanthropus erectus. Proceedings Kon. Akademie van Wetenschappen te Amsterdam. Vol. 27. Nos. 3 and 4, pp. 265—278. — Figures of the calvarium and endocranial cast, a fragment of the mandible and three teeth of Pithecanthropus erectus. Ibid. Nos. 5 and 6, pp. 459—464. Eleven Plates of telephotographic reproductions, natural size. — On the principal characters of the femur of Pithecanthropus erectus. Proceedings Kon. Akademie van Wetenschappen te Amsterdam. Vol. 29, No. 5, pp. 730—743, 33 figures. — Figures of the femur of Pithecanthropus erectus. Ibid. pp. 1275—1277. Four Plates. At the same time plaster casts of all the fossil remains were available for distribution.

orbita (with the eye) and the cranial cavity (with the brain). Besides, in Pithecanthropus the cranial cavity has become more spacious in another way than through the greater size of the body.

"The lower part of the tabular portion of the occipital bone, the pars nuchalis, bends downwards and forwards at an apparently not very obtuse angle. But this obtuse angle was much larger in the intact skull, because at the fossil calvarium the loss of substance greatly increases towards the edge of the fragment, so that this edge only still consists in the knife-like lamina interna. In the intact state of the skull Pithecanthropus resembled the Hylobatidae in the steepness of the planum nuchale.

"The torus occipitalis presents the closest resemblance to that of Symphalangus syndactylus.

p. 269. ... There is also a powerful crista occipitalis interna, which feature distinguishes Pithecanthropus from the Hylobatidae, which have a wide groove, as impression of their round, barer vermis cerebelli, and agrees with the large Simiidae and Man... With respect to the sulcus transversus the parieto-mastoid suture lies exactly as in Hylobates. Also as regards the situation of the internal asterion Pithecanthropus agrees entirely with Hylobates. For the postasterial index (the ratio of the distance between the asterion and the occipital pole and the endocranial length) I find 15.5 in Pithecanthropus, 15.8 in Hylobates, and an average value of 24 in human skulls of different races.

"The form of the skull of Pithecanthropus is on the whole not human; nor is it a transition of any type of manlike apes to the human type. The agreement with the anthropoid canial type, particularly that of the small gibbon species, of the genus Hylobates, may on the other hand be called perfect, taking into consideration the inevitable deviation in the proportions in consequence of the ratio of the volume of the brain and the eye varying with the increasing bulk (weight of the body) and cephalization. For with increasing bulk the eye increases somewhat less in volume than the brain, and on account of the much higher cephalization of Pithecanthropus the brain was besides enlarged far beyond the homoneuric ratio. The fossil cranium is not more highly arched, has no less receding forehead, and the pre-cerebral part of the frontal bone projects equally far forward as in the Apes. The constriction ("Einschnürung") behind the orbitae is also perfectly pithecroid in its depth and its situation at a greater distance from the supraorbital border; so is the place where the external auditory meatus must have been, and the form of the crista supramastoidea. Perfectly pithecoid was further the shape of the torus occipitalis transversus and the value of the angle at which the nuchal plate of the occipital bone bands foreward and downward. In all these points Pithecanthropus is distinguished, no less strongly than the Anthropoid Apes, from Neandertal Man. From the latter character of the fossil skull it may be conluded that also the condyles of the occipital bone were placed in the same way at the skull, so that the head was not equipoised on the spinal column as in modern Man, but was carried by strong nuchal muscles and ligaments as in Apes. It is not to be seen by the structure of the skull that Pithecanthropus deserves the name of (p. 270) *erectus,* assigned to him an account of the features of the femur......

"It is certain that the erect posture of the body, which clearly appears from the shape of the femur, was not such a perfect one as in Man; the correlation, at least, did not extend to the skull.

"Nor can the skull, however, have belonged to an Anthropoid Ape, because the relatively very large skull as regards shape presents a close, nay striking resemblance to the skull of a small Hylobates species, the smallest of the Anthropoid Apes, whereas judging not only from the femur and the molar teeth, but also from the skull itself, Pithecanthropus must have surpassed the size of a large chimpanzee, and very much that of a medium-sized man. Those smallest Manlike Apes distinguish themselves especially by their large neurocranium in the proportion of their splanchnocranium, the facial part of the skull. This is a consequence of the law governing the relation between the quantity of the brain and the bulk of the body in closely allied species. Small species have in general larger brains in comparison with their body weight than large ones of the same genus, sometimes also of the same family, in general than large homoneuric species (species with the same organisation of the nervous system). Judging by the linear dimensions, and as will appear subsequently, by the cranial capacity, Pithecanthropus *as an Anthropoid Ape* would have been a giant of about 300 kg weight, much larger than the heaviest gorilla. But Pithecanthropus was *not* such a giant. This appears not only from the dimensions of the femur, but also in the skull from the great distance of the temporal line, the boundary of the surface origin of the musculus temporalis from the median line, an indication that this masticatory muscle was weak with respect to the size of the neurocranium, though in such a gigantic Anthropoid Ape as Pithecanthropus must have been, it would have had on the contrary a comparatively much larger area of origin, to find sufficient space at the then *relatively* small neurocranium. We may refer to the cranial crests of large male gorillas and orang-utans.

"That the fossil skull bears such striking resemblance to that of Hylobates, this dwarfish genus among the Anthropoid Apes, does not therefore compel us to class Pithecanthropus for this reason among this family, but it also gives support to the view that the Hylobatidae are actually to be considered as genuine and then the most primitive Manlike Apes, though such as are particularly specialized by their long arms and sabre-shaped canine teeth."

What induced VON KOENIGSWALD, who took little notice of the papers from which I have quoted, to his idea indicated in the beginning of this communication, was obviously the only taxonomic sentence, occurring at the end of my paper of 1924, which reads: "The approach of the mandible and the teeth, as also of the femur, to the human type, and the large cranial capacity, added to considerations on the brain-quantities in nearly allied mammalian genera, all this leads me to the conclusion that Pithecanthropus should be considered as a member, but a distinct genus, of the family of the Hominidae."

This sentence was intended to reflect the current views of taxonomists at that time, from which mine, expressed in 1894, and to which I still adhere, is entirely different.

It is most regrettable, that for the interpretation of the important discoveries of human fossils in China and Java, WEIDENREICH, VON KOENIGSWALD and WEINERT were thus guided by preconceived opinions, and consequently did not contribute to, on the contrary they impeded, the advance of knowledge of man's place in nature, what is commonly called human phylogenetic evolution. Real advance appears to depend on obtaining material data in an unbiassed way, such as the *Pithecanthropus* fossil remains and instructing material about the phylogenetic growth of the brain.

Mathematics. — *Strahlenkongruenzen mit einem zweigliedrigen System von quadratischen Regelscharen.* Von O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 30, 1940.)

1. Wir beweisen den folgenden Satz: *die einzigen Strahlenkongruenzen, welche* ∞^2 *quadratische Regelscharen enthalten sind:*

a. die Sehnenkongruenz (1.3) *einer kubischen Raumkurve und die duale Kongruenz* (3.1);

b. eine spezielle Kongruenz (2.2), *nämlich diejenige der Geraden welche eine quadratische Fläche berühren und eine Gerade treffen, die die quadratische Fläche berührt;*

c. die Kongruenz (1.2) *der Geraden, welche einen Kegelschnitt und eine Gerade treffen, welche mit dem Kegelschnitt einen Punkt gemein hat;*

d. die Kongruenzen (1.1).

2. Die Linienkoordinaten p_{ij} einer Geraden genügen der Gleichung

$$p_{12}\,p_{34}+p_{13}\,p_{42}+p_{14}\,p_{23}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Bildet man die Gerade ab auf die Punkte der quadratischen vierdimensionalen Varietät Ω mit der Gleichung (1), dann wird eine Strahlenkongruenz abgebildet auf eine auf Ω gelegene Fläche, eine quadratische Regelschar auf einen auf Ω gelegenen Kegelschnitt. Die Frage nach den Strahlenkongruenzen met ∞^2 quadratischen Regelscharen wird also ersetzt durch die nach denjenigen Flächen auf Ω, welche ∞^2 Kegelschnitte enthalten.

Nun ist nach einem von DARBOUX und SEGRE herrührenden Satz [1]) *jede Fläche, welche* ∞^2 *Kegelschnitte enthält, entweder die* VERONESE*sche Fläche vierter Ordnung in* R_5, *oder eine Projektion dieser Fläche.* Wir müssen also untersuchen ob die VERONESEsche Fläche, bzw. eine Projektion dieser Fläche, auf Ω liegen kann.

3. Sind y_i $(i=1, 2, 3, 4, 5)$ projektive Koordinaten in R_5, dann wird die VERONESEsche Fläche F am einfachsten dargestellt durch die Gleichungen

$$y_1=\lambda^2,\ y_2=\mu^2,\ y_3=\nu^2,\ y_4=\mu\nu,\ y_5=\nu\lambda,\ y_6=\lambda\mu \quad . \quad . \quad . \quad (2)$$

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[1]) Vgl. z.B. BERTINI, Einführung in die projektive Geometrie mehrdimensionaler Räume, 369 (1924); ein einfacher Beweis bei BOMPIANI, Roma Acc. Lincei (5), **30**, 248—251 (1921).

wo λ, μ, ν homogene Parameter sind. F gehört den 6 linear unabhängigen quadratischen Varietäten $y_1 y_2 - y_6^2 = 0$ u.s.w. und $y_1 y_4 - y_5 y_6 = 0$ u.s.w. an, sodasz F auf jedem Exemplar eines linearen Systems von ∞^5 quadratischen Varietäten liegt. Wie man leicht einsieht, sind diese nicht sämtlich Kegel; wir haben damit schon festgestellt, dasz eine Fläche F auf Ω liegen kann. Es zeigt sich auszerdem, dasz es solche Varietäten gibt, welche die Signatur null haben, woraus hervorgeht dasz es auf Ω eine Fläche F geben kann, deren Gleichungen durch eine *reelle* lineare Transformation aus (2) hervorgehen. Wir nehmen folgendes Beispiel:

$$\left.\begin{array}{lll} p_{12} = \mu\nu, & p_{13} = \nu\lambda, & p_{14} = \lambda\mu, \\ p_{34} = \lambda^2 - \lambda\mu, & p_{42} = \mu^2 - \mu\nu, & p_{23} = \nu^2 - \nu\lambda \end{array}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

Die Fläche liegt auf (1). Ist $A = (a_1 a_2 a_3 a_4)$ ein beliebiger fester Punkt, $X = (x_1 x_2 x_3 x_4)$ ein veränderlicher Punkt, dann gehört die Gerade $A\,X$ der Kongruenz (3) an, wenn

$$\begin{array}{lll} a_1 x_2 - a_2 x_2 = \mu\nu & , a_1 x_3 - a_3 x_1 = \nu\lambda & , a_1 x_4 - a_4 x_1 = \lambda\mu \\ a_3 x_4 - a_4 x_3 = \lambda(\lambda - \mu) & , a_4 x_2 - a_2 x_4 = \mu(\mu - \nu) & , a_2 x_3 - a_3 x_2 = \nu(\nu - \lambda) \end{array}$$

woraus nach Elimination der x_i hervorgeht:

$$\left.\begin{array}{r} (a_1 + a_2)\lambda - a_3\mu - a_1\nu = 0 \\ -a_1\lambda + (a_1 + a_3)\mu - a_4\nu = 0 \\ -a_2\lambda - a_1\mu + (a_1 + a_4)\nu = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Diese Gleichungen deren Determinante gleich null ist, haben im allgemeinen *eine* Lösung für die Parameterwerte λ, μ, ν, sodasz die Ordnung der Kongruenz (3) *eins* beträgt. A ist ein singulärer Punkt, wenn das System (4) den Rang eins hat, also wenn

$$\left.\begin{array}{r} a_1^2 + a_1 a_2 + a_2 a_3 = 0 \\ a_1^2 + a_1 a_3 + a_3 a_4 = 0 \\ a_1^2 + a_1 a_4 + a_4 a_2 = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A liegt also auf dem Durchschnitt dreier quadratischen Kegel, welche je zwei eine Erzeugende gemein haben; der Ort von A ist also eine kubische Raumkurve. Ihre Parametergleichungen sind z.B.

$$a_1 = t(t+1),\ a_2 = t^2(t+1),\ a_3 = -(t+1)^2,\ a_4 = -t. \quad . \quad . \quad (6)$$

Die Sehne, welche die Punkte t_1 und t_2 verbindet, ist diejenige Gerade der Kongruenz, wofür gilt

$$\lambda : \mu : \nu = 1 : t_1 t_2 : (t_1 + 1)(t_2 + 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

während umgekehrt die Gerade mit den Parameterwerten λ, μ, ν die Kurve in den zwei Punkten trifft, deren Parameter der Gleichung

$$\lambda t^2 + (\lambda + \mu - \nu) t + \mu = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

genügen. *Die Kongruenz* (3) *ist mithin die Sehnenkongruenz der kubischen Raumkurve* (6) und hat die Klasse *drei.*

Die Träger der ∞^2 quadratischen Regelscharen, welche der Kongruenz angehören, sind die ∞^2 quadratischen Flächen, welche die Kurve enthalten. Die polare Verwandtschaft in Bezug auf Ω ordnet F die VERONESEsche Fläche F^1 zu mit den Gleichungen

$$\left.\begin{array}{lll} p_{12} = \lambda^2 - \lambda\mu \,, & p_{13} = \mu^2 - \mu\nu \,, & p_{14} = \nu^2 - \nu\lambda \\ p_{34} = \mu\nu & , p_{42} = \nu\lambda & , p_{23} = \lambda\mu. \end{array}\right\} \quad . \; . \; . \; . \; . \quad (9)$$

Sie ist die Abbildung einer Kongruenz (3.1).

4. Wir untersuchen jetzt die Kongruenzen, deren Bild auf Ω eine in einem vierdimensionalen Raum gelegene Projektion von F ist und welche also offenbar einem linearen Komplex angehören.

Wird F aus einem Punkte P in R_5 auf einen R_4 projeziert, dann haben wir *drei* Fälle zu berücksichtigen: 1. P ist ein allgemeiner Punkt; 2. P liegt in einer Kegelschnittebene der Fläche F; 3. P ist ein Punkt von F. Die drei möglichen Projektionen wollen wir bzw. mit F_1, F_2 und F_3 bezeichnen.

5. Wird F aus einem allgemeinen Punkt projeziert, dann ist die Projektion eine Fläche vierter Ordnung, die ∞^2 Kegelschnitte und keine Geraden enthält. Ist F durch die Gleichungen (2) vorgelegt und wählen wir P in dem Punkt (111000), der nicht in einer Kegelschnittebene von F liegt und ist $x_1 = 0$ der Projektionsraum, so erhält F_1 die Gleichungen:

$$x_2 = \mu^2 - \lambda^2 \,, \; x_3 = \nu^2 - \lambda^2 \,, \; x_4 = \mu\nu \,, \; x_5 = \nu\lambda \,, \; x_6 = \lambda\mu. \quad . \; . \quad (10)$$

Wie man leicht einsieht gibt es keine quadratische Gleichung in x_2, x_3, x_4, x_5, x_6 welcher (10) identisch genügt. Das heiszt also dasz es keine quadratischen Varietäten gibt, welche F_1 enthalten. F_1 kann also niemals auf Ω liegen. *Es gibt keine Strahlenkongruenzen, deren Bild eine Fläche F_1 ist.*

6. Jetzt nehmen wir das Projektionszentrum P in einem Punkt der in einer Kegelschnittebene von F, aber nicht auf F liegt. Ein solcher Punkt ist z.B. (000001); er liegt nämlich in der Ebene $x_3 = x_4 = x_5 = 0$, die F schneidet in dem Kegelschnitt $x_1 = \lambda^2$, $x_2 = \mu^2$, $x_6 = \lambda\mu$, welcher den Punkt P nicht enthält. Ist $x_6 = 0$ der Projektionsraum, so hat F_2 die folgenden Gleichungen:

$$x_1 = \lambda^2 \,, \; x_2 = \mu^2 \,, \; x_3 = \nu^2 \,, \; x_4 = \mu\nu \,, \; x_5 = \nu\lambda \quad . \; . \; . \; . \quad (11)$$

woraus hervorgeht, dasz F_2 eine Fläche vierter Ordnung ist, welche ∞^2 Kegelschnitte enthält und eine Gerade l, nämlich $x_3 = x_4 = x_5 = 0$.

Während durch F_1 keine quadratischen Varietäten möglich waren, finden wir dasz F_1 gelegen ist auf die ∞^1 quadratischen Varietäten eines Büschels. Zwei Exemplare hiervon sind

$$x_1 x_3 - x_5^2 = 0 \, , \, x_2 x_3 - x_4^2 = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (12)$$

Sämtliche Varietäte des Büschels sind Kegel ersten Ranges; l ist der Ort der Spitzen.

Hieraus geht hervor, dasz wenn F_2 auf Ω liegen soll, der vierdimensionale Raum, welcher F_2 enthält, diese Fundamentalvarietät in einem Kegel schneidet. Die Kongruenz, welche F_2 zum Bild hat, gehört also einem *speziellen* Komplex an. Ist a der Träger dieses Komplexes, dann ist der Bildpunkt von a die Spitze des Kegels und da diese Spitze auf F selber liegt, wird die Gerade a auch selbst zur Kongruenz gehören. Diese Kongruenz enthält weiter einen Strahlenbüschel (das Bild der Geraden l); zu diesem Büschel gehört die Gerade a.

Eine Fläche F_2 die auf Ω liegt, kann mittels einer linearen Transformation der Koordinaten aus (11) hervorgehen. Wir nehmen z.B.

$$\left.\begin{array}{l} p_{12} = \nu^2 \qquad , p_{13} = \nu(\lambda - \mu) \; , p_{14} = \mu^2 + \lambda^2 \\ p_{34} = \mu^2 - \lambda^2 \, , p_{42} = \nu(\lambda + \mu) \, , p_{23} = 0 \end{array}\right\} \quad . \; . \; . \; . \quad (13)$$

Der spezielle Komplex, welchem die Kongruenz angehört, hat die Gleichung $p_{23} = 0$; der Träger a ist also die Gerade $X_1 X_4$, wenn $X_i \; (i = 1, 2, 3, 4)$ die Eckpunkte des Koordinatentetraeders sind. Die Gerade auf F_2 hat die Gleichungen $p_{12} = p_{13} = p_{42} = p_{23} = 0$, der bezügliche Strahlenbüschel hat das Zentrum X_4, liegt in der Ebene $x_2 = 0$ und enthält a. Weil F_2 die Basisfigur eines Büschels quadratischer Varietäten ist, gehören die Strahlen der Kongruenz einem quadratischen Komplex an. Dieser hat die Gleichung

$$p_{13}^2 + p_{42}^2 - 2 p_{12} p_{14} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

Die Kongruenz ist also der Durchschnitt eines linearen und eines quadratischen Komplexes und deshalb eine Kongruenz (2.2). Weil sie eine singuläre Gerade hat, nämlich a, berühren ihre Strahlen eine quadratische Fläche[2]. Diese Fläche findet man als Ort der Punkte, für welche die zwei hindurchgehenden Kongruenzstrahlen zusammenfallen. Die Gleichung von Q ist:

$$x_1^2 - x_3^2 + 2 x_2 x_4 = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (15)$$

Die Gerade a berührt diese Fläche in (0001); der Strahlenbüschel der Kongruenz ist diejenige der Tangenten der Fläche in diesem Punkt.

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[2] STURM, Liniengeometrie, II, 328 (1893).

Wir finden also dasz *die Kongruenz welche* F_2 *als Bild hat eine spezielle* (2.2) *Kongruenz ist und erzeugt wird durch diejenigen Tangenten einer quadratischen Fläche Q, welche eine Gerade a treffen, welche Q berührt.*

Eine lineare Gleichung in den Parametern λ, μ, ν stellt auf der VERONESEsche Fläche (2) einen Kegelschnitt dar und dasselbe gilt für die Fläche F (11). Wir erhalten also eine quadratische Regelschar unsrer Kongruenz indem wir in (13) $\nu = a\lambda + b\mu$ substituieren (a und b nicht beide null). Für die Gleichung des Trägers dieser Regelschar finden wir nach kurzer Rechnung

$$\left.\begin{aligned} 2(a+b)\,x_1 x_2 - (a^2-b^2)\,x_1 x_3 + \\ + 2x_2^2 - 2(a-b)\,x_2 x_3 - (a+b)^2 x_2 x_4 + (a^2+b^2)\,x_3^2 = 0. \end{aligned}\right\} \quad . \quad (16)$$

Auf jeder dieser Flächen liegt augenscheinlich die Gerade a ($x_2 = x_3 = 0$). Dasz jede Fläche (17) die Fläche Q berührt, geht aus der Tatsache hervor, dasz man (17) folgendermaszen schreiben kann

$$\{(a+b)\,x_1 + 2x_2 - (a-b)\,x_3\}^2 - (a+b)^2 (x_1^2 - x_3^2 + 2x_2 x_4) = 0. \quad (17)$$

Wir haben also: *die Träger der* ∞^2 *Regelscharen der durch Q und a bestimmten Kongruenz sind die* ∞^2 *quadratischen Flächen, welche a enthalten und Q berühren in einem Kegelschnitt welcher durch den Berührungspunkt von a und Q geht.*

7. Wird die VERONESEsche Fläche F aus einem ihrer Punkten P projeziert, dann entsteht eine Fläche dritter Ordnung F_3 die auszerhalb ∞^2 Kegelschnitte ∞^1 Gerade enthält. Diese sind die Projektionen der Kegelschnitte auf F, welche durch P gehen. F_3 hat auszerdem eine Leitgerade b, welche sämtliche Erzeugende trifft und die Schnittgerade ist des Projektionsraumes und die Tangentenebene von F in P. Wählen wir P in dem Punkt (001000), mit den Parameterwerten $\lambda = 0$, $\mu = 0$, $\nu = 1$ und ist $x_3 = 0$ der Projektionsraum, dann wird F_3 dargestellt durch

$$x_1 = \lambda^2,\ x_2 = \mu^2,\ x_4 = \mu\nu,\ x_5 = \nu\lambda,\ x_6 = \lambda\mu \quad . \quad . \quad . \quad (18)$$

mit den Leitgeraden $x_1 = x_2 = x_6 = 0$. F_3 liegt auf einem linearen System von ∞^2 quadratischen Varietäten, wovon drei linear unabhängige sind

$$x_1 x_2 - x_6^2 = 0,\ x_1 x_4 - x_5 x_6 = 0,\ x_2 x_5 - x_4 x_6 = 0. \quad . \quad . \quad (19)$$

Die Varietäten des Systems sind sämtlich Kegel und zwar im allgemeinen solche ersten Ranges mit der Spitze auf F_3. Eine Fläche F_3 kann also auf Ω liegen, aber der Durchschnitt des vierdimensionalen Raumes welcher sie enthält und Ω ist ein Kegel und die bezügliche Kongruenz gehört also wie im vorhergehenden Falle einem speziellen linearen Komplex an. Die Kongruenz enthält weiterhin ∞^1 Stralenbüschel und jede Gerade der Kongruenz gehört einem dieser Büschel an. Die Gleichungen der Kongruenz sind z.B.

$$\left.\begin{array}{l} p_{12}=\lambda^2\ ,\ p_{13}=\lambda\mu\ \ ,\ p_{14}=\mu^2 \\ p_{34}=\mu\nu\ ,\ p_{42}=-\nu\lambda\ ,\ p_{23}=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Der spezielle lineare Komplex $p_{23}=0$ hat den Träger a, die Gerade $X_1\,X_4$. Die Leitgerade b von F_3 hat die Gleichungen $p_{12}=p_{13}=p_{14}=0$, der zugeordnete Strahlenbüschel hat das Zentrum X_4 und liegt in der Ebene $x_1=0$.

Eine Kongruenzgerade, welche nicht zu diesem Büschel gehört, trifft die Ebene des Büschels in dem Punkt $(0, p_{12}, p_{13}, p_{14})$ also in $(0, \lambda^2, \lambda\mu, \mu^2)$; der Ort der Schnittpunkte ist der Kegelschnitt K mit den Gleichungen $x_1=0$, $x_2\,x_4-x_3^2=0$, welcher durch x_4 geht.

Die Kongruenz ist also die (1.2) *Kongruenz der Geraden welche den Kegelschnitt K und die Gerade a treffen, wobei K und a einen gemeinschaftlichen Punkt haben.*

Sie ist also ein Sonderfall der Sehnenkongruenz einer kubischen Raumkurve. Die ∞^2 quadratischen Flächen, welche die Regelscharen der Kongruenz tragen sind diejenigen des Netzes welches die zerfallene Kurve als Basisfigur hat. Man kann die Kongruenz auch als einen Sonderfall der (2.2) Kongruenz auffassen: die quadratische Fläche Q ist jetzt ein Kegelschnitt.

Die duale Figur der Kongruenz ist die (2.1) Kongruenz der Geraden, welche a treffen und einen quadratischen Kegel berühren, welche von der Gerade a berührt wird. Sie kann als einen Sonderfall der Kongruenz (3.1) aber auch als einen der Kongruenz (2.2) betrachtet werden: die Fläche Q ist jetzt ein Kegel.

8. Eine Kongruenz, welche ein in einem Raum von weniger als vier Dimensionen gelegenes Bild hat, ist eine (1.1) Kongruenz. Sie enthält ∞^3 Regelscharen.

Mathematics. — *On associated lines in* S_4. By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 30, 1940.)

1. If in fourdimensional space we have four lines in general position there exists a fifth line which has the property that it is met by all planes meeting the four lines. The figure of these *five associated lines* and the ∞^2 intersecting planes is well-known. W. VAN DER WOUDE [1]) has considered the case that the four lines l_1, l_2, l_3 and l_4 have a common transversal l and he has found the remarkable fact, that this special case far from giving trivial results, leads to some interesting theorems. His main results are the following: there are ∞^1 associated lines, each of them meeting the ∞^2 planes which meet the four lines; the ∞^1 lines build up a surface V_3^2 of dimension two and degree three; the ∞^1 lines are determined by an arbitrary quadruple of them; each of the ∞^1 lines meets the transversal l; each of the ∞^2 intersecting planes meets the V_3^2 in a conic; the ∞^2 planes build up a variety of the third class.

In what follows we shall give some supplementary notes on VAN DER WOUDE's article: the equations for the varieties get a simpler form if one chooses another system of coordinates; for the V_2^3 a geometrical construction will be given, the variety shown to be connected with VERONESE's surface, the group of collineations which leave the associated lines invariant discussed.

2. The common transversal l of the four lines l_1, l_2, l_3, l_4 is taken as $x_3 = x_4 = x_5 = 0$; one of the planes meeting these lines (and not containing l) the plane U, is taken as $x_1 = x_2 = 0$. The line l_i meets the line l in the point A_i and U in the point B_i. The coordinates (x_1, x_2) on l are chosen in such a way that we have: $A_1 = (1, 0)$, $A_2 = (0, 1)$, $A_3 = (1, 1)$, $A_4 = (a, 1)$; the coordinates (x_3, x_4, x_5) of B_i in U, these being four points in general position, can be chosen arbitrarily; we take $B_1 = (1, 0, 0)$, $B_2 = (0, 0, 1)$, $B_3 = (1, 1, 1)$, the coordinates of B_4 are for the present (p, q, r). A point P_1 on l_1 is then given by $(\lambda_1\, 0\, 1\, 0\, 0)$, P_2 on l_2 by $(0\, \lambda_2\, 0\, 0\, 1)$, P_3 on l_3 by $(\lambda_3\, \lambda_3\, 1\, 1\, 1)$; the plane through $P_1 P_2 P_3$ has the equations:

$$x_1 = k\lambda_1 + m\lambda_3,\ x_2 = l\lambda_2 + m\lambda_3,\ x_3 = k + m,\ x_4 = m,\ x_5 = l + m,$$

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[1]) VAN DER WOUDE, Ueber vier Gerade in R_4. Proc. Ned. Akad. v. Wetensch., Amsterdam, **43**, 946—954 (1940).

