KONINKLIJKE NEDERLANDSCHE AKADEMIE VAN WETENSCHAPPEN

PROCEEDINGS OF THE SECTION OF SCIENCES

VOLUME XLIX
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VOLUME XLIX

No. 6

President: J. VAN DER HOEVE
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Aerodynamics. — *Some problems of the motion of interstellar gas clouds.* I. By J. M. BURGERS. (Mededeling No. 48 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of May 25, 1946.)

1. *Introduction.* — The following considerations have arisen from discussions with prof. dr. J. H. OORT at Leiden, who suggested most of the problems and to whose interest and advice I am greatly indebted in preparing these notes [1]). They must be taken, nevertheless, as a very preliminary attempt towards the application of aerodynamics in analysing some of the intricate riddles that are presented by the results of astronomical observations on the interstellar material within our Galaxy. Apart from the difficulties inherent in aerodynamics itself or arising when it is applied to matter in a state of such extreme dilution, astronomical observation, notwithstanding the powerful methods of photography and spectroscopy and the beauty of all what has already been revealed, still gives us only very scanty information about the structure and the motion of the cosmic clouds. In attempting to make a few calculations many guesses must be introduced, as otherwise it is impossible to construct simple pictures which can serve as starting points for a provisional theory. The reader therefore must take for granted that the cases treated in the following pages are no more than examples, which it is hoped may elucidate a few features, and perhaps point to a way for further work.

In the calculations the interstellar gas is considered as an ideal gas, to which the ordinary equations of aerodynamics can be applied. Although various elements and even certain chemical combinations are present (Ca, Na, K, Ca^+, Ti^+, Fe^+, CH, CH^+, CN), the bulk (about 99,9 % of the number of atoms) is formed by hydrogen. We will assume hydrogen atoms to be the only constituent, and take the gas law in the form: $p/\varrho = R\,T/m$, with $p =$ pressure, $\varrho =$ density, $T =$ absolute temperature, $R =$ molecular gas constant $= 8{,}315.10^7$ (cm-gram-second units will be applied throughout), and $m =$ molecular weight $= 1$ for ordinary monatomic hydrogen. The ratio of the specific heats is taken as $k = {}^5/_3$.

Dust particles ("smoke") are present in the interstellar gas, and it is mainly due to their presence that clouds become visible, either as dark

[1]) OORT also kindly put at my disposal the manuscript of the George Darwin lecture "*Some phenomena connected with interstellar matter*", delivered by him before the Royal Astronomical Society in London on May 10, 1946, and the results of his discussions with prof. dr. H. A. KRAMERS at Leiden. Some data used in the present communication have been taken from this manuscript, which is to be published in the Monthly Notices Roy. Astron. Society 1946.

masses, screening off the light of the stars, or as luminous patches when stars of sufficient luminosity are near. (Light emitted by the gas itself is visible in some cases, probably in consequence of an important rise in temperature produced by collisional effects as will be considered below.) No account, however, has been taken of the dust particles in the calculations. They will give a small contribution to the density and perhaps might have some influence upon the viscosity of the gas.

The average density of the interstellar material is of the order of 3.10^{-24} gr/cm³. It seems probable that a considerable fraction of the material is concentrated in more or less separate clouds, with an average density of the order of 10^{-22} gr/cm³, an average radius of 20 parsecs $\cong 6 \,.\, 10^{19}$ cm (1 parsec $= 3{,}083 \,.\, 10^{18}$ cm), and consequently a mass of approximately $M = 9 \,.\, 10^{37}$ gr (46000 times the mass of the sun); and distances apart of the order of 65 parsecs $\cong 2 \,.\, 10^{20}$ cm (or more). In the intermediate regions the density then will be much below $3 \,.\, 10^{-24}$. These clouds must take part in the general rotation of the Galaxy so as approximately to balance the gravitational attraction towards the galactic centre. At the sun's distance from the centre (approx. 30.000 lightyears $= 3{,}2 \,.\, 10^{22}$ cm) this rotational velocity is 300 to 320 km/sec $= 3{,}0$ à $3{,}2 \,.\, 10^{7}$ cm/sec. Superposed upon this general motion the clouds have peculiar motions of the order of 15 to 20 km/sec ($1{,}5$ à $2{,}0 \,.\, 10^{6}$ cm/sec).

A density of 10^{-22} gr/cm³ corresponds to $n = 60$ hydrogen atoms per cm³ (mass of a H-atom : $1{,}66 \,.\, 10^{-24}$ gram). Taking as the collisional diameter in the sense of the classical kinetic theory of gases the diameter of the first BOHR orbit $\sigma = 1{,}1 \,.\, 10^{-8}$ cm, the old formula for the free path gives: $l = 1/(\pi n \sigma^2) = 4{,}38 \,.\, 10^{-9}/\varrho$, which for $\varrho = 10^{-22}$ leads to $l \cong 4{,}4 \,.\, 10^{13}$ cm (the distance earth-sun is nearly $1{,}5 \,.\, 10^{13}$ cm).

It will be evident that the possibility of obtaining a Maxwellian velocity distribution and the normal relation between pressure, density and temperature can be found only in elements of volume with dimensions large compared with l. In the case of the cloud around the star Merope of the Pleiades the diameter of the illuminated portion is about 500″, which at a distance of 100 parsecs amounts to about $7{,}5 \,.\, 10^{17}$ cm. This cloud shows stratifications or waves with a thickness of approximately $8'' = 1{,}2 \,.\, 10^{16}$ cm. In the Cygnus nebulae NGC 6960 and 6992 there are extremely thin strips or sheets, with an apparent thickness from 1″ to 5″. Taking the distance to be 350 parsec $= 1{,}08 \,.\, 10^{21}$ cm, these thicknesses amount from 0,5 to $2{,}5 \,.\, 10^{16}$ cm. The density is supposed to be somewhat smaller than 10^{-22} and to correspond to 30 atoms per cm³, giving $\varrho = 5 \,.\, 10^{-23}$ and $l = 8{,}7 \,.\, 10^{13}$ cm. In both cases the applicability of the ideal gas laws to the thinnest observable sheets seems possible.

The temperature of the interstellar gas is assumed to be of the order of 10.000°. The mean molecular velocity then is given by: $c_m = \sqrt{3RT/m} = 1{,}58 \,.\, 10^{6}$ cm/sec. The velocity of sound and velocities of expansion are of the same order of magnitude. In certain cases (clouds produced by the explosion of a nova) velocities of the order of 1000 km/sec $= 10^8$ cm/sec have been observed. In all these cases there is no need for applying relativity corrections.

As the general gravitational field in the Galaxy regulates the motion of a cloud as a whole, it will not have much effect upon expansion or compression phenomena. The gravitational field of a cloud itself perhaps may be of some importance. Taking by way of example a spherical cloud of mass $M = 9 \,.\, 10^{37}$ gr, having a mean density $\bar{\varrho} = 10^{-22}$, application of EMDEN's results for the case of static "polytropic" equilibrium [2]) gives for the density at the centre $\varrho_0 = 6{,}0 \,.\, 10^{-22}$; for the temperature at the centre $T_0 = 650°$; and for the pressure at the centre $p_0 = 3{,}24 \,.\, 10^{-11}$ (the radius comes out as $6 \,.\, 10^{19}$ cm as should be). Hence the pressure due to the gravitational attraction inside the spherical mass in the neighbourhood of the centre becomes comparable with the pressure $8{,}3 \,.\, 10^{-11}$ calculated for a temperature of 10.000° and a density of 10^{-22}. — At the surface of this spherical cloud the value of g is equal to $1{,}67 \,.\, 10^{-9}$. In the case of the expansion problem considered in the next section we will make an estimate of its influence.

Viscosity, heat conduction and radiation in the first instance will be left out of account. A few remarks concerning their possible effects will be made in connection with some results.

Finally the calculations will refer to motions in one dimension only, so as to obviate the mathematical difficulties connected with problems of spherical expansion.

2. *Equations of motion. — Application to a case of simple expansion in a vacuum.* — The equations for the one-dimensional motion of an ideal gas, in the absence of gravity, have the form:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = -\frac{1}{\varrho}\,\frac{\partial p}{\partial x} \quad \ldots \quad (1)$$

$$\frac{D\varrho}{Dt} = \frac{\partial \varrho}{\partial t} + u\,\frac{\partial \varrho}{\partial x} = -\varrho\,\frac{\partial u}{\partial x} \quad \ldots \quad (2)$$

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} = -(k-1)\,T\,\frac{\partial u}{\partial x} \quad \ldots \quad (3)$$

$$p = RT\,\varrho/m \quad \ldots \quad (4)$$

Viscosity, heat conduction and radiation have been neglected; u is the velocity of the gas, while D/Dt in the usual way indicates the total derivative with respect to t for an observer moving with a particular element of volume. For such an observer combination of (2) and (3) gives: $T/\varrho^{k-1} =$ constant,

[2]) R. EMDEN, Gaskugeln (Leipzig 1907), Tab. 4, p. 79, for the case $k = 5/3$, and the formulae of p. 69, pp. 96/97.

expressing POISSON's isentropic relation for an individual element of volume. This relation holds so long as the differential equations can be applied; it breaks down, however, in so-called shock waves where $\partial u/\partial x$ and $\partial T/\partial x$ assume such large values that viscosity and heat conduction become important and lead to an increase of entropy.

As the density in a stellar cloud is much larger than that in the space around it, we must expect that it will expand, with an accompanying compression of the surrounding matter. The simplest case is that of an originally homogeneous mass, expanding into a vacuum, assuming that for $t<0$ we had: $p=p_0$, $\varrho=\varrho_0$ for $x<0$, and $p=0$, $\varrho=0$ for $x>0$. Introducing the velocity of sound

$$c=\sqrt{k\,p/\varrho}=\sqrt{k\,RT/m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(counted positive in the same direction as u) and making use of the relation $p/\varrho^k =$ constant, the solution can be put into the form [3]:

$$\left.\begin{aligned} u &= \frac{2}{k+1}\,\frac{x}{t}+\frac{2}{k+1}\,c_0 \\ c &= \frac{k-1}{k+1}\,\frac{x}{t}-\frac{2}{k+1}\,c_0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This solution is limited to a region defined by two values of x/t. The first one characterizes the front of the expansion wave; here:

$$(x/t)'=2\,c_0/(k-1)=u;\ c=0 \text{ (expansion to zero density);}$$

the second one determines the back of the wave, where

$$(x/t)''=-c_0;\ u=0.$$

In the case $k={}^5/_3$ the front moves outward with 3 times the velocity of sound in the original state of the gas, while the back of the wave moves inward with just this velocity. For $T=10.000^\circ$, $m=1$, $k={}^5/_3$ we obtain: $c_0=1{,}18\,.\,10^6$ cm/sec; $u_{\text{front}}=3{,}53\,.\,10^6$ cm/sec (35,3 km/sec). In a year $=3{,}16\,.\,10^7$ sec the front moves outward over $1{,}12\,.\,10^{14}$ cm, which at a distance of 100 parsecs would correspond to 0,075″ and thus perhaps might be just appreciable. Nevertheless it will be evident that a cloud with a diameter of $7{,}5\,.\,10^{17}$ cm for many centuries will retain its appearance.

Estimation of the influence of gravity. — In order to take gravity into account, a term $-g$ must be added to the right hand side of eq. (1). In connection with the restriction to one-dimensional motion we take g to be a

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[3]) Compare: J. M. BURGERS, Over de eendimensionale voortplanting van drukstoringen in een ideaal gas, Versl. Nederl. Akademie v. Wetenschappen, afd. Natuurkunde, **52**, 478 (1943).

constant. Introducing u, c as dependent variables, eqs. (1) and (2) can be transformed into the system:

$$\left\{\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right\}\left(u+\frac{2}{k-1}c\right)=-g \qquad \left\{\frac{\partial}{\partial t}+(u-c)\frac{\partial}{dx}\right\}\left(u-\frac{2}{k-1}c\right)=-g \quad . \; . \; . \; . \; (7)$$

This system can be solved with the aid of its characteristics, which are determined by $(dx/dt)_I = u + c$; $(dx/dt)_{II} = u - c$. Passing over the details and taking for simplicity the same initial state as considered above (although this cannot be an equilibrium state), the solution is found to be:

$$u=\frac{2}{k+1}\frac{x}{t}+\frac{2}{k+1}c_0-\frac{k}{k+1}gt \qquad c=\frac{k-1}{k+1}\frac{x}{t}-\frac{2}{k+1}c_0+\frac{k-1}{2(k+1)}gt \quad . \; . \; . \; . \; . \; (8)$$

The front of the wave, determined again by $c = 0$, moves according to the equation:

$$x' = 2\,c_0\,t\,/\,(k-1) - \tfrac{1}{2}\,gt^2 = 3\,c_0\,t - \tfrac{1}{2}\,gt^2 . \quad . \; . \; . \; . \; . \; (9)$$

while the back, where $c = -c_0$, is to be found at:

$$x'' = -c_0\,t - \tfrac{1}{2}\,gt^2 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (10)$$

At the back of the wave $u = -gt$; likewise for all values of x to the left of those determined by (10) $u = -gt$, which means that here the whole mass of the gas is moving with the acceleration $-g$ just as a solid body would do.

This latter result is of no particular importance, as it is a consequence of the condition assumed for $t = 0$, no account having been taken of the possibility of a compression (which would require the introduction of a boundary condition for some negative value of x). It will be evident, however, that the influence of gravity will be of importance only when gt becomes equal to, say c_0. With $g = 1{,}67 \cdot 10^{-9}$ as found in the example at the end of section **1**, this will be the case after a lapse of time of $7 \cdot 10^{14}$ sec $= 2{,}2 \cdot 10^7$ year. Hence no great error is made by neglecting the influence of gravity in expansion and compression phenomena of clouds during comparatively short periods.

3. *Expansion with compression of the surrounding gas of low density.* — When for $t < 0$ we have a gas (I) with a density 10^{-22} to the left of $x = 0$, and to the right a gas (II) with a density of, say, $3 \cdot 10^{-24}$, both gases originally being at the same temperature T_0, the expansion of the denser gas (I) will produce a compression wave in gas (II), which compression wave, owing to the high velocities involved, will take the form of a shock wave. The expansion of gas (I) will extend to a pressure equal to that produced by the passage of the shock wave through gas (II). As the front velocity $(x/t)'$ of the expansion region is smaller than the gas velocity u when the expansion does not proceed to zero density, a region of constant velocity, density and pressure appears between the front of the expansion region and the boundary separating gas I from gas II.

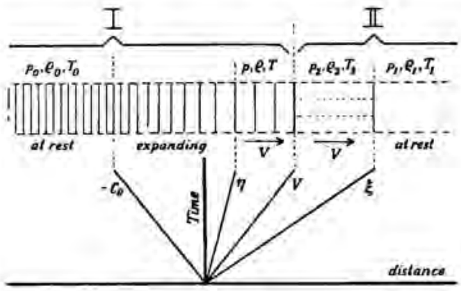

Fig. 1.

The propagation of the waves has been represented schematically in fig. 1. Starting from the right hand side, the shock wave in gas II (original state: p_1, ϱ_1, T_1; zero velocity) moves with the velocity ξ. Behind this wave gas II moves with the constant velocity V, its state being defined by p_2, ϱ_2, T_2. We then have [4]):

$$\xi = \frac{k+1}{4} V + \sqrt{c_1^2 + \frac{(k+1)^2 V^2}{16}}\,; \quad p_2 = p_1 + \varrho_1 V \xi\,. \quad . \quad (11)$$

The boundary between the two gases moves with the velocity V and so does gas I in the region between this boundary and the front of the expansion wave, the latter moving with the velocity $(x/t)' = \eta$. At this front eqs. (6) give:

$$u = \frac{2}{k+1}(\eta + c_0)\,; \; c = \frac{k-1}{k+1}\eta - \frac{2}{k+1} c_0. \quad . \quad . \quad . \quad (12)$$

At the same time:

$$p = \left(\frac{\varrho_0^k}{p_0}\right)^{1/(k-1)} \left(\frac{c^2}{k}\right)^{k/(k-1)} ; \; \varrho = \left(\frac{\varrho_0^k}{p_0}\right)^{1/(k-1)} \left(\frac{c^2}{k}\right)^{1/(k-1)} \quad . \quad . \quad (13)$$

At this front we must have $u = V$; moreover with regard to the boundary between gas I and gas II the condition $p = p_2$ must be fulfilled.

The equations can be solved numerically. With the data:

(gas I) $\varrho_0 = 100.10^{-24}$; $T_0 = 10^4$; $p_0 = 83{,}2.10^{-12}$; $c_0 = 1{,}18.10^6$;
(gas II) $\varrho_1 = 3.10^{-24}$; $T_1 = 10^4$; $p_1 = 2{,}5.10^{-12}$; $c_1 = 1{,}18.10^6$,

[4]) See the paper mentioned in footnote [3]), p. 477, eqs. (9) and (A).

we obtain:

$$\eta = 0{,}42.10^6;\ u = V = 1{,}20.10^6;\ \xi = 2{,}22.10^6,$$

and for the state of gas I in the region of constant velocity $u = V$:

$$p = 10{,}5.10^{-12};\ \varrho = 29.10^{-24};\ T = 4370°;\ c = -0{,}78.10^6,$$

while for gas II in the region behind the shock wave:

$$p_2 = 10{,}5.10^{-12};\ \varrho_2 = 6{,}5.10^{-24};\ T_2 = 19360°.$$

Owing to the compression the rarer gas takes a much higher temperature than the expanding denser gas. The region of this high temperature increases in width at the rate $\xi - V = 1{,}02 \,.\, 10^6$ cm/sec (10,2 km/sec).

With regard to the shock wave front it must be remarked that when account is taken of viscosity and heat conduction, it is found that the sudden change of state in reality is a continuous one, although with very steep gradients. The thickness of the transition layer is of the order $\mu/\varrho V$, where μ is the viscosity [5]). Assuming the viscosity to be given by $\mu = \frac{1}{3}\ \varrho\, l\, c_m$, with $l = 4{,}3 \,.\, 10^{-9}/\varrho$, we find, for $T = 10000°$, $c_m = 1{,}58 \,.\, 10^6$ cm/sec: $\mu \cong 0{,}0023$ [6]). The thickness of the layer becomes of the order of magnitude of l. This generally will be inappreciable from the observational point of view. In the transition region the Maxwellian velocity distribution will not hold; a kinetic treatment of the phenomena in this region would be necessary, which, however, will be extremely difficult.

4. *Collision of two clouds.* — The examples given in the preceding sections will appear artificial, and the reader may ask how the state assumed for $t = 0$ can have arisen. This, however, is a reflection of our general lack of knowledge about the fields of motion in the interstellar gas, which makes it necessary to start from imagined cases. The examples were meant to illustrate that when regions of highly different densities occur side by side, expansion and compression phenomena certainly will be present; their order of magnitude will be apparent from the results obtained.

Clouds of a much higher density than that of the average surrounding gas can be produced in star explosions. Such clouds in the first instants (the first few years or perhaps decades of their existence) will move with enormous velocities, compared to which the velocity of expansion often

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5) Compare *e.g.*: R. BECKER, Stosswelle und Detonation, Zeitschr. f. Physik **8**, p. 339, 1922; G. I. TAYLOR and J. W. MACCOLL, The mechanics of compressible fluids, section 6 (W. F. DURAND, Aerodynamic Theory, Berlin 1935, vol. III, Div. H, p. 218).

6) The value of μ also can be calculated with the aid of SUTHERLAND's formula: $\mu = \mu_0 \sqrt{T/273} \,.\, (1 + C/273)/(1 + C/T)$. Here μ_0 (at $T = 273°$) is proportional to $\sqrt{m}/\sigma^2$. For H_2 ($m = 2$; $\sigma = 2{,}730 \,.\, 10^{-8}$ cm) $\mu_0 = 0{,}000085$, $C = 72$; for He ($m = 4$; $\sigma = 2{,}174 \,.\, 10^{-8}$ cm) $\mu_0 = 0{,}000189$, $C = 80$. Estimating for H_1 ($m = 1$; $\sigma = 1{,}1 \,.\, 10^{-8}$ cm) $\mu_0 = 0{,}00037$, $C = 76$, the formula gives $\mu = 0{,}0029$ at $T = 10000°$, which is slightly larger than the value found above.

may appear negligible. In certain cases such clouds perhaps may impinge upon each other; or they may impinge upon a portion of the interstellar gas with more than minimum density; also the gradual loss of velocity experienced in moving through the interstellar gas deserves attention. Certain problems illustrating these possibilities will be treated now.

The simplest case is that of a "head on" collision of two clouds, having both the same density and temperature, with parallel boundary planes. In each cloud a shock wave will appear. When we introduce a system of coordinates with respect to which the two clouds have equal and opposite velocities of absolute magnitude V, the situation is a symmetrical one (comp. fig. 2). The two shock waves move outwards with the absolute velocity ξ; between them the gases are at rest. Indicating the original state of each gas by p_0, ϱ_0, T_0; and the compressed state between the shock waves by p', ϱ', T', we have [7]:

$$\left.\begin{array}{l} V + \xi = \dfrac{k+1}{4} V + \sqrt{c_0^2 + \dfrac{(k+1)^2 V^2}{16}} \\ p' = p_0 + \varrho_0 V (V + \xi);\ \varrho' = \varrho_0 (V + \xi) / \xi \end{array}\right\} \quad . \quad . \quad . \quad (14)$$

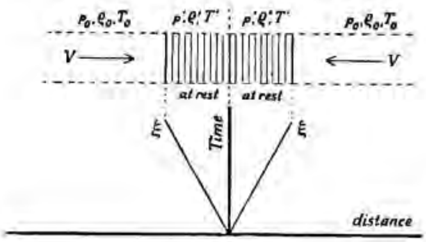

Fig. 2.

When V is large compared with c_0, we may use the approximations:

$$\xi \simeq \tfrac{1}{2}(k-1)\, V;\ p' \simeq \tfrac{1}{2}(k+1)\, \varrho_0\, V^2;\ \varrho' \simeq (k+1)\, \varrho_0 / (k-1) \quad (14a)$$

from which:

$$RT' \simeq \tfrac{1}{2}(k-1)\, V^2 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

By way of example we take $\varrho_0 = 10^{-22}$; $T_0 = 10000°$; $V = 10^7$ cm/sec (100 km/sec). With $k = 5/3$ as before we find: $\xi \cong V/3$; $p' \cong 1{,}33 \,.\, 10^{-8}$; $\varrho' \cong 4\, \varrho_0$; $T' \cong 400.000°$.

In the more general case indicated in fig. 3, where gas I, moving with the velocity V_1, overtakes gas II (velocity V_2), the solution still can be obtained by means of straightforward algebra. In the case of large

[7]) These formulae follow from eqs. (9), (A), (B) of the paper mentioned in footnote [3]), if it is observed that relatively to the uncompressed gas the shock wave moves with the velocity $V + \xi$ and the compressed gas with the velocity V.

velocities the following approximations are obtained, where $\varepsilon = \sqrt{\varrho_2/\varrho_1}$ (supposed to be < 1):

$$\left.\begin{array}{l} p' = p'' \simeq \frac{1}{2}(k+1)\,\varrho_2\,(V_1 - V_2)^2/(1+\varepsilon)^2 \\ \qquad \varrho' \simeq (k+1)\,\varrho_1/(k-1) \quad ; \quad \varrho'' \simeq (k+1)\,\varrho_2/(k-1) \\ RT' \simeq \frac{1}{2}(k-1)\varepsilon^2(V_1-V_2)^2/(1+\varepsilon)^2;\ RT'' \simeq \frac{1}{2}(k-1)(V_1-V_2)^2/(1+\varepsilon)^2 \end{array}\right\} \quad (16)$$

so that $T''/T' \cong \varrho_1/\varrho_2$. — Particular cases, *e.g.* $V_2 = 0$, can be deduced from these formulae. It will be seen that the temperature of the rarer gas increases to a much higher degree than that of the denser gas. The thickness of the zone of high temperature in gas II increases at the rate: $\xi_2 - V^* \cong \frac{1}{2}(k-1)\,(V_1 - V_2)/(1+\varepsilon)$.

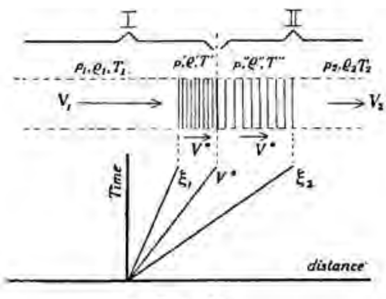

Fig. 3.

[It must be observed that with the very high temperatures and consequent high molecular velocities obtained in collisions between clouds with velocities of 50 km/sec and more, dissociation (ionization) of the hydrogen atoms must be taken into account. In all collisions with sufficiently high relative velocity of the colliding atoms, there will be a loss of energy in consequence of excitation or of the production of ions, which energy afterwards is radiated out in the form of light quanta. There is thus a continuous loss of energy in the compressed gas, at a rate which depends upon the number of favourable collisions. In those cases where this loss of energy is considerable, the temperature of the gas behind the shock wave will decrease, and a temperature gradient will be set up, which will influence the density and the pressure.] [8])

[8]) In the case of the Nebula south of ζ Orionis (IC 434) the impression is gained of a dark cloud of greater density penetrating into a luminous cloud of smaller density, while bright fringes appear just along the boundary. These fringes are rather sharply limited on the side of the dark cloud (in particular around the "Dark Bay"), while they shade off gradually towards the side of the smaller density. Similar features are observed in other cases. With the "Crab Nebula" in Taurus a thin luminous layer seems to precede at some distance the expanding central luminous mass. The spectrum of this thin layer shows many "forbidden" lines, which makes it probable that excitation here is due to collisions in a gas of not too small density.

When an explanation of these features is asked for, the picture developed in the text is too simple and presents the difficulty that it leads to clear cut frontiers limiting the

5. *Cloud with high velocity impinging upon gas at rest.* — We will consider what will happen when a cloud moving with a very high velocity, after having passed through a region of extremely low and negligible density, suddenly impinges upon a gas at rest of sufficient density to make itself felt. Such a case according to OORT seems to be present with a part of the cloud produced in the explosion of Nova Persei in 1901. It is estimated that this cloud moves with a velocity $V_1 = 1200$ km/sec $=$ $= 1{,}2 \,.\, 10^8$ cm/sec.

During the first instants after the collision of the cloud with interstellar gas at rest of appreciable density, the solution appropriate to the case is the same as the one considered in the second part of section 4, with $V_2 = 0$. In the interstellar gas a shock wave appears, moving with the velocity: $\xi_2 = \frac{1}{2}(k+1)\, V_1/(1+\varepsilon) = \frac{4}{3}\, V_1/(1+\varepsilon)$, while the boundary separating the cloud from the interstellar gas moves with the velocity: $V^* = V_1/(1+\varepsilon)$. Hence the region of compressed interstellar gas broadens at the rate: $\xi_2 - V^* = \frac{1}{2}(k-1)\, V_1/(1+\varepsilon) = \frac{1}{3}\, V_1/(1+\varepsilon)$. The temperature in this region, calculated without taking account of radiation, would rise to the value: $T'' = \frac{1}{2}(k-1)\,(V_1^2/R)/(1+\varepsilon)^2 = 5{,}77 \,.\, 10^7/(1+\varepsilon)^2$. — The shock wave in the cloud has the forward velocity

$$\xi_1 = V_1 - \tfrac{1}{2}(k+1)\, \varepsilon\, V_1/(1+\varepsilon),$$

and thus relatively to the material of the cloud moves backward with the velocity:

$$V_1 - \xi_1 = \tfrac{1}{2}(k+1)\, \varepsilon\, V_1/(1+\varepsilon) = \tfrac{4}{3}\, \varepsilon\, V_1/(1+\varepsilon) \quad . \quad . \quad . \quad (22)$$

When the cloud has a finite thickness L and is assumed to have constant

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regions of compression and of increased temperature. However, it has been mentioned already that heat conduction and viscosity turn shock waves into continuous transition regions; at the boundary where the denser and the rarer gas meet with equal pressures and mass velocities (but with different temperatures), also diffusion will play a part. Now in those cases where on one side we have a high density combined with low temperature and on the other side a low density with higher temperature, it is possible that somewhere in the transition region optimal conditions for the emission of radiation will be found. This perhaps could give an explanation of the appearance of narrow bright fringes. In those cases where a shock wave would bring a theoretical increase of temperature of 100.000° or more, optimal conditions for radiation might be found possibly not in the region of high temperature, but in a sheet of the transition layer with some intermediate temperature.

It must be observed nevertheless that as soon as kinetic energy in collisions of atoms is lost in consequence of excitation and ionization (leading to emission of radiant energy) the phenomena become much more complicated. It is possible that owing to this loss of energy no constant condition is obtained behind the shock wave; in certain cases perhaps even the shock wave itself may be absent. Before a satisfactory description can be given of the relations to be expected in such cases and the problem of sharp and diffuse boundaries can be attacked, a more penetrating investigation therefore will be necessary, based upon equations in which due account is taken of radiation and diffusion phenomena, etc.

values of p, ϱ, T throughout, this shock wave will reach the back of the cloud after the interval $\tau = L/(V_1 - \xi_1)$. When tentatively we estimate the thickness of the Nova Persei cloud at $L = 0{,}5'' = 4{,}1 \,.\, 10^{15}$ cm (assuming a distance of 550 parsecs $= 1{,}7 \,.\, 10^{21}$ cm), then with $V_1 = 1{,}2 \,.\, 10^8$, $\varrho_2/\varrho_1 = 0{,}01$, $\varepsilon = 0{,}1$ we find $V_1 - \xi_1 = 1{,}45 \,.\, 10^7$ and $\tau = 2{,}83 \,.\, 10^8$ sec $= 9$ years.

At the end of this interval the whole cloud has undergone a compression; it moves as a whole with the velocity $V^* = 1{,}09 \,.\, 10^8$ and has obtained the temperature T'. Its thickness has been reduced to: $L - \tau(V_1 - V^*) = (k-1)\,L/(k+1) = 1{,}03 \,.\, 10^{15}$, that is in the inverse ratio of the increase in density (ϱ'/ϱ_1).

At the back of the cloud the shock wave is reflected as an expansion wave. The front of this wave has the velocity $c'' = \sqrt{k\,R\,T''} = 8{,}1 \,.\, 10^7$ relatively to the gas, and thus after a further interval of time of approximately 0,4 year reaches the front boundary of the cloud. As soon as the expansion wave has reached the front of the cloud, the velocity and the pressure will decrease here, so that also the compression produced in the interstellar gas will become less. From now onward the whole phenomenon assumes a rather complicated form [9]). To work it out in details does not seem promising in view of the uncertainties of the available data. In particular the assumption of a homogeneous state in the original cloud is an oversimplification of the case.

In view of this situation OORT put forward the question whether a stationary solution could be found, describing a case where owing to the general slowing down of the cloud a "barometric" gradient of the pressure and the density is set up. OORT supposes that such cases may be found in the very thin cloud strips, observed in NGC 6960 in Cygnus. An approximate solution of this type will be constructed in the next section.

(To be continued.)

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[9]) Certain phenomena of this nature have been investigated in the paper mentioned in footnote [3]).

Aerodynamics. — *Some problems of the motion of interstellar gas clouds.* II *). By J. M. BURGERS. (Mededeling No. 48 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft.)

(Communicated at the meeting of May 25, 1946.)

6. *Quasi-stationary solution with "barometric" pressure gradient.* — Our starting point again will be eqs. (1) — (4). In order to be able to take a formal account of radiation loss, of ψ erg per cm^3 and per sec, we multiply eq. (3) by $R/(k-1)$ and to the right hand side add the term $-\psi/\varrho$, so that it takes the form:

$$\frac{R}{k-1}\frac{DT}{Dt}=\frac{R}{k-1}\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}\right)=-RT\frac{\partial u}{\partial x}-\frac{\psi}{\varrho}\ldots \quad (3a)$$

The most convenient way for constructing an appropriate solution is to start from the *streamlines*, for which tentatively the following equation is assumed:

$$x=\varphi_0(t)-s\,\varphi_1(t)\ldots \quad (17)$$

A streamline is defined by $s=$ constant. We take $s=0$ as the front of the moving gas layer and restrict s to positive values; further at $t=0$ (present epoch) we take: $\varphi_1(0)=1$. Hence the motion of the front is defined by: $x=\varphi_0(t)$, and s measures the distance behind the front at $t=0$. With constant t we have for any function of x and t:

$$\partial/\partial x=-(1/\varphi_1)\,.\,\partial/\partial s.\ldots \quad (18)$$

From (17) we obtain for the velocity:

$$u=(dx/dt)_{\text{for constant } s}=\dot\varphi_0-s\,\dot\varphi_1 \ldots \quad (19)$$

(using dots to denote derivatives with respect to t, while an accent will be used for derivatives of functions of s). This gives: $\partial u/\partial x=\dot\varphi_1/\varphi_1$.

For the density the following function is taken:

$$\varrho=\varrho^*(s)/\varphi_1(t) \ldots \quad (20)$$

which satisfies the equation of continuity. Making use of (4), eq. (1) then gives:

$$\ddot\varphi_0-s\,\ddot\varphi_1=\frac{\varrho^{*\prime}}{\varrho^*}\frac{RT}{\varphi_1}+\frac{R}{\varphi_1}\frac{\partial T}{\partial s} \ldots \quad (21)$$

Tentatively for ϱ^* we assume the formula:

$$\varrho^*=\varrho_m\,e^{-\beta s}\ldots \quad (22)$$

*) Part I has appeared in these Proceedings **49** (1946), p. 589.

which gives the desired exponential decrease of the density with s. The coefficient ϱ_m determines the maximum value (ϱ_m/φ_1) of the density at the front of the advancing layer. Equation (21) now can be considered as a differential equation for T as a function of s; its integral is [10]):

$$RT = -\ddot{\varphi}_0\,\varphi_1/\beta + (1+\beta s)\,\ddot{\varphi}_1\,\varphi_1/\beta^2 \quad . \quad . \quad . \quad . \quad . \qquad (23)$$

The pressure at the front of the advancing layer: $p_{fr} = R\,T_{fr}\,\varrho_{fr} = -\varrho_m(\ddot{\varphi}_0/\beta - \ddot{\varphi}_1/\beta^2)$, must be equal to the pressure generated in the interstellar gas into which our layer is penetrating. We will not enter into a detailed investigation of the shock waves and other phenomena set up in this gas, but assume that the "impact pressure" is given by $\frac{1}{2}(k+1)\,\varrho_0\,u_{fr}^2$, where ϱ_0 is the original density of the interstellar gas, while u_{fr}, the front velocity of the advancing layer, is equal to $\dot{\varphi}_0$. Hence we obtain:

$$\ddot{\varphi}_0/\beta - \ddot{\varphi}_1/\beta^2 = -\tfrac{1}{2}\,(k+1)\,(\varrho_0/\varrho_m)\,\dot{\varphi}_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

Introduction of (23) into (3a) finally gives, after multiplication by $-(k-1)$:

$$\frac{\ddot{\varphi}_0\,\varphi_1}{\beta}\left(\frac{\dddot{\varphi}_0}{\ddot{\varphi}_0} + k\,\frac{\dot{\varphi}_1}{\varphi_1}\right) - \frac{(1+\beta s)\,\ddot{\varphi}_1\,\varphi_1}{\beta^2}\left(\frac{\dddot{\varphi}_1}{\ddot{\varphi}_1} + k\,\frac{\dot{\varphi}_1}{\varphi_1}\right) = (k-1)\,\frac{\psi}{\varrho}\,. \qquad (25)$$

In general ψ/ϱ will be a complicated function of s, and it will not be possible to satisfy (25) in an exact way, which shows that the assumption (22) apparently was too restricted. We therefore start by investigating the case $\psi = 0$, and provisionally assume $\ddot{\varphi}_1/\beta \ll |\ddot{\varphi}_0|$. Then to a first approximation (24) gives:

$$\ddot{\varphi}_0/\beta \simeq -\tfrac{1}{2}\,(k+1)\,(\varrho_0/\varrho_m)\,\dot{\varphi}_0^2 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24a)$$

from which:

$$u_{fr} = \dot{\varphi}_0 = \frac{2\,\varrho_m}{(k+1)\,\varrho_0}\,\frac{1}{\beta\,(t+t_0)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

t_0 being an integration constant. At the same time eq. (25) reduces to:

$$k\,\dot{\varphi}_1/\varphi_1 \simeq -\dddot{\varphi}_0/\ddot{\varphi}_0 = 2/(t+t_0). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25a)$$

from which, taking account of the condition $\varphi_1(0) = 1$:

$$\varphi_1 = \{(t+t_0)/t_0\}^{2/k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

With the same degree of approximation (23) becomes:

$$RT \simeq -\frac{\ddot{\varphi}_0\,\varphi_1}{\beta} = \frac{2\,\varrho_m}{(k+1)\,\varrho_0}\,\frac{1}{\beta^2\,t_0^2}\left(\frac{t_0}{t+t_0}\right)^{2-2/k} \quad . \quad . \quad . \quad . \qquad (28)$$

[10]) A term proportional to $e^{\beta s}$ might be added to this solution, which term could be written $\varphi_1\varphi_2\,e^{\beta s}$, where φ_2 is an arbitrary function of t. This term, however, has been discarded. The discrepancy observed in eq. (25) cannot be removed by introducing such a term. — As will be seen below we suppose that the second term of (23) is of minor importance.

As the function φ_1 determines a gradual expansion of the moving gas layer, (28) expresses the adiabatic decrease of T consequent upon this expansion, which is obtained as radiation is neglected.

7. *Continuation. — Numerical data for a thin strip or sheet of cloud in NGC 6960. — Influence of radiation loss.* — From data supplied by OORT (in view of their vagueness slightly adjusted so as to obtain a satisfactory fit in calculating the temperature) we take: $u_{fr} = 7{,}0 \,.\, 10^6$ (70 km/sec) at the present epoch ($t = 0$); $\varrho_m = 5{,}2 \,.\, 10^{-23}$ (31,5 H-atoms per cm^3); $\varrho_0 = 10^{-24}$. Then (26) gives: $u_{fr} = 39/\beta(t + t_0)$, so that for $t = 0$: $\beta t_0 = 5{,}57 \,.\, 10^{-6}$. The timescale evidently is proportional to the linear scale, which according to (22) is fixed by $1/\beta$. Taking $1/\beta$ equal to $2{,}5 \,.\, 10^{16}$ cm (the estimated visual thickness of a sheet is about 5″), we find $\beta = 4{,}0 \,.\, 10^{-17}$, giving: $t_0 = 1{,}39 \,.\, 10^{11}$ sec $= 4410$ year.

From (26) and (27) we now calculate for $t = 0$: $\ddot{\varphi}_0 = -5{,}03 \,.\, 10^{-5}$; $\ddot{\varphi}_1 = +1{,}24 \,.\, 10^{-23}$; $\ddot{\varphi}_1/\beta = 3{,}1 \,.\, 10^{-7}$, so that the assumption $\ddot{\varphi}_1/\beta \ll |\ddot{\varphi}_0|$ appears to be satisfied. The expansion determined by φ_1 is very gradual; with $t = 4410$ year we have $\varphi_1 = 2{,}3$.

Further from (28) we deduce, again for $t = 0 : T = 15100°$, which is in accordance with the astronomical estimate.

We now turn back to the radiation loss. From data supplied by OORT it follows that in the present problem the radiation loss cannot be neglected. We might try to obtain a better approximation than that given by (25*a*) by taking for ψ/ϱ a function of the time alone, calculated with an average value of the density, replacing (25) by:

$$k\,\frac{\dot{\varphi}_1}{\varphi_1} = -\frac{\dddot{\varphi}_0}{\ddot{\varphi}_0} + \frac{(k-1)\,\beta}{\ddot{\varphi}_0\,\varphi_1}\,\frac{\psi}{\varrho}. \qquad (25b)$$

Without integrating this equation it will be evident that the radiation loss tends to decrease the rate of expansion and even may turn it into a contraction. The assumption $\ddot{\varphi}_1/\beta \ll |\ddot{\varphi}_0|$ does not seem to be impaired by the correction. As the temperature is given by $R\,T \cong -\ddot{\varphi}_0\varphi_1/\beta$, the decrease of T with time becomes faster than that given previously by (28).

Although the solution is incomplete, the following conclusions seem possible. The fact that the astronomical data for u_{fr}, ϱ_m, ϱ_0 lead to the correct value for T, can be taken as an indication that the assumption concerning the "impact pressure" experienced in moving through the interstellar gas is not far from the truth. The result concerning φ_1 gives an acceptable explanation of the mechanism by which the very thin sheets of cloud for long times can retain their elegant appearance. The estimate for the timescale depends directly upon the assumption concerning the impact pressure. It appears rather short when considered in relation to the distance between the two nebulae NGC 6960 and 6992 ($155' = 48{,}7 \,.\, 10^{18}$ cm), which are believed to have originated from a single explosion and thus should have travelled each about $24{,}4 \,.\, 10^{18}$ cm. As formula (26) cannot be applied to velocities approaching that of light, we may ask what would be the distance covered in the interval of time evolved since u_{fr} decreased from 10.000 km/sec (10^9 cm/sec; $t + t_0 = 30{,}9$ year) until its present value of 70 km/sec ($7 \,.\, 10^6$ cm/sec; $t + t_0 = 4410$ year). This distance amounts to $4{,}83 \,.\, 10^{18}$ cm, which is less than $^1/_5$ of the total amount travelled since the explosion (to cover the remaining $19{,}6 \,.\, 10^{18}$ cm at 10^9 cm/sec would require ca. 620 year). Evidently eq. (26) must be considered as an approximation valid for the present epoch only, and the decrease of velocity with time originally must have taken place at a slower rate [11]).

8. *Possibility of surface waves on interstellar clouds.* — Thus far we have restricted to the consideration of motion in one dimension only. It is evident that the real motions and currents in interstellar space will be of a much more complicated character. A comparison with atmospheric movements in many cases will have impressed itself on the mind of the observer. In particular the question has been brought forward whether certain types of structure, resembling cirrus clouds in the earth's atmosphere, may indicate the presence of waves or perhaps of vortices, such as can arise owing to the instability of the motion of fluid layers sliding over each other with different velocities. As an instance of such structures the clouds around the Pleiades must be mentioned; in particular the cloud around the star Merope shows a marked periodic structure. In certain parts of these clouds, *e.g.* in those around Alcyone and between Maia and Merope, several systems of periodic structure seem to be present, crossing each other at rather large angles. It is probable that the illuminated patches all belong to one single cloud, extending at least over about $1^\circ = 5{,}4 \,.\, 10^{18}$ cm, and that the periodic structures in the various patches are connected with each other.

When it is attempted to explain such structures by assuming a wave

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[11]) The relations from which the timescale has been deduced, can be put in the form:

$$p_{fr} = R\, T_{fr}\, \varrho_{fr} = \tfrac{1}{2}\,(k+1)\,\varrho_0\, u_{fr}^2 = -\, M_1\,(d\,u_{fr}/dt),$$

where $M_1 = \varrho_{fr} \,.\, L$ (M_1 being the mass of the whole layer per unit area and L its effective thickness). This equation gives: $d\,u_{fr}/dt = -\,R\,T_{fr}/L$, and thus fixes the present timescale from very simple data.

With the formulae of the text: $\varrho_{fr} = \varrho_m/\varphi_1$; $L = \varphi_1/\beta$. The assumption of a smaller value of β (leading to thinner sheets of cloud) reduces both the timescale and the distance covered. Larger values of β are in contradiction with observation. The assumption of a substantially smaller value of ϱ_0, which would lead to a greater value of $\beta\, t_0$, would change the value of the temperature in the same ratio, and thus seems unlikely. It may be thought that perhaps originally the clouds have moved through a portion of the interstellar gas of smaller density, but the problem then presents itself whether both nebulae have suffered the same adventure.

No account has been taken of the possibility of lateral motions (a divergent motion in the plane of the sheet might produce a continuous reduction of thickness), but in view of the elementary character of the relation between temperature, density and impact pressure, such motions are not likely to lead to an appreciably different timescale.

formation of similar nature as observed with certain atmospheric clouds. several difficulties present themselves. The wave motion in the atmosphere, arising between layers of air sliding over each other, is controlled by gravity. Furthermore in atmospheric clouds the wave pattern becomes visible in consequence of condensation of water vapour in those elements of volume which have suffered a reduction in temperature through their upward displacement. In the case of the interstellar gas an increased condensation of dust particles perhaps may be connected with a local increase in density and thus could make visible periodic changes of the density.

As to gravity, local gravitational fields due to separate clouds, as have been considered in section 1, may bring with them values of g of the order of $1{,}67 \, . \, 10^{-9}$. An estimate of the general gravitational field of the Galaxy can be obtained by taking the mass inside a sphere described with the distance from the centre of the Galaxy to the sun as radius (ca. $3 \, . \, 10^{22}$ cm) equal to $2{,}4 \, . \, 10^{11}$ times the mass of the sun, giving $M_{\text{Gal}} = 4{,}7 \, . \, 10^{44}$ gr, which makes the value of g in our neighbourhood $3{,}5 \, . \, 10^{-8}$. The general gravitational field, however, will be compensated by the centripetal acceleration of the clouds, so long as these move freely. Residual effects might be found when clouds collide with each other and produce abnormal accelerations in their surface layers. These accelerations even may exceed considerably the value of g calculated from the gravitational attraction: in the example treated in the preceding section the acceleration in the thin sheet is $du/dt = -u/(t + t_0) = -5 \, . \, 10^{-5}$ at the present epoch. It is not easy to decide whether fields produced in this way can regulate the appearance cf wave motions, supposing that the colliding masses at the same time should have a tangential velocity with respect to each other. As one of the two colliding masses will suffer a deceleration and the other an acceleration, the equivalent g-vectors will have opposite directions, pointing towards the surface of separation. This produces a situation different from that found in ordinary wave motion, which appears in stratified material where gravity everywhere acts in the same direction, pointing from the region of lower density to that of higher density.

A comparison between interstellar and atmospheric motions might be made by applying the theory of similarity, according to which similar motions can exist when in the first place there is geometrical similarity in pattern of motion and in density distribution, while in the second place REYNOLDS' number $Re = \varrho U \lambda / \mu$, FROUDE's number $Fr = U^2 / g\lambda$ and MACH's number $Ma = U/c$ must have the same values in the cases to be compared (U: a characteristic value of the velocities to be considered; λ: wavelength; ϱ: density, for the interstellar gas approx. 10^{-22}, for air at the height of say 4 km 0,0008; μ: viscosity, for the interstellar gas approx. 0,0023, for air 0,00017; c: velocity of sound). All ordinary cases of wave motion are such in which Ma is far below unity and then is of no importance. We therefore take FROUDE's number first, and consider a case of stationary wave motion along the surface of separation between two layers of material, having the densities ϱ, ϱ', the surface of separation being perpendicular to the local gravity vector with density increasing in the direction to which the vector points, the layers sliding over each other with velocities U, U' (measured with respect to the wave pattern). Then the wavelength must satisfy the relation [12]:

$$(\varrho - \varrho')\, g\, \lambda / 2\pi = \varrho\, U^2 + \varrho'\, U'^2 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

provided λ is small compared with the thickness of the layers. In this case $Fr = (\varrho - \varrho')/2\,\pi\varrho - \varrho' U'^2/\varrho g \lambda$, from which appears $Fr < 1/2\,\pi$. Hence in order to obtain a wavelength of $3 \,.\, 10^{16}$ cm as found in the Merope cloud, we must have $U^2/g < 5 \,.\, 10^{15}$, which would require either rather small values of the velocity (*e.g.*: $U = 2 \,.\, 10^3$ with g of the order 10^{-9}) or very high values of g, of the order 10^{-7} and more, provided of course we have to do with stationary (stable) wave motion.

With larger values of U^2/g the wave motion is unstable and in course of time increases in amplitude. This in itself is not impossible, but when U^2/g considerably exceeds the value mentioned the observed wavelength must be determined by some other cause; one might think of the thickness of some intermediate layer. Problems referring to wave motion in stratified systems have been treated by RAYLEIGH, TAYLOR, and others [13]. But as there is no direct clue to the case which should be chosen as a basis for comparison and as we miss any trustworthy datum about the value of g, no promising way for the construction of an appropriate solution as yet is to be seen.

It might be that we should discard any reference to the influence of gravity, and should look exclusively to features of the velocity distribution in order to find an explanation for the observed value of λ. The question arises if viscosity can play a part. If we assume a velocity U of $5 \,.\, 10^5$ cm/sec, the value of REYNOLDS' number $\varrho U \lambda / \mu$ comes out as 650. This appears rather small when compared with the results of some theoretical investigations on the stability of laminar motion when viscosity is effective, which would make us to expect a value above 15000 [14]. The supposition of a much higher value of U, however, would bring us to velocities above

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12) See H. LAMB, Hydrodynamics (Cambridge 1932), p. 377 (Art. 234, form. 5).

13) RAYLEIGH, Theory of Sound (London 1945), Vol. II, Ch. XXI (p. 376 seqq.). — G. I. TAYLOR, Effect of variation in density on the stability of superposed streams of fluid, Proc. Roy. Soc. (London) A **132**, p. 499, 1931. — See also: V. BJERKNES, J. BJERKNES, H. SOLBERG und T. BERGERON, Physikalische Hydrodynamik (Berlin 1933), Kap. VIII, IX u. X (pp. 305—421).

14) See H. SCHLICHTING, Zur Entstehung der Turbulenz bei der Plattenströmung. Göttinger Nachrichten Math.-physik. Kl., 1933 (II, no. 38), p. 181. From fig. 3 (p. 197) the minimum value of $U_m \delta^*/\nu$ is read off as 575 (comp. p. 202; $\nu = \mu/\varrho$) with $\alpha\delta^* = 0{,}23$, where $\alpha = 2\,\pi/\lambda$; this gives $U_m \lambda/\nu = 15700$. The maximum value of $\alpha\delta^*$ is about 0,28 with $U_m \delta^*/\nu =$ ca. 860; $U_m \lambda/\nu = 19300$. In both cases the waves are just on the limit between stability and instability; experimental observations point to the appearance of waves well inside the domain of instability, with still greater values of $U_m \lambda/\nu$.

the speed of sound, in which case the problem takes a wholly different character. Not much is known concerning the formation of waves in this case [15]. Further investigations are necessary before the question raised in this section can be settled.

9. *Considerations on the motion of the interstellar gas in the Galaxy as a whole.* — OORT has raised the problem whether the erratic motions and forms of the interstellar clouds may be due to effects depending upon the variation of galactic rotation with the distance from the centre of the Galaxy. It may be supposed that rotational motion with a regular velocity distribution ("laminar" rotation) would prove to be unstable, and that a form of turbulence should set in. In this connection one may first make an estimate of the magnitude of REYNOLDS' number for the rotation of the gas in the Galaxy. Assuming a radius of 50000 light years $= 5 \cdot 10^{22}$ cm, a circumferential velocity of 260 km/sec $= 2{,}6 \cdot 10^{7}$ cm/sec, $\varrho = 3 \cdot 10^{-24}$ and $\mu = 0{,}0023$, we obtain: $Re = 1{,}7 \cdot 10^{9}$. This value is high, though not excessive. In experimental situations (rotating basin filled with water) values of $8 \cdot 10^{6}$ probably can be reached; in atmospheric cyclones with a radius of (often far) over 100 km values of the order 10^{9} certainly occur. The boundary conditions in the latter cases, however, are wholly different from those existing in the Galaxy, and this prevents a direct comparison.

It is extremely difficult to investigate the stability of laminar rotation, in particular as account must be taken of changes in density and temperature, while even the changes in the gravitational field may be of some importance. According to JEANS gravitational instability in a homogeneous gas may arise for disturbances of a period λ_g exceeding a certain limit of the order $c\sqrt{\pi/\gamma\varrho}$ (c: velocity of sound; γ: gravitation constant $= 6{,}67 \cdot 10^{-8}$; ϱ_0: original density) [16]. With $T = 10.000°$, $c = 1{,}18 \cdot 10^{6}$, $\varrho_0 = 3 \cdot 10^{-24}$ this limit becomes $4{,}7 \cdot 10^{21}$ cm. In view of the dimensions of the Galaxy such a form of instability consequently may play its part. Its timescale is of the order $\lambda_g/c = 4{,}0 \cdot 10^{15}$ sec $= 1{,}3 \cdot 10^{8}$ year. As the distances between the interstellar clouds and their linear dimensions are much smaller than the value of λ_g just mentioned, other causes must be operative as well, and these evidently should be found in the rotation.

We cannot attack this problem at present, but a few remarks may be added. A laminar rotation of the interstellar gas as a whole would require a nearly complete balance between gravitational attraction and centrifugal force, as otherwise extremely large pressure gradients would be necessary. Estimating the value of g in our neighbourhood as $\gamma M_{\mathrm{Gal}}/r^2 = 3{,}13 \cdot 10^{37}/r^2$, we find $U = 5{,}6 \cdot 10^{18}/\sqrt{r}$. Viscous friction in

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[15]) See for some provisional results (which rather would point to a greater stability of surfaces of separation with velocities exceeding that of sound): J. ACKERET, Ueber Luftkräfte bei sehr grossen Geschwindigkeiten, Helv. Phys. Acta **1**, p. 301, 1928.

[16]) J. H. JEANS, Astronomy and Cosmogony (Cambridge, 1928), p. 340.

this case will produce a resultant force per unit volume of magnitude: $-0{,}75\,\mu U/r^2$, which at the sun's distance from the centre of the Galaxy ($r = 3 \,.\, 10^{22}$) amounts to $1{,}92 \,.\, 10^{-48}\, U$. With the density $3 \,.\, 10^{-24}$ this would lead to an acceleration of $-6{,}4 \,.\, 10^{-25}\, U$. The timescale thus arrived at is so enormous that we must conclude that viscous friction is inefficient to regularize the rotation.

A force will also be produced by the resistance of the stars. Taking as example the sun, and writing U_{rel} for its motion relatively to the gas, we find, curiously enough, that REYNOLDS' number for this relative motion $\varrho U_{rel}\, R_{sun}/\mu$ (with $R_{sun} = 7 \,.\, 10^{10}$ cm) even for $U_{rel} = 300$ km/sec $=$ $= 3 \,.\, 10^7$ cm/sec comes out as $2{,}74 \,.\, 10^{-3}$, i.e. far below unity. Hence we must apply STOKES' law of resistance. With the star density in the neighbourhood of the sun $N = 1/(3 \,.\, 10^{56})$ the force per unit volume becomes: $6\pi\mu R_s\, U_{rel}\, N = 10^{-47}\, U_{rel}$. Although comparable with the internal viscous friction, this force just as little can play an important part.

Mathematics. — *On the theory of linear integral equations.* VI. By A. C. ZAANEN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of May 25, 1946.)

§ 1. *Introduction.*

We suppose the reader to be acquainted with the contents of the papers I, II and IV, bearing the same title [1]). Furthermore we shall make use of V, Theorem 8, stating that a Hermitean kernel $A(x,y) \in L_2^{(2m)}$ is general if and only if $Af = \int_\triangle A(x,y) f(y)\, dy = 0$ implies $f(x) = 0$. In this paper we shall consider linear integral equations

$$\int_\triangle K(x,y) f(y)\, dy - \lambda f(x) = g(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in the space $L_2^{(m)}(\triangle)$ with kernel

$$K(x,y) = \int_\triangle A(x,z)\, H(z,y)\, dz,$$

where both kernels $A(x,y)$ and $H(x,y)$ belong to the space $L_2^{(2m)}(\triangle)$ and are Hermitean, while moreover $H(x,y)$ is supposed to be positive (positive means here: of positive type). A. J. PELL [2]) has discussed the equation of this kind, and, denoting the characteristic values of (1) by $\lambda_i\,(i = 1, 2, \ldots)$, and a corresponding H-orthonormal system of characteristic functions by $\psi_i(x)\,(i = 1, 2, \ldots)$, she found under the assumptions that $A(x,y)$ is continuous, while $H(x,y)$ is bounded, the following main result:

If $a_i = \int_{\triangle \times \triangle} H(x,y)\, \overline{\psi_i(x)}\, f(y)\, dx\, dy$ for an arbitrary $f(x) \in L_2$, then

$$\int_\triangle K(x,y) f(y)\, dy = \Sigma\, \lambda_i\, a_i\, \psi_i(x) + p(x),$$

uniformly in x, where $Hp = \int_\triangle H(x,y)\, p(y)\, dy = 0$.

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[1]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **49**, 194—204, 205—212 and 409—423 (1946).

[2]) A. J. PELL, Applications of biorthogonal systems of functions to the theory of integral equations, Transactions of the Am. Math. Soc. **12**, 165—180 (1911).

We shall prove, under the sole assumption that the Hermitean kernel $A(x,y)$ and the positive Hermitean kernel $H(x,y)$ belong to $L_2^{(2m)}$, that

$$K(x,y) - p(x,y) \backsim \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}, \quad \ldots \ldots \quad (2)$$

where

$$\chi_i(x) = \int_\triangle H(x,y) \psi_i(y)\, dy \qquad (i = 1, 2, \ldots)$$

and

$$\int_\triangle H(x,z)\, p(z,y)\, dz = 0$$

almost everywhere in $\triangle \times \triangle$. Furthermore

$$K_n(x,y) \backsim \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \qquad (n \geqslant 2); \quad \ldots \ldots \quad (3)$$

$$\int_\triangle K_n(x,x)\, dx = \sum_i \lambda_i^n \qquad (n \geqslant 2). \quad \ldots \ldots \quad (4)$$

Moreover, writing $a_i = \int_\triangle f(x) \overline{\chi_i(x)}\, dx$ for an arbitrary $f(x) \in L_2$, we have

$$\int_\triangle K(x,y) f(y)\, dy \backsim \Sigma \lambda_i a_i \psi_i(x) + p(x), \quad \ldots \ldots \quad (5)$$

$$\int_\triangle K_n(x,y) f(y)\, dy \backsim \sum_i \lambda_i^n a_i \psi_i(x) \qquad (n \geqslant 2), \quad \ldots \quad (6)$$

where

$$Hp = \int_\triangle H(x,y)\, p(y)\, dy = 0.$$

Thereafter we shall find sufficient conditions under which the convergence in mean in (2), (3), (5) and (6) may be replaced by uniform convergence (it will be proved that the result of A. J. PELL holds under very general conditions (cf. Theorem 4)), and the functions $p(x,y)$ add $p(x)$ in (2) and (5) vanish.

§ 2. *The equation with kernel* $K(x,y) = \int_\triangle A(x,z)\, H(z,y)\, dz$.

Let

$$K(x,y) = \int_\triangle A(x,z)\, H(z,y)\, dz,$$

where $A(x, y)$ and $H(x, y)$ are Hermitean kernels, belonging to the space $L_2^{(2m)}(\triangle)$, while, moreover, $H(x, y)$ is positive (of positive type). Then the linear transformations A and H in the space $L_2^{(m)}(\triangle)$, defined by

$$Af = \int_{\triangle} A(x, y) f(y)\, dy,$$

$$Hf = \int_{\triangle} H(x, y) f(y)\, dy,$$

are completely continuous and self-adjoint, while H is, moreover, positive. The completely continuous transformation $K = AH$ is then determined by

$$Kf = AHf = \int_{\triangle \times \triangle} A(x, z) H(z, y) f(y)\, dz\, dy = \int_{\triangle} K(x, y) f(y)\, dy.$$

As we know, the transformation K is symmetrisable relative to H (since $HK = HAH$ is self-adjoint), and we observe that every $f(x) \in L_2$, satisfying $Hf = 0$, satisfies also $Kf = AHf = 0$. The kernel $K(x, y)$ is therefore what we have called in IV a Marty-kernel. Supposing that

$$\| H(x, y) \|_{2m}^2 = \int_{\triangle \times \triangle} | H(x, y) |^2\, dx\, dy \neq 0,$$

so that H is not identical with the nulltransformation O, the theorems proved in IV may therefore be applied to the equation (1). We shall not repeat them all here, and only pay attention to IV, Theorems 4, 6 and 11, since these may be replaced by stronger theorems. Instead of IV, Theorem 4, we have

Theorem 1. *(Expansion Theorem). Writing* $a_i = (f, \chi_i) = \int_{\triangle} f(x) \overline{\chi_i(x)}\, dx$ *for an arbitrary* $f(x) \in L_2$, *we have*

$$\int_{\triangle} K(x, y) f(y)\, dy \backsim \Sigma \lambda_i a_i \psi_i(x) + p(x), \quad \text{. . . . . (5)}$$

$$\int_{\triangle} K_n(x, y) f(y)\, dy \backsim \sum_i \lambda_i^n a_i \psi_i(x) \qquad (n \geqslant 2), \quad \text{. . . . (6)}$$

where the function $p(x)$ *satisfies the relation*

$$Hp = \int_{\triangle} H(x, y)\, p(y)\, dy = 0$$

for almost every $x \in \triangle$.

Proof. Follows from I, Theorem 15.

Instead of IV, Theorem 6, we have

Theorem 2. *Let $\lambda \neq 0$, and let $g(x) \in L_2$ be H-orthogonal to all characteristic functions of* (1), *belonging to the characteristic value λ. (If λ is no characteristic value, $g(x)$ is therefore arbitrary). Then every solution of* (1) *satisfies a relation of the form*

$$f(x) \backsim -\frac{g(x)}{\lambda} - \Sigma' \frac{\lambda_i}{\lambda(\lambda-\lambda_i)} a_i \psi_i(x) + q(x),$$

where $a_i = \int_\Delta g(x) \overline{\chi_i(x)} \, dx$ *for* $\lambda_i \neq \lambda$, $\int_\Delta H(x, y) q(y) \, dy = 0$ *for almost every* $x \in \Delta$, *and where Σ' denotes that for those values of i for which $\lambda_i = \lambda$ the coefficient of $\psi_i(x)$ has the value* $\int_\Delta f(x) \overline{\chi_i(x)} \, dx$. *For every set of arbitrarily prescribed values of the latter coefficients there exists a solution of* (1).

Proof. Follows from I, Theorem 17.

Instead of IV, Theorem 11 we have

Theorem 3. (*Expansion Theorem for the kernels*). *We have*

$$K(x, y) - p(x, y) \backsim \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}, \quad \ldots \ldots \quad (2)$$

where

$$\int_\Delta H(x, z) p(z, y) \, dz = 0$$

almost everywhere in $\Delta \times \Delta$;

$$K_n(x, y) \backsim \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \qquad (n \geqslant 2); \quad \ldots \ldots \quad (3)$$

$$\int_\Delta K_n(x, x) \, dx = \sum_i \lambda_i^n \qquad (n \geqslant 2) \quad \ldots \ldots \quad (4)$$

Proof. The formulae (2) and (3) will follow from IV, Theorem 9, if only we prove that $\Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}$ converges in mean. For this purpose we observe that the system $\Psi_i = H^{1/2} \psi_i$ is orthonormal, so that by BESSEL's inequality

$$\left. \begin{array}{l} \Sigma |(H \psi_i, f)|^2 = \Sigma |(\psi_i, Hf)|^2 = \Sigma |(Hf, \psi_i)|^2 = \\ \quad \Sigma |(H^{1/2} f, \Psi_i)|^2 \leqslant \|H^{1/2} f\|^2 = (Hf, f) \leqslant \|Hf\| \cdot \|f\| \leqslant \|H\| \cdot \|f\|^2 \end{array} \right\} . \quad (7)$$

for any $f(z) \in L_2$. Taking now $f(z) = \overline{A(x, z)}$, this function belongs to L_2 for almost every $x \in \Delta$, so that

$$\lambda_i \psi_i(x) = \int_\Delta K(x, y) \psi_i(y) \, dy = \int_\Delta A(x, z) \left[\int_\Delta H(z, y) \psi_i(y) \, dy \right] dz = (H \psi_i, f)$$

for these values of x. Hence, by (7),

$$\Sigma \lambda_i^2 |\psi_i(x)|^2 \leqslant \|H\| \,.\, \|f\|^2 = \|H\| \,.\int_\triangle |A(x,z)|^2\, dz, \quad . \quad . \quad (8)$$

so that

$$\Sigma \lambda_i^2 \int_\triangle |\psi_i(x)|^2\, dx \leqslant \|H\| \,.\int_{\triangle\times\triangle} |A(x,z)|^2\, dx\, dz. \quad . \quad . \quad . \quad (9)$$

The system $\Psi_i(x) = H^{1/2}\psi_i$ being orthonormal, we have for arbitrary complex numbers a_i,

$$\|\sum_{i=p}^{q} a_i \chi_i(y)\|^2 = \|H^{1/2}\sum_{i=p}^{q} a_i \Psi_i(y)\|^2 \leqslant \|H^{1/2}\|^2 \,.\, \|\sum_{i=p}^{q} a_i \Psi_i(y)\|^2 =$$
$$= \|H^{1/2}\|^2 \sum_{i=p}^{q} |a_i|^2,$$

or, taking $a_i = a_i(x) = \lambda_i \overline{\psi_i(x)}$, and integrating then over x,

$$\int_{\triangle\times\triangle} |\sum_{i=p}^{q} \lambda_i \psi_i(x) \overline{\chi_i(y)}|^2\, dx\, dy \leqslant \|H^{1/2}\|^2 \sum_{i=p}^{q} \lambda_i^2 \int_\triangle |\psi_i(x)|^2\, dx.$$

By (9), the expression on the right hand side tends to 0 as $p, q \to \infty$, the same is therefore true of the expression on the left hand side, which shows the convergence in mean of $\Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}$.

From this convergence in mean of $\Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)}$ follows now, by IV, Theorem 10, that

$$\int_\triangle K_2(x,x)\, dx - \int_\triangle r(x,x)\, dx = \Sigma \lambda_i^2,$$

$$\int_\triangle K_n(x,x)\, dx = \sum_i \lambda_i^n \quad (n \geqslant 3),$$

where

$$r(x,y) = \int_\triangle K(x,z)\, p(z,y)\, dz.$$

The formula (4) will be proved, therefore, if we can show that $r(x,x) = 0$ for almost every $x \in \triangle$. This follows from the fact that

$$\int_\triangle H(y,z)\, p(z,x)\, dz = 0$$

for almost every $x \in \triangle$, which implies

$$r(x,x) = \int_{\triangle} K(x,z)\, p(z,x)\, dz = \int_{\triangle \times \triangle} A(x,y)\, H(y,z)\, p(z,x)\, dy\, dz =$$

$$= \int_{\triangle} A(x,y) \left[\int_{\triangle} H(y,z)\, p(z,x)\, dz \right] dy = 0$$

for almost every $x \in \triangle$.

§ 3. *The case that $A(x,y)$ is continuous in mean.*

Definition. We shall say that the Hermitean kernel $A(x,y) \in L_2^{(2m)}$ is *continuous in mean* if

$$\int_{\triangle} |A(x,y)|^2\, dy = \int_{\triangle} |A(y,x)|^2\, dy$$

is finite for every $x \in \triangle$, and

$$\lim_{x_2 \to x_1} \int_{\triangle} |A(x_2,y) - A(x_1,y)|^2\, dy = 0$$

for $x_1, x_2 \in \triangle$.

It is immediately to be seen that whenever $A(x,y)$ is continuous in mean, the function $h(x) = \int_{\triangle} A(x,y) f(y)\, dy$ is continuous in $\triangle$ for every $f(x) \in L_2$, so that the same is true of

$$g(x) = \int_{\triangle} K(x,y) f(y)\, dy = \int_{\triangle} A(x,y) \left[\int_{\triangle} H(y,z) f(z)\, dz \right] dy$$

for every $f(x) \in L_2$. In particular the characteristic functions

$$\psi_i(x) = \lambda_i^{-1} \int_{\triangle} K(x,y)\, \psi_i(y)\, dy$$

are continuous in $\triangle$.

Theorem 4. (*Expansion Theorem*). *If $A(x,y)$ is continuous in mean, and $a_i = (f, \chi_i) = \int_{\triangle} f(x) \overline{\chi_i(x)}\, dx$ for an arbitrary $f(x) \in L_2$, we have*

$$\int_{\triangle} K(x,y) f(y)\, dy = \Sigma\, \lambda_i\, a_i\, \psi_i(x) + \tilde{p}(x),$$

$$\int_{\triangle} K_n(x,y) f(y)\, dy = \sum_i \lambda_i^n\, a_i\, \psi_i(x) \qquad (n \geqq 2).$$

uniformly in $\triangle$, *where* $\tilde{p}(x)$ *is continuous in* $\triangle$ *and satisfies the relation*

$$\int_{\triangle} H(x, y)\, \tilde{p}(y)\, dy = 0$$

for almost every $x \in \triangle$. *In the case that* $H(x, y)$ *is a general kernel,* $\tilde{p}(x)$ *vanishes identically.*

Proof. In the proof of the preceding theorem we have seen that

$$\Sigma \lambda_i^2 |\psi_i(x)|^2 \leqslant \|H\| \cdot \int_{\triangle} |A(x, z)|^2 dz$$

for those values of x, for which $g(z) = \overline{A(x, z)}$ belongs to L_2. It is not difficult to prove that now, since $A(x, y)$ is continuous in mean, there exists a constant M such that

$$\int_{\triangle} |A(x, z)|^2 dz \leqslant M^2,$$

so that $g(z) = \overline{A(x, z)}$ belongs to L_2 for every $x \in \triangle$. This shows that

$$\Sigma \lambda_i^2 |\psi_i(x)|^2 \leqslant \|H\| \cdot M^2$$

for all $x \in \triangle$. Furthermore, since $a_i = (f, \chi_i) = (f, H^{1/2} \Psi_i) = (H^{1/2} f, \Psi_i)$ and the system $\Psi_i(x)$ is orthonormal, we have

$$\Sigma |a_i|^2 \leqslant \|H^{1/2} f\|^2,$$

so that, if $\varepsilon > 0$ is given,

$$\sum_{i=p} |a_i|^2 \leqslant \varepsilon^2 / \|H\| \cdot M^2$$

for sufficiently large p. The inequality

$$\sum_{i=p} |\lambda_i a_i \psi_i(x)| \leqslant (\sum_{i=p} |a_i|^2)^{1/2} \cdot (\sum_{i=p} \lambda_i^2 |\psi_i(x)|^2)^{1/2} \leqslant \varepsilon$$

shows then that the series $\Sigma \lambda_i a_i \psi_i(x)$ converges uniformly in $\triangle$. Since both its sum and $\int_{\triangle} K(x, y) f(y)\, dy$ are continuous in $\triangle$, we may write

$$\int_{\triangle} K(x, y) f(y)\, dy = \Sigma \lambda_i a_i \psi_i(x) + \tilde{p}(x),$$

where $\tilde{p}(x)$ is continuous in $\triangle$. From Theorem 1 it follows that $p(x) = p(x)$ for almost every $x \in \triangle$, where $p(x)$ is the function occurring in that theorem, satisfying therefore $Hp = 0$. Then we have also $H\tilde{p} = 0$. This completes the proof of the first part.

That $\sum_i \lambda_i^n a_i \psi_i(x)$ $(n \geq 2)$ converges also uniformly in $\triangle$ is now trivial. On account of Theorem 1 its sum is equal to $\int_\triangle K_n(x, y) f(y)\, dy$ for almost every $x \in \triangle$. But, since both this sum and this function are continuous in $\triangle$, equality holds for every $x \in \triangle$, hence

$$\int_\triangle K_n(x, y) f(y)\, dy = \sum_i \lambda_i^n a_i \psi_i(x) \qquad (n \geq 2),$$

uniformly in $\triangle$.

Assuming now that $H(x, y)$ is a general kernel, it follows from $H\tilde{p} = 0$ that $\tilde{p}(x) = 0$ for almost every $x \in \triangle$. The continuity of $\tilde{p}(x)$ implies then that $\tilde{p}(x)$ vanishes identically.

Theorem 5. *Let $A(x, y)$ be continuous in mean, and $H(x, y)$ be general. Let furthermore $\lambda \neq 0$, and $g(x) \in L_2$ be H-orthogonal to all characteristic functions of $K(x, y)$, belonging to the characteristic value λ (If λ is no characteristic value, $g(x)$ may be any function belonging to L_2). Then the solution of (1) is given by*

$$f(x) = -\frac{g(x)}{\lambda} - \sum{}' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x),$$

where $a_i = \int_\triangle g(x) \overline{\chi_i(x)}\, dx$ for $\lambda_i \neq \lambda$, Σ' denotes that for those values of i for which $\lambda_i = \lambda$ the coefficient of $\psi_i(x)$ is arbitrary, and the series

$$\sum{}' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x)$$

converges uniformly in $\triangle$.

Proof. By the preceding theorem we have

$$g(x) + \lambda f(x) = \int_\triangle K(x, y) f(y)\, dy = \sum \lambda_i b_i \psi_i(x),$$

uniformly in $\triangle$, where $b_i = (f, \chi_i)$. Since

$$\lambda_i b_i = \lambda_i (f, \chi_i) = \lambda_i (f, H\psi_i) = (f, HK\psi_i) = (HKf, \psi_i) =$$
$$= (Kf, H\psi_i) = (g + \lambda f, \chi_i) = a_i + \lambda b_i,$$

we find $b_i = -a_i/(\lambda - \lambda_i)$ for $\lambda_i \neq \lambda$. Furthermore, the solution $f(x)$ being determined except for a characteristic function of $K(x, y)$, belonging to the characteristic value λ, the coefficients b_i may be taken arbitrary for those values of i for which $\lambda_i = \lambda$. Hence

$$f(x) = -\frac{g(x)}{\lambda} - \sum{}' \frac{\lambda_i}{\lambda(\lambda - \lambda_i)} a_i \psi_i(x),$$

uniformly in $\triangle$.

For later purposes we prove

Theorem 6. *If $A(x,y)$ is continuous in mean, the series $\Sigma\lambda_i^2|\psi_i(x)|^2$ converges uniformly in $\triangle$.*

Proof. In the same way as we have proved (8), we may find

$$\Sigma\lambda_i^2|\psi_i(x_2)-\psi_i(x_1)|^2\leqslant\|H\|\int_\triangle|A(x_2,z)-A(x_1,z)|^2\,dz,$$

hence

$$\lim_{x_2\to x_1}\Sigma\lambda_i^2|\psi_i(x_2)-\psi_i(x_1)|^2=0.$$

By MINKOWSKI's inequality we see that

$$|(\Sigma\lambda_i^2|\psi_i(x_2)|^2)^{1/2}-(\Sigma\lambda_i^2|\psi_i(x_1)|^2)^{1/2}|\leqslant(\Sigma\lambda_i^2|\psi_i(x_2)-\psi_i(x_1)|^2)^{1/2};$$

the sumfunction of the series $\Sigma\lambda_i^2|\psi_i(x)|^2$ is therefore a continuous function. Hence, on account of DINI's well-known Theorem, since the functions $\lambda_i^2|\psi_i(x)|^2$ are non-negative and continuous, the uniform convergence of $\Sigma\lambda_i^2|\psi_i(x)|^2$.

§ 4. *The case that $A(x,y)$ and $H(x,y)$ are both continuous in mean.*

Theorem 7. *If $H(x,y)$ (but not necessarily $A(x,y)$) is continuous in mean, the series $\Sigma\lambda_i^2|\chi_i(x)|^2$ converges uniformly in $\triangle$.*

Proof. We observe first that, if $H(x,y)$ is continuous in mean, the functions $\chi_i(x)=\int_\triangle H(x,y)\,\psi_i(y)\,dy$ are continuous in $\triangle$. For every $f(y)\in L_2$ we have $(f,A\chi_i)=(Af,H^{1/2}\Psi_i)=(H^{1/2}Af,\Psi_i)$, hence, the system $\Psi_i(x)$ being orthonormal,

$$\Sigma|(A\chi_i,f)|^2=\Sigma|(f,A\chi_i)|^2\leqslant\|H^{1/2}Af\|^2\leqslant\|H\|\,.\,\|A\|^2\,.\,\|f\|^2\,. \quad (10)$$

on account of BESSEL's inequality. Taking now $f(y)=\overline{H(x,y)}$, there exists, since $H(x,y)$ is continuous in mean, a constant M such that

$$\|f\|^2=\int_\triangle|H(x,y)|^2\,dy\leqslant M$$

for every $x\in\triangle$. Furthermore

$$(A\chi_i,f)=(AH\psi_i,f)=(K\psi_i,f)=\lambda_i(\psi_i,f)=\lambda_i\int_\triangle H(x,y)\,\psi_i(y)\,dy=\lambda_i\chi_i(x),$$

so that by (10)

$$\Sigma\lambda_i^2|\chi_i(x)|^2\leqslant\|H\|\,.\,\|A\|^2\,.\,M$$

for every $x\in\triangle$.

Substituting now $f(y)=\overline{H(x_2, y)-H(x_1, y)}$ in the relation (10), we find in the same way

$$\Sigma \lambda_i^2 |\chi_i(x_2)-\chi_i(x_1)|^2 \leqslant \|H\| \,.\, \|A\|^2 \int_{\triangle} |H(x_2, y)-H(x_1, y)|^2 dy,$$

hence

$$\lim_{x_2 \to x_1} \Sigma \lambda_i^2 |\chi_i(x_2)-\chi_i(x_1)|^2 = 0.$$

By MINKOWSKI's inequality follows now again that the sumfunction of $\Sigma \lambda_i^2 |\chi_i(x)|^2$ is a continuous function, which implies, on account of DINI's Theorem, the uniform convergence of $\Sigma \lambda_i^2 |\chi_i(x)|^2$.

Theorem 8. (*Expansion Theorem for the iterated kernels*). *If both* $A(x, y)$ *and* $H(x, y)$ *are continuous in mean, then*

$$K_n(x, y) = \sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)} \qquad (n \geqslant 2),$$

uniformly in $\triangle \times \triangle$.

Proof. It is not difficult to prove that, if both $A(x, y)$ and $H(x, y)$ are continuous in mean, all kernels $K_n(x, y)$ $(n \geqslant 1)$ are continuous in $\triangle \times \triangle$. Since, [by the Theorems 6 and 7, the series $\Sigma \lambda_i^2 |\psi_i(x)|^2$ and $\Sigma \lambda_i^2 |\chi_i(x)|^2$ are converging uniformly in $\triangle$, the inequality

$$\sum_i |\lambda_i^n \psi_i(x) \overline{\chi_i(y)}| \leqslant (\sum_i \lambda_i^2 |\psi_i(x)|^2)^{1/2} \,.\, (\sum_i \lambda_i^{2n-2} |\chi_i(y)|^2)^{1/2}$$

shows that, for $n \geqslant 2$, the series $\sum_i \lambda_i^n \psi_i(x) \overline{\chi_i(y)}$ converges uniformly in $\triangle \times \triangle$. By Theorem 3 (3) the sumfunction is equal to $K_n(x, y)$ almost everywhere in $\triangle \times \triangle$. But, both $K_n(x, y)$ and this sumfunction being continuous in $\triangle \times \triangle$, equality is seen to hold for every point $(x, y) \in \triangle \times \triangle$. This completes the proof.

§ 5. *The case that* $A(x, y)$ *is continuous in mean and* $H(x, y)$ *is continuous.*

Theorem 9. (*Expansion Theorem for the kernel*). *If* $A(x, y)$ *is continuous in mean and* $H(x, y)$ *is continuous, then*

$$K(x, y) - p(x, y) = \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$, *where*

$$q(x, y) = \int_{\triangle} H(x, z)\, p(z, y)\, dz = 0$$

for every point $(x, y) \in \triangle \times \triangle$.

Proof. [3]) Since $H(x, y)$ is continuous, the transformation $H^{1/2}$ is, on account of II, Theorem 10, given by

$$H^{1/2} f = \int_{\triangle} H_{1/2}(x, y) f(y)\, dy.$$

[3]) The beginning of this proof is identical with the beginning of the proof of V, Theorem 5. For the sake of completeness, however, we shall give the proof in full.

where $\int\limits_{\triangle} |H_{1/2}(x, y)|^2 \, dy$ is bounded. For almost every $x_1 \in \triangle$ and almost every $x_2 \in \triangle$ we have now

$$\int\limits_{\triangle} |H_{1/2}(x_2, y) - H_{1/2}(x_1, y)|^2 \, dy =$$

$$\int\limits_{\triangle} \{H_{1/2}(x_2, y) - H_{1/2}(x_1, y)\} \, \{H_{1/2}(y, x_2) - H_{1/2}(y, x_1)\} \, dy =$$

$$= H(x_2, x_2) - H(x_2, x_1) - H(x_1, x_2) + H(x_1, x_1);$$

consequently, there exists for every $\varepsilon > 0$ a number $\delta(\varepsilon) > 0$ such that

$$\int\limits_{\triangle} |H_{1/2}(x_2, y) - H_{1/2}(x_1, y)|^2 \, dy \leqslant \varepsilon \quad . \quad . \quad . \quad . \quad . \quad (11)$$

for almost every $x_1 \in \triangle$ and $x_2 \in \triangle$, if only the distance $\varrho(x_1, x_2)$ of the points x_1 and x_2 satisfies $\varrho(x_1, x_2) < \delta$. Furthermore we have

$$\chi_i(x_2) - \chi_i(x_1) = \int\limits_{\triangle} \{H_{1/2}(x_2, y) - H_{1/2}(x_1, y)\} \, \Psi_i(y) \, dy;$$

hence by BESSEL's inequality (the system $\Psi_i(x)$ is orthonormal) and (11)

$$\sum_{i=1}^{p} |\chi_i(x_2) - \chi_i(x_1)|^2 \leqslant \int\limits_{\triangle} |H_{1/2}(x_2, y) - H_{1/2}(x_1, y)|^2 \, dy \leqslant \varepsilon \quad . \quad (12)$$

for almost every $x_1 \in \triangle$ and $x_2 \in \triangle$, if only $\varrho(x_1, x_2) < \delta$. Since however the functions $\chi_i(x)$ are continuous in $\triangle$, the left hand side of (12) is continuous in x_1 and x_2, so that (12) holds for all $x_1, x_2 \in \triangle$, satisfying $\varrho(x_1, x_2) < \delta$. Observing now that p is arbitrary, we obtain

$$\lim_{x_2 \to x_1} \Sigma |\chi_i(x_2) - \chi_i(x_1)|^2 = 0,$$

so that, by the device used already several times, we see that $\Sigma |\chi_i(x)|^2$ converges uniformly in $\triangle$. By Theorem 6 the same is true of $\Sigma \lambda_i^2 |\psi_i(x)|^2$; the inequalities

$$\Sigma |\lambda_i \, \psi_i(x) \, \overline{\chi_i(y)}| \leqslant (\Sigma \lambda_i^2 |\psi_i(x)|^2)^{1/2} \, . \, (\Sigma |\chi_i(y)|^2)^{1/2},$$

$$\Sigma |\chi_i(x) \, \overline{\chi_i(y)}| \leqslant (\Sigma |\chi_i(x)|^2)^{1/2} \, . \, (\Sigma |\chi_i(y)|^2)^{1/2}$$

show then that the series $\Sigma \lambda_i \, \psi_i(x) \, \overline{\chi_i(y)}$ and $\Sigma \chi_i(x) \, \overline{\chi_i(y)}$ converge uniformly in $\triangle \times \triangle$. Writing

$$H(z, y) - p_1(z, y) = \Sigma \chi_i(z) \, \overline{\chi_i(y)} \, , \quad . \quad . \quad . \quad . \quad . \quad (13)$$

we see that $p_1(z, y)$ is continuous in $\triangle \times \triangle$. Multiplying (13) with

$A(x, z)$, and writing $p(x, y) = \int_\triangle A(x, z)\, p_1(z, y)\, dz$, we find, since we may use term by term integration on account of the uniform convergence,

$$K(x, y) - p(x, y) = \Sigma \int_\triangle A(x, z)\, \chi_i(z)\, \overline{\chi_i(y)}\, dz = \Sigma\, \lambda_i\, \psi_i(x)\, \overline{\chi_i(y)}$$

for every point $(x, y) \in \triangle \times \triangle$. It follows from Theorem 3 that

$$q(x, y) = \int_\triangle H(x, z)\, p(z, y)\, dz = 0$$

for almost every point $(x, y) \in \triangle \times \triangle$. But, as may be seen easily, $q(x, y)$ is continuous in $\triangle \times \triangle$, so that $q(x, y) = 0$ everywhere in $\triangle \times \triangle$. This completes the proof.

Theorem 10. *If $A(x, y)$ is continuous in mean, and $H(x, y)$ is continuous and general, then*

$$K(x, y) = \Sigma\, \lambda_i\, \psi_i(x)\, \overline{\chi_i(y)}, \qquad (14)$$

uniformly in $\triangle \times \triangle$.

Proof. By the preceding theorem we have

$$K(x, y) - p(x, y) = \Sigma\, \lambda_i\, \psi_i(x)\, \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$, where

$$p(x, y) = \int_\triangle A(x, z)\, p_1(z, y)\, dz, \qquad (15)$$

$p_1(x, y)$ is continuous in $\triangle \times \triangle$, and

$$\int_\triangle H(x, z)\, p(z, y)\, dz = 0.$$

Since $H(x, y)$ is now general, $Hp = 0$ implies $p = 0$, so that, for every $y \in \triangle$, $p(x, y) = 0$ for almost every $x \in \triangle$. It follows however from (15) that $p(x, y)$ is continuous in x for every $y \in \triangle$; hence $p(x, y) = 0$ in $\triangle \times \triangle$, which proves (14).

Theorem 11. *If $A(x, y)$ is continuous in mean and general, while $H(x, y)$ is continuous and general, then*

$$H(x, y) = \Sigma\, \chi_i(x)\, \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$.

Proof. We know already (cf. the proof of Theorem 9) that $\Sigma \chi_i(x)\, \overline{\chi_i(y)}$ converges uniformly in $\triangle \times \triangle$, and by the preceding theorem we have

$$K(z, y) - \Sigma\, \lambda_i\, \psi_i(z)\, \overline{\chi_i(y)} = \int_\triangle A(z, x)\, \{ H(x, y) - \Sigma\, \chi_i(x)\, \overline{\chi_i(y)} \}\, dx = 0.$$

Hence, since $A(z, x)$ is general, for every $y \in \triangle$,

$$H(x, y) - \Sigma \chi_i(x) \overline{\chi_i(y)} = 0 \quad \ldots \ldots \quad (16)$$

for almost every $x \in \triangle$. Both $H(x, y)$ and $\Sigma \chi_i(x) \overline{\chi_i(y)}$ being however continuous in $\triangle \times \triangle$, the relation (16) holds for every point $(x, y) \in \triangle \times \triangle$, so that

$$H(x, y) = \Sigma \chi_i(x) \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$.

Remark. It is not difficult to prove that, if $A(x, y)$ is not general, (16) is not necessarily true.

§ 6. *An example.*

We shall illustrate the theorems, proved in the preceding paragraphs, by an example showing that the functions $\tilde{p}(x)$ and $p(x, y)$, occurring in the Theorems 4 and 9, need not vanish identically.

Let, for this purpose, $\triangle$ be the linear interval $0 \leqslant x \leqslant 2\pi$, and $t_i(x)$ the orthonormal trigonometrical system, hence

$$t_1(x) = (2\pi)^{-1/2},\ t_{2n}(x) = \pi^{-1/2} \cos nx\ (n \geqslant 1),\ t_{2n+1}(x) = \pi^{-1/2} \sin nx\ (n \geqslant 1).$$

It is well-known that the system $t_i(x)$ is complete in the space $L_2(0, 2\pi)$ of all functions $f(x)$ for which $|f(x)|^2$ is summable over $\triangle$. We determine now the sequences $\mu_3, \mu_4, \ldots$ and $\nu_3, \nu_4, \ldots$ of positive numbers such that

$$\lim \mu_i = \lim \nu_i = 0,\ \mu_3 > \mu_4 > \ldots,\ \nu_3 > \nu_4 > \ldots,\ \sum_3 \mu_i < \infty$$

and $\sum_3 \nu_i < \infty$. The Hermitean kernel $A(x, y)$ and the positive Hermitean kernel $H(x, y)$ are then defined by

$$A(x, y) = \tfrac{1}{2}\{t_1(x) + t_2(x)\}\{t_1(y) + t_2(y)\}$$
$$- \tfrac{1}{2}\{t_1(x) - t_2(x)\}\{t_1(y) - t_2(y)\} + \sum_3^\infty \nu_i t_i(x) t_i(y),$$

$$H(x, y) = t_1(x) t_1(y) + \sum_3^\infty \mu_i t_i(x) t_i(y).$$

It is clear that both series are converging uniformly in $\triangle \times \triangle$, both $A(x, y)$ and $H(x, y)$ are therefore continuous in $\triangle \times \triangle$. We find now

$$K(x, y) = \int_\triangle A(x, z) H(z, y)\, dz = t_2(x) t_1(y) + \sum_3^\infty \nu_i \mu_i t_i(x) t_i(y),$$

and we observe that

$$Ht_1 = t_1,\quad Ht_2 = 0,\quad Ht_i = \mu_i t_i \qquad (i = 3, 4, \ldots),$$
$$At_1 = t_2,\quad At_2 = t_1,\quad At_i = \nu_i t_i \qquad (i = 3, 4, \ldots),$$

so that

$$AHt_1 = t_2,\quad AHt_2 = 0,\quad AHt_i = \nu_i \mu_i t_i \qquad (i = 3, 4, \ldots).$$

To find the characteristic functions of $K(x, y)$, belonging to characteristic values $\neq 0$, we write $AHf = \lambda f$ for $f = \sum_{i=1}^{\infty} a_i t_i$ and $\lambda \neq 0$. From this we derive

$$a_1 t_2 + \sum_3 \nu_i \mu_i a_i t_i = \sum_1 \lambda a_i t_i;$$

hence $a_1 = a_2 = 0$ and $\nu_i \mu_i a_i = \lambda a_i \ (i = 3, 4, \ldots)$. Since $\nu_i \mu_i \neq \nu_k \mu_k$ for $i \neq k$ we have therefore $\lambda = \nu_k \mu_k$ for a certain value of $k\ (\geqslant 3)$ and $a_i = 0$ for $i \neq k$, which shows that the functions $a_k t_k(x)\ (k \geqslant 3)$ are the only characteristic functions with characteristic values $\neq 0$. Making them H-normal, we obtain $a_k = \mu_k^{-1/2}$, so that, for $i = 1, 2, \ldots$,

$$\lambda_i = \nu_{i+2}\, \mu_{i+2},$$

$$\psi_i(x) = \mu_{i+2}^{-1/2}\, t_{i+2}(x),$$

$$\chi_i(x) = H \psi_i = \mu_{i+2}^{-1/2} H t_{i+2} = \mu_{i+2}^{1/2}\, t_{i+2}(x),$$

hence

$$\Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)} = \sum_3 \nu_i \mu_i t_i(x) t_i(y) = K(x, y) - t_2(x) t_1(y)$$

or

$$K(x, y) - p(x, y) = \Sigma \lambda_i \psi_i(x) \overline{\chi_i(y)},$$

uniformly in $\triangle \times \triangle$, where

$$p(x, y) = t_2(x)\, t_1(y).$$

Evidently

$$\int_\triangle H(x, z)\, p(z, y)\, dz = 0,$$

as required by Theorem 9.

Furthermore, by Theorem 4,

$$\int_\triangle K(x, y) f(y)\, dy = \Sigma \lambda_i a_i \psi_i(x) + \tilde{p}(x),$$

where $a_i = \int_\triangle f(x) \overline{\chi_i(x)}\, dx$ and $H\tilde{p} = \int_\triangle H(x, y)\, \tilde{p}(y)\, dy = 0$. Taking $f(x) = t_1(x)$, we have

$$\int_\triangle K(x, y) f(y)\, dy = K t_1 = A H t_1 = t_2(x)$$

and

$$a_i = (t_1, \chi_i) = \mu_{i+2}^{1/2} (t_1, t_{i+2}) = 0;$$

hence

$$t_2(x) = \tilde{p}(x).$$

Evidently $H\tilde{p} = H t_2 = 0$, as required.

Mathematics. — *Lattice points in n-dimensional star bodies II. (Reducibility Theorems.)* By K. MAHLER. (Fourth communication.) (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of April 27, 1946.)

§ 15. *The star body* $|(x_1^2+x_2^2)x_3|<1$ *in* R_3.

By another theorem of H. DAVENPORT [16], the star body

$$K: \qquad |(x_1^2+x_2^2)x_3| \leqslant 1$$

is of determinant

$$\triangle(K) = \sqrt{(23)}/2.$$

Let α, β be the two *complex conjugate* roots and γ the *real* root of

$$t^3 - t - 1 = 0.$$

Then the lattice Λ_0 defined by

$$\Lambda_0:\ x_1+x_2 i = u_1+\alpha u_2+\alpha^2 u_3,\ x_1-x_2 i = u_1+\beta u_2+\beta^2 u_3,\ x_3 = u_1+\gamma u_2+\gamma^2 u_3$$
$$(u_1, u_2, u_3 = 0, \mp 1, \mp 1, \ldots)$$

is a critical lattice of K, and every other critical lattice Λ of K may be written in the form [17] $\Lambda = \Omega\Lambda_0$ where Ω is one of the automorphisms

$$\Omega: \qquad x_1 = t_1 x_1' - t_2 x_2',\ \ \mp x_2 = t_2 x_1' + t_1 x_2',\ \ x_3 = t_3 x_3'$$

of K, and where t_1, t_2, t_3 are real numbers such that

$$(t_1^2+t_2^2)\,t_3 = \mp 1.$$

Theorem N: *The star body* K: $(x_1^2+x_2^2)\,|x_3| \leq 1$ *is boundedly reducible.*

Proof: It again suffices to show that Λ_0 is a strongly critical lattice of K. Now Λ_0 contains the point

$$P_0 = (1, 0, 1) \qquad (u_1 = 1, u_2 = u_3 = 0)$$

on the boundary of K, and it is evident that every neighbouring point can be transformed, by one of the automorphisms Ω near to the unit automorphism, into a point collinear with O and P_0.

Hence the assertion is proved if we can show the following result: *There exists a bounded star body* K^* *contained in* K *such that*

$$d(\Lambda^*) \geqslant d(\Lambda_0)$$

[16]) Proc. Lond. Math. Soc. **45** (2), 98—125 (1939).

[17]) This result is proved, in slightly different form, in L. J. MORDELL, Journ. Lond. Math. Soc. **17**, 107—115 (1942).

for every K^*-admissible lattice Λ^* which is (i) *sufficiently near to Λ_0, and which* (ii) *contains a point P^* arbitrarily near to P_0 and collinear with O and P_0.*

Now every latice Λ^* near to Λ_0 can be written in the form

$$\Lambda^*: \quad x_1 = v_1 + \frac{\alpha+\beta}{2} v_2 + \frac{\alpha^2+\beta^2}{2} v_3, \; x_2 = \frac{\alpha-\beta}{2i} v_2 + \frac{\alpha^2-\beta^2}{2i} v_3,$$
$$x_3 = v_1 + \gamma v_2 + \gamma^2 v_3,$$

with

$$\begin{aligned} v_1 &= u_1 + (a_{11} u_1 + a_{12} u_2 + a_{13} u_3), \\ v_2 &= u_2 + (a_{21} u_1 + a_{22} u_2 + a_{23} u_3), \qquad (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots), \\ v_3 &= u_3 + (a_{31} u_1 + a_{32} u_2 + a_{33} u_3), \end{aligned}$$

where the coefficients a_{hk} are real numbers such that

$$a = \max_{h, k = 1, 2, 3} |a_{hk}|$$

is less than any given constant. The point P^* corresponding to P_0 is $P^* = (x_1^*, x_2^*, x_3^*)$ where

$$x_1^* = 1 + a_{11} + \frac{\alpha+\beta}{2} a_{21} + \frac{\alpha^2+\beta^2}{2} a_{31}, \; x_2 = \frac{\alpha-\beta}{2i} a_{21} + \frac{\alpha^2-\beta^2}{2i} a_{31},$$
$$x_3 = 1 + a_{11} + \gamma a_{21} + \gamma^2 a_{31},$$

and so P^* is collinear with O and P_0 if and only if

(a): $$a_{21} = a_{31} = 0,$$

because the three points

$$(1, 0, 1), \left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2i}, \gamma\right), \left(\frac{\alpha^2+\beta^2}{2}, \frac{\alpha^2-\beta^2}{2i}, \gamma^2\right)$$

are linearly independent. We consider from now on only lattices satisfying (a).

Put for shortness,

$$\begin{aligned} S(U) &= (u_1 + \alpha u_2 + \alpha^2 u_3)(u_1 + \beta u_2 + \beta^2 u_3)(u_1 + \gamma u_2 + \gamma^2 u_3) = \\ &= (u_1^3 + u_2^3 + u_3^3) + u_3^2 u_1 + (-u_2 u_3^2 + 2 u_3 u_1^2 - u_1 u_2^2) - 3 u_1 u_2 u_3, \end{aligned}$$

so that

$$(x_1^2 + x_2^2) x_3 = S(U)$$

for the point of Λ_0 belonging to $U = (u_1, u_2, u_3)$. For the corresponding point of Λ^*,

$$(x_1^2 + x_2^2) x_3 = S(V),$$

or on replacing $V = (v_1, v_2, v_3)$ by its expression in U,

$$(x_1^2 + x_2^2) x_3 = S(U) + T(U).$$

Here

$$T(U) = (A_1 u_1^3 + A_2 u_2^3 + A_3 u_3^3) + (B_1 u_2^2 u_3 + B_2 u_3^2 u_1 + B_3 u_1^2 u_2) + \\ + (C_1 u_2 u_3^2 + C_2 u_3 u_1^2 + C_3 u_1 u_2^2) + D\, u_1 u_2 u_3,$$

with the coefficients,

$$\begin{aligned}
A_1 &= 3\mathsf{a}_{11} + O(\mathsf{a}^2),\\
A_2 &= 3\mathsf{a}_{22} - \mathsf{a}_{12} + O(\mathsf{a}^2),\\
A_3 &= 3\mathsf{a}_{33} - \mathsf{a}_{23} + \mathsf{a}_{13} + O(\mathsf{a}^2),\\
B_1 &= -3\mathsf{a}_{12} + 3\mathsf{a}_{23} - 2\mathsf{a}_{32} - \mathsf{a}_{13} + O(\mathsf{a}^2),\\
B_2 &= \mathsf{a}_{11} + 2\mathsf{a}_{33} - 3\mathsf{a}_{23} + 4\mathsf{a}_{13} + O(\mathsf{a}^2),\\
B_3 &= 3\mathsf{a}_{12} + 3\mathsf{a}_{32} + O(\mathsf{a}^2),\\
C_1 &= -\mathsf{a}_{22} - 2\mathsf{a}_{33} + \mathsf{a}_{12} + 3\mathsf{a}_{32} - 3\mathsf{a}_{13} + O(\mathsf{a}^2),\\
C_2 &= 4\mathsf{a}_{11} + 2\mathsf{a}_{33} + 3\mathsf{a}_{13} + O(\mathsf{a}^2),\\
C_3 &= -\mathsf{a}_{11} - 2\mathsf{a}_{22} - 3\mathsf{a}_{32} + O(\mathsf{a}^2),\\
D &= -3\mathsf{a}_{11} - 3\mathsf{a}_{22} - 3\mathsf{a}_{33} + 4\mathsf{a}_{12} - 2\mathsf{a}_{23} + 2\mathsf{a}_{32} + O(\mathsf{a}^2).
\end{aligned}$$

In all these formulae, the O-term consists of the products of two or three of the a_{hk}. If

$$A = \max(|A_1|, |A_2|, |A_3|, |B_1|, |B_2|, |B_3|, |C_1|, |C_2|, |C_3|, |D|),$$

then these formulae imply, in particular, that

$$A = O(\mathsf{a}).$$

On solving for the coefficients a_{hk}, we find that

$$\begin{aligned}
3\mathsf{a}_{11} &= A_1 + O(\mathsf{a}^2),\\
69\mathsf{a}_{22} &= 2A_1 + 27A_2 + 9B_3 + 6C_3 + O(\mathsf{a}^2),\\
69\mathsf{a}_{33} &= -A_1 + 27A_3 - 9B_2 + 3C_2 + O(\mathsf{a}^2),\\
23\mathsf{a}_{12} &= 2A_1 + 4A_2 + 9B_3 + 6C_3 + O(\mathsf{a}^2),\\
23\mathsf{a}_{23} &= -11A_1 - 2A_3 - 7B_2 + 10C_2 + O(\mathsf{a}^2),\\
23\mathsf{a}_{32} &= -3A_1 - 6A_2 - 2B_3 - 9C_3 + O(\mathsf{a}^2),\\
23\mathsf{a}_{13} &= -10A_1 - 6A_3 + 2B_2 + 7C_2 + O(\mathsf{a}^2),
\end{aligned}$$

and also obtain the three identities,

$$\begin{aligned}
B_1 &= -A_1 - B_2 - B_3 + C_2 + O(\mathsf{a}^2),\\
C_1 &= A_1 - A_2 - C_2 - C_3 + O(\mathsf{a}^2),\\
D &= -A_2 - A_3 + B_2 + B_3 - C_2 + O(\mathsf{a}^2),
\end{aligned}$$

and the inequality,

$$\mathsf{a} = O(A), \quad O(\mathsf{a}^2) = O(A^2).$$

So far, the star body K^* has not yet been defined. Let then K^* be a star body K^t where t is so large that all points of Λ_0 for which

$$S(U) = 1, \quad |u_1| \leqslant 3, \quad |u_2| \leqslant 3, \quad |u_3| \leqslant 3$$

belong to K^t. Then the ten points of Λ_0 given by

$$U = (1,0,0),\ (0,1,0),\ (0,0,1),\ (0,1,1),\ (-1,0,1),\ (1,1,0),$$
$$(0,-1,1),\ (2,0,-1),\ (-1,1,0),\ (1,1,1),$$

satisfy the equation,

$$S(U) = 1.$$

If Λ^* is K^*-admissible, then the points of Λ^* belonging to the same U cannot be *inner* points of $K^* = K^t$. The numbers

$$\alpha_1, \alpha_2, \alpha_3,\ \beta_1, \beta_2, \beta_3,\ \gamma_1, \gamma_2, \gamma_3,\ \delta$$

defined by

$$\begin{array}{llll} T(1,0,0) = \alpha_1, & T(0,1,1) = \beta_1, & T(0,-1,1) = \gamma_1, & \\ T(0,1,0) = \alpha_2, & T(-1,0,1) = \beta_2, & T(2,0,-1) = \gamma_2, & T(1,1,1) = \delta, \\ T(0,0,1) = \alpha_3, & T(1,1,0) = \beta_3, & T(-1,1,0) = \gamma_3, & \end{array}$$

are then non-negative since

$$(x_1^2 + x_2^2)\,x_3 = S(U) + T(U) = 1 + T(U) \geqslant 1$$

for these points.

Hence, on substituting in $T(U)$,

$$\begin{aligned}
\alpha_1 &= A_1, \\
\alpha_2 &= A_2, \\
\alpha_3 &= A_3, \\
\beta_1 &= A_2 + A_3 + B_1 + C_1, \\
\beta_2 &= -A_1 + A_3 - B_2 + C_2, \\
\beta_3 &= A_1 + A_2 + B_3 + C_3, \\
\gamma_1 &\ \ - A_2 + A_3 + B_1 - C_1, \\
\gamma_2 &= 8A_1 - A_3 + 2B_2 - 4C_2, \\
\gamma_3 &= -A_1 + A_2 + B_3 - C_3, \\
\delta &= A_1 + A_2 + A_3 + B_1 + B_2 + B_3 + C_1 + C_2 + C_3 + D,
\end{aligned}$$

and conversely,

$$\begin{aligned}
A_1 &= \alpha_1, \\
A_2 &= \alpha_2, \\
A_3 &= \alpha_3, \\
B_1 &= -\alpha_3 + \tfrac{1}{2}\beta_1 + \tfrac{1}{2}\gamma_1, \\
B_2 &= 2\alpha_1 + \tfrac{3}{2}\alpha_3 - 2\beta_2 - \tfrac{1}{2}\gamma_2, \\
B_3 &= -\alpha_2 + \tfrac{1}{2}\beta_3 + \tfrac{1}{2}\gamma_3, \\
C_1 &= -a_2 + \tfrac{1}{2}\beta_1 - \tfrac{1}{2}\gamma_1, \\
C_2 &= 3\alpha_1 + \tfrac{1}{2}\alpha_3 - \beta_2 - \tfrac{1}{2}\gamma_2, \\
C_3 &= -\alpha_1 + \tfrac{1}{2}\beta_3 - \tfrac{1}{2}\gamma_3, \\
D &= -5\alpha_1 + \alpha_2 - 2\alpha_3 - \beta_1 + 3\beta_2 - \beta_3 + \gamma_2 + \delta.
\end{aligned}$$

We deduce from these formulae that

$$\begin{aligned}\beta_1 &= -\alpha_1 + \alpha_2 - \tfrac{1}{2}\alpha_3 + 2\beta_2 - \beta_3 + \tfrac{1}{2}\gamma_2 + O(a^2),\\ \gamma_1 &= \alpha_1 + \alpha_2 + \tfrac{1}{2}\alpha_3 - \tfrac{1}{2}\gamma_2 - \gamma_3 + O(a^2),\\ \delta &= 3\alpha_1 - 2\alpha_2 + \tfrac{3}{2}\alpha_3 - 2\beta_2 + \tfrac{1}{2}\beta_3 - \tfrac{1}{2}\gamma_2 + \tfrac{1}{2}\gamma_3 + O(a^2).\end{aligned}$$

Hence, if

$$\alpha = \max(|\alpha_1|, |\alpha_2|, |\alpha_3|, |\beta_1|, |\beta_2|, |\beta_3|, |\gamma_1|, |\gamma_2|, |\gamma_3|, |\delta|),$$

then all three numbers a, A, α are of the same order,

$$\alpha = O(a) = O(A), \quad O(a^2) = O(\alpha^2), \quad O(A^2) = O(\alpha^2).$$

Finally, the lattice Λ^* is of determinant

$$d(\Lambda^*) = d(\Lambda_0) \begin{vmatrix} 1 + a_{11} & a_{12} & a_{13} \\ 0 & 1 + a_{22} & a_{23} \\ 0 & a_{32} & 1 + a_{33} \end{vmatrix} = d(\Lambda_0)(1 + \sigma),$$

and here

$$\begin{aligned}\sigma &= a_{11} + a_{22} + a_{33} + O(a^2),\\ &= \tfrac{1}{23}(8A_1 + 9A_2 + 9A_3 - 3B_2 + 3B_3 + C_2 + 2C_3) + O(A^2),\\ &= \tfrac{1}{23}(3\alpha_1 + 6\alpha_2 + 5\alpha_3 + 5\beta_2 + \tfrac{5}{2}\beta_3 + \gamma_2 + \tfrac{1}{2}\gamma_3) + O(\alpha^2).\end{aligned}$$

We find therefore, just as in the last proof, that either

$$\sigma > 0, \quad d(\Lambda^*) > d(\Lambda_0),$$

or

$$\sigma = 0, \quad d(\Lambda^*) = d(\Lambda_0),$$

and that the second case can hold only if $\alpha = a = A = 0$, that is, if Λ^* coincides with Λ_0; whence the assertion.

§ 16. *Some further examples.*

I have applied the method of the last paragraphs to three further star bodies in R_3 and R_4. From the well-known results of A. OPPENHEIM [18]) on the minima of the indefinite quadratic forms in three and four variables, I have so deduced that

the star body K_1: $|x_1^2 + x_2^2 - x_3^2| \leq 1$ in R_3 with $\triangle(K_1) = \sqrt{\tfrac{3}{2}}$,
the star body K_2: $|x_1^2 + x_2^2 + x_3^2 - x_4^2| \leq 1$ in R_4 with $\triangle(K_2) = \sqrt{\tfrac{7}{4}}$,
and the star body K_3: $|x_1^2 + x_2^2 - x_3^2 - x_4^2| \leq 1$ in R_4 with $\triangle(K_3) = \tfrac{3}{2}$,

are each one boundedly reducible. As before, Theorem L is the basis of the proof; since no new ideas are used, I omit this proof.

18) See L. E. DICKSON, Studies in the theory of numbers (Chicago 1930), chapters 8 and 9.

In all these examples of boundedly reducible star bodies, it would be of great interest to obtain irreducible star bodies of equal determinants contained in them.

§ 17. *Applications.*

The following theorems show that the preceeding results can be useful for other purposes.

Theorem O: *There exists a positive constant c with the following property: If α_1, α_2 are real numbers, and t is a positive number, then there exist integers u_1, u_2, u_3 not all zero such that*

$$\{(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2\} \, |u_3| \leqslant \frac{2}{\sqrt{23}},$$

$$(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2 \leqslant \frac{c}{t}, \quad |u_3| \leqslant t.$$

Hence if α_1, α_2 are irrational, then there are arbitrarily large integers u_1, u_2, u_3 such that [19])

$$\left(\frac{u_1}{u_3} - \alpha_1\right)^2 + \left(\frac{u_2}{u_3} - \alpha_2\right)^2 \leqslant \frac{2}{\sqrt{23}\,|u_3|^3}.$$

Proof: By Theorem N, a positive number r exists such that

K^*: $\qquad (x_1^2 + x_2^2)\,|x_3| \leqslant 1, \quad x_1^2 + x_2^2 + x_3^2 \leqslant r^2$

is of the same determinant as

K: $\qquad (x_1^2 + x_2^2)\,|x_3| \leqslant 1,$

namely $\Delta(K) = \Delta(K^*) = \sqrt{(23)}/2$. On applying the transformation

Ω: $\qquad x_1 = \tau x_1', \; x_2 = \tau x_2', \; x_3 = \tau^{-2} x_3' \quad (\tau > 0),$

of K, we find that

K_τ^*: $\qquad (x_1^2 + x_2^2)\,|x_3| \leqslant 1, \quad \tau^2 (x_1^2 + x_2^2) + \tau^{-4} x_3^2 \leqslant r^2$

is likewise of determinant $\Delta(K_\tau^*) = \Delta(K) = \sqrt{(23)}/2$. Let Λ be the lattice

$$\Lambda: \; x_1 = u_1 - \alpha_1 u_3, \; x_2 = u_2 - \alpha_2 u_3, \; x_3 = \frac{\sqrt{23}}{2} u_3 \;\; (u_1, u_2, u_3 = 0, \mp 1, \mp 2, \ldots).$$

Since $d(\Lambda) = \sqrt{(23)}/2$, at least one point $P \neq O$ of Λ lies inside or on the boundary of K_τ^*; let this be the point belonging to the integers u_1, u_2, u_3 not all zero. Then

$$\{(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2\}\,|u_3| \leqslant \frac{2}{\sqrt{23}},$$

$$\tau^2 \{(u_1 - \alpha_1 u_3)^2 + (u_2 - \alpha_2 u_3)^2\} + \frac{23}{4\tau^4} u_3^2 \leqslant r^2.$$

---

[19]) A slightly less exact result is proved in a joint paper by DAVENPORT and myself, in DUKE Math. Journal **13**, 105—111 (1946).

hence

$$(u_1-a_1 u_3)^2+(u_2-a_2 u_3)^2 \leqslant \frac{r^2}{\tau^2}, \quad |u_3| \leqslant \frac{2\tau^2 r}{\sqrt{23}}.$$

If now

$$c=\frac{2r^3}{\sqrt{23}}, \quad \tau=\left(\frac{23 t^2}{4 r^2}\right)^{1/4},$$

then

$$(u_1-a_1 u_3)^2+(u_2-a_2 u_3)^2 \leqslant \frac{c}{t}, \quad |u_3| \leqslant t,$$

as asserted. — Assume next that a_1 is irrational and that t tends to infinity. Then u_3 is different from zero for sufficiently large t, and since $|u_1 - a_1 u_3|$ tends to zero, $|u_3|$ tends to infinity.

In a similar way, Theorem M leads to the following result:

Theorem P: *There exists a positive constant γ with the following property: If β_1, β_2 are real numbers and t is a positive number, then integers v_1, v_2, v_3 not all zero exist such that*

$$|v_1 v_2 (\beta_1 v_1 + \beta_2 v_2 + v_3)| \leqslant \tfrac{1}{7},$$
$$|v_1| \leqslant t, \quad |v_2| \leqslant t, \quad |\beta_1 v_1 + \beta_2 v_2 + v_3| \leqslant \frac{\gamma}{t^2}.$$

Assume further that β_1, β_2 have the following stronger properties: (i) *β_1, β_2, 1 are linearly independent over the rational field.* (ii) *When the integers v, v', v'' tend to infinity in any way, then*

$$\lim v^2 |\beta_1 v + v'| = \infty, \quad \lim v^2 |\beta_2 v + v''| = \infty.$$

Under these conditions, there exist an infinity of triples of integers v_1, v_2, v_3 all different from zero such that

$$0 < |\beta_1 v_1 + \beta_2 v_2 + v_3| \leqslant \frac{1}{7|v_1 v_2|}.$$

The results on boundedly reducible star bodies are also of use for obtaining asymptotic formulae for the determinants of certain star bodies depending on a parameter [20]). For instance, it is easy to deduce from Theorem M that when $a > 0$ tends to zero, then the star body

K_1: $|x_1|^\alpha + |x_2|^\alpha + |x_3|^\alpha \leqslant 1$ is of determinant $\triangle(K_1) = \frac{1}{7} e^{-2/\alpha}(1+O(\alpha))$,

and the star body

K_2: $(|x_1|^\alpha + |x_2|^\alpha)|x_3|^{\alpha/2} \leqslant 1$ is of determinant $\triangle(K_2) = \frac{1}{7} e^{-1/\alpha}(1+O(\alpha))$.

I remark, finally, that the just given examples of boundedly reducible star bodies in R_3 and R_4 are all automorphic, and even satisfy the stronger

[20]) For a special case, see my paper Proc. Cambr. Phil. Soc. **40**, 116—120 (1944), in particular the proof of Theorem 4.

conditions of Theorem 23 of Part I. This suggests that the following problem has an affirmative answer:

Problem 10: *Is it true that every automorphic star body is boundedly reducible if it satisfies the conditions of Theorem 23 of Part I?*

§ 18. *An addition to Theorem 9 of Part I.*

In Theorem 9 of Part I, $\triangle(K)$ was proved to depend continuously on K if K varied in a rather restricted way. To conclude this Part II, we prove a more general continuity property of $\triangle(K)$.

Theorem Q: *Let $F(X)$ and $F_r(X)$ $(r=1, 2, 3, \ldots)$ be distance functions in R_n such that*

$$\lim_{r\to\infty} F_r(X) = F(X)$$

uniformly in X on the unit sphere $|X|=1$ [21]*; and let the star bodies*

$$K:\ F(X) \leqslant 1, \quad \text{and} \quad K_r:\ F_r(X) \leqslant 1 \quad (r=1,2,3,\ldots)$$

be of the finite type. Then

$$\liminf_{r\to\infty} \triangle(K_r) \geqslant \triangle(K).$$

Proof: Let $\varepsilon > 0$ be arbitrarily small. By the Corollary to Theorem 10 of Part I, there exists a positive number t such that the determinant of the star body

$$K^t: \qquad F(X) \leqslant 1, \qquad |X| \leqslant t$$

satisfies the inequalities,

$$(1-\varepsilon)\triangle(K) \leqslant \triangle(K^t) \leqslant \triangle(K).$$

There is further an integer $r_0 = r_0(\varepsilon)$ such that

$$F_r(X) \leqslant 1+\varepsilon \text{ for the points } X \text{ of } K^t \qquad \text{if } r \geqslant r_0;$$

hence K^t is contained in $(1+\varepsilon)K_r$ if $r \geqq r_0$. This implies

$$\triangle(K^t) \leqslant (1+\varepsilon)^n \triangle(K_r) \qquad \text{if } r \geqslant r_0,$$

whence

$$\triangle(K_r) \geqslant (1+\varepsilon)^{-n} \triangle(K^t) \geqslant \frac{1-\varepsilon}{(1+\varepsilon)^n} \triangle(K) \qquad \text{if } r \geqslant r_0.$$

For $\varepsilon \to 0$, the assertion is obtained.

In the result

$$\liminf_{r\to\infty} \triangle(K_r) \geqslant \triangle(K)$$

of Theorem Q, the sign "$\geqq$" cannot always be replaced by the equality sign, as the following example shows.

---

[21]) This implies the uniform convergence in every bounded set.

Theorem R: *For every dimension n, there exist star bodies K and K_r $(r = 1, 2, 3, \ldots)$ in R_n satisfying the hypothesis of Theorem Q, but such that*

$$\lim_{r\to\infty} \triangle(K_r) \text{ exists and is greater then } \triangle(K).$$

Proof: Denote by $c > 0$ a constant which is so large that the sphere

$$H: \qquad |X| \leqslant c$$

is of greater determinant than the star body

$$K: \qquad F(X) \leqslant 1, \qquad \text{where} \quad F(X) = |x_1 x_2 \ldots x_n|^{1/n};$$

denote further by $r = 1, 2, 3, \ldots$ the sequence of all positive integers. The distance function

$$F_r(X) = \min \left\{ F(X), \frac{1}{c} \left(\frac{x_1^2 + \ldots + x_{n-1}^2}{r^2} + r^{2(n-1)} x_n^2 \right)^{1/2} \right\}$$

defines a star body K_r: $F_r(X) \leq 1$ which contains K and is easily seen to be of the finite type. The automorphisms of K,

$$\Omega_r: \qquad x_1 = r x_1', \ldots, x_{n-1} = r x_{n-1}', \; x_n = r^{-(n-1)} x_n',$$

change K_r into K_1; hence

$$\triangle(K_1) = \triangle(K_2) = \triangle(K_3) = \ldots = \lim_{r\to\infty} \triangle(K_r).$$

On the other hand,

$$\triangle(K_r) \geqslant \triangle(H) > \triangle(K)$$

since K_r contains H; hence

$$\lim_{r\to\infty} \triangle(K_r) > \triangle(K).$$

Consider now $F_r(X)$ on the unit sphere $|X| = 1$. Here

$$F(X) \leqslant 1, \text{ and } \frac{1}{c} \left(\frac{x_1^2 + \ldots + x_{n-1}^2}{r^2} + r^{2(n-1)} x_n^2 \right)^{1/2} \geqslant \frac{r^{n-1} |x_n|}{c},$$

and so

$$F_r(X) = F(X) \text{ unless } |x_n| \leqslant c\, r^{-(n-1)}.$$

If further

$$|x_n| \leqslant c\, r^{-(n-1)}, \quad |X| = 1,$$

then

$$F(X) \leqslant (c\, r^{-(n-1)})^{1/n}, \quad 0 \leqslant F_r(X) \leqslant F(X),$$

whence

$$|F_r(X) - F(X)| \leqslant F(X) \leqslant (c\, r^{-(n-1)})^{1/n},$$

and here the right-hand side tends to zero uniformly in X, as asserted.

Theorem Q leaves many interesting questions unsolved. For instance, the star domain

$$K_\lambda: \qquad |(x_1^2-x_2^2)(x_1^2-\lambda x_2^2)| \leq 1$$

is easily proved to be of the finite type; is $\triangle(K_\lambda)$ a continuous function of λ?

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Mathematics Department,
University of Manchester.

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(Communicated at the meeting of April 27, 1946.)

§ 10. Second expansion formula.

Theorem 2. Assumptions: *k, l, m, n, p and q are integers with*

$$q \geqq 1, \quad 0 \leqq l-1 \leqq n \leqq p \leqq q \text{ and } 0 \leqq m \leqq k \leqq q; \quad . \quad . \quad (111)$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_k$ *fulfil the conditions* (1), (99) *and* (100); r *is an arbitrary integer which satisfies the inequality*

$$r \geqq \operatorname{Max}(0, k+l-m-n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

Assertion:

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) &= A^{m,n-l+1}_{\ \ \ k} \sum_{s=0}^{r-1} \Omega^{m,n-l+1}_{\ \ \ k}(s)\, G_{p,q}^{k,l-1,n}\left(z\,e^{(k+l-m-n-2s-1)\pi i}\right) \\ &+ \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \triangle^{m,n-l+1}_{\ \ \ k}(t)\, G_{p,q}^{k,l,n}\left(z\,e^{(k+l-m-n-2r)\pi i} \,\|\, a_t\right). \end{aligned}\right\} (113)$$

Proof. We may distinguish three cases:

First case:

$$1 \leqq l \leqq n \leqq p \leqq q, \quad k+l-n \leqq m \leqq k \leqq q, \quad r \geqq 0.$$

Formula (113) can be established by induction. If $r=0$, then (113) reduces to (102) with $\lambda = 0$. We may therefore suppose $r \geqq 1$ and assume that (113) with $r-1$ instead of r has yet been proved.

Now we have by (58)

$$\begin{aligned} G_{p,q}^{k,l,n}\left(z\,e^{(k+l-m-n-2r+2)\pi i} \,\|\, a_t\right) &= e^{2\pi i a_t} G_{p,q}^{k,l,n}\left(z\,e^{(k+l-m-n-2r)\pi i} \,\|\, a_t\right) \\ &\quad -2\pi i\, e^{\pi i a_t} G_{p,q}^{k,l-1,n}\left(z\,e^{(k+l-m-n-2r+1)\pi i}\right). \end{aligned}$$

If this is substituted on the right-hand side of (113) with $r-1$ instead of r, the sum $\sum\limits_{t=1}^{n-l+1}$ not only gives the corresponding sum in (113) but also

$$-2\pi i\, G_{p,q}^{k,l-1,n}\left(z\,e^{(k+l-m-n-2r+1)\pi i}\right) \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r-1)\pi i a_t} \triangle^{m,n-l+1}_{\ \ \ k}(t)$$

and this expression may by means of (59) be reduced to

$$A^{m,n-l+1}_{\ \ \ k}\, \Omega^{m,n-l+1}_{\ \ \ k}(r-1)\, G_{p,q}^{k,l-1,n}\left(z\,e^{(k+l-m-n-2r+1)\pi i}\right),$$

since $\Omega^{m,n-l+1}_{\ \ \ k}(k+l-m-n-r) = 0$, because of $k+l-m-n-r \leqq -1$.

It follows therefore that the sum $\sum\limits_{s=0}^{r-2}$ on the right-hand side of (113) with $r-1$ instead of r reduces to the sum $\sum\limits_{s=0}^{r-1}$ in (113). Thus the first case is finished.

Second case:

$$1 \leqq l \leqq n \leqq p \leqq q,\ 0 \leqq m \leqq k+l-n,\ r \geqq k+l-m-n.$$

This case may also be proved by induction. Owing to the first case formula (113) is true if $m = k+l-n$. We may therefore suppose $0 \leq m \leq k+l-n-1$ and assume that (113) with $m+1$ instead of m has yet been proved.

Now it follows from (113) with $m+1$ instead of m, $z\,e^{-\pi i}$ instead of z, $r-1$ instead of r and s replaced by $s-1$

$$e^{\pi i b_{m+1}}\, G_{p,q}^{m+1,n}(z\,e^{-\pi i}) = e^{\pi i b_{m+1}}\, A^{m+1,n-l+1}_{k} \sum_{s=1}^{r-1} \Omega^{m+1,n-l+1}_{k}(s-1)\, G_{p,q}^{k,l-1,n}(z e^{(k+l-m-n-2s-1)\pi i})$$
$$+ \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t}\, e^{\pi i (b_{m+1}-a_t)}\, \triangle^{m+1,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}(z\, e^{(k+l-m-n-2r)\pi i} \,\|\, \mathbf{a}_t).$$

We further have by (113) with $m+1$ instead of m and $z\,e^{\pi i}$ instead of z

$$e^{-\pi i b_{m+1}}\, G_{p,q}^{m+1,n}(z\,e^{\pi i}) = e^{-\pi i b_{m+1}}\, A^{m+1,n-l+1}_{k} \sum_{s=0}^{r-1} \Omega^{m+1,n-l+1}_{k}(s)\, G_{p,q}^{k,l-1,n}(z\, e^{(k+l-m-n-2s-1)\pi i})$$
$$+ \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t}\, e^{\pi i (a_t-b_{m+1})}\, \triangle^{m+1,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}(z\, e^{(k+l-m-n-2r)\pi i} \,\|\, \mathbf{a}_t).$$

From these two relations and (55) it appears

$$\left.\begin{aligned}
G_{p,q}^{m,n}(z) = &-\frac{e^{-\pi i b_{m+1}}}{2\pi i}\, A^{m+1,n-l+1}_{k}\, \Omega^{m+1,n-l+1}_{k}(0)\, G_{p,q}^{k,l-1,n}(z\,e^{(k+l-m-n-1)\pi i}) \\
&-\frac{e^{-\pi i b_{m+1}}}{2\pi i}\, A^{m+1,n-l+1}_{k} \sum_{s=1}^{r-1} \{ \Omega^{m+1,n-l+1}_{k}(s) - e^{2\pi i b_{m+1}}\, \Omega^{m+1,n-l+1}_{k}(s-1) \} \times \\
&\qquad\qquad \times G_{p,q}^{k,l-1,n}(z\,e^{(k+l-m-n-2s-1)\pi i}) \\
&+\frac{1}{\pi} \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \sin(b_{m+1}-a_t)\pi \,.\, \triangle^{m+1,n-l+1}_{k}(t)\, G_{p,q}^{k,l,n}(z e^{(k+l-m-n-2r)\pi i} \,\|\, \mathbf{a}_t).
\end{aligned}\right\} \quad (114)$$

Now it is obvious on account of the definition of the coefficients A

$$-\frac{e^{-\pi i b_{m+1}}}{2\pi i}\, A^{m+1,n-l+1}_{k} = A^{m,n-l+1}_{k}.$$

Moreover we find without difficulty in view of the definition of the coefficients Ω

$$\Omega^{m+1,n-l+1}_{k}(0) = \Omega^{m,n-l+1}_{k}(0)$$

and

$$\Omega^{m+1,n-l+1}_{\ \ k}(s)-e^{2\pi i b_{m+1}}\,\Omega^{m+1,n-l+1}_{\ \ k}(s-1)=\Omega^{m,n-l+1}_{\ \ k}(s).$$

Finally it follows from the definition of the coefficients $\triangle$ that

$$\frac{1}{\pi}\sin(b_{m+1}-a_l)\,\pi\,.\,\triangle^{m+1,n-l+1}_{\ \ k}(t)=\triangle^{m,n-l+1}_{\ \ k}(t).$$

Formula (114) is therefore equivalent to (113). So the second case is also finished.

Third case:

$q\geqq 1,\ n=l-1,\ \ 0\leqq l-1\leqq p\leqq q,\ \ 0\leqq m\leqq k\leqq q,\ \ r\geqq 1+k-m.$

From the definition of the function G we easily deduce

$$G^{m,l-1}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)=G^{m,l}_{p+1,q+1}\left(z\,\middle|\,\begin{matrix}\alpha,a_1,\dots,a_p\\ b_1,\dots,b_k,\alpha,b_{k+1},\dots,b_q\end{matrix}\right); \quad (115)$$

herein is α an arbitrary number.

To the function $G^{m,l}_{p+1,q+1}(z)$ on the right-hand side of this relation we may apply (113) with $n=l$, $k+1$ instead of k, $p+1$ instead of p and $q+1$ instead of q. Now it is clear, on account of the definitions of the coefficients A, Ω and $\triangle$ and the function G, that in the particular case under consideration ($\alpha, a_1, \dots, a_p$ instead of $a_1, a_2, \dots, a_{p+1}$ and $b_1, \dots, b_k, \alpha, b_{k+1}, \dots, b_q$ instead of $b_1, \dots, b_{q+1}$)

$$A^{m,1}_{\ \ k+1}=A^{m,0}_{\ \ k},\ \ \Omega^{m,1}_{\ \ k+1}(s)=\Omega^{m,0}_{\ \ k}(s),$$
$$G^{k+1,l-1,l}_{p+1,q+1}(\zeta)=G^{k,l-1,l-1}_{p,q}(\zeta)$$

and

$$\triangle^{m,1}_{\ \ k+1}(1)=0.$$

We therefore get (113) with $n=l-1$ when we apply (113) to the right-hand side of (115).

With this the theorem has been completely proved.

§ 11. Third expansion formula.

Theorem 3. Assumptions: *k, l, m, n, p and q are integers with*

$$q\geqq 1,\ 0\leqq l-1\leqq n\leqq p\leqq q,\ 0\leqq m\leqq k\leqq q \text{ and } m+n\leqq k+l;$$

the numbers $a_1, \dots, a_n$ and $b_1, \dots, b_k$ fulfil the conditions (1), (99) *and* (100); *r is an arbitrary integer which satisfies the inequality*

$$0\leqq r\leqq k+l-m-n.$$

Assertion:

$$
\left.\begin{aligned}
G^{m,n}_{p,q}(z) = {} & A^{m,n-l+1}_{k} \sum_{s=0}^{r-1} \Omega^{m,n-l+1}_{k}(s)\, G^{k,l-1,n}_{p,q}\left(z\, e^{(k+l-m-n-2s-1)\pi i}\right) \\
& + \overline{A}^{m,n-l+1}_{k} \sum_{\tau=0}^{k+l-m-n-r-1} \overline{\Omega}^{m,n-l+1}_{k}(\tau)\, G^{k,l-1,n}_{p,q}\left(z\, e^{(m+n-k-l+2\tau+1)\pi i}\right) \\
& + \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \triangle^{m,n-l+1}_{k}(t)\, G^{k,l,n}_{p,q}\left(z\, e^{(k+l-m-n-2r)\pi i} \,\|\, a_t\right).
\end{aligned}\right\} \quad (116)
$$

Proof. The theorem can be established by induction. The formula is true if $r = k + l - m - n$, since (116) with $r = k + l - m - n$ is equivalent to (113) with $r = k + l - m - n$. We may therefore suppose $0 \leqq r \leqq k + l - m - n - 1$ and assume that (116) with $r + 1$ instead of r has yet been proved. Now it follows from (57), if $n \geqq l$,

$$
\begin{aligned}
G^{k,l,n}_{p,q}\left(z\, e^{(k+l-m-n-2r-2)\pi i} \,\|\, a_t\right) = {} & e^{-2\pi i a_t}\, G^{k,l,n}_{p,q}\left(z\, e^{(k+l-m-n-2r)\pi i} \,\|\, a_t\right) \\
& + 2\pi i\, e^{-\pi i a_t}\, G^{k,l-1,n}_{p,q}\left(z\, e^{(k+l-m-n-2r-1)\pi i}\right).
\end{aligned}
$$

If this is substituted on the right-hand side of (116) with $r + 1$ instead of r, the sum $\sum\limits_{t=1}^{n-l+1}$ not only yields the sum $\sum\limits_{t=1}^{n-l+1}$ in (116) but besides

$$
2\pi i\, G^{k,l-1,n}_{p,q}\left(z\, e^{(k+l-m-n-2r-1)\pi i}\right) \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r+1)\pi i a_t} \triangle^{m,n-l+1}_{k}(t) \quad {}^{36)}
$$

and this expression is by (59) equal to

$$
\begin{aligned}
-G^{k,l-1,n}_{p,q}\left(z\, e^{(k+l-m-n-2r-1)\pi i}\right) \big\{ & A^{m,n-l+1}_{k}\, \Omega^{m,n-l+1}_{k}(r) \\
& - \overline{A}^{m,n-l+1}_{k}\, \overline{\Omega}^{m,n-l+1}_{k}(k+l-m-n-r-1) \big\}.
\end{aligned}
$$

The sums $\sum\limits_{s=0}^{r}$ and $\sum\limits_{\tau=0}^{k+l-m-n-r-2}$ on the right-hand side of (116) with $r + 1$ instead of r reduce therefore to the sums $\sum\limits_{s=0}^{r-1}$, respect. $\sum\limits_{\tau=0}^{k+l-m-n-r-1}$ in (116). So the theorem is established.