$\lambda_1, \lambda_2, \lambda_3$ being constants and k, l, m homogeneous parameters. If this plane meets the line l_4, in the point $(\lambda a, \lambda, p, q, r)$ say, the following equations are due:

$$m = q,\ l = r - q,\ k = p - q,\ (r-q)\lambda_2 + q\lambda_3 = \lambda,\ (p-q)\lambda_1 + q\lambda_3 = a\lambda$$

Eliminating k, l, m, λ we have

$$(p-q)\lambda_1 - a(r-q)\lambda_2 + (1-a)q\lambda_3 = 0 \quad \ldots \ldots \quad (1)$$

In order to give this relation a simple form, we determine p, q, r in the following way: $p = a^2$, $q = a$, $r = 1$; a being $\neq \infty, 0, 1$ the points B_i **are** of general position. The relation between $\lambda_1, \lambda_2, \lambda_3$ becomes

$$\lambda_3 = \lambda_1 + \lambda_2 \quad \ldots \ldots \ldots \ldots \quad (2)$$

The parameter equations of a plane, meeting the four lines (but not containing l and thus belonging to VAN DER WOUDE's *second system*) are thus:

$$x_1 = (k+m)\lambda_1 + m\lambda_2,\ x_2 = m\lambda_1 + (l+m)\lambda_2,\ x_3 = k+m,\ x_4 = m,\ x_5 = l+m$$

so that these planes are given by the equations:

$$x_1 = \lambda_1 x_3 + \lambda_2 x_4, \quad x_2 = \lambda_1 x_4 + \lambda_2 x_5 \,. \quad \ldots \ldots \quad (3)$$

the constants λ_1, λ_2 showing that there are ∞^2 planes.

Now the point is if there are other lines which intersect all these planes and those of the first system. Such a line m meets l. If $A = (t\ 1\ 0\ 0\ 0)$ is the point of intersection with l, $B = (0\ 0\ x_3\ x_4\ x_5)$ that with U, according to (1) we must have

$$(x_3 - x_4)\lambda_1 - t(x_5 - x_4)\lambda_2 + (1-t)x_4\lambda_3 = 0$$

or

$$(x_3 - x_4)\lambda_1 - t(x_5 - x_4)\lambda_2 + (1-t)x_4(\lambda_1 + \lambda_2) = 0$$

or

$$\lambda_1(x_3 - t x_4) + \lambda_2(x_4 - t x_5) = 0.$$

The line m meets all the planes if this equation holds for all values of λ_1 and λ_2, thus if

$$x_3 : x_4 : x_5 = t^2 : t : 1. \quad \ldots \ldots \ldots \quad (4)$$

If A varies, the locus of B is indeed a *conic* in the plane U. It is clear that it passes through the points B_1, B_2, B_3 and B_4, for $t = \infty, 0, 1$ and a respectively. It is uniquely determined by the property that it is the conic of the pencil of conics through B_i on which the points B_1, B_2, B_3 and B_4 have the same harmonic ratio, viz. a, as the points A_1, A_2, A_3 and A_4 have on l.

As the plane U is an arbitrary one of the ∞^2 planes, we have found a determination of the conic in each of the planes, using the points of intersection of such a plane with the original four lines.

Moreover we have got the following *construction of the set of the* ∞^1 *associated lines*: if in one of the planes intersecting the four lines in B_i, we determine the conic in the way mentioned just now, fixing a projective connection between the points of l and the points of the conic, so that the conjugated points of A_1, A_2, A_3 respectively, are B_1, B_2, B_3 (and thus the conjugate of A_4 is B_4) it follows that an associated line is a line which joins an arbitrary point of l with the conjugate point on the conic.

It is obvious from this construction, that *the set of associated lines is determined by each quadruple of them.*

3. An associated line joining the point $A = (t\,1\,0\,0\,0)$ on l and the point $B = (0\,0\,t^2\,t\,1)$ on U, the surface generated by the ∞^1 associated lines is given by

$$x_1 = \lambda t, \quad x_2 = \lambda, \quad x_3 = t^2, \quad x_4 = t, \quad x_5 = 1 \quad . \quad . \quad . \quad . \quad (5)$$

where λ, t are parameters. By means of intersection with an arbitrary plane the degree of the surface is actually shown to be *three*. An equation $t =$ constant gives one of the associated lines. The conics on the surface are found by intersection with one of the planes (3):

$$x_1 = a\,x_3 + b\,x_4, \quad x_2 = a\,x_4 + b\,x_5$$

so that they are given by a relation

$$\lambda = a\,t + b$$

The equation $\lambda = \infty$ gives the line l, $\lambda = 0$ the plane U. Through a point (λ, t), not on l, pass one associated line and ∞^1 conics; each conic meets each line as is due; two conics have one point in common; through two points (λ_1, t_1) and (λ_2, t_2) not on the same line $(t_1 \neq t_2)$, passes one conic:

$$\lambda = \frac{\lambda_1 - \lambda_2}{t_1 - t_2}\,t + \frac{\lambda_2 t_1 - \lambda_1 t_2}{t_1 - t_2}.$$

The intersection of the surface with a solid is generally a twisted cubic, which meets each of the lines once and each of the conics twice. If the solid passes through one of the lines the cubic breaks up in this line and one of the conics; the intersection with a solid through l consists of l and of two of the associated lines.

4. If in *five*-dimensional space S_5 we consider VERONESE's surface Ω:

$$x_0 = \lambda^2, \quad x_1 = \lambda t, \quad x_2 = \lambda, \quad x_3 = t^2, \quad x_4 = t, \quad x_5 = 1 \quad . \quad . \quad (6)$$

it is easy to see, that *our surface is the projection of the surface of*

VERONESE *on a fourdimensional space* ($x_0=0$)*, the centre of projection being chosen on this surface* (viz. in the point O with the coordinates $x_0=1$, $x_1=x_2=x_3=x_4=x_5=0$)[2]. The tangent plane in O has the equations $x_3=x_4=x_5=0$ and produces the line l. Ω contains ∞^2 conics, ∞^1 passing through an arbitrary point, two conics having a point in common. The projections of the conics not passing through 0 are the ∞^2 conics of the surface V_2^3; *the projections of the conics through O are the ∞^1 associated lines.* Ω being of degree four and being projected from one of its points, it is clear that the projection is of the third degree.

5. A quadric passing through the surface V_2^3 has the equation

$$\alpha\,(x_4^2-x_3x_5)+\beta\,(x_1x_4-x_2x_3)+\gamma\,(x_1x_5-x_2x_4)=0. \quad . \quad . \quad (7)$$

all these ∞^2 quadrics are cones. As VAN DER WOUDE has already pointed out, the V_2^3 may be considered to be the intersection of two such cones, e.g. of $x_1x_4-x_2x_3=0$ and $x_1x_5-x_2x_4=0$ if the plane $x_1=x_2=0$ is left out of account. The vertex of the cone (7) is the point $x_1=\alpha\gamma$, $x_2=-\alpha\beta$, $x_3=\gamma^2$, $x_4=\gamma\beta$, $x_5=\beta^2$, being a point of the surface, viz. $\lambda=\frac{\alpha}{\beta}$, $t=\frac{-\gamma}{\beta}$; the cone with its vertex in the point (λ, t) thus being

$$\lambda\,(x_4^2-x_3x_5)-(x_1x_4-x_2x_3)+t\,(x_1x_5-x_2x_4)=0 \quad . \quad . \quad . \quad (8)$$

There is one cone of the second order:

$$x_4^2-x_3x_5=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

its line of vertices is l and each of the conics can be taken as its basic curve. The set (7) contains ∞^2 pencils of cones, the locus of the vertices being either one of the associated lines or one of the conics, thus showing the two species of pencils of cones, which are possible in S_4[3].

6. If we have four lines in general position, there are ∞^2 planes, which meet these four lines (and the associated fifth line). The locus of these planes is a variety of the third class; that is to say: the condition that a solid with the equation $u_1x_1+u_2x_2+u_3x_3+u_4x_4+u_5x_5=0$ passes through one of the planes, leads to an equation of the third degree for the coordinates u_i. Through a general point of S_4 pass two of the planes, the locus of the points where the two planes coincide with one another is a variety V_3^4 of dimension three and degree four.

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[2] The following theorem is well-known: a surface, in space of any number of dimensions, which contains ∞^2 conics, is the surface of VERONESE or one of its projections; cf. BERTINI, Einführung in die projektive Geometrie mehrdimensionaler Räume, 369 (1924).

[3] BERTINI, l.c. S. 174.

In our special case there are ∞^2 planes just as well; they form two systems, one built up by the planes containing the line l, the other given by the equations (3). A solid passing through the plane $x_1 = a x_3 + b x_4$, $x_2 = a x_4 + b x_5$, has coordinates which satisfy:

$$a u_1 + u_3 = b u_2 + u_5 = b u_1 + a u_2 + u_4 = 0,$$

so that if we eliminate a and b, we obtain the following equation for the variety:

$$u_1^2 u_5 - u_1 u_2 u_4 + u_2^2 u_3 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

which is also satisfied by the planes of the first system: $u_1 = u_2 = 0$. Just as in the general case the variety is of the third class.

Through a point in S_4 pass two planes, one of the first system and one of the second, as is seen from (3). The points where the two planes coincide, are the points of the planes belonging to both systems. It is easy to see, that they are the points of the cone (9). The V_3^4 mentioned above in the general case, degenerates here in a quadratic cone of the second order, counted twice.

7. As four lines in general position have no absolute invariants, there is always a collineation in S_4 which interchanges the lines of a set of five associated lines. Consequently there is a group of 5! collineations, which leaves the set as a whole invariant. There are no collineations (besides identity) which leave each of the four (or five) lines invariant.

In our case the four lines have clearly one invariant: the anharmonic ratio of the four points A_1, A_2, A_3, A_4. There are ∞^3 collineations, which leave each of the four lines (and each of the associated lines) invariant, viz.:

$$\left.\begin{array}{r} x_1^1 = p x_1 + q x_3 + r x_4, \quad x_2^1 = p x_2 + q x_4 + r x_5, \\ x_3^1 = x_3, \quad x_4^1 = x_4, \quad x_5^1 = x_5 \end{array}\right\} \quad . \quad . \quad (11)$$

or

$$\lambda^1 = p \lambda + q t + r, \quad t^1 = t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Each point of l is invariant and each plane of the first system too. As regards the planes of the second system, the one with the equations $x_1 = a x_3 + b x_4$, $x_2 = a x_4 + b x_5$ is conjugated to the plane $x_1 = a' x_3 + b' x_4$, $x_2 = a' x_4 + b' x_5$ where $a' = pa + q$, $b' = pb + r$. Each collineation leaves one plane of the second system invariant, namely the plane

$$a = \frac{q}{1-p}, \quad b = \frac{r}{1-p}.$$

Moreover there are collineations which interchange the four lines according to the *Vierergruppe*; if $(A_1 A_2 A_3 A_4)$ is harmonic there are eight, if it is aequianharmonic there are twelve collineations interchanging the four lines.

If we consider the whole set of associated lines, there is a group of ∞^6 collineations leaving them invariant, namely the group induced by the parametertransformations:

$$\lambda^1 = \frac{p\lambda + qt + r}{st + m}, \quad t^1 = \frac{nt + l}{st + m} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

that is

$$\left.\begin{aligned}
x_1^1 &= pn\, x_1 + pl\, x_2 + qn\, x_3 + (ql + rn)\, x_4 + rl\, x_5 \\
x_2^1 &= ps\, x_1 + pm\, x_2 + qs\, x_3 + (qm + rs)\, x_4 + rm\, x_5 \\
x_3^1 &= n^2 x_3 + 2\,nl\, x_4 + l^2 x_5 \\
x_4^1 &= ns\, x_3 + (nm + ls)\, x_4 + lm\, x_5 \\
x_5^1 &= s^2 x_3 + 2\,ms\, x_4 + m^2 x_5
\end{aligned}\right\} \quad . \qquad (14)$$

where $p(nm - ls) \neq 0$. The group is isomorphic with the group of affinities of the plane.

The planes of the second system are transformed according to the equations:

$$a^1 = \frac{pm\,a - ps\,b + qm - rs}{nm - ls}, \quad b^1 = \frac{-pl\,a + pn\,b - ql + rn}{nm - ls}. \qquad (15)$$

Mathematics. — *Die Gleichberechtigung gleichförmig beschleunigter Beobachter für die elektromagnetischen Erscheinungen.* Von J. HAANTJES. (Communicated by Prof. J. A. SCHOUTEN.)

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Zusammenfassung.

Die Konforminvarianz der MAXWELLschen Gleichungen ist von mehreren Autoren bewiesen worden. Zuerst von BATEMAN und CUNNINGHAM 1) im Jahre 1910. Im ersten Paragraphen dieser Arbeit wird gezeigt dass auch die Formel (1.5) für das Berechnen des Feldes einer Punktladung konforminvariant ist. Es wird weiter für den Fall der speziellen Relativitätstheorie eine physikalische Interpretation der Konforminvarianz angegeben. Es sei B' ein Beobachter, der in bezug auf ein kartesisches System, das System des Beobachters B, in gleichförmig beschleunigter Translationsbewegung ist. Sodann existiert ein euklidischer Fundamentaltensor $'g_{hi} = \sigma^2 g_{hi}$, für welchen B' ebenso ein bevorzugter Beobachter ist, wie B in bezug auf g_{hi}. Die Beschleunigung lässt sich also mittels einer konformen Transformation wegtransformieren. Aus der Konforminvarianz der elektromagnetischen Gesetze folgt dann, dass die Beobachter B und B' gleichberechtigt sind. Die Raum-Zeit-Welt muss daher, allerdings nur was die elektromagnetischen Erscheinungen anbelangt, nicht als ein euklidischer Raum, sondern als ein konformeuklidischer Raum betrachtet werden. Nicht die Gruppe der LORENTZtransformationen aber die konforme Transformationsgruppe bestimmt die Geometrie der Raum-Zeit-Welt.

Am Schlusz wird gezeigt dass B und B' mittels eines geladenen Massenteilchens dasselbe elektromagnetische Feld messen, wenn angenommen wird dass m bei der konformen Transformation $g_{hi} \rightarrow \sigma^2 g_{hi}$ einen Faktor σ^{-1} erhält. Dies bringt mit sich, dass die Dimension $[ML]$ bei konformen Transformationen invariant ist, was mit der Konstanz von h (Dimension $[ML^2 T^{-1}]$) im Einklang ist.

§ 1. *Die Konforminvarianz der elektromagnetischen Gleichungen.*

Bekanntlich kann das elektromagnetische Feld $F_{ij} (h, i, j, \ldots = 1, 2, 3, 4)$ von einem Potential φ_i abgeleitet werden

$$F_{ij} = 2\,\partial_{[i}\,\varphi_{j]} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1.1)$$

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1) E. CUNNINGHAM, Proc. London Math. Soc., 8, 77 (1910).
H. BATEMAN, Proc. London Math. Soc., 8, 223 (1910).

In einer Raum-Zeit-Welt mit einer konformen Metrik, d.h. in einem Raum in dem g_{ih} überhaupt nur bis auf einen beliebigen (nicht konstanten) Zahlenfaktor gegeben ist, existiert eine Tensordichte $\mathfrak{G}_{ih}$ vom Gewicht $-{}^1/_2$, die man folgendermassen aus g_{hi} erhält:

$$\mathfrak{G}_{ih} = (-\mathfrak{g})^{-\frac{1}{4}}\, g_{ih} \ ; \ \mathfrak{g} = \mathrm{Det}\,(g_{ih}) \quad . \quad . \quad . \quad . \quad . \quad (1.2)$$

Das elektromagnetische Feld genügt nun den folgenden Gleichungen

$$\left.\begin{aligned} &a) \quad \partial_{[j}\, F_{ih]} = 0 \\ &b) \quad \partial_j\, \mathfrak{F}^{jh} = -\mathfrak{s}^h \ ; \ \mathfrak{F}^{hi} = \mathfrak{G}^{hj}\, \mathfrak{G}^{il}\, F_{jl} \\ &c) \quad \partial_j\, \mathfrak{s}^j = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (1.3)$$

wo $\mathfrak{s}^h$ steht für die Stromvektordichte vom Gewicht $+1$. Dies sind die MAXWELLschen Gleichungen in der konforminvarianten Gestalt [2]).

In der *speziellen Relativitätstheorie* ist das Linienelement euklidisch und das Bezugssystem lässt sich derart wählen dass gilt

$$g_{11} = g_{22} = g_{33} = -g_{44} = -1 \ , \ g_{hi} = 0 \ (h \neq i). \quad . \quad . \quad . \quad (1.4)$$

In diesem Falle ist es möglich den Potentialvektor φ_i als Lösung von (1.3*b*) näher anzugeben. Jede Punktladung bestimmt nämlich ein Feld und das Gesamtfeld erhält man durch Addition (oder Integration) dieser Felder. Es genügt daher einen Ausdruck für das Feld einer einzigen Punktladung e anzugeben.

Dazu betrachten wir einen beliebigen Punkt P. Es sei Q den Schnittpunkt der Weltlinie der Punktladung mit dem Nullkegel der Vergangenheit von P und u^h ein Tangentialvektor dieser Weltlinie in Q. Der Feldwert des durch die Ladung e hervorgerufenen Potentials φ_i im Punkte P ist nun bekanntlich

$$\varphi_i = -\frac{e}{4\pi}\,\frac{u_i}{u_h\, R^h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.5)$$

Dabei ist R^h der Radiusvektor $\overrightarrow{QP}$, also ein Nullvektor ($R^h R_h = 0$). Das elektromagnetische Feld der Punktladung lässt sich dann aus (1.5) mittels (1.1) berechnen.

Wir werden jetzt zeigen, dass auch die Formel (1.5) konforminvariant ist. Damit ist hier folgendes gemeint:

Wird eine konforme Transformation $'g_{ih} = \sigma^2 g_{ih}$ ausgeführt, derart, dass der Fundamentaltensor euklidisch bleibt, und wird in bezug auf ein

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[2]) J. A. SCHOUTEN und J. HAANTJES, Ueber die konforminvariante Gestalt der Maxwellschen Gleichungen und der elektromagnetischen Impulsenergiegleichungen, Physica I 869—872 (1934). Vgl. auch J. A. SCHOUTEN und J. HAANTJES, Ueber die konforminvariante Gestalt der relativistischen Bewegungsgleichungen, Proc. Kon. Akad. v. Wetensch., Amsterdam, **39** 1059—1065 (1936).

zu $'g_{ih}$ gehöriges kartesisches Koordinatensystem (h'), (d.h. ein System, für welches $'g_{ih}$ den Bestimmungszahlen (1. 4) hat) der Potentialvektor

$$\varphi'_{i'} = -\frac{e}{4\pi}\frac{'g_{i'j'}\,u^{j'}}{'g_{h'j'}\,u^{h'}\,'R^{j'}} \quad \ldots\ldots \quad (1.6)$$

gebildet, so unterscheidet sich φ'_i von φ_i nur um einen Gradientvektor. φ'_i und φ_i führen daher zu dasselbe elektromagnetische Feld. Anders gesagt: Die Formeln (1. 5) für das Berechnen des elektromagnetischen Feldes sind konforminvariant.

Die Transformation, die das zu g_{hi} gehörige euklidische Bezugssystem (h) in ein zu $'g_{hi}$ gehöriges euklidisches Bezugssystem (h') überführt, ist eine konforme Transformation. Nun lässt sich jede konforme Transformation aus Inversionen und LORENTZtransformationen zusammensetzen[3]). Wir brauchen daher nur die oben genannte Invarianz für eine Inversion zu beweisen denn die Invarianz bei LORENTZtransformationen ist unmittelbar klar aus (1. 5). Dazu betrachten wir die folgende Inversion (hier als Koordinatentransformation aufgefasst):

$$x^{h'} = \frac{x^h}{x^i\,x_i}\,\delta^{h'}_h \quad \ldots\ldots \quad (1.7)$$

Differentiation von (1. 7) führt zu den Transformationskoeffizienten eines Vektors

$$A^{h'}_h = \delta^{h'}_h\,\frac{1}{x^i\,x_i} - \frac{2\,x_h\,x^j}{(x^i\,x_i)^2}\,\delta^{h'}_j \quad \ldots\ldots \quad (1.8)$$

In bezug auf (h') hat g_{ih} die Bestimmungszahlen

$$g_{i'h'} = A^{i\;h}_{i'h'}\,g_{ih} \quad \ldots\ldots \quad (1.9)$$

Eine kleine Rechnung zeigt

$$g_{i'h'} = (x^j\,x_j)^2\,g_{ih}\,\delta^i_{i'}\,\delta^h_{h'} \quad \ldots\ldots \quad (1.10)$$

In bezug auf (h') hat der transformierte Fundamentaltensor die Bestimmungszahlen (1. 4), also

$$'g_{i'h'} = g_{ih}\,\delta^i_{i'}\,\delta^h_{h'}, \quad \ldots\ldots \quad (1.11)$$

woraus sich in Verbindung mit (1. 10) ergibt

$$'g_{ih} = \sigma^2\,g_{ih}\;:\;\sigma^2 = (x^i\,x_i)^{-2}. \quad \ldots\ldots \quad (1.12)$$

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[3]) Vgl. J. HAANTJES, Conformal representations of an *n*-dimensional euclidean space with a non-definite fundamental form on itself, Proc. Kon. Akad. v. Wetensch., Amsterdam, **40**, 700—705 (1937). Vgl. auch SCHOUTEN–STRUIK, Einführung II, Noordhoff, S. 210.

Wir werden zuerst den Vektor $'R^{h'}$ berechnen. Die Koordinaten von P bzw. Q seien x^h bzw. y^h. Es ist dann

$$R^h = x^h - y^h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.13)$$

mit

$$R^h R_h = x^h x_h - 2\, x^h y_h + y^h y_h = 0 . \quad . \quad . \quad . \quad . \quad (1.14)$$

Die Bestimmungszahlen von $'R^{h'}$ in bezug auf (h') sind die Koordinatendifferenzen von P und Q in bezug auf (h')

$$'R^{h'} = x^{h'} - y^{h'} = \left(\frac{x^h}{x^i x_i} - \frac{y^h}{y^i y_i} \right) \delta^{h'}_h . \quad . \quad . \quad . \quad (1.15)$$

Es sei nun $y^h = y^h(\tau)$ die Weltlinie der Punktladung; τ sei ein beliebiger Parameter. Der Tangentialvektor u^h in Q hat in bezug auf (h') die Bestimmungszahlen (vgl. (1.8))

$$u^{h'} = (A^{h'}_h)_Q\, u^h = \left(\delta^{h'}_h - \frac{2\, y_h y^j}{y^i y_i} \delta^{h'}_j \right) \frac{1}{y^i y_i} \frac{dy^h}{d\tau} \; ; \; \left(u^h = \frac{dy^h}{d\tau} \right) . \, (1.16)$$

Aus (1.11), (1.15) und (1.16) finden wir den Nenner von (1.6)

$$\left. \begin{aligned} 'g_{h'i'}\, 'R^{h'} u^{i'} &= g_{lm}\, \delta^l_{h'}\, \delta^m_{i'} \left(\frac{x^h}{x^j x_j} - \frac{y^h}{y^j y_j} \right) \delta^{h'}_h \left(\delta^{i'}_i - \frac{2\, y_i y^k}{y^j y_j} \delta^{i'}_k \right) \frac{1}{y^j y_j} \frac{dy^i}{d\tau} \\ &= g_{hi} \left(\frac{x^h}{x^j x_j} - \frac{y^h}{y^j y_j} \right) \frac{1}{y^j y_j} \left(\frac{dy^i}{d\tau} - 2\, y_l \frac{y^i}{y^j y_j} \frac{dy^l}{d\tau} \right) \\ &= \frac{1}{(x^i x_i)(y^i y_i)} \left\{ x_h \frac{dy^h}{d\tau} - \frac{x^i x_i}{y^i y_i} y_h \frac{dy^h}{d\tau} - \frac{2\, x_i y^i}{y^i y_i} y^h \frac{dy^h}{d\tau} + \frac{2\, x^i x_i}{y^i y_i} y_h \frac{dy^h}{d\tau} \right\} \end{aligned} \right\} . \, (1.17)$$

Dieser Ausdruck lässt sich mittels (1.14) vereinfachen. Wir erhalten

$$'g_{h'i'}\, 'R^{h'} u^{i'} = \frac{1}{(x^i x_i)(y^i y_i)} \left(x_h \frac{dy^h}{d\tau} - y_h \frac{dy^h}{d\tau} \right) = \frac{R^h u_h}{(x^i x_i)(y^i y_i)} . \quad . \quad . \quad (1.18)$$

Für die Berechnung von φ'_i muss der Vektor $u^{h'}$ parallel (im Sinne von $'g_{ih}$) von Q nach P verschoben werden. In bezug auf das System (h') änderen sich die Bestimmungszahlen bei dieser pseudoparallelen Verschiebung nicht. Wir brauchen nach (1.6) die mittels $'g_{ij}$ gebildeten kovarianten Bestimmungszahlen des pseudoparallel verschobenen Vektors. Diese sind nach (1.16)

$$\left. \begin{aligned} u'_{i'} = {}'g_{i'j'}\, u^{j'} &= g_{ij}\, \delta^i_{i'}\, \delta^j_{j'} \left(\delta^{j'}_h - \frac{2\, y_h y^m}{y^l y_l} \delta^{j'}_m \right) \frac{1}{y^l y_l} \frac{dy^h}{d\tau} \\ &= \frac{1}{y^l y_l} \delta^i_{i'} \left(g_{ih} - \frac{2\, y_h y_i}{y^l y_l} \right) \frac{dy^h}{d\tau} . \end{aligned} \right\} . \quad (1.19)$$

Die Bestimmungszahlen in bezuf auf (h) lassen sich nun unter Benutzung von (1. 8), (1. 14) und (1. 19) finden. Es ergibt sich

$$'u_i = (A_i^{i'})_P \, 'u_{i'} = \frac{1}{(x^j x_j)(y^j y_j)} \left\{ g_{ih} + \frac{2 y_h (x_i - y_i)}{y^j y_j} - \frac{2 x_i (x_h - y_h)}{x^j x_j} \right\} \frac{dy^h}{d\tau}. \quad (1.20)$$

Zu jedem Punkte $P(x^h)$ gehört ein bestimmter Punkt Q auf der Kurve $y^h = y^h(\tau)$, also ein bestimmter Wert von τ. Man kann daher τ als Funktion von x^h betrachten. Diese Funktion folgt aus (vgl. 1. 14))

$$x^i x_i + y^i(\tau) y_i(\tau) - 2 x^i y_i(\tau) = 0. \quad . \quad . \quad . \quad . \quad (1.21)$$

Differentiation nach x^i ergibt

$$x_i - y_i - (x_h - y_h) \frac{dy^h}{d\tau} \frac{\partial \tau}{\partial x^i} = 0. \quad . \quad . \quad . \quad . \quad . \quad (1.22)$$

also (vgl. (1. 16))

$$\frac{\partial \tau}{\partial x^i} = \frac{x_i - y_i}{R^h u_h} \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.23)$$

Wir kehren jetzt nach (1. 18) und (1. 20) zurück und bilden den neuen zu $'g_{hi}$ gehörigen Potentialvektor φ_i'. Wir betrachten die Bestimmungszahlen des neuen Potentialvektors in bezug auf das alte Bezugssystem (h). Es ist nach (1. 6), (1. 18), (1. 20) und (1. 23)

$$\left.\begin{aligned} \varphi_i' &= -\frac{e}{4\pi} \frac{'u_i}{'g_{h'i'} \, 'R^{h'} u^{i'}} \\ &= -\frac{e}{4\pi} \left\{ \frac{u_i}{R^h u_h} + \frac{2 y_h}{y^j y_j} \frac{dy^h}{d\tau} \frac{\partial \tau}{\partial x^i} - \frac{2 x_i}{x^j x_j} \right\} \\ &= \varphi_i - \frac{e}{4\pi} \partial_i \log (y^j y_j) + \frac{e}{4\pi} \partial_i \log (x^j x_j) \\ &= \varphi_i + \frac{e}{8\pi} \{ \partial_i \log (\sigma)_Q - \partial_i \log (\sigma)_P \}. \end{aligned}\right\} \quad . \quad . \quad (1.24)$$

φ_i' und φ_i unterscheiden sich also nur um einen Gradientvektor, woraus hervorgeht

$$F_{ij}' = F_{ij}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.25)$$

Wir haben daher bewiesen:

Wird das elektromagnetische Feld in der speziellen Relativitätstheorie ausgehend von einem euklidischen Fundamentaltensor $'g_{ij} = \sigma^2 g_{ij}$ *statt von* g_{ij} *mittels derselben Formeln berechnet, so erhält man dasselbe Resultat.* Die Konforminvarianz der MAXWELLschen Gleichungen mit berücksichtigend erhalten wir den folgenden Satz:

Die elektromagnetischen Gleichungen haben in bezug auf die zu $'g_{ih}$ *gehörigen kartesischen Bezugssysteme dieselbe Form als in bezug auf die zu* g_{ih} *gehörigen kartesischen Systeme. Diese Bezugssysteme sind also alle gleichberechtigt.*

§ 2. *Die zu einem gleichförmig beschleunigten Beobachter gehörige konforme Transformation.*

Betrachten wir jetzt die Weltlinie eines Punktes (oder eines Beobachters B'), der in bezug auf ein kartesisches Bezugssystem B in gleichförmig beschleunigter Translationsbewegung ist. Zur Zeit $t=0$ sei B' relativ zu B in Ruhe. Wir wählen die Richtung der x^1-Achse derart, dass diese mit der Bewegungsrichtung von B' zusammenfällt. Die Gleichung der Weltlinie von B' ist dann wie im Folgenden verifiiert wird

$$a\,\{(x^1)^2-(x^4)^2\}+2\,x^1=0\ ,\ x^2=x^3=0\ . \quad . \quad . \quad . \qquad (2.\,1)$$

Diese Weltlinie ist eine Hyperbel, derer Asymptotenrichtungen Nullrichtungen sind. Die Gleichung der Hyperbel lautet in Parameterform

$$x^1=\frac{2}{a\,(\lambda^2-1)}\ ,\ x^4=\frac{2\,\lambda}{a\,(\lambda^2-1)}\ ,\ x^2=x^3=0. \quad . \quad . \quad . \qquad (2.\,2)$$

Daraus geht hervor

$$\frac{dx^1}{ds}=\frac{2\,\lambda}{\lambda^2-1},\ \frac{dx^4}{ds}=\frac{\lambda^2+1}{\lambda^2-1},\ ds^2=(dx^4)^2-(dx^1)^2=\frac{4}{a^2}\frac{(d\,\lambda)^2}{(\lambda^2-1)^2}, \qquad (2.\,3)$$

also

$$\frac{d^2\,x^1}{ds^2}=\frac{a\,(\lambda^2+1)}{\lambda^2-1},\ \frac{d^2\,x^4}{ds^2}=\frac{2\,a\,\lambda}{\lambda^2-1}. \quad . \quad . \quad . \quad . \quad . \qquad (2.\,4)$$

Für die Länge des Vektors $\frac{d^2\,x^h}{ds^2}$ finden wir

$$\left\{\left(\frac{d^2\,x^1}{ds^2}\right)^2-\left(\frac{d^2\,x^4}{ds^2}\right)^2\right\}^1=|\,a\,|. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.\,5)$$

Die Beschleunigung von B' ist daher in jedem Punkte P der Weltlinie in bezug auf einen Beobachter, der mit konstanter Geschwindigkeit (in bezug auf B) bewegt und dieselbe Geschwindigkeit hat wie B' in P, gleich

$$a\,c^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.\,6)$$

denn für diesen Beobachter ist $ds^2=c^2\,dt^2$.

Wir fragen jetzt nach der konformen Koordinatentransformation

$$x^{h'}=f^{h'}\,(x^i). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2.\,7)$$

welche so beschaffen ist, dass die Weltlinie von B' in bezug auf das System (h') die Gleichung

$$x^{1'} = 0 \ , \ x^{2'} = 0 \ , \ x^{3'} = 0 \quad . \ . \ . \ . \ . \ . \ . \quad (2.8)$$

hat. Es gibt mehrere solche Transformationen, doch unter bestimmten physikalisch plausibelen Einschränkungen nur eine. Wir werden nämlich voraussetzen, dass die Transformation folgenden Forderungen genügt:

I. Der Ursprung des Bezugssystems sei invariant.

II. Die Transformation ändere sich nicht wenn man entweder

a) x^2 durch $-x^2$ ersetzt
oder *b*) x^3 durch $-x^3$ ersetzt
oder *c*) x^4 durch $-x^4$ ersetzt (also Vergangenheit und Zukunft verwechselt).

III. In bezug auf (h') habe g_{ih} im Ursprung dieselbe Bestimmungszahlen als in bezug auf (h), also $(-1, -1, -1, +1)$. Anders gesagt: Im Ursprung sei $\sigma = 1$.

Die allgemeinste infinitesimale konforme Transformation die den Forderungen I und III genügt ist

$$\left.\begin{aligned}
x^{1'} &= \frac{1}{N}\{x^1 + a_{12}x^2 + a_{13}x^3 + a_{14}x^4 + a_{15}g_{ij}x^i x^j\}\\
x^{2'} &= \frac{1}{N}\{-a_{12}x^1 + x^2 + a_{23}x^3 + a_{24}x^4 + a_{25}g_{ij}x^i x^j\}\\
x^{3'} &= \frac{1}{N}\{-a_{13}x^1 - a_{23}x^2 + x^3 + a_{34}x^4 + a_{35}g_{ij}x^i x^j\}\\
x^{4'} &= \frac{1}{N}\{+a_{14}x^1 + a_{24}x^2 + a_{34}x^3 + x^4 + a_{45}g_{ij}x^i x^j\},
\end{aligned}\right\} \quad (2.9)$$

wo

$$N = 1 - 2a_{15}x^1 - 2a_{25}x^2 - 2a_{35}x^3 + 2a_{45}x^4 \quad . \ . \ . \quad (2.10)$$

ist und die Koeffizienten $a_{\lambda\varkappa}$ $(\lambda, \varkappa = 1, \ldots, 5)$ infinitesimal sind. Die Forderung II führt sodann zu der folgenden infinitesimalen Transformation:

$$\left.\begin{aligned}
x^{1'} &= \frac{x^1 + \frac{1}{2}b\,g_{ij}x^i x^j}{1 - b\,x^1} &&\text{oder} = x^1 + b\{(x^1)^2 + \tfrac{1}{2}g_{ij}x^i x^j\}\\
x^{2'} &= \frac{x^2}{1 - b\,x^1} &&\text{oder} = x^2 + b\,x^1 x^2\\
x^{3'} &= \frac{x^3}{1 - b\,x^1} &&\text{oder} = x^3 + b\,x^1 x^3\\
x^{4'} &= \frac{x^4}{1 - b\,x^1} &&\text{oder} = x^4 + b\,x^1 x^4,
\end{aligned}\right\} \quad (2.11)$$

bei Vernachlässigung von b^2. Die zugehörige endliche Transformation lässt sich finden als Lösung des Differentialgleichungssystems

$$\left.\begin{aligned} &a)\quad \frac{dx^1}{db} = (x^1)^2 + \tfrac{1}{2}\, g_{ij}\, x^i\, x^j \\ &b)\quad \frac{dx^a}{db} = x^1\, x^a,\ (a = 2, 3, 4). \end{aligned}\right\} \quad \ldots\ldots \quad (2.\,12)$$

Aus (2. 12) geht hervor

$$\frac{d\,(g_{ij}\, x^i\, x^j)}{db} = 2\, x^1 \left\{ (x^4)^2 - (x^3)^2 - (x^2)^2 - (x^1)^2 - \tfrac{1}{2}\, g_{ij}\, x^i\, x^j \right\} = x^1\, g_{ij}\, x^i\, x^j, \quad (2.\,13)$$

also (mit (2. 12 a))

$$\frac{d^2\, x^1}{db^2} = 2\, x^1 \frac{dx^1}{db} + \tfrac{1}{2}\, x^1\, g_{ij}\, x^i\, x^j = 3\, x^1 \frac{dx^1}{db} - (x^1)^3. \quad .. \quad (2.\,14)$$

Die Lösung dieser Differentialgleichung wird am einfachsten erhalten, wenn man zuerst $x^1 = -\dfrac{d}{db} \log y$ substituiert. Diese Substitution führt zu $y''' = 0$, woraus hervorgeht

$$x^1 = \frac{a^1 - \beta\, b}{1 - a^1\, b + \tfrac{1}{2}\, \beta\, b^2}, \quad \ldots\ldots\ldots \quad (2.\,15)$$

wo a^1 und β Konstanten sind. Substitution dieses Ausdrucks in (2. 12*b*) ergibt

$$x^a = \frac{a^a}{1 - a^1\, b + \tfrac{1}{2}\, \beta\, b^2},\ (a = 2, 3, 4). \quad \ldots \quad (2.\,16)$$

Aus (2. 12a), (2. 15) und (2. 16) folgt dann

$$\beta = -\tfrac{1}{2}\, g_{ij}\, a^i\, a^j. \quad \ldots\ldots\ldots \quad (2.\,17)$$

Die Konstanten a^i werden aus den Anfangswerten von x^h bestimmt. Es ergibt sich $a^h = (x^h)_0$. Nennen wir die Anfangswerte wieder x^h und die transformierten Bestimmungszahlen $x^{h\prime}$, so lautet die *endliche* konforme Transformation (*b* ist hier nicht mehr infinitesimal)

$$\left.\begin{aligned} x^{1\prime} &= \frac{x^1 + \tfrac{1}{2}\, b\, g_{ij}\, x^i\, x^j}{1 - b\, x^1 - \tfrac{1}{4}\, b^2\, g_{ij}\, x^i\, x^j}, \\ x^{a\prime} &= \frac{x^a}{1 - b\, x^1 - \tfrac{1}{4}\, b^2\, g_{ij}\, x^i\, x^j},\ (a = 2, 3, 4). \end{aligned}\right\} \quad .\ . \quad (2.\,18)$$

Die inverse Transformation erhält man, indem man in (2. 18) den Parameter b durch $-b$ ersetzt.

In bezug auf (h') hat die Weltlinie (2. 1) von B' die Gleichung

$$(a+b)\{(x^{1'})^2-(x^{4'})^2\}+2\,x^{1'}=0,\ x^{2'}=x^{3'}=0. \quad . \quad . \quad (2.\,19)$$

Wählen wir daher $b=-a$, so erhält diese Weltlinie die erwünschte Gestalt

$$x^{1'}=x^{2'}=x^{3'}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.\,20)$$

Die Transformation (2. 18) *mit* $b=-a$ *ist also die einzige konforme Transformation, welche den drei oben genannten Forderungen genügt und zu der Gleichung* (2. 20) *führt.*

Betreffs der Transformationen (2. 18) können wir noch folgendes bemerken.

1. Die Transformationen (2. 18) bilden eine einparametrige Gruppe mit b als Parameter. Bezeichnen wir die zum Parameter b gehörige Transformation mit T_b, so lässt sich das Multiplikationsgesetz schreiben als

$$T_b\,.\,T_c=T_{b+c}\ ;\ T_0=I\ \text{(Identität)}. \quad . \quad . \quad . \quad . \quad . \quad (2.\,21)$$

Der Beweis dieses Gesetzes werden wir hier unterdrücken.

2. In der schon oben zitierten Arbeit [4]) ist die folgende konforme Transformation abgeleitet:

$$x^{h'}=\frac{x^h-\frac{1}{2}\,b^h\,x^i\,x_i}{1-b^i\,x_i},\quad b^i\,b_i=0. \quad . \quad . \quad . \quad . \quad . \quad (2.\,22)$$

Die Transformation (2. 18) ist nun das Produkt zweier solcher Transformationen, die eine mit $b^h\left(-\frac{b}{2},0,0,-\frac{b}{2}\right)$, die andere mit $b^h\left(-\frac{b}{2},0,0,\frac{b}{2}\right)$.

3. Die Bestimmungszahlen des Fundamentaltensors g_{ij} in bezug auf (h') lassen sich mittels der Formel $g_{h'i'}=A^{h\,i}_{h'i'}\,g_{hi}$ berechnen. Es ergibt sich

$$\left.\begin{aligned} g_{1'1'}=g_{2'2'}=g_{3'3'}=-g_{4'4'}=-(1-b\,x^1-\tfrac{1}{4}\,b^2\,g_{ij}\,x^j\,x^i)^2,\\ g_{h'i'}=0\,(h'\neq i').\end{aligned}\right\}\quad (2.\,23)$$

§ 3. *Die physikalische Folgerung.*

Der *speziellen Relativitätstheorie* liegt folgendes Postulat zu Grunde. Wird ein Koordinatensystem B so gewählt, dass in bezug auf dasselbe die physikalischen Gesetze in ihrer einfachsten Form gelten (bevorzugte Koordinatensysteme), so gelten *dieselben* Gesetze auch in bezug auf jedes andere Koordinatensystem B', das relativ zu B in *gleichförmiger*

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[4]) L.c. [3]). S. 704.

Translationsbewegung begriffen ist. Es gibt bevorzugte Beobachter und zu jedem bevorzugten Beobachter gehört in der Raum-Zeit-Welt ein bestimmtes kartesisches Koordinatensystem. Diese Beobachter haben relativ zu einander konstante Geschwindigkeiten, während die zugehörigen Bezugssysteme aus einander hervorgehen mittels LORENTZtransformationen. Für jeden Beobachter ist die Vakuum-Lichtgeschwindigkeit gleich c, also konstant. Die bevorzugten Beobachter sind alle *gleichberechtigt.*

Wir können nun aber, wenigstens was die elektromagnetischen Gesetze der Physik anbelangt, etwas weiter gehen und die Klasse der bevorzugten Beobachter erweitern. Auch die Beobachter, die in bezug auf ein kartesisches System in *gleichförmiger beschleunigter Translationsbewegung* sind, können wir als bevorzugt betrachten. Das sind also die Beobachter, deren Weltlinien verallgemeinerte Kreise sind. Wir haben nämlich im zweiten Paragraphen gezeigt, dass man die Beschleunigung mittels einer konformen Transformation wegtransformieren kann, wie man eine Geschwindigkeit mittels einer LORENTZtransformation wegtransformieren kann und im ersten Paragraphen ist gezeigt, dass die elektromagnetischen Gesetze konforminvariant sind, d.h. sich bei konformen Transformationen nicht ändern.

Wir können dieses Resultat etwas näher präzisieren. Betrachten wir dazu den Beobachter B', der in bezug auf ein kartesisches System (h), das System des Beobachters B, die folgende Weltlinie hat (vgl. (2. 1))

$$a\{(x^1)^2-(x^4)^2\}+2x^1=0,\ x^2=x^3=0. \quad . \quad . \quad . \quad . \quad (3.\,1)$$

B' ist relativ zu B also in gleichförmiger beschleunigter Bewegung mit der Beschleunigung ac^2. Das zu B' gehörige bevorzugte Bezugssystem wird, wie gezeigt ist, aus (h) erhalten mittels der Transformation (vgl. (2. 18))

$$\left.\begin{aligned} x^{1'} &= \frac{x^1-\frac{1}{2}a\,g_{ij}\,x^i x^j}{1+a\,x^1-\frac{1}{4}a^2 g_{ij}\,x^i x^j} \\ x^{a'} &= \frac{x^a}{1+a\,x^1-\frac{1}{4}a^2 g_{ij}\,x^i x^j} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.\,2)$$

In bezug auf (h') ist die Weltlinie von B'

$$x^{1'}=x^{2'}=x^{3'}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.\,3)$$

Im ersten Paragraphen haben wir gezeigt dass für B' die elektromagnetischen Gleichungen in bezug auf (h') dieselbe Form haben. Für ihn ist (h') das bevorzugte System. Auch die Vakuum-Lichtgeschwindigkeit ist für B' gleich c. B' benutzt aber nicht denselben Fundamentaltensor wie B. Sein Fundamentaltensor hat in bezug auf (h') dieselben Bestimmungszahlen als g_{ih} in bezug auf (h). Man erhält den Fundamentaltensor von B' aus dem Fundamentaltensor von B durch Multiplikation mit einem Faktor (vgl. (2. 23)). Was die elektromagnetischen Erscheinungen anbelangt sind B und B' also auch gleichberechtigt.

Dies bringt mit sich, dass wir unsere Auffassung betreffs der Raum-Zeit-Welt etwas ändern. *Die Raum-Zeit-Welt ist was die elektromagnetischen Erscheinungen anbelangt kein euklidischer Raum, aber ein konformeuklidischer Raum.* Denn B und B' sind gleichberechtigt, also auch die beiden zugehörigen Fundamentaltensoren. *Die Transformationen* (3. 2) *bilden zusammen mit den* LORENTZ*transformationen die konforme Gruppe, welche die Geometrie der Raum-Zeit-Welt bestimmt.*

Bis jetzt ist noch keine Masse in Betracht genommen. Das Feld F_{ij} wird aber mit Hilfe eines geladenen Massenteilchens gemessen, z.B. durch die Beschleunigung des Teilchens, welche das Feld hervorbringt. Dazu misst man die Beschleunigung des Teilchens einmal mit Feld und einmal ohne Feld. Es handelt sich also um die *Aenderung* der Beschleunigung, die das Feld F_{ij} verursacht. Wir können jetzt noch beweisen, dass *die zwei Beobachter B und B' auf diese Weise das selbe Massresultat erhalten, wenn angenommen wird, dass die Masse m bei der Transformation* $g_{ih} \rightarrow \sigma^2 g_{ih}$ *einen Faktor* σ^{-1} *bekommt.* Dies bringt also mit sich, dass sich $m(-\mathfrak{g})^{\frac{1}{4}}$ bei der konformen Transformation nicht ändert.

Die Bewegungsgleichung eines geladenen Teilchens für den Beobachter B:

$$m \frac{d^2 x^h}{d\tau^2} = \ldots \qquad (3.4)$$

enthält nämlich das Glied

$$\frac{e}{c} \frac{dx^i}{d\tau} F_{ij} g^{hj}. \qquad (3.5)$$

Dabei ist m die Masse für B während τ mittels $g_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 1$ bestimmt ist. Dieser Ausdruck hat in bezug auf ein bevorzugtes System (h') des Beobachters B' die Gestalt

$$\frac{e}{c} \frac{dx^{i'}}{d\tau} F_{i'j'} g^{h'j'} = \sigma^3 \frac{e}{c} \frac{dx^{i'}}{d\tau'} F_{i'j'}{}' g^{h'j'}. \qquad (3.6)$$

Das linke Glied der Bewegungsgleichung (3. 4) lautet in bezug auf (h')

$$m \frac{d^2 x^{h'}}{d\tau^2} + \ldots = m \sigma^2 \frac{d^2 x^{h'}}{(d\tau')^2} + \ldots \; ; \; (d\tau')^2 = {}'g_{hi}\, dx^i\, dx^h = \sigma^2 (d\tau)^2, \qquad (3.7)$$

wo nur die Terme, welche die zweite Ableitung enthalten, angeschrieben sind. Die Beobachter sind daher auch was diese Messung anbelangt gleichberechtigt, wenn

$$m' = \sigma^{-1} m \qquad (3.8)$$

ist, wo die Masse für den Beobachter B' mit m' bezeichnet ist. B und B' messen in diesem Falle natürlich nicht dieselbe Beschleunigung aber dieselbe Aenderung der Beschleunigung.

Die Formel (3. 8) bedeutet, dass die Masse m für zwei Beobachter, welche eine Beschleunigung in bezug auf einander haben, verschieden ist. Bei der konformen Transformation $g_{ih} \rightarrow \sigma^2 g_{ih}$ enthält m einen Faktor σ^{-1}. Transformation der Längen mit σ muss also Transformation der Massen mit σ^{-1} mit sich bringen. Die Dimension $[ML]$ ist daher invariant, was man auch aus der Konstanz der PLANCKschen Konstante h (Dimension $ML^2 T^{-1}$) erschliessen könnte[5].

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[5] Vgl. J. A. SCHOUTEN und J. HAANTJES, Ueber die konforminvariante Gestalt der relativistischen Bewegungsgleichungen. Proc. Kon. Akad. v. Wetensch., Amsterdam, **39**, 1063 (1936).

Geology. — *Over een polyëdrische pisoliet van Palimanan bij Cheribon.* Door B. G. ESCHER. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of November 30, 1940.)

Het Rijksmuseum van Geologie en Mineralogie te Leiden ontving in 1937 van Dr. G. H. R. VON KOENIGSWALD een collectie gesteenten van Java, waaronder het in fig. 1 afgebeelde stuk. De vindplaats ervan is: Nabij de warme bronnen bij Dessa Palimanan, Goenoeng Kromong, ten W. van Cheribon.

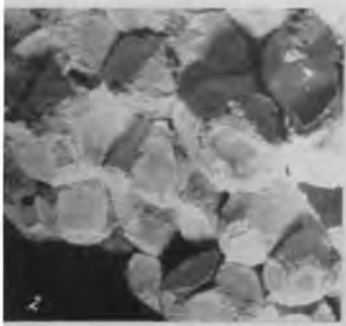
Bij vluchtige beschouwing maakt het gesteente den indruk van een agglomeraat van pseudomorfosen volgens een regulair mineraal. Het wisselend aantal der vlakken en de variatie in de grootte der tweevlakshoeken verraden echter, dat hier van een kristalvorm geen sprake is.

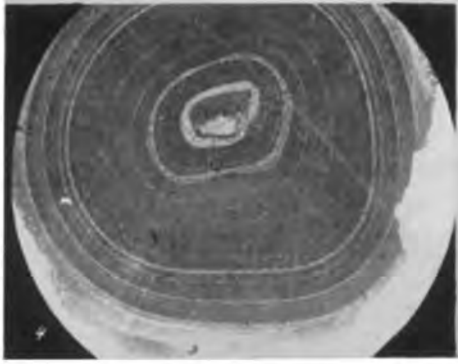
Na slijpen en polijsten van een plat vlak aan het agglomeraat, verraadt zich de pisolitische structuur door een concentrische gelaagdheid van ronde korrels, die door een secundairen neerslag van calciet aan elkaar zijn gekit. Deze secundaire calciet-omhulsels zijn van elke korrel uit gekristalliseerd en raken elkaar in platte vlakken, wier aantal wisselt met het aantal der elkaar omgevende korrels. Op de contactvlakken is de verkitting veelal onvolledig, waardoor het agglomeraat meestal langs deze platte aanrakingsvlakken afbreekt (fig. 2) en slechts bij uitzondering dwars door de korrels of langs concentrische schalen (fig. 3). De grootte der polyëdrische korrels varieert van 2 tot 10 mm in den pisoliet van Palimanan. De kern der korrels wordt door een aggregaat van mikrokristallijnen calciet met onregelmatige begrenzing gevormd. Hieromheen liggen concentrische laagjes van afwisselend mikrokristallijnen en kryptokristallijnen calciet, welke laatste in dunne doorsneden ondoorzichtig is (fig. 4). Op een polijstvlak (fig. 5) zijn de mikrokristallijnen ringen donker, de kryptokristallijnen licht. De reactie van MEIGEN viel negatief uit, evenals de zilverreactie volgens FEIGL en LEITMEIER, die beide door Mej. Dr. C. M. KOOMANS werden uitgevoerd.

De korrels bestaan dus geheel uit calciet en zoowel aragoniet als ktypeïet (LACROIX, 1898 en 1922) ontbreken. Dr. J. A. A. KETELAAR was zoo vriendelijk Röntgenopnamen van dezen pisoliet te maken, zoowel van de korrels als van de kitmassa, waaruit bleek, dat beide uit calciet bestaan.

Is er uitsluitend $CaCO_3$ neergeslagen, dan zijn bij opvallend licht de mikrokristallijnen ringen grijs en de kryptokristallijnen, die het grootste oppervlak innemen, wit. Is er wat kleurstof, vermoedelijk een ijzerverbinding, mee neergeslagen, dan zijn de mikrokristallijnen ringen donker geel-bruin en de kryptokristallijnen heel licht bruin getint.

B. G. ESCHER: OVER EEN POLYËDRISCHE PISOLIET VAN PALIMANAN BIJ CHERIBON.





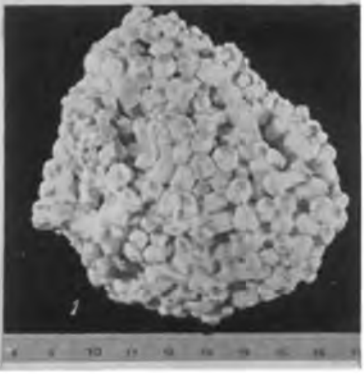


Fig. 1. Polyëdrische pisoliet van Palimanan.

Fig. 2. Het door platte vlakken begrensde breukvlak van dezen pisoliet.

Fig. 3. De concentrische structuur der doorgebroken korrels.

Fig. 4. Microfoto van een korrel in doorvallend licht; donker = kryptokristallijnen, licht = mikrokristallijnen calciet.

Fig. 5. Polijstvlakje van den pisoliet in opvallend licht; licht = kryptokristallijnen, donker = mikrokristallijnen calciet.

Fig. 6. Karlsbad-erwtensteen met ingedeukte korrels.

Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

Een radiaal-vezelige structuur ontbreekt zoowel in de korrels als in de kitmassa.

Polyëdrische pisolieten schijnen zeldzaam te zijn. LACROIX (1898 en 1922) wijst op het verschijnsel, dat de korrels elkaar soms hebben ingedeukt, hetgeen waarschijnlijk maakt, dat het $CaCO_3$ aanvankelijk colloïdaal werd neergeslagen. Dit verschijnsel is o.a. aan de bekende erwtensteen van *Karlsbad* te zien (fig. 6), die echter desondanks langs concentrische schalen afbreekt, en dan ook geen polyëdrischen indruk maakt.

Voorzoover ik kan nagaan, zijn polyëdrische pisolieten slechts door R. R. SHROCK (1930) beschreven van een zwerfsteen, die in *N. Indiana* gevonden is. Hij vond daarin vormen die varieeren van bijna volmaakte bollen tot ellipsoïden, onvolledige cubussen en asymmetrische polyëders met 9 of minder tot 15, veelal 12 vlakken [1]) van verschillende grootte. Die vlakken waren plat, convex of concaaf. De ronde korrels waren klein, met een middellijn van 2.5 tot 3 mm, de polyëdrische hadden een diameter van 6 à 7 mm. De structuur der korrels werd gekarakteriseerd door radiair gerangschikte vezels of prisma's van $CaCO_3$. De kern was rond met licht geel-bruine ringen, die vermoedelijk door ferri-oxyde gekleurd waren. Terwijl de pisoliet van *Karlsbad* op het breukvlak glanzende schalen vertoont, is het breukvlak van den pisoliet van Palimanan dof, met een krijtachtig uiterlijk.

De genese van den pisoliet van Palimanan is vermoedelijk als volgt: In een bronkanaal heerschte een vrij sterke opwaartsche beweging van het warme bronwater. Om kernen van mikrokristallijnen calciet, die in het water zweefden, werden concentrische lagen van $CaCO_3$ afgezet. Nadat het bronkanaal verstopt was en het bronwater een anderen uitweg gevonden had, bezonken de ronde korrels, die vervolgens in min of meer stilstaand bronwater door een secundaire afzetting van calciet verkit werden.

Summary.

The described polyhedral pisolite has been found in the immediate neighbourhood of hot springs. The grains are rounded and have a concentric structure of alternating layers of microcristalline and cryptocristalline calcite. They are surrounded by a secondary precipitation of calcite, which gives the grains their polyhedral shape. Aragonite as well as Ktypeite are absent. The grains have a diameter of 2 to 10 mm.