§ 12. Extension of theorem 3.

In the same manner as formula (113) we may prove the formula conjugate to (113)

$$
\left.\begin{aligned}
G^{m,n}_{p,q}(z) = {} & \overline{A}^{m,n-l+1}_{k} \sum_{s=0}^{r-1} \overline{\Omega}^{m,n-l+1}_{k}(s)\, G^{k,l-1,n}_{p,q}\left(z\, e^{(m+n-k-l+2s+1)\pi i}\right) \\
& + \sum_{t=1}^{n-l+1} e^{(k+l-m-n-2r)\pi i a_t} \triangle^{m,n-l+1}_{k}(t)\, G^{k,l,n}_{p,q}\left(z\, e^{(m+n-k-l+2r)\pi i} \,\|\, a_t\right).
\end{aligned}\right\} \quad (117)
$$

---

36) This is still true if $n = l - 1$, since the sums $\sum\limits_{t=1}^{n-l+1}$ then vanish.

This relation holds, provided that the conditions (111), (112), (1), (99) and (100) are satisfied.

We now replace r by $k+l-m-n-r$ and s by τ. Then formula (117) reduces to

$$\left.\begin{aligned} G^{m,n}_{p,q}(z) = {} & \overline{A}^{m,n-l+1}_{k} \sum_{\tau=0}^{k+l-m-n-r-1} \overline{\Omega}^{m,n-l+1}_{k}(\tau)\, G^{k,l-1,n}_{p,q}\left(z e^{(m+n-k-l+2\tau+1)\pi i}\right) \\ & + \sum_{t=1}^{n-l+1} e^{(m+n-k-l+2r)\pi i a_t} \triangle^{m,n-l+1}_{k}(t)\, G^{k,l,n}_{p,q}\left(z\, e^{(k+l-m-n-2r)\pi i} \,\|\, a_t\right); \end{aligned}\right\} \quad (118)$$

herein is r an arbitrary integer which satisfies the inequality

$$r \leqq \mathrm{Min}\,(0, k+l-m-n).$$

We may now show that formula (116) holds under conditions which are much more general than those of theorem 3. Indeed, I will prove:

Theorem 4. *Suppose that k, l, m, n, p and q are integers which satisfy the conditions* (111); *further that the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ fulfil the conditions* (1), (99) *and* (100); *finally that r is an arbitrary integer (positive, negative or zero).*

Then formula (116) *is valid.*

Proof. Observing that $\Omega^{m,n-l+1}_{k}(s)$ and $\overline{\Omega}^{m,n-l+1}_{k}(s)$ vanish for $s=-1,-2,-3,\ldots$, we may distinguish six cases [37]):

First case: $m+n \geqq k+l$, $r \geqq 0$. Formula (116) reduces to (113).

Second case: $m+n \geqq k+l$, $k+l-m-n \leqq r \leqq 0$. Formula (116) reduces to (102) with $\lambda = -r$.

Third case: $m+n \geqq k+l$, $r \leqq k+l-m-n$. Formula (116) reduces to (118).

Fourth case: $m+n \leqq k+l$, $r \geqq k+l-m-n$. Formula (116) reduces to (113).

Fifth case: $m+n \leqq k+l$, $0 \leqq r \leqq k+l-m-n$. This is the case of theorem 3.

Sixth case: $m+n \leqq k+l$, $r \leqq 0$. Formula (116) reduces to (118).

§ 13. Some more lemmas.

Lemma 19. *Suppose that $k, l, p, q, \varkappa$ and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ \varkappa \geqq 1,\ 0 \leqq \nu \leqq k \leqq q \ \text{ and } \ l+\nu-1 \leqq p \leqq q;$$

suppose further that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions

$$a_j - b_h \neq 1, 2, 3, \ldots \ (j = \nu+1, \ldots, l+\nu-1;\ h = 1, \ldots, k). \quad (119)$$

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \ (j = 1, \ldots, \nu;\ t = 1, \ldots, \nu;\ j \neq t). \quad (120)$$

[37]) Comp. also definition 4.

Then the following formula holds [38]:

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) = &- \sum_{h=1}^{k-\nu+\varkappa-1} \Omega^{0,\nu}_{k}(h)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{-2h\pi i}) \\ &- \frac{1}{A^{0,\nu}_{k}} \sum_{\tau=1}^{\nu} e^{(k-\nu+2\varkappa-1)\pi i a_\tau} \triangle^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \,\|\, a_\tau). \end{aligned}\right\} \quad (121)$$

Proof. If we put $m = 0$, $n = l + \nu - 1$, $r = k - \nu + \varkappa$ and $z = \zeta\, e^{(\nu-k)\pi i}$ in (113) and suppose that $\nu \leqq k$, then we find (121), because of (5) and (50).

Lemma 20. *Suppose that $k, l, p, q, \varkappa, \lambda$ and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ 1 \leqq \lambda \leqq \varkappa,\ 0 \leqq \nu \leqq k \leqq q \quad \textit{and} \quad l + \nu - 1 \leqq p \leqq q;$$

suppose further that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) and (120).

Then the following formula holds:

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) = &\sum_{h=1}^{k-\nu+\varkappa-\lambda} \Phi^{k,0}_{\nu,k}(h;\lambda)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2h-2\lambda+2)\pi i}) \\ - \frac{1}{A^{0,\nu}_{k}} \sum_{\tau=0}^{\lambda-1} \Phi^{k,0}_{\nu,k}(1;\tau) &\sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-2\tau-1)\pi i a_\sigma} \triangle^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \quad (122)$$

Proof. From (74) (with $\lambda = 1$) and (73) (with $\lambda = 0$) it follows

$$\Phi^{k,0}_{\nu,k}(h;1) = -\Omega^{0,\nu}_{k}(h)\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

We further have by (74) if $\lambda \neq 1 - h$

$$\Phi^{k,0}_{\nu,k}(h;\lambda) = -\sum_{\tau=0}^{\lambda-2} \Phi^{k,0}_{\nu,k}(1;\tau)\, \Omega^{0,\nu}_{k}(h+\lambda-\tau-1) - \Phi^{k,0}_{\nu,k}(1;\lambda-1)\, \Omega^{0,\nu}_{k}(h);$$

in view of (74) we find therefore if $\lambda \neq 1 - h$

$$\Phi^{k,0}_{\nu,k}(h;\lambda) = \Phi^{k,0}_{\nu,k}(h+1;\lambda-1) - \Phi^{k,0}_{\nu,k}(1;\lambda-1)\, \Omega^{0,\nu}_{k}(h). \quad . \quad (124)$$

From (123) and (73) (with $\lambda = 0$) it appears that (122) with $\lambda = 1$ reduces to (121). Hence we may suppose $2 \leqq \lambda \leqq \varkappa$ and assume that (122) with $\lambda - 1$ instead of λ has already been proved.

Now formula (122) with $\lambda - 1$ instead of λ may be written in the following way

$$\left.\begin{aligned} G_{p,q}^{k,l-1,l+\nu-1}(\zeta) = &\ \Phi^{k,0}_{\nu,k}(1;\lambda-1)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2\lambda+2)\pi i}) \\ &+ \sum_{h=1}^{k-\nu+\varkappa-\lambda} \Phi^{k,0}_{\nu,k}(h+1;\lambda-1)\, G_{p,q}^{k,l-1,l+\nu-1}(\zeta\, e^{(-2h-2\lambda+2)\pi i}) \\ - \frac{1}{A^{0,\nu}_{k}} \sum_{\tau=0}^{\lambda-2} \Phi^{k,0}_{\nu,k}(1;\tau) &\sum_{\sigma=1}^{\nu} e^{(k-\nu+2\varkappa-2\tau-1)\pi i a_\sigma} \triangle^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(\zeta\, e^{(2\nu-2k-2\varkappa+1)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \quad (125)$$

[38]) The products $\triangle^{0,\nu}_{k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(w \,\|\, a_\sigma)$ on the right of (121) must be defined by a limiting process when $a_\sigma - b_h = 1, 2, 3, \ldots$ $(1 \leqq h \leqq k)$; comp. the Remark at the end of § 9.

The first term on the right-hand side of this relation is because of (121) with $\zeta e^{(-2\lambda+2)\pi i}$ instead of ζ and $\varkappa-\lambda+1$ instead of $\varkappa$ equal to

$$\Phi^{k,0}_{\nu,k}(1;\lambda-1)\,G^{k,l-1,l+\nu-1}_{p,q}(\zeta\,e^{(-2\lambda+2)\pi i})$$

$$=-\sum_{h=1}^{k-\nu+\varkappa-\lambda}\Phi^{k,0}_{\nu,k}(1;\lambda-1)\,\Omega^{0,\nu}_{k}(h)\,G^{k,l-1,l+\nu-1}_{p,q}(\zeta\,e^{(-2h-2\lambda+2)\pi i})$$

$$-\frac{\Phi^{k,0}_{\nu,k}(1;\lambda-1)}{A^{0,\nu}_{k}}\sum_{\sigma=1}^{\nu}e^{(k-\nu+2\varkappa-2\lambda+1)\pi i a_\sigma}\triangle^{0,\nu}_{k}(\sigma)\,G^{k,l,l+\nu-1}_{p,q}(\zeta\,e^{(2\nu-2k-2\varkappa+1)\pi i}\,\|\,a_\sigma).$$

If this is substituted on the right-hand side of (125), then (125) reduces in virtue of (124) to (122), so that the lemma has been proved.

Lemma 21. *Suppose that k, l, p, q, r and ν are integers with*

$$l\geqq 1,\ q\geqq 1,\ r\geqq 1,\ 0\leqq\nu\leqq k\leqq q \text{ and } l+\nu-1\leqq p\leqq q;$$

suppose further that the numbers $a_1,\dots,a_{l+\nu-1}$ and $b_1,\dots,b_k$ satisfy the conditions (119) and (120).

Then the following formula holds:

$$\left.\begin{aligned}G^{k,l-1,l+\nu-1}_{p,q}(\zeta)=\sum_{h=1}^{k-\nu}\Phi^{k,0}_{\nu,k}(h;r)\,G^{k,l-1,l+\nu-1}_{p,q}(\zeta\,e^{(-2h-2r+2)\pi i})\\ -\frac{1}{A^{0,\nu}_{k}}\sum_{\sigma=1}^{\nu}e^{(k-\nu+1)\pi i a_\sigma}\,\Theta^{k,0}_{\nu}(\sigma;r-1)\triangle^{0,\nu}_{k}(\sigma)\,G^{k,l,l+\nu-1}_{p,q}(\zeta\,e^{(2\nu-2k-2r+1)\pi i}\,\|\,a_\sigma).\end{aligned}\right\}\quad(126)$$

P r o o f. From (80) it follows

$$\sum_{\tau=0}^{r-1}e^{(2r-2\tau-2)\pi i a_\sigma}\,\Phi^{k,0}_{\nu,k}(1;\tau)=\Theta^{k,0}_{\nu}(\sigma;r-1).$$

We therefore find (126) if we put $\varkappa=\lambda=r$ in (122).

R e m a r k. Formula (122) is also valid if the following conditions are satisfied: k, l, p, q, $\varkappa$, λ and ν are integers with

$$l\geqq 1,\ q\geqq 1,\ 0\leqq k\leqq q,\ \nu\geqq 0,\ l+\nu-1\leqq p\leqq q,\ \lambda\leqq 0 \text{ and } \varkappa\geqq 1+\nu-k;$$

the numbers $a_{\nu+1},\dots,a_{l+\nu-1}$ and $b_1,\dots,b_k$ fulfil the condition (119).

For, if $\lambda\leqq 0$, the sum $\sum\limits_{h=1}^{k-\nu+\varkappa-\lambda}$ on the right-hand side of (122) is because of (73) and (75) equal to $G^{k,l-1,l+\nu-1}_{p,q}(\zeta)$; since $\Phi^{k,0}_{\nu,k}(1;\tau)=0$ for $\tau<0$, the sum $\sum\limits_{\tau=0}^{\lambda-1}$ is zero for $\lambda\leqq 0$ (comp. definition 4).

Similarly formula (126) is also true under the following conditions: k, l, p, q, r and ν are integers with

$$l\geqq 1,\ 1\leqq 1+\nu\leqq k\leqq q,\ 1+\nu-k\leqq r\leqq 0 \text{ and } l+\nu-1\leqq p\leqq q;$$

the numbers $a_{\nu+1},\dots,a_{l+\nu-1}$ and $b_1,\dots,b_k$ fulfil the condition (119).

Lemma 22. *Suppose that k, l, m, n, p, q and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ 0 \leqq m \leqq k \leqq q,\ 0 \leqq n-l+1 \leqq \nu \leqq k \text{ and } l+\nu-1 \leqq p \leqq q;$$

further that λ is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned} &\sum_{s=0}^{\lambda-1} \Omega^{m,n-l+1}_{\ k}(s)\, G^{k,l-1,l+\nu-1}_{p,q}(w\,e^{-2s\pi i}) \\ =&\sum_{h=1}^{k-\nu} \{ \Phi^{m,n-l+1}_{\nu,k}(h;\lambda) - \Omega^{m,n-l+1}_{\ k}(h+\lambda-1)\}\, G^{k,l-1,l+\nu-1}_{p,q}(w\,e^{(-2h-2\lambda+2)\pi i}) \\ -&\frac{1}{A^{0,\nu}_{\ k}} \sum_{\tau=1}^{\nu} e^{(k-\nu+1)\pi i a_\tau}\, \Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1)\, \triangle^{0,\nu}_{\ k}(\sigma)\, G^{k,l,l+\nu-1}_{p,q}(w\,e^{(2\nu-2k-2\lambda+1)\pi i} \,||\, a_\tau). \end{aligned}\right\} \quad (127)$$

P r o o f. We first suppose $\lambda \leq 0$. Then the left-hand side of (127) vanishes since $\Omega^{m,n-l+1}_{\ k}(s) = 0$ for $s < 0$. Because of $\Phi^{m,n-l+1}_{\nu,k}(1;\tau) = 0$ if $\tau < 0$, it appears from (71)

$$\Phi^{m,n-l+1}_{\nu,k}(h;\lambda) - \Omega^{m,n-l+1}_{\ k}(h+\lambda-1) = 0 \text{ for } \lambda \leqq 0.$$

We further have by (54)

$$\Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1) = 0 \text{ for } \lambda \leqq 0.$$

Hence formula (127) is certainly true if $\lambda \leqq 0$.

We now consider the case with $\lambda > 0$. Because of (77) we have

$$\sum_{s=0}^{\lambda-1} \Omega^{m,n-l+1}_{\ k}(s)\, \Phi^{k,0}_{\nu,k}(h;\lambda-s) = \Phi^{m,n-l+1}_{\nu,k}(h;\lambda) - \Omega^{m,n-l+1}_{\ k}(h+\lambda-1); \quad (128)$$

besides it follows from (79)

$$\sum_{s=0}^{\lambda-1} \Omega^{m,n-l+1}_{\ k}(s)\, \Theta^{k,0}_{\nu}(\sigma;\lambda-s-1) = \Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1) \quad . \quad . \quad (129)$$

If we replace in (126) ζ by $w\,e^{-2s\pi i}$ and r by $\lambda - s$ and use (128) and (129), we easily find (127).

Lemma 23. *Suppose that k, l, m, n, p, q, μ and ν are integers with*

$$l \geqq 1,\ q \geqq 1,\ 0 \leqq m \leqq k \leqq q,$$
$$0 \leqq n-l+1 \leqq \nu,\ 0 \leqq \mu \leqq k-\nu \text{ and } l+\nu-1 \leqq p \leqq q;$$

further that λ is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) *and* (120).

Then the following formula holds:

$$\left.\begin{aligned} & A^{m,n-l+1}_{\;k} \sum_{s=0}^{\lambda-1} \Omega^{m,n-l+1}_{\;k}(s)\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{-2s\pi i}) \\ = {} & A^{m,n-l+1}_{\;k} \sum_{h=1}^{k-\nu-\mu} \{\Phi^{m,n-l+1}_{\nu,k}(h;\lambda) - \Omega^{m,n-l+1}_{\;k}(h+\lambda-1)\}\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(-2h-2\lambda+2)\pi i}) \\ & - \overline{A}^{0,\nu}_{\;k}\, B^{m,n-l+1}_{\nu} \sum_{\varkappa=1}^{\mu} \Psi^{m,n-l+1}_{\nu,k}(\varkappa;\lambda)\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\varkappa)\pi i}) \\ & - B^{m,n-l+1}_{\nu} \sum_{\sigma=1}^{\nu} e^{(k-\nu-2\mu+1)\pi i a_\sigma}\, \Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1)\, \triangle^{0,\nu}_{\;k}(\sigma)\, G^{k,l,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu+1)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \quad (130)$$

P r o o f. From the definitions 5 and 6 it follows

$$\frac{A^{m,n-l+1}_{\;k}}{A^{0,\nu}_{\;k}} = B^{m,n-l+1}_{\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

Formula (130) with $\mu = 0$ is therefore equivalent to (127). Hence we may suppose $1 \leqq \mu \leqq k - \nu$ and assume that (130) with $\mu - 1$ instead of μ has already been proved.

Now it follows from (57), if $\nu \geqq 1$,

$$\left.\begin{aligned} G^{k,l,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu-1)\pi i} \,\|\, a_\sigma) = {} & e^{-2\pi i a_\sigma}\, G^{k,l,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu+1)\pi i} \,\|\, a_\sigma) \\ & + 2\pi i\, e^{-\pi i a_\sigma}\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}). \end{aligned}\right\} \quad (132)$$

If this is substituted on the right-hand side of (130) with $\mu - 1$ instead of μ, the expression $-B^{m,n-l+1}_{\nu} \sum\limits_{\sigma=1}^{\nu}$ not only yields the corresponding expression in (130) but besides

$$\begin{aligned} & -2\pi i\, B^{m,n-l+1}_{\nu}\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \times \\ & \qquad \times \sum_{\sigma=1}^{\nu} e^{(k-\nu-2\mu+2)\pi i a_\sigma}\, \Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1)\, \triangle^{0,\nu}_{\;k}(\sigma) \end{aligned}$$ [39])

and this expression is by virtue of (80) equal to

$$\begin{aligned} & -2\pi i\, B^{m,n-l+1}_{\nu}\, G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \times \\ & \qquad \times \sum_{\tau=0}^{\lambda-1} \Phi^{m,n-l+1}_{\nu,k}(1;\tau) \sum_{\sigma=1}^{\nu} e^{(k-\nu+2\lambda-2\mu-2\tau)\pi i a_\sigma}\, \triangle^{0,\nu}_{\;k}(\sigma) \\ & \qquad = G^{k,l-1,l+\nu-1}_{p,q}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \sum_{\tau=0}^{\lambda-1} \Phi^{m,n-l+1}_{\nu,k}(1;\tau) \times \\ & \times \{A^{0,\nu}_{\;k}\, B^{m,n-l+1}_{\nu}\, \Omega^{0,\nu}_{\;k}(k-\nu+\lambda-\mu-\tau) - \overline{A}^{0,\nu}_{\;k}\, B^{m,n-l+1}_{\nu}\, \overline{\Omega}^{0,\nu}_{\;k}(\mu+\tau-\lambda)\} \end{aligned}$$

(after (59)).

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[39]) This is still true for $\nu = 0$, since the sums $\sum\limits_{\sigma=1}^{\nu}$ then vanish.

The right-hand side of this relation is on account of (131), (71) (with $h = k - \nu - \mu + 1$) and (81) equal to

$$G_{p,q}^{k,l-1,l+\nu-1}(w\, e^{(2\nu-2k-2\lambda+2\mu)\pi i}) \times$$
$$\times [A^{m,n-l+1}_{k} \{ \Omega^{m,n-l+1}_{k}(k-\nu+\lambda-\mu) - \Phi^{m,n-l+1}_{\nu,k}(k-\nu-\mu+1;\lambda)\}$$
$$- \overline{A}^{0,\nu}_{k} B^{m,n-l+1}_{\nu} \Psi^{m,n-l+1}_{\nu,k}(\mu;\lambda)].$$

It appears therefore that the sums $\sum_{h=1}^{k-\nu-\mu+1}$ and $\sum_{\varkappa=1}^{\mu-1}$ on the right-hand side of (130) with $\mu - 1$ instead of μ reduce by the substitution (132) to the corresponding sums $\sum_{h=1}^{k-\nu-\mu}$ and $\sum_{\varkappa=1}^{\mu}$ in (130). This establishes the lemma.

Mathematics. — *Ueber die Beziehungen zwischen den Theorien der Parallelübertragung in Finslerschen Räumen.* By L. BERWALD †. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of May 25, 1946.)

Es gibt im Wesentlichen zwei Theorien der Parallelübertragung in Finslerschen Räumen [1]), von denen die eine auf E. NOETHER [2]) zurückgeht und 1925 vom Verfasser [3]) entwickelt wurde, während die andere von E. CARTAN [4]) (1933) stammt. Herr CARTAN ist wiederholt auf die Beziehungen zurückgekommen, die zwischen den beiden Theorien bestehen. Er bemerkt in dieser Hinsicht unter Anderem: „Il y aurait intérêt à établir effectivement les formules permettant de retrouver les tenseurs de l'une des théories par les procédés de l'autre théorie" [5]). Dieser Zusammenhang zwischen den wichtigsten Tensoren der beiden Theorien soll im Folgenden auseinandergesetzt werden.

1. Der Kürze halber benutzen wir die Bezeichnungen der Abhandlung [4]) von CARTAN. Wir ersetzen ferner die Ableitung $\frac{\partial \dots}{\partial x'^i}$ der Theorie des Verfassers durch die CARTANsche Ableitung

$$\dots_{\|i} = L \frac{\partial \dots}{\partial x'^i} \quad \dots \dots \dots \quad (1)$$

Weiter erinnern wir an die Gleichungen

$$l^i = \frac{x'^i}{L}, \qquad l_i = \frac{\partial L}{\partial x'^i}, \qquad A_{ihj} = \tfrac{1}{2}\, g_{ih\|j} \quad \dots \quad (2)$$

die Tensoren der CARTANschen Theorie durch solche der Theorie des Verfassers ausdrücken. Endlich ist der Tensor

$$g_{ik} = \frac{1}{2} \frac{\partial^2 (L^2)}{\partial x'^i \, \partial x'^k} \quad \dots \dots \dots \quad (2a)$$

beiden Theorien gemeinsam.

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[1]) Die Parallelübertragung von J. L. SYNGE, A generalisation of the Riemannian line-element. Trans. Amer. Math. Soc. 27 (1925), S. 61 und J. H. TAYLOR, A generalization of Levi-Civita's parallelism and the Frenet formulas, ebenda, S. 246, kommt hier nicht in Betracht, weil ihr keine Krümmungstheorie des Raumes entspricht.

[2]) E. NOETHER, Invarianten beliebiger Differentialausdrücke. Göttinger Nachr. 1918, S. 37.

[3]) Vgl. insbesondere L. BERWALD, Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus. Math. Zeitschrift, 25 (1926), S. 40.

[4]) E. CARTAN, Les espaces de Finsler. Actualités scientifiques et industrielles 79. Hermann et Cie., Paris 1934.

[5]) E. CARTAN, Les espaces de Finsler. Abh. a. d. Seminar f. Vektor- und Tensoranalysis. IV. Moskau und Leningrad 1937, S. 70; bes. S. 81.

Zunächst besprechen wir, wie die beiden *Parallelübertragungen* zusammenhängen. Die Parameter der Parallelübertragung des Verfassers sind die I^k_{ij}, die der Parallelübertragung von CARTAN (wenn das Stützelement parallel mitübertragen wird), die Γ^{*k}_{ij}. Herr CARTAN selbst hat bereits die grundlegende Beziehung

$$\boxed{I_{ihj} = \Gamma^{*}_{ihj} + A_{ihj|0} \quad ; \quad I_i{}^j{}_k = \Gamma^{*}_i{}^j{}_k + A_i{}^j{}_{k|0}} \quad . \quad . \quad . \quad (3)$$

aufgestellt [6]). Aus ihr folgt, dass das kovariante Differential des Einheitsvektors in beiden Theorien dasselbe ist.

Um auch umgekehrt die Γ^{*}_{ihj} durch geometrische Objekte auszudrücken, die in der Theorie des Verfassers auftreten, bezeichnen wir die kovariante Ableitung im Sinne des Verfassers durch einen Strichpunkt; z.B.

$$g_{ih;j} = \frac{\partial g_{ih}}{\partial x^j} - \frac{\partial g_{ih}}{\partial x'^m}\frac{\partial I^m}{\partial x'^j} - I_{ihj} - I_{hij}. \quad . \quad . \quad . \quad . \quad (4)$$

Dann gilt insbesondere

$$L_{;p} = 0, \quad l_{i;j} = 0. \text{ [7])}$$

Wegen der Identität $g_{ih|j} = 0$ und der Symmetrie der A_{ihj} ist

$$\tfrac{1}{2}\, g_{ih;j} = -A_{ihj|0}. \text{ [8])} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Die gesuchte inverse Relation lautet also

$$\boxed{\Gamma^{*}_{ihj} = I_{ihj} + \tfrac{1}{2}\, g_{ih;j}.} \text{ [9]).} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Mittels der Gleichungen (3), (5), (6) können wir unmittelbar einige Tensoren der einen Theorie durch solche der anderen ausdrücken. Wegen der Symmetrie der $g_{ih||j}$, $g_{ih;j}$ [10]) ergibt sich

$$A_{ihj|k} = \tfrac{1}{2}\, g_{ih|j;k} - \tfrac{1}{4}\, g^{mp}\,(g_{ih||m}\, g_{jp;k} + g_{hj||m}\, g_{ip;k} + g_{ji||m}\, g_{hp;k}) \quad (7)$$

und umgekehrt

$$\tfrac{1}{2}\, g_{ih|j;k} = A_{ihj|k} - A_i{}^m{}_k\, A_{jmk|0} - A_h{}^m{}_j\, A_{imk|0} - A_j{}^m{}_i\, A_{hmk|0}. \quad (8)$$

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[6]) a.a.O. [4]), Nr. 17.

[7]) a.a.O. [3]), Formeln (25), (35).

[8]) Siehe J. A. SCHOUTEN und J. HAANTJES, Ueber die Festlegung von allgemeinen Massbestimmungen und Uebertragungen in Bezug auf ko- und kontravariante Vektordichten, Monatshefte f. Math. u. Phys. 43 (1936), S. 161; bes. Formel (94).

[9]) Eine äquivalente Gleichung bei A. KAWAGUCHI, Beziehung zwischen einer metrischen linearen Uebertragung und einer nicht-metrischen in einem allgemeinen metrischen Raume. Diese Proceedings 40 (1937), S. 596, Nr. 3.

[10]) Vgl. die erste Gleichung (9).

Ferner gelten die Formeln [11])

$$\left.\begin{aligned} g_{ih;k} &= L\, l_p \frac{\partial^3 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k}; \\ g_{ih;k\|j} &= L\, g_{jp} \frac{\partial^3 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k} + L^2 \frac{\partial^4 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k\, \partial x'^j}; \\ g_{ih\|j;k} - g_{ih;k\|j} &= L \left(g_{ip} \frac{\partial^3 I^p}{\partial x'^h\, \partial x'^j\, \partial x'^k} + g_{hp} \frac{\partial^3 I^p}{\partial x'^j\, \partial x'^i\, \partial x'^k} \right), \end{aligned}\right\} \quad (9)$$

mit deren Hilfe der Tensor $A_{ihj|k}$ durch die Tensoren $L \dfrac{\partial^3 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k}$ und $L^2 \dfrac{\partial^4 I^p}{\partial x'^i \partial x'^h \partial x'^j \partial x'^k}$ dargestellt werden kann.

Um umgekehrt den Tensor $L \dfrac{\partial^3 I^p}{\partial x'^i\, \partial x'^k\, \partial x'^h}$ durch Tensoren der CARTANschen Theorie auszudrücken, leite man zunächst die zweite Identität (3) nach x'^h ab und ersetze sodann $L \dfrac{\partial\, I^{*j}_{i\;k}}{\partial x'^h}$ mittels der Gleichung (44) der Arbeit [4]) von CARTAN durch die $A_i{}^j{}_k$ und ihre kovarianten Ableitungen. Ferner beachte man die Gleichungen [12])

$$\left.\begin{aligned} A_{imk|0\|h} &= A_{ikh\|m|0} - l_m A_{ikh|0} + A_{ikm|h} - A_i{}^p{}_k A_{mph|0} - A_k{}^p{}_m A_{iph|0} \\ &\quad - A_m{}^p{}_i A_{kph|0}, \\ A_i{}^j{}_{k|0\,|h} &= -2 A_m{}^j{}_h A_i{}^m{}_{k|0} + g^{mj} A_{imk|0\|h}. \end{aligned}\right\} \quad (10)$$

Man erhält dann schliesslich

$$\boxed{\begin{aligned} L \frac{\partial^3 I^j}{\partial x'^i\, \partial x'^k\, \partial x'^h} &= g^{mj} (A_{ikh\|m|0} - A_{ikh|m}) - \\ &- l^j A_{ikh|0} + A_i{}^j{}_{k|h} + A_k{}^j{}_{h|i} + A_h{}^j{}_{i|k} \\ &- 2 (A_i{}^j{}_m A_k{}^m{}_{h|0} + A_k{}^j{}_m A_h{}^m{}_{i|0} + A_h{}^j{}_m A_i{}^m{}_{k|0}). \end{aligned}} \quad \ldots \quad (11)$$

Wegen (7), (9), (11) sind die Identitäten

$$A_{ikh|m} = 0 \quad \text{und} \quad \frac{\partial^3 I^j}{\partial x'^i\, \partial x'^k\, \partial x'^h} = 0$$

äquivalent. Jede von ihnen kennzeichnet die „affin-zusammenhängenden Räume" des Verfassers [13]).

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[11]) a.a.O. [3]) Formeln (41), (37), (25).

[12]) Zur Ableitung der ersten Gleichung (10) führt man am Besten die Grössen

$$C_{ijh} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x'^h} = \frac{1}{4} \frac{\partial^3 (L^2)}{\partial x'^i\, \partial x'^j\, \partial x'^h}, \quad C_{ijhk} = \frac{\partial C_{ijh}}{\partial x'^k}$$

ein. Vgl. a.a.O. [4]), S. 11 und 19.

[13]) L. BERWALD, a.a.O. [3]), S. 47; E. CARTAN, a.a.O. [4]), Nr. 41.

Endlich schreiben wir noch die Beziehung

$$l_{i\|j;k} = g_{ij;k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

an, die es gestattet, die $g_{ij;k}$ durch Ableitungen der l_i zu ersetzen.

2. Wir gehen jetzt zu den *Krümmungstensoren* über.

Der „Grundtensor der Krümmung" des Verfassers [14]), der durch

$$K^j_{.kh} = \frac{\partial^2 I^j}{\partial x'^k \partial x^h} - \frac{\partial^2 I^j}{\partial x'^h \partial x^k} - I_k{}^j{}_r \frac{\partial I^r}{\partial x'^h} + I_h{}^j{}_r \frac{\partial I^r}{\partial x'^k} \quad . \quad . \quad (13)$$

gegeben ist, tritt auch in der CARTANschen Theorie auf. Es ist nämlich

$$\boxed{K^j_{.kh} = L R_0{}^j{}_{kh}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Durch Ableitung des Tensors $K^j_{.kh}$ nach x'^i erhält man den Krümmungstensor $K_{i.kh}^{\;j}$ des Verfassers [14]):

$$K_{i.kh}^{\;j} = \frac{\partial K^j_{.kh}}{\partial x'^i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Aus (14), (15) folgt leicht

$$K_{i.kh}^{\;j} = l_i R_0{}^j{}_{kh} + R_0{}^j{}_{kh\|i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

und

$$K_{ijkh} = R_{ijkh} - 2 A_i{}^m{}_j R_{0mkh} + l^m R_{mjkh\|i} \quad . \quad . \quad . \quad . \quad (17)$$

Aus jeder der beiden Formeln (16), (17) ergibt sich

$$K_{0jkh} = R_{0jkh} \quad , \quad K_{i0kh} = R_{i0kh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

wenn

$$l^m K_{mjkh} = K_{0jkh} \quad , \quad l^m K_{imkh} = K_{i0kh}$$

gesetzt wird. Ferner zeigt (17), dass der Tensor

$$T_{ijkh} = -(g_{ij;k;h} - g_{ij;h;k}) = \frac{\partial g_{ij}}{\partial x'^p} K^p_{.kh} + K_{ijkh} + K_{jikh}, \quad ^{15)} \quad (19)$$

der in der Theorie des Verfassers als „Tensor der Streckenkrümmung" auftritt, durch

$$T_{ijkh} = l^m (R_{mjkh\|i} + R_{mikh\|j}) - 2 A_i{}^m{}_j R_{0mkh} \quad . \quad . \quad . \quad (20)$$

gegeben ist.

Um umgekehrt den Krümmungstensor R_{ijkh} der CARTANschen Theorie durch Tensoren der Theorie des Verfassers auszudrücken, kann man von der Formel (XIX) der Abhandlung [4]) von CARTAN ausgehen. Mittels (6), (13), (14), (19) findet man durch eine Rechnung, die keine Schwierigkeit bietet:

$$\boxed{R_{ijkh} = \tfrac{1}{2} (K_{ijkh} - K_{jikh}) - \tfrac{1}{4} g^{mp} (g_{im;k} g_{jp;h} - g_{im;h} g_{jp;k}).} \quad (21)$$

[14]) a.a.O. [3]), S. 46.

[15]) a.a.O. [3]), Formeln (36) und (54).

Aus $R_{ijkh} = 0$ folgt wegen (17) $K_{ijkh} = 0$, aber das Umgekehrte gilt im Allgemeinen nicht.

Die Gleichung (17) kann durch eine andere Formel ersetzt werden. Um sie abzuleiten, hat man die „BIANCHIsche Identität"

$$\left.\begin{aligned} R_{mjkh\|l} + R_{jrkh} A_m{}^r{}_l - R_{mrkh} A_j{}^r{}_l + P_{mjhl|k} - P_{mjkl|h} \\ + P_{mjkr} A_h{}^r{}_{l|0} - P_{mjhr} A_k{}^r{}_{l|0} + S_{mjrl} R_0{}^r{}_{hk} = 0 \end{aligned}\right\} \quad (22)$$

der CARTANschen Theorie heranzuziehen und die Gleichungen [16])

$$A_0{}^r{}_i = 0 \quad , \quad S_{0jmq} = 0 \quad , \quad P_{0jkr} = A_{jkr|0} \quad . \quad . \quad . \quad (23)$$

zu berücksichtigen. Man erhält so

$$\boxed{\begin{aligned} K_{ijkh} = R_{ijkh} + A_{imk|0} A_j{}^m{}_{h|0} - A_{imh|0} A_j{}^m{}_{k|0} - A_i{}^m{}_j R_{0mkh} \\ + A_{ijk|0|h} - A_{ijh|0|k}. \end{aligned}} \quad (24)$$

Es ist also

$$\left.\begin{aligned} \tfrac{1}{2}(K_{ijkh} - K_{jikh}) &= R_{ijkh} + A_{imk|0} A_j{}^m{}_{h|0} - A_{imh|0} A_j{}^m{}_{k|0}, \\ \tfrac{1}{2} T_{ijkh} &= A_{ijk|0|h} - A_{ijh|0|k}. \end{aligned}\right\} \quad (25)$$

Schliesslich werde noch für den zweiten Krümmungstensor P_{ijkh} von CARTAN der Ausdruck

$$\boxed{P_{ijkh} = \tfrac{1}{2}(g_{kh;i\|j} - g_{kh;j\|i}) - \tfrac{1}{4} g^{mp}(g_{ik;m}\, g_{jh\|p} - g_{jk;m}\, g_{ih\|p})} \quad (26)$$

angegeben, den man erhält, wenn man von Gleichung (XVII) der Arbeit [4]) von CARTAN ausgeht und die Formeln (3), (7), (9) benutzt.

3. Die vorstehenden Formeln lassen sich vor Allem auf *die Finslerschen Räume* anwenden, *in denen die Parallelübertragung eines Linienelementes, wenn sein Zentrum einen gegebenen Weg beschreibt, die Winkelmetrik der Linienelemente* (*im Sinn von* CARTAN [17])) *ungeändert lässt. Diese Räume sind nach* CARTAN [17]) *durch* $A_{ihj|0} = 0$, also *durch das Zusammenfallen der beiden betrachteten Parallelübertragungen gekennzeichnet.* Sie sind für $n = 2$ mit den „LANDSBERGschen Räumen" des Verfassers [18]) identisch. Nach (5) und (9) können sie auch durch $g_{ih;j} = 0$ oder durch

$$l_p \frac{\partial^3 I^p}{\partial x'^i\, \partial x'^h\, \partial x'^k} = 0$$

[16]) a.a.O. [4]), Nr. 9, 10, 36, 37.

[17]) a.a.O. [4]), Nr. 11, 40.

[18]) L. BERWALD, Ueber zweidimensionale allgemeine metrische Räume. Journal f. d. reine u. angew. Mathematik 156 (1927), S. 191; bes. S. 208 f., und § 5; E. CARTAN, Sur un problème d'équivalence et la théorie des espaces métriques généralisés. Mathematica, Cluj 4 (1930), S. 114; bes. S. 133 f.

charakterisiert werden. Der Rang der Matrix

$$\left(\frac{\partial^3 I^1}{\partial x'^i \,\partial x'^h \,\partial x'^j}, \frac{\partial^3 I^2}{\partial x'^i \,\partial x'^h \,\partial x'^j}, \ldots \frac{\partial^3 I^n}{\partial x'^i \,\partial x'^h \,\partial x'^j}\right), (i, h, j = 1, 2 \ldots n)$$

von der wir nur eine Zeile angeschrieben haben, ist für diese Räume also höchstens $n - 1$. Der Ausdruck (26) zeigt ferner, *dass für die in Rede stehenden Räume der Krümmungstensor* P_{ijkh} *Null ist.* Endlich gilt für sie

$$R_{ijkh} = \tfrac{1}{2}(K_{ijkh} - K_{jikh}),\; K_{ijkh} = R_{ijkh} - A_i{}^m{}_j\, R_{0mkh}.$$

Die betrachteten Räume gehören zu der umfassenderen Klasse der *Finslerschen Räume, für die der Tensor* T_{ijkh} *verschwindet. Diese Räume* (*die „Räume ohne Streckenkrümmung"* in der Theorie des Verfassers) *sind durch*

$$A_{ijk|0|h} - A_{ijh|0|k} = 0, \qquad (27)$$

gekennzeichnet.

Applied Mechanics. — *Large distortions of circular rings and straight rods.* I. By A. VAN WIJNGAARDEN. (Nationaal Luchtvaartlaboratorium.) (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of May 25, 1946.)

1. *Introduction.* In a series of three papers that recently have appeared in these Proceedings [1]), BIEZENO and KOCH dealt with the problem of the distortions of a closed circular ring, loaded by an arbitrary equilibrium-system of forces in its own plane. At first sight, this problem might seem to be rather elementary, but the real aim of their investigation was to give a solution, valid also for loads of such a size, that the influence of the distortions on the distribution of the bending moments in the ring, that is neglected in the elementary theory, is appreciable. On the other hand, however, their treatment of the problem is still limited, a fact, the authors are of course fully aware of. Indeed, the authors use the differential-equation of the slightly curved beam to describe the distortions, an approximation, that can only be made, if the difference of the slope of the elastic line in its deformed and its undeformed shape is small. So, their solution, though it tends farther than the solution, obtained by the use of elementary means, f.i. CASTIGLIANO's theorem, is still limited to not too large distortions. Moreover, the distortions, as calculated by their method, appear as an infinite series of terms. As the calculation of the separate terms is rather tedious, we have, for practical reasons, to confine ourselves to a small number of terms. The authors f.i. use only two terms. So there might be an uncontrolable influence of the neglected terms.

It might be interesting therefore, to give an exact treatment of the same problem, partly for having the exact results for themselves, partly for providing a check to the approximate results. But it is obvious, that the exact solution of the general problem — with arbitrarily varying, continuous, tangential and normal loads — might be extremely difficult, if not impossible. As however both the particular problems, the authors deal with, viz. the principal one, treated in the third paper, which led the authors to their investigation, as well as the introducing problem, treated in the second paper, are problems, in which only discrete forces are applied at

[1]) C. B. BIEZENO and J. J. KOCH, The generalized buckling problem of the circular ring, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48**, 447—468 (1945).

The circular ring under the combined action of compressive and bending loads, ibid., **49**, 1—8 (1946).

On the non-linear deflection of a semi-circular ring, clamped at both ends, ibid., **49**, 139—145 (1946).

the ring, we too will confine ourselves here to such problems. It will be shown, that we can treat them in an exact way indeed. We shall apply our theory to the two problems, just mentioned, so as to provide the required check.

The method used in this paper, is principally based on the same ideas, as used already by some authors in the classical period of the theory of elasticity for elastica-problems. The method is not at all restricted to closed circular rings, and many other problems of large distortions can be solved by the same means. As an illustration, we shall give the exact solutions of two problems of a straight beam. The first of these problems has been treated allready approximately by SONNTAG [2]) and afterwards graphically by BIEZENO [3]).

2. *The distortion of a circular ring.* Let the problem be to determine the distortion of a part of a circular ring loaded only at its ends. At the one end P_0, the load consists of a couple M_0, a normal force N_0 and a shearing force D_0 (see fig. 1), and at the other end P_1, it consists of a

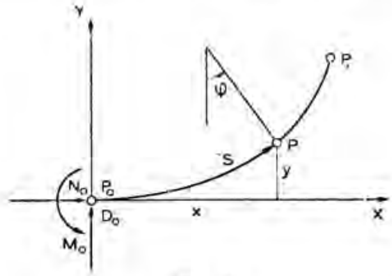

Fig. 1.

couple M_1, a normal force N_1 and a shearing force D_1, so that the ring is in equilibrium. A system of axes $X — Y$, with its origin in P_0 is introduced such that the X-axis coincides with the tangent in P_0 (in the distorted state) in the direction of P_1, and the Y-axis coincides with the normal in P_0 in that direction, in which (in the undistorted state) the centre of the circle lies. Then it follows from reasons of equilibrium, that the bending moment M (positive, if it decreases the curvature) in an arbitrary point $P(x, y)$ of the ring is:

$$M = M_0 - D_0\, x + N_0\, y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, if r is the radius of the unbent ring, R the radius of curvature of the bent ring and EI the flexural rigidity, then the differential equation of

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[2]) R. SONNTAG, Der beiderseits gestützte, symmetrisch belastete gerade Stab mit endlicher Durchbiegung und seine Stabilität, Ing. Arch., **12**, 283—306 (1943).

[3]) C. B. BIEZENO, On a special case of bending, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **45**, 438—442 (1942).

the ring, if the distortions due to tension and shearing are neglected against those due to bending, is [4]):

$$M = EI\left(\frac{1}{r} - \frac{1}{R}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We pass from the XY-system to a system of natural coordinates s and φ, where s is the length of the arc P_0P, and φ is the angle between the Y-axis and the normal in P. As $1/R = d\varphi/ds$, $dx/ds = \cos\varphi$ and $dy/ds = \sin\varphi$, we find by combining (1) and (2) and differentiating with respect to s:

$$EI\frac{d^2\varphi}{ds^2} - D_0\cos\varphi + N_0\sin\varphi = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the following nondimensional quantities are introduced

$$a = \frac{2\,D_0\,r^2}{EI},\ \tan\beta = \frac{D_0}{N_0} = \frac{1}{\nu},\ \sigma = \frac{s}{r},\ \xi = \frac{x}{r},\ \eta = \frac{y}{r} \quad . \quad . \quad (4)$$

equation (3) can be written as:

$$\frac{d^2\varphi}{d\sigma^2} + \frac{a}{2\sin\beta}\sin(\varphi - \beta) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

After multiplication by $d\varphi/d\sigma$ integration of this equation leads to:

$$\left(\frac{d\varphi}{d\sigma}\right)^2 - \frac{a}{\sin\beta}\cos(\varphi - \beta) = \frac{a}{\sin\beta}\gamma, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where γ is a constant of integration. It follows that:

$$\frac{d\varphi}{d\sigma} = \pm\sqrt{\frac{a}{\sin\beta}}\sqrt{\gamma + \cos(\varphi - \beta)}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The ambigious sign is of much importance. If the load is small, the shape of the elastic line does not differ much from the original circle and there will be no inflexions. In this case, φ increases monotonously with s from zero to its final value. With higher loads, it may however occur that there are inflexions in the elastic line. Then for instance, φ decreases from zero to a lower limit φ_{min}, increases from that value to an upper limit φ_{max}, and decreases again to its final value. In this case successively the minus-, plus- and minus-sign must be used in (7). We shall deal with this question in more detail further on, but denote from now all integrations with respect to φ by the symbol $\int\, !\, d\varphi$, as a warning that the appropiate sign should be attributed to the integrand.

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[4]) In the corresponding formula (19) of the first paper in note [1]) on page 450, the factor r^2 has to be deleted.

With these precautions, the solution of equation (7) can be given in the following form:

$$\left.\begin{aligned} \sigma &= \sqrt{\frac{\sin\beta}{\alpha}} \int_0^{\varphi} \frac{d\varphi'}{\sqrt{\gamma + \cos(\varphi' - \beta)}} \\ \xi &= \sqrt{\frac{\sin\beta}{\alpha}} \int_0^{\varphi} \frac{\cos\varphi'\, d\varphi'}{\sqrt{\gamma + \cos(\varphi' - \beta)}} \\ \eta &= \sqrt{\frac{\sin\beta}{\alpha}} \int_0^{\varphi} \frac{\sin\varphi'\, d\varphi'}{\sqrt{\gamma + \cos(\varphi' - \beta)}} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (8)$$

From the boundary conditions of each particular problem, β, γ and the position of the eventual inflexions, should be determined as functions of α.

3. *The reduction of the integrals, involved in eqs.* (8). We reduce the integrals, involved in eqs. (8) to elementary functions and the standard elliptic integrals:

$$\left.\begin{aligned} E(k, \varphi) &= \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \psi}\, d\psi \quad , \quad \mathsf{E}(k) = E\left(k, \frac{\pi}{2}\right), \\ F(k, \varphi) &= \int_0^{\varphi} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} \quad , \quad \mathsf{K}(k) = F\left(k, \frac{\pi}{2}\right), \end{aligned}\right\} \quad . \quad (9)$$

by substituting $\psi = \frac{\varphi' - \beta}{2}$ and $k = \sqrt{\frac{2}{1+\gamma}}$. The result is:

$$\left.\begin{aligned} \sigma &= \sqrt{\frac{2\sin\beta}{\alpha}}\; f_1(k, \varphi) \Big|_{-\beta/2}^{(\varphi - \beta)/2} \\ \xi &= \sqrt{\frac{2\sin\beta}{\alpha}}\; \{\cos\beta\, f_2(k, \varphi) + \sin\beta\, f_3(k, \varphi)\} \Big|_{-\beta/2}^{(\varphi - \beta)/2} \\ \eta &= \sqrt{\frac{2\sin\beta}{\alpha}}\; \{\sin\beta\, f_2(k, \varphi) - \cos\beta\, f_3(k, \varphi)\} \Big|_{-\beta/2}^{(\varphi - \beta)/2} \end{aligned}\right\} \quad . \quad (10)$$

where

$$\left.\begin{aligned} f_1(k, \varphi) &= k\, F(k, \varphi) \\ f_2(k, \varphi) &= \frac{2\, E(k, \varphi) - (2 - k^2)\, F(k, \varphi)}{k} \\ f_3(k, \varphi) &= \frac{2}{k} \sqrt{1 - k^2 \sin^2 \varphi} \end{aligned}\right\} \quad . \; . \; . \; . \; . \quad (11)$$

The elliptic integrals $E(k, \varphi)$ and $F(k, \varphi)$ are tabulated for the case of real k and $|k| \leqq 1$, and real φ and $0 \leqq \varphi \leqq \pi/2$. But these conditions are not always satisfied in our problems. We shall see that, on the contrary, γ also assumes values $\gamma < -1$, so that k is purely imaginary, as well as values $-1 < \gamma < 1$, so that k is real, but $|k| > 1$. Furthermore φ can have values $\varphi < 0$ as well as $\varphi > \pi/2$. We therefore have to introduce some simple transformations as to be able to use the tabulated values.

If $|k| > 1$, we find, by substituting $k \sin \psi = \sin \psi'$ in eqs. (9), that

$$\left.\begin{aligned} &E(k, \varphi) = \frac{1}{k^*} E(k^*, \varphi^*) - \frac{1-k^{*2}}{k^*} F(k^*, \varphi^*) \\ &F(k, \varphi) = k^* F(k^*, \varphi^*) \\ &\text{with } k^* = 1/k \text{ and } \varphi^* = \sin^{-1} k \sin \varphi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (12)$$

So we have reached our aim, if, at least, $|k \sin \varphi| \leqq 1$; this holds certainly true in our case, as otherwise $E(k, \varphi)$ and $F(k, \varphi)$ would be complex, in contradiction with the fact that σ, ξ and η are real. From (12) we find for the functions $f_j(k, \varphi)$ if $|k| > 1$:

$$\left.\begin{aligned} &f_1(k, \varphi) = F(k^*, \varphi^*) \\ &f_2(k, \varphi) = 2\,E(k^*, \varphi^*) - F(k^*, \varphi^*) \\ &f_3(k, \varphi) = 2\,k^* \cos \varphi^* \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If k is purely imaginary, we substitute in eqs. (9): $\psi = \frac{\pi}{2} - \psi'$. Then we obtain the result:

$$\left.\begin{aligned} &E(k, \varphi) = \frac{1}{\sqrt{1-k^{**2}}} \;\{\mathsf{E}(k^{**}) - E(k^{**}, \varphi^{**})\} \\ &F(k, \varphi) = \sqrt{1-k^{**2}} \;\{\mathsf{K}(k^{**}) - F(k^{**}, \varphi^{**})\} \\ &\text{with } k^{**} = \frac{-ik}{\sqrt{1-k^2}} \text{ and } \varphi^{**} = \frac{\pi}{2} - \varphi \end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

Now k^{**} is real and $|k^{**}| < 1$. So we find from (14), if k is purely imaginary

$$\left.\begin{aligned} &f_1(k, \varphi) = ik^{**} \;\{\mathsf{K}(k^{**}) - F(k^{**}, \varphi^{**})\} \\ &f_2(k, \varphi) = \frac{-i}{k^{**}} \;\{2\,\mathsf{E}(k^{**}) - (2-k^{**2})\,\mathsf{K}(k^{**}) - 2\,E(k^{**}, \varphi^{**}) + \\ &\qquad\qquad + (2-k^{**2})\,F(k^{**}, \varphi^{**})\} \\ &f_3(k, \varphi) = \frac{-2i}{k^{**}} \sqrt{1-k^{**2}\sin^2 \varphi^{**}} \end{aligned}\right\} . \quad (15)$$

If φ is real, but does not satisfy the condition $0 \leqq \varphi \leqq \pi/2$, we can always reduce the integrals to the tabulated ones by means of the self-evident relations:

$$\left.\begin{array}{ll} E(k,\varphi) = -E(k,-\varphi) \quad , & E(k,\varphi) = 2\,\mathsf{E}(k) - E(k,\pi-\varphi) \\ F(k,\varphi) = -F(k,-\varphi) \quad , & F(k,\varphi) = 2\,\mathsf{K}(k) - F(k,\pi-\varphi) \end{array}\right\} . \quad (16)$$

4. *Complications due to the inflexions of the elastic line.* We shall investigate now in more detail the question of the inflexions. From equation (7) we learn that for the extreme values φ_{ext} of φ, for which inflexions occur, holds:

$$\gamma + \cos(\varphi_{ext} - \beta) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It is obvious that inflexions can only occur, if $-1 \leqq \gamma \leqq 1$, so if $k \geqq 1$. Therefore it recommends itself to express our results in terms of k^*. We verify easily that the values φ_{min} resp. φ_{max} corresponding to the "lower" resp. "upper" inflexions, are given by

$$\left.\begin{array}{l} \varphi_{min} = \beta - 2\sin^{-1} k^* \\ \varphi_{max} = \beta + 2\sin^{-1} k^* \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Attention should be given to the fact that the quantities φ_{min} and φ_{max}, as defined by (18), are only mathematical quantities without a physical sense, unless inflexions really occur. So with sufficiently small α, we are certain that both φ_{min} and φ_{max} are lying outside the range of φ from zero to its final value (the latter value varies in general with α) and that they have no meaning for the problem. But of course we can calculate them, and see how they vary with increasing values of α. Then we see for instance φ_{min} increasing with α, until it reaches a maximum value equal to zero, and then decreasing again. From that moment it has a physical sense.

The quantity $(\varphi - \beta)/2$, that occurs in the limits of integration, takes for these special values of φ the form:

$$\left.\begin{array}{l} (\varphi_{min} - \beta)/2 = -\sin^{-1} k^* \\ (\varphi_{max} - \beta)/2 = \sin^{-1} k^* \end{array}\right\} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

We can now distinguish several different cases of integration according to the number and kind of inflexions. With a view to the applications that we shall make of our theory, we shall confine ourselves to those cases, in which no more than two inflexions occur. Then there are four possibilities. In case I, there is no inflexion at all in the part of the ring under consideration. In case II, there is one lower inflexion, in case III one upper inflexion, and in case IV, there are two inflexions, a lower and an upper one. In case I, φ' increases directly from 0 to φ. In case II, φ' first decreases from 0 to φ_{min} (where the integrand has the negative sign) and then it increases from φ_{min} to φ. In case III, φ' increases from 0 to φ_{max} and then decreases from φ_{max} to φ, where the integrand has the negative sign. In case IV, φ' decreases from 0 to φ_{min}, where the integrand has the negative sign, then increases from $\varphi_{\min}$ to $\varphi_{\max}$, and at last decreases again from $\varphi_{\max}$ to φ, where the integrand has the negative sign.

Now, we have to see, what are the consequences of these considerations with respect to the limits of integration in the eqs. (10). If we denote the three functions f_1, f_2 and f_3, for a moment by f, then we have in the different cases:

$$f(k,\varphi)\Bigg|_{-\beta/2}^{(\varphi-\beta)/2\ !} = \left\{\begin{array}{ll} f\left(k, \frac{\varphi-\beta}{2}\right) - f\left(k, -\frac{\beta}{2}\right) & \text{in case I} \\ f\left(k, \frac{\varphi-\beta}{2}\right) + f\left(k, -\frac{\beta}{2}\right) - 2f(k, -\sin^{-1}k^*) & \text{in case II} \\ -f\left(k, \frac{\varphi-\beta}{2}\right) - f\left(k, -\frac{\beta}{2}\right) + 2f(k, \sin^{-1}k^*) & \text{in case III} \\ -f\left(k, \frac{\varphi-\beta}{2}\right) + f\left(k, -\frac{\beta}{2}\right) + 2f(k, \sin^{-1}k^*) - & \\ \qquad -2f(k, -\sin^{-1}k^*) & \text{in case IV} \end{array}\right\}. \quad (20)$$

But the functions $f_1(k,\varphi)$ and $f_2(k,\varphi)$ are odd functions of φ, whereas $f_3(k,\varphi)$ is an even function of φ. Moreover, the values assumed by the functions f for $\varphi = \sin^{-1}k^*$ are very special ones. Indeed, we have:

$$\left.\begin{array}{l} f_1(k, \sin^{-1}k^*) = \mathsf{K}(k^*) \\ f_2(k, \sin^{-1}k^*) = 2\,\mathsf{E}(k^*) - \mathsf{K}(k^*) \\ f_3(k, \sin^{-1}k^*) = 0 \end{array}\right\} \quad . \ . \ . \ . \ . \ . \quad (21)$$

So we find at last:

$$f_1(k,\varphi)\Bigg|_{-\beta/2}^{(\varphi-\beta)/2\ !} = \left\{\begin{array}{ll} f_1\left(k, \frac{\varphi-\beta}{2}\right) + f_1\left(k, \frac{\beta}{2}\right) & \text{in case I} \\ f_1\left(k, \frac{\varphi-\beta}{2}\right) - f_1\left(k, \frac{\beta}{2}\right) + 2\,\mathsf{K}(k^*) & \text{in case II} \\ -f_1\left(k, \frac{\varphi-\beta}{2}\right) + f_1\left(k, \frac{\beta}{2}\right) + 2\,\mathsf{K}(k^*) & \text{in case III} \\ -f_1\left(k, \frac{\varphi-\beta}{2}\right) - f_1\left(k, \frac{\beta}{2}\right) + 4\,\mathsf{K}(k^*) & \text{in case IV} \end{array}\right\} \quad . \ . \ . \quad (22)$$

$$f_2(k,\varphi)\Bigg|_{-\beta/2}^{(\varphi-\beta)/2\ !} = \left\{\begin{array}{ll} f_2\left(k, \frac{\varphi-\beta}{2}\right) + f_2\left(k, \frac{\beta}{2}\right) & \text{in case I} \\ f_2\left(k, \frac{\varphi-\beta}{2}\right) - f_2\left(k, \frac{\beta}{2}\right) + 4\,\mathsf{E}(k^*) - 2\,\mathsf{K}(k^*) & \text{in case II} \\ -f_2\left(k, \frac{\varphi-\beta}{2}\right) + f_2\left(k, \frac{\beta}{2}\right) + 4\,\mathsf{E}(k^*) - 2\,\mathsf{K}(k^*) & \text{in case III} \\ -f_2\left(k, \frac{\varphi-\beta}{2}\right) - f_2\left(k, \frac{\beta}{2}\right) + 8\,\mathsf{E}(k^*) - 4\,\mathsf{K}(k^*) & \text{in case IV} \end{array}\right\} \quad (23)$$

$$f_3(k,\varphi)\Big|_{-\beta/2}^{(\varphi-\beta)/2} = \left\{\begin{array}{ll} f_3\left(k,\frac{\varphi-\beta}{2}\right) - f_3\left(k,\frac{\beta}{2}\right) & \text{in case I} \\ f_3\left(k,\frac{\varphi-\beta}{2}\right) + f_3\left(k,\frac{\beta}{2}\right) & \text{in case II} \\ -f_3\left(k,\frac{\varphi-\beta}{2}\right) - f_3\left(k,\frac{\beta}{2}\right) & \text{in case III} \\ -f_3\left(k,\frac{\varphi-\beta}{2}\right) + f_3\left(k,\frac{\beta}{2}\right) & \text{in case IV} \end{array}\right\} . \quad (24)$$

In the equations of this and the preceding section we have all means at hand, by which our problems can be solved. Only in the case, that $\alpha \langle\langle 1$ some difficulties remain which will be considered in the following section.

5. *Expansions for small α.* A look at the equations (8) suffices to see, that difficulties arise for very small values of α. Indeed, β will generally not be small for small α, as it depends on the ratio of the normal and the shearing force. So, with small α, γ will be very large, and the fractions in (10) will all have the indefinite form 0/0. To overcome this difficulty, we develop the integrals in (8) in suitable series, keeping in mind, that (α being small) certainly no inflexions are present. The integrals, therefore, can be taken directly between the limits 0 and φ.

We introduce the variable $\delta = \sqrt{\frac{\sin\beta}{\gamma}}$ which is small with α. From (8) we find:

$$\left.\begin{array}{l} \sigma = \frac{\delta}{\sqrt{\alpha}} \int\limits_0^{\varphi} \frac{d\varphi'}{\sqrt{1+\frac{\cos(\varphi'-\beta)}{\sin\beta}\,\delta^2}} \\ \xi = \frac{\delta}{\sqrt{\alpha}} \int\limits_0^{\varphi} \frac{\cos\varphi'\,d\varphi'}{\sqrt{1+\frac{\cos(\varphi'-\beta)}{\sin\beta}\,\delta^2}} \\ \eta = \frac{\delta}{\sqrt{\alpha}} \int\limits_0^{\varphi} \frac{\sin\varphi'\,d\varphi'}{\sqrt{1+\frac{\cos(\varphi'-\beta)}{\sin\beta}\,\delta^2}} \end{array}\right\} \quad . \; . \; . \; . \; . \; : \quad (25)$$

As $\cot\beta = \nu$, we have:

$$\left.\begin{array}{l} \left\{1+\frac{\cos(\varphi'-\beta)}{\sin\beta}\,\delta^2\right\}^{-1/2} = 1 - \frac{1}{2}\,\delta^2(\nu\cos\varphi+\sin\varphi) + \\ \quad + \frac{3}{8}\,\delta^4(\nu^2\cos^2\varphi + 2\nu\cos\varphi\sin\varphi + \sin^2\varphi) - \ldots \end{array}\right\} . \; . \quad (26)$$

If this expression is substituted into (25), we have only to evaluate elementary integrals. For the applications in view the case $\varphi = \pi/2$ is particularly interesting. It leads to:

$$\left.\begin{aligned} \sigma(\pi/2) &= \frac{\delta}{\sqrt{a}}\left\{\frac{\pi}{2} - \frac{1}{2}\delta^2(\nu+1) + \frac{3}{8}\delta^4\left(\frac{\pi}{4}\nu^2 + \nu + \frac{\pi}{4}\right) - \dots\right\} \\ \xi(\pi/2) &= \frac{\delta}{\sqrt{a}}\left\{1 - \frac{1}{2}\delta^2\left(\frac{\pi}{4}\nu + \frac{1}{2}\right) + \frac{3}{8}\delta^4\left(\frac{2}{3}\nu^2 + \frac{2}{3}\nu + \frac{1}{3}\right) - \dots\right\} \\ \eta(\pi/2) &= \frac{\delta}{\sqrt{a}}\left\{1 - \frac{1}{2}\delta^2\left(\frac{\nu}{2} + \frac{\pi}{4}\right) + \frac{3}{8}\delta^4\left(\frac{1}{3}\nu^2 + \frac{2}{3}\nu + \frac{2}{3}\right) - \dots\right\} \end{aligned}\right\} . \quad (27)$$

Next we develop both δ and ν with respect to a:

$$\left.\begin{aligned} \delta &= \delta_0\sqrt{a} + \delta_1 a\sqrt{a} + \delta_2 a^2\sqrt{a} + \dots \\ \nu &= \nu_0 + \nu_1 a + \nu_2 a^2 + \dots \end{aligned}\right\} \quad (28)$$

Then it follows from (27):

$$\left.\begin{aligned} \sigma(\pi/2) &= \frac{\pi}{2}\delta_0 + \left(\frac{\pi}{2}\delta_1 - \frac{1+\nu_0}{2}\delta_0^3\right)a + \\ &\quad \left\{\frac{\pi}{2}\delta_2 - \frac{3(1+\nu_0)}{2}\delta_0^2\delta_1 + \frac{3}{8}\left(\frac{\pi}{4}\nu_0^2 + \nu_0 + \frac{\pi}{4}\right)\delta_0^5 - \frac{\nu_1}{2}\delta_0^3\right\}a^2 + \dots \\ \xi(\pi/2) &= \delta_0 + \left(\delta_1 - \frac{\pi\nu_0 + 2}{8}\delta_0^3\right)a + \\ &\quad \left\{\delta_2 - \frac{3(\pi\nu_0+2)}{8}\delta_0^2\delta_1 + \frac{2\nu_0^2 + 2\nu_0 + 1}{8}\delta_0^5 - \frac{\pi}{8}\nu_1\delta_0^3\right\}a^2 + \dots \\ \eta(\pi/2) &= \delta_0 + \left(\delta_1 - \frac{2\nu_0 + \pi}{8}\delta_0^3\right)a + \\ &\quad \left\{\delta_2 - \frac{3(2\nu_0+\pi)}{8}\delta_0^2\delta_1 + \frac{\nu_0^2 + 2\nu_0 + 2}{8}\delta_0^5 - \frac{\nu_1}{4}\delta_0^3\right\}a^2 + \dots \end{aligned}\right\} \quad (29)$$

Applications of these formulas will be given later on.

Applied Mechanics. — *Large distortions of circular rings and straight rods.* II. By A. VAN WIJNGAARDEN. (Nationaal Luchtvaartlaboratorium.) (Communicated by Prof. C. B. BIEZENO.)

(Communicated at the meeting of May 25, 1946.)