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[1]) Zie over het optreden van 12 vlakken de volgende mededeeling, waarin ook naar de literatuur verwezen is.

Geology. — *Over de regulaire en hexagonale dichtste bolstapelingen en de deformatie hunner bollen tot dodekaëders tengevolge van compactie.* Door B. G. ESCHER. (Communicated by Prof. L. RUTTEN.)

(Communicated at the meeting of November 30, 1940.)

Sommige gedeelten der bekende pisolieten (erwtensteenen) van Karlsbad bezitten ingedeukte bollen (ESCHER, 1940), die door min of meer platte vlakken begrensd zijn. Zij maken den indruk colloïdaal als bolletjes gevormd te zijn, en, terwijl zij nog plastisch waren, door druk gedeformeerd te zijn (LACROIX (1922).

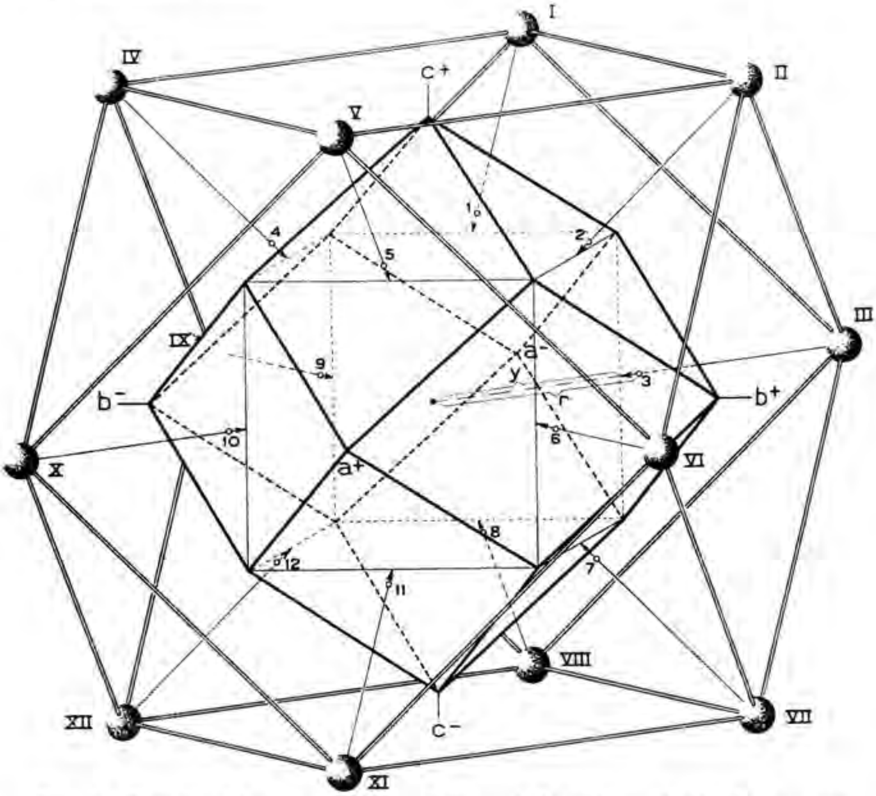
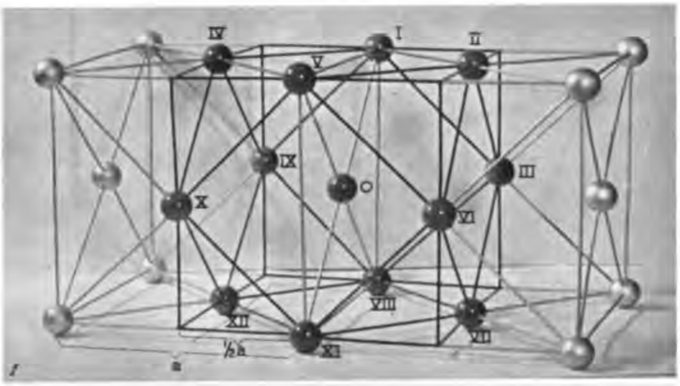


Fig. 4. Draadmodel der regulaire dichtste bolstapeling (loodrecht op een viertallige as) waarvan de hoekpunten I t/m XII de middelpunten van even groote bollen (met straal $= r$) zijn, die een centralen bol in de punten 1 t/m 12 raken. De compactie vindt plaats in de richting der pijlen, waardoor de centrale bol in een rhombendodekaëder verandert, waarvan de zwaartepunten der vlakken met de pijlpunten samenvallen. De lengte van de normalen, uit het middelpunt op de vlakken neergelaten, bedraagt $y = 0.90472\, r$.

B. G. ESCHER: OVER DE REGULAIRE EN HEXAGONALE DICHTSTE BOLSTAPELINGEN EN DE DEFORMATIE HUNNER BOLLEN TOT DODEKAËDERS TENGEVOLGE VAN COMPACTIE.



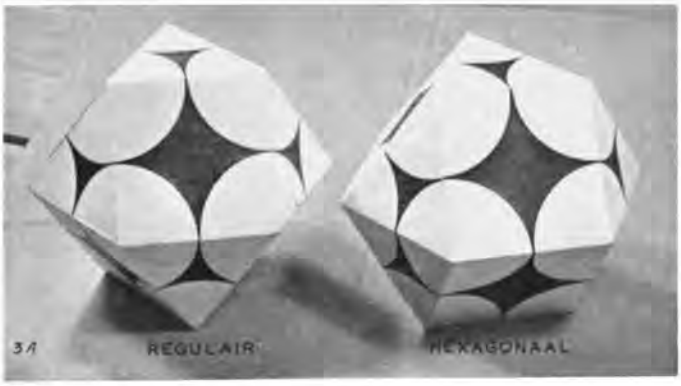


Fig. 1. Rangschikking van deeltjes volgens den cubus met gecenterde vlakken (hoog a, diep a, lang 2a). De punten I t/m XII vormen met 0 een rangschikking der middelpunten van even groote bollen, volgens de regulaire dichtste bolstapeling.

Fig. 2. Regulaire dichtste bolstapeling afgesneden door de middelpunten der bollen 1 t/m XII van fig. 1, met een viertallige as loodrecht geplaatst.

Fig. 3*A*. Regulaire en hexagonale dichtste bolstapeling met drietallige as loodrecht geplaatst.

Fig. 3*B*. Opengelegd model der regulaire dichtste bolstapeling.

Proc. Ned. Akad. v. Wetensch., Amsterdam, Vol. XLIII, 1940.

Polyëdrische pisolieten zijn door R. R. SHROCK (1930) beschreven uit een zwerfsteen in N. Indiana. Hij merkt hierbij op, dat het aantal vlakken der polyëders 9 of minder tot 15 bedraagt, terwijl veelal 12 vlakken optreden. Deze vlakken zijn plat, convex of concaaf.

Hierbij kan worden opgemerkt, dat 12 vlakken moeten optreden, wanneer plastische bollen van dezelfde grootte in dichtste bolstapeling liggen en zij door alzijdigen druk gedeformeerd worden.

Een der bekendste vormen van presse-papiers bezit een bijzondere aantrekkelijkheid voor den kristallograaf, want in de eerste plaats vertoont hij de combinatie van hexaëder en octaëder „in evenwicht", in de tweede plaats vormen de hoekpunten met het middelpunt een der afbeeldingen van het regulaire tralie van BRAVAIS met gecentreerde vlakken (fig. 1) en tenslotte vormen diezelfde punten de middelpunten van bollen in dichtste regulaire bolstapeling om een centralen bol (fig. 2).

Het verband tusschen de dichtste regulaire en hexagonale bolstapeling wordt op eenvoudige wijze gedemonstreerd, door den beschreven polyëder op een octaëdervlak te plaatsen (fig. 3 A en B). Wordt dan de bovenste helft 60° in een horizontaal vlak gedraaid, zoo ontstaat de hexagonale dichtste bolstapeling, waarbij $a : c = 1 : 1{,}633$ $\left(= 1 : \frac{2}{3} a \sqrt{6}\right)$.

Tot welke polyëders worden de bollen nu gedeformeerd, indien zij,

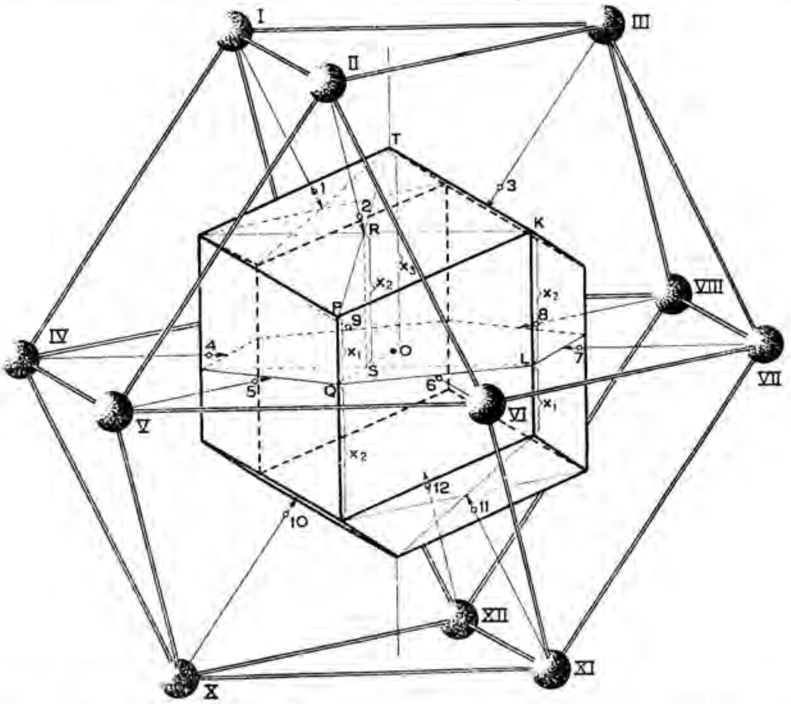


Fig. 5. Draadmodel der regulaire dichtste bolstapeling loodrecht op een drietallige as, met het door compactie gevormde rhombendodekaëder.

geheel plastisch verondersteld, aan compactie worden onderworpen, zoodanig, dat het poriënvolume geheel verdwijnt. Het is duidelijk, dat de centrale bol, die door 12 even groote bollen omringd is, in de regulaire dichtste bolstapeling in een dodekaëder verandert. De drukrichtingen staan in de richtingen der stralen, die de middelpunten der 12 omringende bollen met het middelpunt van den centralen bol verbinden. In de regulaire dichtste bolstapeling (fig. 4 en 5) worden deze richtingen aangegeven door:

$$[101],\quad [011],\quad [\bar{1}01],\quad [0\bar{1}1]$$
$$[110],\quad [\bar{1}10],\quad [\bar{1}\bar{1}0],\quad [1\bar{1}0]$$
$$[10\bar{1}],\quad [01\bar{1}],\quad [\bar{1}0\bar{1}],\quad [0\bar{1}\bar{1}]$$

Loodrecht op deze richtingen staan de twaalf vlakken van den polyëder, waarvan dus het symbool {101} is, waarmede tevens bewezen is, dat de polyëder het *rhombendodekaëder* uit het regulaire stelsel is (afgekort R).

Het andere twaalfvlak is minder regelmatig (fig. 6). De hexagonale dichtste bolstapeling komt overeen met twee in elkaar geplaatste hexagonale tralies van BRAVAIS met een translatie $\frac{2}{3}$ a, $\frac{1}{3}$ a, $\frac{1}{2}$ c waarbij a : c = 1 : 1,633.

Het trigonale dodekaëder, dat door compactie van deze bolstapeling ontstaat, is een combinatie van een hexagonaal prisma met een trigonale bipyramide.

Van de 12 drukrichtingen liggen er 6 in een horizontaal vlak, met de indices:

$$[11\bar{2}0],\quad [\bar{1}2\bar{1}0],\quad [\bar{2}110],\quad [\bar{1}\bar{1}20],\quad [1\bar{2}10],\quad [2\bar{1}\bar{1}0].$$

De vlakken, die normaal op deze drukrichtingen staan bezitten het symbool $\{11\bar{2}0\}$ en vormen samen een hexagonaal prisma van de 2e soort.

De drie drukrichtingen van de bovenste helft naar het centrum hebben de indices:

$$[ho\bar{h}l],\ [\bar{h}hol]\ \text{en}\ [o\bar{h}hl],$$

de drie van de onderste helft:

$$[ho\bar{h}\bar{l}],\ [\bar{h}ho\bar{l}]\ \text{en}\ [o\bar{h}h\bar{l}].$$

De zesvlakken, die normaal op deze drukrichtingen staan, bezitten het symbool $\{ho\bar{h}l\}$ en vormen een trigonale bipyramide van de 1e soort.

Deze combinatie (afgekort T) van hexagonaal prisma van de 2e soort met trigonale bipyramide van de 1e soort treedt op in de ditrigonaal-dipyramidale klasse.

De eenheid op de a, b en d-assen bedraagt y (fig. 6), die op de c-as $\frac{1}{2}$ y $\sqrt{6}$.

Uit fig. 7 volgt dat OB = 2y of 2 eenheden op de horizontale as.

De notatie van een der 6 vlakken is dus volgens WEISS $2 : \infty : \bar{2} : 1$ en het indexsymbool van BRAVAIS–MILLER is $\{10\bar{1}2\}$.

Het poriënvolume der dichtste bolstapelingen bedraagt 25,951 % van het totale volume van bollen plus poriën. Het wordt gevonden uit het verschil van de inhouden van een parallelopipedum en een bol:

$$\frac{\frac{1}{2}\,d^3\sqrt{2} - \frac{1}{6}\,\pi\,d^3}{\frac{1}{2}\,d^3\sqrt{2}} = 25{,}951\,\%$$

Bij volledige compactie der plastische bollen verdwijnt het poriënvolume en er blijft $\frac{74{,}049}{100}$ van het oorspronkelijke volume over.

Lineair bedraagt de compactie dus $x = \sqrt[3]{\frac{74{,}049}{100}} = 0{,}90472.$

Dit wil zeggen, dat wanneer de afstand der middelpunten der bollen oorspronkelijk 1 bedroeg, de afstand der middelpunten der polyëders, na

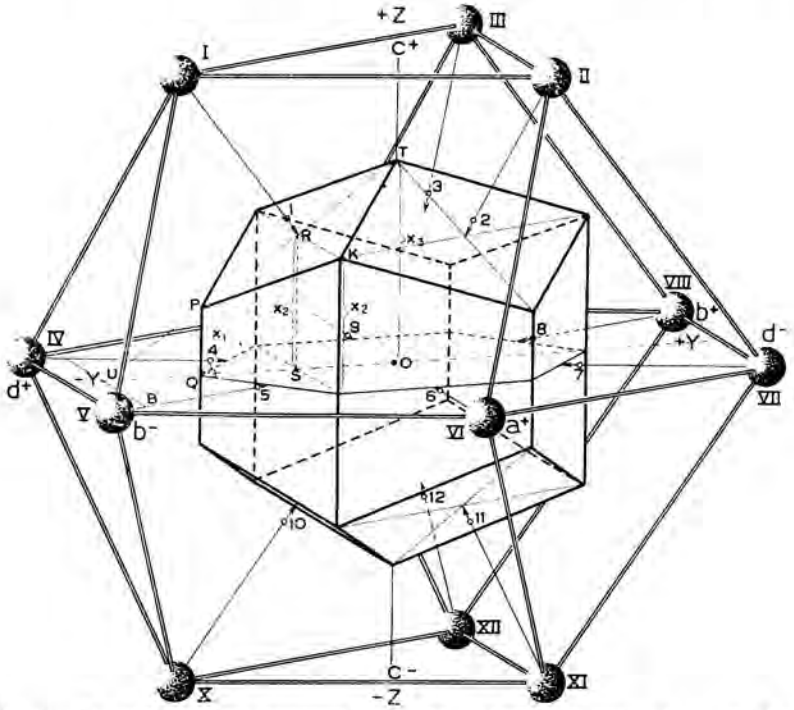


Fig. 6. Draadmodel der hexagonale dichtste bolstapeling loodrecht op de drietallige as, met het door compactie gevormde trigonale dodekaëder, een combinatie van {1120} en {1012}.

volledige compactie, 0,90472 geworden is. In fig. 7 is dus **$y = 0{,}90472\, r$.**

In y uitgedrukt kan het volume van het rhombendodekaëder als volgt worden bepaald (fig. 4):

Inh. rhombendodekaëder = Inh. cubus + 6 × Inh. Vierzijdige pyramide

= 2 × Inh. cubus.

y = halve diagonaal van een cubusvlak, dus is de zijde van den cubus $= y\sqrt{2}$ waaruit volgt: *Inh. rhomdendodekaëder* $= 2 \times (y\sqrt{2})^3 = 4y^3\sqrt{2}$.

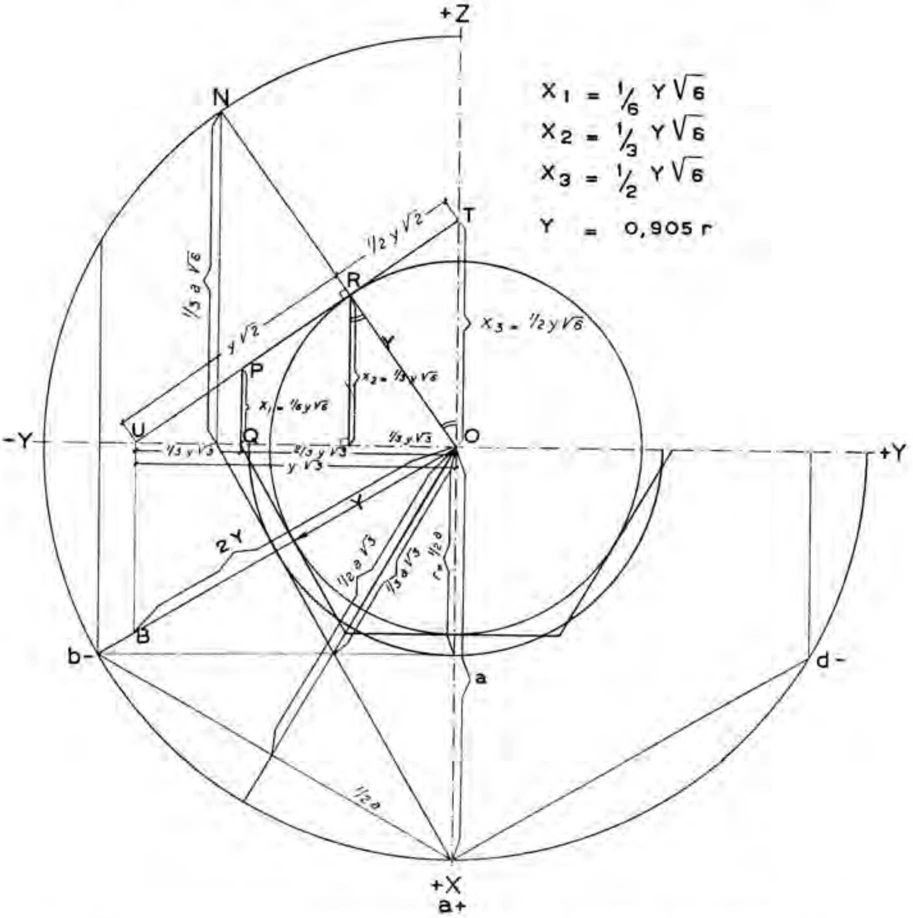


Fig. 7. Horizontale en verticale projectie voor de bepaling der grootheden van het trigonale dodekaëder.

Het volume van de combinatie T van $\{11\bar{2}0\}$ en $\{10\bar{1}2\}$ bestaat uit 12 maal den inhoud van het afgesneden driezijdig prisma OLQTKP (fig. 6). De voor deze berekening noodige grootheden zijn in fig. 7 geschreven.

De hoogtelijn van het middelpunt van den zeshoek, op de zijde neergelaten, bedraagt y. De zijde van den zeshoek bedraagt dus $\frac{2}{3}\,y\sqrt{3}$. Het oppervlak van den driehoek bedraagt dus $\frac{1}{3}\,y^2\sqrt{3}$.

Voorts wordt gevonden:

$$x_1=\frac{1}{6}\,y\sqrt{6},\; x_2=\frac{1}{3}\,y\sqrt{6},\; x_3=\frac{1}{2}\sqrt{6}.$$

Inh. dodekaëder

$$T = 12 \text{ x Inh. afgesneden driezijdig prisma} =$$

$$= 12 \text{ x } \frac{1}{3}\,(x_1+x_2+x_3)\,\frac{1}{3}\,y^2\sqrt{3} =$$

$$= 12 \text{ x } \frac{1}{3}\left(\frac{1}{6}\,y\sqrt{6}+\frac{1}{3}\,y\sqrt{6}+\frac{1}{2}\,y\sqrt{6}\right)\frac{1}{3}\,y^2\sqrt{3} =$$

$$= 4\,y^3\sqrt{2}.$$

Aangezien de beide dodekaëders denzelfden inhoud als de bol bezitten, moet

$$4\,y^3\sqrt{2} = 4{,}18879\,r^3$$

waaruit wederom volgt:

$$\mathbf{y = 0{,}90472\ r.}$$

Met beide dodekaëders kan de ruimte volledig gevuld worden, dit volgt uit de wijze waarop zij hier zijn afgeleid.

Nu zijn er nog andere dichtste bolstapelingen mogelijk en de vraag dringt zich op, of hieruit door compactie andere dodekaëders zouden kunnen ontstaan. Dit komt mij voor niet het geval te zijn.

Het verschil tusschen de regulaire en hexagonale dichtste bolstapeling kan ook als volgt beschreven worden.

Alle horizontale lagen zijn in hun opbouw identiek; zij bestaan uit een dichtste rangschikking van even groote bollen met straal r in een plat vlak en worden gekarakteriseerd doordat elke bol door 6 bollen geraakt wordt, die elkaar onderling eveneens raken. De volgende laag kan echter op twee wijzen op of onder de eerste gestapeld worden, na een translatie over $t=\frac{2}{3}\,r\sqrt{3}$. De hoek tusschen beide translatie-richtingen bedraagt 60° (fig. 8). Indien de onderste en bovenste laag een translatie in de evengenoemde verschillende richtingen hebben ondergaan, is de stapeling regulair; vond de translatie van deze lagen in dezelfde richting plaats, dan is de stapeling hexagonaal.

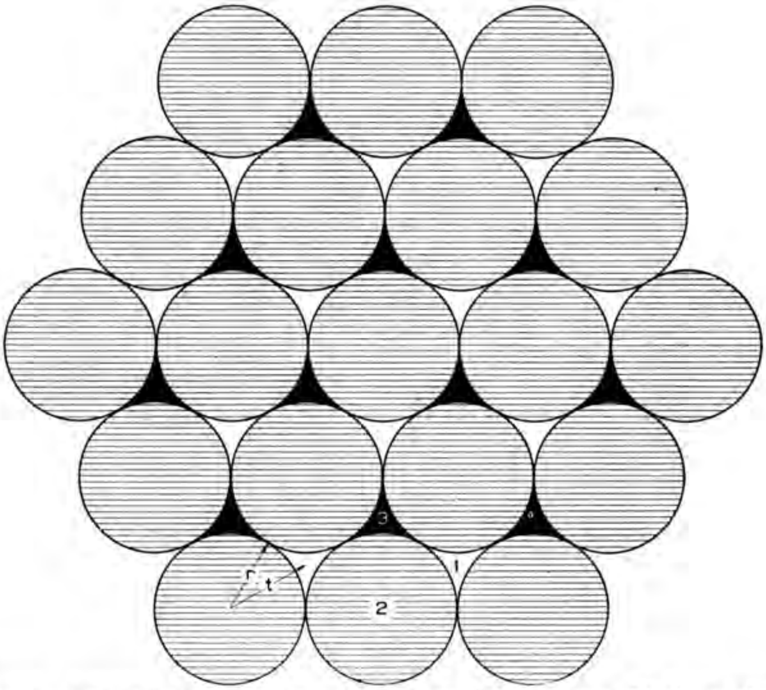


Fig. 8. Schematische voorstelling der dichtste bolstapelingen. Bij de regulaire liggen de middelpunten der onderste laag van bollen in de vakken 1, die der bovenste laag in de vakken 3. Bij de hexagonale liggen de middelpunten der onderste en bovenste lagen of beide in 1 of beide in 3. De middelpunten van de middelste laag liggen in 2; $t = \frac{2}{3} r \sqrt{3}$.

                                                            3
In het regulaire geval noemen wij de lagen 2, in het hexagonale geval
                                                            1
1   1   2
2 of 3 of 3
1   1   2
                    3   1
Naast regulair 2 is regulair 2 mogelijk:
                    1   3
                    1   2   1   3   2   3
naast hexagonaal 2 ook 1, naast 3 ook 1 en naast 3 ook 2
                    1   2   1   3   2   3

In verticalen zin kan een groot aantal lagen op vele wijzen gestapeld worden, manieren, die uit een combinatie der evengenoemde stapelingen van drie lagen worden afgeleid.

G. MENZER (1938, p. 391) noemt ze dichtste bolstapelingen van hoogere orde. Als hoogste orde definieert hij een volkomen onregelmatige opeenvolging der drie lagen, terwijl talrijke combinaties mogelijk zijn, waarbij een of meer lagen als symmetrievlak optreden.

Eenige voorbeelden van gemengde stapelingen zijn:

Met symmetrie			Onregelmatig
		1	3
		2 R	2 T
		3 T	3 R
		2 T	1 T
		3 T	3 R
		2 R	2 R
1	1	1 T	1 T
2 R	2 R	2 R	2 R
3 T	3 R	3 T	3 R
2 R	1 T	2 T	1 R
1 T	3 R	3 T	2 T
2 R	2 R	2 R	1 R
3 T	1 T	1 T	3 R
2 R	2 R	2 R	2 T
1 T	3 R	3 T	3 R
2 R	1 T	2 T	1 T
3 T	3 R	3 T	3 R
2 R	2 R	2 R	2 R
1	1	1	1

Het is duidelijk, dat R optreedt, wanneer een laag tusschen twee lagen met verschillend nummer ligt, en T, wanneer onder en boven een laag, lagen met hetzelfde nummer optreden. Er ontstaan dus hetzij regulaire rhombendodekaëders R hetzij trigonale combinaties T van {1120} en {1012}. Andere vormen zijn niet mogelijk.

E. 1 Nov. 1940.

Summary.

R. R. SHROCK has described a polyhedral pisolite and remarks, that the grains are often bound by 12 faces. The pisolite of Karlsbad (Bohemia) shows here and there idented spheres, which suggest, that they were formed in a colloidal state, and afterwards deformed by compaction. When the compaction goes so far as to annihilate the pore space between the spheres, the latter change into dodecahedrons; the cubic close packing giving the regular dodecahedron, the hexagonal close packing a trigonal

combination of the second order prism {1120} with the trigonal pyramid {10$\bar{1}$2}. The pore space of close packed spheres being 25.950 %, the remaining volume after complete compaction amounts to 74.049 % of the original content, and the compaction consequently shortens the linear dimensions to 0.90472 of the original ones. Both dodecahedrons fill the space completely and no other polyhedrons are possible by compaction of other imaginable close packings (of equal spheres) with or without planes of symmetry.

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G. MENZER, Zwillingsgesetze und regelmässige Verwachsungen der dichtesten Kugelpackungen. Zeitschr. f. Kristallographie, **99**, 378—409 (1938).

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Physiology. — *Problems of foetal respiration in the cow.* By J. ROOS and C. ROMIJN. (From the Laboratory for Veterinary Physiology of the State University, Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of November 30, 1940.)

Second Part.

I. *The oxygen- and the carbon dioxide content of maternal and foetal blood.*

An important point in the problem of foetal respiration is the volume of oxygen and carbon dioxide contained by the foetal blood before passing the placental vessels, and after having been ventilated there.