6. *The closed circular ring, loaded by two diametral forces.* We shall apply our theory-first to the case of a closed circular ring, loaded diametrically by two radial forces P, the first example, treated by BIEZENO and KOCH in their second paper [5]). We attribute the positive sign to P if it is a compressive force and accordingly we attribute the positive sign to the radial deflection u, if it is directed inwards. We write for shortness $\lambda = u/r$. The forces P may be applied at the points $\sigma = 0$ and $\sigma = \pi$ respectively. From reasons of symmetry it follows immediately, that for these points $\varphi = 0$ and $\varphi = \pi$ respectively, and that to the point $\sigma = \pi/2$ corresponds $\varphi = \pi/2$. Further it is evident that $D_0 = P_0/2$, and $N_0 = 0$. So $\nu = 0$ and $\beta = \pi/2$.

If we substitute $\varphi = \pi/2$ in the eqs. (10), they become:

$$\left.\begin{aligned} \sqrt{\alpha} &= \frac{2\sqrt{2}}{\pi} f_1(k,\varphi) \Big|_{-\pi/4}^{0} \\ \lambda(0) &= 1 - \sqrt{\frac{2}{\alpha}}\, f_2(k,\varphi) \Big|_{-\pi/4}^{0} \\ \lambda\left(\frac{\pi}{2}\right) &= 1 - \sqrt{\frac{2}{\alpha}}\, f_3(k,\varphi) \Big|_{-\pi/4}^{0} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (30)$$

Assuming a number of values of k, we calculate from the first of the equations (30) the corresponding values of α, and then from the second and third equation the corresponding of $\lambda(0)$ and $\lambda\left(\frac{\pi}{2}\right)$. For small positive values of α, γ is large and $k < 1$. For $\gamma = 1$ we have $k = 1$, and for smaller values of γ, i.e. larger values of α, evidently k is > 1, so that eqs. (13) must be used to calculate the elliptic integrals. To $\gamma = 0$ corresponds $k = \frac{1}{2}\sqrt{2}$ and consequently $\varphi_{\min} = 0$. A lower inflexion occurs, and we use the eqs. (22), (23) and (24) for case II. For negative values of α, γ is < -1, and k is imaginary; so we use eqs. (15). A survey of the values of k, k^*, k^{**} and $\varphi_{\min}$ as functions of $\lambda(0)$, which itself varies within the limits $1 - \pi/2$

[5]) Compare also R. SONNTAG, Die Kreisringfeder, Ing. Arch., **13**, 380—397 (1943). The author solves the same problem, at least, as far as no inflexions occur. His method is exact but somewhat akward and unnecessarily intricate.

and $1 + \pi/2$, is given in fig. 2. The dotted part of the graph for $\varphi_{\min}$ is that one, where $\varphi_{\min}$, though real, has no physical sense.

For very small positive or negative values of a, we use the expansions of number **5.** As $\nu = 0$, it follows from (28) that $\nu_0 = \nu_1 = \nu_2 = \ldots = 0$. Furthermore $\sigma\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$; so from the first of the eqs. (29) it follows that $\delta_0 = 1$, $\delta_1 = 1/\pi$ and $\delta_2 = 3/\pi^2 - {}^3/_{16}$, and further:

$$\left.\begin{aligned} \lambda(0) &= \left(\frac{\pi}{8} - \frac{1}{\pi}\right) a + \left(\frac{5}{16} - \frac{3}{\pi^2}\right) a^2 \ldots = 0{,}074389\, a + 0{,}008536\, a^2 \ldots \\ \lambda\left(\frac{\pi}{2}\right) &= -\left(\frac{1}{\pi} - \frac{1}{4}\right) a - \left(-\frac{1}{16} - \frac{3}{4\pi} + \frac{3}{\pi^2}\right) a^2 \ldots = -0{,}068310\, a - 0{,}002731\, a^2 \ldots \end{aligned}\right\} \quad (31)$$

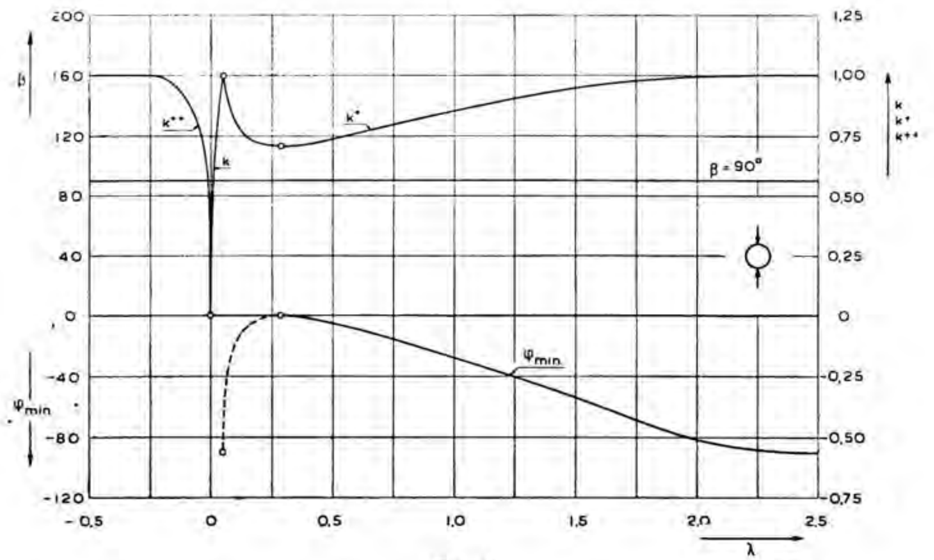

Fig. 2.

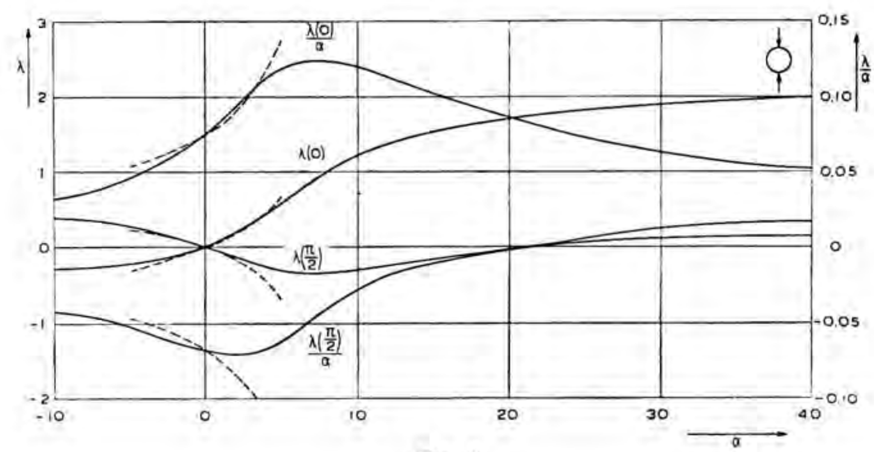

Fig. 3.

These exact developments can be compared with those, derived from the corresponding expressions of BIEZENO and KOCH. The coefficients of α are of course the same, as this term is the "elementary" one; as a matter of fact the authors did not calculate this term separately, but took it from the elementary theory. For the coefficients of α^2 they find 0,006289 and — 0,006425 respectively.

In fig. 3, $\lambda(0)$, $\lambda(\pi/2)$, $\lambda(0)/\alpha$ and $\lambda(\pi/2)/\alpha$ are given as functions of α. The drawn lines give the exact values. The dotted ones represent the approximate values of BIEZENO and KOCH.

7. *The semicircular ring, loaded by a radial force.* We consider the problem of a semicircular ring, clamped at both ends, and loaded in its middle by a radial force P. We attribute signs as in number **6**. Now the shearing force D_0 for $\sigma = 0$ ($\varphi = 0$) is again $P/2$, but the normal force N_0 is not zero, and, moreover, unknown. On the other hand, apart from the condition $\sigma = \pi/2$ for $\varphi = \pi/2$, also the condition $\xi = 1$ for $\varphi = \pi/2$ has to be fulfilled. So, the eqs (10) take the form:

$$\left.\begin{aligned}
\sqrt{\frac{\alpha}{2\sin\beta}} &= \frac{2}{\pi} f_1(k,\varphi)\Big|_{-\beta/2}^{\pi/4-\beta/2} \\
\sqrt{\frac{\alpha}{2\sin\beta}} &= \{\cos\beta\, f_2(k,\varphi) + \sin\beta\, f_3(k,\varphi)\}\Big|_{-\beta/2}^{\pi/4-\beta/2} \\
\lambda(0) &= 1 - \sqrt{\frac{2\sin\beta}{2}}\{\sin\beta\, f_2(k,\varphi) - \cos\beta\, f_3(k,\varphi)\}\Big|_{-\beta/2}^{\pi/4-\beta/2}
\end{aligned}\right\} . \quad (32)$$

In this case we first take a certain value of k and together with this value, some different values of β. Then we calculate the difference of the expressions at the right hand side of the first and the second equation (32). By interpolation we determine that value of β, for which this difference is zero, (as it should be, because both expressions must be equal to the same quantity at the left hand side). If in this way we have found a pair k, β, we calculate α, and from the third equation $\lambda(0)$. The physical interpretation of this proces is, that the ring of section **6** is considered, but now loaded by a second pair of diametral forces at right angles to the first pair, the magnitude of which is determined in such a way, that the displacement of the points $\sigma = \pm\,\pi/2$ is zero.

To start with the process, we solve first the problem for small α, by means of the results of number **5**. If we substitute in the first of the eqs. (29): $\sigma(\pi/2) = \pi/2$, and in the second one: $\xi(\pi/2) = 1$, the values of δ_k and ν_k can be calculated. The first values are:

$$\left.\begin{aligned}
\delta_0 &= 1 \\
\delta_1 &= \frac{\pi - 2}{\pi^2 - 8} = 0{,}610606 \\
\delta_2 &= \frac{-3\pi^6 + 124\pi^4 - 256\pi^3 - 640\pi^2 + 1792\pi - 512}{16(\pi^2 - 8)^3} = 0{,}556041 \\
\nu_0 &= \frac{8 - 2\pi}{\pi^2 - 8} = 0{,}918277 \\
\nu_1 &= \frac{-\pi^5 - 8\pi^4 + 76\pi^3 - 64\pi^2 - 448\pi + 768}{2(\pi^2 - 8)^3} = 0{,}007376
\end{aligned}\right\} . \quad (33)$$

With these values, we find from the third of the eqs. (29):

$$\left.\begin{aligned}
\lambda(0) = {} & \frac{\pi^3 - 20\pi + 32}{8(\pi^2 - 8)}\,\alpha + \\
& + \frac{-5\pi^6 + 6\pi^5 + 204\pi^4 - 680\pi^3 - 576\pi^2 + 4608\pi - 4608}{16(\pi^2 - 8)^3}\,\alpha^2 - \ldots = \\
& \qquad 0{,}0116618\,\alpha + 0{,}0007136\,\alpha^2 - \ldots
\end{aligned}\right\} . \quad (34)$$

This can be compared with the result of BIEZENO and KOCH, where in stead of the coefficient 0,0007136 appears 0,00047.

The value of β for $\alpha = 0$ proves to be 47,4397°, so that we have a preliminary idea, what the value of β amounts to for small values of α. The numerical work is much more complicated than in the case of the foregoing section. The elastic behaviour of the ring appears however to be highly interesting. For $\lambda(0) = 0{,}0974$ and $\alpha = 5{,}5206$ a lower inflexion appears, followed (for $\lambda(0) = 0{,}1236$ and $\alpha = 7{,}1617$) by an upper in-

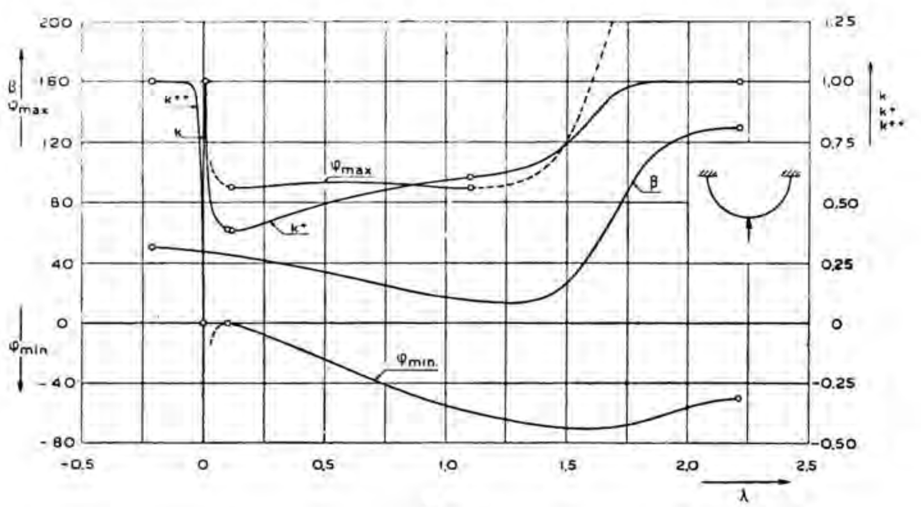

Fig. 4.

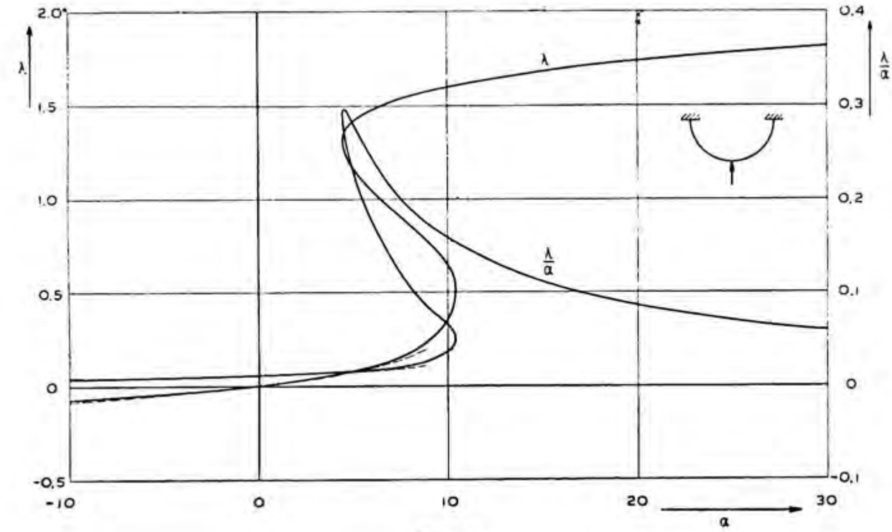
λ
λ/α
α
2.0
1.5
1.0
0.5
0
-0.5
0,4
0,3
0,2
0,1
0
-0,1
-10
0
10
20
30

Fig. 5.

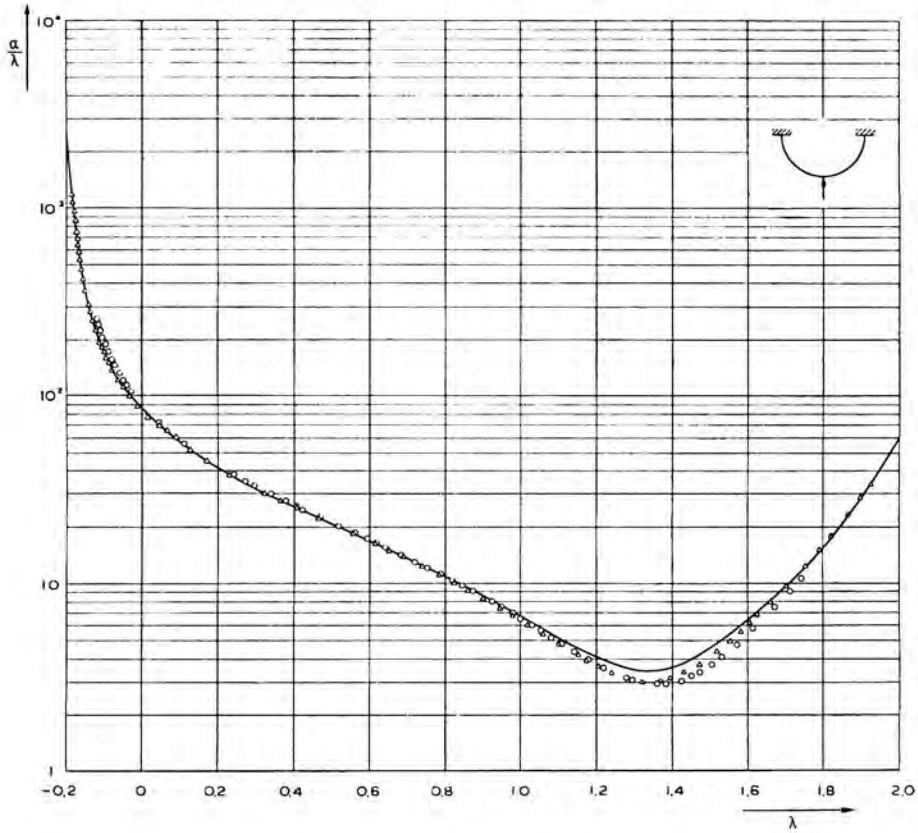
α/λ
λ
10⁴
10³
10²
10
1
-0,2
0
0,2
0,4
0,6
0,8
1,0
1,2
1,4
1,6
1,8
2,0

Fig. 6.

flexion. In the mean time the flexability, as defined by the ratio of displacement and loading force (λ/α) has increased extraordinarily. If $\lambda(0) = 0{,}48$, α assumes a maximum value 10,45. With increasing λ, α then decreases. This region therefore is an unstable one. For $\lambda(0) = 1{,}1000$ and $\alpha = 5{,}5637$ the upper inflexion disappears again. For $\lambda(0) = 1{,}27$, α reaches its minimum value 4,45 and up from here it increases rapidly, to become infinite for $\lambda(0) = 1 + \frac{1}{2}\sqrt{\pi^2 - 4}$. For negative values of $\lambda(0)$, $|\alpha|$ increases rapidly with $|\lambda|$, to become infinite for $\lambda(0) = 1 - \frac{1}{2}\sqrt{\pi^2 - 4}$.

In fig. 4, the values of β, k (or k^*, k^{**}), φ_{min} and φ_{max} are given as functions of $\lambda(0)$, which itself varies within the limits $1 - \frac{1}{2}\sqrt{\pi^2 - 4}$ and $1 + \frac{1}{2}\sqrt{\pi^2 - 4}$. The dotted parts of the lines for φ_{min} and φ_{max} represent again the regions, where these quantities have no physical sense.

In fig. 5 the quantities $\lambda(0)$ and $\lambda(0)/\alpha$ are given as functions of α. The drawn lines are the exact results from our theory; the dotted lines represent the approximate results of BIEZENO and KOCH.

Some experiments were made on two semicircular rings, made of steel strip, with a width of 30 mm and a thickness of 0,2 mm and 0,3 mm respectively. The radius of the ring was $r = 300$ mm. The results are plotted in fig. 6. The drawn line is the theoretical curve and the small triangles and circles represent the experimental values.

8. *The straight beam, supported in two points and loaded by a force in its middle.* We shall now give two other applications of our theory, dealing with a straight beam, supported in two points, and loaded by a normal force P in its middle.

As a first possibility, we assume, that the beam is free to move over its supports, which are supposed to remain at a fixed distance. The length of the deflected beam between the supports is then larger than this distance and increases with the deflection. In this simple example the load P again shows a maximum P_{cr} at a critical deflection u_{cr}, as was allready shown by SONNTAG.

The standard of length, which in the foregoing examples we choose equal to the radius r of the ring, is now taken equal to l, viz. the half distance of the supports (see fig. 7). These supports are supposed to exert on the

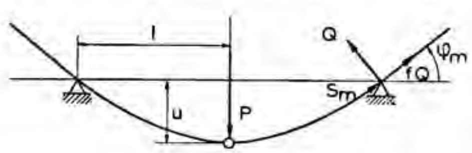

Fig. 7.

beam a normal reaction force Q and a tangential force, due to friction, fQ, where $f = tg\,\varphi_f$ is the coefficient of friction. The sign of f must be chosen in accordance with the fact that the load P is reached in an increasing or decreasing way. The slope of the beam at the supports may be φ_m. Then it follows from considerations of symmetry and equilibrium, that $D_0 = -P/2$ and $N_0 = \frac{1}{2}P \tan(\varphi_m - \varphi_f)$, so that $\beta = -\frac{1}{2}\pi + \varphi_m - \varphi_f$. As for $\varphi = \varphi_m$, we have $d\varphi/d\sigma = 0$, it follows from equation (7) that $\gamma = -\sin\varphi_f$ and consequently $k = 1/\sin(\frac{1}{4}\pi - \frac{1}{2}\varphi_f) \geqq 1$. We therefore have to use the transformation (13) with $k^* = \sin(\frac{1}{4}\pi - \frac{1}{2}\varphi_f)$.

We know, that for $\varphi = \varphi_m$, $\xi = 1$, and so we can find from the second equation (10) for a number of values of φ_m the corresponding α and from this we calculate the relative lengthening $\mu = \sigma_m - 1$, and the deflection $\lambda = u/l$ with the aid of the first and third equation (10).

The calculations have been actually performed for the case that no friction was present $\varphi_f = 0$) and no inflexions occurred in the elastic line. The results are plotted in fig. 8. The approximate results of SONNTAG and

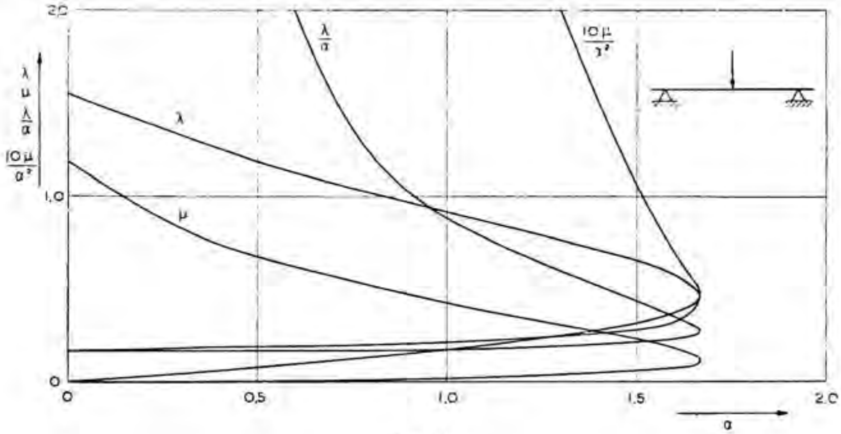

Fig. 8.

those of BIEZENO are very well confirmed. Of course we can give more accurate figures. So we find for the critical load $P_{cr} = 1{,}667\ EI/l^2$, and the corresponding values $\varphi_{m\,cr} = 38{,}30^\circ$, $u_{cr} = 0{,}4766\ l$, and $s_{m\,cr} = 1{,}125\ l$. SONNTAG's values are 1,70; 39°, and 0,48 respectively, the critical length being not explicitly given.

As a last example, we assume that the beam does not move over its supports, so that its length between the supports remains constant ($2l$), but that on the other hand, one of the supports is free to move towards the other (see fig. 9). We suppose this movement not to be influenced by

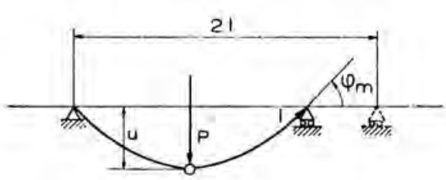

Fig. 9.

friction. The considerations are similar to those in the foregoing example. For $\varphi = \varphi_m$ we now have $\sigma = 1$. So we find α from the first equation (10) and calculate the deflection $\lambda = u/l$ and the contraction $\mu = 1 - \xi_m$ from the second and the third equation. The results are plotted in fig. 10.

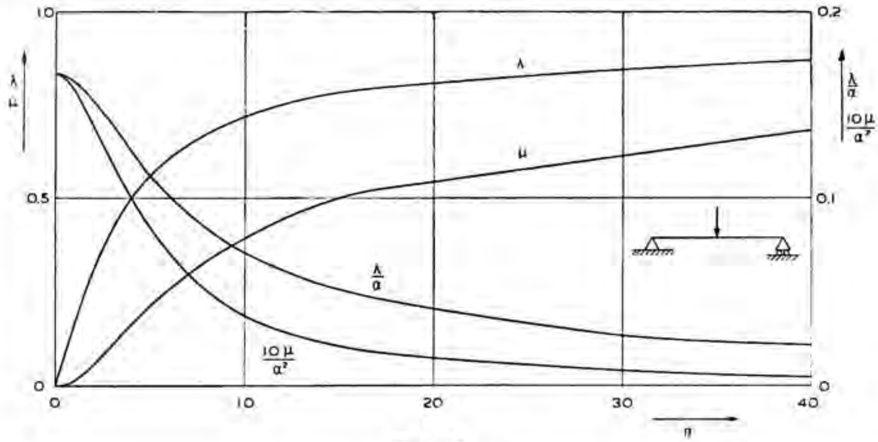

Fig. 10.

Anatomy. — *On the dependence of the weight of the brain on the $^2/_9$, resp. the $^5/_9$ power of the weight of the body.* By R. BRUMMELKAMP. (Communicated by Prof. C. U. ARIËNS KAPPERS.)

(Communicated at the meeting of May 25, 1946.)

It has appeared from DUBOIS' investigations that the weight of the brain E_i with individuals of the same sex and the same kind is proportional to the $^2/_9$ power of the weight of the body P_i,

$$E_i \, (:) \, P_i^{2/9}$$

and with average individuals of closely related kinds (Cat, Lynx, Poema, Panther and Lion; Mouse and Rat; Cavia and Rabbit) the weight of the brain E_g is proportional to the $^5/_9$ power of the weight of the body P_g.

$$E_g \, (:) \, P_g^{5/9}$$

This phenomenon might be explained, if we could assume that the average weight (resp. volume) of every cell-element, ē, from which the brain has been built, is proportional to the $^2/_9$ power of the weight of the body and the number of these elements with individuals of the same sex and the same kind is constant, while with related kinds of animals this number varies according to the $^1/_3$ power of the weight of the body (resp. according to the length of the body) of the average individual of the kind.

From countings of cells by SUGITA, CONKLIN, DONALDSON and LEVI it has become very obvious that the number of neural-cells, N_i, with individuals of the same kind is constant,

$$N_i = \text{constant},$$

while it could be shown by me, with data of my own and those taken from VAN ERP TAALMAN KIP and CATTANEO that the number of neural-cells with closely related kinds, N_g, is proportional to the length of the spinal marrow, resp. the $^1/_3$ power of the weight of the body, P_g, of the average individual of the kind,

$$N_g \, (:) \, P_g^{1/3}$$

It remains for us to make it plausible that the weight (volume) of the average cell-element, ē, is proportional to the $^2/_9$ power of the weight of the body.

It is a difficult matter, however, to determine the volume of a cell-element. We must understand by it the volume of a neurone with the offshoots belonging to it (axis-cylinder and dendrites), glia and intercellular substance.

Fortunately we can reach our goal in an indirect manner. For it has appeared, by weighing and measuring, that the weight of the brain E is proportional to the surface of the cortex S,

$$E\,(:)\,S.$$

(BRODMANN; DAVISON, KRAUS and WEIL; ARIËNS KAPPERS; BRUMMELKAMP.)

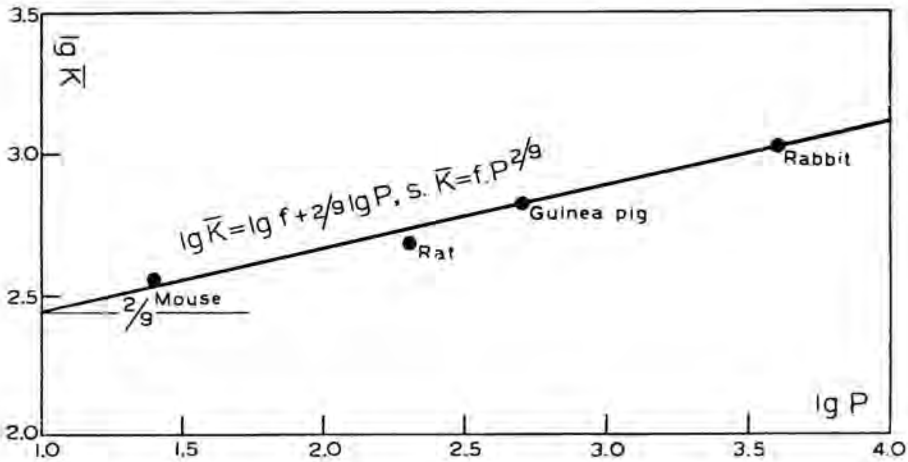

Fig. 1.

Further, it is known [1]) that the sum of all nucleus-volumes, K_z, found in columns of the same base, going perpendicularly through the cortex from pia to marrow, in a cortex not irregularly built by curving or otherwise, is nearly constant in one and the same individual and with individuals of closely related kinds (BRUMMELKAMP; BRUMMELKAMP and VAN VEEN),

$$K_z = \text{constant}.$$

As the total nucleus-volume of the whole cortex, K_t, is equal to the product of the surface of the cortex and the total nucleus-volume per column, it follows from this that with closely related kinds and within the same kind the total nucleus-volume of the cortex is equal to the weight of the brain,

$$K_t\,(:)\,E.$$

For the total nucleus-volume we can also write the product of the number of nuclei, N, and the average nucleus-volume $\bar{K}$,

$$K_t = N\,.\,\bar{K}.$$

[1]) It should be mentioned here that Dr. VAN ALPHEN does not share our opinion in this respect. To this I may add that I measured again the in cytotectonic respect very different cortex of the area calcarina and praecentralis, and found my former conclusions to be right.

so that also

$$E\,(:)\,N\,.\,\bar{K}.$$

It goes without saying that the weight of the brain is equal to the product of the number of elements, N, and the average weight of one element, $\bar{e}$,

$$E = N\,.\,\bar{e},$$

so that it must also be true that

$$\bar{e}\,(:)\,\bar{K}.$$

If we can show that $\bar{K}$ is proportional to the $^{2}/_{9}$ power of the weight of the body, this will also have to obtain for $\bar{e}$.

I have examined this for the mouse, the rat, the cavia and the rabbit. The weight of the body of these animals amounted to resp. 25 gr., 200 gr., 500 gr., and 4000 gr. The average nucleus-volume, $\bar{K}$, was determined in the following way. Going through the cortex from pia to marrow, the largest outline of all the nuclei of the neural-cells that became visible was drawn in Nissl-preparations 20 μ thick, with the help of the microscope (ocular Leitz 8; objective Leitz, oil-immersion 16, fluorite-system) and the camera lucida of Abbe. If we consider the nuclei as ellipsoids of revolution (BOK), we can establish from the dimensions of the long and the short axis, a and b, because $I = {}^{1}/_{6}\,\pi\, a\,.\,b^{2}$, the volume of each nucleus separately and by summation and succeeding division come to the average nucleus-volume, $\bar{K}$; see table. Every average is the result of more than 2000 nucleus-measurements.

	Average nucleus-volume $\bar{K}$	Weight of the body P
Mouse	361	25
Rat	496	200
Cavia	644	500
Rabbit	1.081	4.000

If we draw in the diagram lg $\bar{K}$ against lg P, the points of relation appear to lie on a right line with a direction $^{2}/_{9}$, from which follows immediately that for these kinds of animals it can be said, at least approximately, that

$$\bar{K}\,(:)\,P^{2/9}$$

from which follows also, in connection with the preceding, that

$$\bar{e}\,(:)\,P^{2/9}$$

or in words, that the weight (volume) of the average cell-element, $\bar{e}$, is indeed proportional to the $^{2}/_{9}$ power of the weight of the body, P.

Two questions arise now. The first is: whence this proportion between the number of brain-cells, N, and the $^1/_3$ power of the weight of the body, resp. first power of the length of the body? We shall try to find an answer to this.

It has been known since SPEMANN that the differentiation of the neural plate is dependent on the chorda dorsalis lying beneath the ectoderm. KINGSBURY, BRUNS, LEHMANN a.o. showed that this dependence is of a quantitative kind and that the differentiation only extends over a distance over which direct contact is present between the chorda and the ectoderm. RAVEN could make it plausible that the transferring of a chemical agens plays an important part here. So there is a direct proportion between the length of the chorda dorsalis and that of the neural plate. It is reasonable to suppose that with closely related kinds always the same fraction of the available number of neural cells appears as matrix of the great brain. The consequence of this is a proportion between the number of these matrix-cells (and, with equal division-frequency, of the number of brain-cells) and the number of cells that build up the neural plate. For the same reason, because of the longitudinal arrangement of the cells of the neural plate the length of this plate will also be proportional to the number of brain-cells. As, however, on the other hand the length of the neural plate is determined by the length of the chorda dorsalis and this itself determines the length of the stem of the body, resp. the length of the body, we can also understand that there is a direct proportion between the number of cells of the neural plate and the length of the body, resp. the $^1/_3$ power of the weight of the body and therefore also of the number of brain-cells and the above-mentioned body-quantities.

Secondly, we ask ourselves whether a plausible explanation can be given of the fact that with closely related kinds the average nucleus-volume $\bar{K}$ is proportional to the $^2/_9$ power of the weight of the body, resp. the $^2/_3$ power of the length of the body. The following consideration may be useful. Neurones with long offshoots have a large cell-volume; inversely, neurones with short offshoots have a small cell-volume. Long offshoots belong to a great length of the body, short offshoots of homologous cells to a small length of the body. Probably the volume of homologous neural cells varies directly proportional to the length of its offshoots, resp. to the length of the body of the individual in question.

From measurements of LEVI, taken on the spinal ganglia and the cells of the ventral horn of a great number of kinds of animals, varying much in length of body, it can at any rate be shown convincingly that the volume of these cells, C, is proportional to the $^1/_3$ power of the weight of the body, resp. the first power of the length of the body, L,

$$C \, (:) \, L.$$

BOK showed that in one and the same individual the $^2/_3$ power of the cell-volume is proportional to the nucleus-volume,

$$C^{2/3} (:) K.$$

If a general significance must be ascribed to the two relations, it also follows that

$$\bar{K} (:) L^{2/3}, \text{ resp. } P^{2/9}.$$

Apart from these considerations the fact remains, established by measurement and weighing, that with the above-mentioned kinds of animals the number of brain-cells (N) is proportional to the first power of the stem-length (c.q. the $^1/_3$ power of the weight of the body) and that the average volume (c.q. weight), $\bar{e}$, of a cell-element is proportional to the $^2/_9$ power of the weight of the body, from which follows that the weight of the brain ($E = N \cdot \bar{e}$) is proportional to the $^5/_9$ power of the weight of the body in average individuals of related kind and porportional to the $^2/_9$ power individuals of the same kind.

Summary.

An explanation is given of the dependence of the weight of the brain on the $^2/_9$ power of the weight of the body in individuals of one and the same kind and on the $^5/_9$ power of the weight of the body with the average individuals of closely related kinds.

It appears that with closely related kinds the number of cell-elements from which the brain has been built is proportional to the (stem-)length, resp. the $^1/_3$ power of the weight of the body and is constant with individuals of one and the same kind. It is shown in an indirect way that the average size of a cell-element is proportional to the $^2/_9$ power of the weight of the body. It follows immediately from this that with individuals of one and the same kind the weight of the brain is dependent on the $^2/_9$ power of the weight of the body and with the average individuals of closely related kinds on the $^5/_9$ power of the weight of the body. We should understand by the volume of a cell-element the volume of a neurone with the offshoots belonging to it, glia and intercellular substance. We mean by the volume of a neural cell the volume of the body of a neural cell without its offshoots.

The proportion between the number of cell-elements and the $^1/_3$ power of the weight of the body, resp. the (stem-)length becomes plausible when we point to the direct interdependence of the length of the chorda (the future body-stem) and the neural plate (of which a certain fraction, probably constant with closely related kinds, becomes matrix of the brain).

The dependence of a cell-element on the $^2/_3$ power of the stem-length, resp. the $^2/_9$ power of the weight of the body, is plausible when we consider, on the one hand that the average size of a cell-element is proportional to the average nucleus-size of the neural cell belonging to it and on the other

hand that the average nucleus-size is probably proportional to the $^2/_3$ power of the average nucleus-volume, which itself is again directly proportional to the length of the offshoots belonging to it and is thus also proportional to the length of the body-stem; the average nucleus-size will then be dependent on the $^2/_3$ power of the stem-length, resp. of the $^2/_9$ power of the weight of the body. This last fact could be confirmed experimentally for the Mouse, the Rat, the Cavia and the Rabbit.

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Palæontology. — *The evolution of the skeleton of Rhinoceros sondaïcus Desmarest.* By D. A. HOOIJER. (Communicated by Prof. S. T. BOK.)

(Communicated at the meeting of May 25, 1946.)

In an earlier paper (HOOIJER, 1946) I described the fossil remains of *Rhinoceros sondaïcus* Desmarest from the Pleistocene of Java collected by EUG. DUBOIS some fifty years ago. I had many recent skulls and also four skeletons of the species for comparison. The fossil teeth and many of the limb and foot bones proved to be larger than the recent, with the exception of the humerus, femur and tibia which present smaller dimensions than the recent. I did not, then, especially emphasize this point, and merely stated (l.c., p. 76) that the femora might belong to some small variety. There is nothing peculiar in the fact that fossil bones and teeth of a still living species average larger than the recent; on the contrary this is a common thing to students of Pleistocene and prehistoric Mammals. Many animals have diminished in size, both on continents and on islands, since the Pleistocene, and the purpose of the present paper is to make it evident that the reduction in size may affect different parts of the skeleton to a different degree. This is shown by the rhinoceros material I have worked upon, and the explanation will be offered below.

Let us turn now to the facts. In the following table I give the observed ranges and means for the dimensions of upper toothrow and limb and foot bones of recent and fossil *Rhinoceros sondaïcus* Desmarest, extracted from my paper of 1946. Much more convenient than to compute many indices it is to follow SIMPSON (1941) in constructing ratio diagrams of the dimensions of the different bones. This principle has been fully explained by SIMPSON (l.c., pp. 23—25), but a short explanation will not be out of place here.

Rhinoceros sondaïcus Desmarest.

Length of	Recent		Fossil	
	Range	Mean	Range	Mean
$pd^1 - M^3$	242—255	249	267—272	270
Humerus	[illegible]	392	386 389	388
Radius	318 329	324	343—359	349
Metacarpal III	170—173	171	187	187
Femur	440—495	474	438 476	459
Tibia	323 335	328	320 337	330
Metatarsal III	150 155	153	165	165

First the direct measurements are converted to their logarithms, and then the differences are calculated from some one standard, for which I selected the logarithms of the means of the measurements found in the fossil specimens. They are set in a straight vertical line, the larger observations fall to the right of this line, and the smaller to the left. The more nearly the line, connecting the means of the corresponding values in some other material, approaches a vertical line, the closer the similarity in proportions throughout the parts measured. It goes without saying that this will do regardless of absolute size which is ignored here; the differences between the logarithms represent the logarithms of the ratios. Size is of no importance; I have shown that the fossil remains are doubtless specifically identical with the recent Javan rhinoceros.

A glance at fig. 1 will show that the humerus and the femur, and, to a lesser extent also the tibia, have disproportional dimensions in the recent skeletons as compared to that of their forerunner in the Pleistocene. The fossil animal had the radius, tibia and distal limb segments longer relative to humerus and femur than the recent. Why should fore arm and manus, leg and pes have become shortened in the course of time?

The explanation presents itself immediately. It is exactly the same trend of evolution observed in some phyla of the brontotheres (Titanotheres) of North America, viz., the transformation from a mediportal to a graviportal type (OSBORN, 1929, especially Chapter IX). Humerus and femur lengthen, radius, tibia, and metapodials shorten when passing from swift-moving to slow-moving animals. In our example the tibia is shortened to a lesser degree than the radius, and the metatarsal seems to abbreviate less than the metacarpal. I was desirous to know whether the other recent rhinoceroses present proportions throughout the parts of their skeleton similar either to the recent or to the fossil Javan rhinoceros or not. Skeletons of *Dicerorhinus sumatrensis* (Fischer) and *Diceros bicornis* (L.) are in the Leiden Museum, and I took the measurements of *Rhinoceros unicornis* L. from CUVIER (1822) (with a slight correction for the length of the humerus which he measured in another way than I did). The ratio diagram, with the same standard of comparison as that in fig. 1, is given in fig. 2. It shows that the proportions for *unicornis* are more like the recent *sondaïcus* than those of the African *bicornis*, which has the maxima in the fore arm and leg instead of in the proximal limb segments. *D. sumatrensis* has an especially short radius but for the rest comes nearest to the fossil *sondaïcus* in the comparative proportions of its limb segments. It must be kept in mind that these similarities in ratios imply no genetic relations but represent only parallelisms in adaptation to speed and weight.

D. sumatrensis is regarded by OSBORN (1898) and others as the most primitive among the living species of rhinoceroses. The subfossil humerus from Sumatra described and figured by me (HOOIJER, 1946, pp. 26—27, pl. X fig. 6) constitutes all we know of the early history of the postcranial skeleton of the Sumatran form. When plotting this specimen against the log difference scale in the diagram, the point is seen to fall much to the right of the standard line, while the recent specimens all remain to the

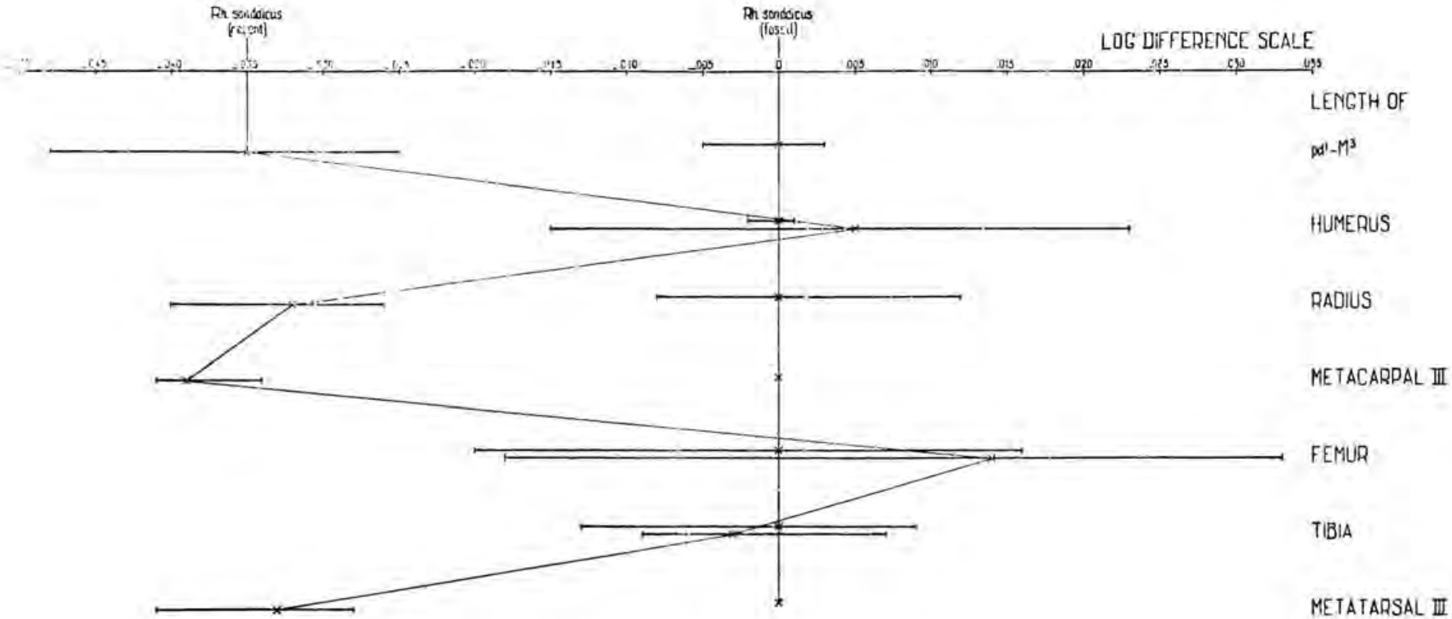

Fig. 1.

Ratio diagram comparing dimensions of recent and fossil teeth and bones of *Rhinoceros sondaïcus* Desmarest. The means of the proportions found in the fossil specimens are taken as the standard of comparison and are set in a straight vertical line. The observed ranges are represented as horizontal lines, with a cross at the mean value.

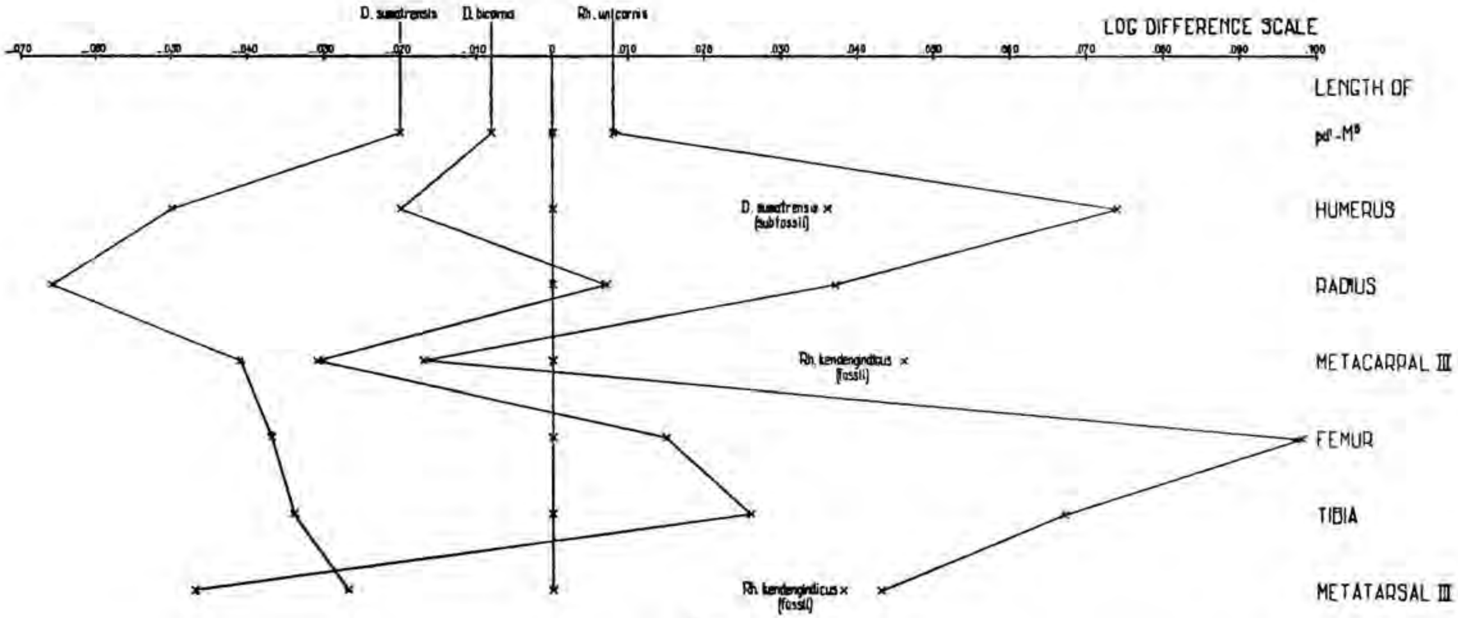

Fig. 2.

Ratio diagram comparing dimensions of *Dicerorhinus sumatrensis* (Fischer), *Diceros bicornis* (L.) and *Rhinoceros unicornis* L. Same standard of comparison as in fig. 1. Isolated crosses represent the subfossil humerus of *D. sumatrensis* (Fischer) from Sumatra and the metapodials of *Rh. kendengindicus* Dubois from the Pleistocene of Java in the Dubois Collection.

left of it. The teeth were larger too, but it is certain that the Sumatran rhino had a different story than the Javan in which latter the humerus remained of equal size or rather lengthened since the Pleistocene.

Several bones have been found associated with teeth in a cave deposit in Sarawak, Borneo. They might very well belong to *sumatrensis* (see HOOIJER, l.c., p. 10), but this is uncertain until the specimens will turn up again in the British Museum collection.

Apart from an uncertain astragalus from the Narbada beds (HOOIJER, l.c., p. 83) we know nothing about the post-cranial skeleton of *Rh. unicornis* L. in prehistoric or Pleistocene times. From Java I have described and figured two complete metapodials as belonging to a species, *Rh. kendengindicus* Dubois, which is distinguished from *unicornis* only by its less hypsodont teeth, more molariform premolars, and the upper molars being comparatively narrower posteriorly. The post-cranial remains comprise fragments of humerus and femur of the same size as recent *unicornis*, and a third metacarpal and metatarsal which are likewise larger than the corresponding bones in recent *sondaïcus*. The latter have been plotted in the diagram; they are not so much different in size as the corresponding bones in *unicornis* and fall near a single vertical line with the Sumatran humerus. It would be very interesting to know how the other parts of the skeleton of *kendengindicus* are; the teeth only indicate that the species combines progressive and primitive characters relative to *unicornis* which also dates from the Pleistocene.

Thus *sondaïcus* is the only Asiatic species of rhinoceros which is represented by a fair amount of material, which enables us to follow its history since the end of the Tertiary. The species is now very near complete extinction; probably less than seventy of this, one of the rarest and most famous of the large Mammals (HARPER, 1945, p. 381), are in existence at the present day (LOCH, 1937, p. 146). Recently COLBERT (1942) has postulated that *sondaïcus* (of which he examined only recent skulls) truly is a persisting primitive form and anatomically may be regarded as at about a lower Pleistocene or perhaps an upper Pliocene stage of development; it is, he says, a true living fossil. These conclusions are based on the comparison with skulls of the lower Pliocene genus *Gaindatherium*, of the Pleistocene *Rh. sivalensis* and *Rh. sinensis*, and of the recent *Rh. unicornis*. *Rh. sondaïcus* is shown to be intermediate in its skull characters between *Gaindatherium* and the more advanced Pleistocene and recent species of *Rhinoceros* mentioned above. COLBERT states that every distinguishing character shows an advance in the Indian rhinoceros over its expression in the Javan form, and surmises that the same holds for the post-cranial skeleton. This I am able to confirm, but matters turn out to be more complicated than first supposed.

When we trace *sondaïcus* back into the Pleistocene its limb and foot structure becomes more different from that of a graviportal type as the recent *unicornis* and is transformed to a type like *sumatrensis*, the only recent species of rhinoceros which is regarded as mediportal (OSBORN, 1929, pp. 749, 780). In its progression into the graviportal or slow-moving type *sondaïcus* is not so advanced as the recent Indian rhinoceros, as shown also in the following table of indices which I have computed to enable the direct comparison with the tables of OSBORN (l.c., pp. 735—739). Of

	Tibio-femoral	Metatarso-femoral	Radio humeral	Metacarpo-humeral
Rh. unicornis L.	67	32	83	39
Rh. sondaïcus Desm. recent	61	32	83	44
Rh. sondaïcus Desm. fossil	72	36	90	48
D. sumatrensis (Fischer)	72	37	89	50

course this series does not represent a phylogenetic sequence but only a sequence of adaptive types.

The present contribution shows the mode of evolution known up to now only from certain series of species in successive stages to be found also within a species. Within one and the same species, for, as I have shown (HOOIJER, 1946) tooth for tooth and bone for bone the Pleistocene *Rhinoceros sondaïcus* Desmarest is identical with the living Javan rhinoceros. Should subspecific names be required, the Pleistocene form must be named *Rh. sondaïcus sivasondaicus* Dubois, and the recent *Rh. sondaïcus sondaïcus* Desmarest.

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Comparative Physiology. — *The presence of α- and β-amylase in the saliva of man and in the digestive juice of Helix pomatia.* I. By L. ANKER and H. J. VONK. (From the laboratory of Comparative Physiology, University of Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of April 27, 1946.)

In 1889 WIJSMAN showed that malt-extract contains two enzymes causing cleavage of starch. These are now called α-amylase and β-amylase, according to KUHN. These names were introduced by KUHN, because with the activity of α-amylase the maltose first is formed in the α-form, with that of the β-amylase in the β-form [1]). We know chiefly also through WIJSMAN that with the activity of α-amylase the residual product is not coloured by iodine, while with that of β-amylase the residual product is coloured with iodine. For the separation of these enzymes WIJSMAN used the separation-method of two substances discovered by BEIJERINCK, by means of a gelatine-plate often used in bacteriology. When he brought a drop of malt-extract on such a plate which contained soluble starch, after a few days, by colouring with a solution of iodine in potassium-iodide, he found at the place of the drop an uncoloured field surrounded by a purple ring, round which a very narrow clear-blue ring formed the transition to the dark blue colour of the rest of the medium. This observation led him to suppose that at the height of the purple ring, by a quicker diffusion, there was only one enzyme, which decomposes starch, not only into maltose, but also into a product that still gives a purple colour with iodine. He proved this, because part of this ring not yet coloured with iodine, transferred to another plate, there only gave a diffusion-field coloured purple by iodine. This enzyme is the β-amylase and the coloured disintegration-product is erythro-granulose which is decomposed further by α-amylase into products (chiefly maltose) that can no longer be coloured with iodine.

In the same way in 1934 GIESBERGER examined, among other things, the saliva of man. He found a weak purple ring round the uncoloured field and thought that by this he had demonstrated the presence of β-amylase beside that of the α-amylase, chiefly present (as appeared from the big uncoloured field). However he could not carry out the critical experiment, the transferring of part of the ring to another plate, the ring being too narrow for this purpose.

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[1]) This is shown with the polarimeter. After some time the normal equilibrium mixture of 36 % α-maltose and 64 % β-maltose arises. KUHN found that pancreasamylase (at least, chiefly) is an α-amylase. Some experiments with the polarimetric method will be discussed in a following communication.

In the same year, however, PURR, with a method which makes use of the connection between the degree of saccharification and the colouring with iodine, confirmed the current opinion that saliva-amylase is the purest α-type which is known; in this investigation he also showed that pancreas-juice of the pig contains β-amylase, but in an inactive form, which he could activate with vitamin C.

In the following year GIESBERGER published a rectification to his work of 1934. For when repeating the diffusion-experiments, he had only succeeded a few times in getting a purple ring. In these cases also the starch in the rest of the plate was not coloured dark-blue, but purple-blue. In most of the cases he found a clear blue ring round the colourless diffusion-field. Because of these new results he thought he could not maintain his original statement that saliva-amylase is a mixture of two enzymes. The clear blue ring would mean that as a consequence of a weak enzyme-concentration, unchanged starch is still to be found there, while the purple ring would point to a product of disintegration already present in the starch and not to an activity of β-amylase from the saliva.

That this ultimate conclusion is insufficiently founded may appear from the following theoretical considerations and from experiments taken to test them.

GIESBERGER does not mention a p_H at which he made his experiments, so that we may suppose that he worked at the acidity of the gelatine-starch plate. In such weakly acid surroundings (as appeared from our measurements 5,8) according to the p_H-optimumcurves of OHLSSON and those of VAN KLINKENBERG, the activity of β-malt-amylase has already been reduced by the p_H which is high for this enzyme, while α-malt-amylase acts about optimally here [2]). For the amylases of pancreas-juice and saliva (the former of which is doubtless and the latter probably α-amylase) the optimum lies between p_H 6.2 and 6.8 (dependent on the buffer used). Though the situation of the p_H-optima of enzymes is influenced by additional mixtures (the optimum for saliva-amylase does lie a little higher than for that of malt), it is not unreasonable to suppose that also for a β-amylase if present in the saliva or the pancreas-juice, the optimum will lie lower than that of the α-amylase. If this supposition is right, we may expect that under the circumstances of GIESBERGER's experiments the activity of a β-amylase if present in a very small quantity, becomes less clearly visible; for the few residual products of this disintegration that can be coloured with iodine will be changed relatively quickly into the products that are not coloured any more, by the α-amylase.

A comparison of the p_H-optimumcurves of α- and β-amylases will no more make us expect a purple ring at a higher p_H. On the other hand, the possibility of discovering β-amylase in the saliva at p_H-values lower than

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[2]) The optimum for α-malt-amylase lies at p_H 5.75, that for β-malt-amylase at p_H 4.9. For the former enzyme the optimum-curve runs steeply upward, for the latter it is flatter.

those of an unbuffered gelatine-plate, increases, in the first place because this enzyme becomes then more active and secondly because the resulting residual products will be disintegrated more slowly by α-amylase. Anyhow, it is wrong to conclude that β-amylase is absent, if the p_H was not sufficiently varied when the experiments were made.

The following experiments confirm that these expectations are reasonable (Table I). The experiments 1—3 of this table are repetitions of those made by GIESBERGER. The plates [3]) remained resp. 2, 3 and 4 days in the thermostat (we shall say more presently about the importance of this indication of time). In these experiments the purple ring was also absent; the white diffusion-field was surrounded by a clear blue ring. By this a reproduction of GIESBERGER's last results was obtained. In experiment 4 a weak NaOH-solution was added to the saliva and in experiment 5 a weak HCl-solution. In the letter case the colourless field was indeed surrounded by a narrow purple ring. Round this there lay a clear blue ring, about as broad, the presence of which can also be known in advance, as the residual products formed when the β-amylatic disintegration takes place can be coloured blue with iodine a considerable time after the beginning of the enzyme-activity (SAMEC, HANES) and only become purple via blue-violet and violet towards the time that the disintegration-limit is reached. So this result of experiment 5 is a first argument for β-amylase being present in the saliva. In order to be as certain as possible that this β-amylase did not come from food-remnants or other pollutions of the mouth we always used saliva which had been produced before breakfast after thorough cleansing of the mouth-cavity.

TABLE I.

Experiment	Digestive juice	Diffusiontime in days	Result
1	Saliva	2	—
2	„	3	—
3	„	4	—
4	„ with NaOH	2	—
5	„ „ HCl	2	+

GIESBERGER is right in not ascribing the purple ring found by him in some cases, to the activity of β-amylase, because in those cases the unchanged soluble starch also had a purple-blue colour in the plate, which might point to disintegration-products already being present. But he is wrong in taking the blue ring for the place where there is starch still unchanged as a consequence of a weak concentration of α-amylase. For if one lets the diffusion-experiments take place on an alkaline medium (at p_H 8

[3]) They contain 8 % gelatine (powder-gelatine Twee Torens, Delft) and 0.25 % soluble starch (amylum solubile of KAHLBAUM) without addition of buffer. For the preparation cf. the dissertation of VAN KLINKENBERG.

to 9) the uncoloured field is immediately surrounded by the blue of the unchanged starch.

It follows from this that α-amylase by itself (the β-amylase of saliva is probably also inactive at this p_H), does not cause a clear blue ring round the uncoloured diffusion-field; in other words the blue ring is an indicator for the presence of β-amylase.

This inequality of the diffusion-field appearing after colouring with iodine, which must be based on the fact that the α- and the β-amylase affect the starch-molecule differently, is in complete accordance with the characteristics that have been given of these enzymatic changes. The α-amylatic disintegration is characterised by a quickly disappearing colouring with iodine, because according to modern views this enzyme causes the starch-molecule to break up into fragments that can no longer be coloured, already in the first stage of the development. With the β-amylatic disintegration the colouring blue with iodine continues to exist a long time and only changes into purple when the disintegration-limit is reached. For this enzyme separates one maltose-molecule each time from the extremities of the starch-molecule, so that the disintegration is more gradual than that caused by the α-amylase.

WIJSMAN also mentions a blue ring round the purple one. In view of what was known at that time of enzymatic starch-disintegration it can be understood that he takes this ring for the place where the amylum, by the β-amylase, is made more suitable to be coloured by iodine.

At present, according to us, the most likely explanation of the phenomena is this:

a) a colourless diffusion-field, surrounded immediately by the blue of the unaffected starch, points to exclusive presence or activity of α-amylase,

b) a colourless diffusion-field, surrounded by a clear blue ring, points to a low concentration or little activity of β-amylase, present in the solution of α-amylase.

c) a colourless diffusion-field, surrounded by a purple and a blue ring, points to a higher concentration or a higher activity of β-amylase present in the solution of α-amylase.

In order to determine the p_H where the ring is clearest the experiments 6—17 were made (Table II) [4]. When in the following we speak of a ring, we mean by this the purple ring. From this series of experiments appears that the ring is clearest at $p_H = 4{,}5$—$4{,}7$ both for saliva and for the digestive juice of HELIX.

This confirms the supposition expressed above that β-amylase will be demonstrated best at a weakly acid reaction.

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[4]) The plates contain 8 % gelatine and 0.25 % soluble starch as in the experiments of table I. They were here buffered with potassium-biphtalate and NaOH (according to CLARK and LUBS) in various proportions.

A factor which may have a considerable influence on the results of the experiments is the time during which one allows the diffusion to proceed. In order to get the rings sharply divided from one another, it appeared necessary to let the diffusion take place at a low temperature (in a thermostat at 3° C) viz. during two days. Also at room-temperature one can make the phenomenon appear, but then iodine should be added sooner, as the diffusion takes a quicker course at a higher temperature. It is easy to see that when the period of diffusion too short, there is no sufficient differentiation. But also if one makes the period too long, the phenomenon becomes unclear. This is illustrated by the experiments 6—11 of Table II. It follows,

TABLE II.

Experiment	Digestive juice	Diffusiontime in days	Results					
			3.8	4.5	4.7	4.9	5.1	5.3
6	Saliva	2		+	+	+?	+?	—
7	,,	3		+?	+?	—	—	—
8	,,	4		+?	+?	—	—	—
9	dig. juice Helix	2		++	++	++	++	+?
10	,,	3		+	+	+	+	+?
11	,,	4		+	+	+	+	+?
12	,,	2	+		++			
13	,,	2	+		++		+	
14	,,	2		++			++	
15	,,	2		++	+	+	—	
16	Saliva with 60 % aethanol	2	+	+	+?			
17	Saliva (filtered)	2	+?	+	—			

— = no purple ring.
\+ = purple ring.
++ = broad purple ring.
+? = hardly visible purple ring.

among other things, from these experiments

a) that the diffusion at 3° C should not be allowed to take longer than two days,

b) that the digestive juice of *Helix pomatia* gives a clearer blue ring than the saliva of man, from which follows that the former probably contains more β-amylase.

When these experiments were carried out, care was taken that the gelatine-plates were of exactly the same thickness (equally large Petri-dishes with an equal number of cm^3 of gelatine). As it was often very difficult to judge the breadth of the ring (so that mostly we had our observations checked by two persons), it is important to compare the rings at different p_H-values only when this requirement that the plates are equally thick, is met. The p_H values were measured with the Hellige-comparator. In the whole studied p_H-region the degree of acidity of liquid gelatine appeared to lie 0,2 higher than that of solid gelatine. The influence of saliva and digestive juice of Helix (the p_H of which appeared to lie resp. at 6,8 and 5,3) on the degree of acidity of the plates was not examined.

In view of the results of PURR, who, it appeared, could activate the β-amylase from the pancreas of pigs by the addition of vitamin C, we examined what influence this vitamin has on the ring. A tablet of "Cebion" of E. MERCK, Darmstadt, was dissolved in 2, 4 and 6 cm^3 of water. Such a tablet contains 0,05 gr. of crystalline ascorbic acid. To one drop of saliva one drop of vitamin-solution was added. In all the three cases the ring was clearer than in the controls. If the vitamin C was brought into the gelatine-media in equally strong concentrations, it appeared to have no influence. These experiments took place at p_H 4,5. In an experiment following upon this at p_H 6,4 a broad purple-blue ring arose with gradual transitions, both in- and outside. No opinion can as yet be offered on the interpretation of this phenomenon.

Finally, in imitation of WIJSMAN, the enzyme-solution (the saliva) was heated for *ten* minutes at 70° C in order to render (analogous to WIJSMAN's experiments) the β-amylase, if present, inactive. It appeared that the activity of the α-amylase was also so strongly reduced by this treatment that a light-blue field at the place of the drop was the only visible result. The diffusion-field of a saliva-solution heated for *five* minutes appeared to agree much with that on an alkaline medium (so only activity of α-amylase). This is a new argument for the opinion, formed because of the diffusion-experiments, that in the saliva beside the α-amylase a β-amylase is present, the latter of which is rendered inactive by 5 minutes' heating, while the former remains intact. The transition between the colourless field and the blue of the uneffected starch was somewhat less sharp than in the experiments on an alkaline plate, but there was hardly any blue ring left.

The facts stated in the preceding experiments form together a very strong indication for the presence of a small quantity of β-amylase in the saliva.

Summary.

According to the diffusion-method introduced by WIJSMAN in enzymology we examined whether β-amylase was present in the saliva of man and in digestive juice of *Helix pomatia*. The following results:

1) the presence of a purple and a clear-blue ring round the colourless diffusion-field in acid surroundings (optimal at $p_H = 4{,}5$) and their absence in alkaline surroundings,

2) the broadening of the purple ring by the addition of vitamin C,

3) the absence of the rings after 3 minutes' heating at 70° are all in favour of the supposition that β-amylase occurs beside the chiefly present α-amylase in the saliva of man, while a corresponding, even clearer diffusion-picture at a lower p_H points to the presence of β-amylase beside the chiefly present α-amylase in the digestive juice of *Helix pomatia*.

The presence of β-amylase in the digestive juices can have a certain biological significance. For the starch is decomposed further and especially more quickly by the α-amylase, which is chiefly present, in the presence of β-amylase. The activity of vitamin C to render β-amylase active might play a part here, as vitamin C is always present in the intestinal canal of herbivorous animals.

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Comparative Physiology. — *The determination of the coagulation-time of bloodplasma by means of apparatuses of* LINDERSTRØM–LANG *adapted to this purpose.* By H. J. VONK, A. STOLK and C. H. NUYTEN. (From the laboratory of Comparative Physiology, University of Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of April 27, 1946.)

Numerous different methods have been indicated to determine the blood-coagulation-time. A simple one is the sucking up of blood in a long glass capillary, which is put horizontally (if possible in a thermostat), after which every other minute small pieces of this capillary are broken off. When with this breaking off a fibrin-thread arises, the coagulation is assumed to be completed (SABRAZÈS, 1906). This method is rather primitive. Even though the capillaries are put in horizontal tubes which run through a water-thermostat, they must be touched and again and again taken out of their surroundings with a constant temperature. An advantage of this method is that a small quantity of blood is sufficient.

One can also pull a white horse-hair, which has been cleaned with alcohol and ether, through a glass capillary filled with blood. If the hair comes out no longer white but red, the point where the coagulation begins has been reached. After total coagulation the hair again comes out white (VIERORDT, 1878). There are about as many advantages and drawbacks as with the preceding method.

The blowing out of blood from glass capillaries can also be used to determine the coagulation-time (WRIGHT, 1893). Also the rising in them which of course does not take place after coagulation (LÖWENTHAL).

FONIO has given a short summary of these and other methods [1]). Later WOLVIUS [2]), FESTEN [3]) and PROOST [4]) have worked out a very useful method, based on the absorption of light by the coagulating blood sample. To determine the light-absorption WOLVIUS used an extinction-meter of MOLL, FESTEN and PROOST a Pot-galvanometer of Cambridge combined with a photo-cell. The only drawback of the method according to FESTEN and PROOST seems to me that the sample is not kept at a constant temperature. An advantage is that the blood is not set in motion, as is the case with so many other methods. FESTEN and PROOST do not give figures

[1]) A. FONIO, Die Gerinnung des Blutes, Hndb. d. norm. u. path. Physiologie, VI, 1 p. 307—411, especially p. 358—364 (1928).

[2]) J. WOLVIUS, Diss. Utrecht 1923 (Een objectieve methode ter bepaling van het verloop der bloedstolling).

[3]) H. FESTEN, Nederl. Tijdschr. v. Geneesk. **83**, I, 396 (1939).

[4]) J. B. PROOST, Diss. Utrecht 1941 (Het phosphatiden- en cholesterinegehalte van het serum tijdens senium en puerperium en het eventueel verband met den stollingstijd van het bloed).

from which it appears that the method can be reproduced on one and the same blood-sample. A certain drawback of WOLVIUS' method is that a very expensive extinction-meter is necessary for it.

In the method which we are now going to describe, use is made of part of the apparatuses which LINDERSTRØM–LANG [5]) indicated for the carrying out of his micro-titrations. Here he uses a small test-tube, the content of which is about 1 or 2 cm³. The sharply drawn-out point of the microburette is put in the titration-liquid. The solution with which the titration is done is driven out of the microburette by screwing up a mercuric pile. As with this way of titration the liquid in the vessel cannot be shaken, LINDERSTRØM–LANG brings about the mixture by making a small glass ball, in which a little Ferrum reductum (or a small piece of iron wire) has been melted, jump up and down by means of an electromagnet provided with a mercury-interruptor. It seemed to us that this method of stirring could also be made to serve the determination of the coagulation-time of blood, at least of recalcified oxalate-plasma or of oxalate-plasma with thrombine etc. The titration-vessel may then be filled with oxalate-plasma, which can be made to coagulate by adding a solution of calcium-chloride or with a mixture of oxalate-plasma and bloodserum (which still contains a remnant of thrombine). On the latter mixture the p_H optimum of the coagulation can very well be determined, when buffers or HCl and NaOH are added.

Every 30 or 60 seconds, by switching on the magnet, we try to see whether the little ball can still be set in motion [6]). If this is no longer the case, the coagulation is ended. Permanent motion of the ball is not desirable, as this has a strong influence on the coagulation-time and often no gel arises either, but a flock of fibrin, so that the coagulation-time cannot be determined.

The figure on page 687 shows how the experiments are arranged. It represents a horizontal section of the apparatus at A, a vertical one at B. The electromagnet (1) with nucleus (2) has been fixed on a little table (3) which, by means of a serrated path (4) and screw (5) can be moved up and down and adjusted at the right height. A second table (6) which can also be put higher and lower by means of a serrated path (7) and screw (8) carries a cork-plate (9) on which are two plates of asbestos screwed together (10 and 11) between which a heating-element (12) has been fixed (13 and 14 are its supply- and outlet-wires). On the upper asbestos plate stands a *thin* crystallizing-dish (15) (Jena-glass) with a

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[5]) K. LINDERSTRØM–LANG u. H. HOLTER, Zs. physiol. Chem. **201**, 9 (1931).

[6]) Before use the small glass balls should be boiled with diluted nitric acid, after which in the used acid one reacts with KCNS to iron. If no colouring red occurs, the balls can be used.

[7]) The mercury-interruptor and the switch in front of it have been left out in the drawing.

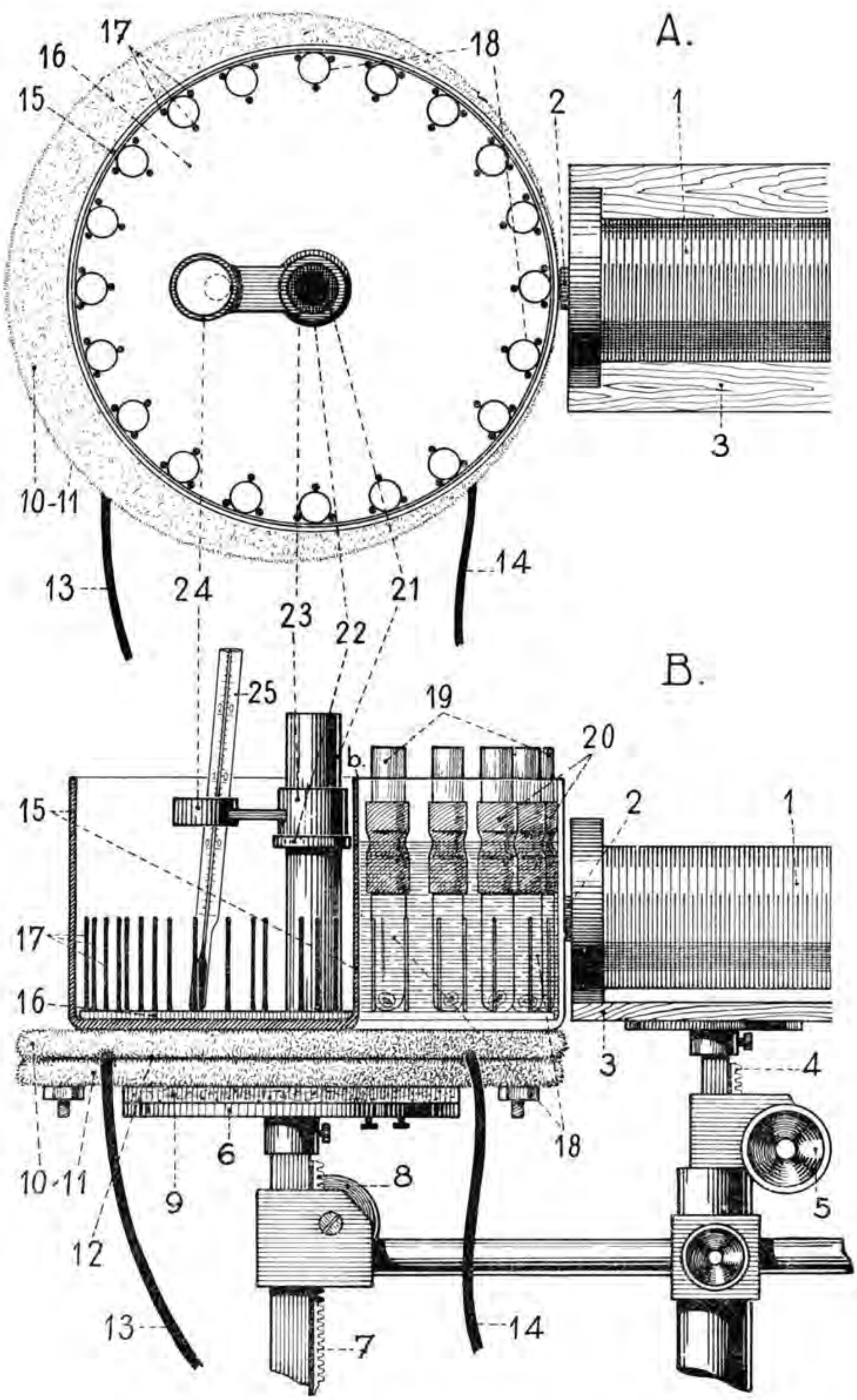
[8]) In fig. *B* on the right of *b* the crystallizing-dish is represented as filled with water, on the left of *b* empty and with the wall taken out.

height of about 6 cm and a diameter of 10 cm [8]). This can be shifted or turned with the hand. On the bottom of this crystallizing-dish there is a brass plate (16), on which along the edge twenty times three small brass bars (17) have been fixed, between which 20 glass tubes (18) can be put. These glass tubes (the titration-vessels of LINDERSTRØM–LANG) contain the liquid of which the coagulation-time must be examined. They are closed by glass stoppers (19), which are fastened on the tubes by pieces of rubber-tube (20). In the vertical section the small glass ball with iron nucleus is drawn in the tubes on the right of *b*. On the middle of the brass plate (16) a brass bar (21) has been attached, which serves the purpose of removing the brass plate easily from the dish. On this bar there is a stud (22), on which rests a loose ring (23), to which a second ring (24) has been fastened, which serves as holder for a thermometer (25). The dish can be turned with the hand, so that every time another tube comes to stand before the magnet-nucleus. The heating-element can be regulated by switching some carbon filament lamps of 5, 10 and 16 candle-units. Thus at a room-temperature that does not vary too much the temperature of the bath can be easily kept constant for an hour within a limit of 0.5°.

With the apparatuses of LINDERSTRØM–LANG adapted in this way the coagulation-time of bloodserum for experiments can be determined to the nearest 1 to 0,5 minutes. The technics are less satisfactory for the determination of the coagulation-time of blood itself, as the glass ball is not clearly visible. This also obtains, however, for the (up to now most exact) experiments of WOLVIUS, FESTEN and PROOST, who also used oxalate-plasma instead of blood, to which at a certain moment a certain quantity of pure calcium-chloride was added. Many experiments, which show the various phenomena on which the classical blood-coaguation theory is founded, in a simple way were carried out by us successfully in this manner.

We must state most emphatically that the test-tubes should be cleaned very carefully indeed. The coagulates should be removed from them mechanically and then the tubes should be boiled in strong nitric acid or aqua regia. After this they are rinsed many times with tap-water and finally a few times with distilled water and dried. It appeared that the inconstant values found at the beginning of the experiments were entirely due to insufficient cleaning of the tubes. As is well-known, the character of the surface strongly influences the blood-coagulation time. To this and to traces of enzyme, possibly adsorbed to the wall in former experiments, the occurring of such-like irregularities must be ascribed.

In order to investigate the accuracy of the method we determined the coagulation-time of one and the same test-liquid several times. This was mostly made up of a mixture of 1 cm^3 of oxalate-plasma and 1 cm^3 of serum (to which the same quantity of oxalate had been added). Then the thrombine is present in the serum, the fibrinogen in the oxalate-plasma. The addition of oxalate to the serum takes away the calcium in it. When joined



Apparatus to determine the blood-coagulation time.
Explanation in the text.

to the plasma this might otherwise cause the formation of new thrombine, so that irregularities might arise.

The following results were obtained. A coagulation-time of 11 min. was found in 7 experiments at 23° for a mixture of 1 cm^3 of oxalate-plasma and 1 cm^3 of serum. 8 min. was found 6 times for 2 cm^3 of plasma and 3 cm^3 of serum. Differences below 1 minute could not be observed in this way (switching of the magnet every other minute). A few times the differences appeared to be greater than 1 min., but this was the case with a long coagulation-time (serum and plasma, brought together in a proportion 2 to 3, were both 1 day old, so that the activity of the thrombine becomes considerably weaker, probably by destruction of the enzyme). Thus we found: 58¾, 60¾, 58¾, 59¾ and 59¾ min. Average 59.5 min. The mean error of a single determination was then (rounded off) ¾ min., that of the average $^3/_8$ min. In spite of the somewhat larger dispersion, the determination, as for percentage, becomes more exact when the time is longer.

In experiments with serum and oxalate-plasma joined together, in various proportions of the quantities of the components, a minimal coagulation-time is found. The cause of this is the following. When there is a constant quantity of oxalate-plasm, at first the coagulation-time will become shorter as the quantity of added serum increases, because the quantity of the enzyme is enlarged. As it is only the plasma that contains fibrinogen, the concentration of the fibrinogen decreases as the quantity of added serum increases, so that the formation of a gel will be delayed and finally this will not appear at all. The following experiment shows this:

No.	Quantity of ox. plasma	Quantity of serum	Temp.	Coag. time
1	1 cm^3	0.5 cm^3	22°	50 min.
2	1 cm^3	0.7 cm^3	22°	30 min.
3	1 cm^3	0.9 cm^3	22°	26 min.
4	1 cm^3	1.0 cm^3	22°	22 min.
5	1 cm^3	1.5 cm^3	22°	18 min.
6	1 cm^3	2.0 cm^3	22°	14 min.
7	1 cm^3	2.5 cm^3	22°	13 min.
8	1 cm^3	3.0 cm^3	22°	15 min.
9	1 cm^3	3.5 cm^3	22°	15 min.
10	1 cm^3	4.0 cm^3	22°	16 min.

Tube no. 7 shows the minimal coagulation-time. It appeared however, that after 24 hours the coagulate could be shaken out of the tube without difficulty, while this was least easy with the coagulates in number 3 and 4. So we shall be able to determine the coagulation-time in the most exact manner when the proportion of the mixture is 1 : 1. For the more solid the congelation that has been formed, the more unmistakably shall we be able to determine the moment when the motion of the ball ceases.

The method of WOLVIUS, where the absorption-curve is completely fixed and measured, is undoubtedly the most exact. But it requires very expensive apparatuses. A further advantage is that it can also be carried out at a constant temperature and that the liquid is not set in motion, except when we mix with calciumchloride. The methods of FESTEN and PROOST also have the latter advantage. But they don't work at a constant temperature, which is a drawback especially for a longer period of time, while our method does work at a constant temperature. As for accuracy, they are about equal to our method. The motion of the glass ball every other minute has with our method a certain influence on the coagulation. On the other hand, a special mixture of the liquids by a spatula, needed for the optical methods, is superfluous here. This drawback is removed, however, for comparative experiments, when the coagulation-times do not differ too much. (If one wishes to determine the absolute coagulation-time by means of our method, one can make three similar experiments. In the first tube one lets the ball move every other minute. Only when the liquid in this tube has coagulated one lets the ball in number 2 jump every other minute. When the coagulation of this tube is completed, then that in no. 3. In this way one approximates the true coagulation time.)

Further, our method has the advantage that a great number of determinations can be made at the same time. So those who have the titration-apparatuses of LINDERSTRØM–LANG at their disposal, can make them very suitable in a very simple way, for the determining of the blood-coagulation time. The description of this method seemed important to us, as in many modern investigations (e.g. of ASTRUP c.s. [9]) rather subjective methods are still used to determine this coagulation-time (observation of the first fibrinous flock or moving of the tube with the hand).

Summary.

A method is described to determine the blood-coagulation-time, founded on an application of the way in which in the micro-titrations of LINDERSTRØM–LANG the mixture of the liquids takes place.

[9]) ASTRUP, GALSMAR, VOLKERT, Acta physiol. Scand, 8 (1944).

Zoology. — *Over het oriëntatieprobleem bij vogels* [1]). (Preliminary communication.) By S. DIJKGRAAF. (Communicated by Prof. W. H. ARISZ.)

(Communicated at the meeting of April 27, 1946.)

Inleiding.

Het is bekend, dat vogels over merkwaardige oriëntatievermogens beschikken. Dit volgt zoowel uit het gedrag van trekvogels, als uit transportproeven met postduiven en in het wild levende vogelsoorten. In het laatste geval werden de dieren meestal in den broedtijd gevangen, zoo snel mogelijk naar elders vervoerd, geringd en weer vrijgelaten. Een zeker percentage van de vogels bleek dan op het nest terug te keeren („homing"), ook in die gevallen, waarin het als vrijwel zeker kon worden beschouwd, dat de dieren het gebied van vrijlating tevoren nooit hadden bezocht. De vraag dringt zich op: hoe weet een vogel vanaf een willekeurig punt in *onbekend* terrein zijn nest terug te vinden?

Er doen zich ter verklaring drie mogelijkheden voor:

1. De vogel vliegt zoolang zoekend rond, tot hij — al of niet — ergens zijn „woongebied" raakt (het gebied dus, dat hem door ervaring bekend is), waarna hij verder rechtstreeks naar het nest kan terugkeeren.

2. De vogel kent het gebied van vrijlating weliswaar niet door individueele ervaring, doch bezit er een zekere *aangeboren* kennis van.

3. De vogel richt zich op den terugweg naar prikkels, die op den heenweg, tijdens het transport naar het punt van vrijlating, op hem inwerkten.

Wat de eerste mogelijkheid betreft kan men zeggen, dat terugkeer wel in *vele* gevallen op deze wijze tot stand kan zijn gekomen, echter *niet in alle*. Het percentage terugkeerende vogels is vaak zóó hoog, de tijd tusschen vrijlating en aankomst op het nest zóó kort, dat men wel genoodzaakt is een min of meer rechtstreeks naar het woon- of broedgebied gerichte vlucht aan te nemen. Deze opvatting wordt bevestigd door het feit, dat *alle* vindplaatsen van vogels, die op den terugweg verongelukten, in de buurt van de rechte verbindingslijn bleken te liggen.

De tweede mogelijkheid doet aan als een uit de lucht gegrepen veronderstelling. Toch is iets dergelijks bij trekvogels gerealiseerd. Wanneer in sommige gevallen de jonge vogels zonder geleide van oudere soortgenooten voor de eerste maal trekken en daarbij een voor de soort specifieke route volgen is het duidelijk, dat de dieren een zekere aangeboren „kennis" van den te volgen weg en zijn kenmerken moeten bezitten. Bij „homing"-proeven met

[1]) "On the Orientation Problem in Birds." With a summary in English, French and German.

trekvogels werd intusschen ook dan vaak terugkeer op het nest geconstateerd, wanneer het gebied van vrijlating ver buiten de normale trekroute lag. Dat ook in deze gevallen en bij de transportproeven met standvogels een aangeboren kennis van het terrein een rol zou hebben gespeeld, lijkt weinig waarschijnlijk [2]).

Tenslotte de derde mogelijkheid: oriëntatie tijdens het transport. In de meeste gevallen werden de dieren op zoodanige wijze vervoerd, dat visueele oriëntatie — bij vogels ongetwijfeld van overheerschend belang — reeds bij voorbaat uitgeschakeld was. Een normaal functionneeren van den zgn. draaiïngszin (gelocaliseerd in de booggangen van het labyrinth) werd in vele gevallen opzettelijk verhinderd, doordat men de vogels tijdens het vervoer liet roteeren. Het terugkeerpercentage werd daardoor niet beïnvloed. Evenmin was dat het geval, wanneer de vogels op de heenreis in narcose werden gehouden (KLUYVER 1935).

Van de drie genoemde mogelijkheden ter verklaring van „homing" vanuit onbekend gebied blijkt dus de eerste slechts in een deel der gevallen in aanmerking te komen, terwijl voor de tweede vermoedelijk hetzelfde geldt; de derde mogelijkheid is in geen enkel geval gerealiseerd. Dit leidt ertoe, een vierde mogelijkheid te overwegen: het bestaan van speciale, „onbekende" zintuigelijke vermogens.

Onder den indruk van de verrassende „homing"-prestaties van wilde vogelsoorten en het gedrag van trekvogels onder bepaalde omstandigheden (gericht vliegen bij duisternis, in dichte mist, boven open zee) waren de meeste ornithologen reeds lang tot de overtuiging gekomen, dat de oriëntatie van vogels op grond van de bekende zintuigen alléén niet bevredigend verklaard kan worden (STRESEMANN 1935, RÜPPELL 1936/1937, VAN OORDT 1936/1943, KNIERIEM 1942 e.a.). Het phenomeen werd op verschillende wijze aangeduid. SCHÜZ (1931) sprak van „Empfindung für die geographische Lage", BIERENS DE HAAN (1934) onderscheidt „kompaszin" en „plaatszin" (waarvan hij alleen de eerste gerealiseerd acht), VAN OORDT (1943) spreekt van „richtingszin" (bij trek) en „oriëntatiezin" (bij „homing"). Wij kunnen hier op deze termen en hun beteekenis niet nader ingaan.

Het vraagstuk boeit behalve de ornithologen uiteraard ook de zintuigphysiologen. Zij zijn over het algemeen minder geneigd het bestaan van nieuwe, „onbekende" zintuigvermogens aan te nemen (zie b.v. KOEHLER 1942; GRIFFIN 1944). Niettemin staan ook zij voor de moeilijkheid, het gedrag van de vogels bevredigend te verklaren. Tot dusver is er ondanks een groot aantal experimenteele gegevens nog geen algemeen aanvaarde oplossing van het „oriëntatieprobleem" gevonden. Dit is voornamelijk een

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[2]) Het blijft niettemin denkbaar, en men heeft deze verklaringsmogelijkheid dan ook m.i. ten onrechte tot dusver geheel buiten beschouwing gelaten. Evenals bij trekvogels de trekroute, zouden bij standvogels bepaalde (constante) kenmerken van het landschap, waarin de soort voorkomt, de dieren aangeboren „bekend" kunnen zijn (b.v. bergketens, groote rivieren, kustlijnen, vegetatietypen enz.).

gevolg van het feit, dat bij proeven en waarnemingen in de vrije natuur allerlei niet of moeilijk te controleeren factoren een rol kunnen spelen, zoodat het gedrag van de vogels verschillend kan worden uitgelegd en het trekken van scherp bepaalde conclusies meestal onmogelijk wordt gemaakt.

Een en ander bracht mij op het denkbeeld, het probleem langs geheel anderen weg te benaderen, namelijk door een onderzoek in het laboratorium, waarbij men de experimenteele omstandigheden volkomen beheerscht. De hieronder beschreven proeven vormen een eerste poging in

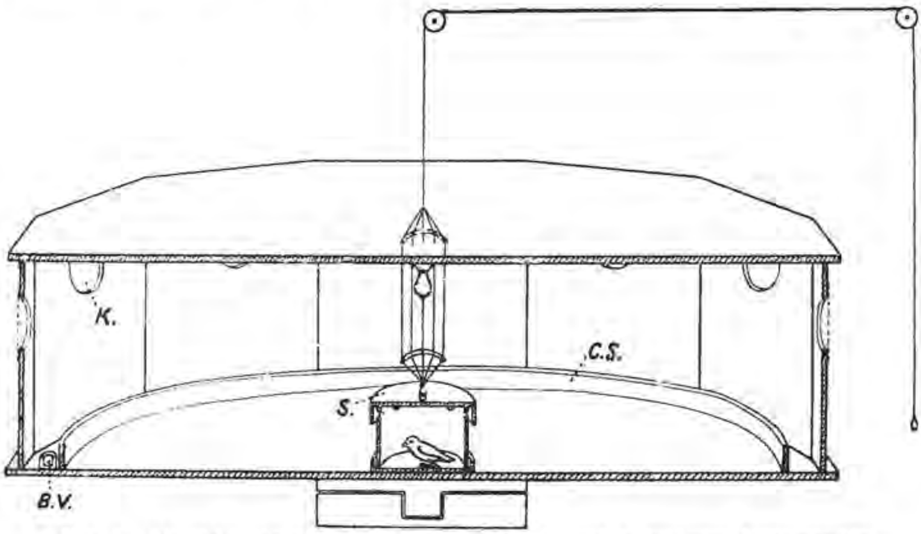

Fig. 1. Toestel voor de richtingdressuur. Het toestel is mediaan doorgesneden; slechts één helft is afgebeeld. De middellijn van het bodemvlak bedroeg 120 cm. B.V. bakje met voedsel; C.S. cirkelvormig schot (reep linoleum); K. kijkgat; S. stolp.

deze richting. Er werd getracht het bestaan van den zoo vaak gepostuleerden „richtingszin" met behulp van de dressuurmethode exact aan te toonen. Een positief resultaat leek nauwelijks te verwachten, zou echter omgekeerd van zoo groot belang zijn en zóóveel perspectieven openen voor een voortgezette analyse, dat de poging mij wel gerechtvaardigd scheen. De proeven werden verricht gedurende de maanden November 1941—Maart 1942, waarna het onderzoek ten gevolge van de tijdsomstandigheden vóórtijdig moest worden afgebroken. Daar het onzeker is, wanneer hervatting mogelijk zal zijn, worden de verkregen gegevens hieronder voorloopig medegedeeld.

Methode van onderzoek.

Het voor de richtingdressuur ontworpen toestel bestond uit een draaibaar opgestelde twaalfzijdige kist, die van binnen homogeen grijs was geverfd, teneinde visueele oriëntatie tegen te gaan (fig. 1). Het toestel was in een donkere kamer opgesteld, die tevens vrij goed tegen geluiden van buiten af geïsoleerd was. Het werd van binnen belicht door een melkglasbol (40 W).

die in het midden van de bovenzijde was gemonteerd. In de twaalf zijwanden waren door vitragestof afgesloten kijkgaten aangebracht, waardoorheen het dier ongemerkt kon worden gadegeslagen. De vogels werden één voor één in het proeftoestel gebracht en na de dressuurvoedering weer in hun eigen kooi teruggezet. Tijdens de overbrenging werd de vogel door een stolp met luchtsluis omgeven, zoodat hij ongestoord kon ademen, doch niet naar buiten kon kijken. Deze stolp werd midden op de bodem van het dressuurtoestel geplaatst, het toestel gesloten, waarna de stolp van buiten af door middel van een dun touw omhoog kon worden geheschen (vgl. fig. 1). De vogel kreeg daardoor gelegenheid, zich naar den wand van het toestel te begeven, waar een bakje met het begeerde voedsel was opgesteld, voor de vogel verborgen achter een 4 cm hoog, cirkelvormig rondom de bodem loopend schot. Het bakje werd steeds aan de zuidzijde van het toestel geplaatst en nu nagegaan, of de proefvogel zou leeren, vanaf het centrum ineens de juiste richting — het Zuiden — te kiezen.

Het dressuurtoestel was optisch zooveel mogelijk homogeen gehouden; teneinde visueele oriëntatie volkomen uit te schakelen werd het toestel tusschen twee dressuurvoederingen telkens in een anderen stand gebracht, zoodat op onregelmatig wisselende wijze beurtelings alle twaalf zijwanden de zuidzijde vormden en eventueele optische kenmerken het dier niet van nut konden zijn. Ook de plaats van den waarnemer werd met het oog op mogelijke geluidsprikkels (ritselen van kleeding e.d.) op onregelmatige wijze gevariëerd. Tenslotte werd de vogel tijdens het transport van zijn kooi naar het proeftoestel (een weg van ± 20 m door het gebouw) opzettelijk willekeurig gedraaid.

Als proefdieren fungeerden een groenling (*Chloris chloris* L.), een sijsje (*Chrysomitris spinus* L.) en — met het oog op de proeven van RÜPPELL — een spreeuw (*Sturnus vulgaris* L.) [3]). Het voedsel bestond bij de dressuurvoederingen, die meestal tweemaal daags plaats vonden, uit de resp. lievelingskostjes (hennepzaad, koolzaad en meelwormen). Er werd verder voor gezorgd, dat de dieren bij de proeven voldoende hongerig waren.

Resultaat der dressuurproeven.

1. Om de dieren aan de situatie in het proeftoestel te wennen, werd het cirkelvormige schot aanvankelijk weggelaten, zoodat het bakje met voedsel zichtbaar was. De dieren gingen dan steeds rechtstreeks op het voedsel af. Zij deden dit ook na plaatsing van een schot van 1 of 2 cm hoogte, waarachter het bakje nog gedeeltelijk zichtbaar was. Zoodra door gebruik van een hooger schot het bakje geheel aan het gezicht onttrokken was liepen de vogels naar een willekeurig punt van het schot en gluurden er overheen, gingen er vervolgens opzitten en bewogen zich huppend in één richting op het schot voort, tot zij het bakje in het oog kregen. Vanaf het begin van

[3]) De groenling werd bij een Groninger vogelhandelaar gekocht. Sijs en spreeuw waren in October 1941 op den herfsttrek in Zuid-Holland gevangen en mij door vriendelijke bemiddeling van den Heer L. TINBERGEN toegezonden.

de proeven bleek daarbij elk van de drie vogels een vaste gewoonte te volgen: de groenling wendde zich bij het bereiken van het schot steeds naar rechts, sijs en spreeuw naar links. Bereikte het sijsje het schot b.v. aan de zuidoostelijke zijde, dan volgde het den geheelen cirkel en kwam via oost-, noord- en westzijde bij het bakje met voedsel terecht.

Het gedrag van de dieren bleef bij voortzetting van de proeven ongewijzigd. Er was na resp. 39, 29 en 40 dressuurvoederingen met groenling, sijs en spreeuw nog geen spoor van voorkeur voor de juiste richting of zelfs maar voor de zuidelijke helft van het toestel te bekennen.

2. Teneinde de storende gewoonte van de vogels, zich eerst naar het schot en dan daarop naar het voedsel te begeven tegen te gaan, werd er een viertal radiair geplaatste, verticale schotten in het toestel aangebracht (fig. 2). De schotten reikten van bodem tot bovenzijde en van den wand

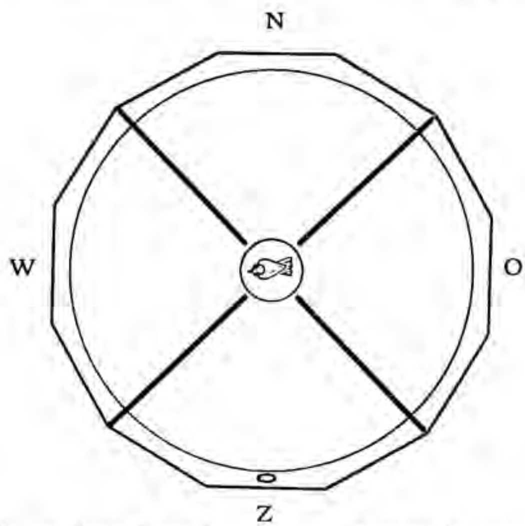

Fig. 2. Het proeftoestel van boven gezien (bovenzijde weggenomen) met de vier radiair geplaatste schotten.

van het toestel tot dicht bij de stolp in het centrum. Het inwendige van het dressuurtoestel was zoodoende in vier vakken verdeeld, die slechts centraal met elkaar communiceerden. — Ondanks dezen maatregel bleef het gedrag van de dieren in wezen gelijk: een vak werd geïnspecteerd, en indien er geen voedsel in werd aangetroffen volgden één voor één de overige vakken, wederom in een vaste volgorde. Opmerkelijk was hierbij slechts, dat zoowel groenling als sijs thans een richting volgden, tegengesteld aan die van de eerste proefreeks. De spreeuw daarentegen koos nog steeds de volgorde O → N → W → Z.

Het aantal proeven met groenling, sijs en spreeuw bedroeg resp. 53, 42 en 78; in resp. 17, 11 en 18 van deze gevallen werd dadelijk het juiste vak (zuidzijde) gekozen. Het percentage „goede" keuzen ten bedrage van resp. 32, 26 en 23 % wijkt dus nauwelijks af van hetgeen bij een zuiver toevallige verdeeling te verwachten ware (25 %).

3. In verband met de veel besproken mogelijkheid van magnetische invloeden werden alle ijzeren onderdeelen van het proeftoestel vervangen door indifferent materiaal en het geheele toestel buiten de stad in een schuur op het platteland opgesteld. De verdeeling in vakken door middel van de vier verticale schotten werd ditmaal achterwege gelaten. — De spreeuw koos in 55 proeven 15 maal, d.w.z. in 27 % der gevallen, den zuidelijken sector (begrensd door drie van de twaalf zijwanden). Ook onder deze omstandigheden kon dus geen voorkeur voor de dressuurrichting worden geconstateerd.

4. In de volgende serie proeven (weer in het laboratorium) werd ook het cirkelvormige schot weggelaten, daar het de dieren steeds sterk aantrok, hen tot optische inspectie prikkelde en daardoor mogelijk de aandacht van eventueele richtingsprikkels afgeleid hield. Onderin elk der twaalf zijwanden was een deurtje aangebracht in den vorm van een hangend klapluikje, dat naar buiten toe kon worden opengeduwd (in fig. 1 niet weergegeven). Het bakje met voedsel was thans buiten het toestel geplaatst, vlak achter een der deurtjes.

Een inleidende reeks proeven diende om de dresseerbaarheid van het proefdier (spreeuw) onder deze omstandigheden na te gaan. Het dier werd daartoe op een geluidsprikkel gedresseerd, namelijk licht krabben aan den buitenwand van het toestel, vlak naast een der deurtjes. Reeds na 6 à 8 voederingen traden de eerste positieve reacties op: het in of bij het centrum staande of loopende dier wendde zich, zoodra er gekrabd werd, meteen rechtstreeks naar het juiste deurtje en pikte er tegenaan. In het vervolg bleek, dat de spreeuw het geluid verrassend goed wist te localiseeren. De belichting in het toestel werd door middel van weerstanden geleidelijk verzwakt en tenslotte geheel uitgeschakeld. Ook in volslagen duisternis kwam de vogel soms snel en zeker op de geluidsbron af en pikte aan het juiste deurtje [4]).

Nadat uit deze proeven de dresseerbaarheid van het dier was gebleken werd opnieuw getracht een richtingdressuur te verwezenlijken. Daartoe werden alle deurtjes van buiten gegrendeld, met uitzondering van het aan de zuidzijde gelegen deurtje, waarachter het bakje met voedsel was geplaatst. Na het opheffen van de stolp werd de vogel, nadat hij eenigen tijd „besluiteloos" had rondgeloopen, door wat krabgeluid naar het zuidelijke deurtje geleid. Bij herhaling van de proef bleef het gedrag van het dier echter ongewijzigd. Indien de spreeuw erg hongerig of „ongeduldig" was, pikte hij ook wel aan de deurtjes vóór er gekrabd werd; van eenige voorkeur voor de zuidzijde was daarbij echter geen sprake (30 proeven).

Het voornemen, deze proefreeks met uitgeschakelde belichting te herhalen kon niet meer ten uitvoer worden gebracht.

5. Hetzelfde geldt van voorgenomen proeven, waarbij de spreeuw in

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[4]) Dit werd door beluisteren vastgesteld. — Over het vermogen tot geluidslocalisatie bij vogels is nog weinig bekend. Het was tot dusver slechts aangetoond bij de kip (ENGELMANN 1928).

plaats van door geluid, door optische bakens in de juiste richting zou worden geleid. De bakens zouden dan *geleidelijk* verkleind of de belichting verzwakt kunnen worden, teneinde een eventueele richtingszin naar voren te doen komen.

Als inleiding hiertoe werd de vogel gedresseerd op een met de vulpen op een der deurtjes aangebrachte stip van 1½ mm middellijn. Hoewel het dier reeds bij de derde voedering het gemerkte deurtje koos, duurde het nog tot de 24ste proef, voor de dressuur volledig gelukt was, d.w.z. steeds ineens juist gekozen werd. Tot mijn verrassing bleek, dat het dier ook na verwijdering van de stip nog juist bleef kiezen. Het had zich bij zijn keuze blijkbaar minder naar de voor een menschelijken waarnemer zoo opvallende stip gericht, dan naar andere (minimale) kenmerken van het deurtje of de omgeving.

Discussie.

Naar reeds in de inleiding werd uiteengezet heeft men om twee redenen aangenomen, dat er behalve de gewone zintuigen bij vogels nog een soort „richtingszin" moet voorkomen: *a*) „homing" vanuit onbekend gebied; *b*) gerichte vlucht bij trekvogels, ook waar de mogelijkheid tot visueele oriëntatie schijnbaar ontbreekt.

Geen der beide redenen blijkt bij nadere beschouwing steekhoudend. Zoo weet men bij „homing"-proeven met wilde vogels nooit *volkomen* zeker, of het gebied van vrijlating de dieren onbekend is. In gevangenschap in een ruime kooi opgekweekte en daar nestelende spreeuwen keerden na transport over 114 km niet op hun nest terug, in tegenstelling met de in het wild opgegroeide en daarna even lang in dezelfde kooi gevangen gehouden contrôle-spreeuwen (RÜPPELL en SCHEIN 1941). — Bij proeven met postduiven worden in den regel vele dieren gelijktijdig losgelaten; de vogels voor wie het terrein onbekend is zouden zich naar de ervaren duiven kunnen richten. Toen de dieren één voor één in onbekend gebied werden vrijgelaten was het resultaat dan ook volkomen negatief (HEINROTH 1941).

Bij trekvogels is het de vraag, of de mogelijkheid tot optische oriëntatie ooit *volkomen* uitgesloten is. Boven open zee b.v. zou de richting van de golven het dier als baken kunnen dienen. Dat trekvogels in sommige gevallen inderdaad een bepaalde kompasrichting schijnen te volgen is o.a. gebleken uit de proeven van SCHÜZ (1934) met ooievaars. Hij bracht een groot aantal jonge ooievaars uit Oost-Pruisen over naar West-Duitschland. De hoofdmassa van deze dieren, die in Oost-Pruisen ZO-waarts (naar den Bosporus) zouden zijn getrokken, bleek ook thans ZO-waarts te trekken (een deel vloog over de Alpen naar Noord-Italië), ondanks het geheel verschillende terrein en in tegenstelling met de gewoonte van West-Duitsche ooievaars, die ZW-waarts (naar Gibraltar) plegen te trekken. De dieren wisten de hun blijkbaar aangeboren trekrichting dus onafhankelijk van de kenmerken van het landschap te bepalen. Op welke wijze dit geschiedde is een open vraag. Men kan er een aanwijzing voor het bestaan van een „richtingszin" in zien; het ligt echter meer voor de hand aan oriëntatie ten opzichte van den zonnestand e.d. te denken.

Ook onze dressuurproeven hadden tot dusver een negatief resultaat, hetgeen uiteraard de mogelijkheid van het bestaan van een richtingszin onaangetast laat. In dit verband zij aan de „homing"-prestaties van visschen en vleermuizen herinnerd, waar zich een soortgelijk probleem voordoet. Wellicht zal een onderzoek naar vermogen en reikwijdte van de *bekende* zintuigen — in menig opzicht nog terra incognita — veel ophelderen, van wat thans onverklaarbaar schijnt.

Samenvatting.

De stand van het oriëntatieprobleem bij vogels wordt besproken. De kernvraag luidt: zijn de bekende zintuigen toereikend om het gedrag van de dieren in alle gevallen bevredigend te verklaren? Vele ornithologen zijn overtuigd van het bestaan van een „richtingszin", zonder dat deze opvatting tot dusver strikt bewezen kon worden. Met het oog hierop werd een poging ondernomen, vogels in een speciaal daartoe geconstrueerd toestel op een bepaalde kompasrichting te dresseeren. Eenige voorkeur voor de dressuurrichting kon tot dusver niet worden vastgesteld, hoewel dressuren op acustische en optische prikkels in hetzelfde toestel gemakkelijk gelukten. Het onderzoek is nog niet afgesloten. Als bijkomstig resultaat kan worden vermeld, dat een spreeuw geluiden uitstekend wist te localiseeren.

Zusammenfassung.

Der Stand des Problems der Fernorientierung bei Vögeln wird besprochen. Die Kernfrage lautet: reichen die bekannten Sinne aus um das Verhalten der Tiere in allen Fällen befriedigend zu erklären? Viele Ornithologen sind von der Existenz eines „Richtungssinnes" überzeugt, obwohl ein strikter Nachweis für diese Auffassung bisher nicht erbracht werden konnte. Im Hinblick darauf wurde der Versuch unternommen, Vögel in einem speziell zu diesem Zweck konstruierten Apparat auf eine bestimmte Kompassrichtung zu dressieren. Irgendeine Bevorzugung der Dressurrichtung konnte bisher nicht festgestellt werden, obwohl Dressuren auf akustische und optische Reize im gleichen Apparat leicht gelangen. Die Versuche sind noch nicht abgeschlossen. Nebenbei ergab sich, dass ein Star Geräusche ausgezeichnet zu lokalisieren verstand.

Summary.

The present state of our knowledge of the orientation problem with birds is discussed. The main question is whether it is possible to explain the behaviour of birds in all circumstances on the basis of the known senses only. Many ornithologists are convinced of the existence of a "sense of direction", though this opinion is not yet strictly proved. Therefore an

attempt was made to train birds to choose an appointed direction by putting them in a special apparatus. So far no preference for the training direction could be stated, though training on acoustic and optic stimuli in the same apparatus succeeded well. The experiments are not yet finished. As an accessory result it can be stated that a starling could locate sounds very well.

Résumé.

l'Etat actuel du problème de l'orientation lointaine chez les oiseaux est discussié. Il s'agit principalement de la question s'il est possible d'expliquer la conduite des oiseaux par les sens connus seuls. Bien des ornithologistes sont convaincus de l'existence d'un „sens de la direction", quoiqu'il en manque jusqu'ici la preuve stricte. En vue de cette question un appareil spécial fut construit, dans lequel les oiseaux furent dressés à choisir une certaine direction. Jusqu'ici aucune préférence de la direction a pu être constatée, quoiqu'il était simple de dresser un oiseau à des stimulants acoustiques ou optiques dans le même appareil. Les expériences ne sont pas encore terminées. Comme résultat accessoire soit mentionné, qu'un sansonnet pouvait très bien localiser des sons.

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CONTENTS

Mathematics. — *On* MINKOWSKI's *fundamental theorem in the geometry of numbers.* By J. G. VAN DER CORPUT and H. DAVENPORT[1]).

(Communicated at the meeting of June 29, 1946.)

MINKOWSKI's fundamental theorem asserts that if an open convex body in n dimensions, symmetrical about the origin O, contains no point other than O of a lattice of determinant 1, then the volume of the body does not exceed 2^n. It is known[2]) that if the volume equals 2^n, the body must be a polyhedron with not more than $2(2^n-1)$ faces. In two and three dimensions such polyhedra can be realised, with exactly this number of faces.

The object of this note is to obtain some improvements on MINKOWSKI's theorem. We prove first:

Theorem 1. *Let K be an open convex body in n dimensional space, symmetrical about O, which contains no point other than O of a certain lattice of determinant* 1. *Then there exists an open convex polyhedron K', symmetrical about O, with the following properties:* (i) *K' contains K,* (ii) *K' has not more than $2(2^n-1)$ faces,* (iii) *K' contains no lattice point other than O,* (iv) *each face of K' contains at least one lattice point in its interior.*

As an immediate consequence, we have:

Theorem 2. *For any closed convex body K, symmetrical about O, let V' denote the minimum volume of any circumscribing polyhedron, symmetrical about O, with not more than $2(2^n-1)$ faces. If $V' \geqq 2^n$, then K contains a point other than O of every lattice of determinant* 1.

In the two-dimensional case, a simple condition by which one can ensure a definite improvement on the constant 2^n of MINKOWSKI's theorem is that the boundary of K consists of a curve with a continuous radius of curvature ϱ, which is bounded below by a positive number ϱ_0. We prove:

Theorem 3. *If $\varrho \geqq \varrho_0 > 0$, and the area of the closed convex symmetrical figure K is at least equal to $4-(2\sqrt{3}-\pi)\varrho_0^2$, then K contains a point other than O of every lattice of determinant* 1.

The constant $2\sqrt{3}-\pi$ is the best possible, as one easily sees from the case of a circle and a regular hexagonal lattice.

In the n dimensional case, the notion of radius of curvature is not so

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1) The authors wish to express their thanks to the British Council, without whose help in arranging VAN DER CORPUT's visit to England this joint work would not have been possible.

2) MINKOWSKI, *Geometrie der Zahlen,* § 32, 33, 34.

appropriate, and we introduce restrictions on the body K of the following kind. For any plane R (that is, any $n-1$ dimensional linear space) which has at least one point in common with K, we denote by $d(R)$ the distance of R from the nearest parallel tangent plane (or tac-plane) to K, and by $U(R)$ the $n-1$ dimensional volume of the common part of R and K. We may call $U(R)$ the surface area which K cuts off on R. The limitations which we impose are that either

$$U(R) \leqslant (\varrho_1\, d(R))^{\frac{n-1}{2}} \text{ for every } R, \quad \ldots \ldots \quad (1)$$

or

$$U(R) \geqslant (\varrho_2\, d(R))^{\frac{n-1}{2}} \text{ for every } R, \quad \ldots \ldots \quad (2)$$

where ϱ_1, ϱ_2 are positive numbers, the former being not too small and the latter not too large. The first condition can be regarded as precluding K from having any plane faces, the second as precluding K from having any corners. We prove

Theorem 4. *If K is a convex body, symmetrical about O, which satisfies condition* (1), *where $\varrho_1 \geq 1$, and if K contains no point other than O of a certain lattice of determinant* 1, *then*

$$V(K) < 2^n - \frac{c_1}{\varrho_1},$$

where $c_1 > 0$ depends only on n.

Theorem 5. *If K is a convex body, symmetrical about O, which satisfies condition* (2), *where $0 < \varrho_2 \leq 1$, and if K contains no point other than O of a certain lattice of determinant* 1, *then*

$$V(K) < 2^n - c_2\, \varrho_2^n,$$

where $c_2 > 0$ depends only on n.

Proof of Theorem 1. Let K be the given open convex body, symmetrical about O, which contains no lattice point other than O. For every lattice point P, other than O, we define a strip of space $S(P)$ as follows. Let OP meet the surface of K at P'. Then $S(P)$ is the strip lying strictly between two planes through P and $-P$, each parallel to a tangent plane to K at P'. Plainly $S(P)$ contains K for every lattice point P other than O.

We define K_1 to be the common part of the strips $S(P)$, where P runs through all lattice points other than O. Then K_1 is convex and symmetrical about O; K_1 contains K, and K_1 contains no lattice point other than O. We observe that K_1 is bounded; in fact every convex body which contains K, and has a volume not exceeding 2^n, is plainly contained in a bounded region depending only on K (and n).

It is easy to prove that K_1 is a polyhedron. For the perpendicular distance from O to a tangent plane to K at any point P' of K is at least $\lambda \, . \, OP'$, where λ is a positive number depending only on K. Hence the breadth of any strip $S(P)$ is at least $2\lambda . OP$, and the strip can be omitted from the definition of K_1 if P is at a sufficiently great distance from O. Thus K_1 is bounded by a finite number of planes, *i.e.* K_1 is a polyhedron.

If every face of K_1 has a lattice point on it, in its interior, we take $K' = K_1$. If not, we consider any particular face which has no lattice point on it in its interior, and move this face, parallel to itself, away from O, carrying out the corresponding operation on the symmetrical face. We continue this movement until one of two things happens:

(a) the area of the face becomes zero:

(b) the face has a lattice point on it, in its interior.

One of these two things must happen, for throughout this operation the polyhedron has no lattice point other than O in its interior, and therefore, as already remarked, it is contained in a bounded region depending only on K.

Let K_2 denote the open polyhedron obtained from K_1 by this operation. Then K_2 is convex and symmetrical about O, and $K_2 \supset K_1 \supset K$, and K_2 contains no lattice point other than O. In both case (a) and case (b), the number of faces of K_2 which do not have a lattice point in their interior is less than the corresponding number for K_1.

After a finite number of such operations, we reach an open polyhedron K' which satisfies (i), (iii) and (iv) of Theorem 1. The assertion (ii) now follows by an argument due to MINKOWSKI (*Geometrie der Zahlen*, 80). Let $\pm P_1, \ldots, \pm P_m$ be lattice points on the surface of K', one in the interior of each face. If $m \geqq 2^n$, there must be two of the points $O, P_1, P_2, \ldots, P_m$ such that the midpoint of the line joining them is a lattice point, and this is impossible since this midpoint is inside K'. Hence $m \leqq 2^n - 1$.

Proof of Theorem 3. Let K be a convex figure in the plane, symmetrical about O, bounded by a curve with a continuous radius of curvature ϱ satisfying $\varrho \geqq \varrho_0 > 0$. By Theorem 2, it suffices to prove that any circumscribing hexagon, symmetrical about O, has an area which exceeds that of K by at least $(2\sqrt{3} - \pi)\varrho_0^2$. Let L and M be two consecutive points of contact of the hexagon, and let the tangents at L and M meet at an angle α. We take the tangent and normal at L as coordinate axes. If $P(x, y)$ is any point on the curve between L and M, we denote by $l = l(x, y)$ the length intercepted on the tangent at P between P and the x-axis. Then the area between the arc LM and the tangents at L and M is

$$\frac{1}{2}\int_0^\alpha l^2\, d\psi = \frac{1}{2}\int_0^\alpha \frac{y^2}{\sin^2\psi}\, d\psi.$$

Here

$$\frac{dy}{d\psi} = \varrho \sin\psi,$$

hence

$$y \geqslant \int_0^{\psi} \varrho_0 \sin \Phi \, d\Phi = 2\varrho_0 \sin^2 \tfrac{1}{2}\psi.$$

Thus the area in question is

$$\geqslant 2\varrho_0^2 \int_0^{\alpha} \frac{\sin^4 \frac{1}{2}\psi}{\sin^2 \psi} \, d\psi = \varrho_0^2 (\tan \tfrac{1}{2}\alpha - \tfrac{1}{2}\alpha).$$

Let the angles between consecutive sides of the hexagon be α_1, α_2, α_3, α_1, α_2, α_3; then $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. The area of the hexagon exceeds the area of K by at least

$$2\varrho_0^2 \sum_{r=1}^{3} (\tan \tfrac{1}{2}\alpha_r - \tfrac{1}{2}\alpha_r).$$

Putting $t_r = \tan \frac{1}{2}\alpha_r$, we have

$$t_2 t_3 + t_3 t_1 + t_1 t_2 = 1,$$

whence

$$(t_1 + t_2 + t_3)^2 \geqslant 3(t_2 t_3 + t_3 t_1 + t_1 t_2) = 3.$$

Finally, the area in question is at least

$$2\varrho_0^2 (\sqrt{3} - \tfrac{1}{2}\pi),$$

as stated.

Proof of Theorem 4. By Theorem 1, there exists a polyhedron $\overline{K}$, obtained by expanding K' if necessary, such that $\overline{K}$ is convex and symmetrical about O, $\overline{K} \supset K$, $\overline{K}$ has at most $2(2^n - 1)$ faces, and

$$V(\overline{K}) = 2^n. \qquad (3)$$

We have to prove that

$$V(\overline{K}) - V(K) > \frac{c_1}{\varrho_1}. \qquad (4)$$

Let F be any face of the polyhedron $\overline{K}$, and let $W(F)$ denote its area. We assert that there exists a face F for which

$$W(F) > c_3, \qquad (5)$$

where c_3 (and similarly $c_4, \ldots, c_{10}$) is a positive number depending only on n. For $\sum_F W(F)$ is the surface area of $\overline{K}$, and by the well known isoperimetric theorem

$$\frac{(\sum_F W(F))^n}{(V(\overline{K}))^{n-1}}$$

is greater than or equal to the corresponding ratio, say c_4, for an n dimensional sphere. Thus

$$\sum_F W(F) \geqq (c_4\, 2^{n(n-1)})^{\frac{1}{n}},$$

and since the number of terms in the sum is at most $2(2^n - 1)$, the result follows.

Let F be any face satisfying (5), and let h be the perpendicular distance from O to a tangent plane to K which is parallel to F. Suppose first that

$$(\varrho_1 h)^{\frac{n-1}{2}} \leqslant \tfrac{1}{2} W(F). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Then for every plane R parallel to F, whose distance from O does not exceed h, we have $d(R) \leqq h$ in (1), and so the area $U(R)$ cut off on such a plane by K satisfies

$$U(R) \leqslant (\varrho_1 h)^{\frac{n-1}{2}} \leqslant \tfrac{1}{2} W(F).$$

On the other hand, any plane parallel to F, lying between F and its image in O, cuts off from $\bar{K}$ an area at least equal to $W(F)$. Thus, in this case, we have

$$V(K) \leqq \tfrac{1}{2} V(\bar{K}),$$

and (4) is trivial, since $\varrho_1 \geqq 1$.

Now suppose that (6) is not satisfied, so that

$$(\varrho_1 h)^{\frac{n-1}{2}} > \tfrac{1}{2} W(F).$$

We consider planes R parallel to F for which

$$O \leqslant d(R) \leqslant (\tfrac{1}{2} W(F))^{\frac{2}{n-1}} \varrho_1^{-1},$$

the number on the right being less than h. For every such plane,

$$U(R) \leqslant (\varrho_1 d(R))^{\frac{n-1}{2}} \leqslant \tfrac{1}{2} W(F),$$

and on the other hand, the area cut off on such a plane by $\bar{K}$ is at least $W(F)$. Hence

$$V(\bar{K}) - V(K) \geqslant \tfrac{1}{2} W(F)\ (\tfrac{1}{2} W(F))^{\frac{2}{n-1}} \varrho_1^{-1},$$

and, in view of (5), this proves (4).

For the proof of Theorem 5, we require the following simple lemma.

Lemma. *Let $l_1, \ldots, l_\nu$ be any directions in n dimensional space. Then there exists a direction l such that the angles α_r between l and l_r satisfy*

$$|\cos \alpha_r| > C \qquad (r = 1, 2, \ldots, \nu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $C > 0$ depends only on n and ν.

Proof. Each direction is represented by a point on the surface of a sphere of radius unity. The points which correspond to directions violating a particular inequality of the form (7) lie in a band on the sphere, whose area depends only on n and c, and is small with c. Choosing c so that the sum of ν such areas is less than the surface area of the sphere, we obtain the result.

Proof of Theorem 5. As before, there exists a convex polyhedron $\overline{K}$, symmetrical about O, with not more than $2(2^n - 1)$ faces, such that $\overline{K} \supset K$ and satisfies (3). We have to prove that

$$V(\overline{K}) - V(K) > c_2 \varrho_2^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In the Lemma, we take $l_1, \ldots, l_\nu$ to be the directions of the edges of the polyhedron $\overline{K}$. Then $\nu < c_5$, and the Lemma tells us that there exists a direction l for which

$$|\cos \alpha_r| > c_6 \qquad (r = 1, 2, \ldots, \nu). \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Let Π be a plane through O perpendicular to l, and let P_0 be one of the two opposite vertices of $\overline{K}$ whose distance from Π is greatest. Let P_0Z be the perpendicular from P_0 on Π, and let $P_0Z = h$. If P_0P_r is any edge of the polyhedron $\overline{K}$ through P_0, the angle α_r between P_0P_r and P_0Z satisfies (9). Hence P_0P_r meets the plane Π at Q_r, where

$$Q_r Z = h\,|\tan \alpha_r| < c_7\, h.$$

Thus the various faces of $\overline{K}$ which meet at P_0 cut off on Π a convex polygon whose area is A, where

$$A < c_8\, h^{n-1}.$$

Let Π_0 be a tangent plane to K, parallel to Π, between Π and P_0, and let h_0 be the perpendicular distance from P_0 to Π_0. The faces of $\overline{K}$ which meet at P_0 cut off on Π_0 an area A_0, where

$$\frac{A_0}{h_0^{n-1}} = \frac{A}{h^{n-1}} < c_8. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

We observe that if $h_0 \geq \frac{1}{2} h$, then (8) is trivial. For then the region $\overline{K} - K$ contains a prism whose height is h_0, and the area of whose base is A_0, so that

$$V(\overline{K}) - V(K) \geqslant \frac{1}{n}\, h_0\, A_0.$$

On the other hand, the half of $\overline{K}$ which lies on the same side of the plane Π as P_0 is contained in a prism with height h and area of base A, so that

$$V(\overline{K}) \leqslant \frac{2}{n}\, h\, A.$$

If $h_0 \geq \frac{1}{2} h$, we have

$$V(\bar{K}) - V(K) \geq \frac{1}{n} h_0 A_0 = \left(\frac{h_0}{h}\right)^n \frac{1}{n} h A \geq \frac{1}{2^{n+1}} V(\bar{K}) = \tfrac{1}{2},$$

which implies (8), since $\varrho_2 \leq 1$.

We can therefore suppose that $h_0 < \frac{1}{2} h$. Let R be a plane parallel to Π for which $d(R) = h_0$. This cuts off from K an area $U(R)$, where

$$U(R) \geq (\varrho_2 h_0)^{\frac{n-1}{2}}.$$

On the other hand, the faces of K which meet at P_0 cut off from R an area A_1, where

$$\frac{A_1}{(2 h_0)^{n-1}} = \frac{A_0}{h_0^{n-1}}.$$

Hence

$$2^{n-1} A_0 = A_1 \geq U(R) \geq (\varrho_2 h_0)^{\frac{n-1}{2}}, \qquad (11)$$

whence, by (10),

$$2^{n-1} c_8 h_0^{n-1} > (\varrho_2 h_0)^{\frac{n-1}{2}},$$

and consequently

$$h_0 > c_9 \varrho_2. \qquad (12)$$

The region $\bar{K} - K$ contains the prism of volume $\frac{1}{n} h_0 A_0$, hence, by (11) and (12),

$$\begin{aligned} V(\bar{K}) - V(K) &\geq \frac{1}{n} h_0 A_0 \\ &\geq \frac{1}{n} h_0 2^{1-n} (\varrho_2 h_0)^{\frac{n-1}{2}} \\ &> c_{10} \varrho_2^n. \end{aligned}$$

This completes the proof of (8), and so of Theorem 5.

Mathematics. — *Rhythmic systems.* By J. G. VAN DER CORPUT. (First communication.)

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§ 1. *Rhythmic functions.*

In this paper everything is real.

$a \equiv b \pmod 1$ means that $a - b$ is an integer.

Let $f(x)$ be defined for any integer x. We are only interested in the behaviour of $f(x) \pmod 1$, i.e. in the remainder which we get when dividing $f(x)$ by 1.

Consider a number, say N, of consecutive integers $x_h (h = 1, \ldots, N)$ and the corresponding points $P_h = (x_h, y_h)$ defined by $y_h \equiv f(x_h) \pmod 1$ and $0 \leqq y_h < 1$. $(h = 1, \ldots, N)$.

We say that the points $P_1, \ldots, P_N$ form a wave of the function $f(x)$ with length $N - 1$. We say that this wave repeats itself, apart from ε (where $0 < \varepsilon < 1$), if we can find a wave $Q_h = (x'_h, z_h)$ $(h = 1, \ldots, N)$ of $f(x)$ with the same length such that

$$| y_h - z_h - e_h | < \varepsilon \ (h = 1, \ldots, N),$$

where $e_h = 0$ or ± 1.

Definition: *A function $f(x)$, defined for all integers x, is called rhythmic if to every positive number $\varepsilon < 1$ and to every wave w of $f(x)$ corresponds a length $L = L(\varepsilon, w)$, such that in any interval of length L the wave w repeats itself, apart from ε.*

It is easy to show that a constant function and a periodic function with rational period are rhythmic; also that $f(x) + a$ is rhythmic, if $f(x)$ is rhythmic and a denotes a constant.

If $f(x)$ is rhythmic and if the Diophantine inequality

$$\alpha < f(x) < \beta \pmod 1$$

possesses at least one integral solution, then the inequality possesses an infinite number of integral solutions and there exists a length L, such that every interval with length L contains at least one integral solution of the inequality.

The inequality means, that an integer y can be found with

$$\alpha < f(x) - y < \beta.$$

Proof. Be $\bar{x}$ a solution. Hence

$$\alpha < f(\bar{x}) < \beta \pmod 1.$$

Then there exists a positive $\varepsilon < 1$, such that

$$\alpha + \varepsilon < f(\bar{x}) < \beta - \varepsilon \pmod 1.$$

Consider the wave, consisting of only one point, namely the point $(\bar{x}, \bar{y})$, where $\bar{y} \equiv f(x) \pmod 1$. Since $f(x)$ is rhythmic, there exists a length L, such that this wave repeats itself, apart from ε, in every interval with length L. So in every interval with length L we can find an integer x with

$$-\varepsilon < f(x) - f(x) < \varepsilon \pmod 1,$$

For this x we obtain

$$\alpha < f(x) < \beta \pmod 1,$$

which gives the required result.

If $f(x)$ is rhythmic, then to any given positive number $\varepsilon < 1$ and to every length λ a length Λ corresponds, such that every wave w of length λ repeats itself, apart from ε, in each interval of length Λ.

The difference between this theorem and the definition of a rhythmic function is, that the theorem states that the length L can be chosen independently of the wave w, if the length of w is given.

Proof. Be k the smallest integer $\geqq \frac{2}{\varepsilon}$ and $\geqq 1 + \lambda$. We say that the wave $(P_1, \ldots, P_k)$ of $f(x)$ belongs to the class $(u_1, \ldots, u_k)$, where $P_h = (x_h, y_h)$ and u_h is an integer $\geqq 0$ and $< k$ $(h = 1, \ldots, k)$, if it is possible to define the integers $e_1, \ldots, e_k$ such that

$$\frac{u_h}{k} \leqq f(x_h) - e_h < \frac{u_h + 1}{k} \quad (h = 1, \ldots, k). \quad \ldots \quad (1)$$

Every wave of $f(x)$ with length $k - 1$ belongs to a certain class K and the number of all these classes is at most k^k. In each class K we choose a wave w' of $f(x)$. Since $f(x)$ is rhythmic, a length $L = L(\frac{1}{2}\varepsilon, w')$ can be found such that in each interval of length L the wave w' repeats itself, apart from $\frac{\varepsilon}{2}$. The number of these lengths is at most k^k. Be Λ the greatest of all these lengths. We shall prove that Λ possesses the required property.

Since $k \geqq 1 + \lambda$, w is a part of a wave w^* of $f(x)$, consisting of k points. This wave belongs to a class K. In this class we chose a wave w' to which corresponds the length $L = L(\frac{1}{2}\varepsilon, w') \leqq \Lambda$. In every interval of length L, hence certainly in every interval of length Λ we can find a wave w'', which, apart from $\frac{\varepsilon}{2}$, equals w'. Since w' and w^* belong to the same class and $k \geqq \frac{2}{\varepsilon}$, it follows from (1), that w' and w^* are equal, apart from $\frac{1}{k}$, hence apart from $\frac{\varepsilon}{2}$. Consequently w'' and w^* are equal, apart from ε. Hence the wave w^*, and therefore also the wave w, repeats itself, apart from ε, in every interval of length Λ.

Definition. *A function $\psi(v)$ is called periodic (mod 1), if from $v \equiv w$ (mod 1) follows $\psi(v) \equiv \psi(w)$ (mod 1).*

A function $\psi(v)$ defined for every real v is called continuous (mod 1) in a point v_0, if to any v in the vicinity of v_0 an integer $e(v)$ corresponds, such that $\psi(v) - e(v)$ is continuous in v_0.

First continuity theorem. *If $f(x)$ is rhythmic and $\psi(v)$, defined for all real v, is periodic (mod 1) and continuous (mod 1), then $\psi(f(x))$ is also rhythmic.*

Proof. Consider an arbitrary wave w of $\psi(f(x))$, possessing the abscissae $x_1, \ldots, x_N$. Put $f(x_h) = v_h$ $(h = 1, \ldots, N)$. Since $\psi(v)$ is continuous (mod 1), to every positive $\varepsilon < 1$ corresponds a positive δ such that from

$$|v - v_h| < \delta \ (h = 1, \ldots, N) \text{ follows } |\psi(v) - \psi(v_h)| < \varepsilon \ (\text{mod } 1);$$

here δ depends on ε, $v_1, \ldots, v_N$ and ψ.

Be w' the wave of the function $f(x)$ possessing the abscissae $x_1, \ldots, x_N$. As $f(x)$ is rhythmic, a length $L = L(\delta, w')$ exists, such that every interval with length L contains a wave w''' of $f(x)$, which is equal to w', apart from δ. Be w'' the wave of $\psi(f(x))$, lying at the same place as w'''. As the difference between the waves w' and w''' is at most δ, the difference between the waves w and w'' is at most ε; hence the wave w of $\psi(f(x))$ repeats itself in each interval of length L, apart from ε.

Of the condition that $\psi(v)$ is continuous (mod 1) we only used the fact that $\psi(v)$ is continuous (mod 1) in those points $f(x)$, where x is an integer. Therefore we may say:

If $f(x)$ is rhythmic and the function $\psi(v)$, defined for all real v, is periodic (mod 1) and moreover continuous (mod 1) in those points $f(x)$, where x is an integer, then $\psi(f(x))$ is rhythmic.

By $\triangle f(x)$ we denote $f(x+1) - f(x)$.

Main theorem. *A function $f(x)$ is rhythmic, if and only if $\triangle f(x)$ is rhythmic.*

It is easy to show, that $\triangle f(x)$ is rhythmic, if $f(x)$ is rhythmic.

So we have only to show, that $f(x)$ is rhythmic, if $\triangle f(x)$ is rhythmic.