Such determinations have been carried out by various workers from which COHNSTEIN and ZUNTZ (1884) were the first, and using various experimental animals: the sheep, the goat and the cat. Also for the new born child, born either normally or delivered by Caesarian section, some figures have been given. Most of this work was discussed in an earlier paper (1938).

Attention was also paid to the considerable differences in oxygen saturation of arterial foetal blood, amounting to some hundreds p.c., which had been found by the different authors, whilst especially the low figures for oxygen saturation found in literature had been mentioned. The lowest degree of oxygen saturation found in literature was 17—18 p.c. STEELE and WINDLE (1939) found in the foetal cat a still lower oxygen content, ranging between 0.3 and 4.8 vol. p.c. for arterial blood. Their highest percentage of saturation remained below 45 p.c. In the greater part of their experiments it was below 18 p.c. and even figures below 3 p.c. saturation are given. In a recent paper BARCROFT, KENNEDY and MASON (1940) put attention to the same points. Their criticism is justified for the greater part. The review, given by them, shows that the highest values had been obtained by us in 1938. We had measured in the foetal arterial blood of the cow an oxygen saturation up to 90 p.c., 81 p.c. being the highest value found by others.

Besides this the relation between maximum and minimum, found by the same author in various experiments, by which relation the constancy of observation is expressed to a degree, varied considerably in some papers. Whereas in our work it was as 1.7 : 1, the ratio up to 5 : 1 was found in the work of others. Also the values found for the goat by BARCROFT, FLEXNER and MCCLURKIN (1934), which work had not been mentioned by us in a previous paper, exhibited differences of more than 100 %, the maximum and minimum being 8.50 vol. p.c. and 3.93 vol. p.c. respectively.

Nevertheless our results had not satisfied us fully and new experiments in the cow seemed desirable for many reasons. In the first place because of technical difficulties, experienced in the experimental work. The foetus had been taken out of the uterus, its cord remaining in connection with the placenta. In this way the uterus had been brought under unphysiological conditions. It will also come to contraction and the state of the capillaries of maternal and foetal cotyledones may thus be altered materially. Furthermore it was an open question how the general blood pressure was affected by opening the abdomen and by exposure of the abdominal organs, the uterus especially, to the temperature of the room and to exsiccation. In his fine work on the goat and the sheep BARCROFT has conquered many of the difficulties by emerging the mother in a bath of saline, maintained at a temperature between 37° and 38° C. For the cow, which is a favourable experimenting animal in other respects, this method of working is not practicable. Another technique had to be searched therefore.

Technique.

The technique, used in the experiments which will be described here, was as follows. After the pregnant cow had been placed on its left side the abdominal wall was opened under local anaesthesia. As we had also done in earlier experiments this form of anaesthesia was prefered to other ones on account of the condition of the nervous system remaining unaffected in this way. The place of incision was chosen in correlation with the position of the foetus, determined by palpation; in most experiments the wound was made about parallel to the midline and about 10 cm higher, that is at the right side of it. It was kept as small as possible, long enough to permit the hand to pass in order to detect the position of the foetus. Then it was lengthened to about 20 cm. In all instances the great curvature of the gravide uterus horn was found to lie near the abdominal wound.

Abdominal wound and uterus were irrigated continually by a 0.9 p.c. salt solution of a temperature between 38° and 39° C. during all the experimenting time. The first sample of blood was withdrawn from a uterus vein, which could be traced through the peritoneum of the uterus. This was done without difficulties. In some experiments another sample was taken from a cotyledone vein which had been exposed at the base of a maternal cotyledone. Then uterus and amnion were opened at the great curvature by an incision between the cotyledones, 6—8 cm long which gave part of the amnion liquor occasion to escape. The further procedure was determined by the situation met. It was our intention to bring part of the umbilical cord at the level of the uterus opening in order to insert a canula. Sometimes however it was not possible to do so without enlarging the wound and bringing one of the foetal hindlegs, sometimes both, outside the uterus. If this was necessary, opening of the allantois could not be avoided. When the latter could be left undamaged the advantage was gained, that a considerable part of the uterus liquor remained inside. As is generally known, the volume of the amnion fluid is reduced gradually after the midst of gestation, whereas the allantois liquor increases and reaches a maximum (up to 10 liter) at the end of pregnancy. At this moment there is still about 4 liter of liquor contained by the amnion.

In some experiments a blood sample was taken from the foetal placenta vessels at the same time. We preferred to take a sample of blood from the umbilical cord because of this blood being mixed and being therefore less susceptible to local alterations in the placental circulation which could have their localisation in some cotyledones or even in a single one.

The insertion of a canula into a vessel of the umbilical cord can not have any undesired effect, since in the cow these vessels, arteries and veins, are paired, and their diameter is such that circulation is not impeded by this procedure in any way. Cooling as well as exsiccation of the cord were avoided by irrigation and a visible contraction of one of the umbilical vessels was never observed.

Further, maternal blood was drawn from the art. carotis and from the jugular vein. As a rule the samples of arterial blood were taken together from mother and foetus; the same was done with respect to the venous blood.

In one experiment (no. 32) in which the calf had been taken out of the uterus and brought under warm salt solution, the uterus remaining under continual irrigation, the results were less satisfactory. In an other one (no. 34) the abdomen was opened at nearly its highest point, viz. in the flank region. Then the uterus, after having been fixed by sutures to the margins of the abdominal wound, was opened. In this way the loss of fluid was restricted to a minimum. The umbilical cord was quickly brought into the wound and samples were taken. The results however were not so good as could have been expected.

An experience, repeatedly made, was that the first factor which was obviously decisive for the results in this work was the tempo of the experiment. The best results were obtained in those experiments which had been carried out with the most push. The time, elapsing between the moment of the opening of the abdomen and the sampling of the foetal blood, was in the good experiments never longer than 10 minutes, sometimes shorter than 5 minutes. Samples, taken 15 min. later intentionally, revealed a lower oxygen saturation. The experience demonstrates an experimental difficulty.

The blood was collected under paraffine oil and thoroughly mixed with either the anti-coagulant powder or the solution, in the manner described in the First Part of this paper (1940). The blood samples were kept in the refrigerator up to the moment of analysis, which was carried out within an hour. The blood was analysed in VAN SLYKE's constant volume apparatus, using 2 cc of blood.

Results.

Eight animals were used for the experiments. In two of them arterial blood pressure was recorded during the experiment. The surprising result was obtained that the pressure was not altered in any way neither by opening the abdomen nor by manipulations in the uterus. An average pressure of 150 mm Hg was maintained.

Four experiments which beforehand we had thought to be the best on behalf of the fluent course of the operation (group A) will be reported here. Besides these the figures, measured in two other cases will be given in table 1 together with those, obtained in earlier experiments.

TABLE 1. Oxygen in the foetal blood of the cow.

Group	Number of animal	Age of foetus in months	O_2 in vol. p.c.		O_2 — Capacity	Saturation p.c.		Ratio $\frac{\text{umb. art.}}{\text{umb. vein}}$
			umb. vein	umb. art.		umb. vein	umb. art.	
A	38	7	9.07	2.39	10.39	87.3	23	0.27
	36	8	6.76	3.66	10.82	62.5	33.8	0.54
	37	8	4.84	2.25	9.76	50.0	23.1	0.46
	39	8	7.85	4.28	13.10	60.0	32.6	0.54
	Average	—	7.13	3.15	11.02	65.0	28.1	0.43
	33	4	3.43	0.78	11.38	30.1	7	0.23
	34	7	5.19	1.05	9.50	54.6	11	0.20
B	1	7	8.92	2.41	12.15	73.4	19.7	0.27
	2	8	7.48	4.39	11.68	64.0	37.6	0.56
	3	8	6.20	3.46	11.49	54.6	30.4	0.56
	4	8	7.61	2.87	15.57	48.8	18.4	0.38
	5	8½	9.86	5.07	11.35	90.0	46.3	0.51
	Average	—	8.01	3.64	12.45	64.3	29.2	0.45
Sheep 18—21 weeks (Barcroft)			10.5	5.8	16.6	63.3	34.9	0.55

The oxygen saturation, found in experiment 33, was so low that we do not rely on it, though the experimental work had proceeded well. In experiment 34 the uterus had not lost liquor, since it had been attached to the abdominal wall. Nevertheless the results did not satisfy.

Experiment 38 was performed in an animal with hydrallantois, a condition in which an uncommon great volume of liquor is contained by the allantois membrane. The uterus was in a condition of inertia and it contracted only slightly after the foetus had been taken out of it after the experiment. The foetus was a little more slender than it is commonly but it was normally developed.

The oxygen saturation, observed in the experiments 36, 37 and 39, in which the foetus had the age of 8 months, agree rather well for the arterial as well as for the venous blood. On the whole the figures found for saturation are not higher than those observed in earlier work (group B). The degree of oxygen saturation, measured by Barcroft, Kennedy and Mason in the sheep at a corresponding stage of pregnancy, does not differ notably from ours, though their figures for oxygen content are somewhat higher. We found an average saturation of 65 p.c. for the arterial and of 28.1 p.c. for the venous umbilical blood, which values are 63.3 and 34.9 respectively in the sheep.

For foetus of the same age the ratio $\frac{\text{saturation venous blood}}{\text{saturation arterial blood}}$ given in the last column of the table is rather constant.

The figures for the carbon dioxide content are collected in table 2.

TABLE 2. Carbon dioxide content of the foetal blood in the cow.

Number of animal	Age of foetus in months	CO_2 in vol. p.c.		Difference in vol. p.c.
		umb. vein	umb. art.	
38	7	52.78	55.82	3.04
36	8	49.48	51.48	2.00
37	8	41.81	46.44	4.63
39	8	44.72	46.18	1.46
33	4	58.43	60.37	1.94
34	7	52.76	57.07	4.31

Under the conditions of the experiment the oxygen saturation of the arterial blood of the mother varied from 81 p.c. to 93.9 p.c. in the eight animals. The saturation of the blood taken from the jugular vein may differ considerably. It ranged between 23 and 64.7 p.c., 2 observations remaining below 40 p.c. A higher degree of saturation was found in the blood taken from a uterus vein than in that drawn from a cotyledone vein of the same animal. These results are given in table 3.

Discussion.

How far may the values measured in the foetus be esteemed reliable as approximating physiological conditions?

TABLE 3. Oxygen saturation of maternal blood in the cow.

Number of animal	Oxygen saturation in p.c.			
	a. carotis	v. jugul.	v. uterin.	v. cotyledon
38	93.4	58.8	41.2	—
36	93.9	40.0	57.0	—
37	89.0	41.5	71.1	56.6
39	91.0	51.2	60.5	43.0

This question so far as the oxygen is concerned has also been discussed by BARCROFT, KENNEDY and MASON in their recent paper (1940), in which the authors reported experiments made under controlled conditions in the sheep during a great part of the period of pregnancy, covering from the 63rd to the 145th day. (The period of gestation for the sheep takes 21 weeks.)

Our experiments give only information for the two last months of gestation. The values, found in 3 foetus of 8 months, agree very well; they differ 25 p.c. maximally. The range for the venous blood is somewhat greater but also here the agreement is satisfactory. It even exceeds the degree of constancy found by BARCROFT et al. during the corresponding stage of pregnancy. Neither was there more agreement between their duplicated measurements in younger foetus, in which a higher degree of constancy was stated by the authors. In BARCROFT's experiments, which were made under almost ideal conditions, the blood samples were drawn without taking the foetus out of the uterus and without touching the umbilical vessels which, in the sheep, had been found to be very sensitive for mechanical insult.

A still more valuable standard to judge the value of an experiment is the degree of oxygen saturation found. For duplication of measurements does not exclude the possibility that successive experiments involve the same experimental error. We therefore considered always those experiments to have the greatest chance to be reliable, which gave the highest percentages of oxygen saturation, not only for the arterial blood but also for the venous, and stated already two years ago (1938) that in our experiments the foetal blood, venous as well as arterial, was saturated to a rather high degree. For this reason we gave in the last column of table 1 the ratio $\frac{\text{oxygen saturation in art. umb.}}{\text{oxygen saturation in v. umb.}}$. In the 8 months' foetus mentioned under group A this ratio is rather high and fairly constant. Taking also into consideration the experiments of the B-group we have two observations in foetus 7 months old, in which not only the figures for oxygen saturation are rather high and agree in a satisfactory way, but moreover both ratio's given in the last column, agree rather fairly. For both it is 0.27.

Although we think it justified to consider the figures given here approximating physiological values we remain fully aware of the difficulties in this field of work.

It seems that oxygen saturation of the arterial foetal blood decreases in the 8th month of pregnancy. BARCROFT found the same in the sheep for the last three weeks of intra-uterine life. In the venous umbilical blood however we found a lower saturation in the foetus of 7 months than in those one month older.

A question which still has to be considered, is this. Is it reasonable to assume that the umbilical blood, which is oxygenised in the placenta, will be saturated there to the same percentage, when all foetal conditions as age, weight, position, structure of the placental vessels, foetal oxygen dissociation curve are the same, and supposing that the venous foetal blood arrives in the placenta in completely the same state in all instances? Especially for the ox the question has to be answered in the negative, because of the wide dimensions of the field withtin which the maternal oxygen d.c.s have their position. If the maternal blood is oxygenised in the lungs to a distinct percentage of 93 p.c. for instance, its oxygen pressure, may vary between 56 and 76 mm Hg. It must therefore be concluded that arterialisation of the foetal blood will distinctly experience the consequences of this wide range, and the degree of oxygen saturation of the foetal blood will thereby show variations due to this circumstance. This is another reason for which we deem duplicated measurements though useful of relative value.

II. *The tensions of the blood gases.*

Since the oxygen dissociation curves had been determined for both maternal and foetal blood (see First Part (1940)) the partial oxygen pressures on either side of the placenta can be found by interpolating in these curves the percentages of the oxygen saturation measured. The carbon dioxide pressures may be determined in an analogous way, starting from the carbon dioxide dissociation curves. The figures thus obtained are given in table 4.

In Fig. 1 a scheme is given for the relations established for animal 39, at a gestation of 8 months. Two sets of figures are given for the maternal venous blood, for blood taken from a uterus vein and from a cotyledon vein respectively. The figures for the latter have been placed between brackets.

The scheme given here would also fit the experiments 36 and 37. The foetal aerated blood which leaves the placenta through the umbilical veins has a carbon dioxide pressure, still higher than that in the venous maternal blood, whilst the latter is still able to give off oxygen to the foetal aerated blood because of its oxygen pressure being considerably

TABLE 4. Oxygen and carbon dioxide pressure in maternal and foetal blood.

Number of animal	Age of foetus in months	Blood taken from	Oxygen saturation in p.c.	Oxygen pressure in mm Hg.	Carbon dioxide volume p.c.	Carbon dioxide pressure in mm Hg.
38	7	a. carot.	93.4	83	44.67	26
		v. uterin.	41.2	29	47.94	37
		v. umbil.	87.3	34	52.78	51
		a. umbil.	23.0	14	55.82	60
36	8	a. carot.	93.9	67½	46.57	38
		v. uterin.	57.0	36½	47.11	40
		v. umbil.	62.5	23	49.48	41½
		a. umbil.	33.8	14	51.48	44
37	8	a. carot.	89.0	58½	51.92	32
		v. uterin.	71.1	39	55.42	42
		v. umbil.	50.0	25½	41.81	56
		a. umbil.	23.1	15½	46.44	68
39	8	a. carot.	91.0	70	42.39	38
		v. uterin.	60.5	38	47.56	55
		v. cotyl.	43.0	31	49.60	62
		v. umbil.	60.0	24	44.72	70
		a. umbil.	32.6	18½	46.18	73½

higher. So the gas exchange may proceed all during the time that foetal blood passes along the maternal through the placental capillaries. As is well known both capillary systems are separated in the cow by two layers of epithelial cells, from which the maternal is perforated only at the top of the crypt; here the connective tissue comes in contact wiht the tropho-

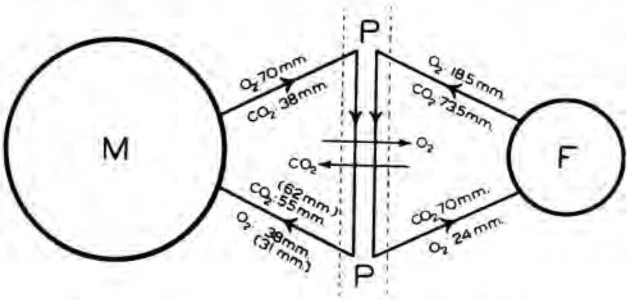

Fig. 1. Gas tensions in the maternal (*M*) and the foetal blood (*F*). Age of the foetus 8 months. *P* = placenta.

blast. So there is not a true placenta but a pseudo-placenta, called placenta syndesmochoriale. A thin layer of uterine milk is found between both epithelian layers.

If the direction in which the blood passes through the foetal capillary vessels should be contrary to that drawn in Fig. 1 the conditions for a continual gas exchange would not be less favourable. Both directions would fit in the scheme and the simplest relation was chosen therefore.

In experiment 38, however, the situation is a different one. Here the conditions, favourable for a good change of gases between maternal and foetal blood are available only then, if the arrangement of the capillary circulation is otherwise.

If we assume that maternal and foetal circulation in the placentae are thus arranged, that foetal ventilation goes on as long as maternal and foetal blood pass each other, we are brought to the scheme of Fig. 2. An other relation is not compatible with the oxygen pressures determined.

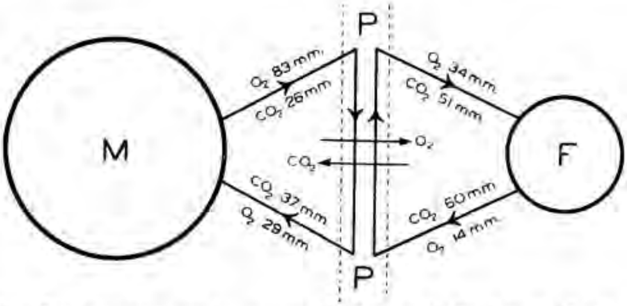

Fig. 2. Gas tensions in the maternal (*M*) and the foetal blood (*F*) in experiment 38. Age of the foetus 7 months. *P* = placenta.

This conclusion is of such importance that it forces us another time to consider critically the figures, from which it is derived. Obviously the direction of the arrows has become imperative by the relative high oxygen pressure in the umbilical veins, which is materially higher than in all other experiments. The figures, found for the carbon dioxide, do not have any preference as to the direction of the arrows.

What may be the reason for this high oxygen pressure? The arterial foetal blood was saturated to a high degree, viz. 87.3 p.c. Probably the state of hydrallantois of the mother was favourable, since there had been little or no tendency to contraction here.

There is still another point of importance for the degree of foetal oxygen saturation: the high oxygen pressure in the arterial maternal blood. It was exceptionally high (84 mm) and surpassed the figures found in all other experiments, which showed an average of 60 mm. This fact finds its origin in two circumstances. Firstly maternal arterial blood was saturated to a rather high degree, 93.4 p.c. In the ox, and the same is probably the case in other animals who do not have a well trained pulmonary ventilation, a lower degree of saturation is commonly found. Experience showed us that it fluctuates between 80 p.c. and 94 p.c., provided the animal is in lying position, as in the experiments described. When the animal is on its feet other results are obtained. Secondly in consequence of the shape of the oxygen dissociation curve, which has been given in Fig. 10 (o) of the First Part, a high pressure is exerted by the oxygen in the region of high saturation. If for instance the same degree of saturation had been found in experiment 36 (Fig. 11) (△) First Part, the oxygen pressure of maternal blood would have been no more than 65 mm, whereas in the experiment in question a pressure of 84 mm is exerted. So it is conceivable that by the high maternal oxygen pressure foetal oxygenation is performed to a high degree.

On the ground of this experiment one is inclined to correct the direction of the arrows, given in Fig. 1 and to presume that in the cow the arrangement of maternal and foetal capillaries in the placentae is thus, that blood circulation occurs as shown by Fig. 2. It remains reserved to further work, to settle this problem. Here we have only to refer the reader to the anatomical work by MOSSMAN (1926) who, by pure anatomical examination, came to the conclusion that in the rabbit placental capillaries are arranged in the manner indicated by the scheme of Fig. 2, so that in the foetal capillaries of the placenta the blood passes in the opposite direction as in the adjacent maternal. According to MOSSMAN, TAFANI would have found before him the same relation in the placenta of the cat. To our regret TAFANI's paper was not obtainable for us.

Summary of the First Part (1940) and Second Part of this paper.

Various points of foetal respiration have been examined in the cow. The results are as follows.

(1) During the period of pregnancy the maternal oxygen dissociation curve becomes steeper, and starting from the 4th month its inflection increases. The curve does not leave the field determined for normal non-pregnant cows; its lower most inflected part only may fall outside of it. These changes are not produced by changes in reserve alkali.

(2) In these respects the curves differ from those determined for the goat and for man.

(3) As a rule the reserve alkali of the pregnant animal does not fall below the level of normal non-pregnant animals.

(4) The foetal oxygen dissociation curve differs from the maternal by both position and shape. It is found to the left of the maternal and to the left of the area, established for normal non-pregnant cows.

(5) The gap between maternal and foetal curve, though brought about without a displacement of the maternal curve, is a considerable one so that a notable difference in oxygen avidity between maternal and foetal blood is expressed by it.

(6) Foetal and maternal carbon dioxide curves have been determined and discussed. The dislocation of the foetal oxygen dissociation curve is counteracted by a fall of alkali reserve in the foetal blood during the last months of foetal life.

(7) The foetus has a haemoglobin with other properties than the maternal.

(8) A fairly constant oxygen saturation of the blood in the umbilical vessels was found in foetus, 7 and 8 months old. It varied between 50 and 87 p.c. for the arterial blood and between 23 and 32 p.c. for the venous. The content of carbon dioxide was also determined. At the same time corresponding measurements have been carried out for the maternal arterial and venous blood in the placenta. The reserve alkali in both mother and foetus has been discussed.

(9) The tension of the gases in both maternal and foetal blood was determined by interpolating the figures for oxygen saturation and carbon dioxide content in the respective dissociation curves. So it was made probable that the arrangement of the capillaries in maternal and foetal placenta is thus that the blood which passes on either side in order to change gases and other materials goes in directions, opposite to each other.

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Psychology. — *Die menschlichen Kommunikationsformen und die sog. Tiersprache.* II [1]). Von G. RÉVÉSZ. (Communicated by Prof. A. DE KLEYN.)

(Communicated at the meeting of November 30, 1940.)

II. *Die sog. Tiersprache.*

4. *Das hypostasierte Wortverständnis der Tiere.*

Die Ansicht, nach der gewisse Tiere imstande wären unsere Sprache zu verstehen, findet ihre Stütze in der Erfahrung, dass gewisse Tiere wie Hunde, Katzen, ferner in der menschlichen Gesellschaft erzogene Affen auf eine Anzahl an sie gerichteter Worte im Sinne der Dressur oder Gewöhnung eindeutig reagieren. Hunde führen die Befehle ihrer Herren aus, Bücher, Stöcke, Zeitungen heranzuholen, die Stube zu verlassen. Auch Affen reagieren richtig auf Befehle wie „Iss!", „Herunter!", „Komm herein!", „Oeffne die Tür!", selbst unter sorgfältiger Vermeidung der Armbewegungen und des Mienenspiels. Man begeht aber einen Irrtum, wenn man aus dem Umstand, dass Tiere auf bestimmte phonetisch differentierte Lautgebilde (Dressurworte) im Sinne der Dressur reagieren, schliesst, dass sie die Sprache *verstehen*, d.h. die *Bedeutung* der Worte begreifen. Gerade darauf kommt es aber an: nur unter dieser Bedingung darf den Tieren rechtmässig Sprach- oder Wortverständnis zugesprochen werden.

Wir können ohne Einschränkung den Satz aufstellen, dass *kein Tier unsere Sprache versteht*. Die Worte, die sie angeblich begreifen, sind für sie nichts weiter als einfache Lauteindrücke, die infolge individueller Erfahrung oder Dressur mit gewissen Handlungen oder Unterlassungen assoziiert sind. Das Lautwort beim Tier gewinnt niemals eine bezeichnende, benennende oder symbolische Bedeutung; es stellt nur einen wirksamen Reiz dar, genauso wie ein beliebiger anderer Eindruck (Farbe, Figur, Geruch, Geräusch), insofern dieser mit einer bestimmten Verhaltungsweise assoziativ verknüpft ist.

Tierpsychologische Versuche weisen unzweideutig darauf hin, dass das Tier zwischen Laut und Gegenstand (bezw. Tätigkeit) keine sinnvolle Verbindung zu stiften vermag; für das Tier ist der Laut ein Nebeneindruck, der den Befehl begleitet und es zu einer bestimmten Reaktion veranlasst. Das Pferd weiss nicht, was das Wort „Galopp" *bedeutet*, und doch reagiert

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[1]) Teil I über die menschlichen Kommunikationsformen siehe Proc. Ned. Akad. v. Wetensch. Amsterdam, **43**, 1078 (1940).

es darauf richtig. Dass hierbei nicht das *Wort*, sondern ausschliesslich der *Laut* bestimmend ist, lässt sich daraus entnehmen, dass der Lautkomplex „Galopp" nur unter *bestimmten* Umständen (z.B. in der Manege unter dem Kommando des Reitlehrers) und nur bei *bestimmter* Betonung die erforderte Reaktion auslöst. Nicht das Wort als sprachliches Element, sondern der phonematische Charakter und der Tonfall des Dressurwortes sind für die Reaktion massgebend. Dasselbe zeigt sich bei einem Hund, der dressiert ist auf die in der folgenden Betonung ausgesprochenen Wortlaute: „Gēh sȫfȫrt hīnaus!", das Zimmer zu verlassen. Er wird den Befehl im Falle einer deutlichen Aenderung des Tonfalles, der Betonung, der Klangfarbe, der Reihenfolge der Wörter nicht ausführen, obwohl die Sinnbezogenheit der Dressurworte unverändert blieb. Demgegenüber wird der Hund auch dann der Dressur entsprechend reagieren, wenn das letzte Wort wegfällt, selbst dann, wenn der erste Wortlaut einer akustisch zwar unbedeutenden, sprachlich jedoch entscheidenden Veränderung erleidet, z.B. wenn statt gēh, gēp oder gēl gerufen wird. Die Unterlassung der Reaktion erklärt sich dadurch, dass der dressierte Hund die Aenderung des phonetischen Gepräges am Klangbild beachtet, die Reaktion dagegen dadurch, dass er die Veränderung eben ausser Acht lässt. Daraus folgt, dass weder die eine noch die andere Verhaltungsweise im Sinne eines Wort- bezw. Sprachverständnisses zu deuten ist.

Diese Verhaltungsweise des Hundes gegenüber zugerufenen Wörtern wurde durch BUYTENDIJK und FISCHEL experimentell erforscht ¹). Sie konnten zeigen, dass wenn z.B. das Dressurwort „Spring!" durch solche Wörter vertreten wird, die die ersten Lautbestandteile (s, p und r) in der ursprünglichen Stelle und Reihenfolge (spr-) enthält, der Hund in der überwiegenden Anzahl der Fällen der Dressur entsprechend reagieren wird, weitgehend unabhängig davon, wie sich die letzten Lautbestandteile (ing) verändern. Findet indessen in den Anfangslauten eine Veränderung statt, so bleibt die Reaktion aus.

Durch diese Versuchsergebnisse ist m.E. bewiesen, dass der Hund die Wörter nicht nach ihrem *Sinn*, sondern nur nach ihrem sinnlich-*akustischen* Charakter zu erkennen vermag. Es ist evident, dass das *Wort* „spring" sprachlich nicht durch die *Laute* „spr" bestimmt ist, sonst wäre es nicht möglich spring von spr-ich oder von spr-itz zu unterscheiden. Die unentschiedene Stellungsnahme BUYTENDIJKS inbezug auf diese Frage ist nicht recht verständlich, zumal gerade seine Versuche zeigen, dass das Tier in seinen Reaktionen nur durch die akustische Gestalt des Dressurwortes ²), nicht aber durch den semantischen Wert des Sprachwortes bestimmt ist.

Nicht viel anders steht es mit dem von den KELLOGG's dressierten

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¹) F. J. J. BUYTENDIJK, Wege zum Verständnis der Tiere. 1940.

²) Die Gleichheit des sinnlichen Aehnlichkeitseindrucks beim Mensch und Tier ist in besonders überzeugender Weise durch die Affenversuche von H. KLÜVER erwiesen. (Behavior Mechanisms in Monkey. The Univ. of Chicago Press).

Schimpansenweibchen, das angeblich nach einer Lernzeit von etwa 9 Monaten eine ziemlich grosse Anzahl von Worten „verstand" ¹). Das Tier soll auf die ihr zugerufenen Worte unabhängig von begleitenden Gebärden richtig reagiert haben. (Für den völligen Ausschluss der Bewegung fehlen allerdings die Belege). Wir unterschätzen keineswegs die Leistungen dieses Versuchstieres und am wenigsten die Geduld der Experimentatoren, wenn wir diesen Dressurleistungen im Bezug auf das Wortverständnis nicht die geringste Beweiskraft zusprechen. Es kommt nicht auf die Anzahl der eingeprägten Dressurworte an. Dass die Schimpansin die Worte als solche verstanden hätte, ist von den KELLOGGs nicht bewiesen worden. Sie drücken sich selbst sehr vorsichtig aus, wenn sie sagen, dass „we cannot tell in a strict usage of the term whether the subjects introspectively *comprehend* what is said to them or not. All we can do is to observe whether they are able to react distinctively and individually to separate words and phrases. This, then, must serve as our criterion of „comprehension" which is employed in these pages only in this more objective sense" (p. 289).

Unabhängig von diesen am empirischen Material vollzogenen Ueberlegungen folgt die Unmöglichkeit des Wortverständnisses bei Tieren schon aus der Erkenntnis, dass das Sprach- und Wortverständnis eine Manifestation der *Sprachfunktion* ist, die die Sprachtätigkeit in sich schliesst. Man müsste also erwarten, dass ein Tier, das Worte versteht, auch Worte als solche aktiv hervorzubringen vermag, insofern sein lautmotorischer Apparat es gestattet. Das ist aber — wie wir unten sehen werden — nicht der Fall.

Wollte man gegen diese letzte Bemerkung die Tatsache anführen, dass kleine Kinder schon *vor* der Sprachtätigkeit Worte und Sätze begreifen, so ist darauf zu entgegnen, dass bei Menschen die Sprachfunktion in ihren beiden Aeusserungsformen bereits in der vorsprachlichen Periode als Disposition unzweifelhaft vorhanden ist und der Sprachakt nur infolge des noch unentwickelten Sprachmechanismus und der gehemmten Spontaneität des Kindes nicht in Wirkung tritt. Wir können noch hinzufügen, dass der Zeitpunkt bezüglich des Auftretens des Sprachverständnisses und der Sprachtätigkeit in der Regel nur wenig auseinander fallen. Meistens spricht das Kind schon einige Worte, wenn das Sprachverständnis in deutlicher Form in Erscheinung tritt ²).

5. *Die vermeintliche imitative Sprachtätigkeit der Tiere.*

Nach diesen Ausführungen scheint es überflüssig zu sein auf die Frage einzugehen, ob Tiere befähigt sind, sich die menschliche Sprache anzueignen. Abgesehen von den Tierfabeln und Mythen wird nirgends

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¹) W. N. & L. A. KELLOGG, The Ape and the Child. 1933.

²) Handbook of Child Psychology, ed. K. Murchison (1931); A. FEYEUX, L'acquisition du langage et ses retard; W. STERN, Psychologie der frühen Kindheit (1930); R. RASMUSSEN, Die Entw. eines Mädchens (1934).

behauptet, dass Tiere imstande wären, sich in menschlicher Sprache auszudrücken. Man könnte demnach ruhig verzichten, die Sprachtätigkeit der Tiere zu diskutieren, wenn nicht immer wieder berichtet würde, dass es endlich gelungen ist, hochorganisierten Tieren das „Aussprechen" einiger Wörter beizubringen und zwar in der Weise, dass sie die Wörter angeblich in ihrer Bedeutung erfassten. Obgleich das imitative Nachsprechen von Wortlauten noch keine Sprachtätigkeit darstellt, könnte man in diesen Leistungen wenigstens die artikulatorische Möglichkeit des Sprechens bewiesen sehen. Nicht einmal diese Behauptung trifft zu. Denn wenn man die Berichte nebst der akustischen Analyse der phonographisch aufgenommenen Laute von sog. sprechenden Hunden und Affen untersucht, so stellt sich heraus, dass Tiere ganze Wörter richtig nachzuahmen nicht im stande sind. Mängel der Artikulationsorgane können daran nicht Schuld sein, da die Einzellaute zuweilen richtig ausfallen.

Wenn trotz alledem immer wieder versucht wird, Affen einige Wörter beizubringen, so weist das deutlich darauf hin, dass die Experimentatoren sich weder über das Wesen der Sprache noch über das Zustandekommen und die Bedeutung der Nachahmung lautlicher Gebilde Rechenschaft gegeben haben 1). Sie haben sich nicht einmal darüber eine richtige Vorstellung gebildet, dass durch blosse Nachahmung überhaupt keine Sprache zu erlernen ist, worauf die laut-imitierenden Vögel deutlich hinweisen. Das Kind würde sich die Muttersprache niemals aneignen, wenn es sich ausschliesslich auf die Nachahmung der Sprachlaute seiner Umgebung beschränken würde. Würde das Spracherlernen des Kindes nur auf Nachahmung beruhen, dann wäre es nicht zu verstehen, wie das Kind spontan Sätze und Wortverbindungen bildet, gelegentlich sogar eigene Sprachschöpfungen hervorbringt, die es dann konsequent anwendet. Der eventuell vorzubringende Einwand, dass das taubstumme Kind die Lautsprache imitativ erlernt, lässt sich mit dem Hinweis entkräftigen, dass sich der Taubstumme nicht die Sprache als solche, sondern nur eine besondere Art der Sprachtätigkeit aneignet. Der Taubstumme spricht doch die Gebärdensprache, bevor er zum Artikulationsunterricht übergeht.

Die Frage betreffs der *Gebärden* lässt sich kurz beantworten. Kein Tier führt Gebärden aus. Ich habe Gelegenheit gehabt verschiedene Tiere lange Zeit hindurch zu beobachten, ausserdem liess ich eine grosse Anzahl von niederen und menschähnlichen Affen durch meine Schüler wochenlang beobachten. Es ist nicht gelungen auch nur ein einziges Mal eine solche Bewegung zu sehen, die als Gebärde gelten konnte. Das lässt sich auch nicht anders erwarten, wenn man bedenkt, dass die Ausführung von

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1) Nachdem es H. FURNESS (Observations on the mentality of the Chimpanzee and Orang-Outans, Proc. Am. Phil. Soc. Philadelphia 1916) gelang, nach 6 Monaten einem Orang das Wort „Papa" beizubringen, sagt er: „I think this showed conclusively that there was a glimmering idea of the connection of the word with the object and with her desire." Mit dem Wort „cup" hat er kein Resultat erreicht. — Auch KELLOGG's kamen mit ihrem Schimpansen nicht weiter als bis zur „Vokalisierung" von pa-pa.

Gebärden die Sprachfunktion bereits voraussetzt [1]). Entwicklungsgeschichtlich kommt dies darin zum Ausdruck, dass die erste Form der Gebärden, nämlich das *Weisen* auf etwas, beim Kind erst in der ersten Periode der sprachlichen Entwicklung auftritt, wenn nämlich das Wortverständnis und die Sprachtätigkeit bereits in Aktion getreten sind [2]). Betrachtet man die weisende Gebärde als Ausdruck der Sprachfunktion, dann braucht nicht einmal die Möglichkeit einer nachahmenden, bezw. autochthonen Gebärdensprache bei Tieren diskutiert zu werden [3]).

6. *Die Frage nach der autochthonen Tiersprache.*

Wenn wir unsere tierpsychologischen Erfahrungen von der Definition der Sprache aus betrachten, so kann es nicht zweifelhaft sein, dass Tiere nicht über solche Kommunikationsformen verfügen, die unserer Sprache gleichen. Andererseits wird gelegentlich, sogar von anerkannten Forschern, die Existenz spezifischer „Tiersprachen" behauptet. Dabei wird in erster Linie auf Aeusserungen, Bewegungen, Laute, Haltungen hingewiesen, die bei gewissen Insekten, Vögeln und Säugetieren, bei ständigen und periodischen Vergesellschaftungen die gemeinsame Arbeit, die Hilfeleistung, den sexuellen Verkehr, die Wanderung u.s.w. kollektiv bestimmen. In diesem Sinne spricht man von den eigenen „Sprachen" der Bienen, Ameisen, Pferde, Hunde, Affen.

Die Richtigkeit aller dieser Beobachtungen voraussetzend, stellen wir die Frage, ob die Tatsachen der tierischen Kommunikation die Annahme *spezifischer Tiersprachen* rechtfertigen. Das ist umsomehr nötig, da die Anwendung des Begriffes der Sprache die Vermutung nahelegt, dass man zwischen den Kommunikationsformen der Tiere und der menschlichen Sprache eine nahe Beziehung, gar eine innere Verwandtschaft zu statuieren geneigt ist. Diese Meinung tritt nicht einmal so selten — ausdrücklich oder verdeckt — in der tierpsychologischen Literatur zum Vorschein.

Ausgehend von den vorangehenden Ausführungen über die Sprache, dürfen wir unter dem Begriff der Sprache nur dann tierische Kommunikationen bringen, wenn sie nachweisbar konstituierende Merkmale der menschlichen Sprache oder deren Derivate aufweisen. Nur in diesem

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[1]) G. RÉVÉSZ, La fonction sociologique de la main humaine et de la main animale. Journal de Psychologie, p. 46 (1938).

[2]) G. RÉVÉSZ, Het probleem van den oorsprong der taal. Nederl. Tijdschr. v. Psychologie, **8** (1940); R. VUYK, Wijzen en spreken in de ontwikkeling van het kleine kind. Ned. Tijdschr. v. Wijsbegeerte en Psychologie, **33** (1940).

[3]) Es ist ein Irrtum zu glauben, dass die hinweisende Bewegung sich allmählich aus der Greifbewegung entfaltet hätte. Diese allmähliche, fortschreitende Entwicklung zeigt sich nur bei Menschen, bei denen die erforderlichen geistigen Vorbedingungen eben erfüllt sind. (Siehe dazu E. CASSIRER: Philosophie der symbolischen Formen, S. 126). Nicht die Greifbewegungen entwickeln sich zu hinweisenden Gebärden, sondern der Mensch entwickelt sich und modifiziert die ursprüngliche Greifbewegung zu Kommunikationszwecken.

Falle sind wir berechtigt von einer Art der „Sprache" überhaupt zu reden. Dementsprechend wird man von einer autochthonen *„Sprache"* der Tiere nur dann reden können, wenn eine Tierart über ein solches Laut- oder Bewegungs*system* verfügt, welches mit einer der menschlichen Spracharten *Uebereinstimmungen von prinzipieller Art* zeigt. Wenn es also bei einer Tierart gelingen würde festzustellen, dass sie zum Zwecke der Kommunikation mit ihren Artgenossen ein *System* von Lauten und Bewegungen verwendet, wo z.B. jeden Laut bezw. Lautkomplex oder jeder Bewegung bezw. jedem Bewegungskomplex ein bestimmter Sinn zugeordnet ist, oder wenn es sich zeigen liesse, dass sie zum Zwecke der Mitteilung über ein wohlgeordnetes Zeichensystem verfügt, dann hätten wir das Recht von einer *autochthonen* Tier*sprache* zu reden und diese zu der menschlichen Sprache in Analogie zu bringen. Man dürfte zwar auch in diesem Falle nicht behaupten, dass das Tier im eigentlichen Sinne *spricht*, denn diese tierische Mitteilungsform würde sicherlich eine Anzahl der wesentlichen Merkmale der menschlichen Sprache entbehren, sie würde immerhin mit der menschlichen Sprache das gemein haben, dass sie aus einer bestimmten Anzahl von konstanten und sich auf bestimmte innere und äussere Ereignisse beziehenden *Zeichen* bestünde, die dem Zwecke der *Mitteilung* dienen. Liege ein solcher Fall vor, dann könnte man per analogiam von Tiersprachen sprechen, z.B. von einer autochthonen Sprache der Schimpansen, Gibbons, Hunde, Katzen u.s.w.

Untersucht man indessen die sogenannten Tiersprachen einerseits bezüglich ihrer Funktionen, innerer Struktur und äusserer Gestalt, andererseits bezüglich ihres Entstehens, so kommen wir unzweifelhaft zu negativem Resultat.

Bei der Begründung unserer Anschauung wollen wir uns auf die tierischen Lautäusserungen beschränken, erstens weil „Gebärdensprache" bei Tieren niemals vorkommt, zweitens weil die Stimmlaute der Tiere sowohl der natürlichen Beobachtung wie der physikalischen Registrierung zugänglich sind.

Betreffs der *Funktionen* lässt sich mit Sicherheit sagen, dass die Tierlaute keine darstellende Funktion haben, es fehlt also gerade jene Funktion, die nach BÜHLER mit Recht als das wesentliche Kriterium der menschlichen Sprache angesehen werden muss. Dass mit dem Fehlen der darstellenden Funktion auch andere wichtige Teilfunktionen, wie die benennende, die symbolische, die logisch-grammatikalische ausfallen, liegt auf der Hand. Die Tierlaute bezeichnen nichts, sie stellen nichts dar, sie haben keine symbolische Bedeutung. Es liegt ihnen auch keine Mitteilungsabsicht zu Grunde, ohne welche eine Sprache, selbst im weitesten Sinnes des Wortes, nicht möglich ist.

Was die innere *Struktur* anbelangt, zeigen die Lautäusserungen der Tiere kein einziges Merkmal, das der menschlichen Sprache eigen ist. Um auf einige fehlende Merkmale hinzuweisen: sie bilden kein phonologisches System, es fehlt der eindeutige Verband zwischen Ausdruck und Bedeutung, die Einzelgebilde fügen sich nicht zu einer Verbindung zusammen, die ihrerseits etwas Anderes ausdrücken würde als die Einzelgebilde, sie zeigen nicht die geringste Spur von der Sonderung in Redeteile, geschweige denn von einer grammatikalischen Struktur.

Phonetisch betrachtet sind die sog. Sprechlaute der höheren Tierarten, wie die der Hunde, Katzen, Affen, die als besonders ansprechende Beispiele angeführt zu werden pflegen, unserem akustischen Eindruck nach gestaltlose, unmelodiöse Geräuschtöne, Kreisch-, Stoss- und Blöklaute, also Laute ohne phonematischen Charakter. Sie lassen sich in kein Lautsystem einordnen, daher sind sie nicht notierbar. Die schriftliche Festlegung der Lautäusserungen der Tiere durch unser Alphabet ist wissenschaftlich gänzlich unbrauchbar. Würde man die Klanganalyse der Affenlaute und ihrer Nachahmung durch menschliche Laute ausführen, so würde man erfahren, was für ein schlechtes Aufnahmeorgan unser Gehör ist und welch unzuverlässiges Abgabeorgan unser Stimmapparat für derartige Laute ist. Die „Konsonanten" sind nicht unsere Konsonanten, die „Vokale" nicht unsere Vokale. Das phonetische Bild des „K" oder „A" in der Affensprache ist mit dem Konsonanten „K" und Vokal „A" nicht identisch. Diese tierischen Laute sind nicht einmal mit den Lallwörtern der Kinder zu vergleichen. Ausserdem sind sie nicht artikuliert. Die Artikulation ist ein geistiges Moment, infolgedessen fehlt sie bei Tierlauten und bei der Lautimitation der Tiere (W. v. HUMBOLDT).

Vom *entwicklungspsychologischen* Standpunkte aus ist bemerkenswert, dass die „Tiersprache" nicht wie die menschliche Sprache als Produkt einer langen, allmählich sich differenzierenden Entwicklung anzusehen ist. Ganz im Gegenteil: sie repräsentiert ein vererbtes, unveränderliches, starres Ausdrucksmittel. Die Lautäusserungen der Tiere haben keine Geschichte und entwickeln und differenzieren sich auch während des individuellen Lebens nicht. Das Tier erlernt die Ausdruckszeichen nicht von seiner Umgebung wie wir die Sprache, sondern bringt sie im wesentlichen fertig mit sich. Das kann auch nicht anders sein, wenn man sich vergegenwärtigt, dass die tierischen Aeusserungen auch dann, wenn sie kommunikativen Zwecken dienen, nichts anderes sind als *unmittelbare Reaktionen innerer Zustände* des Tierindividuums [1]).

Aus diesen Feststellungen folgt, dass die sogenannten Tiersprachen keine gemeinsame Basis mit der Sprache haben, was so viel heisst, dass *die Tierlaute, wie ausdrucksvoll sie auch sein mögen, zu der menschlichen Sprache nicht in Beziehung gesetzt werden können.*

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[1]) Tierlaute ebenso wie affektbetonte menschliche Laute wie etwa „Ai" oder „Oh" symbolisieren nicht den Schmerz bezw. die Freude, sondern sie *drücken* diese Zustände nur *aus*. Sie sind unmittelbare körperliche Folgeerscheinungen dieser Lust- und Unlustzustände, nicht aber sprachliche Gebilde. (Vergl. dazu W. KÖHLER, Zur Psychologie der Chimpansen. Psychol. Forschung 1921; K. BÜHLER, Geistige Entwicklung des Kindes. 1921).

Um die Behauptungen durch zuverlässige Tatsachen zu stützen, gebe ich hier eine Analyse der Beobachtungen von YERKES und LEARNED, die sie anlässlich des Studiums der „Affensprache" mitgeteilt haben. Die Analyse dieser Versuche wird zugleich ein Licht auf die Unzulänglichkeit und Gefährlichkeit der von anthropomorphistischen Gesichtspunkten geleiteten tierpsychologischen Experimente werfen.

YERKES und LEARNED haben die „Vocal Expressions" zweier Schimpansen nach dem Gehör aufgenommen und eine Liste von den spontan ausgestossenen Lautäusserungen zusammengestellt (List of words or elements of speech) [1]). Sie fanden 32 „Wörter", von denen 16 von dem Schimpansen A, 14 von dem Schimpansen B und 2 von beiden benutzt worden sind. Die „Wörter" wurden dem Beginnlaut entsprechend in 5 Gruppen geordnet. Zu der ersten und zweiten Gruppe gehörten die Wörter beginnend mit dem gutturalen „G" und „K", „whispered and vocalized", zu der ditten Gruppe die mit „H"-aspirate, zu der vierten die Wörter mit Nasal- und Labiallauten, schliesslich zu der fünften die mit Vokalen.

Wenn man die Liste genau untersucht, so zeigen sich folgende Verhältnisse:

1. Alle Lautkomplexe ohne Ausnahme beziehen sich auf *eigene* Gefühlszustände. Soweit auf Grund der Angaben eine Klassifizierung möglich ist, kann ich die zu Grunde liegenden Gefühlszustände in 5 Gruppen einteilen und zwar:

a. Hungergefühl (Nahrungsbedürfnis),
b. Heftigkeit, Ungeduld, Unruhe,
c. Bedrängnis, Angst,
d. körperlicher Schmerz,
e. Freude, Zufriedenheit.

Die ersten 4 Gruppen sind unlustbetonte, die letzte Gruppe lustbetonte Gefühle. Die Gruppen stimmen im wesentlichen mit den Ausdruckslauten überein, die bereits BOUTAN bei einem weiblichen Gibbon aufgenommen hat. Auch nach ihm haben sich alle Lautäusserungen der Affen auf Affektzustände der Tiere bezogen und zwar auf den Zustand der Zufriedenheit und des Wohlseins, des Krankseins, der Furcht, der Freude und auf den Zustand grosser Erregung [2]).

2. Es zeigte sich ferner, dass derselbe Affektzustand oder dasselbe gefühlsbetonte Bedürfnis mit *verschiedenen* Lautkomplexen verbunden wird. So kommt der Nahrungsbedürfnis in 9 verschiedenen Lauten zur Aeusserung, die zu den vier ersten phonetischen Klassen gehören. Erregtheit und Unruhe wird mit Lautkomplexen aus der Gruppe *a*, *b*, *d* und *e*

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[1]) R. M. YERKES and B. W. LEARNED, Chimpanzee intelligence and its vocal expressions. 1925.

[2]) BOUTAN, Le pseudo-langage. Act. Soc. Linnée, Bordeaux. 1913.

ausgedrückt. Es ist also deutlich, dass nicht einmal zwischen Laut und Gefühlsbewegung eine eindeutige Beziehung besteht.

3. Es gibt nur eine sehr geringe Anzahl phonetisch deutlich zu unterscheidender Lautkomplexe bei Affen. Soweit man den nach dem Gehör notierten Aufzeichnungen Vertrauen schenken kann, (wobei noch zu berücksichtigen ist, dass kleine Unterschiede auf mangelhafte Intonation und unexakte Stimmgebung zurückgeführt werden können), bleiben nicht mehr als 6—8 phonetisch einigermassen verschiedene Laute, wobei die erste Gruppe (Typ: Gakh, Kgak) bereits 45 % der angegebenen 32 „Wörter" oder „Elemente" umfasst. Die weiteren Typen sind: Gho — Kah — Kuku — Wah — Huh und schliesslich einige ungenaue Vokaltöne wie Ao, Ai, Ae, Ooh. Eine schärferer Trennung zwischen den Lauten lässt sich höchstens bei den Lautreaktionen der Lust- und Unlustgefühle machen, aber auch hier kann das nicht ohne Willkür geschehen.

Diese Analyse klärt uns darüber auf, dass Affen, deren Lautäusserungen nach der Ansicht von zahlreichen Zoologen und Tierpsychologen besonders ausdrucksvoll und mannigfaltig sein sollen, nur über ein äusserst dürftiges Lautmaterial verfügen, und ihre Aeusserungen in *keiner* Hinsicht den Namen einer autochthonen Tiersprache verdienen. Es ist eine vollkommen aus der Luft gegriffene Behauptung, die Schimpansen und die Gibbons hätten eine *eigene* Sprache, und einen noch grösseren Irrtum begeht man, falls man in diesen Aeusserungen Ansätze zu der menschlichen Sprache zu finden meint. Ganz im Gegenteil: die Versuche von YERKES und LEARNED, ebenso die von BOUTAN weisen ausdrücklich darauf hin, dass die Lautäusserungen der Anthropoiden nicht nur zu der menschlichen Sprache, sondern überhaupt zu der Sprachfunktion in keiner Beziehung stehen[1]). Zu demselben Resultat gelangt man, wenn man die Liste der

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[1]) Obgleich YERKES die spontanen Lautäusserungen der Schimpansen zwar nicht als Sprache, auch nicht als Schimpansensprache betrachtet, fühlt er sich doch zu der Annahme berechtigt, dass „although there certainly is a useful substitute (!) which might readily be developed or transformed into a true language, if the animal would be induced to imitate sounds persistently." (YERKES and LEARNED, p. 66). Aus solchen Bemerkungen muss man den Schluss ziehen, dass die Forscher von der prinzipiellen Unmöglichkeit der Tiersprache nicht überzeugt sind. Diese Meinung muss ich aufrecht halten, trotz einem gegenteiligen Ausspruche von YERKES. Er sagt nämlich in seinem schönen Buch „Great Apes", S. 179 (1929): „Everything seems to indicate that their vocalisations do not constitute true language in the sense in which BOUTAN uses the term." „Their word-like sounds always lack ideational meaning." Dieser Ausspruch wird aber durch eine weitere Bemerkung aufgehoben, wo YERKES sich gegenüber der prinzipiellen Möglichkeit der Affensprache sehr positiv äussert, indem er die Ansicht vertritt, dass, wenn die Anthropoiden die Sprechtöne so gut imitieren könnten wie die Papageien, sie dann die menschliche Sprache erlernen können müssten. Damit gibt YERKES zu, dass die Anthropoiden Anlage für Sprachfähigkeit haben und dass bloss die unentwickelte sprachtechnische Fähigkeit im Wege stehe, sich mit uns zu unterhalten. (Von dieser Anschauung konnte sich auch der vorsichtige H. DELACROIX (Psychologie du Langage. 1933) nicht ganz emanzipieren).

Stimmäusserungen, die sog. Vokabularien anderer Tierarten einer Analyse unterwirft.

Um jedes Missverständnis auszuschliessen soll nochmals betont werden, dass wir bei dem Begriff autochthone Tiersprache das Gewicht auf „Sprache" und nicht auf „autochthone" legen. Denn, dass Tiere arteigene lautliche Kommunikationsformen besitzen, haben wir nicht in Abrede gestellt. Es handelt sich allein darum, mit aller Entschiedenheit hervorzuheben, dass tierische Kommunikationsformen mit der menschlichen *Sprache* nichts gemein haben, auch dann, wenn sie unter den Begriff Kommunikation fallen. Daraus, dass Tiere nur über elementare Kontaktmittel verfügen, lässt sich auf eine Inferiorität nicht schliessen, jedenfalls nicht in Bezug auf die Befriedigung ihrer Bedürfnisse und auf die Gestaltung ihrer Lebensweise. Für die Existenz der Tiere genügen die geschilderten Kommunikationsmittel; sie beherrschen ihre Umwelt anscheinend besser *ohne* Sprache, als wir unsere *mit* Sprache.

Die Sprache entspringt aus einer anderen Quelle und ist durch andere Tendenzen bestimmt und durch andere Gesetze beherrscht als die tierischen Aeusserungen. Die letzteren sind ohne Ausnahme Auswirkungen von Trieb- und Affektzuständen, stellen daher ebensowenig Wörter oder andere sinnerfüllte Symbole dar wie die unmittelbaren Reaktionen unserer Trieberregungen.

Die vorangegangenen Erörterungen und Analysen deuten auf eine wichtige sprachpsychologische Tatsache hin, nämlich, dass die wirkliche Sprache, *die Sprache als solche, erst dort einsetzt, wo lebende Wesen nicht mehr ausschliesslich durch Instinkte und Affekte beherrscht, sondern durch gewisse Absichten und Ziele und durch die Einsicht in die zu ihrer Verwirklichung geeigneten Mittel geleitet werden. Gerade diese Art des Gerichtet-Seins, welches als Voraussetzung der ganzen menschlichen Kultur gilt, trennt das menschliche Leben vom tierischen Dasein* [1]).

Die Feststellung, dass Tiere jeder Form der Sprache entbehren, hat eine weittragende entwicklungsgeschichtliche Bedeutung. Wenn man nämlich das Wesensmerkmal des Menschen in allererster Linie in der *Sprache* sieht, — und das wird mit vollem Recht seit der griechischen Philosophie bis in unsere Zeit gelehrt — dann sind die Anthropoiden, trotz ihrer anatomisch-morphologischen Verwandtschaft mit den Menschen, nicht weniger *Tiere* als die unähnlichen in der Entwicklungsreihe viel tiefer stehenden animalischen Wesen.

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[1]) Es handelt sich hier nicht um eine terminologische Streitfrage, um den Begriff „Sprache" bei Tieren ex definitione auszuschliessen. Es geht hier vielmehr auf sachlich-psychologische Argumente an, die die definitorischen Auseinandersetzungen erst veranlassten.

Comparative Physiology. — *Die Regulierung der Herztätigkeit bei Helix pomatia durch extracardiale Nerven und das Zentralnervensystem.*

Von N. VAN TIEL. (Aus dem Institut für vergleichende Physiologie der Universität zu Utrecht.) (Communicated by Prof. H. J. JORDAN.)

(Communicated at the meeting of November 30, 1940.)