Be w an arbitrary wave of the function $f(x)$. If we translate this wave upwards over a distance u and consider the remainder (mod 1), we get a wave, which we call $w + u$. Be λ a natural number. By $w(\lambda)$ we denote the wave of $f(x)$, the abscissae of which are $\geqq -\lambda$ and $\leqq \lambda$; let S be the system of points u, such that $f(x)$ possesses for any natural number λ and for each positive number δ a wave, which is equal to $w(\lambda) + u$, apart from δ. The system S contains the origin; if S contains a point u, it contains every point $\equiv u$ (mod 1).

If S contains two points u and u', it contains also their sum $u + u'$. In fact, $f(x)$ possesses a wave w', which equals $w(\lambda) + u$, apart from $\frac{1}{2}\delta$; choose the natural number $\lambda' \geqq \lambda$ such that $w(\lambda')$ contains the wave w';

then $f(x)$ possesses a wave, which equals $w(\lambda') + u'$, apart from $\frac{1}{2}\delta$, and this wave contains a wave, which equals $w(\lambda) + u + u'$, apart from δ.

To every positive number ε a natural number $\lambda^* = \lambda^*(\varepsilon)$ and a positive number $\delta^* = \delta^*(\varepsilon) \leqq \frac{1}{4}\varepsilon$ correspond, such that each integer $\Lambda \geqq \lambda^*$ satisfies the following condition: If $f(x)$ possesses a wave, which equals $w(\Lambda) + u$, apart from δ^*, then the distance of u from S is less than $\frac{1}{4}\varepsilon$. In fact, if that were not the case, to any natural number λ' and to any positive number δ' would correspond a number $u \geqq 0$ and < 1, which has a distance $\geqq \frac{1}{4}\varepsilon$ from S, such that $f(x)$ possesses a wave, which equals $w(\lambda') + u$, apart from δ'. These points u form a bounded set, which possesses at least one limit point v. This point v would have from S a distance $\geqq \frac{1}{4}\varepsilon$. Nevertheless it belongs to S, for to any natural number λ and to any positive number δ corresponds a point u, satisfying the inequality $|u - v| < \frac{1}{2}\delta$, such that $f(x)$ possesses a wave, which equals $w(\lambda) + u$, apart from $\frac{1}{2}\delta$; this wave equals $w(\lambda) + v$, apart from δ, so that v belongs to S.

If the positive number ε is given, it is possible to find in S a finite number of points $u_1, \ldots, u_s$, each $\geqq 0$ and $\leqq 1$, such that to any point $u \geqq 0$ and $\leqq 1$ of S corresponds at least one point u_σ, which has from u a distance $< \frac{1}{8}\varepsilon$.

To any point u_σ corresponds a point u_τ, satisfying the inequality

$$-\tfrac{1}{4}\varepsilon < u_\sigma + u_\tau < \tfrac{1}{4}\varepsilon \pmod 1. \quad \ldots \ldots \quad (2)$$

In fact, the inequality

$$-\tfrac{1}{8}\varepsilon < h u_\tau < \tfrac{1}{8}\varepsilon \pmod 1)$$

is satisfied by an appropriate integer $h > 1$, and $(h-1)u_\sigma$ belongs to S, since u_σ belongs to it; the system $u_1, \ldots, u_s$ contains at least one point u_τ, such that

$$-\tfrac{1}{8}\varepsilon < (h-1)\,u_\sigma - u_\tau < \tfrac{1}{8}\varepsilon \pmod 1,$$

whence follows (2).

Be w an arbitrary wave of $f(x)$. This wave is a part of $w(\lambda')$, if the natural number λ' is chosen sufficiently large. Since $u_1, \ldots, u_s$ belong to S, the function $f(x)$ possesses waves, which are equal to $w(\lambda') + u_\tau$, $(\tau = 1, \ldots, s)$, apart from $\frac{1}{4}\varepsilon$. All these waves belong to $w(\Lambda)$, if the integer $\Lambda \geqq \lambda^*(\varepsilon)$ is sufficiently large. Be W the wave of $\triangle f(x)$ situated at the same place as $w(\Lambda)$, i.e. possessing the same abscissae x. Since $\triangle f(x)$ is rhythmic, a length $L = L\left(\dfrac{\delta^*}{2\Lambda+1}, W\right)$ exists, such that each interval of length L contains at least one wave W' of $\triangle f(x)$ which equals W, apart from $\dfrac{\delta^*}{2\Lambda+1}$, where $\delta^* = \delta^*(\varepsilon)$. Be y the smallest abscissa occurring in the wave W and y' the smallest abscissa occurring in the wave W'; then for all abscissae x and x', occurring in W and W', we have

$$f(x) - f(y) = \sum_{t=y}^{x-1} \triangle f(t) \quad \text{and} \quad f(x') - f(y') = \sum_{t'=y'}^{x'-1} \triangle f(t'),$$

where

$$|\Delta f(t') - \Delta f(t)| < \frac{\delta^*}{2\Lambda+1} \pmod 1.$$

As each of these sums contains at most $2\Lambda + 1$ terms, we find for corresponding x and x'

$$|f(x') - f(x) - f(y') + f(y)| < \delta^* \pmod 1.$$

Putting $f(y') - f(y) \equiv u \pmod 1$ $(0 \leqq u < 1)$, we find, that the wave w' of $f(x)$, lying at the same place as W', is equal to $w(\Lambda) + u$, apart from δ^*, thus certainly apart from $\frac{1}{4}\varepsilon$. In virtue of $\Lambda \geqq \lambda^*$ the distance of u from S is less than $\frac{1}{4}\varepsilon$, so that at least one point u_σ has a distance $< \frac{1}{2}\varepsilon$ from u. Thus w' is equal to $w(\Lambda) + u_\sigma$, apart from $\frac{1}{2}\varepsilon$, and therefore also equal to $w(\Lambda) - u_\tau$, apart from $\frac{3}{4}\varepsilon$, if u_τ satisfies (2). The wave $w(\Lambda)$ contains the wave $w + u_\tau$, apart from $\frac{1}{4}\varepsilon$, so that w' contains a wave, which is equal to w, apart from ε. Any interval of length L contains a wave W', and therefore also a wave w', which proves the theorem.

Corollary. *Every polynomial is rhythmic.*

§ 2. *Rhythmic systems.*

Be $f_\nu(x) = f_\nu(x_1, \ldots, x_m)$ $(\nu = 1, \ldots, n)$ defined for each lattice point $x = (x_1, \ldots, x_m)$ of the m-dimensional space. Here $m \geqq 1$ and $n \geqq 1$. Consider in the m-dimensional space a parallel cube C (by which we mean a cube, the edges of which are parallel to the coordinate axes).

The system $(f_1(x), \ldots, f_n(x))$, where x runs through all lattice points of C, is called a wave of the system (f_ν); the edge of the cube C is called the edge of the wave.

We say that this wave repeats itself, apart from ε (where $0 < \varepsilon < 1$), if a parallel cube C' can be found, as large as C, such that

$$-\varepsilon < f_\nu(x') - f_\nu(x) < \varepsilon \pmod 1;$$

here ν runs through $1, \ldots, n$, whereas x and x' are arbitrary corresponding lattice points of C and C'.

Definition. *A system (f_ν) of functions $f_\nu(x)$, defined in every lattice point $x = (x_1, \ldots, x_m)$ of m-dimensional space, is called rhythmic, if to every positive number $\varepsilon < 1$ and to every wave w corresponds a length $L = L(\varepsilon, w)$ such that the wave w repeats itself, apart from ε, in each parallel cube with edge L.*

The next theorems have analogue proofs as the corresponding theorems of the preceding section. We only prove therefore the main theorem.

If the system (f_ν) of functions $f_\nu(x)$ is rhythmic, and the system of Diophantine inequalities

$$\alpha_\nu < f_\nu(x) < \beta_\nu \pmod 1 \quad (\nu = 1, \ldots, n)$$

has at least one integral solution, then this system of inequalities has an infinite number of integral solutions. There even exists a length L, such *that every parallel cube with edge L contains at least one lattice point x, which satisfies the n inequalities.*

If (f_ν) is a rhythmic system, then to each positive number $\varepsilon < 1$ and to each length λ corresponds a length Λ, such that each wave with edge λ repeats itself, apart from ε, in each parallel cube with edge Λ.

A function $\psi(v) = \psi(v_1, \ldots, v_n)$ is called periodic (mod 1) if from $v_\nu \equiv w_\nu$ (mod 1) $(\nu = 1, \ldots, n)$ follows $\psi(v) \equiv \psi(w)$ (mod 1).

A function $\psi(v)$, defined in all points v of n-dimensional space, is called continuous (mod 1) in a point v', if to any point v in the vicinity of v' corresponds an integer $e(v)$, such that $\psi(v) - e(v)$ is continuous in v'.

Generalisation of the first theorem of continuity. *If the functions $f_\nu(x)$ $(\nu = 1, \ldots, n)$ form a rhythmic system, and if $\psi(v_1, \ldots, v_n)$, defined for each $v = (v_1, \ldots, v_n)$ is periodic (mod 1) and continuous (mod 1), then the $n+1$ functions $f_\nu(x)$ and $\psi(f_1(x), \ldots, f_n(x))$ also form a rhythmic system.*

By $\triangle_\mu f(x)$ we denote $f(x_1, \ldots, x_{\mu-1}, x_\mu + 1, x_{\mu+1}, \ldots, x_m) - f(x)$.

Main theorem. *A system of functions $f_\nu(x)$ is rhythmic if and only if the $m\,n$ functions $\triangle_\mu f_\nu(x)$ form a rhythmic system.*

Proof. By $w(\lambda)$ we denote the wave of (f_ν), the coordinates of which are $\geq -\lambda$ and $\leq \lambda$; let S be the system of points u in n-dimensional space, such that the system (f_ν) possesses for any natural number λ and for each positive number δ a wave, which is equal to $w(\lambda) + u$, apart from δ. The system S contains the origin.

If S contains a point u, it contains any point, each coordinate of which is congruent (mod 1) to the corresponding coordinate of u.

If S contains two points u and u', it contains also their sum $u + u'$. This and the following property are proved in the same way as the corresponding properties in § 1.

To any positive number ε a natural number $\lambda^* = \lambda^*(\varepsilon)$ and a positive number $\delta^* = \delta^*(\varepsilon) \leq \frac{1}{4}\varepsilon$ correspond, such that each integer $\Lambda \geq \lambda^*$ satisfies the following condition: If (f_ν) possesses a wave, which equals $w(\Lambda) + u$, apart from δ^*, then the distance of u from S is less than $\frac{1}{4}\varepsilon$.

If the positive number ε is given, it is possible to find in S a finite number of points $u_1, \ldots, u_s$, each lying in the n-dimensional cube $0 \leq u^{(\nu)} \leq 1$ $(\nu = 1, \ldots, n)$, such that to any point u of S, lying in that cube, corresponds at least one point u_σ, which has from u a distance $< \frac{1}{8}\varepsilon$.

To any point $u_\sigma = (u_\sigma^1, \ldots, u_\sigma^{(n)})$ corresponds a point $u_\tau = (u_\tau^1, \ldots, u_\tau^{(n)})$, satisfying the inequalities

$$\tfrac{1}{4}\varepsilon < u_\sigma^{(\nu)} + u_\tau^{(\nu)} < \tfrac{1}{4}\varepsilon \pmod 1 \quad (\nu = 1, \ldots, n), \quad \ldots \quad (3)$$

since the system of inequalities

$$-\frac{\varepsilon}{8} < h\,u_\sigma^{(\nu)} < \frac{\varepsilon}{8} \pmod 1 \quad (\nu = 1, \ldots, n)$$

is satisfied by an appropriate integer $h > 1$.

Be w an arbitrary wave of $f(x)$. This wave is a part of $w(\lambda')$, if the natural number λ' is chosen sufficiently large. Since $u_1, \ldots, u_s$ belong to S, the system (f_ν) possesses waves which are equal to $w(\lambda') + u_\tau$ $(\tau = 1, \ldots, s)$, apart from $\frac{\varepsilon}{4}$. All these waves belong to $w(\Lambda)$, if the integer $\Lambda \geq \lambda^*(\varepsilon)$ is sufficiently large.

Be W the wave of $(\triangle_\mu f_\nu)$ lying at the same place as $w(\Lambda)$. Since $(\triangle_\mu f_\nu)$ is rhythmic, a length $L = L\left(\frac{\delta^*}{(2\Lambda+1)m}, W\right)$ exists, such that each interval of length L contains at least one wave W' of $(\triangle_\mu f_\nu)$, which equals W, apart from $\frac{\delta^*}{(2\Lambda+1)m}$, where $\delta^* = \delta^*(\varepsilon)$.

Be y the lattice point with the smallest coordinates occurring in the wave w and be y' the lattice point with the smallest coordinates occurring in w'. Then we have for each lattice point x of w and for each lattice point x' of w'

$$f_\nu(x) - f_\nu(y) = \Sigma \triangle_\mu f_\nu(t) \quad \text{and} \quad f_\nu(x') - f_\nu(y') = \Sigma \triangle_\mu f_\nu(t');$$

each sum contains at most $(2\Lambda + 1)m$ terms, where t and t' denote lattice points. For each set of corresponding lattice points t and t' we have

$$|\triangle_\mu f_\nu(t') - \triangle_\mu f_\nu(t)| < \frac{\delta^*}{(2\Lambda+1)m} \pmod 1,$$

hence we obtain for two corresponding lattice points x and x'

$$|f_\nu(x') - f_\nu(x) - f_\nu(y') + f_\nu(y)| < \delta^* \pmod 1.$$

Putting $f_\nu(y') - f_\nu(y) \equiv u^{(\nu)} \pmod 1$ $(0 \leq u^{(\nu)} < 1)$, we find that the wave w' of (f_ν), lying at the same place as W', is equal to $w(\lambda) + u$, apart from δ^*, thus certainly apart from $\frac{1}{4}\varepsilon$; here u denotes the point $(u', \ldots, u^{(n)})$. In virtue of $\Lambda \geq \lambda^*$ the distance of u from S is less than $\frac{1}{4}\varepsilon$, so that at least one point u_σ has a distance $\leq \frac{1}{2}\varepsilon$ from u. Thus w' is equal to $w(\Lambda) + u_\sigma$, apart from $\frac{\varepsilon}{2}$, and therefore also equal to $w(\Lambda) - u_\tau$, apart from $\frac{3}{4}\varepsilon$, if u_τ satisfies (3). The wave $w(\Lambda)$ contains the wave $w + u_\tau$, apart from $\frac{1}{4}\varepsilon$, so that w' contains a wave, which is equal to w, apart from ε. Each interval of length L contains a wave W' and therefore also a wave w'. This establishes the proof.

§ 3. *Absolutely rhythmic systems.*

Definition. *A function $f(x) = f(x_1, \ldots, x_m)$ is called absolutely rhythmic, if every rhythmic system (f_ν) of functions $f_\nu(x)$ remains rhythmic if the function $f(x)$ is added to the system (f_ν).*

It is easy to show, that a rhythmic system remains rhythmic, if we add an arbitrary number of rhythmic functions to it.

A finite number of absolutely rhythmic functions form a rhythmic system.

An absolutely rhythmic function is rhythmic.

Theorem of addition. *The sum of two absolutely rhythmic functions is also absolutely rhythmic.*

Proof. Consider an arbitrary rhythmic system, consisting of r functions $\varphi_\varrho(x) = \varphi_\varrho(x_1, \ldots, x_m)$. If $f_1(x)$ and $f_2(x)$ are absolutely rhythmic, the $r+2$ functions $\varphi_\varrho(x)$, $f_1(x)$ and $f_2(x)$ form a rhythmic system. To each positive number $\varepsilon < 1$ and to each wave w of this system corresponds a length $L = L\left(\frac{\varepsilon}{2}, w\right)$, such that in every parallel cube with edge L the wave repeats itself, apart from $\frac{\varepsilon}{2}$. The $r+1$ functions $\varphi_\varrho(x)$, $f_1(x) + f_2(x)$ form then a system of which the wave, situated at the same place as w, repeats itself in that cube, apart from ε. Hence this system is rhythmic and $f_1(x) + f_2(x)$ is absolutely rhythmic.

It is easy to show, that also the difference of two absolutely rhythmic functions is absolutely rhythmic; and similarly, that an absolutely rhythmic function remains so, if multiplied by an integer.

The proofs of the next three theorems are also obvious.

Second theorem of continuity. *If the functions $f_\nu(x) = f_\nu(x_1, \ldots, x_m)$ $(\nu = 1, \ldots, n)$ are absolutely rhythmic, then the function $\psi(f_1(x), \ldots, f_n(x))$ is absolutely rhythmic, if the function $\psi(v_1, \ldots, v_n)$, defined for each point $v = (v_1, \ldots, v_n)$, is periodical* (mod 1) *and continuous* (mod 1).

Here we may replace the condition, that $\psi(v)$ be continuous (mod 1) by the condition which only postulates, that $\psi(v)$ be continuous (mod 1) in any point $(f_1(x), \ldots, f_n(x))$, where x is an arbitrary lattice point of n-dimensional space.

A function $f(x) = f(x_1, \ldots, x_m)$ is absolutely rhythmic if and only if the m functions $\triangle_\mu f(x)$ are absolutely rhythmic.

Every polynomial is absolutely rhythmic.

If s is a natural number and $\sigma_1, \ldots, \sigma_m$ are integers, then the system

$$\alpha_\nu < f_\nu(x) < \beta_\nu \ (\text{mod } 1) \ (\nu = 1, \ldots, n); \ x_\mu \equiv \sigma_\mu \ (\text{mod } s) \ (\mu = 1, \ldots, m), \quad (4)$$

where the functions $f_\nu(x) = f_\nu(x_1, \ldots, x_m)$ form a rhythmic system, has either no integral solution or an infinite number of integral solutions $x = (x_1, \ldots, x_m)$. In the latter case there even exists a length L, such that every parallel cube with edge L contains at least one lattice point x, satisfying the inequalities.

For (4) is identical with

$$\alpha_\nu < f_\nu(x) < \beta_\nu \ (\text{mod } 1) \ (\nu = 1, \ldots, n); \ -\frac{1}{s} < \frac{x_\mu - \sigma_\mu}{s} < \frac{1}{s} \ (\text{mod } 1) \ (\mu = 1, \ldots, m);$$

the linear polynomials $\frac{x_\mu - \sigma_\mu}{s}$ are absolutely rhythmic, so that the $n+m$ functions $f_\nu(x)$ and $\frac{x_\mu - \sigma_\mu}{s}$ form a rhythmic system.

A periodic function $f(x)$ of one variable x, the period of which is equal to a natural number s, is absolutely rhythmic.

In fact, r arbitrary rhythmic functions $\varphi_1(x), \ldots, \varphi_r(x)$ form with the absolutely rhythmic function $\frac{x}{s}$ a rhythmic system $\left(\varphi_\varrho, \frac{x}{s}\right)$. To any positive $\varepsilon < \frac{1}{s}$ and to any wave w of this rhythmic system corresponds a length $L = L(\varepsilon, w)$, such that every interval of length L contains a wave w' of $\left(\varphi_\varrho, \frac{x}{s}\right)$, which equals w, apart from ε. Let W and W' be the waves of (φ_ϱ, f) situated at the same place as w and w'. Obviously these two waves are equal, apart from ε. This proves the theorem.

§ 4. *The polynomial theorem.*

I introduce the smallest set R, each element of which is a function $f(x)$, defined at each lattice point x in m-dimensional space (here m denotes a fixed natural number), such that R contains the function which is identically equal to 0, and that R possesses the following properties:

I. If R contains two functions $f(x)$ and $F(x)$, it contains also their sum $f(x) + F(x)$.

II. Let $\varphi(x)$ be an arbitrary function, which takes at each lattice point x an integral bounded value, such that the product $c\,\varphi(x)$ is absolutely rhythmic for every constant c. If R contains a function $f(x)$, it contains also the product $f(x)\varphi(x)$.

III. If the m difference functions $\triangle_\mu f(x)$ of a function $f(x)$ occur in R, then R contains also the function $f(x)$.

Lemma 1. *A function $\varphi(x)$, which satisfies the condition mentioned in* II, *has the property, that $f(x)\,\varphi(x)$ is absolutely rhythmic, if $f(x)$ is an absolutely rhythmic function.*

We must prove, that the system $(\varphi_\varrho, f\varphi)$ is rhythmic, if the system (φ_ϱ) is rhythmic. The absolute value of the bounded function $\varphi(x)$ is $\leqq c$, where c denotes a suitable number $\geqq 1$. Since $f(x)$ and $\frac{\varphi(x)}{4c}$ are absolutely rhythmic, they form with the functions $\varphi_\varrho(x)$ a rhythmic system. Consider an arbitrary wave w of that system. There exists a number $\eta \geqq 1$, depending on w, such that every value of the function $f(x)$, which occurs in w, has an absolute value $\leqq \eta$. To every positive number $\varepsilon < 1$ corresponds a length $L = L\left(\frac{\varepsilon}{8c\eta}, w\right)$, such that every parallel cube with edge L contains a wave w', which is equal to w, apart from $\frac{\varepsilon}{8c\eta}$. If x and x' denote two corresponding lattice points occurring in the waves w and w', we obtain

$$-\frac{\varepsilon}{8c} < f(x') - f(x) < \frac{\varepsilon}{8c} \pmod 1 \quad \ldots \ldots \quad (5)$$

$$-\frac{\varepsilon}{8c\eta} < \frac{1}{4c}\{\varphi(x') - \varphi(x)\} < \frac{\varepsilon}{8c\eta} \pmod 1. \quad \ldots \ldots \quad (6)$$

Each of the terms $\frac{\varphi(x)}{4c}$ and $\frac{\varphi(x')}{4c}$ has an absolute value $\leq \frac{1}{4}$, so that the absolute value of their difference is $\leq \frac{1}{2} < 1 - \frac{\varepsilon}{8c\eta}$. Hence the inequality (6) remains valid, if we omit "mod 1", i.e.

$$-\frac{\varepsilon}{2\eta} < \varphi(x') - \varphi(x) < \frac{\varepsilon}{2\eta}.$$

By (5) there exists an integer $p(x)$, such that

$$-\frac{\varepsilon}{8c} < f(x') - f(x) - p(x) < \frac{\varepsilon}{8c}.$$

The function $q(x) = p(x) \cdot \varphi(x')$ assumes only integral values and has the property

$$\begin{aligned} &|f(x')\varphi(x') - f(x)\varphi(x) - q(x)| \\ &= |\{f(x') - f(x) - p(x)\}\varphi(x') + \{\varphi(x') - \varphi(x)\}f(x)| < \frac{\varepsilon}{8c}c + \frac{\varepsilon}{2\eta}\eta < \varepsilon. \end{aligned}$$

Denote by W the wave of the system $\left(\varphi_\varrho, f, \frac{\varphi}{4c}, f\varphi\right)$ lying at the same place as w; similarly by W' the wave of the same system lying at the same place as w'. It follows from the obtained result, that W' is equal to W, apart from ε. Hence the last system, and therefore also the system $(\varphi_\varrho, f\varphi)$ is rhythmic.

Lemma 2. *Each function, belonging to the set R, is absolutely rhythmic.*

By the definition of R we obtain each function of R by applying, if necessary, the following three operations, starting from the function, which is identically 0:

I. In the first operation we take the sum of two functions, occurring in R.

II. In the second operation we take the product of a function, occurring in R, and a function $\varphi(x)$, possessing the property, mentioned in condition II.

III. In the third operation we construct a function $f(x)$, the difference functions $\triangle_\mu f(x)$ of which occur in R.

Above we have shown, that each of these three operations, applied on absolutely rhythmic functions, produces an absolutely rhythmic function, so that R contains only such functions.

Lemma 3. *If R contains two functions $f(x)$ and $F(x)$, it contains also the product $f(x)\,F(x)$.*

For the proof it is recommendable to show:

If a and b are two lattice points in the m-dimensional space, and if R contains two functions $f(x)$ ad $F(x)$, then it contains also the product $f(x+a)\,F(x+b)$.

In fact, starting from the function, which is identically equal to 0, we obtain the functions $f(x)$ and $F(x)$ by repeatedly applying the operations I, II and III, mentioned in lemma 2. In this manner $f(x)$ occurs as last function of a finite system of functions; these functions, $f(x)$ itself excepted, are called the preceding functions of $f(x)$. Similarly the function $F(x)$ possesses also a finite number of preceding functions. Since the assertion is valid for the functions, which are identically equal to 0, we may assume, that the proof is already established if at least one of the functions $f(x)$ and $F(x)$ is replaced by one of its preceding functions. At least one of the three following cases occurs:

1°: $f(x)$ is the sum of two preceding functions $f_1(x)$ and $f_2(x)$. By our assumption R contains the functions $f_1(x+a)\,F(x+b)$ and $f_2(x+a)\,F(x+b)$, hence also their sum $f(x+a)\,F(x+b)$.

2°: The function $f(x)$ is equal to a preceding function $f_1(x)$, multiplied by a function $\varphi(x)$, possessing the property, mentioned in condition II. By our assumption R contains the product $f_1(x+a)\,F(x+b)$, hence also the product $f_1(x+a)\,F(x+b)\,\varphi(x+a) = f(x+a)\,F(x+b)$.

3°: The m difference functions $\triangle_\mu f(x)$ $(\mu = 1, \ldots, m)$ preceed $f(x)$. Here three cases are possible:

3, 1: $F(x)$ is the sum of two preceding functions. This case is the same as case 1°, except that $f(x)$ and $F(x)$ are exchanged.

3, 2: $F(x)$ is equal to a preceding function, multiplied by a function $\varphi(x)$, possessing the property, mentioned in condition II. Here we proceed as in case 2°.

3, 3: The m difference functions $\triangle_\mu F(x)$ preceed $f(x)$. We have:

$$\triangle_\mu \{f(x+\mathrm{a})\,F(x+b)\} = F(x+b)\,\triangle_\mu f(x+\mathrm{a}) + f(x+\mathrm{c}_\mu)\,\triangle_\mu F(x+b), \quad (7)$$

where

$$\mathrm{c}_\mu = (\mathrm{a}_1, \ldots, a_{\mu-1},\ a_\mu + 1,\ \mathrm{a}_{\mu+1}, \ldots, a_m).$$

By our assumption the two terms, occurring at the right hand side of (7), denote functions of R, so that R contains also their sum. Hence R contains the m difference-functions of $f(x+a)\,F(x+b)$ and therefore also this last function itself.

Now it is obvious, that R is a ring, and that R contains each polynomial in m variables.

Lemma 4. *If R contains a function $f(x)$, which does not assume an integral value at any lattice point x, then R contains also $[f(x)]$, where $[u]$ denotes the greatest integer $\leq u$.*

I show first, that $k[f(x)]$ is absolutely rhythmic for every constant k. Since $f(x)$ belongs to R, and is therefore absolutely rhythmic by lemma 2, and since $k(v - [v])$ is continuous (mod 1) at every point $v = f(x)$, where x denotes an arbitrary lattice point in the m-dimensional space, the function $k(f(x) - [f(x)])$ is absolutely rhythmic by the second continuity theorem (§ 3). R contains $kf(x)$, so that this function is absolutely rhythmic. Hence $k[f(x)]$, as difference of two absolutely rhythmic functions, is absolutely rhythmic.

Now I omit the condition that $f(x)$ does not assume an integral value at any lattice point x, so that $f(x)$ is an arbitrary function occurring in R, and I show: If the constant c is chosen in such a manner, that $f(x)+c$ does not take an integral value at any lattice point x, then R contains also $[f(x)+c]$. Since this assertion is valid for the function which is identically 0, we may assume that the proof is already established if $f(x)$ is replaced by one of its preceding functions. Three cases are possible:

1°: $f(x)$ is the sum of two preceding functions $f_1(x)$ and $f_2(x)$. Choose the numbers c_1 and c_2 such that neither $f_1(x)+c_1$ nor $f_2(x)+c_2$ assumes an integral value at any lattice point x. By our assumption R contains the functions $[f_1(x)+c_1]$ and $[f_2(x)+c_2]$. Then by our above argument the functions $k[f(x)+c]$, $k[f_1(x)+c_1]$ and $k[f_2(x)+c_2]$ are absolutely rhythmic for every constant k.

If we put

$$[f(x)+c]=[f_1(x)+c_1]+[f_2(x)+c_2]+\varphi_1(x),$$

the function $k\varphi_1(x)$ is absolutely rhythmic for every constant k. By condition II, applied with 1 instead of $f(x)$ and with $\varphi_1(x)$ instead of $\varphi(x)$, R contains $\varphi_1(x)$, so that $[f(x)+c]$ is the sum of three functions, belonging to R, and occurs therefore in R.

2°: The function $f(x)$ is equal to a preceding function $f_1(x)$, multiplied by a function $\varphi(x)$, possessing the property, mentioned in condition II. If the number c_1 is chosen in such a manner, that $f_1(x)+c_1$ does not assume an integral value at any lattice point x, then R contains by assumption $[f_1(x)+c_1]$ and therefore also $[f_1(x)+c_1]\varphi(x)$. Then by our above argument the functions $k[f(x)+c]$ and $k[f_1(x)+c_1]$ are absolutely rhythmic for every constant k, so that $k[f_1(x)+c_1]\varphi(x)$ is absolutely rhythmic by lemma 1. The function

$$\varphi_2(x)=[f(x)+c]-[f_1(x)+c_1]\varphi(x),$$

multiplied by an arbitrary constant, produces a function, which is the difference of two absolutely rhythmic functions and therefore absolutely rhythmic. The function $\varphi_2(x)$ is bounded and assumes only integral values, since that is also the case with $\varphi(x)$ and since $f(x)=f_1(x)\,.\,\varphi(x)$. By condition II, applied with $f(x)=1$ and $\varphi(x)=\varphi_2(x)$, the function $\varphi_2(x)$ belongs to R, so that

$$[f(x)+c]=[f_1(x)+c_1]\varphi(x)+\varphi_2(x)$$

is the sum of two functions, belonging to R, and belongs therefore to R.

3°: The m difference functions $\triangle_\mu f(x)$ preceed $f(x)$. If we choose the numbers $c_1, \ldots, c_m$ in such a manner, that $\triangle_\mu f(x)+c_\mu$ does not assume an integral value at any lattice point x, then by assumption R contains the m

functions $[\triangle_\mu f(x) + c_\mu]$. Then by our above argument the $m+1$ functions $k[f(x)+c]$ and $k[\triangle_\mu f(x) + c_\mu]$ are absolutely rhythmic for every constant k. Then the m functions $k\triangle_\mu[f(x)+c]$ are absolutely rhythmic (compare § 3). The function

$$\Phi_\mu(x) = \triangle_\mu[f(x)+c] - [\triangle_\mu f(x) + c_\mu]$$

is bounded, assumes only integral values and produces, if multiplied by an arbitrary constant, a function which is absolutely rhythmic as difference of two absolutely rhythmic functions. By condition II, applied with $f(x) = 1$ and $\varphi(x) = \Phi_\mu(x)$, the function $\Phi_\mu(x)$ belongs to R, so that

$$\triangle_\mu[f(x)+c] = [\triangle_\mu f(x) + c_\mu] + \Phi_\mu(x)$$

belongs to R as a sum of two functions belonging to R. Consequently by condition III R contains the function $[f(x)+c]$.

The polynomial theorem. *Consider the system*

$$\left.\begin{array}{l} \alpha_1 < f_1(x_1, \dots, x_m) - y_1 < \beta_1 \\ \alpha_2 < f_2(x_1, \dots, x_m, y_1) - y_2 < \beta_2 \\ \dots\dots\dots\dots\dots\dots \\ \alpha_n < f_n(x_1, \dots, x_m, y_1, \dots, y_{n-1}) - y_n < \beta_n \end{array}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

where $f_1, \dots, f_n$ denote polynomials, and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ denote constants. If this system of inequalities possesses at least one integral solution (i.e. a solution with integers $x_1, \dots, x_m, y_1, \dots, y_n$), then the system possesses an infinite number of integral solutions, and there exists even a length L, such that each parallel cube with edge L, situated in m-dimensional space, contains at least one lattice point $(x_1, \dots, x_m)$, which satisfies the system, if the integers $y_1, \dots, y_n$ are suitably chosen.

Proof. Without loss of generality, we may assume $\beta_\nu - \alpha_\nu < \frac{1}{2}$. Choose the numbers $c_1, \dots, c_n$ such that the inequalities

$$\tfrac{1}{4} - \tfrac{1}{2}(\alpha_\nu + \beta_\nu) < c_\nu < \tfrac{3}{4} - \tfrac{1}{2}(\alpha_\nu + \beta_\nu) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

are valid, and that $f_\nu(x_1, \dots, x_m, y_1, \dots, y_{\nu-1}) + c_\nu$ (where ν runs through the values $1, \dots, n$) does not assume an integral value for any system of integers $x_1, \dots, x_m, y_1, \dots, y_{\nu-1}$. Put

$$\varphi_\nu(x) = [f_\nu(x_1, \dots, x_m, \varphi_1(x), \dots, \varphi_{\nu-1}(x)) + c_\nu] \quad (\nu = 1, \dots, n-1)$$

To prove, that (8) possesses the same integral solutions as the system

$$\alpha_\nu < f_\nu(x_1, \dots, x_m, \varphi_1(x), \dots, \varphi_{\nu-1}(x)) - y_\nu < \beta_\nu \quad (\nu = 1, \dots, n), \quad (10)$$

I distinguish two cases:

1°: If $(x_1, \dots, x_m, y_1, \dots, y_n)$ denotes an integral solution of (8), then (9) implies

$$0 < f_\nu(x_1, \dots, x_m, y_1, \dots, y_{\nu-1}) + c_\nu - y_\nu < 1.$$

Hence it follows successively that $y_\nu = \varphi_\nu(x)$ $(\nu = 1, \ldots, n)$, so that the considered solution satisfies also (10).

2°: If $(x_1, \ldots, x_m, y_1, \ldots, y_n)$ denotes an integral solution of (10), then (9) implies

$$0 < f_\nu(x_1, \ldots, x_m, \varphi_1(x), \ldots, \varphi_{\nu-1}(x)) + c_\nu - y_\nu < 1 \quad (\nu = 1, \ldots, n).$$

Hence it follows successively that

$$y_\nu = [f_\nu(x_1, \ldots, x_m, \varphi_1(x), \ldots, \varphi_{\nu-1}(x)) + c_\nu] = \varphi_\nu(x) \quad (\nu = 1, \ldots, n-1),$$

so that the considered solution satisfies also (8).

Now we know that (10) possesses an integral solution, and it is sufficient to prove that this system possesses an infinity of integral solutions and that each sufficiently large parallel cube contains at least one lattice point $(x_1, \ldots, x_m)$ satisfying the system. Consequently it is sufficient to show that the n functions $f_\nu(x_1, \ldots, x_m, \varphi_1(x), \ldots, \varphi_{\nu-1}(x))$ are absolutely rhythmic. The ring R contains the polynomial $f_1(x_1, \ldots, x_m) + c_1$, therefore by lemma 4 also the function $\varphi_1(x)$. If R contains $\varphi_1(x), \ldots, \varphi_{\nu-1}(x)$, it contains also $f_\nu(x_1, \ldots, x_m, \varphi_1(x), \ldots, \varphi_{\nu-1}(x)) + c_\nu$; consequently, again by lemma 4, it contains $\varphi_\nu(x)$. Hence R contains $\varphi_1(x), \ldots, \varphi_{n-1}(x)$, and therefore also the n functions $f_\nu(x_1, \ldots, x_m, \varphi_1(x), \ldots, \varphi_{\nu-1}(x))$. Consequently these n functions are absolutely rhythmic.

Mathematics. — *On the fundamental theorem of algebra.* (First communication.) By J. G. VAN DER CORPUT.

(Communicated at the meeting of June 29, 1946.)

§ 1. *Introduction* [1]).

There exist several versions of the fundamental theorem of algebra. One of these versions is as follows:

If Ω be the field of the real numbers, then any polynomial

$$F(X) = f_0 + f_1 X + \dots + f_\mu X^\mu$$

of degree $\mu \geqq 1$, the coefficients of which belong to Ω and the highest coefficient of which is equal to 1, *possesses exactly μ roots $x_1, \dots, x_\mu$, belonging to the field $\Omega(i)$, where $i = \sqrt{-1}$*; in other words it is possible to write $F(X)$ in the form

$$F(X) = (X - x_1)\ (X - x_2) \dots (X - x_\mu),$$

where $x_\varrho = a_\varrho + ib_\varrho$ $(\varrho = 1, \dots, \mu)$ and a_ϱ and b_ϱ denote elements of Ω.

It is not possible to give a purely algebraic proof of this theorem, because this proposition involves the notion of real numbers and therefore the notion of a limit, which does not belong to algebra. It is not appropriate to call the theorem in this form the fundamental theorem of algebra, because by far the greatest part of algebra does not require the theorem in this form at all. The simplest proof of the theorem in this form is given by J. E. LITTLEWOOD [2]), who uses the fact (belonging to analysis), that a continuous function $\geqq 0$, given on a bounded closed set, assumes at one point of this set at least a minimum value.

Now the second version that may justly be called the fundamental theorem of algebra:

If Ω be an arbitrary commutative field, then to any polynomial $F(X)$ of degree $\mu \geqq 1$, the coefficients of which belong to Ω and the highest coefficient of which is equal to the unit element e of Ω, corresponds a commutative field Ω_1, containing all elements of Ω, such that $F(X)$ possesses exactly μ roots, all belonging to Ω_1.

The proof of this theorem is purely algebraic. Here $\sqrt{2}$ is a symbol, for which addition, subtraction, multiplication and division are defined in such a manner, that the usual rules remain valid and that the square of this symbol equals 2.

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1) Lecture given at the Manchester University, May 28th 1946.

2) J. E. LITTLEWOOD, Mathematical notes (14): "Every polynomial has a root". J. London Math. Soc. **16**, 95—98 (1941).

The arguments, applied in this part of mathematics, do not permit us to distinguish between $\sqrt{2}$ and $-\sqrt{2}$. Any rational relation with rational coefficients, involving $\sqrt{2}$, remains valid if $\sqrt{2}$ is replaced by $-\sqrt{2}$. On this fact, suitably generalised, is based the whole Galois theory.

In this theory we suppose that it is always possible to decide, whether a given polynomial $F(X)$, the coefficients of which belong to the given commutative field Ω, is reducible or not. The polynomial $F(X)$ of degree μ is called reducible (with respect to Ω), if it is possible to write $F(X)$ as a product $F_1(X) \cdot F_2(X)$ of two polynomials of degree $< \mu$, the coefficients of which belong to Ω. There are many fields, f.i. the field of the rational numbers, which satisfy this condition, but there are exceptions. And even if it is theoretically possible, then the calculations are so long, that practically nobody gets through them. Nevertheless this investigation is often necessary, even for the very simplest problem.

Let α be a root of $F(X)$, so that α is an element of Ω_1. Consider a polynomial

$$G(X) = g_0 + g_1 X + \ldots + g_\nu X^\nu,$$

of degree $\nu \geqq 1$, the coefficients of which belong to Ω (and therefore to Ω_1) and the highest coefficient of which is the unit element of Ω (and therefore also of Ω_1). According to the fundamental theorem of algebra, applied with Ω_1 instead of Ω, there exists a commutative extension Ω_2 of Ω, such that $G(X)$ possesses exactly ν roots, all belonging to Ω_2. If β be a root of $G(X)$, then what do we know about $\alpha + \beta$ and $\alpha\beta$? We can construct in the following manner two polynomials $U(X)$ and $V(X)$, both of degree $\mu\nu$, such that $\alpha + \beta$ is a root of $U(X)$ and $\alpha\beta$ is a root of $V(X)$. The products

$$\Pi(X - Y_\varrho - Z_\sigma) \quad \text{and} \quad \Pi(X - Y_\varrho Z_\sigma), \quad \ldots \ldots \quad (1)$$

where ϱ runs over $1, 2, \ldots, \mu$ and σ over $1, 2, \ldots, \nu$ and where

$$X, Y_1, \ldots, Y_\mu, Z_1, \ldots, Z_\nu$$

denote indeterminates, are integral rational symmetrical functions of the indeterminates $Y_1, \ldots, Y_\mu$ and also of the indeterminates $Z_1, \ldots, Z_\nu$. Hence these products may be written as integral rational functions of X, the elementary symmetrical functions of $Y_1, \ldots, Y_\mu$, and the elementary symmetrical functions of $Z_1, \ldots, Z_\nu$. If we replace the elementary symmetrical functions $\Sigma Y_1, \Sigma Y_1 Y_2, \ldots, Y_1 Y_2 \ldots Y_\mu$ successively by

$$-f_{\mu-1}, f_{\mu-2}, \ldots, (-1)^\mu f_0$$

and similarly the elementary symmetrical functions

$$\Sigma Z_1, \Sigma Z_1 Z_2, \ldots, Z_1 Z_2, \ldots, Z_\nu$$

successively by $-g_{\nu-1}, g_{\nu-2}, \dots, (-1)^\nu g_0$, then the products in question transform into polynomials

$$U(X, f_0, \dots, f_{\mu-1}, g_0, \dots, g_{\nu-1}) \text{ and } V(X, f_0, \dots, f_{\mu-1}, g_0, \dots, g_{\nu-1})$$

in the X, f_ϱ en g_σ ($\varrho = 0, \dots, \mu-1$; $\sigma = 0, \dots, \nu-1$).

These polynomials, both of degree $\mu\nu$ in X, and both uniquely determined by the polynomials $F(X)$ and $G(X)$, will be denoted by $F(X) \mathbin{\boldsymbol{+}} G(X)$ and $F(X) \mathbin{\boldsymbol{\times}} G(X)$. It is easy to see that $\alpha + \beta$ is a root of the first and $\alpha\beta$ is a root of the second polynomial.

For instance, if

$$F(X) = X^2 - 2 \text{ and } G(X) = X^2 - 2X - 1,$$

then we find

$$F(X) \mathbin{\boldsymbol{+}} G(X) = (X-1)^2\,(X^2 - 2X - 7)$$
$$F(X) \mathbin{\boldsymbol{\times}} G(X) = (X^2 + 4X + 2)\ \ (X^2 - 4X - 2).$$

This example shows, that the properties of $\alpha + \beta$ are not completely determined by the fact, that $\alpha + \beta$ is a root of $F(X) \mathbin{\boldsymbol{+}} G(X)$, for $\alpha + \beta$ may be equal to 1, or $\alpha + \beta$ may be a root of $X^2 - 2X - 7$, and the roots of the last polynomial do not have the same proporties as the number 1.

To find the properties of $\alpha + \beta$ we must decompose $F(X) \mathbin{\boldsymbol{+}} G(X)$ into irreducible factors (generally a tiresome problem) and then we must know which of these factors has $\alpha + \beta$ as a root.

To a third polynomial $H(X)$ corresponds a commutative extension Ω_3 of Ω, containing the roots of $F(X)$, $G(X)$ and $H(X)$, and so we can go on.

If the number of elements of the given commutative field is enumerable, then we find in this manner after an infinite number of steps a commutative extension Ω'' of Ω, containing all roots of each polynomial, whose coefficients belong to Ω.

The last features of this theory, on which I draw the attention, is that it is not possible to distinguish here beween real and non-real roots, and that we may not say, that $\sqrt{2}$ is situated between 1 and 2; in fact if that were the case, then $-\sqrt{2}$ would be negative, and, as we have said, it is impossible in this theory to distinguish between $\sqrt{2}$ and $-\sqrt{2}$. It is therefore impossible to approximate $\sqrt{2}$ by rational numbers.

Let us now consider a third version of the fundamental theorem of algebra. Let Ω be a commutative Archimedeanly ordered field, i.e. I assume that it is possible for any couple of elements a and b to decide whether $a = b$, $a > b$ or $a < b$; furthermore an arbitrary element a being given, a natural number ν can be found, such that a is less than the sum of ν terms, each of which is equal to the unit-element e of the field.

In a purely algebraic manner I will show that it is possible to construct in one step a commutative Archimedeanly ordered extension Ω' of Ω with the following property:

Any polynomial of degree $\mu \geq 1$, the coefficients of which belong to Ω and the highest coefficient of which equals the unit element e of Ω, possesses exactly μ roots, all belonging to a third field $\Omega'(i)$, where $\Omega'(i)$ denotes the field, formed by adjoining to Ω' the number $i = \sqrt{-e}$.

If Ω is the field of the rational numbers, then Ω' is the field of the real algebraic numbers. If Ω is the field of the real algebraic numbers, then Ω' is identical with Ω. If we take for Ω the field of the real numbers, we depart from algebra because then we want the notion of limits. Every real number belongs in this case to Ω', since Ω' is an extension of Ω. Conversely each element of Ω' can be approximated [3]) by elements of Ω', i.e. by real numbers and is therefore a real number itself. Hence Ω' is identical with Ω.

Now some remarks about the proof. By an interval Φ we mean the set of elements x of the given ordered field Ω, satisfying the inequalities $a \leq x \leq b$, where a and b are elements of Ω with $a \leq b$. The elements a and b may coincide; in that case the interval consists of only one point.

By the characteristic divisor $F^*(X)$ of $F(X)$ we mean the quotient

$$F^*(X) = \frac{F(X)}{\left(F, \frac{dF}{dX}\right)},$$

where $\left(F, \frac{dF}{dX}\right)$ denotes the greatest common divisor of the polynomial $F(X)$ and its derivative $\frac{dF(X)}{dX}$; if we put the highest coefficient of this greatest common divisor equal to the unit element of Ω, this divisor is uniquely determined.

I say that the polynomial $F(X)$, the coefficients of which belong to Ω, changes sign in the interval Φ, if Φ contains two elements u and v of Ω, satisfying the inequalities

$$F^*(v) \leq 0 \leq F^*(u);$$

if u and v coincide, we have $F^*(u) = 0$, hence $F(u) = 0$ [4]).

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[3]) As will appear presently, each element γ of Ω' has the form (Γ, C), where C denotes a polynomial and Γ an interval, which may be taken arbitrarily small. The endpoints of Γ, which belong to Ω, give an arbitrarily precise approximation of γ.

[4]) If the polynomial $F(X)$ and the interval Φ are given, it is possible to decide in a finite number of steps, whether Φ contains two elements u and v which satisfy the above named inequalities. In fact by § 3 the interval Φ can be divided into a finite number of subintervals, such that throughout each of these subintervals the polynomial $F^*(X)$ has either a fixed sign or is an increasing or a decreasing function of X. If $F^*(X)$ has the same sign in the endpoints of all these subintervals, then $F^*(X)$ has a fixed sign in the interval Φ. Otherwise Φ contains two elements u and v, which satisfy the above named inequalities; in fact one of the subintervals in question has the property that $F^*(X)$ takes a value ≥ 0 at one endpoint and a value ≤ 0 at the other endpoint.

I say, that $F(X)$ changes sign in Φ more than once, if Φ contains three elements, u, v and w of Ω, with $u < v < w$, such that either

$$F^*(u) \leqq 0, \quad F^*(v) \geqq 0, \quad F^*(w) \leqq 0$$

or

$$F^*(u) \geqq 0, \quad F^*(v) \leqq 0, \quad F^*(w) \geqq 0.$$

Now the main point:

Let us consider couples (Γ, C), where $C = C(X)$ denotes a polynomial in Ω with highest coefficient $= e$, that changes sign only once in the interval Γ. If Γ contains only one point w, then the polynomial $C(X)$ vanishes at that point w; in this case we identify (Γ, C) with that point w. The set Ω' formed by couples (Γ, C) is therefore an extension of the given field Ω.

We write $(\Gamma, C) = (\triangle, D)$ if and only if the greatest common divisor of $C(X)$ and $D(X)$ changes sign in the common part $(\Gamma, \triangle)$ of the intervals Γ and $\triangle$; in that case this greatest common divisor changes sign only once in $(\Gamma, \triangle)$, as I will show in § 2.

It is easy to show that this notion of equality is reflexive and symmetrical, i.e. we have $(\Gamma, C) = (\Gamma, C)$ and the relation $(\Gamma, C) = (\triangle, D)$ implies $(\triangle, D) = (\Gamma, C)$. In § 2 I show, that this notion of equality is also transitive, i.e. $(\Gamma, C) = (\triangle, D)$ and $(\triangle, D) = (\Lambda, L)$ implies $(\Gamma, C) = (\Lambda, L)$.

Now we have to define the sum of two couples (Γ, C) and $(\triangle, D)$. If u runs through the interval Γ and v runs through the interval $\triangle$, then $u + v$ runs through an interval which we denote by $\Gamma \boldsymbol{+} \triangle$. In § 3 I show that the above defined polynomial $C(X) \boldsymbol{+} D(X)$ changes sign in the interval $\Gamma \boldsymbol{+} \triangle$. It is possible, that this polynomial changes sign in this interval more than once, but we can find a subinterval Γ_1 of Γ, in which $C(X)$ changes sign, and a subinterval $\triangle_1$ of $\triangle$, in which $D(X)$ changes sign, in such a manner that $C \boldsymbol{+} D$ changes sign only once in the interval $\Gamma_1 \boldsymbol{+} \triangle_1$. (This last condition is satisfied, if the subintervals Γ_1 and $\triangle_1$ are small enough.) In that case we have by definition

$$(\Gamma, C) = (\Gamma_1, C) \text{ and } (\triangle, D) = (\triangle_1, D).$$

In § 3 I will show, that then the couple $(\Gamma_1 \boldsymbol{+} \triangle_1, C \boldsymbol{+} D)$ is uniquely determined by the couples (Γ, C) and $(\triangle, D)$ (also independant of the choice of the subintervals Γ_1 and $\triangle_1$). I put:

$$(\Gamma, C) + (\triangle, D) = (\Gamma_1 \boldsymbol{+} \triangle_1, C \boldsymbol{+} D).$$

Let us consider the special case in which Γ consists of only one point u, and $\triangle$ of only one point v. Then $\Gamma_1 = \Gamma$ and $\triangle_1 = \triangle$, so that $\Gamma_1 \boldsymbol{+} \triangle_1$ consists of the point $u + v$ only. The polynomial $C \boldsymbol{+} D$, which changes sign in that interval, vanishes therefore at the point $u + v$ and we have:

$$(\Gamma, C) = u; \quad (\triangle, D) = v; \quad (\Gamma_1 \boldsymbol{+} \triangle_1, C \boldsymbol{+} D) = u + v.$$

Hence it appears, that the above definition of addition of two couples is allowed.

In a similar way we define the product of two couples. If u runs through the interval Γ and v through the interval $\triangle$, then uv runs through an interval, which we denote by $\Gamma \times \triangle$. The polynomial $C(X) \times D(X)$ changes sign in that interval, but perhaps more than once. It is possible to replace again Γ and $\triangle$ by subintervals Γ_1 and $\triangle_1$ with $(\Gamma, C) = (\Gamma_1, C)$ and $(\triangle, D) = (\triangle_1, D)$, such that $C \times D$ changes sign only once in the interval $\Gamma_1 \times \triangle_1$; then the couple $(\Gamma_1 \times \triangle_1, C \times D)$ is uniquely determined (also independent of the choice of the sub-intervals) and I put:

$$(\Gamma, C)\ (\triangle, D) = (\Gamma_1 \times \triangle_1, C \times D).$$

Similarly as above it is obvious, that this definition is allowed.

It is easy to show, as shall be done in § 4, that the couples (Γ, C) form a commutative field Ω'. It is therefore possible to calculate the value, which the polynomial $F(X)$ assumes, if we replace the indeterminate X by a couple (Φ, F). We find that F vanishes in that case; hence (Φ, F) is a root of the polynomial $F(X)$. Therefore I call Ω' the field of the real algebraic numbers with respect to the given field Ω.

To give an example: If Φ is the interval with the endpoints 1 and 2 and if we put $F(X) = X^2 - 2$, then we have to show that $(\Phi, F)^2 = 2$. Here $F \times F = (X^2 - 4)^2$ and $\Phi \times \Phi$ is the interval Ψ with endpoints 1 and 4. If Ψ_1 consists of only the number 2, we have

$$(\Phi, F)^2 = (\Phi \times \Phi, (X^2-4)^2) = (\Psi, (X^2-4)^2) = (\Psi_1, (X^2-4)^2) = 2.$$

The proof of the formula $\sqrt{2}\sqrt{3} = \sqrt{6}$ proceeds as follows: If Φ is again the interval with endpoints 1 and 2, hence $\Psi = \Phi \times \Phi$ the interval with endpoints 1 and 4, we have

$$(\Phi, X^2 - 2)\ (\Phi, X^2 - 3) = (\Psi, (X^2 - 6)^2) = (\Psi, X^2 - 6).$$

In the field Ω', formed by the elements, which are real algebraic with respect to Ω, the element 0 is the couple (Π, P), where Π contains the element 0 of Ω and P is a polynomial, which vanishes in that point. In fact $(\Pi, P) = (\Pi_0, X)$ by definition, where Π_0 is the interval which consists of only the element 0; if (Π_0, X) is added to an arbitrary element $(\triangle, D)$, we get again $(\triangle, D)$, for $\triangle + \Pi_0 = \triangle$ and $D + X = D$.

To order the field Ω' it is sufficient to distinguish whether a couple $\gamma = (\Gamma, C)$, which is not equal to the element 0, is positive or negative. Since $\gamma \neq 0$, it is impossible that Γ contains the element 0 and that at the same time $C(0) = 0$. Hence only two cases are possible:

1°: $C(X)$ changes sign in the interval, formed by the elements $X \geqq 0$ of Γ; in this case we put γ positive.

2° $C(X)$ changes sign in the interval, formed by the elements $X \leqq 0$ of Γ; in that case we put γ negative.

It is obvious that the sign of (Γ, C) is independant of the choice of Γ and C i.e. if $(\Gamma, C) = (\triangle, D)$, then $(\triangle, D)$ is positive, 0 or negative according to whether (Γ, C) is positive, 0 or negative.

The two conditions imposed on an ordered field are satisfied here, viz.:

1°: For every couple γ one and only one of the relations $\gamma = 0$, $\gamma > 0$ and $\gamma < 0$ is valid. In fact, put $L(X) = (-1)^{\mu} C(-X)$, where μ denotes the degree of the polynomial $C(X)$. Be Λ the interval of the elements $-x$ of Ω, where x runs through the interval Γ. Then we have $-\gamma = (\Lambda, L)$, which means that $(\Gamma, C) + (\Lambda, L)$ is the element 0 of Ω'. It is obvious that, if γ is not equal to the element 0, only one of the couples (Γ, C) and (Λ, L) is positive.

2°: If $\gamma = (\Gamma, C)$ and $\delta = (\triangle, D)$ are positive, then also their sum and product are positive. In fact, we may suppose, that all elements both of Γ and $\triangle$ are $\geqq 0$, consequently also all elements of $\Gamma + \triangle$ and $\Gamma \times \triangle$.

In this manner we have ordered Ω'.

The axiom of ARCHIMEDES is valid, for to a given couple γ corresponds an element $c > \gamma$ of Ω and Ω being Archimedeanly ordered, a natural number ν exists, such that the sum of ν terms, each equal to the unit element of Ω, is greater than c, hence greater than γ.

By means of the arbitrary commutative Archimedeanly ordered field Ω we have constructed a new Archimedeanly ordered field Ω' consisting of the elements, which are real-algebraic with respect to Ω. Repeating our argument with Ω' in stead of Ω we find the commutative Archimedeanly ordered field, formed by the elements, which are real-algebraic with respect to Ω'. In order to prove that this new field is identical with Ω', it is sufficient to show, that an arbitrary element (Φ', F') of the new field belongs to Ω'; here Φ' denotes an interval, formed by elements of Ω'; the polynomial $F'(X)$ changes sign only once in the interval Φ' and the coefficients of this polynomial belong to Φ'. The coefficients of the polynomial F' belong to Ω' and therefore are real algebraic with respect to the original field Ω'. As we show in § 5 it is possible to construct a polynomial $F(X)$, not identically $= 0$, such that the coefficients of $F(X)$ belong to Ω and that $F'(X)$ is a divisor of $F(X)$. The polynomial $F(X)$ changes sign in Φ', but perhaps more than once; however it is possible to find a subinterval Φ'_1 of Φ', such that $F'(X)$ and $F(X)$ both change sign only once in Φ'_1. By definition we have

$$(\Phi', F') = (\Phi'_1, F).$$

If the characteristic divisor of the polynomial $F(X)$ vanishes at one endpoint of Φ'_1, then (Φ'_1, F) is by definition equal to that endpoint, and therefore equal to an element of Ω'. Hence we may assume, that this characteristic divisor does not vanish at either of the endpoints of the interval. Then it is possible to find a subinterval Φ'_2 of Φ'_1, such that the endpoints a and b of Φ'_2 belong to Ω and that F changes sign only once in that interval Φ'_2. So we obtain

$$(\Phi', F') = (\Phi'_1, F) = (\Phi'_2, F).$$

Be Φ the interval formed by the elements of Ω, belonging to Φ'_2; hence

Φ is formed by the elements $\geqq a$ and $\leqq b$ of Ω. The couple (Φ, F) is an element $\geqq a$ and $\leqq b$ of Ω' and belongs consequently to Φ'_2. Moreover the polynomial $F(X)$ vanishes, if the indeterminate X is replaced by (Φ, F). By definition $(\Phi', F') = (\Phi'_2, F)$ is equal to the element (Φ, F) of Φ'_2, and this element belongs to Ω'.

Hence it is not possible to extend the field Ω' in the specified manner.

Now we pass to the third version of the fundamental theorem of algebra, which we state as follows:

Be Ω an arbitrary commutative Archimedeanly ordered field, Ω' the ordered commutative field, formed by the elements which are real-algebraic with respect to Ω. Then any polynomial of degree $\mu \geqq 1$, the coefficients of which belong to $\Omega'(i)$, and the highest coefficient of which is equal to e, possesses exactly μ roots and these roots belong to $\Omega'(i)$.

In order to give a proof we show first, that the polynomial $F(X)$ possesses at least one root, belonging to $\Omega'(i)$. Let us first consider the case $F(X) = X^2 - q$, where q denotes a positive element of Ω'. Then it is possible to find a positive element b of Ω', such that $b^2 > q$. If Φ denotes the interval with the endpoints 0 and b, formed by elements of Ω', then $X^2 - q$ changes sign only once in that interval; hence the couple $x_1 = (\Phi, X^2 - q)$ is a root of the polynomial $X^2 - q$. Consequently this polynomial possesses in Ω' the root x_1 and the polynomial $X^2 + q$ possesses in $\Omega'(i)$ the root $x_1 i$. Hence each quadratic polynomial $X^2 + pX + q$, with highest coefficient $=$ e and the coefficients of which belong to Ω', possesses at least one root, belonging to $\Omega'(i)$, for this polynomial is identical with

$$\left(X + \frac{p}{2}\right)^2 - \left(\frac{p^2}{4} - q\right).$$

Consider now the case that $F(X)$ is a polynomial of odd degree with coefficients, belonging to Ω'. Then it is possible to find two elements a and b of Ω', such that the polynomial changes sign in the interval with the endpoints a and b. Perhaps it changes sign more than once in that interval, but it is always possible to find a subinterval Φ, such that F changes sign only once in Φ. Hence the polynomial $F(X)$ possesses a root (Φ, F) belonging to Ω'.

So we have shown that each quadratic polynomial and also each polynomial of odd degree with coefficients belonging to Ω' possesses at least one root, belonging to $\Omega'(i)$. As GAUSS has shown in his second proof of the fundamental theorem of algebra, herefrom it follows, that each polynomial with highest coefficient equal to e and the coefficients of which belong to $\Omega'(i)$, possesses at least one root belonging to $\Omega'(i)$.

If x_1 is a root of $F(X)$, belonging to $\Omega'(i)$, the coefficients of $\dfrac{F(X)}{X - x_1}$ belong also to $\Omega'(i)$, so that the argument may be repeated with this quotient instead of $F(X)$. Continuing in this manner we find the number of roots to be equal to the degree of the polynomial.

In the last section (§ 6) I give a proof of the following lemma.

Be $0 \leqq \lambda \leqq \mu$. Suppose that the polynomials $A(X) = a_0 + \ldots + a_\mu X^\mu$ and $B(X) = b_0 + \ldots + b_\mu X^\mu$, the coefficients of which belong to $\Omega'(i)$, satisfy the inequalities

$$\sum_{\varrho=0}^{\lambda} |a_\varrho| \geqq \frac{a}{u};\ \sum_{\varrho=\lambda}^{\mu} |a_\varrho| \geqq \frac{a}{u};\ \sum_{\varrho=0}^{\lambda} |b_\varrho| \geqq \frac{b}{u};$$

$$\sum_{\varrho=\lambda}^{\mu} |b_\varrho| \geqq \frac{b}{u};\ |b_\varrho - a_\varrho| \leqq a\, r^{\mu!}\ (\varrho = 0, \ldots, \mu);$$

here a denotes the sum $\neq 0$ of the absolute values of the coefficients of $A(X)$ and b the sum $\neq 0$ of the absolute values of the coefficients of $B(X)$, whereas u and r are positive elements of Ω.

1° Then Ω contains a positive element v depending only on μ and u with the following property:

$A(X)$ may be written in the form

$$A(X) = a'(X - x_1) \ldots (X - x_\lambda)(e - x_{\lambda+1} X) \ldots (e - x_\mu X), \quad . \quad . \quad (2)$$

where

$$|x_\varrho| \leqq v\ (\varrho = 1, \ldots, \mu) \text{ and } |a'| \leqq a v. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

2° To any decomposition of $A(X)$ of the form (2), satisfying the inequalities (3), corresponds a decomposition of $B(X)$ of the form

$$B(X) = b'(X - y_1) \ldots (X - y_\lambda)\ (e - y_{\lambda+1} X) \ldots (e - y_\mu X),$$

such that

$$|y_\varrho - x_\varrho| < w\, r\ (\varrho = 1, \ldots, \mu) \text{ and } |b' - a'| < w\, a\, r,$$

where w depends only on μ, u and v.

The intuitionist does not object to the above arguments, since each consists of a finite number of steps. For instance, in the preceding lemma, it is possible to evaluate v in a finite number of steps, if μ and u are given. One has however to take into consideration that, according to the intuitionist, the set of the real numbers does not posses the property imposed on the field Ω, viz. that it is possible for any couple of elements a and b of Ω to decide in a finite number of steps, which of the three cases $a = b$, $a > b$ or $a < b$ occurs. Therefore it is in the intuitionistic mathematics not allowed to take for Ω the set of the real numbers. Nevertheless it is possible to give in a few lines a purely intuitionistic proof of the fundamental theorem of algebra. I prove even this theorem in the following stronger form, due to L. E. J. BROUWER [5]).

[5]) Compare: H. WEYL, Randbemerkingen zu Hauptproblemen der Mathematik, Mathematische Zeitschrift **19**, 131—150 (1924).

B. DE LOOR, Die hoofstelling van die algebra van intuïsionistiese standpunt, Dissertation Amsterdam, 1925, 63 p.

L. E. J. BROUWER and B. DE LOOR, Intuitionistischer Beweis des Fundamentalsatzes der Algebra, Proc. Kon. Akad. v. Wetensch., Amsterdam, **27**, 186—188 (1924). The same

The polynomial

$$F(X) = f_0 + \dots + f_\mu X^\mu$$

with complex coefficients, where f_σ *and* f_τ $(0 \leqq \sigma \leqq \tau \leqq \mu)$ *are positively different from* 0, *may be written for any integer* $\lambda \geqq \sigma$ *and* $\leqq \tau$ *in the form*

$$F(X) = a(X - x_1) \dots (X - x_\lambda)(1 - x_{\lambda+1} X) \dots (1 - x_\mu X), \quad . \quad . \quad (4)$$

where a is positively different from 0.

If f_μ is positively different from 0, we may choose σ, τ and λ all equal to μ, so that then we get the decomposition

$$F(X) = f_\mu (X - x_1) \dots (X - x_\mu).$$

For a proof we remark that f_σ and f_τ differ positively from 0; hence a positive rational number $u \geqq \mu + 1$ exists, such that

$$|f_\sigma| \geqq \frac{3f}{u} \text{ and } |f_\tau| \geqq \frac{3f}{u};$$

here f denotes the sum of the absolute values of the coefficients of $F(X)$ and is therefore positively different from 0.