I. *Einleitung. Die nervöse Regulierung des Tonus und der Beweglichkeit bei Gastropoden.* Die Funktion der höchsten Zentren bei den Gastropoden beruht im Wesentlichen darauf, dass das *Ausmass* aller autonomen Reaktionen der lokomotorischen Muskulatur reguliert wird. Jede Regulation beruht auf dem Vermögen der Zentren, die Mittelwerte der autonomen Reaktionen zu erhöhen und zu erniedrigen. Normalerweise überwiegt die *Erniedrigung* der Leistung, d.h. also die *Hemmung*, bei dieser Anpassung peripheren Geschehens an die Anforderungen des gesamten Organismus. Die verschiedenen Hypothesen über die Mechanik der Hemmung mögen hier unbesprochen bleiben. Die Notwendigkeit Versuchsergebnisse zusammenfassend zu beschreiben, zwingt uns zur Benutzung bestimmter hypothetischer Schemata. Es sei aber ausdrücklich darauf hingewiesen, dass es hierbei nur auf die Phaenomenologie der Erscheinungen und nicht auf Erklärung eines Prinzips ankommt.

Bei den Gastropoden kennen wir eine zentrale Regulation der Lokomotion durch das G. cerebrale und des Muskeltonus durch das G. pedale. Reflexerregbarkeit und Lokomotion werden unter gewöhnlichen Umständen (d.h. hauptsächlich in der Ruhe) durch das G. cerebrale *gehemmt*, der tonische Zustand des Hautmuskelschlauches aber durch das G. pedale. Alle Versuche sprechen dafür, dass diese Hemmungen nicht durch eine künstliche Reizung der betreffenden Ganglien hervorgerufen werden können. Reizung beider Ganglien hat einen nichtspezifischen Effekt, nämlich Verkürzung aller Muskeln. Die Regulierung dahingegen ist für das G. cerebrale und das G. pedale spezifisch; sie ist „tonischer" 1) Natur. Die normale hemmende Wirkung der Ganglien offenbart sich durch Ausfallserscheinungen, also dadurch, dass die Funktionen in ihrem Ausmasse stark zunehmen, wenn man das entsprechende Ganglion exstirpiert, oder durch „starke Kokainisierung" ausschaltet. Die Hemmung selbst aber tritt am stärksten zutage, wenn die Ganglien je durch *schwache* Kokainisierung auf ein niederes „Aktivitätsniveau" herabgesetzt worden sind ohne dass

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1) Wir wünschen für Dauerwirkungen der Zentren das Wert Tonus zu vermeiden und sprechen daher in der Regel von Aktivitätszustand und Aktivitätsniveau

ihre Funktion hierbei ausgeschaltet wird. Es ist auffällig, dass die Ganglien im Zustande narkotischer Depression ihre normalste Funktion am besten leisten. Erhöhung des Aktivitätszustandes des Cerebralganglions durch Kochsalzaufbringung erhöht dahingegen die Reflextätigkeit der Peripherie; die gleiche Behandlung des Pedalganglions macht, dass der Widerstand der Muskeln gegen passive Deformierung zunimmt (Tonus). Ein schwach kokainisiertes Pedalganglion erniedrigt auch einen Muskeltonus, den es nicht selbst erzeugt hat, der vielmehr experimentell durch Aufbringen von Kochsalz auf den Schneckenfuss hervorgerufen worden war. (H. J. JORDAN, Frl. C. E. SLUITER und N. POSTMA, 1938). Offenbar sind die Ganglien nicht im üblichen Sinne des Wortes Hemmungszentren, da die Reizung eines solchen die entsprechende Hemmung hervorrufen müsste. Man könnte an das Vorhandensein eines Erregungszentrums neben einem Hemmungszentrum denken, wobei nach Reizung nur das erstere sich geltend machte. Halblähmung mit Kokain müsste aber dieses Erregungszentrum ausschalten, denn sonst bliebe gesteigerte Hemmungswirkung in diesem Zustande unerklärt. Es bliebe nur das Hemmungszentrum übrig und elektrische Reizung müsste dann Hemmung der entsprechenden Funktion hervorrufen; es tritt aber immer nur unspezifische Muskelkontraktion auf. Dass hierbei *Wedensky*hemmung eine Rolle spielt, konnte durch Versuche mit einiger Sicherheit ausgeschlossen werden.

JORDAN hat daher geschlossen, dass ein Gegensatz zwischen Vagus- und Acceleranswirkung beim Vertebratenherz und der Regulierung von Tonus und Beweglichkeit bei Gastropoden besteht. Zur anschaulichen Beschreibung der Tatsachen bediente er sich rein phaenomenologisch eines Schemas: Muskeln mit Nervennetzen, sowie die zentralen Ganglien sind Endpunkte der Erregungsleitung, die sich jeweils in einem bestimmten Aktivitätszustande befinden. Es besteht aber offenbar das Bestreben gegenseitiger Angleichung oder Koordination dieses Aktivitätszustandes. Dieses Schema gibt keine Erklärung für die nervösen Prozesse, die sich zwischen den Leitungsendpunkten abspielen. Allein es erlaubt die Erscheinungen, dass die Gesamtmuskulatur z.B. einer Schnecke, sich jeweils im gleichen Erregbarkeits- und Tonuszustande befindet, dass das erregende und hemmende Eingreifen der Ganglien sich nach dem jeweiligen Zustande der Peripherie richtet und endlich, dass erregende und hemmende Wirkung der Ganglien jeweilig nach Massgabe eigenen Aktivitätszustandes stattfindet, zusammenfassend zu beschreiben. Auf diese „koordinative" Weise kann aber nur eine Zustandsregulierung auftreten, d.h. Erzeugung von abgestufter „Reflexbereitschaft" (MAGNUS). Das bedeutet eine Regulierung der *Beweglichkeit*. Unhemmbare Schwimmbewegung bei beiderseitiger, Kreisbewegungen bei einseitiger Dezerebralisierung (Aplysia) sind Beispiele hierfür. (H. J. JORDAN, 1918, 1929, 1935, JORDAN, C. E. SLUITER und N. POSTMA, 1938.) Das Zustandekommen von komplizierten Handlungen, z.B. bei der Kopulation, dürfte differenziertere Einflüsse und Impulse der höchsten Zentren zur Voraussetzung haben. *Bewegungen* fordern daher offenbar auch eine andere Form der Regulierung, bei der die Ganglien spezielle Hemmungs- und Erregungszentren enthalten müssen, deren Impulse in das Getriebe der Bewegung selbst eingreifen. Dies hat sich neuerdings durch die im folgenden beschriebenen Versuche über Beschleunigung vor allem aber über Hemmung des Schneckenherzens durch zentrale Ganglien und den N. intestinalis ergeben.

II. *Einige Literatur.* Die wenigen Angaben in der Literatur über die Regulierung des Herzrhythmus durch extracardiale Nerven und das Zentralnervensystem bei Gastropoden liessen eine erneute Untersuchung über Beschleunigung und Verzögerung (Hemmung) des Schneckenherzens aussichtsreich erscheinen. Ransom (1884) hatte gefunden, dass Reizung des Nervus intestinalis Stillstand des Schneckenherzens in Diastole verursacht. Carlson (1904, 1905 a, b) meinte seinen Versuchen die Tatsache entnehmen zu können, dass neben Herzhemmung auch regulative Beschleunigung besteht, eine Angabe, der Subkow (1934 a, b) widersprach. Subkow dürfte der einzige Forscher sein, der einige Erfahrungen über die

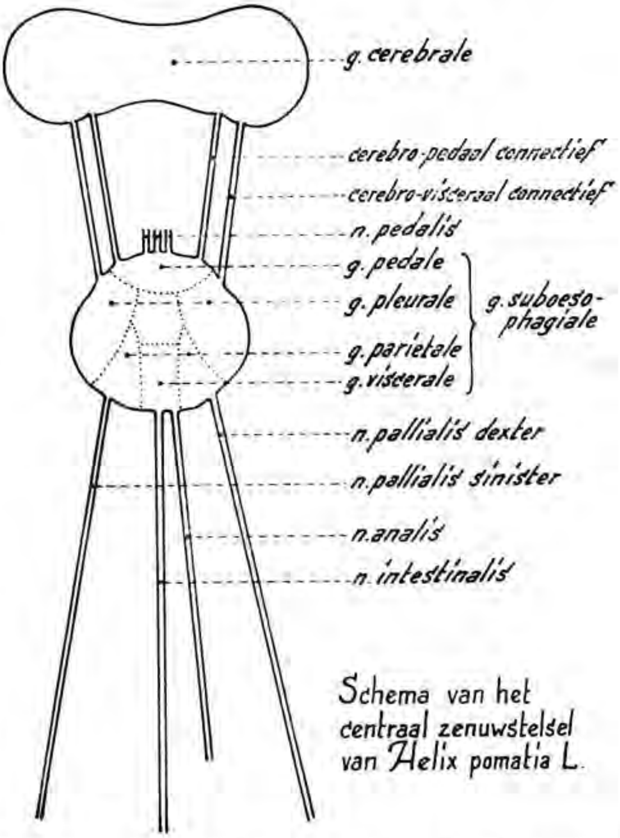

Fig. 1. Schema des zentralen Nervensystems mit dem Ursprung der für uns wichtigen peripheren Nerven bei Helix pomatia L.

regulative Wirkung der Ganglien auf das Helixherz mitgeteilt hat (siehe meine ausführliche Publikation).

III. *Untersuchungsmethode.* Das Herz einer Helix kommt an einen vertikalen isometrischen Hebel (Zeiger), der vor dem Spalt einer Rollfilmkamera steht. Durch starkes Projektionslicht beleuchtet, wirft der Zeiger einen Schatten auf den Spalt, der auf dem ruhenden Film einen Punkt, auf dem laufenden Film eine Linie und bei Bewegung des Zeigers das Bild eine Kurve erzeugt.

IV. *Reizversuche, durch welche bewiesen wird, dass nur der N. Intestinalis regulierende Funktion hat.* Welche periphere (extracardiale) Nerven kommen für die Regulierung des Herzens von Helix pomatia durch die Gangliensysteme in Frage? Auf Figur 1 sieht man, dass aus der suboesophagealen Ganglienmasse 4 Nerven entspringen, die nicht zum System des Pedalganglions gehören. 2 Nerven kommen aus den Parietalganglien, der Nervus pallialis dexter und sinister. Aus dem Ganglion viscerale entspringen der Nervus analis und der Nervus intestinalis. Faradisierung jedes der 4 mit der Ganglienmasse verbundenen Nerven verursacht Herzstillstand in Diastole (Fig. 2). Wenn man dahingegen die 4 Nerven, da, wo sie die Ganglienmasse verlassen, durchschneidet, erhält man Stillstand nur bei Reizung des Nervus intestinalis. In der erstgenannten Versuchsserie war der Stillstand bei Reizung der beiden palialen und des analen Nerven reflektorisch auf den N. intestinalis übertragen worden. Der N. intestinalis ist daher der Herznerv. *Jede Reizstärke erzeugt bei Reizung des N. intestinalis Herzstillstand in Diastole.*

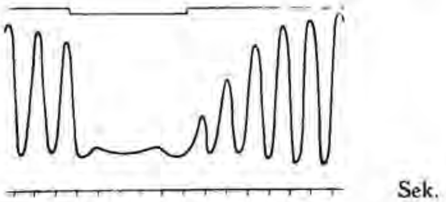

Fig. 2. Herzschlag bei Helix pomatia. Hemmung (Stillstand in Diastole) durch Reizung des N. intestinalis. Oben Reizsignal, zu unterst die Zeit in Sekunden.

V. *Die refraktäre Periode der Herzhemmung.* Man reizt den N. intestinalis während des spontanen Herzrhythmus. Nur während bestimmter Teile einer Phase treten Hemmungserscheinungen auf. In andern ist das Herz für die Hemmungswirkung mehr oder weniger refraktär (Fig. 3).

a. Reize von kurzer Dauer, welche das Herz während des ersten Beginnes einer Systole treffen, unterdrücken die noch nicht eigentlich begonnene Systole.

b. Reize, die etwas später fallen, sodass die Systole schon angefangen hat, unterdrücken diese nicht mehr vollständig. Sie wird nur kleiner als eine normale Systole, und zwar macht sich die Hemmung desto weniger geltend (grössere Systole), je später der Reiz eintrifft, bis zu einer Grenze, von der an der Hemmungsreiz wirkungslos wird. Dann hat die betreffende Systole normale Grösse und es tritt bei kurzer Reizdauer überhaupt keinerlei Hemmung auf.

c. Auch während der Diastole hat kurze Reizung keinen sichtbaren Effekt.

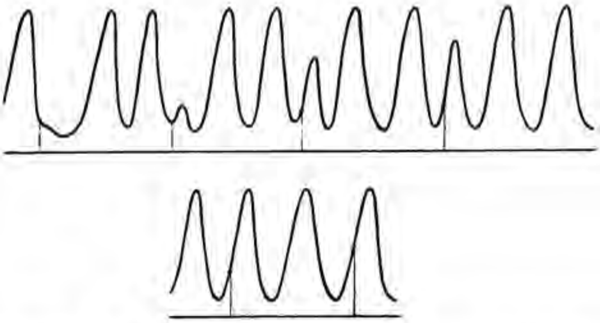

Fig. 3. Herzschlag bei Helix pomatia. Kurze Reizung des N. intestinalis an verschiedenen Stellen der Systole, die Stelle je durch eine Ordinate angegeben. „Refraktäre Periode der Hemmung" des Herzens.

VI. *Einfluss der Ganglien auf die Regulierung der Herztätigkeit.*

a. Uebersicht über die Erscheinungen und ihre zunächst hypothetische Erklärungen. Die Versuchsergebnisse lassen sich am besten erklären, wenn man im Ganglion cerebrale das Vorhandensein eines einzigen Zentrums, und im G. viscerale zweier Zentra annimmt. Von den visceralen Zentren ist eins ein Hemmungszentrum („Vaguszentrum"), das andere ein Erregungs- oder Acceleranszentrum. Das Hemmungszentrum hat eine niedere Reizschwelle, das Erregungszentrum eine hohe Schwelle. Das Herzzentrum des G. cerebrale ist mit beiden visceralen Zentren verbunden und wirkt auf beide gleichsinnig, n.l. hemmend, sodass vom Cerebralganglion aus sowohl die hemmenden als die erregenden Wirkungen des G. viscerale herabgesetzt werden können. Dieses vorläufig zum Teil noch hypothetische Schema dient hier lediglich, um die Versuchsergebnisse übersichtlich vortragen zu können. Genauere Argumentierung des hypothetischen Teiles dieses Beschreibungsschemas sei späteren Publikationen vorbehalten.

b. Versuche mit dem Ganglion cerebrale, Ausschaltung und Erregung. 1. *Ausschaltung des G. cerebrale durch Vergiftung mit Kokain* (Fig. 4). Auftragen von Kokainpulver hat zuerst unspezifische Reizwirkung: Stillstand des Herzens in Diastole. Sodann stellt sich die Herztätigkeit wieder her. In der Zwischenzeit hat das Kokain aber das Ganglion völlig gelähmt, was sich durch Verzögerung des Herzschlages, im Momente seines Wiedereinsetzens, zu erkennen gibt. Normalerweise überwiegt die Hemmung durch das viscerale Hemmungszentrum, da dieses die niedrigste Reizschwelle hat („Vagotonus"). Diese übertriebene Hemmung wird durch den Einfluss des Ganglion cerebrale zur Normalen reduziert. Ausschaltung des G. cerebrale verursacht daher übertriebene Hemmung, d.h. die Frequenz (und Amplitüde) des Rhythmus wird vermindert. Ein Einfluss der Ausschaltung auf das Acceleranszentrum ist nicht nachzuweisen, da offenbar die normale hemmende Wirkung des G. cerebrale zu schwach ist, um sich auf das Acceleranszentrum des G. viscerale, mit seiner hohen Schwelle, nachweislich geltend zu machen.

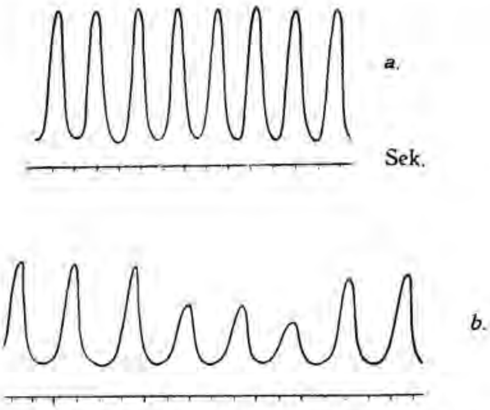

Fig. 4. Herzschlag bei Helix pomatia. Einfluss des durch Kokain gelähmten G. cerebrale auf den Herzrhythmus. *a.* Normaler Rhythmus. *b.* Verzögerter Rhythmus, nachdem der Einfluss das G. cerebrale ausgeschaltet worden ist. (Die Aenderung der Amplitüde des Herzschlages ist ein nebensächliches Phänomen, auf welches wir hier nicht weiter eingehen.)

2. *Erregung des G. cerebrale durch Aufbringen von kristallinischem Kochsalz* (Fig. 5). Auch Kochsalz wirkt zuerst unspezifisch als Reiz und verursacht Herzstillstand in Diastole. Nach Abklingen der Reizwirkung übt Kochsalz eine starke, dauernde *erregende* Wirkung auf das Cerebralganglion aus. Sie ist so stark, dass nunmehr auch das Acceleranszentrum soweit gehend gehemmt wird, dass der Herzrhythmus verlangsamt wird. Aber ähnlich wie beim Vertebratenherzen, stellt sich allmählich die Frequenz des Rhythmus des Helixherzens einigermassen wieder her, trotzdem das Kochsalz nicht vom G. cerebrale entfernt wird. Die Frequenz nimmt aber wieder ab, wenn man neues Kochsalz auf das G. cerebrale bringt. Zusammenfassend können wir sagen: *Reizung* hat immer den gleichen unspezifischen Effekt: Stillstand in Diastole. Dauererregung aber ruft Erscheinungen mit spezifischen Merkmalen hervor. Trotz der Heftigkeit der *Erregung* durch Kochsalz wird der Rhythmus nur verzögert, nicht völlig unterdrückt. Die Hemmung durch erste *Reizung* behauptet sich nicht. Aehnlich wie beim Einfluss des G. cerebrale und

G. pedale auf die Lokomotionsmuskulatur, hat Erregung des G. cerebrale und seine totale Ausschaltung gleichsinnigen Einfluss auf das periphere Geschehen, in diesem Falle auf das Herz: Nur können wir die Erscheinungen am Helixherzen nicht mit dem Koordinationsschema beschreiben. Niemals erzeugt bei der Lokomotionsmuskulatur Reizung des höchsten Zentrums Hemmung. Die Ganglien hemmen auf Grund ihres normalen Aktivitätszustandes und erregen wenn dieser gesteigert wird. Auch tritt peripherische Mehraktivität auf, wenn die zentrale Hemmung durch Ganglienexstirpation wegfällt. Beim Herzen ergibt sich das Gegenteil. Ausschaltung (mit Kokain) und Erregung (mit Kochsalz) des G. cerebrale haben zwar auch gleichsinnige Wirkung, diese verursacht aber Verzögerung des Herzrhythmus, also das Gegenteil von gegenseitiger Angleichung. Unterschiede prinzipieller Art zwischen den Wirkungen geringerer und höherer Kokainkonzentration konnten beim Herzen (im Gegensatz zur Lokomotionsmuskulatur) nicht nachgewiesen werden. Daher wollen wir für die zusammenfassende Beschreibung das in der Physiologie der Vertebraten übliche Schema für die Regulierung des Herzrhythmus benutzen. Bei dem Koordinationsschema der Lokomotionsmuskulatur kann man die Annahme bestimmter gegensätzlich wirkender Zentren entbehren. Jedes Ganglion reguliert ja nur eine einzige Funktion. Bei der Regulierung des

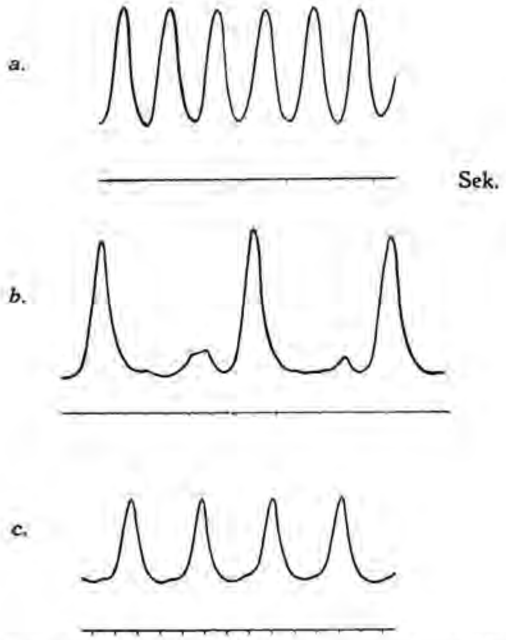

Fig. 5. Herzschlag bei Helix pomatia. Einfluss des durch NaCl erregten G. cerebrale auf den Herzrhyhtmus. *a*. Normaler Rhythmus. *b*. Verzögerter Rhythmus, nachdem das G. cerebrale in Erregungszustand versetzt worden ist. *c*. Partielle Wiederherstellung des Rhythmus durch Adaptation des Ganglions an den Erregungszustand.

Herzrhythmus müssen wir aber, wie oben schon gesagt wurde, das Vorhandensein zweier visceraler Zentra annehmen, von denen das eine den Rhythmus erhöht, das andere hingegen den Rhythmus verzögert. Das G. cerebrale aber wirkt auf beide Zentra hemmend: es hemmt die Hemmung und die Beschleunigung. Die paradoxale Gleichsinnigkeit der Wirkung von Kokain und Kochsalz auf das G. cerebrale wird durch einen Unterschied in der Reizschwelle beider visceraler Zentra erklärt: Lähmung schaltet den normalen cerebralen Einfluss aus, der zu schwach ist um das Beschleunigungszentrum mit seiner hohen Schwelle nachweislich zu hemmen, dem aber das Hemmungszentrum mit seiner niederen Schwelle unterworfen ist. Daher fällt bei Kokainisierung Hemmung der Hemmung fort, die Hemmung wird stärker. Bei Erregung des G. cerebrale durch Kochsalz wird die Schwelle des Beschleunigungszentrums durch die cerebrale Hemmung überschritten: auch hierdurch wird der Herzrhythmus verzögert; daher die prinzipielle Uebereinstimmung zwischen zwei gegensätzlichen Eingriffen.

c. Das Ganglion viscerale. 1. *Ausschaltung durch Kokain* (Fig. 6). Auch hier tritt erst eine Reizwirkung auf, mit Stilstand des Herzens in Diastole. Sobald aber die narkotische Wirkung einsetzt, wird erst das Acceleranszentrum mit seiner hohen Schwelle ausgeschaltet, denn normale Erregungen werden für dieses Zentrum unterschwellig, daher verlangsamter Rhythmus. Später aber, bei fortschreitender Kokainnarkose, werden Erregungen auch für das Hemmungszentrum unterschwellig, sodass die Herzfrequenz wieder zunimmt.

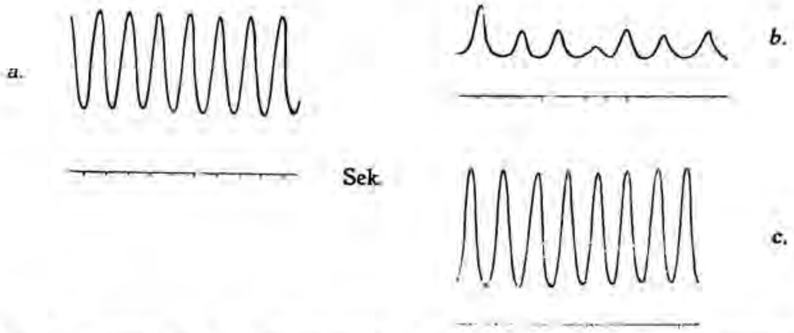

Fig. 6. Herzschlag bei Helix pomatia. Wirkung der Lähmung des G. viscerale durch Kokain auf den Herzrhythmus. *a.* Normaler Rhythmus. *b.* Verzögerter Rhythmus, nachdem das viscerale „Acceleranszentrum" ausgeschaltet worden ist. c. Beschleunigter Rhythmus, nachdem auch das viscerale Hemmungszentrum ausgeschaltet worden ist.

2. *Aufbringung von Kochsalz auf das G. viscerale* (Fig. 7). *a.* Wenig Kochsalz. Das Hemmungszentrum mit der niedersten Schwelle wird erst stark gereizt. Daher tritt Stillstand in Diastole, als nichtspezifische *Reiz-*

wirkung auf. Danach macht Reizung der Dauer*erregung* durch das nämliche Kochsalz Platz, sodass das Herz wieder zu schlagen beginnt, nun aber mit verzögertem Rhythmus.

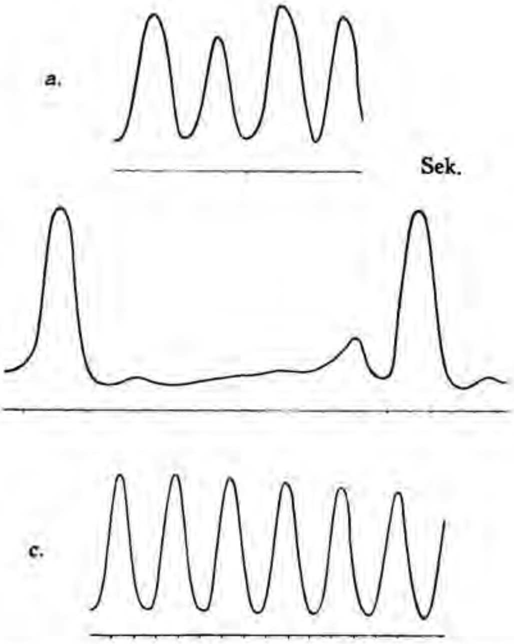

Fig. 7. Herzschlag bei Helix pomatia. Einfluss des von NaCl erregten G. viscerale auf den Herzrhythmus. *a*. Normaler Rhythmus. *b*. Verzögerter Rhythmus, nachdem das viscerale Hemmungszentrum in Erregungszustand versetzt worden ist. *c*. Beschleunigter Rhythmus, nachdem auch das viscerale Acceleranszentrum in Erregungszustand versetzt worden ist.

b. Mehr Kochsalz. Nun wird auch das Beschleunigungszentrum mit der höheren Schwelle erregt. Es tritt, verglichen mit der normalen, erhöhte Frequenz des Herzschlages auf.

c. Das Kochsalz wird weggewaschen. Die Frequenz nimmt wieder ab, denn die Erregung sinkt unter die Schwelle des Beschleunigungszentrums.

d. Kochsalz wird wieder hinzugefügt, die Frequenz steigt wieder.

Die Reproduzierbarkeit aller dieser Versuche ist naturgemäss nicht vollkommen. In erster Linie haben wir es mit der Reaktion höchster Zentren zu tun, wobei man vollständige Reproduzierbarkeit nicht erwarten kann. Wichtiger aber ist, dass bei dem cerebralen Einfluss auf zwei Zentren mit entgegengesetzter Wirkung, kleine Verschiebungen der Schwelle, die für unser Objekt so charakteristisch sind, das Verhältnis zwischen Hemmung und Erregung verschieben können. Es wird die Aufgabe der ausführlichen Veröffentlichung sein, statistisches Material zu geben und die hypothetische Grundlage dieser Auseinandersetzungen, die hier, wie gesagt, lediglich zur Ordnung des Materials benutzt wurde, auf sichereren Boden zu bringen.

VII. *Schlussbetrachtung.* Die Regulierung des Tonus und der Beweglichkeit durch die höchsten Zentren geschieht bei den Gastropoden in vielen Beziehungen anders, als bei den Wirbeltieren. Um so auffallender ist die weitgehende Uebereinstimmung in der Regulierung der Herztätigkeit bei beiden Tiergruppen. Auch bei den Gastropoden kann man Vagus- und Acceleranswirkung, Vago- und Sympathicotonus nachweisen. Das Vaguszentrum und seine Regulierung durch cerebrale Hemmung spielt bei alledem wegen seiner niederen Schwelle, die Hauptrolle.