Consider a positively convergent series $r_1 + r_2 + \dots$, consisting of positive, rational, decreasing numbers $r_\nu \leqq 1$, such that

$$\tfrac{1}{4} u \, r_\nu^{\mu!} < 1.$$

To any natural number ν corresponds a polynomial

$$A_\nu(X) = a_{\nu 0} + \dots + a_{\nu\mu} X^\mu$$

with rational complex coefficients, such that

$$|a_{\nu\varrho} - f_\varrho| < \frac{f}{4} r_\nu^{\mu!} \qquad (\varrho = 0, \dots, \mu).$$

Hence

$$\sum_{\varrho=0}^{\mu} |a_{\nu\varrho}| < \sum_{\varrho=0}^{\mu} |f_\varrho| + \tfrac{1}{4}(\mu + 1) f r_\nu^{\mu!} < 2f.$$

Moreover we obtain

$$\sum_{\varrho=0}^{\lambda} |a_{\nu\varrho}| \geqq |a_{\nu\tau}| \geqq |f_\tau| - \tfrac{1}{4} f r_\nu^{\mu!} \geqq \frac{3f}{u} - \frac{f}{u} = \frac{2f}{u} > \frac{1}{u} \sum_{\varrho=0}^{\mu} |a_{\nu\varrho}|$$

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L. E. J. BROUWER, Intuitionistische Ergänzung des Fundamentalsatzes der Algebra, Proc. Kon. Akad. v. Wetensch., Amsterdam, **27**, 631—634 (1924). The same paper in Dutch: Intuitionistische aanvulling van de hoofdstelling der algebra, Versl. Kon. Akad. v. Wetensch., Amsterdam, **33**, 459—462 (1924).

and similarly

$$\sum_{=\lambda}^{\mu} | a_{\nu\varrho} | > \frac{1}{u} \sum_{\varrho=0}^{\mu} | a_{\nu\varrho} |.$$

Finally we get, from $r_{\nu+1} < r_\nu$,

$$| a_{\nu+1,\varrho} - a_{\nu\varrho} | < \tfrac{1}{2} f r_\nu^{\mu!} < r_\nu^{\mu!} \sum_{\varrho=0}^{\mu} | a_{\nu\varrho} |.$$

Hence it appears that the conditions of the lemma are satisfied with $A(X) = A_\nu(X)$, with $B(X) = A_{\nu+1}(X)$ and with $r = r_\nu$. Consequently we may write $A_1(X)$ in the form

$$A_1(X) = a_1' (X - x_{11}) \ldots (X - x_{1\lambda}) (1 - x_{1,\lambda+1} X) \ldots (1 - x_{1\mu} X).$$

To this form corresponds a decomposition of $A_2(X)$

$$A_2(X) = a_2' (X - x_{21}) \ldots (X - x_{2\lambda}) (1 - x_{2,\lambda+1} X) \ldots (1 - x_{2\mu} X),$$

such that

$$| x_{2\varrho} - x_{1\varrho} | < w r_1 \ , \ | a_2' - a_1' | < w r_1 \sum_{\varrho=0}^{\mu} | a_{1\varrho} | < 2 w r_1 f,$$

where w denotes a positive rational number depending only on μ and u. Continuing in this way we find for $A_\nu(X)$ ($\nu = 1, 2, \ldots$) the decomposition

$$A_\nu(X) = a'_\nu (X - x_{\nu 1}) \ldots (X - x_{\nu\lambda}) (1 - x_{\nu,\lambda+1} X) \ldots (1 - x_{\nu\mu} X), \quad (5)$$

such that

$$| x_{\nu+1,\varrho} - x_{\nu\varrho} | < w r_\nu \text{ and } | a'_{\nu+1} - a'_\nu | < 2 w r_\nu f.$$

Since the series $r_1 + r_2 + \ldots$ is positively convergent, the numbers $x_{\nu 1}, \ldots x_{\nu\mu}, a'_\nu$ tend positively to limits $x_1, \ldots, x_\mu, a$, so that (5) gives (4) by a passage to the limit.

Zoology. — *Notes on a specimen of Lernaeodiscus squamiferae Pérez (Crustacea Rhizocephala)*. By H. BOSCHMA.

(Communicated at the meeting of June 29, 1946.)

GERBE (1862) was the first to record a Rhizocephalan parasite of *Galathea squamifera* Leach, obtained from some locality on the Atlantic Ocean, and regarded by him as a species of *Sacculina* or *Peltogaster*. Later this parasite was mentioned again by PÉREZ (1922), who found it near Villefranche in the Mediterranean, and named it *Lernaeodiscus squamiferae* after its host. Hitherto the species never was sufficiently described, so that in a previous paper (BOSCHMA, 1928) I gave as my opinion that this parasite might belong to the same species as that of *Galathea intermedia* Lillj., viz., *Triangulus galatheae* (Norman & Scott). As the latter species is known to infest more than one species of hosts VAN BAAL (1937) also supposed that probably the parasite occurring on *Galathea squamifera* is not distinct from that of *Galathea intermedia*. Recently VEILLET (1945) again remarked that PÉREZ found the parasite of *Galathea squamifera* at Villefranche, but as he had no material of this animal at his disposal he could not give any particulars of its shape and internal characters.

In the collection of the Berlin Zoological Museum there is a specimen of a Rhizocephalan parasite of *Galathea squamifera* which is described in the present paper. It was collected in Norway by SCHILLING, apparently many years ago, it has the register number 1093.

The symmetrical arrangement of the internal organs proves that the parasite belongs to the genus *Lernaeodiscus*; in its characters it shows a close resemblance to *L. ingolfi* Boschma, as especially results from the elaborate description of this species by BRINKMANN (1936).

As but a single specimen is available, which, moreover, is not in an altogether excellent state of preservation, it is difficult to decide whether the peculiarities of the specimen constitute the specific characters. Provisionally for this parasite of *Galathea squamifera* the name *Lernaeodiscus squamiferae* may be used. More and better preserved material is needed to decide whether it really is specifically distinct from *L. ingolfi*.

The length (height) of the specimen (antero-posterior diameter) is 4 mm, the breadth (larger diameter) 5 mm, the thickness (dorso-ventral diameter) 2.5 mm. It is of oval shape, symmetrical, the surface of the mantle shows some grooves which divide the lateral parts of the body into broad lobes. The mantle opening is extremely wide, it closely surrounds a part of the visceral mass which protrudes through this opening (fig. 1).

The external cuticle of the mantle in some parts of the body is quite smooth (fig. 2*c*), in other parts its surface is more or less uneven (fig. 2*d*) or covered with minute excrescences of various shape (fig. 2e—*h*). It is interesting that the external cuticle of the part of the visceral mass which

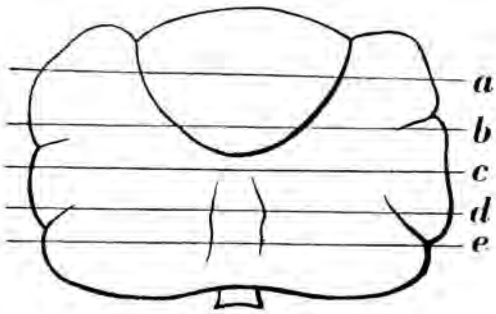

Fig. 1.
Lernaeodiscus squamiferae, dorsal surface. The lines correspond with the sections of fig. 4, indicated with the same letters. × 10.

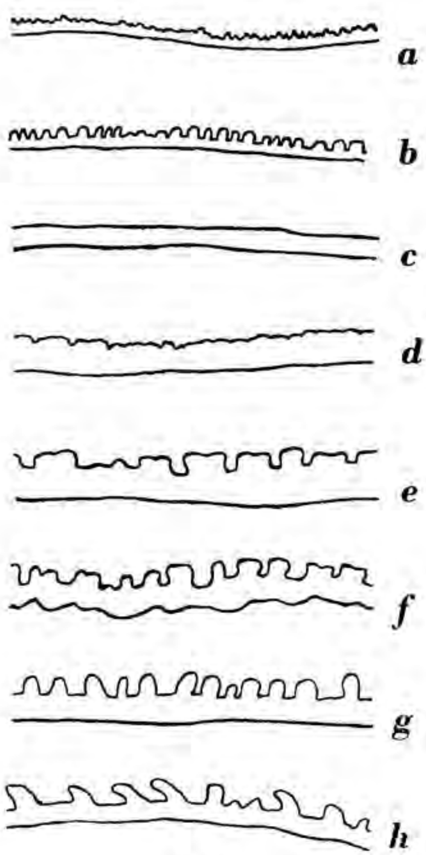

Fig. 2.
Lernaeodiscus squamiferae, sections of the external cuticle of the visceral mass (*a*, *b*) and of the mantle (*c*—*h*). × 530.

protrudes through the mantle opening also bears these excrescences (fig. 2a, b). The length of the excrescences of the external cuticle of the mantle does not exceed 7 μ.

The external cuticle of the stalk is much thicker than that of the mantle, here the chitin forms numerous rather long and slender excrescences which penetrate towards the interior of the stalk (fig. 3). Quite similar chitinous excrescences were described in *Lernaeodiscus ingolfi* by BRINKMANN (1936).

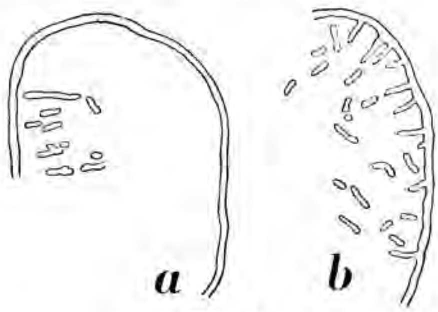

Fig. 3.
Lernaeodiscus squamiferae, transverse sections of the stalk, showing chitinous excrescences; *b* is from a more distal region than *a*. × 45.

The transverse sections represented in fig. 4 show the chief particulars of the internal anatomy of the specimen. The figures show that the state of preservation of the specimen is not altogether excellent. The external cuticle in the greater part of the body has lost its connection with the epithelium and the muscle layers, and only part of the visceral mass has remained undamaged. The male organs, however, are quite distinct and occupy a completely symmetrical position.

Fig. 4*a* represents a section through the anterior part of the body, showing the extremely wide mantle opening through which a large part of the visceral mass is protruding, thereby reducing the actual mantle opening tc a narrow slit. The mantle itself is rather thick and muscular. In the mantle cavity, which throughout the body is rather narrow, there are a quantity of eggs, proving that the specimen is adult. Moreover the mantle cavity contains some parasitic Isopods, indicated with × in the figures.

In the section of fig. 4*b* the mantle opening has already become much narrower, here again this opening is almost completely closed by the visceral mass.

Fig. 4*c* shows how the mantle is attached to the visceral mass by a narrow, strongly muscular ventral mesentery, and by an extremely broad dorsal mesentery.

The two mesenteries are still visible in the section of fig. 4*d*, which moreover shows the terminal parts of the vasa deferentia. The male openings are found at a slightly more anterior level than that of this section, close to the ventral mesentery.

The testes are found in the posterior part of the body to the right and the left of the plane of symmetry, their closed ends occupying the most lateral position. They are rather compact, short and thick (fig. 4e), whilst the vasa deferentia have a distinctly tortuous course, their last coils being close together at each side of the plane of symmetry. In this region of the body the ventral mesentery gradually becomes broader, so that posteriorly at last the mantle cavity is reduced to two small lateral clefts.

As a result of the insufficient state of preservation of the specimen no trace of the colleteric glands was found. This is not altogether astonishing, for in the Lernaeodiscidae these glands as a rule are very small and of simple structure.

It is by no means certain that *Lernaeodiscus squamiferae* is specifically distinct from *L. ingolfi*, the characters of which were described in detail by BRINKMANN (1936). The male organs of *L. squamiferae* have approximately the same structure as those of *L. ingolfi*. Both parasites possess the peculiar excrescences of the external cuticle in the region of the stalk; BRINKMANN did not find these excrescences in the other species of Lernaeodiscidae studied by him. Moreover the extremely wide mantle opening constitutes a character which hitherto was known only in *Lernaeodiscus ingolfi* (cf. BRINKMANN, 1936, pl. I figs. 1, 2, 3*a*); in the latter species also a part of the visceral mass may protrude through the mantle opening so as to close it almost entirely (l.c., pl. I fig. 1).

The occurrence of the parasites on different hosts gives an indication that they may be specifically distinct. *Lernaeodiscus squamiferae* lives on *Galathea squamifera* Leach, whilst *Lernaeodiscus ingolfi* is known to occur on *Munida tenuimana* Sars, *M. sarsi* Brinkmann, and *M. bamffia* (Pennant). Moreover there is a difference in size. The specimen which may be regarded as the type of *L. ingolfi* (BOSCHMA, 1928, fig. 8*g, h*) has a larger diameter of 15 mm, those represented by BRINKMANN (1936) on figs. 1 and 2 of his plate I have a larger diameter of 17 mm, whilst those of figs. 3 and 4 of the same plate according to the explanation of the plate are 6 times enlarged, so that their larger diameter would amount to 9 mm (as these specimens are adult it is possible that a slight mistake has occurred here). However this may be, the specimen of *L. squamiferae*, which is adult and has a wide mantle opening, has a larger diameter of 5 mm. Specimens of *L. ingolfi* according to BRINKMANN attain maturity at a size of 5 mm, but at this stage their mantle opening is still extremely small (cf. BRINKMANN, 1936, pl. I fig. 5).

Although the arguments given above may point to the probability of a specific distinction between the two parasites, this distinction as yet is by no means proven. More material is needed to obtain a definite result. On the other hand it is shown that the parasite of *Galathea squamifera* is entirely different from that of *G. intermedia*.

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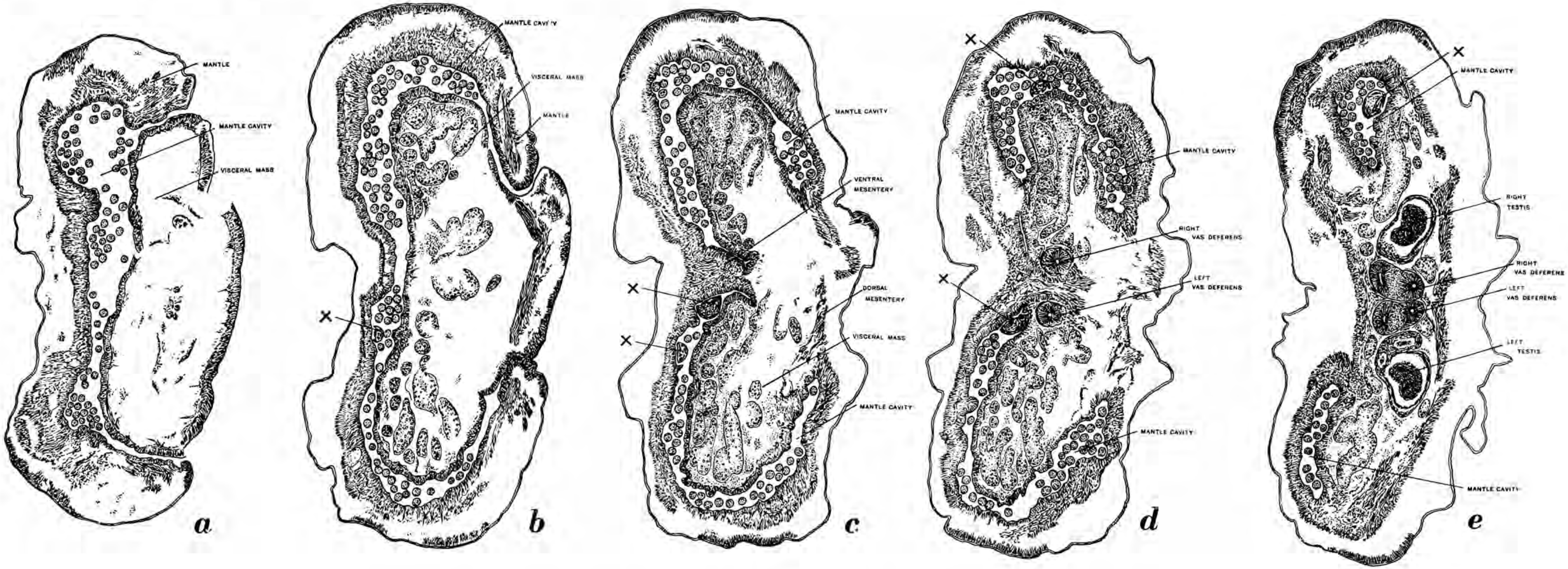

Fig. 4.
Lernaeodiscus squamiferae, transverse sections of the body, corresponding with the lines in fig. 1. ×, parasitic Isopods in the mantle cavity. × 25.

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Mathematics. — *De projectieve invarianten van vier G_d in G_{2d}.* By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of June 29, 1946.)

In het onderstaande wordt aangetoond, dat voor vier G_d, projectieve $(d-1)$-dimensionale ruimten, gelegen in één $(2d-1)$-dimensionale ruimte, symbolisch voorgesteld door a^d, α^d, p^d, π^d, het stelsel van $d+4$ invarianten bestaande uit

zes invarianten van het type $A_{12} = (a^d\, \alpha^d)$ **en**

$d-2$ **invarianten** $I_{1234}^{(d-\lambda)} = (a^d\, \alpha^{d-\lambda}\, p^\lambda)\,(p^{d-\lambda}\, \alpha^\lambda\, \pi^d)$, $\quad \lambda = 1, 2 \ldots d-2$,

een kleinste volledig invariantensysteem vormt.

I. *Elke invariant kan in den symbolischen vorm* $I = (a^d\, \alpha^\lambda\, p^{d-\lambda})\,(p^\lambda \ldots) \ldots$ *worden gebracht.*

Bewijs:

$I = (a^d \quad x\, y\, z \ldots t)\ (\ldots)$ bevat onder de $x, y, z \ldots t$ minstens $\left[\dfrac{d}{3}\right]$ aequivalente rijen. Men kan dus onderstellen:

$$I = (a^d\, \alpha_1\, \alpha_2 \ldots \alpha_\lambda\, x\, y \ldots t)\ (\ldots) \ \ldots \text{ met } x, y, z, \ldots t \not\equiv a, \alpha.$$

Door identisch vervormen kan men alle rijen α_1 in den eersten haakfactor brengen; hierbij treden hoogstens $d-\lambda$ rijen a uit, zoodat een som van invarianten van het type

$$I = (a^d\, \alpha^\mu\, x\, y\, z \ldots t)\,(\ldots) \qquad \mu \geqslant \lambda \geqslant \left[\frac{d}{3}\right],\ x, y, z, t \not\equiv a, \alpha$$

ontstaat, welke invarianten hier van de klasse μ genoemd worden.

Zijn niet alle $x, y, z, \ldots t$ aequivalent dan is dus

$$I = (a^d\, \alpha^\mu\, p_1^{\varrho_1} \ldots p_i^{\varrho_i}\ \ \pi_1^{\tau_1} \ldots \pi_\varkappa^{\tau_\varkappa})\,(\pi_1^{d-\tau_1} \ldots)\,(\ldots)$$

daar men alle rijen π, welke niet in den eersten haakfactor voorkomen, door identisch vervormen in den tweeden factor kan brengen. Brengt men nu *alle* rijen π_1 in den tweeden factor dan doen zich de volgende gevallen voor:

1. Een rij $\beta \equiv \alpha$ komt in den eersten factor, de term is identiek nul;
2. Een rij $b \equiv a$ komt in den eersten factor. Men werkt dan alle rijen b in den eersten factor en er ontstaan termen met

$$T = (b^d\, \alpha^\nu\, p \ldots \pi \ldots)\,(\ldots) \text{ met } \nu > \mu;$$

3. Er treden slechts rijen p, π uit en er ontstaan termen van den vorm

$$T = (a^d\, \alpha^\mu\, p_1^{\varrho_1}\, p_2^{\varrho_2} \ldots \pi_2^{\tau_2} \ldots \pi_\mu^{\tau_\mu})\,(\pi_1^d \ldots)\,(\pi_2^{d-\tau_2} \ldots) \ldots .$$

I wordt dus door identisch vervormen verkregen als som van invarianten van een klasse ν, $\nu > \mu$ èn invarianten waarbij de rij π_1 in een haakfactor $(\pi_1{}^d \ldots)$ afgesplitst is. Herhaalt men dit proces, dan kan men alle rijen π uit den eersten factor werken en terugkomen tot invarianten van een klasse ν, $\nu > \mu$ èn

$$I = (a^d\, a^\mu\, p_1^{\varrho_1}\, p_2^{\varrho_2} \ldots p_i^{\varrho_i})\, (\pi_1^d \ldots)\, (\pi_2^d \ldots) \ldots (\pi_\varkappa^d \ldots)\, (\ldots).$$

Is nu *niet* $\varrho_1 = d - \mu$ dan bestaan er twee mogelijkheden:

1. òf alle p_1 komen voor in de haakfactoren met $\pi_1, \pi_2, \ldots \pi_\varkappa$, waaruit volgt, dat I op den vorm

$$I = (a^d\, a^\mu\, p_1^{\varrho_1}\, p_2^{\varrho_2} \ldots p_i^{\varrho_i})\, (\pi_1^d\, p_1^{d-\varrho_1} \ldots)\, (\ldots)$$

te brengen is, waarin wegens $\varrho_1 < d - \mu$, $d - \varrho_1 > \mu$ wordt, zoodat I van een klasse $> \mu$ is; òf dit is voor één der p_m uit den eersten haakfactor zoo;

2. òf de rij p_1 komt nog in een anderen haakfactor voor, waaruit volgt

$$I = (a^d\, a^\mu\, p_2^{\varrho_2} \ldots p_i^{\varrho_i}\, xy \ldots)\, (\pi_1^d \ldots)\, (\pi_2^d \ldots) \ldots (\pi_k^d \ldots)\, (p_1^d \ldots)\, (\ldots) \quad x, y, \not\equiv a, \alpha.$$

Bevat hier de $x, y, \ldots$ een rij π dan isoleert men deze in een nieuwen haakfactor; rijen p die niet alle in de factoren met $\pi_1{}^d, \pi_2{}^d, \ldots \pi_k{}^d \ldots$ voorkomen brengt men samen in een nieuwen factor en dit proces herhaalt men totdat het geval onder 1. zich voordoet, waarbij reductie op invarianten met klasse $> \mu$ ontstaat òf alle rijen p in den eersten haakfactor gelijk geworden zijn.

Elke invariant kan dus vervormd worden tot een som van termen van de gedaante

$$I = (a^d\, \alpha^\lambda\, p^{d-\lambda})\, (p^\lambda\, xy \ldots zt)\, (\ldots).$$

II. *Elke invariant kan worden vervormd tot een som van λ-ketens.*

$$I = (p^{d-\lambda}\, a^d\, \alpha^\lambda)\, (\alpha^{d-\lambda}\, \pi^d\, q^\lambda)\, (\ldots).$$

Bewijs:

A. Komt in

$$I = (a^d\, \alpha^\lambda\, p^{d-\lambda})\, (p^\lambda\, xy \ldots zt)\, (\ldots)$$

onder de $x, y, \ldots z, t$ gèèn π voor, dan werkt men alle rijen p in den tweeden factor en er ontstaat reductie op invarianten van klasse $> \lambda$ bij uittreden van rijen $\equiv \alpha$, termen identiek gelijk nul voor rijen $\equiv a$. Treden slechts rijen $\equiv p$ uit, dan ontstaan termen

$$(a^d\, \alpha^\lambda\, q_1 \ldots q_{d-\lambda})\, (p^d \ldots)\, (\ldots)$$

en dus volgens het onder I aangegeven proces

$$(a^d\, \alpha^\lambda\, q^{d-\lambda})\, (p^d \ldots)\, (q^\lambda\, xy \ldots zt).$$

Iteratie put òf alle rijen $\equiv p$ uit en er ontstaat reductie op klasse $> \lambda$ òf er treden tenslotte rijen π onder de $x, y, \ldots z, t$ op, dus reductie op B:

B. Komt onder de $x, y, \ldots z, t$ een π voor, dan werkt men alle rijen π in den tweeden factor. Er ontstaan termen

$$T = (a^d \alpha^\lambda p^{d-\lambda})(p^\mu \pi^d \ldots)(p^{\lambda-\mu} \ldots)(\ldots).$$

Is hierbij $\mu < \lambda$ dan werkt men alle rijen p uit den tweeden in den derden factor. Door iteratie put men dan òf alle rijen π uit en verkrijgt reductie volgens A òf men komt uitsluitend tot termen met $\mu = \lambda$:

$$T = (a^d \alpha^\lambda p^{d-\lambda})(p^\lambda \pi^d x y \ldots z t)(\ldots),$$

waarmede λ-ketens ontstaan, tenzij reductie op klasse $> \lambda$ optreedt.

III. *Ketens met meer dan twee haakfactoren zijn reduceerbaar.*

Bewijs:

A. Men kan in een keten steeds een volgorde van het type 12 — 34 — 12 — 34 ... aanbrengen.

Stel, de keten begint met:

$$(a^\lambda \alpha^d p^{d-\lambda})(p^\lambda \pi^d \beta^{d-\lambda})(\beta^\lambda b^d \varrho^{d-\lambda}) \ldots \quad \varrho \equiv p, \pi$$

dan levert het inwerken van alle β in den derden factor bij uittreden van ϱ òf $\equiv 0$ òf reductie op invarianten van klasse $> \lambda$, òf uitsluitend b treedt uit, waarmede

$$(a^\lambda \alpha^d p^{d-\lambda})(p^\lambda \pi^d b^{d-\lambda})(b^\lambda \beta^d \varrho^{d-\lambda}) \ldots$$

verkregen is.

B. Bevat de keten een *oneven* aantal haakfactoren, dan ontstaat

$$(a^\lambda \alpha^d p^{d-\lambda})(p^\lambda \ldots) \ldots (b^\lambda \beta^d a^{d-\lambda})$$

en dus bij inwerken van de a uit den eersten factor in den laatsten afsplitsing van $(a^d \beta^d)$.

C. Bevat de keten een *even* aantal haakfactoren, dan ontstaat

$$(a^\lambda \alpha^d p^{d-\lambda})(p^\lambda \pi^d b^{d-\lambda}) \ldots (p_1^\lambda \pi_1^d a^{d-\lambda}).$$

Werkt men nu alle a uit den laatsten in den tweeden factor, dan ontstaat daar π niet kan uittreden zonder termen $\equiv 0$ te leveren, en p niet kan uittreden zonder reductie op een klasse $> \lambda$ te leveren, afsplitsing van

$$(a^\lambda \alpha^d p^{d-\lambda})(p^\lambda \pi^d a^{d-\lambda})$$

òf reductie op klasse $> \lambda$.

Een volledig invariantensysteem bevat dus slechts invarianten van het type

$$A_{ik} = (i^d k^d)$$

$$I_{klmn}^{(d-\lambda)} = (k^d l^{d-\lambda} m^\lambda)(m^{d-\lambda} l^\lambda n^d)$$

waarbij (k, l, m, n) overeenkomt met het stelsel (a, α, p, π).

Evident is nu:

$$1.\quad I^{(d-\lambda)}_{klmn} = I^{(d-\lambda)}_{nmlk},$$

$$2.\quad I^{(d-\lambda)}_{klmn} = I^{(\lambda)}_{kmln},$$

$$3.\quad I^{(d-\lambda)}_{klmn} = \sum_{i=0}^{\lambda} c_i\, I^{(d-\lambda+i)}_{klnm} \qquad c_i = \text{constante},$$

zooals blijkt door de λ rijen n uit den eersten haakfactor van den invariant in het linkerlid, in den tweeden haakfactor te brengen.

Hieruit volgt dat elke $I^{(d-\lambda)}_{klmn}$ een lineaire combinatie is van

$$I^{(d-\lambda)}_{1234} = (a^d\, \alpha^{d-\lambda}\, p^{\lambda})\,(p^{d-\lambda}\, \alpha^{\lambda}\, \pi^d) \quad \lambda = 1, 2, 3, \ldots d-1,$$

en van de producten $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$, waarbij de indices 1, 2, 3, 4 correspondeeren respectievelijk met a^d, α^d, p^d, π^d.

IV. *De invarianten* A_{ik}, $I^{(d-\lambda)}_{1234}$ $\lambda = 1, 2, 3, \ldots \underline{d-2}$, *vormen een kleinste volledig invariantensysteem.*

Bewijs:

A. Stel $H^{(d-\lambda)}_k = (a^d\, \alpha^{d-\lambda}\, p^{\lambda-k}\, \pi^k)\,(p^{d-\lambda+k}\, \alpha^{\lambda}\, \pi^{d-k})$ en vervorm identiek:

$$(d-\lambda+k+1)\, H^{(d-\lambda)}_k = (-1)^d \left\{ (d-k)\, H^{(d-\lambda)}_{k+1} - \lambda\, H^{d-\lambda+1}_k \right\},$$

waaruit door herhaling wordt verkregen

$$\frac{(d-\lambda+k+p)!}{(d-\lambda+k)!}(-1)^{pd}\, H^{(d-\lambda)}_k = \sum_{\mu=0}^{p} (-1)^{\mu d} \binom{p}{\mu}\binom{\lambda}{\mu} \frac{\mu!\,(d-k)!}{(d-k-p+\mu)!}\, H^{(d-\lambda+\mu)}_{k+p-\mu}.$$

Specialiseert men met $k = 0$ $p = \lambda$ dan ontstaat wegens

$$H^{(d-\lambda)}_0 = I^{(d-\lambda)}_{1234},\quad H^{(d-\lambda)}_{\lambda} = I^{(d-\lambda)}_{1243}$$

$$(-1)^{(d-\lambda)}\, I^{(d-\lambda)}_{1234} = \sum_{\mu=0}^{\lambda} (-1)^{\mu d} \frac{\binom{\lambda}{\mu}^2}{\binom{d-\lambda+\mu}{\mu}}\, I^{(d-\lambda+\mu)}_{1243}.$$

In het bijzonder voor

$$\lambda = 1 \quad (-1)^{d-1}\, I^{(d-1)}_{1234} = I^{(d-1)}_{1243} + \frac{(-1)^d}{d}\, I^{(d)}_{1243},$$

$$\lambda = (d-1) \quad -I^{(1)}_{1234} = \sum_{\mu=0}^{d-1} (-1)^{\mu d} \frac{\binom{d-1}{\mu}^2}{\mu+1}\, I^{(\mu+1)}_{1243}.$$

Verwisselt men hierin 4 met 2 dan ontstaat:

$$-I^{(1)}_{1432} = -I^{(d-1)}_{1342} = (-1)^d I^{(d-1)}_{1324} + \frac{1}{d} I^{(d)}_{1324} =$$

$$= \sum_{\mu=0}^{d-3} (-1)^{\mu d} \frac{\binom{d-1}{\mu}^2}{\mu+1} I^{(d-\mu-1)}_{1243} + \frac{(-1)}{d} I^{(0)}_{1243} + (-1)^d (d-1) I^{(1)}_{1243}$$

Zoodat tenslotte:

$$(-1)^d d I^{(1)}_{1234} = -\frac{1}{d} I^{(d)}_{1324} + \frac{1}{d} I^{(0)}_{1243} + \sum_{\mu=0}^{d-3} (-1)^{\mu d} \frac{\binom{d-1}{\mu}^2}{\mu+1} I^{(d-\mu-1)}_{1243} +$$

$$-(-1)^d (d-1) \sum_{\varrho=1}^{d-2} (-1)^{\varrho d} \frac{\binom{d-1}{\varrho}^2}{\varrho+1} I^{(\varrho+1)}_{1234}$$

waarmede $I^{(1)}_{1234}$ uit te drukken is in A_{ik} en $I^{(\lambda)}_{1234}$ met $\lambda > 1$.

B. Kiest men een punt P in p^d, verbindt men dit met $\alpha^d = (\xi_1 \dots \xi_d)$ en snijdt men deze G_{d+1} met π^d dan is het snijpunt

$$(P \alpha^d \pi^{d-1}) (\pi u') = 0 \text{ of } (S_4 u') \equiv A_{24} (P u') + \Sigma \pm (P \pi^d \xi_1 \dots \xi_d)_k (\xi_k u') = 0$$

als $(P \pi^d \xi_1 \dots \xi_d)_k$ van alle ξ_i slechts ξ_k niet bevat;

Snijden van dezelfde G_{d+1} met a^d levert een snijpunt

$$(S_1 u') = A_{21} (P u') + \Sigma \pm (P a^d \xi_1 \dots \xi_d)_k (\varrho_k u') = 0.$$

De punten $P = S_3$, S_1 en S_4 liggen op één rechte als

$$(P \pi^d \xi_1 \dots \xi_d)_k = \lambda (P a^d \xi_1 \dots \xi_d)_k \quad k = 1, 2, \dots d.$$

In dit geval stellen namelijk

$$\Sigma \pm (P \pi^d \xi_1 \dots \xi_d)_k (\xi_k u') = 0 \text{ en } \Sigma \pm (P a^d \xi_1 \dots \xi_d)_k (\xi_k u') = 0$$

hetzelfde punt S_2 uit α^d voor.

De dubbelverhouding van de vier snijpunten op een transversaal der vier G_d is gegeven door

$$D = DV (S_3 S_2 S_1 S_4) = \frac{\lambda A_{21}}{A_{24}},$$

waarbij λ voldoet aan

$$\det |(P_i \pi^d \xi_1 \dots \xi_d)_k - \lambda (P_i a^d \xi_1 \dots \xi_d)_k| = 0$$

als P_i $i = 1, 2, \dots d$ een stel vrij gelegen punten uit p^d is.

Uitwerking van dezen determinant levert:

$$\sum_{k=0}^{d} c_k (a^d p^{d-k} \alpha^k) (\alpha^{d-k} p^k \pi^d) A_{12}^{d-k-1} A_{42}^{k-1} (-\lambda)^{d-k} = 0, \; c_k = \text{constant},$$

waaruit volgt voor de dubbelverhoudingen op de transversalen

$$D^d + \sum_{k=1}^{d} c'_k \frac{I^{(k)}_{1234}}{A_{13} A_{24}} D^{d-k} = 0.$$

De absolute invarianten

$$\frac{I^{(k)}_{1234}}{A_{13} A_{24}} \quad k = 1, 2, \ldots d$$

zijn dus onafhankelijke grootheden. Op grond van de reduceerbaarheid van $I^{(1)}_{1234}$ op A_{ik} en $I^{(k)}_{1234}$ $k \neq 1$ volgt, dat de absolute invarianten

$$\frac{A_{12} A_{34}}{A_{13} A_{24}}, \quad \frac{A_{14} A_{23}}{A_{13} A_{24}}, \quad \frac{I^{(k)}_{1234}}{A_{13} A_{24}} \; k = 2, 3, \ldots d-1$$

onderling onafhankelijk zijn voor $d \geqq 2$, waaruit het gestelde volgt.

Voor $d = 2$ vindt men de reduceerbaarheid van de vier-ketens voor rechten uit G_4 terug.

Voor $d = 1$ wordt wegens $(ac)\ (bd) \equiv (bc)\ (ad) + (ab)\ (cd)$ slechts één absolute invariant gevonden.

Mathematics. — *Sur les mouvements presque périodiques.* By F. LOONSTRA.
(Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of June 29, 1946.)

§ 1. *Introduction.*

Dans la partie de la mécanique ordinairement comprise sous le nom théorie des perturbations, on rencontre souvent des fonctions presque périodiques dont l'étude a été inaugurée par H. BOHR [1]).

Ce sont les séries du type

$$\Sigma a_{n_1, n_2, \ldots, n_k} \cos (n_1 \theta_1 + n_2 \theta_2 + \ldots + n_k \theta_k)$$

où les $n_1, n_2, \ldots, n_k$ sont entiers, $\theta_r = \lambda_r t + \varepsilon_r$, et où les coefficients a, λ et ε sont constants, qui ont été reconnues depuis longtemps comme les séries les plus convenables à developper des coordonnées dans le domaine de l'astronomie. Déjà DELAUNAY [2]) avait indiqué l'importance de ces séries pour les coordonnées de la lune; NEWCOMB [3]) les appliquait aux coordonnées des planètes, pendant que plusieurs autres auteurs comme LINDSTEDT, TISSERAND et POINCARÉ découvraient des applications semblables pour le problème des trois corps. En physique on rencontre des problèmes pareils depuis l'établissement de la théorie atomique de N. BOHR: l'influence d'un champ électrique extérieur et celle des forces d'inertie relativistes sur les courbes de KEPLER d'un atome de l'hydrogène est calculable par la méthode des perturbations séculaires. Or, en général, les fonctions presque périodiques qui se présentent en astronomie comme solutions d'équations différentielles de la théorie des perturbations, sont d'un caractère spécial.

Dans le présent travail nous nous occupons d'abord des mouvements presque périodiques en général; ensuite nous traiterons en particulier la question suivante: Quels mouvements presque périodiques sont (spécialement pour le cas de deux dimensions) réalisables physiquement?

§ 2. H. BOHR a étudié sous le nom de fonction presque périodique une fonction $f(t)$ univoque et continue d'une variable réelle jouissant des propriétés suivantes:

A tout nombre ε positif arbitrairement petit, on peut faire correspondre un nombre $L = L(\varepsilon)$ dépendant de ε et de $f(t)$ tel que tout intervalle de longueur L contienne au moins une presque-période τ, pour laquelle on a, quel que soit t,

$$|f(t+\tau) - f(t)| < \varepsilon.$$

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[1]) H. BOHR, Zur Theorie der fastperiodischen Funktionen. Acta math. **45** (1925), **46** (1925), **47** (1926).

Fastperiodische Funktionen. Ergebnisse der Math. und ihrer Grenzgebiete (Springer, 1932).

[2]) DELAUNAY: Théorie du mouvement de la lune, Paris, 1860.

[3]) Newcomb.: Smithsonian Contributions, 1874.

On appelle l'ensemble des nombres τ un ensemble de nombres d'inclusion Considérons un nombre fini de fonctions presque périodiques

$$x_1 = f_1(t), x_2 = f_2(t), \ldots, x_n = f_n(t);$$

alors on a:

2.1 A tout nombre ε positif arbitrairement petit, on peut faire correspondre un nombre Δ dépendant de ε et des f_i tel que tout intervalle de longueur Δ contienne une presque-période τ, pour laquelle on a, quel que soit t,

$$|f_i(t+\tau) - f_i(t)| < \varepsilon \qquad (i = 1, 2, \ldots, n).$$

La démonstration est une extension immédiate de celle de BOHR pour le cas $n = 2$.

Définition. Un mouvement est dit presque périodique s'il est représenté par n fonctions presque périodiques

$$x_1 = f_1(t), x_2 = f_2(t), \ldots, x_n = f_n(t),$$

qui admettent en outre les dérivées successives.

Il est d'importance que toutes les n coordonnées $x_1, x_2, \ldots, x_n$ reprennent à peu près leurs valeurs originales au bout d'un espace de temps τ. Par rapport aux cas particuliers nous mentionnons que le mouvement total $x_i = f_i(t)$ $(i = 1, 2, \ldots, n)$, dans lequel les $f_i(t)$ sont des fonctions périodiques continues avec la période p_i admettant des dérivées successives représente un mouvement périodique au seul cas que le rapport d'une paire de périodes quelconques est rationnel; cependant il est possible que les p_i se divisent en groupes maximaux de rapports rationnels; alors on parle de mouvements conditionnellement périodiques [4]). Si le rapport de tout couple de périodes est irrationnel, le mouvement est presque périodique au sens absolu.

§ 3. Pour considérer les mouvements presque périodiques possibles, nous nous occupons du théorème suivant:

3.1 Si $x = x(t)$ est une fonction presque périodique, chaque fonction uniformément continue $F(x)$ de la variable x est également une fonction presque périodique.

Démonstration: Si $x = x(t)$ est presque périodique et si $F(x)$ est uniformément continue, on sait que pour tout $\varepsilon > 0$ il existe un $\eta > 0$ de sorte que

$$|F(x_1) - F(x_2)| \leq \varepsilon \quad \text{si} \quad |x_1(t) - x_2(t)| \leq \eta$$

et puis que d'après la définition il existe pour tout $\eta > 0$ un l_η de sorte que chaque intervalle $a < t < a + l_\eta$ de longueur l_η contient un nombre d'in-

[4]) Par exemple: CHARLIER, Die Mechanik des Himmels, Bd. I.

clusion τ de $x(t)$; alors τ est un nombre d'inclusion de $F(x)$ correspondant à ε. Si $x(t)$ est en particulier une fonction périodique de période p, $F(x)$ est également une fonction périodique de la même période.

Considérons la suite finie $x_1 = x_1(t)$, $x_2 = x_2(t)$, ..., $x_n = x_n(t)$ de fonctions presque périodiques; la continuité d'une fonction $F(x_1, x_2, \ldots, x_n)$ par rapport aux $x_i (i = 1, 2, \ldots, n)$ est dite uniforme, si à chaque $\varepsilon > 0$ on peut faire correspondre un $\delta > 0$ de manière que

$$| F(x_1', x_2', \ldots, x_n') - F(x_1'', \ldots, x_n'') | \leqq \varepsilon \text{ pour } | x_i' - x_i'' | \leqq \delta \quad (i = 1, 2, \ldots, n).$$

3.2 Toute fonction $F(x_1, x_2, \ldots, x_n)$ uniformément continue par rapport aux fonctions presque périodiques $x_i = x_i(t)$ est aussi une fonction presque périodique.

En effet, nous avons vu dans ce qui précède qu'à tout nombre $\varepsilon > 0$ on peut faire correspondre un nombre $\delta > 0$ de manière que

$$| F(x_1', x_2', \ldots, x_n') - F(x_1'', x_2'', \ldots, x_n'') | \leqq \varepsilon \text{ pour } | x_i' - x_i'' | \leqq \delta \quad (i = 1, 2, \ldots, n),$$

tandis que d'après 2.1 on peut faire correspondre à tout $\delta > 0$ un ensemble de nombres d'inclusion de $x_1, x_2, \ldots, x_n$. Par conséquent $F(x_1, x_2, \ldots, x_n)$ est aussi unefonction presque périodique de t. D'après **3.2** on obtient un théorème important pour ce qui suit:

3.3 Supposons $x_1 = x_1(t)$, $x_2 = x_2(t)$, ..., $x_n - x_n(t)$ les équations d'un mouvement presque périodique et

$$\begin{aligned} y_1 &= F_1(x_1, x_2, \ldots, x_n), \\ y_2 &= F_2(x_1, x_2, \ldots, x_n), \\ &\ldots \\ y_n &= F_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

des fonctions uniformément continues par rapport à $x_i (i = 1, 2, \ldots, n)$; à condition qu'elles admettent des dérivées, $y_1, y_2, \ldots, y_n$ sont aussi des fonctions presque périodiques définissant un mouvement presque périodique. D'après ce qui précède ce mouvement peut être dans un cas particulier un mouvement périodique ou un mouvement conditionnellement périodique.

§ 4. Dans ce qui suit nous étudierons les mouvements presque périodiques en les soumettant à une transformation uniformément continue; en particulier nous étudierons la structure topologique des trajectoires transformées. D'abord nous citerons quelques définitions et théorèmes topologiques [5]).

4.1 Une transformation univoque de M en N sera dite uniformément continue, si, pour chaque $\varepsilon > 0$ on peut faire correspondre un nombre $\delta > 0$

[5]) A consulter par exemple: V. KERÉKJÁRTÓ, Vorlesungen über Topologie (Springer, 1923); ALEXANDROFF–HOPF, Topologie (Springer, 1935).

de sorte que la distance $P'Q'$ d'une paire quelconque de points de N est plus petite que ε, si $PQ < \delta$.

4.2 Toute transformation univoque et continue d'un intervalle borné et fermé est uniformément continue.

4.3 Un ensemble K sera dit courbe continue, si K est une transformation univoque et continue de l'intervalle $0 \leqq x \leqq 1$.

4.4 La condition nécessaire et suffisante pour qu'un ensemble K soit une courbe continue est que K soit un continu localement connexe (HAHN-MAZURKIEWICZ).

4.5 Une transformation qui est à la fois univoque et continue de M en N et de N en M, autrement dite biunivoque et bicontinue, est appelée transformation topologique.

Comme une fonction presque périodique est bornée, le mouvement presque périodique, représenté par

$$x_1 = x_1(t),\ x_2 = x_2(t),\ \dots,\ x_n = x_n(t)$$

aura lieu dans le fini: l'ensemble des points des trajectoires forme un ensemble borné; alors, nous considérons les transformations univoques et continues de $x_i (i = 1, 2, \dots, n)$, parce que ces transformations sont aussi uniformément continues. Pour conserver le sens physique, nous supposons toujours que les transformations des trajectoires admettent des dérivées successives.

§ 5. *A.* Sur les mouvements, dont la trajectoire est un segment fermé.

5.1 Il y a des mouvements périodiques dont la trajectoire du point mouvant est un segment fermé, par exemple le mouvement, désigné par $x = \sin t$ avec la trajectoire $-1 \leqq x \leqq +1$.

5.2 Il y a des mouvements presque périodiques dont la trajectoire est un segment fermé.

Démonstration: Le mouvement

$$x = \sin \{2\pi \,.\, (\sin \pi t + \sin \pi t \sqrt{2})\}$$

est, comme nous verrons, un mouvement presque périodique et la trajectoire, c'est le segment fermé $[-1;\ +1]$. Supposons $\xi = \sin \pi t + \sin \pi t \sqrt{2}$, nous démontrerons que les valeurs de ξ restent comprises entre -2 et $+2$, les extrémités exclues. Nous supposons comme étant connu, que ξ soit une fonction presque périodique; $\sin \xi$ est une fonction uniformément continue par rapport à ξ, alors x est également une fonction presque périodique admettant des dérivées quelconques et qui présente un mouvement presque périodique. Il est clair que ξ n'admettra jamais la valeur $+2$ (ou -2): $\sin \pi t = 1$ pour $t = 4n + 1$, $\sin \pi t \sqrt{2} = 1$ pour $t = \dfrac{4m+1}{\sqrt{2}}$, où m et n

sont entiers, tandis que jamais $4n + 1 = \frac{4m+1}{\sqrt{2}}$. Néamoins, à tout nombre ε positif arbitrairement on peut faire correspondre des nombres m et n, tels que

$$\left|(4n+1) - \frac{4m+1}{\sqrt{2}}\right| < \frac{\varepsilon}{\sqrt{2}} \text{ ou } \left|(n\sqrt{2} - m) + \frac{\sqrt{2}-1}{4}\right| < \frac{\varepsilon}{4}; \text{(a)}$$

puisque l'ensemble $n\sqrt{2} - m$ est dense en C, où C est le continu linéaire, et puisque l'ensemble $n\sqrt{2} - m + \frac{\sqrt{2}-1}{4}$ est aussi dense en C, il y a une infinité de paires de nombres entiers m et n satisfaisant la condition (a), (qui forment en outre un ensemble relativement dense).

Il s'ensuit qu'il y a des nombres n, de sorte que sin $\pi t\sqrt{2}$ ne diffère que d'un nombre infiniment petit de 1, alors ξ ne diffère que d'un nombre infiniment petit de 2. Une argumentation semblable nous apprend que ξ admet aussi des valeurs qui ne diffèrent que d'un peu de -2. En vertu de la continuité, ξ admettra toutes les valeurs entre -2 et $+2$, par conséquent l'intervalle $[-2; +2]$ sera rempli à peu près (les extrémités exclues) par ξ pour $-\infty < t < +\infty$.

Puisque $x = \sin 2\pi\xi$ et que $2\pi\xi$ admet l'intervalle $(-4\pi; +4\pi)$ (les extrémités exclues), on a $-1 \leqq x \leqq +1$. Par conséquent il existe des mouvements périodiques et presque périodiques dont la trajectoire est un segment fermé.

En nous servant du théorème qu'une transformation univoque et continue d'un intervalle borné et fermé est un continu localement connexe, il s'ensuit:

5.3 Un continu borné et localement connexe, étant une transformation univoque, continue et différentielle d'un intervalle borné et fermé, peut être une trajectoire possible d'un mouvement presque périodique ou périodique.

5.4 Un arc simple, étant même une transformation topologique d'un segment borné et fermé, peut être une trajectoire possible d'un mouvement presque périodique.

Parmi les trajectoires possibles se trouvent des courbes comme une ellipse, un cercle, des courbes „lemniscatiens", etc. En physique on peut réaliser ces formes de mouvements; supposons par exemple

$$x = a \,.\, \sin 2\pi \{\sin \pi t + \sin \pi t\sqrt{2}\} = a \,.\, \sin 2\pi\xi,$$
$$y = b \,.\, \cos 2\pi \{\sin \pi t + \sin \pi t\sqrt{2}\} = b \,.\, \cos 2\pi\xi;$$

x et y sont des fonctions presque périodiques et différentielles; par conséquent le tout est un mouvement presque périodique. En outre ξ admet toutes les valeurs entre -2 et $+2$, alors toute l'ellipse représente la trajectoire du mouvement. Si $a = b$, on a un exemple d'un mouvement presque périodique circulaire. Il s'ensuit:

5.5 Chaque courbe simple continue, ou courbe de JORDAN, étant une transformation topologique d'une circonférence d'un cercle, peut être une trajectoire possible d'un mouvement presque périodique.

Les équations

$$x = a \,.\, \sin \pi \,\{\sin \pi t + \sin \pi t \sqrt{2}\} = a \,.\, \sin \pi \xi$$
$$y = b \,.\, \sin 2\pi \,\{\sin \pi t + \sin \pi t \sqrt{2}\} = b \,.\, \sin 2\pi \xi$$

représentent les équations d'un mouvement presque périodique dont la trajectoire admet un point double. On peut facilement construire d'autres exemples.

B. Sur les mouvements, dont la trajectoire est un segment ouvert.

Le mouvement $x = \sin t + \sin t \sqrt{2}$ est presque périodique et la trajectoire est le segment ouvert $-1 < x < +1$. Considérant que la trajectoire est dense sur le segment fermé et considérant qu'on peut étendre chaque transformation uniformément continue d'un segment ouvert à une transformation continue du segment fermé, qui est aussi uniformément continue, on a:

5.6 Si un mouvement presque périodique, dont la trajectoire K est un segment ouvert, sera transformé par une transformation uniformément continue et différentielle, la trajectoire transformée K' est un ensemble borné et K' est, à 0,1 ou 2 points près, une transformation continu du segment fermé. Par exemple:

$$x = \sin \{\tfrac{1}{2}\pi (\sin t + \sin t \sqrt{2})\},$$
$$y = \cos \{\tfrac{1}{2}\pi (\sin t + \sin t \sqrt{2})\}.$$

La trajectoire de ce mouvement presque periodique est à un point près toute la circonférence d'un cercle.

C. Sur les mouvements, dont la trajectoire est un ensemble partout dense sur un ensemble borné d'un plan euclidien.

Considérons d'abord le mouvement presque périodique, défini par

$$x = \sin 2\pi t, \quad y = \sin 2\pi\lambda t,$$

où λ est un nombre irrationnel. Nous allons démontrer que cette trajectoire est un ensemble partout dense sur le rectangle $-1 \leq x \leq +1$; $-1 \leq y \leq +1$. Soit $(x_0; y_0)$ un point situé dans le rectangle, nous supposons $x = x_0$ pour $t = t_0 + n$ et $y = y_0$ pour $t = t_1 + \dfrac{\lambda}{m}$. Quand il y a des nombres entiers m et n, pour lesquels $t_0 + n = t_1 + \dfrac{\lambda}{m}$, les coordonnées $(x_0; y_0)$ seront atteintes encore une fois. Sinon, on démontre la propriété suivante:

A tout nombre ε positif arbitrairement petit, on peut faire correspondre un nombre $L = L(\varepsilon)$ tel que tout intervalle $a < t < a + L$ de longueur L

contient une presque période τ, pour laquelle le point mobile reprend ses coordonnées initiales $(x_0; y_0)$ à peu près, c'est à dire:

$$|x - x_0| < \frac{\varepsilon}{\sqrt{2}}, \quad |y - y_0| < \frac{\varepsilon}{\sqrt{2}}.$$

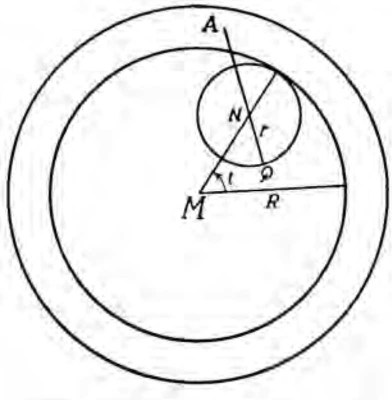
Evidemment il y a des nombres entiers m, n formant un ensemble relatif dense avec la condition

$$\left| (t_0 + n) - \left(t_1 + \frac{m}{\lambda} \right) \right| < \eta$$

où $\eta > 0$ est un nombre arbitrairement petit. En vertu de la continuité de la fonction sinusoïdale les coordonnées x et y ne diffèrent également que d'une grandeur arbitrairement petite de x_0 et y_0. Il existe des mouvements presque périodiques ayant la propriété que la trajectoire est dense sur l'interne d'un cercle. Pour cela on étudie des équations

$$x = \cos t - \cos (\lambda - 1)t \text{ et } y = \sin t + \sin (\lambda - 1)t,$$

où λ est irrationnel; ce mouvement se produit en faisant rouler un cercle $(N; r)$ d'un centre N et d'un rayon r le long de l'intérieur de la circonférence



d'un cercle fixé $(M; R)$, pendant que le prolongement d'un diamètre du petit cercle contient un point A de sorte que $QA = R$. Supposant $R/r = \lambda$ et $R - r = 1$, il n'est pas difficile à vérifier les équations cités plus haut. On a

$$d^2 = x^2 + y^2 = 2(1 - \cos \lambda t),$$

et par conséquent la distance jusqu'au centre M est au maximum 4 et au minimum 0, pendant que le cercle $(M; 4)$ est rempli partout dense par la trajectoire de A. On s'assure que le point A n'atteint jamais deux fois le même point du cercle extérieur; en outre A passe régulièrement par M.

En demandant la transformation uniformément continue et différentielle de la trajectoire K d'un mouvement presque périodique où K est un ensemble partout dense sur un ensemble borné E, nous savons que la courbe transformée K' est bornée, parce qu'il est possible d'étendre la transformation uniformément continue de la trajectoire à une transformation continue de l'ensemble borné fermé E. La trajectoire transformée K' est alors bornée et elle est partout dense sur l'ensemble borné et fermé E'. Pour quelques cas particuliers nous nous prononcerons plus nettement:

5.7 La trajectoire b d'un mouvement presque périodique est partout dense sur l'intérieur d'un cercle C; une transformation topologique transforme C en une courbe fermée simple K, l'intérieur de C en celui de K, pendant que la transformation de b en k (remplissant l'intérieur de K partout dense) est en outre différentielle; alors K est la trajectoire d'un mouvement presque périodique et k est partout dense sur le domaine intérieur de K.

D. Sur les mouvements dont la trajectoire est partout dense sur le domaine entre deux courbes de JORDAN.

A l'aide d'un exemple nous établirons une seconde possibilité de remplir partout dense: un cercle d'un rayon r roule le long de l'extérieur d'un cercle fixé d'un rayon R, de sorte que le rapport des rayons est irrationnel; alors un point de la circonférence du cercle roulant décrit une trajectoire, qui remplit le domaine entre deux cercles concentriques partout denses.

Les équations de mouvement sont:

$$x = a \cdot \cos t - \cos a \cdot t; \; y = a \cdot \sin t - \sin a \cdot t$$

où a est un nombre irrationnel. Il s'ensuit aussitôt

$$x^2 + y^2 = a^2 + 1 - 2a \cdot \cos (a - 1)t$$

de sorte que

$$a-1 \leqq \sqrt{x^2+y^2} \leqq a+1;$$

le rayon du cercle fixé $R = a - 1$, pendant que $a + 1 = R + 2r$.

Il est possible de transformer les cercles concentriques par une transformation topologique en deux courbes fermées simples qui n'ont pas un point d'intersection, de sorte que les domaines intérieurs se transforment l'un en l'autre.

Si la trajectoire entre les cercles est transformée en outre différentiellement en une courbe qui remplit le domaine entre les courbes fermées simples partout denses, alors la courbe qu'on vient de nommer est une trajectoire possible d'un mouvement presque périodique. Comme exemple de cette espèce il y a: le mouvement de Mercure; le mouvement relatif d'un pendule de FOUCAULT poussé centralement, le mouvement relatif à l'écliptique d'un pendule de FOUCAULT lâché sans impulsion, le mouvement relatif à la terre d'un pendule de FOUCAULT lâché sans impulsion.

Mathematics. — *A theorem on uniform convergence.* By J. KOREVAAR.
(Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of June 29, 1946.)

§ 1. From his well-known necessary and sufficient condition [1]) which must be satisfied, in order that it be possible to approximate to a certain continuous function $f(x)$, uniformly on $0 \leq x \leq 1$, by (linear) polynomials in continuous functions belonging to a given set $\{f_n(x)\}$ $(n = 1, 2, 3, \ldots)$, F. RIESZ has deduced the following

Theorem R. *Let the sequence*

$$\{f_n(x)\} \qquad (n = 1, 2, 3, \ldots)$$

have the following properties:

(i) *for each n, $f_n(x)$ is continuous on $0 \leq x \leq 1$;*

(ii) *for each x of $0 \leq x \leq 1$, $f_n(x) \to f(x)$ as $n \to \infty$;*

(iii) *$f(x)$ is continuous on $0 \leq x \leq 1$;*

(iv) *there is a constant F such that for all x of $0 \leq x \leq 1$ and all n, $|f_n(x)| < F$.*

Then there exists a sequence of (linear) polynomials

$$\varphi_n(x) = \sum_{k=1}^{n} a_{nk} f_k(x) \qquad (n = 1, 2, 3, \ldots)$$

in functions of $\{f_n(x)\}$, converging to $f(x)$ uniformly on $0 \leq x \leq 1$ [2]).

D. C. GILLESPIE and W. A. HURWITZ [3]) have given a more direct proof of this theorem, showing at the same time that it is possible to take in $\varphi_n(x)$ $a_{nk} \geq 0$, $\Sigma a_{nk} = 1$.

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[1]) Viz., that the equations

$$\int_0^1 f_n(x)\, dg(x) = 0 \qquad (n = 1, 2, 3, \ldots)$$

imply

$$\int_0^1 f(x)\, dg(x) = 0,$$

whenever

$$\int_0^1 |dg(x)| < \infty.$$

Cf. F. RIESZ, C. R. **150**, 674—677 (1910); S. BANACH, Théorie des opérations linéaires, 73.

[2]) F. RIESZ, Bulletin Calcutta Math. Soc. **20**, 55—58 (1928).

[3]) Trans. Am. Math. Soc. **32**, 527—543 (1930).

§ 2. Theorem R — and a fortiori RIESZ's more general theorem — may be used to prove that certain sets $\{f_n(x)\}$ have the property that it is possible to approximate, uniformly on $0 \leq x \leq 1$, to *every* continuous function $f(x)$ by (linear) polynomials in functions of the set. Sets having this property are said to be *closed* in C, the space of functions continuous on $0 \leq x \leq 1$. By WEIERSTRASZ's theorem on polynomial approximation a set is closed in C when, and only when, it is possible to approximate, uniformly on $0 \leq x \leq 1$, to each of the functions

$$x^n \quad (n = 0, 1, 2, \ldots)$$

by (linear) polynomials in functions of the set.

E. HILLE and O. SZÁSZ [4]) have shown that sets of "LAMBERT-functions" like

$$1, (1-x)\frac{x^n}{1-x^n} \quad (n = 1, 2, 3, \ldots) \quad \ldots \ldots \quad (1)$$

are closed in C. Briefly, their proof is as follows. If $0 \leq x < 1$,

$$x = \sum_{k=1}^{\infty} \mu(k) \frac{x^k}{1-x^k},$$

where $\mu(k)$ is the factor of MÖBIUS [5]). Hence

$$(1-x)\, x^r = \lim_{n\to\infty} (1-x) \sum_{k=1}^{n}{}' \frac{\mu(k)}{k} \frac{k\, x^{kr}}{1-x^{kr}} \qquad (0 \leqslant x < 1)$$

for every positive integer r. Applying summation by parts we get, putting

$$\sum_{1}^{k} \frac{\mu(m)}{m} = g(k),$$

$$(1-x)\, x^r = \lim_{n\to\infty} (1-x) \left[\sum_{1}^{n-1} g(k) \left\{ \frac{k\, x^{kr}}{1-x^{kr}} - \frac{(k+1)\, x^{(k+1)r}}{1-x^{(k+1)r}} \right\} + g(n) \frac{n\, x^{nr}}{1-x^{nr}} \right]$$
$$(0 \leqslant x < 1)$$

$$(1-x)\, x^r = \lim_{n\to\infty} (1-x) \sum_{1}^{n-1} g(k) \left\{ \frac{k\, x^{kr}}{1-x^{kr}} - \frac{(k+1)\, x^{(k+1)r}}{1-x^{(k+1)r}} \right\} \qquad (0 \leqslant x \leqslant 1)$$

$$= \lim_{n\to\infty} f_n(x). \qquad (0 \leqslant x \leqslant 1)$$

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[4]) Annals of Math. (2) **37**, 801—815 (1936).

[5]) $\mu(1) = 1$, $\mu(k) = (-1)^q$ if k is the product of q different primes, $\mu(k) = 0$ if n contains a quadratic factor > 1. A well-known property of the μ's is that

$$\sum_{d \mid r} \mu(d) = \begin{cases} 1 \; if\, r = 1, \\ 0 \; if\, r > 1. \end{cases}$$

(Cf. E. LANDAU, Vorlesungen über Zahlentheorie, I.)

Since $|g(k)| \leq 1$ [6]) and all terms in braces are positive we have for $0 \leq x \leq 1$, $n = 1, 2, 3, \ldots$,

$$|f_n(x)| \leqslant (1-x) \sum_1^{n-1} \left\{ \frac{k\, x^{kr}}{1-x^{kr}} - \frac{(k+1)\, x^{(k+1)r}}{1-x^{(k+1)r}} \right\}$$

$$\leqslant (1-x) \frac{x^r}{1-x^r} \leqslant \frac{1}{r} \leqslant 1.$$

Hence, by theorem R, it will be possible to approximate, uniformly on $0 \leq x \leq 1$, to each of the functions

$$(1-x)\, x^r \qquad (r = 1, 2, 3, \ldots)$$

by polynomials in functions of the set (1). As (1) already contains the functions 1 and x, it will be possible uniformly to approximate to all powers of x: if $r \geq 2$,

$$x^r = x - (1-x)\, x - \ldots - (1-x)\, x^{r-1}.$$

An application of WEIERSTRASZ's theorem completes the proof.

§ 3. The sequence $\{f_n(x)\}$ considered in § 2 converges uniformly on every interval $0 \leq x \leq \lambda$ where $\lambda < 1$. In this special case the statement of theorem R is much easier to prove. We shall find even more:

Theorem 1. *Let the sequence*

$$\{f_n(x)\} \qquad (n = 1, 2, 3, \ldots)$$

have the properties (i)—(iv) *of theorem* R, *and let moreover*

(v) *the convergence under* (ii) *be uniform on every interval* $0 \leq x \leq \lambda$ *where* $\lambda < 1$.

Then there exists a sequence of positive integers $k_1, k_2, \ldots, k_n, \ldots$ *such that*

$$\frac{1}{n} \{f_{k_1}(x) + f_{k_2}(x) + \ldots + f_{k_n}(x)\} \to f(x)$$

as $n \to \infty$, *uniformly on* $0 \leq x \leq 1$.

[6]) It follows from the relation mentioned in [5]) that

$$\sum_{r=1}^{k} \mu(r) \left[\frac{k}{r}\right] = \sum_{m=1}^{k} \sum_{d \mid m} \mu(d) = 1.$$

Since

$$\left| \sum_{r=1}^{k} \mu(r) \left\{ \frac{k}{r} - \left[\frac{k}{r}\right] \right\} \right| \leqslant \sum_{r=1}^{k} \left\{ \frac{k}{r} - \left[\frac{k}{r}\right] \right\} \leqslant k-1,$$

we have

$$\left| \sum_{r=1}^{k} \mu(r) \frac{k}{r} \right| \leqslant k$$

or

$$|g(k)| \leqslant 1.$$

Cf. E. LANDAU, loc. cit. p. 21. It was proved by VON MANGOLDT that $g(k) \to 0$ as $k \to \infty$. (Cf. E. LANDAU, loc. cit.), a result, equivalent to the prime number theorem. If we should use this result we should be able to prove that the sequence $\{f_n(x)\}$ converges uniformly on $0 \leq x \leq 1$.

P r o o f. By (ii) there exists a positive integer k_1 such that

$$|f(1)-f_{k_1}(1)|<1.$$

It follows from (i), (ii) and (iii) that there is a point λ_1 between 0 and 1 such that

$$|f(x)-f_{k_1}(x)|<1 \qquad (\lambda_1 \leqq x \leqq 1).$$

By (v) and (ii) there is a positive integer k_2 such that

$$|f(x)-f_{k_2}(x)|<\tfrac{1}{2} \qquad (0 \leqq x \leqq \lambda_1),$$

and

$$|f(1)-f_{k_2}(1)|<\tfrac{1}{2}.$$

Hence there is a point λ_2 between λ_1 and 1 such that

$$|f(x)-f_{k_2}(x)|<\tfrac{1}{2} \qquad (\lambda_2 \leqq x \leqq 1).$$

We thus find a sequence of integers $k_1, k_2, \ldots, k_n, \ldots$ and a sequence of points $0=\lambda_0<\lambda_1<\lambda_2<\ldots<\lambda_n<\ldots$ $(\lambda_n<1)$ such that always

$$|f(x)-f_{k_n}(x)|<\frac{1}{2^{n-1}} \quad \begin{cases} 0 \leqq x \leqq \lambda_{n-1}, \\ \lambda_n \leqq x \leqq 1. \end{cases}$$

By (ii) and (iv)

$$|f(x)-f_{k_n}(x)|<2F \qquad (\lambda_{n-1}<x<\lambda_n).$$

Hence, for all x of $0 \leqq x \leqq 1$,

$$|nf(x)-\{f_{k_1}(x)+f_{k_2}(x)+\ldots+f_{k_n}(x)\}|<2F+1+\tfrac{1}{2}+\ldots+\frac{1}{2^{n-1}}<2F+2,$$

that is

$$\max_{0 \leqq x \leqq 1} \left| f(x)-\frac{1}{n}\{f_{k_1}(x)+f_{k_2}(x)+\ldots+f_{k_n}(x)\}\right|<\frac{2F+2}{n},$$

more than had to be proved.

I could not decide whether the statement of theorem 1 is true in the general case also, that is, without the hypothesis (v).