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Anatomy. — *Ein Finger-Umfang-Mass.* Von ST. MARTIN. (Communicated by Prof. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of November 30, 1940.)

Die Goldschmiedfirma BRUNO BADER in Pforzheim hat zur Feststellung des Finger-Umfanges für die Anfertigung von Fingerringen ein kleines Instrument herausgebracht, *das Ringmass D.R.G.M.*, das — unbeabsichtigt — auch dem Anthropologen bei der Untersuchung der Finger-Weichteile (Muskel plus Fett) von Nutzen ist.

Man hat sich bisher auf die Feststellung der knöchernen Unterlage der Finger und auf die Hand-Umrisse beschränkt, die ja auch in den Fällen genügt, wenn es sich darum handelt, erbbiologische Verschiedenheiten bezw. Aehnlichkeiten zu untersuchen. Man sollte aber doch nicht die *konstitutionell* bedingten Eigentümlichkeiten der Finger unbeachtet lassen, deren Umfänge nicht nur vom Skelet, sondern ebenso sehr von der individuellen Anlage und nicht zum wenigsten von *peristatischen* Ver-

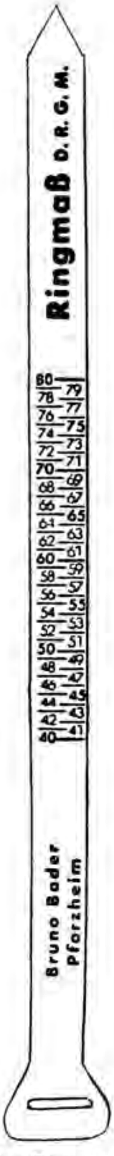
Fingerumfänge (mit dem Ringmass gemessen): Linke Hand einer Frau

	V	IV	III	II	I
Phalanx I (Mitte zw. d. Gelenken)	61 mm	62 mm	71 mm	72 mm	70 mm
Phalanx II (Mitte zw. d. Gelenken)	52	56	60	56	59
Phalanx III (Mitte des Nagels)	42	47	49	45	—
Zusammen:	155	165	180	173	129
Fingerlänge vom *Phalangion* aus:	69	83	97	84	61

$$\text{Index: } \frac{\text{Fingerlänge} \times 100}{\text{Summe der 3 resp. 2 (Daumen) Fingerumfänge}}$$

	V	IV	III	II	I
Schwester d. Vaters (obiges Beispiel)	44.5	50.3	53.8	48.5	47.2
Vater	42.2	50.8	53.0	45.6	42.5
Mutter	60.8	72.2	70.0	62.0	61.7
Sohn, erwachsen	54.2	68.0	63.8	61.0	56.5
Tochter, erwachsen	53.2	66.6	65.6	60.6	58.5

änderungen abhängig sind. Eine nicht zu übersehende Bedeutung hat ferner die Beobachtung der Fingerdicke während des Wachstums und dann als *erbliches* Moment, speziell bei Familienuntersuchungen.



Das Ringmass gibt den Umfang des Fingers in Millimetern an.

(etwa um $^1/_{10}$ vergrössert)

Beispiel: Vater: ausgesprochen fleischige kurze Hand,
Mutter: ausgesprochen knöcherne lange Hand,
Sohn u. Tochter: beide haben die Hand der Mutter geerbt (vergl. die kleine Indextabelle),
während die Mehrzahl der übrigen körperlichen Eigenschaften des Vaters *dominant* waren, wie die Körpergrösse, (dunkle) Haarfarbe, (dunkle) Augenfarbe, Kopf- und Gesichtsform.

Der Vater ist manuell durch seine starken Fingerumfänge ungeschickt; Mutter und Kinder haben mit ihren schlanken Fingern eine besondere handliche Geschicklichkeit, was für die qualitative Bedeutung des Fingerumfang-Längen-Index spricht.

Konstitutionell spielen demnach die Fingerumfänge eine bisher nicht beachtete Rolle. Ein weiterer Faktor ist die Veränderung der Fingerdicke durch bestimmte Betätigung: der Handwerker wird mehr erreichen, wenn er sich an die Fingerumfänge seiner Hand hält und danach die Berufswahl berücksichtigt. Das Handtrainung des Pianisten lässt selten viel Fettpolster aufkommen, kräftigt aber Muskel und Bänder in der Weise, dass die Umfänge im bestimmten Verhältnis zur knöchernen Unterlage stehen. Die Fingergeschicklichkeit des Gynäkologen korreliert mit dem Fingerumfang.

Das Fettpolster der Hand spielt bei der Beurteilung des individuellen Ernährungszustandes eine Rolle; die *Stärke* des Fettpolsters ist aber auch konstitutionell bedingt und hängt somit nicht allein von der Ernährung, sondern von der individuellen Reaktion auf die Umwelt, sowie von der hormonalen Beschaffenheit der Person ab. Bei bestimmten Krankheiten ist die Abmagerung der Hand und besonders der Fingerumfänge, wie überhaupt ihre Veränderung eine Indikation; es wäre also auch für den Arzt wichtig mittels dieses kleinen Geräts, das genau und fein ist, festzustellen, in welchem Mass die Abmagerung bezw. Veränderung nach gewissen Zeiträumen erfolgt. Schliesslich ist die rassiale Bestimmung der Fingerumfänge eine bisher vernachlässigte Arbeit. Man vergleiche z.B. die mageren langen Negerhände mit den plumpen der Ainu. Aber nicht nur die Finger, auch die Zehen lassen sich mit dem kleinen Ringmass messen. Seine Handlichkeit und billige Herstellung erlauben es, eine Menge mit sich zu führen, da begreiflicherweise die Haltbarkeit des Zellophan gering ist. Der Vorteil ist, dass Zellophan sich nicht dehnt — das ist leicht kontrollierbar —, wie ein Stoffbandmass und sich auch sonst durch den Gebrauch nicht ändert, sodass die Fehlerquellen bei der Messung kaum am Gerät liegen. Der Beobachter muss darauf achten, stets an bestimmter Stelle der Finger zwischen den Gelenken das kleine Bandmass anzulegen und bei der Wiederholung strikte dabei zu bleiben. Vielleicht wird durch diese Anregung ein Versuch auf seine Brauchbarkeit gemacht, auch hinsichtlich seiner Brauchbarkeit bei der Umfang-Messung langer Röhrenknochen, für die das Ringmass aber mindestens bis 90 mm graduiert sein muss.

Medicine. — *Biological Properties of Aethinyl-testosterone* *). By L. A. M. STOLTE. (From the Department of Pharmacology, University of Leiden. Director Prof. S. E. DE JONGH.) (Communicated by Prof. J. VAN DER HOEVE.)

(Communicated at the meeting of November 30, 1940.)

II. *Not Progesterone-like properties (oestrone-like, testosterone-like).*

In a previous paper (1) some progesterone-like activities of aeth.-test. were described. We now wish to deal with the other effects, observed with aeth.-test., grouped as follows:

1. oestrone-like,
2. testosterone-like,
3. oestrone and/or testosterone-like?

1. *Oestrone-like activities of aeth.-test.*

a. An indication for this sort of activity we found in already published experiments, concerning the suppression of the oestrone-oestrus in castrated *mice*: with a quantity of aeth.-test., not (yet) causing an inhibition, an earlier appearance of the *oestrus* was obtained. (Table I).

TABLE I.

Treatment	Number of animals in wich oestrus appeared	
	Within 24 hours after last dosage	More than 24 hours after last dosage
0.1 γ oestrone	2	6
0.1 γ oestrone + 12 mgm aeth.-test. *per os*	9	—
0.12 γ oestrone	3	5
0.12 γ oestrone + 12 mgm aeth.-test. *per os*	10	1

This table further shows, that with the combined treatment a larger number of animals reached oestrus than with oestrone alone. The acceleration of the oestrus-effect also occurred in the majority of the animals, that escaped inhibition after a higher dosage aeth.-test. For these reasons it was investigated, whether aeth.-test. alone is capable of producing oestrus in castrated female mice: 10 or more of a series of 20 animals received 12—20 mgm aeth.-test. orally or 6—10 mgm aeth.-test. subcutaneously.

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*) For literature and further details vid. anew Dissertation Leiden 1940.

The rest was used as controls. In the following weeks the groups were interchanged or a pause was shoven in between treatments. We finished with 40 mgm orally or 30 mgm subcut., in the course of 5 days.

In these experiments we sometimes met with a positive oestrus sign, what has to be considered as a proof of the oestrogenic activity of aeth.-test., since oestrus fails to appear in controls, not treated with this substance. The oestrus developed early and was of short duration: obviously the progesterone-like effect predominates very soon. Besides, the same animal could not be brought to oestrus even after weeks. The histological examination yielded the explanation: aeth.-test. had produced mucification as well as cornification in the vaginal wall (sometimes even mucous cells upon a layer of cornification!) The mucification lasted more than a week.

In castrate *rats* too we could show the oestrogenic effect of aeth.-test.: 200 γ aeth.-test. subcut. or 400 γ *per os*, both daily given for 20 days to 4—5 w. old animals gave rise (besides an initial oestrus) to the return of the oestrus on the 12th—13th day in the second group.

Oestrus was also observed in pregnant rats, castrated on the 5th or 10th day and treated with daily 5 mgm aeth.-test. subcut. or 10 mgm orally. The placenta not yet being of importance at this stage of pregnancy and the only other possibility for the production of oestrogenic substances being removed (MC KEOWN, c.s. (2)) we considered this to be a new proof of the oestrogenic power of aeth.-test.

b. The opening of the introitus vaginae, combined with oestrus is another mainly oestrogenic effect of aeth.-test., observed by us in 8 immature rats (16—19 days old), treated with 400 γ aeth.-test. subcut., twice a day for 3 days. Though the influence of the testosterone-like component cannot be wholly denied (testosterone too is capable of producing this phenomenon), the oestrogenic effect must have played the main role here, since an equal dose of testosterone caused a disclosure, appearing *later* and *without* oestrus. Progesterone, that never produces oestrus, also fails to give rise to vaginal opening, even to a later occurring one. 800—1000 γ aeth.-test. *per os*, twice daily resulted in vaginal opening without oestrus.

c. The *changes in the uteri* of the *rats* in the above investigations also pointed to an oestrone-like and partially to a progesterone-like influence of aeth.-test.

The uterus weights in the groups "aeth.-test. subcut." were distinctly higher than those in all other groups. The enlargement of the uteri with aeth.-test. *per os* was of the same range as that with testosterone.

Histologically the "aeth.-test. subcut." uteri showed a loose, vacuolated stroma with blown-up nuclei and an ephithelium of high, cylindrical cells with clear basal nuclei and a high protoplasm seam. Neither testosterone, nor progesterone, nor aeth.-test. *per os* could call forth this picture. It was only found in rats, treated with 400 γ progesterone + 0.25 γ oestrone twice daily for 3 days or 0,25 γ oestrone 2 dd during 6 days, combined with 2 dd 400 γ progesterone the last 3 days.

d. The oestrogenic effect of aeth.-test. on the *uterus* was also seen in immature *rabbits*: 40 mgm aeth.-test. orally, divided over 5 days caused a growth of the organ, somewhat inferior to that, obtained with 10 γ oestrone subc. and somewhat superior to that with 20 mgm testosterone proprionate subc. 20 mgm aeth.-test. subcut. caused a much smaller growth; after 5 mgm progesterone (subcut.) or a combination of 8 γ oestrone and 5 mgm progesterone (subcut.) hardly any growth occurred. Aeth.-test. orally given produced, histologically followed, the same pregravid alterations (less than corresponding with the sign +) as the combination of oestrone and progesterone did.

Moreover, sensibilization of the uterus to 1 mgm progesterone proved to be possible with a precursory gift of 40 mgm aeth.-test. *per os*, just like with oestrone in the CLAUBERG test. This also could be obtained with the combination of oestrone and progesterone given simultaneously.

2. *Testosterone-like effects.*

a. Influence upon seminal vesicles and prostatic gland of the castrated, whether or not hypophysectomized rat. Castrated, 4—5 weeks old rats received 0,5 mgm aeth.-test. daily for 8 days in one or two gifts a day. At autopsy on the 9th day seminal vesicles + coagulation gland and the ventral lobe of the prostata proved to have grown distinctly more than those of the controls.

Though quantitatively much less impressive than after treatment with 200 γ testosterone daily, in qualitative sense the development after aeth.-test. was just the same ("activity" in the histological picture).

Oral administration of twice a subcut. active dosage gave no result. The fourfold caused a slight increase in weight, but no histological changes. (Table II).

TABLE II.

Daily dosage aeth.-test.	Sem. ves. and coag. gl.	Ventr. lobe prost.
2 dd 0.25 mgm subcut.	43.8 mgm (11.3 mgm)	31.6 mgm (6.5 mgm)
1 dd 0.5 mgm „	23.6 mgm (9.5 mgm)	17.8 mgm (9.8 mgm)
2 dd 0.5 mgm *per os*	9.6 mgm (9.4 mgm)	9.8 mgm (8.8 mgm)
1 dd 1 mgm „	8.8 mgm (9.5 mgm)	8.2 mgm (9.8 mgm)
1 dd 2 mgm „	13.3 mgm (9.5 mgm)	13.8 mgm (9.8 mgm)

Av. wt. semin. ves. + coag. gl. and ventr. lobe of the prost., as obtained in 2 experiments. Corresp. control wts. in parenthesis.

These experiments showed — in agreement with the findings of EMMENS' c.s. (3), pertaining to the influence of aeth.-test. on the cock's comb — that aeth.-test. orally given, doesn't unfold but a weak "masculine" activity in the male rat. From the obtained increase of the weight of the seminal vesicles it follows, that this "masculine" activity bears a really testosterone-like character and not a progesterone-like one, as might be expected because of the findings of GREENE c.s. (4). The above increase can be obtained with progesterone in massive dosages only (reaching, when expressed in rabbit U., far beyond the quantities of aeth.-test., used by us. (GREENE c.s.). In this respect progesterone corresponds with "male" hormone (type androsterone) from the urine, aeth.-test. with "male" hormone (type testosterone) from the testicles.

That the "masculine" activity of aeth.-test. belongs to the testosterone-type, also appeared from an experiment, in which 0.25 mgm aeth.-test. twice daily subcutaneously was administered for 8 days to hypophysectomized rats, 9—10 weeks of age and castrated 4 weeks before. On the 9th day the weights of seminal vesicles + coagulation gland and of the prostate were resp. av. 25,6 mgm and 9,3 mgm; of the controls resp. 8 and 4,6 mgm. FREUD and LAQUEUR (5) showed that such an increase in weight of the seminal vesicles only occurs in hypophysectomized, castrate rats, when the testosterone type is used and not with the androsterone type (even not if combined with oestrone).

The stimulation of seminal vesicles and prostata occurred, as a by-phenomenon, in one of our experiments, in which we daily administered during 20 days 400 γ aeth.-test. *per os*, 200 γ aeth.-test. subcut., 200 γ progesterone or 200 γ testosterone. Testosterone gave rise to a strong growth of the above organs, aeth.-test. subcut. to a lesser degree, but still distinct enlargement; the remaining substances had hardly any or no effect.

b. Inhibition of the paradoxical effect in the castrated mouse: The technics of the experiment have been dealt with in our previous paper; it was shown that aeth.-test. can inhibit the paradoxical alterations, that are specially suspectible for the influence of progesterone. However, also the changes, for which testosterone has a certain preference (epithelium of the coagulation gland and of the efferent ducts of seminal vesicles and coagulation gland) were reduced!

The used quantity of aeth.-test. was 1.2—2 mgm *per os* daily. The effect of the quantity of oestrone, used by us, can be inhibited with 6 γ testosterone daily, as has been shown in other experiments in our Laboratory (DE JONGH c.s. (6)). Basing hereupon, the "masculine" activity of aeth.-test. is only ca 1/200—1/300 of that of testosterone. A further sign of this "masculine" property has been found in the "male" aspect of the coagulation gland of the animals, that had received 2 mgm aeth.-test. *per os* daily.

c. Inhibition of the oestrone-oestrus in mice. These experiments too have been described already in the previous communication: aeth.-test., orally given inhibits the oestrone-oestrus and the relation with progesterone concerning the active dosages is rather well the same as is found with the CLAUBERG test. VAN DER WOERD (7), however, showed in experiments, not wholly comparable with ours, that 150 γ testosterone counteracts the oestrus effect of 0,1 γ oestrone; DE FREMERY c.s. (8) suppressed a limit dosage of oestrone with 3 rabbit U. progesterone. When comparing these dosages, one conclusevely finds the relation index (pro mgm), also fixed by ROBSON (9): testosterone inhibition-progesterone inhibition = 15 : 1.

In view of the inhibition of the paradoxical oestrone effect in male mice the activity of aeth.-test. *per os* proved to be only ca 1/200—1/300 of that of testosterone. Hence it may be expected that the oestrone-oestrus can be suppressed with 200—300 × 150 γ = 30—40 mgm aeth.-test. In our experiments 20 mgm aeth.-test. sufficed. The rough calculation basis taken into account, the inhibition of the oestrone-oestrus with aeth.-test. orally given, *might* have been due to a testosterone-like effect.

d. Influence on the rat penis and clitoris: In the above experiments on the influence of aeth.-test. on the vaginal opening in the immature rat aeth.-test. subcut. proved to be capable of causing a *clitoris growth*, stronger (!) than that with an equal dose of testosterone. Twice this quantity of aeth.-test., orally given, caused a slight increase in weight, an equal quantity of progesterone had no effect. (Table III).

TABLE III.

Daily dosage	1st experiment	2nd experiment
800 γ aeth.-test. subcut.	15.5 mgm	9.6 mgm
1600—2000 γ aeth.-test. *per os*	7 mgm	—
800 γ testosterone	8.3 mgm	5.6 mgm
800 γ progesterone	3.2 mgm	—
control	3 mgm	4.5 mgm

Average weight of the clitoris.

In experiments of 20 days duration with 4—5 weeks old, castrated rats 200 γ testosterone daily caused a larger gain in weight than 200 γ aeth.-test. subcut. did; 400 γ aeth.-test. orally caused a slight, 200 γ progesterone no growth.

The growth, in these experiments induced with aeth.-test. subcut. appeared together with a cartilagenous alteration of the clitoris of the same character as is seen in consequence of a treatment with testosterone. In long lasting experiments aeth.-test. *per os* as well as progesterone sometimes caused slight changes in the same direction. In 3 experiments progesterone only did so, when combined with oestrone.

Aeth.-test. subcut. also induced in similar experiments, in the shorter as well as in the long lasting ones, penile growth in 4—5 weeks old, castrated rats, (Table IV), which proved to be preserved in the above mentioned hypophysectomized, castrated rats.

TABLE IV.

Daily Dosage mgm		Penile wt. (mgm)
8 days	0.5 aeth.-test. subc.	48.4
	1 aeth.-test. *per os*	33
	2 aeth.-test. *per os*	34.4
	control	30.5
20 days	0.2 aeth.-test. subc.	24.7
	0.4 aeth.-test. *per os*	27.3
	0.2 testost. subc.	90.3
	0.2 progest. subc.	25
	control	24.7

Av. weight of the rat penis.

The cartilagenous transformation as found in the clitoris, was also observed in the glans penis of those animals, that had received aeth.-test. subcutaneously. To a lesser degree this was also the case after 2 mgm aeth.-test. orally during 8 days and after 400 γ aeth.-test. orally during 20 days; the changes were highly developed after the administration of 200 γ testosterone.

In the os priapi we observed modifications as described by DE JONGH c.s. (10) for testosterone. The whole penile bone was thicker than in the controls, albeit that no distinct separation in 2 layers could be shown; there was an active cartilagenous mantle of the marrow cavity, in which markedly developed trabeculated tissue was present. These changes were obtained much more distinctly with aeth.-test. subcut., than with the double quantity, orally administered.

In the penile bones of hypophysectomized, castrated rats too, changes as described by DE JONGH c.s. for testosterone-proprionate were observed.

3. *Oestrone-like and/or testosterone-like activity.*

In this group a series of changes are reported, that did not fit in the foregoing groups.

a. Influence on the intact pregnancy in the rat. With oestrogens pregnancy can be disturbed in the rat during the first half. Concerning a disturbance with testosterone (propr.) valuable details are only given in literature for the second half of pregnancy (HAIN (11), GREENE (12), etc.).

We investigated the influence of 2 dd 0,625 mgm progesterone, 2 dd 0,625 mgm aeth.-test. subcut. and 2 dd 50 mgm aeth.-test., orally administered from the 4th until and inclusive the 11th day of pregnancy to groups of 4 rats. Laparatomy on the 12th and 24th day after the detection of sperm in the vagina.

Progesterone proved to induce a (slight) lengthening of pregnancy; aeth.-test. subcut. had disturbed pregnancy in 3 and aeth.-test. *per os* in 2 cases (In the remaining cases of aeth.-test *per os* the term of delivery had been postponed!) Testosterone disturbed pregnancy in all cases.

b. Influence on the preputial glands in the rat. VAN DER WOERD (13) reported that oestrone as well as testosterone make the preputial gland grow in male and female rats all alike; with progesterone this occurs to a much lesser degree. In our experiments on male and female rats (during 3, 8 or 20 days) these glands have been weighed almost regularly. Without exception we found a slight increase in weight after aeth.-test.

c. Influence on the pituitary changes after castration. Testosterone and oestrone are capable of preventing the appearance of "castration cells" in the hypophysis, not progesterone (SCHOELLER, DOHRN and HOHLWEG (14)). The female pituitary is more sensible than the male one is. In experiments, in which 4—5 weeks old, castrated rats had been treated with testosterone, progesterone, aeth.-test. subcut. or aeth.-test. *per os*, the hypophysis was controlled at autopsy. In the following table the presence of castration cells is marked with +, absence with —, in case of only slight development ±. (Table V).

TABLE V.

Daily dosage	Castr. cells in ♂ rats	Castr. cells in ♀ rats
200 γ aeth.-test. subc.	±, +, ±	±, +, ±
400 γ aeth.-test. *per os*	±, +, +	—, —, ±
200 γ testosterone	—, —, —	—, —, —
200 γ progesterone	+, +, +	+, +, +
control	+, +, +	+, +, +

In the female (more sensitive!) rats aeth.-test. *per os* prevented the appearance of castration cells. Our material does not allow an explanation of the failure of "aeth.-test. subcut.". In our pregnant rats, however, castrated on the 5th day 5 mgm aeth.-test. subcut. and 10 mgm aeth.-test. *per os*, both daily given, did *not* fail to suppress the development of castration cells!

d. Inhibition of lactation in mice. Oestrone and testosterone are capable of inhibiting lactation (DE JONGH c.s. (15), ROBSON (16)). We took the body weight of the young as an index, but could not show that way any inhibition with 400 γ aeth.-test. subcut., 800 γ aeth.-test. *per os* or 400 γ progesterone subcut., daily given from the 1st until and inclusive the 21st day after parturition. With testosterone, conversely, we could induce a demonstrable inhibition (400 γ daily).

Discussion.

In our previous communication a series of experiments were described, with which a number of progesterone-like properties of aethinyltestosterone were clearly shown, in good agreement with earlier publications on the subject. The present paper deals with a number of different properties of aeth.-test., characterized by us as "oestrone-like" and "testosterone-like", thus treading in the footsteps of EMMENS and PARKES.

The testosterone-like properties, in view of the structural formula of aeth.-test., need not raise our astonishment; the oestrone-like activity, however, was more or less startling news. The spreading of both effects over the various effector organs is irregular, as results from the communicated data. Thus it cannot be said in anticipation of the relative experiment, where and to which degree the oestrone-like resp. testosterone-like powers will be expressed. Neither it will be possible to draw from experiments with animals of one species conclusions for the other and in how far man is susceptible for these influences remains questionable.

In clinical literature, though scarce untofar, no facts are communicated, from which it could be deducted, that the aforementioned non-progesterone-like effects of aeth.-test. also are important for man and eventually might contain an argument against the application of aeth.-test. instead of progesterone in medical practice.

We are aware, that with the progress of clinical experience this state of affairs may change. In every case, however, the combined occurrence of three types of activity in one single sterole derivative seems us to be of sufficient scientific interest, as to justify the publication of the results of our experiments.

Summary.

Besides progesterone-like activity aethinyl-testosterone has some other properties.

A. Oestrone-like effects.

1. Quantities of aeth.-test., inadequate to inhibit oestrus, advance and corroborate the oestrone-oestrus in spayed mice.

2. It is sometimes possible to bring about oestrus in spayed mice with the aid of aeth.-test.; in those cases cornification of the vaginal wall can be shown histologically; because of the progesterone-like activity of aeth.-test., however, this effect is soon overshadowed by mucification. In, wether or not pregnant, spayed rats too, aeth.-test. produces oestrus.

3. Aeth.-test. causes vaginal opening in the immature rat, together with oestrus. This effect may be partially "testosterone-like".

4. In the uterus of immature rats, aeth.-test. brings about alterations, similar to those, induced with oestrone + progesterone in certain combinations.

Aeth.-test. promotes the growth of the immature rabbit uterus (with traces of pregravid changes!) and sensibilizes it to subsequently given progesterone.

B. Testosterone-like effects.

1. Aeth.-test. causes growth of the sex organs (seminal vesicles, coagulation gland, prostata, penis) in the castrated, whether or not hypophysectomized, immature rat, qualitatively in the same sense as testosterone does.

2. The relation progesterone : aeth.-test. subcut. : aeth.-test. orally = 1 : 4 : 8—10. as found in the CLAUBERG test for the rabbit does not hold for the testosterone-like effects in the rat. This effect is after oral administration relatively much less pronounced. The testosterone-like activity of aeth.-test. does *not* rest upon its progesterone-like properties, but is wholly independant.

3. In the immature rat aeth.-test. causes growth of the clitoris, exceeding that, following the administration of the same quantity of testosterone.

4. Aeth.-test. inhibits paradoxical effects of oestrone in the male castrated mouse, that are electively sensitive to testosterone.

The inhibition of the oestrone-oestrus in mice may also be due to testosterone-like activity of aeth.-test.

C. Oestrone- or testosterone-like effects.

1. Aeth.-test. disturbs the intact pregnancy in the rat.

2. Aeth.-test. causes growth of the preputial glands in male and female rats.

3. Aeth.-test. prevents the development of "castration cells" in the pituitary of the spayed rat.

4. 400 γ aeth.-test. daily subcutaneously or 800 γ aeth.-test. daily orally does not inhibit lactation in nursing mice.

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CONTENTS

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ERRATA

in: "Report on Fervorization of Plant Nutrient-Substrata" by A. RADERMACHER and Z. KLAS (see the Proceedings N⁰. 8 of this year, page 1050—1060).

p. 1050, 13th line from top: "V. Vouk's (1)", read: "V. VOUK's [2])".

p. 1050, at the foot of the page, add the following note: "[2]) VOUK, Kohle und Pflanzenwachstum, Denkschr. Akad. Wiss. Wien, 1931".

p. 1057, in Table VII after *Solanum lycopersicum*: "(30. IV.—28. VI. 1939)", read: "(30. IV.—28. VII. 1939)".

p. 1057, in Table VII after *Tagetes erecta*: "(25. VIII.—6. IX. 1939)", read: "(25. VIII.—6. XI. 1939)".

p. 1057, 9th line from bottom: "(Figures on Plate II)", read: "(Figures on Plate III)".

p. 1058, in Table VIII after *Vicia Faba*: "(28. VI.—27. VI. 1939)", read: "(28. IV.—27. VII. 1939)".

p. 1059, in Table X "*Sinapis alba*", read: "*Sinapis alba* III".

p. 1060, 16th line from top: "BEWERLEY", read: "BEWLEY".