§ 4. As an application of theorem 1 we shall prove

Theorem 2. *The set*

$$1, \log(1+x^n) \qquad (n=1,2,3,\ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is closed in C.

To prove theorem 2, we consider the development

$$x-x^2=\sum_1^{\infty}\frac{\mu(k)}{k}\log(1+x^k).$$

valid for $0 \leq x < 1$ [7]. If r is any positive integer we have, proceeding as in § 2,

$$x^r - x^{2r} = \lim_{n \to \infty} \sum_1^n \frac{\mu(k)}{k} \log(1 + x^{kr}) \qquad (0 \leqslant x < 1)$$

$$= \lim_{n \to \infty} \sum_1^{n-1} g(k) \{\log(1 + x^{kr}) - \log(1 + x^{(k+1)r})\} + $$
$$+ \lim_{n \to \infty} g(n) \log(1 + x^{nr}) \quad (0 \leqslant x < 1)$$

$$x^r - x^{2r} = \lim_{n \to \infty} \sum_1^{n-1} g(k) \{\log(1 + x^{kr}) - \log(1 + x^{(k+1)r})\} \qquad (0 \leqslant x \leqslant 1)$$

$$= \lim_{n \to \infty} f_n(x). \qquad (0 \leqslant x \leqslant 1)$$

Since $|g(k)| \leq 1$ [8] the sequence $\{f_n(x)\}$ converges boundedly on $0 \leq x \leq 1$:

$$|f_n(x)| \leqslant \log(1 + x^r) \leqslant \log 2,$$

and uniformly on every interval $0 \leq x \leq \lambda$, $\lambda < 1$:

$$|f_n(x) - f_m(x)| = |\sum_m^{n-1} g(k) \{\log(1 + x^{kr}) - \log(1 + x^{(k+1)r})\}|$$

$$\leqslant \log(1 + x^{mr}) \leqslant \log(1 + \lambda^{mr}) \to 0$$

as $m \to \infty$ $(n > m)$.

Hence, by theorem 1, it will be sufficient to prove that *the set*

$$1, \log(1 + x), \; x^r - x^{2r} \; (r = 1, 2, 3, \dots)$$

is closed in C.

In the above set, *we may replace the function* $\log(1 + x)$ *by* x, for if the set

$$1, x, x^r - x^{2r} \; (r = 1, 2, 3, \dots) \; . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is closed and if $F(x)$ is continuous on $0 \leq x \leq 1$, while ε is an arbitrary positive number, then there exist numbers $a_0, b_0, a_1, a_2, \dots, a_n$ such that

$$\max_{0 \leq x \leq 1} \left| \left\{ F(x) - F(0) - \frac{F(1) - F(0)}{\log 2} \log(1+x) \right\} - \right.$$
$$\left. - \left\{ a_0 + b_0 x + \sum_1^n a_k (x^k - x^{2k}) \right\} \right| < \frac{\varepsilon}{2}.$$

Hence

$$|a_0| < \frac{\varepsilon}{2}, \quad |a_0 + b_0| < \frac{\varepsilon}{2}.$$

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[7]) Cf. [5]). If we should use that $g(k) \to 0$ we could prove that this development is uniformly convergent on $0 \leq x \leq 1$.

[8]) Cf. [6]).

and as a consequence

$$|a_0 + b_0 x| < \frac{\varepsilon}{2},$$

$$\max_{0 \le x \le 1} \left| F(x) - \left\{ F(0) + \frac{F(1) - F(0)}{\log 2} \log(1+x) + \sum_1^n a_k (x^k - x^{2k}) \right\} \right| < \varepsilon.$$

In order to prove that *the set* (3) *is closed* we consider the sequence of linear combinations

$$\begin{aligned} f_n(x) &= (x^k - x^{2k}) + (x^{2k} - x^{4k}) + \ldots + (x^{2^{n-1}k} - x^{2^n k}) \\ &\quad + x - (x - x^2) - (x^2 - x^4) - \ldots - (x^{2^{n-1}} - x^{2^n}) \\ &= x^k - x^{2^n k} + x^{2^n}, \end{aligned}$$

which converges to

$$f(x) = x^k$$

as $n \to \infty$ for every x on $0 \le x \le 1$, boundedly on $0 \le x \le 1$ and uniformly on every interval $0 \le x \le \lambda < 1$. It follows from theorem 1 that it is possible to approximate to all functions x^k ($k = 2, 3, \ldots$) by linear combinations of functions belonging to (3), uniformly on $0 \le x \le 1$. Since the set (3) contains the functions 1 and x, an application of WEIERSTRASZ's theorem again completes the proof.

We may write theorem 2 into the following form:

Theorem 2a. *Let $p(x)$ be continuous and non-negative on $0 \le x \le 1$. Then it is possible to approximate to $p(x)$, uniformly on $0 \le x \le 1$, by finite products of the form*

$$A \prod_1^n (1 + x^k)^{a_k}.$$

Proof. Let $p(x) \le P$ ($0 \le x \le 1$) and let $\delta > 0$. By theorem 2 there exist numbers $\log A$, $a_1, a_2, \ldots, a_n$ such that for all x of $0 \le x \le 1$

$$-\delta < -\log\{p(x) + \delta\} + \log A + \sum_1^n a_k \log(1 + x^k) < \delta$$

or

$$e^{-\delta} < \frac{A \prod_1^n (1 + x^k)^{a_k}}{p(x) + \delta} < e^{\delta}.$$

Hence

$$(1 - e^{-\delta})(P + \delta) > p(x) + \delta - A \prod_1^n (1 + x^k)^{a_k} > (1 - e^{\delta})(P + \delta).$$

It follows that

$$\max_{0 \le x \le 1} |p(x) - A \prod_1^n (1 + x^k)^{a_k}| < (e^{\delta} - 1)(P + \delta) + \delta,$$

which is less than $\varepsilon > 0$ if δ is small enough.

Mathematics. — *A combinatorial problem.* By N. G. DE BRUIJN. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of June 29, 1946.)

1. Some years ago Ir. K. POSTHUMUS stated an interesting conjecture concerning certain cycles of digits 0 or 1, which we shall call P_n-cycles [1]). For $n = 1, 2, 3, \ldots$ a P_n-cycle be an ordered cycle of 2^n digits 0 or 1, such that the 2^n possible ordered sets of n consecutive digits of that cycle are all different. As a consequence, any ordered set of n digits 0 or 1 occurs exactly once in that cycle.

For example, a P_3-cycle is $\overrightarrow{00010111}$ [2]), respectively showing the triples 000, 001, 010, 101, 011, 111, 110, 100, which are all possible triples indeed.

For $n = 1, 2, 3, 4$, the P_n-cycles can easily be found.

We have only one P_1-cycle, viz. 01, and only one P_2-cycle, viz. 0011. There are two P_3-cycles, viz. 00010111 and 11101000, and sixteen P_4-cycles, eight of which are

0000110100101111	1111001011010000
0000100110101111	1111011001010000
0000101100111101	1111010011000010
0000110101111001	1111001010000110

the remainig eight being obtained by reversing the order of these, respectively.

Ir. POSTHUMUS found the number of P_5-cycles to be 2048, and so he had the following number of P_n-cycles for $n = 1, 2, 3, 4, 5$:

$$1 \quad , 1 \quad , 2 \quad , 2^4 \quad , 2^{11} \quad ,$$
$$\text{or} \quad 2^{2^0-1} , \ 2^{2^1-2} , \ 2^{2^2-3} , \ 2^{2^3-4} , \ 2^{2^4-5} .$$

Thus he was led to the conjecture, that the number of P_n-cycles be $2^{2^{n-1}-n}$ for general n. In this paper his conjecture is shown to be correct. Its proof is given in section **3**, as a consequence of a theorem concerning a special type of networks, stated and proved in section **2**. In section **4** another application of that theorem is mentioned.

2. We consider a special type of networks, which we shall call T-nets.

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1) These arise from a practical problem in telecommunication.

2) With this notation, $\overrightarrow{00010111}$, $\overrightarrow{00101110}$, etc., are to be considered as the same cycle. (Properly speaking, the digits must be placed around a circle.) On the other hand we do not identify the cycles $\overrightarrow{00010111}$ and $\overrightarrow{11101000}$, the second of which is obtained by reversing the order of the first one.

Henceforth we simply write 00010111 instead of $\overrightarrow{00010111}$.

A *T-net of order m* will be a network of *m junctions* and *2m one-way roads* (oriented roads), with the property that each junction is the start of two roads and also the finish of two roads. The network need not lie in a plane, or, in other words, viaducts, which are not to be considered as junctions, are allowed. Furthermore we do not exclude roads leading from a junction to that same junction, and we neither exclude pairs of junctions connected by two different roads, either in the same, or in opposite direction. Figs. 1*a* and 1*b* show examples of T-nets, of orders 3 and 6, respectively.

In a T-net we consider closed walks, with the property that any road of the net is used exactly once, in the prescribed direction. Such walks will be called "*complete walks*" of that T-net. Two complete walks are considered to be identical, if, and only if, the sequence of roads [3] gone through in the first walk is a cyclic permutation of that in the second walk. The nets of figs. 1*a* and 1*b* admit 2 and 8 complete walks, respectively.

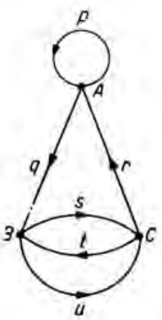

Fig. 1*a*

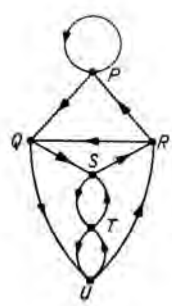

Fig. 1*b*

The number of complete walks of a T-net N be denoted by $|N|$. This number $|N|$ is zero, if N is not connected, that is to say, if N can be divided into two seperate T-nets [4].

We now describe a process, which we call the "*doubling*" of a T-net, and which is illustrated by the relation between the nets of figs. 1*a* and 1*b*. Be N a T-net of order m, with junctions $A, B, C, \ldots$, and roads $p, q, r, \ldots$. Then we construct the "doubled" net N^* by taking $2m$ junctions $P, Q, R, \ldots$, corresponding to the roads of N, respectively. We construct a one-way road from a junction P to a junction Q, if the corresponding roads p and q of N have the property, that the finish of p lies in the same junction of N as the start of q. Thus $4m$ roads are obtained in N^*, and it is easy to see that N^* is a T-net; its order is $2m$.

[3]) If we should replace the word "roads" by "junctions" here, this sentence would get another meaning, since two junctions may be connected by two roads in the same direction.

[4]) The converse is also true: for a connected T-net we have $|N| > 0$. However, we do not need this result in the proof of our main theorem.

A remarkably simple relation exists between the numbers of complete walks of N and N^* [5]):

Theorem. If N is a T-net of order m ($m = 1, 2, 3, \ldots$), and N^* is the doubled net, then we have

$$|N^*| = 2^{m-1} \cdot |N|. \qquad (1)$$

Proof. We first consider two cases, in which (1) is easily established.

Case 1. If N is not connected, the same holds for N, and hence $|N| = |N^*| = 0$.

Case 2. We now consider the case, where each junction of N is connected with itself. For any value of m, only one connected net of this type exists, consisting of junctions $A_1, A_2, \ldots, A_m$, connected by roads $A_1A_2, A_2A_3, \ldots, A_{m-1}A_m, A_mA_1$, and $A_1A_1, A_2A_2, \ldots, A_mA_m$ [6]). For this net we have $|N| = 1$, and some quite trivial considerations show that $|N^*| = 2^{m-1}$.

We prove the general case by induction. For $m = 1$ only one T-net is possible, consisting of one junction A and two roads leading from A to A. This net belongs to case 2 mentioned above, and we have $|N| = |N^*| = 1$.

Now suppose (1) to be valid for all T-nets of order $m - 1$ ($m > 1$), and be N a T-net of order m. We may suppose to be able to choose a junction A, not connected with itself, for otherwise N belongs to case 2. Hence we have four different roads p, q, r, s; p and q leading to A, r and s starting from A.

A net N_1 arises from N by omitting A, p, q, r, s, and constructing two new roads, one from the start of p to the finish of r, and one from the start of q to the finish of s.

A second net N_2 arises in a similar way, but now by combining p with s and q with r. This is illustrated by fig. 2; the parts of the nets, which are not drawn, are equal for N, N_1 and N_2.

A complete walk of N corresponds to a complete walk either of N_1, or of N_2, and so we have

$$|N| = |N_1| + |N_2|. \qquad (2)$$

On doubling the nets N_1 and N_2 we obtain nets N_1^* and N_2^*, respectively.

We shall prove

$$|N^*| = 2|N^*_1| + 2|N^*_2|. \qquad (3)$$

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[5]) This relation can also be interpreted without introducing the doubling process. Namely, a complete walk of N^* corresponds to a closed walk through N, with the property that any road of N is used exactly twice in that walk, and such that at any junction each of the four possible combinations of a finish and a start is taken exactly once. We can give an even simpler interpretation in terms of N^*, for a complete walk of N corresponds to a closed walk through N^*, visiting any junction of N^* exactly once. But, since not every T-net can be considered as a N^*, this does not lead to an essential simplification of our theorem.

[6]) AB denotes a one-way road leading from A to B.

N_1^* and N_2^* arise directly from N^* by simple operations. If P, Q, R, S are the junctions of N^* corresponding to the roads p, q, r, s of N, we obtain N_1^* by omitting the roads PR, PS, QR, QS from N^*, and identifying the

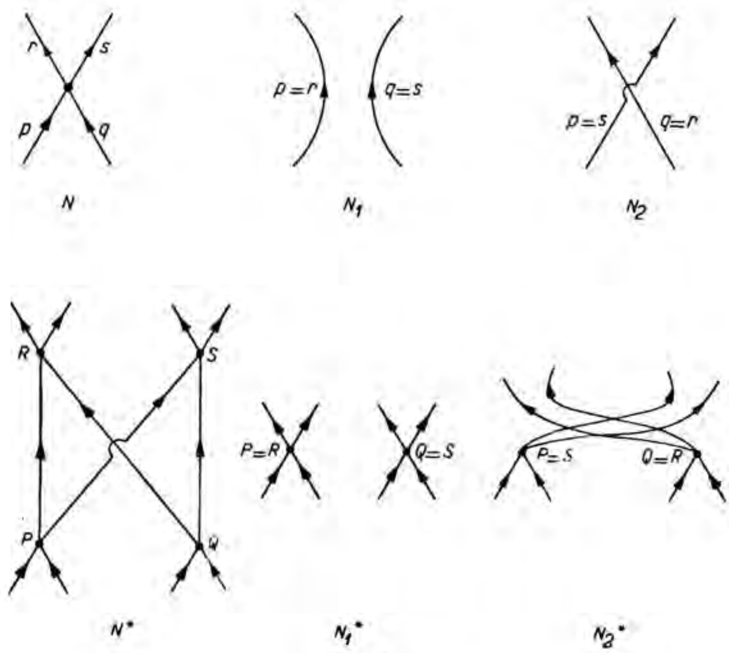

Fig. 2

four junctions two by two: $P = R$ and $Q = S$. N_2^* is obtained analogously ($P = S$ and $Q = R$). Again, fig. 2 shows the corresponding details of N^*, N_1^* and N_2^*.

Henceforth we deal with N^*, N_1^* and N_2^*, and no longer consider N, N_1, N_2.

We now first introduce the term "path". A *path* is an ordered sequence of roads, no two of which are identical, such that the finish of each road is the start of the next one. The last one, however, need not lead to the start of the first one.

A complete walk of N^*, N_1^*, or N^*_2, contains four special paths, each one leading from one of the junctions R, S to one of the junctions P, Q, such that any road of N^*, except PR, PS, QR, QS, belongs to just one of those paths. Choosing a definite set of four paths, according to the conditions just mentioned, we consider all (possibly existing) complete walks of N, N_1 and N_2 containing those paths. The numbers of these complete walks be denoted by n, n_1, n_2, respectively.

The numbers n, n_1, n_2 admit of a simple interpretation. Be N^{**} the net, arising from N^* on replacing each of the four paths by one single road, with the same start and finish as the corresponding path. In the same way nets N_1^{**} and N_2^{**} arise from N_1^* and N_2^*. Evidently $n = |N^{**}|$, $n_1 = |N_1^{**}|$, $n_2 = |N_2^{**}|$. We now show, that

$$n = 2n_1 + 2n_2, \quad \ldots \ldots \quad (4)$$

for which we have to consider two different cases.

Case A (fig. 3). The paths starting from R lead to different junctions.

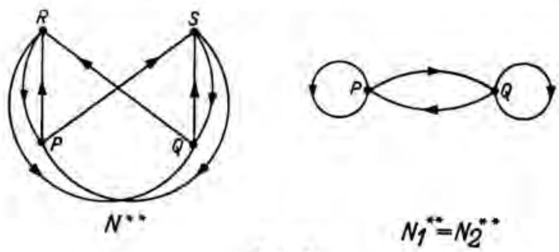

Fig. 3

The four paths thus respectively lead from R to P, from R to Q, from S to P, and from S to Q.

Now N^{**} consists of the junctions P, Q, R, S, with the roads PR, PS, QR, QS, RP, RQ, SP, SQ. This net admits four different complete walks. N_1^{**} consists of only two junctions P and Q, with roads PP, PQ, QP, QQ. This net admits only one complete walk. The net N_2^{**} is equivalent to N_1^{**}. Thus we have obtained $n = 4$, $n_1 = 1$, $n_2 = 1$, and (4) holds true.

Case B (fig. 4). The paths starting from R lead to one and the same

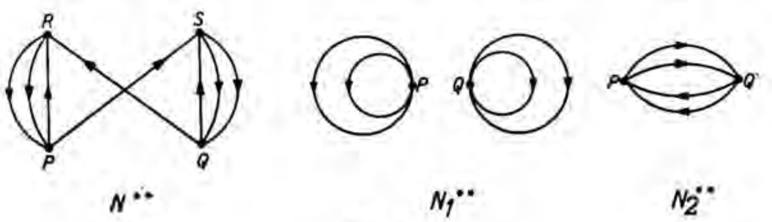

Fig. 4

junction, say P (the same obtains for Q). We now have the four paths RP, RP, SQ, SQ.

Now N^{**} consists of the junctions P, Q, R, S, with roads PR, PS, QR, QS, RP, RP, SQ, SQ. This net admits four complete walks. N_1^{**} consists of two junctions P and Q, with roads PP, PP, QQ, QQ, and so it is not connected. N_2^{**} consists of P, Q, with roads PQ, PQ, QP, QP, admitting two complete walks. Now we have $n = 4$, $n_1 = 0$, $n_2 = 2$, and hence (4) holds also true in case B.

Formula (4) being proved for any admissible system of four paths, the truth of (3) is now evident.

Our theorem is an immediate consequence of (3). Namely, N_1 and N_2 being nets of order $m - 1$, our assumption of induction yields

$$|N_1^*| = 2^{m-2} |N_1|, \qquad |N_2^*| = 2^{m-2} |N_2|,$$

and by (3) and (2) we now have

$$|N^*| = 2|N_1^*| + 2|N_2^*| = 2^{m-1}|N_1| + 2^{m-1}|N_2| = 2^{m-1}|N|.$$

3. The theorem of the preceding section provides a proof of POSTHUMUS' conjecture. For $n \geq 2$, N_n be the following network of order 2^n. As junctions we take the ordered n-tuples of digits 0 or 1, and we connect two n-tuples A and B by a one-way road AB, if the last $n-1$ digits of A correspond to the first $n-1$ digits of B. Fig. 5 shows the nets N_2 and N_3.

On "doubling" this net N_n we obtain the net N_{n+1}. Namely, any road AB of N_n (see N_2 in fig. 5) corresponds to an ordered $(n+1)$-tuple,

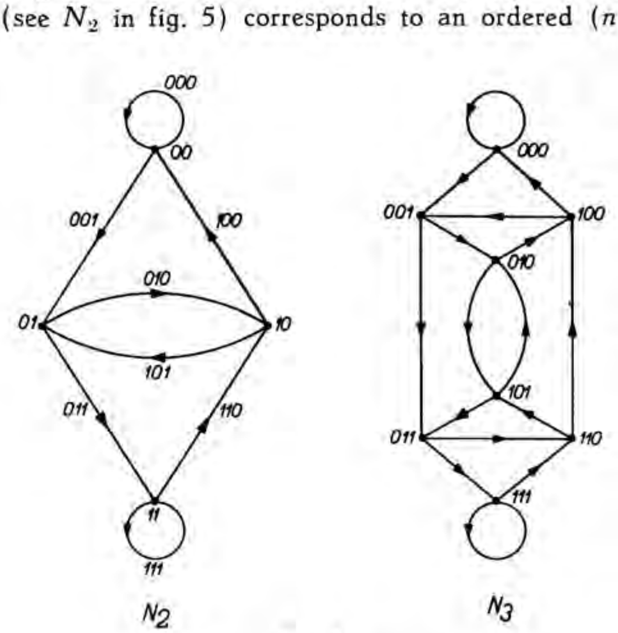

Fig. 5

consisting of the digits of A, followed by the last digit of B (or, what is the same, the first digit of A, followed by the digits of B). Two $(n+1)$-tuples P, Q, turn out to be connected in N_{n+1}, if the last n digits of the first one correspond to the first n digits of the second one, since these n digits characterize the common finish and start of the roads p and q of N_n. Hence $N_n^* = N_{n+1}$.

A complete walk of N_n leads to a P_{n+1}-cycle in the following way. If such a walk consecutively goes through the roads AB, BC, ..., ZA, we write down consecutively, the first digit of A, the first digit of B, ..., the first digit of Z. This sequence, considered as a cycle, is a P_{n+1}-cycle. Namely, on taking the first digits of $n+1$ consecutive junctions $A, B, C, \ldots$ of the walk under consideration, we obtain the $(n+1)$-tuple, belonging to the road AB. The walk in N_n being complete, it is now clear that any $(n+1)$-tuple occurs exactly once in our cycle.

Conversely, any P_{n+1}-cycle arises from a complete walk in N_n by the described process, and different complete walks lead to different P_{n+1}-cycles. Hence the number of P_{n+1}-cycles equals $|N_n|$.

We now prove POSTHUMUS' conjecture by induction. For $n = 1, 2, 3$ its truth is already established in section **1**. Now take $n \geq 3$, and suppose the number of P_n-cycles to be $2^{2^{n-1}-n}$, whence $|N_{n-1}| = 2^{2^{n-1}-n}$. The order of N_{n-1} being 2^{n-1}, the theorem of section **2** yields

$$| N^*_{n-1} | = 2^{2^{n-1}-1} \cdot | N_{n-1} |,$$

and it follows

$$| N_n | = 2^{2^{n-1}-1} \cdot 2^{2^{n-1}-n} = 2^{2^n-n-1}.$$

The number of P_{n+1}-cycles equalling $|N_n|$, POSTHUMUS' conjecture turns out to be true.

4. Another application of section **2** is the following one. We call a n-tuple of digits 0, or 2 admissible, if no two consecutive digits are equal; the last digit, however, may be the same as the first one. The number of admissable n-tuples is easily shown to be $3 \cdot 2^{n-1}$. As a Q_n-cycle we now define an ordered cycle of $3 \cdot 2^{n-1}$ digits 0,1 or 2, such that any admissable n-tuple is represented once by n consecutive digits of the cycle. For instance twelve Q_3-cycles exist. Two of them are 012010202121 and 012020102121, whereas the other ten are found by applying permutations of the symbols 0,1 and 2.

For general $n > 1$, the number of Q_n-cycles amounts to $3 \cdot 2^{3 \cdot 2^{n-2}-n-1}$. A proof can be given completely analogous to that in section **3**.

Eindhoven, June 1946.

Natuurkundig Laboratorium der
N.V. Philips' Gloeilampenfabrieken.

Mathematics. — *On the G-function.* V. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 29, 1946.)

Lemma 24. *Suppose that k, l, m, n, p, q, μ and ν are integers with $l \geqq 1$, $q \geqq 1$, $\mu \geqq 0$, $0 \leqq m \leqq k \leqq q$, $0 \leqq n-l+1 \leqq \nu \leqq k$ and $l+\nu-1 \leqq p \leqq q$; further that λ is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{n-l+1}$, $a_{\nu+1}, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ fulfil the conditions* (100) *and* (119) *and satisfy besides the inequality*

$$a_j - b_h \neq 1, 2, 3, \ldots \quad (j = 1, \ldots, n-l+1;\ h = 1, \ldots, m) \ . \tag{133}$$

Then the following formula holds [40]:

$$\left.\begin{aligned}
&\sum_{t=1}^{n-l+1} e^{(m+n+k-l+2\lambda-2\nu)\pi i a_t} \triangle^{m,\,n-l+1}_{k}(t)\, G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta \,\|\, a_t) \\
&= -A^{m,\,n-l+1}_{k} \sum_{h=k-\nu-\mu+1}^{k-\nu} \Omega^{m,\,n-l+1}_{k}(h+\lambda-1)\, G^{k,\,l-1,\,l+\nu-1}_{p,q}(\zeta e^{(2k-2\nu-2h+1)\pi i}) \\
&+ \overline{A}^{m,n-l+1}_{k} \sum_{\varkappa=1}^{\mu} \overline{\Omega}^{m,n-l+1}_{k}(l-m-n-\lambda+\nu+\varkappa-1)\, G^{k,\,l-1,\,l+\nu-1}_{p,q}(\zeta e^{(2\varkappa-1)\pi i}) \\
&\quad + \sum_{\tau=1}^{n-l+1} e^{(m+n+k-l+2\lambda-2\nu-2\mu)\pi i a_\tau} \triangle^{m,n-l+1}_{k}(\sigma)\, G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta e^{2\mu\pi i} \,\|\, a_\tau).
\end{aligned}\right\} \tag{134}$$

P r o o f. Formula (134) is obvious if $\mu = 0$. We may therefore suppose $\mu \geqq 1$ and assume that (134) with $\mu - 1$ instead of μ has already been proved.

Now it follows from (57), if $\nu \geqq 1$,

$$\begin{aligned}
G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta e^{(2\mu-2)\pi i} \,\|\, a_\tau) = e^{-2\pi i a_\tau} G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta e^{2\mu\pi i} \,\|\, a_\tau) \\
+ 2\pi i e^{-\pi i a_\tau} G^{k,\,l-1,\,l+\nu-1}_{p,q}(\zeta e^{(2\mu-1)\pi i}).
\end{aligned}$$

If we substitute this on the right-hand side of (134) with $\mu - 1$ instead of μ, the sum $\sum\limits_{\tau=1}^{n-l+1}$ not only gives the corresponding sum in (134) but also

$$2\pi i\, G^{k,\,l-1,\,l+\nu-1}_{p,q}(\zeta e^{(2\mu-1)\pi i}) \sum_{\tau=1}^{n-l+1} e^{(m+n+k-l+2\lambda-2\nu-2\mu+1)\pi i a_\tau} \triangle^{m,\,n-l+1}_{k}(\sigma) \text{ [41]}$$

---

[40]) The products $\triangle^{m,n-l+1}_{k}(t)\, G^{k,\,l,\,l+\nu-1}_{p,q}(\zeta \,\|\, a_t)$ in (134) must be defined by a limiting process if $a_t - b_h = 1, 2, 3, \ldots$ $(m+1 \leqq h \leqq k)$.

[41]) This is still true if $\nu = 0$, since then $n - l + 1 = 0$, so that the sums $\sum\limits_{\tau=1}^{n-l+1}$ vanish.

and this expression is by (59) equal to

$$-G_{p,q}^{k,l-1,l+\nu-1}(\zeta e^{(2\mu-1)\pi i})\{A^{m,n-l+1}_{k}\,\Omega^{m,n-l+1}_{k}(k+\lambda-\nu-\mu)$$
$$-\overline{A}^{m,n-l+1}_{k}\,\overline{\Omega}^{m,n-l+1}_{k}(l-m-n-\lambda+\nu+\mu-1)\}.$$

The sums $\sum_{h=k-\nu-\mu+2}^{k-\nu}$ and $\sum_{\varkappa=1}^{\mu-1}$ on the right-hand side of (134) with $\mu-1$ instead of μ reduce therefore to the corresponding sums $\sum_{h=k-\nu-\mu+1}^{k-\nu}$ and $\sum_{\varkappa=1}^{\mu}$ in (134). With this the lemma has been proved.

Lemma 25. *Suppose that k, l, m, n, p, q and ν are integers with $l \geqq 1$, $q \geqq 1$, $0 \leqq m \leqq k \leqq q$, $0 \leqq n-l+1 \leqq \nu$ and $l+\nu-1 \leqq p \leqq q$; further that r is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{n-l+1}$, $a_{\nu+1}, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (100), (119) and (133).*

Then the following formula holds:

$$\left.\begin{aligned}
&G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_{n-l+1},a_{\nu+1},\ldots,a_p\;a_{n-l+2},\ldots,a_\nu\\ b_1,\ldots,b_q\end{matrix}\right)\\
&=A^{m,n-l+1}_{k}\sum_{s=0}^{r-1}\Omega^{m,n-l+1}_{k}(s)\,G_{p,q}^{k,l-1,l+\nu-1}(z\,e^{(k+l-m-n-2s-1)\pi i})\\
&+\overline{A}^{m,n-l+1}_{k}\sum_{\tau=0}^{k+l-m-n-r-1}\overline{\Omega}^{m,n-l+1}_{k}(\tau)\,G_{p,q}^{k,l-1,l+\nu-1}(z\,e^{(m+n-k-l+2\tau+1)\pi i})\\
&+\sum_{t=1}^{n-l+1}e^{(m+n-k-l+2r)\pi i a_t}\,\triangle^{m,n-l+1}_{k}(t)\,G_{p,q}^{k,l,l+\nu-1}(z\,e^{(k+l-m-n-2r)\pi i}\,\|\,a_t).
\end{aligned}\right\}\quad(135)$$

Proof. We consider formula (116) that holds under the general conditions which have been stated in theorem 4. On the left-hand side we replace $G_{p,q}^{m,n}(z)$ by

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\ldots,a_{n-l+1},a_{\nu+1},\ldots,a_p,a_{n-l+2},\ldots,a_\nu\\ b_1,\ldots,b_q\end{matrix}\right).$$

Observing that $G_{p,q}^{k,\lambda}(z)$ is a symmetric function of $a_{\lambda+1}, \ldots, a_p$, it follows easily from the definitions of $G_{p,q}^{k,l-1,n}(z)$ and $G_{p,q}^{k,l,n}(z \,\|\, a_t)$ (see the end of § 1) that the functions $G_{p,q}^{k,l-1,n}(\zeta)$ and $G_{p,q}^{k,l,n}(w \,\|\, a_t)$ on the right of (116) must then be replaced by

$$G_{p,q}^{k,l-1}\left(\zeta\,\middle|\,\begin{matrix}a_{\nu+1},\ldots,a_p,a_1,\ldots,a_\nu\\ b_1,\ldots,b_q\end{matrix}\right),$$

respect.

$$G_{p,q}^{k,l}\left(w\,\middle|\,\begin{matrix}a_t,a_{\nu+1},\ldots,a_p,a_1,\ldots,a_{t-1},a_{t+1},\ldots,a_\nu\\ b_1,\ldots,b_q\end{matrix}\right)$$

(where $1 \leq t \leq n-l+1 \leq \nu$)

and these functions are according to the definitions of $G_{p,q}^{k,l-1,n}(z)$ and $G_{p,q}^{k,l,n}(z \parallel a_t)$ equal to $G_{p,q}^{k,l-1,l+\nu-1}(\zeta)$, respect. $G_{p,q}^{k,l,l+\nu-1}(w \parallel a_t)$.

The coefficients A, $\bar{A}$, Ω, $\bar{\Omega}$ and $\triangle$ take in the new formula the same values as in the original one and the system of conditions (1), (99) needs to be replaced by the system (119), (133). This proves the lemma.

In the same manner as the above lemmas we may prove the conjugate lemmas. Thus the conjugate of lemma 23 is

Lemma 26. *Suppose that k, l, m, n, p, q, δ and ν are integers with*

$$l \geqq 1, \; q \geqq 1, \; 0 \leqq m \leqq k \leqq q,$$
$$0 \leqq n-l+1 \leqq \nu, \; 0 \leqq \delta \leqq k-\nu \text{ and } l+\nu-1 \leqq p \leqq q;$$

further that β is an arbitrary integer; finally that the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ satisfy the conditions (119) and (120).

Then the following formula holds:

$$\left.\begin{aligned}
&A^{m,n-l+1}_{\;\;k} \sum_{\tau=0}^{\beta-1} \bar{\Omega}^{m,n-l+1}_{\;\;k}(\tau)\, G_{p,q}^{k,l-1,l+\nu-1}(w\, e^{2\tau\pi i}) \\
&= -A^{0,\nu}_{\;\;k}\, \bar{B}^{m,n-l+1}_{\nu} \sum_{h=1}^{\delta} \bar{\Psi}^{m,n-l+1}_{\nu,k}(h;\beta)\, G_{p,q}^{k,l-1,l+\nu-1}(w\, e^{(2k-2\nu-2h+2\beta)\pi i}) \\
&+ A^{m,n-l+1}_{\;\;k} \sum_{\varkappa=1}^{k-\nu-\delta} \{\bar{\Phi}^{m,n-l+1}_{\nu,k}(\varkappa;\beta) - \bar{\Omega}^{m,n-l+1}_{\;\;k}(\varkappa+\beta-1)\}\, G_{p,q}^{k,l-1,l+\nu-1}(w\, e^{(2\varkappa+2\beta-2)\pi i}) \\
&- B^{m,n-l+1}_{\nu} \sum_{\sigma=1}^{\nu} e^{(\nu-k+2\delta-1)\pi i a_\sigma}\, \bar{\Theta}^{m,n-l+1}_{\nu}(\sigma;\beta-1)\, \triangle^{0,\nu}_{\;\;k}(\sigma)\, G_{p,q}^{k,l,l+\nu-1}(w\, e^{(2k-2\nu+2\beta-2\delta-1)\pi i} \parallel a_\sigma).
\end{aligned}\right\} \quad (136)$$

§ 14. Fourth expansion formula.

The most important theorem of the present paper is

Theorem 5. Assumptions: *k, l, m, n, p, q and ν are integers with*

$$l \geqq 1, q \geqq 1, 0 \leqq m \leqq k \leqq q, 0 \leqq n-l+1 \leqq \nu \leqq k \text{ and } l+\nu-1 \leqq p \leqq q;$$

the numbers $a_1, \ldots, a_{l+\nu-1}$ and $b_1, \ldots, b_k$ fulfil the conditions (119), (120) and (133); μ is an arbitrary integer which satisfies the inequality

$$0 \leqq \mu \leqq k-\nu;$$

λ is an arbitrary integer.

Assertion:

$$\left.\begin{aligned}
&G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_{n-l+1}, a_{\nu+1}, \ldots, a_p, a_{n-l+2}, \ldots, a_\nu \\ b_1, \ldots, b_q \end{matrix}\right) \\
&= \sum_{h=1}^{k-\nu-\mu} R^{m,n-l+1}_{\nu,k}(h:\lambda)\, G_{p,q}^{k,l-1,l+\nu-1}(z\, e^{(k+l-m-n-2h-2\lambda+1)\pi i}) \\
&+ \sum_{\varkappa=1}^{\mu} \bar{R}^{m,n-l+1}_{\nu,k}(\varkappa; l-m-n-\lambda+\nu)\, G_{p,q}^{k,l-1,l+\nu-1}(z\, e^{(l-k-m-n+2\varkappa-2\lambda+2\nu-1)\pi i}) \\
&+ \sum_{\sigma=1}^{\nu} e^{(k-\nu-2\mu)\pi i a_\sigma}\, T^{m,n-l+1}_{\nu,k}(\sigma;\lambda)\, G_{p,q}^{k,l,l+\nu-1}(z\, e^{(l-k-m-n-2\lambda+2\mu+2\nu)\pi i} \parallel a_\sigma).
\end{aligned}\right\} \quad (137)$$

P r o o f. In formula (135) we replace r by $k+\lambda-\nu$. Then we obtain

$$G_{p,q}^{m,n}\left(z \left| \begin{matrix} a_1,\ldots,a_{n-l+1},a_{\nu+1},\ldots,a_p,a_{n-l+2},\ldots,a_\nu \\ b_1,\ldots,b_q \end{matrix}\right.\right)$$

$$=A^{m,n-l+1}_{\ \ k} \sum_{s=0}^{\lambda-1} \Omega^{m,n-l+1}_{\ \ k}(s)\, G_{p,q}^{k,l-1,l+\nu-1}(z\, e^{(k+l-m-n-2s-1)\pi i})$$

$$+A^{m,n-l+1}_{\ \ k} \sum_{h=1}^{k-\nu} \Omega^{m,n-l+1}_{\ \ k}(h+\lambda-1)\, G_{p,q}^{k,l-1,l+\nu-1}(z\, e^{(k+l-m-n-2h-2\lambda+1)\pi i})$$

$$+\overline{A}^{m,n-l+1}_{\ \ k} \sum_{\tau=0}^{l-m-n-\lambda+\nu-1} \overline{\Omega}^{m,n-l+1}_{\ \ k}(\tau)\, G_{p,q}^{k,l-1,l+\nu-1}(z\, e^{(m+n-k-l+2\tau+1)\pi i})$$

$$+\sum_{t=1}^{n-l+1} e^{(m+n+k-l+2\lambda-2\nu)\pi i a_t} \triangle^{m,n-l+1}_{\ \ k}(t)\, G_{p,q}^{k,l,l+\nu-1}(z\, e^{(l-k-m-n-2\lambda+2\nu)\pi i} \,\|\, a_t).$$

On the right-hand side of this relation we may reduce the sum $\sum\limits_{s=0}^{\lambda-1}$ by means of (130) (with $w=z\,e^{(k+l-m-n-1)\pi i}$), further the sum $\sum\limits_{\tau=0}^{l-m-n-\lambda+\nu-1}$ by means of (136) (with $\beta=l-m-n-\lambda+\nu$, $\delta=k-\nu-\mu$ and $w=z\,e^{(m+n-k-l+1)\pi i}$) and finally the sum $\sum\limits_{t=1}^{n-l+1}$ by means of (134) (with $\zeta=z\,e^{(l-k-m-n-2\lambda+2\nu)\pi i}$). Then we find

$$G_{p,q}^{m,n}\left(z \left| \begin{matrix} a_1,\ldots,a_{n-l+1},a_{\nu+1},\ldots,a_p,a_{n-l+2},\ldots,a_\nu \\ b_1,\ldots,b_q \end{matrix}\right.\right)$$

$$=\sum_{h=1}^{k-\nu-\mu}\{A^{m,n-l+1}_{\ \ k}\,\Phi^{m,n-l+1}_{\nu,k}(h;\lambda)-A^{0,\nu}_{\ \ k}\,\overline{B}^{m,n-l+1}_{\nu}\,\overline{\Psi}^{m,n-l+1}_{\nu,k}(h;l-m-n-\lambda+\nu)\}\times$$

$$\times G_{p,q}^{k,l-1,l+\nu-1}(z\,e^{(k+l-m-n-2h-2\lambda+1)\pi i})$$

$$+\sum_{\varkappa=1}^{\mu}\{\overline{A}^{m,n-l+1}_{\ \ k}\,\overline{\Phi}^{m,n-l+1}_{\nu,k}(\varkappa;l-m-n-\lambda+\nu)-\overline{A}^{0,\nu}_{\ \ k}\,B^{m,n-l+1}_{\nu}\,\Psi^{m,n-l+1}_{\nu,k}(\varkappa;\lambda)\}\times$$

$$\times G_{p,q}^{k,l-1,l+\nu-1}(z\,e^{(l-k-m-n+2\varkappa-2\lambda+2\nu-1)\pi i})$$

$$-\sum_{\sigma=1}^{\nu}\{e^{(k-\nu-2\mu+1)\pi i a_\sigma}\,B^{m,n-l+1}_{\nu}\,\Theta^{m,n-l+1}_{\nu}(\sigma;\lambda-1)$$

$$+e^{(k-\nu-2\mu-1)\pi i a_\sigma}\,\overline{B}^{m,n-l+1}_{\nu}\,\overline{\Theta}^{m,n-l+1}_{\nu}(\sigma;l-m-n-\lambda+\nu-1)\}\times$$

$$\times \triangle^{0,\nu}_{\ \ k}(\sigma)\,G_{p,q}^{k,l,l+\nu-1}(z\,e^{(l-k-m-n-2\lambda+2\mu+2\nu)\pi i}\,\|\,a_\sigma)$$

$$+\sum_{\sigma=1}^{n-l+1} e^{(m+n+k-l+2\lambda-2\nu-2\mu)\pi i a_\sigma}\,\triangle^{m,n-l+1}_{\ \ k}(\sigma)\,G_{p,q}^{k,l,l+\nu-1}(z\,e^{(l-k-m-n-2\lambda+2\mu+2\nu)\pi i}\,\|\,a_\sigma).$$

Because of (84), (85), (86) and (87) the coefficients in this relation are equal to those in (137). This establishes the theorem.

§ 15. On a special transformation formula.

It is sometimes desirable to express a function $M(z)$, that is many-valued with a branch-point at $z=0$, in the following way

$$M(z)=\sum_{h=1}^{k} r_h\, M(z\, e^{(\gamma-2h)\pi i}),$$

or, more general, to write a function $N(z)$ as follows

$$N(z)=\sum_{h=1}^{k} s_h\, M(z\, e^{(\delta-2h)\pi i}).$$

In this § I will establish such relations for the function G. I will prove:

Theorem 6. A s s u m p t i o n s: *k, m, n, p and q are integers with*

$$q \geqq 1,\; 0 \leqq n \leqq p \leqq q \;\; \text{and} \;\; 0 \leqq m \leqq k \leqq q;$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_k$ satisfy the condition

$$a_j - b_h \neq 1, 2, 3, \ldots \qquad (j=1,\ldots,n;\; h=1,\ldots,k);$$

λ is an arbitrary integer.

A s s e r t i o n:

$$G_{p,q}^{m,n}(z)=\sum_{h=1}^{k} R_{0,k}^{m,0}(h;\lambda)\, G_{p,q}^{k,n}(z\, e^{(k-m-2h-2\lambda+2)\pi i}). \quad . \quad . \quad (138)$$

P r o o f. This theorem is a special case of theorem 5. Indeed, it is obvious on account of the definition of $G_{p,q}^{k,l-1,n}(z)$ that

$$G_{p,q}^{k,n,n}(z)=G_{p,q}^{k,n}(z).$$

Hence we find, if we take $l=n+1$ and $\nu=0$ in (137),

$$\left.\begin{aligned} G_{p,q}^{m,n}(z)= & \sum_{h=1}^{k-\mu} R_{0,k}^{m,0}(h;\lambda)\, G_{p,q}^{k,n}(z\, e^{(k-m-2h-2\lambda+2)\pi i}) \\ & + \sum_{\varkappa=1}^{\mu} \overline{R}_{0,k}^{m,0}(\varkappa;1-m-\lambda)\, G_{p,q}^{k,n}(z\, e^{(-k-m+2\varkappa-2\lambda)\pi i}), \end{aligned}\right\} . \quad (139)$$

where $0 \leqq \mu \leqq k$. Putting $\mu=0, 1, \ldots, k$, it seems as if there are $k+1$ different relations, formula (138) being the special case with $\mu=0$. But this is only true in seeming. For (139) may also be written in the following way

$$\left.\begin{aligned} G_{p,q}^{m,n}(z)= & \sum_{h=1}^{k-\mu} R_{0,k}^{m,0}(h;\lambda)\, G_{p,q}^{k,n}(z\, e^{(k-m-2h-2\lambda+2)\pi i}) \\ & + \sum_{h=k-\mu+1}^{k} R_{0,k}^{m,0}(k-h+1;1-m-\lambda)\, G_{p,q}^{k,n}(z\, e^{(k-m-2h-2\lambda+2)\pi i}). \end{aligned}\right\} \quad (140)$$

Subtracting (140) with $\mu+1$ instead of μ from (140), we find for $0 \leqq \mu \leqq k-1$

$$\overline{R}_{0,k}^{m,0}(\mu+1;1-m-\lambda)=R_{0,k}^{m,0}(k-\mu;\lambda).$$

hence for $1 \leq h \leq k$

$$\overline{R}_{0,k}^{m,0}(k-h+1 ; 1-m-\lambda) = R_{0,k}^{m,0}(h ; \lambda),$$

so that (140) is equivalent to (138).

I will now apply formula (138) to MACDONALD's BESSEL function with imaginary argument $K_\nu(z)$ [42], to the BESSEL function of the second kind $Y_\nu(z)$ and to WHITTAKER's function $W_{k,m}(z)$ [43]. These functions may in the following way be expressed in terms of the G-function [44])

$$K_\nu(z) = \tfrac{1}{2}\, G_{0,2}^{2,0}(\tfrac{1}{4} z^2 \mid \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (141)$$

$$Y_\nu(z) = (-1)^t\, G_{1,3}^{2,0}\left(\tfrac{1}{4} z^2 \,\middle|\, \begin{matrix} -\tfrac{1}{2}\nu - t - \tfrac{1}{2} \\ \tfrac{1}{2}\nu, -\tfrac{1}{2}\nu, -\tfrac{1}{2}\nu - t - \tfrac{1}{2} \end{matrix}\right) \quad (t = 0, \pm 1, \pm 2, \ldots), \quad (142)$$

$$e^{-\frac{1}{2} z}\, W_{k,m}(z) = z^{\frac{1}{2}}\, G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \tfrac{1}{2} - k \\ m, -m \end{matrix}\right) \quad . \quad . \quad . \quad . \quad . \qquad (143)$$

Now we have by lemma 16, if $b_1 = -b_2 = \tfrac{1}{2}\nu$,

$$R_{0,2}^{2,0}(1 ; \lambda) = \frac{\sin(\lambda+1)\nu\pi}{\sin \nu\pi} \quad \text{and} \quad R_{0,2}^{2,0}(2 ; \lambda) = -\frac{\sin \lambda\nu\pi}{\sin \nu\pi}.$$

Hence we obtain, if we apply (138) with $m = k = 2$ to (141),

$$K_\nu(z) = \frac{\sin(\lambda+1)\nu\pi}{\sin \nu\pi}\, K_\nu(z e^{-\lambda\pi i}) - \frac{\sin \lambda\nu\pi}{\sin \nu\pi}\, K_\nu(z e^{-(\lambda+1)\pi i});$$

this formula, wherein λ is an arbitrary integer, occurs in a somewhat other form by WATSON [45]; it may be used to obtain asymptotic expansions for $K_\nu(z)$ for large values of $|z|$ with $|\arg z| \geqq \tfrac{3}{2}\pi$ [46]).

Similarly we find, if we apply (138) to (142),

$$Y_\nu(z) = \frac{\sin(\lambda+1)\nu\pi}{\sin \nu\pi}\, Y_\nu(z e^{-\lambda\pi i}) - \frac{\sin \lambda\nu\pi}{\sin \nu\pi}\, Y_\nu(z e^{-(\lambda+1)\pi i}).$$

The corresponding relation for $W_{k,m}(z)$ follows from (138) and (143) [47])

$$W_{k,m}(z) = (-1)^\lambda \left\{ \frac{\sin 2(\lambda+1)m\pi}{\sin 2m\pi}\, W_{k,m}(z e^{-2\lambda\pi i}) + \frac{\sin 2\lambda m\pi}{\sin 2m\pi}\, W_{k,m}(z e^{-2(\lambda+1)\pi i}) \right\}. \quad (144)$$

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[42]) WATSON, [31], 78. The function $K_\nu(z)$ is equal to $\tfrac{1}{2}\pi i e^{\frac{1}{2}\nu\pi i} H_\nu^{(1)}(z e^{\frac{1}{2}\pi i})$, where $H_\nu^{(1)}(z)$ is the first HANKEL function.

[43]) WHITTAKER and WATSON, [32], chapter XVI.

[44]) Comp. [21], 205—206 and [22], 187.

[45]) WATSON, [31], 75, formula (5); comp. also DEBYE, [7], 6.

[46]) The well-known asymptotic expansion

$$K_\nu(z) \backsim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\, {}_2F_0\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu ; -\frac{1}{2z}\right)$$

is valid for $|\arg z| < \tfrac{3}{2}\pi$ (comp. WATSON, [31], 202—203).

[47]) The numbers k and m in (143) are, of course, not the same as those in (138)

§ 16. The particular cases with $l = 1$, $k = q$ of the expansion formulae (102), (113), (116) and (137).

The most important particular cases of the expansion formulae (102), (113), (116) and (137) are those with $l = 1$, $k = q$. The right-hand sides may then because of (10) and (11) be written in a somewhat simpler form. Theorem 1 gives theorem 7; theorem 2 gives theorem 8 A. Theorem 8 B is conjugate to theorem 8 A. Theorem 9 is furnished by theorem 3. Theorem 10 is a particular case of theorem 5 $(l = 1,\ k = q,\ \nu = p)$.

Theorem 7. Assumptions: *m, n, p and q are integers with*

$$1 \leqq n \leqq p \leqq q,\ 1 \leqq m \leqq q \text{ and } m + n \geqq q + 1;$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions*

$$a_j - b_h \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots, n;\ h = 1, \ldots, m), \quad . \quad . \quad (1)$$

$$a_j - a_t \neq 0, \pm 1, \pm 2, \ldots \qquad (j = 1, \ldots, n;\ t = 1, \ldots, n;\ j \neq t); \quad (20)$$

λ *is an arbitrary integer which satisfies the inequality*

$$0 \leqq \lambda \leqq m + n - q - 1.$$

Assertion:

$$G^{m,n}_{p,q}(z) = \sum_{t=1}^{n} e^{(m+n-q-2\lambda-1)\pi i a_t} \triangle^{m,n}_{q}(t)\, G^{q,1}_{p,q}(z\, e^{(q-m-n+2\lambda+1)\pi i} \,||\, a_t). \quad (145)$$

Theorem 8 A. Assumptions: *m, n, p and q are integers with*

$$q \geqq 1,\ 0 \leqq n \leqq p \leqq q \text{ and } 0 \leqq m \leqq q; \quad . \quad . \quad . \quad (146)$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequality*

$$r \geqq \operatorname{Max}(0, q - m - n + 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (147)$$

Assertion:

$$\left.\begin{aligned} G^{m,n}_{p,q}(z) = A^{m,n}_{q} \sum_{s=0}^{r-1} \Omega^{m,n}_{q}(s)\, G^{q,0}_{p,q}(z\, e^{(q-m-n-2s)\pi i}) \\ + \sum_{t=1}^{n} e^{(m+n-q+2r-1)\pi i a_t} \triangle^{m,n}_{q}(t)\, G^{q,1}_{p,q}(z\, e^{(q-m-n-2r+1)\pi i} \,||\, a_t). \end{aligned}\right\} \quad (148)$$

Theorem 8 B. Assumptions: *m, n, p and q are integers which satisfy the conditions* (146);
the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequality* (147).

Assertion:

$$\left.\begin{aligned} G^{m,n}_{p,q}(z) = \bar{A}^{m,n}_{q} \sum_{s=0}^{r-1} \bar{\Omega}^{m,n}_{q}(s)\, G^{q,0}_{p,q}(z\, e^{(m+n-q+2s)\pi i}) \\ + \sum_{t=1}^{n} e^{-(m+n-q+2r-1)\pi i a_t} \triangle^{m,n}_{q}(t)\, G^{q,1}_{p,q}(z\, e^{(m+n-q+2r-1)\pi i} \,||\, a_t). \end{aligned}\right\} \quad (149)$$

Theorem 9. Assumptions: *m, n, p and q are integers with*

$$q \geqq 1,\ 0 \leqq n \leqq p \leqq q,\ 0 \leqq m \leqq q \text{ and } m + n \leqq q + 1;$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); r *is an arbitrary integer which satisfies the inequality*

$$0 \leqq r \leqq q - m - n + 1.$$

Assertion:

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = {} & A^{m,n}_{\ q} \sum_{s=0}^{r-1} \Omega^{m,n}_{\ q}(s)\, G_{p,q}^{q,0}(z\, e^{(q-m-n-2s)\pi i}) \\ & + A^{m,n}_{\ q} \sum_{\tau=0}^{q-m-n-r} \overline{\Omega}^{m,n}_{\ q}(\tau)\, G_{p,q}^{q,0}(z\, e^{(m+n-q+2\tau)\pi i}) \\ & + \sum_{t=1}^{n} e^{(m+n-q+2r-1)\pi i a_t} \triangle^{m,n}_{\ q}(t)\, G_{p,q}^{q,1}(z\, e^{(q-m-n-2r+1)\pi i} \,\|\, a_t). \end{aligned}\right\} \quad (150)$$

Theorem 10. Assumptions: *m, n, p and q are integers with*

$$q \geqq 1,\ 0 \leqq n \leqq p \leqq q \text{ and } 0 \leqq m \leqq q;$$

the numbers $a_1, \ldots, a_p$ *and* $b_1, \ldots, b_m$ *fulfil the conditions*

$$a_j - b_h \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots, n;\ h = 1, \ldots, m). \quad . \quad . \quad (1)$$

$$a_j - a_h \neq 0, \pm 1, \pm 2, \ldots \qquad (j = 1, \ldots, p;\ h = 1, \ldots, p;\ j \neq h); \quad (38)$$

μ *is an arbitrary integer which satisfies the inequality*

$$0 \leqq \mu \leqq q - p; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (151)$$

λ *is an arbitrary integer.*

Assertion:

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) = {} & \sum_{h=1}^{q-p-\mu} R_{p,q}^{m,n}(h;\lambda)\, G_{p,q}^{q,0}(z\, e^{(q-m-n-2h-2\lambda+2)\pi i}) \\ & + \sum_{\varkappa=1}^{\mu} \overline{R}_{p,q}^{m,n}(\varkappa; p-m-n-\lambda+1)\, G_{p,q}^{q,0}(z\, e^{(2p-q-m-n+2\varkappa-2\lambda)\pi i}) \\ & + \sum_{\sigma=1}^{p} e^{(q-p-2\mu)\pi i a_\sigma} T_{p,q}^{m,n}(\sigma;\lambda)\, G_{p,q}^{q,1}(z\, e^{(2p-q-m-n-2\lambda+2\mu+1)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \quad (152)$$

Mathematics. — *Lattice points in some n-dimensional non-convex regions.* I. By L. J. MORDELL. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 29, 1946.)

Let Λ be an n-dimensional lattice of determinant $\triangle > 0$ and denote by $(x_1, x_2, \ldots x_n)$, say (x), any point of Λ. A well-known theorem of MINKOWSKI states that an n-dimensional region R which is closed, convex, symmetrical about the origin O, $(x) = (0)$, and of volume $\geq 2^n \triangle$, contains a point other than O of Λ. No analogue of this result is known when R is non-convex or infinite, but crude results can be found by inscribing convex regions S in R. Thus if R is symmetrical for every reflexion in the x-planes, i.e. R is unaltered by writing $(\pm x_1, \ldots \pm x_n)$ for (x) in the equations defining R, then the tangent plane at a point $(\xi) = (\xi_1, \ldots \xi_n)$ on the boundary of R together with its reflexions forms an n-dimensional octahedron which in some obvious cases is a suitable inscribed convex region. In general, it does not seem too easy to improve the crude result obtained in this way. The first step in this direction was made by BLICHFELDT [1]) by aid of the following lemma due essentially to him but not stated in this form.

Lemma I.

Let K be any closed n-dimensional region of volume V_K and let K' be the region defined as the set of points $P_1 - P_2$ where P_1, P_2 are any points of K. Then K' contains a point of Λ other than O if $\triangle \leq V_K$.

Thus if K is the n-dimensional octahedron obtained from the one above by reducing its dimensions by $\frac{1}{2}$ and keeping its centre at O, K' is the original n-dimensional octahedron which has been supposed to lie in R. The problem is really to enlarge this K to S, say, by adding to it some appropriate region such that the new K', say S', will be still inscribed in R.

This was the idea that BLICHFELDT applied to the problem of simultaneous approximation when he proved

Theorem I.

If $\theta_1, \theta_2 \ldots \theta_n$ are any n real numbers, then an infinity of integers $X, X_1, \ldots X_n$ exist such that $X \neq 0$ and

$$|X^{1/n}(X_r - \theta_r X)| \leq \frac{n}{n+1}\left(1 + \left(\frac{n-1}{n+1}\right)^{n+3}\right)^{-1/n} \quad (r = 1, 2 \ldots n)$$

[1]) "A New Principle in the Geometry of Numbers with some Applications". Transactions of the American Mathematical Society, 15, 227—235 (1914).

Another application of BLICHFELDT's method was given a few years ago by KOKSMA [2]) and MEULENBELD who proved

Theorem II.

An infinity of points of Λ satisfy the inequality

$$(|x_1| + \ldots + |x_\alpha|)^\alpha (|x_{\alpha+1}| + \ldots + |x_n|)^\beta \leq \triangle/k,$$

where α is given $\leq \frac{1}{2} n$ and $\alpha + \beta = n$; and

$$\frac{n!\,k}{2^n} = \frac{1}{\beta^\beta} \sum_{s=0}^{\beta} \binom{n}{s} \left(\beta - \frac{n}{2}\right)^s (n-\beta)^{\beta-s} + \beta \binom{n}{\beta} \sum_{s=\beta+1}^{\infty} \frac{1}{s\,\beta^s} \left(\beta - \frac{n}{2}\right)^s.$$

Their proof is long and complicated.

BLICHFELDT's proof was rather condensed, and in studying it, I have been led to simple theorems, not without a little generality, for some n-dimensional regions. Thus I not only show that his simultaneous approximation result is a special case of a much more general result, and that this also applies to the result of KOKSMA and MEULENBELD, but I also find other results, e.g. of the form

$$|x_1|^\lambda + \ldots |x_n|^\lambda \leq k_1 \triangle^{\lambda/n}, \text{ where } 0 < \lambda < 1,$$

$$\sum_{(r)}^{n} |x_1 x_2 \ldots x_r| \leqq k_2 \triangle^{r/n},$$

where the summation refers to the combinations of $x_1, x_2 \ldots x_n$ taken r at a time.

1. I commence with the two-dimensional case. Let $\bar{R}$ be a region in the first quadrant defined by the inequalities

$$x \geqslant 0, \; y \geqslant 0, \; y \leqslant f(x),$$

where for $x > 0$, $f(x) \geqq 0$ and is a single-valued differentiable function of x. We suppose also that for $x > 0$, $f(x)$ is steadily decreasing, and $f'(x)$ is steadily increasing, as e.g. when the first quadrant region outside $\bar{R}$ is convex.

We now define the region R by $|y| \leq f(|x|)$. We seek for numbers $\triangle_R$ such that all lattices Λ of determinant $\triangle \leq \triangle_R$ should have a point other than O in R. We do not find the best possible results. The values given for $\triangle_R$ are permissable results greater than the obvious ones. A trivial result is easily found as noted above. Thus let $P(a, b)$ be a first quadrant point on $y = f(x)$, and let the tangent at P make intercepts p, q on the axes of x, y. This tangent and its images in the axes define a parallelogram of area $2pq$ inscribed in R. Hence if $2\triangle_R = pq$, the

[2]) "Sur le théorème de MINKOWSKI concernant un système de formes linéaires réelles", Proc. Ned. Akad. v. Wetensch., Amsterdam, 5, 256—362, 354—359, 471—478, 578—584 (1942).

parallelogram, and so the region R, contains a point of Λ other than O. This result can be improved, i.e. a larger value for $\triangle_R$ can be found, by taking P so that pq is a maximum.

Our object, however, is to improve the result in the way suggested above. We first prove

Lemma II.

Either $p < 2a$ or $q < 2b$ for general P except when $f(x) = c/x$, when c is a constant, and then $p = 2a$, $q = 2b$.

For if not, $p \geq 2a$, $q \geq 2b$, and so $p = 2a$, $q = 2b$ since

$$a/p + b/q = 1.$$

The tangent at $P(a, b)$ is

$$y - f(a) = f'(a)(x - a)$$

and so

$$q = f(a) - a f'(a).$$

Then $f(a) + a f'(a) = 0$, i.e. $d\,[a f(a)]\,da = 0$.

Hence by a change of notation if need be, we may suppose $b \geq \frac{1}{2}q$. We then define S as $\bar{S}$, the aggregate of two regions

$$\text{I.}\quad 0 \leqslant x \leqslant a,\ y \geqslant 0,\ x/p + y/q \leqslant \tfrac{1}{2}.$$
$$\text{II.}\quad x \geqslant a,\ y \geqslant 0,\ y + q/2 \leqslant f(x),$$

together with the three images of S in the axes. We prove

Theorem III.

$$\triangle_R = \tfrac{1}{2}\,pq + 4 \iint dx\,dy,$$

where the integral is taken over the region

$$x/p + y/q \geqslant \tfrac{1}{2},\ y \geqslant 0,\ x \geqslant a,\ y + \tfrac{1}{2}\,q \leqslant f(x).$$

When $b = \frac{1}{2}q$, i.e. $f(x) = c/x$, II is merely the point $x = a$, $y = 0$, the integral is zero, and $\triangle_R$ is no improvement on the crude result $\frac{1}{2}pq$.

We can also write Theorem III in the form

Theorem III′

$$\triangle_R = 4\,(V_1 + V_2),$$

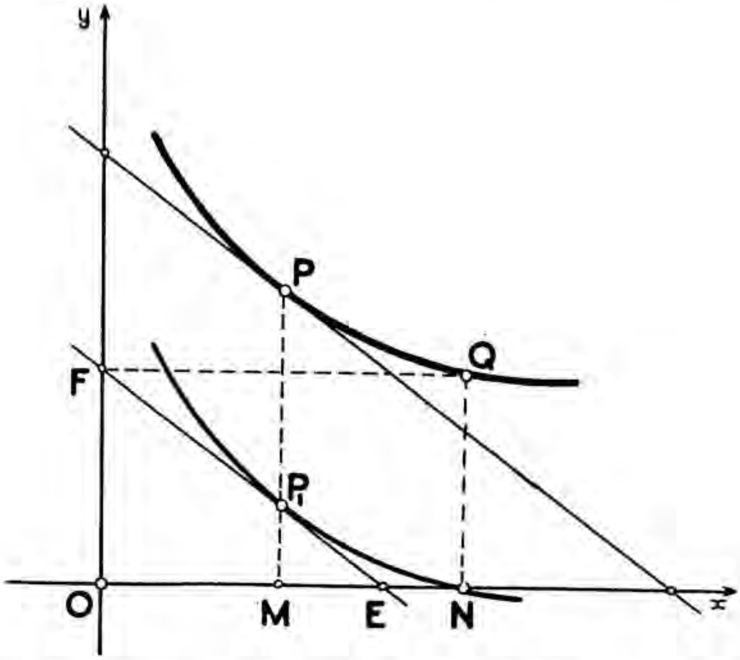
where $V_1 = \iint dx\,dy$ *over* $0 \leqslant x \leqslant a,\ y \geqslant 0,\ x/p + y/q \leqslant \frac{1}{2}$,

$$V_2 = \iint dx\,dy \text{ over } x \geqslant a,\ y \geqslant 0,\ y + q/2 \leqslant f(x).$$

It is clear that if $a \geq \frac{1}{2}p$, theorem III′ holds if we take

for V_1, $0 \leqslant y \leqslant b,\ x \geqslant 0,\ x/p + y/q \leqslant \frac{1}{2}$, and
for V_2, $y \geqslant b,\ x \geqslant 0,\ y \leqslant f(x + p/2)$.

The figure shows the geometrical meaning of I, II. The curve $y + q/2 = f(x)$ meets the ordinate MP at P_1, an inner point of MP since



$b > \frac{1}{2}q$, and touches EF the line $x/p + y/q = \frac{1}{2}$ at the point P_1, and meets the axis of x at N where $ON = \xi$ and $q/2 = f(\xi)$; so that the ordinate NQ at N equals $OF = q/2$. Then I is the trapezium OMP_1F and II is the curvilinear triangle P_1MN.

Let now [3] $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be any two points of S. To show that S' lies in R, i.e. $(x_1 - x_2, y_1 - y_2)$ lies in R, we have to prove that

$$|y_1 - y_2| \leqslant f(|x_1 - x_2|).$$

This is true if

$$|y_1| + |y_2| \leqslant f(|x_1 - x_2|),$$

or since $f(|x|)$ is a decreasing function of $|x|$ if

$$|y_1| + |y_2| \leqslant f(|x_1| + |x_2|).$$

Hence we need only prove that if P_1, P_2 are two points of $\bar{S}$, then $P_1 + P_2$ lies in $\bar{R}$.

Suppose first that P_1, P_2 lie in II so that

$$x_1 \geqslant a,\ y_1 \geqslant 0,\ x_2 \geqslant a,\ y_2 \geqslant 0,\ y_1 + q/2 \leqslant f(x_1),\ y_2 + q/2 \leqslant f(x_2).$$

We have to prove that

$$y_1 + y_2 \leqslant f(x_1 + x_2).$$

[3]) Not the same as P_1 above.

It suffices if, for $x_1 \geq a$, $x_2 \geq a$,

$$F \equiv f(x_1) + f(x_2) - q - f(x_1 + x_2) \leqslant 0.$$

Now

$$\partial F/\partial x_1 = f'(x_1) - f'(x_1 + x_2) \leqslant 0,$$

since $f'(x)$ is an increasing function of x.

Hence F is a decreasing function of x_1, and similarly of x_2, and so has its greatest value when $x_1 = x_2 = a$. It remains then to prove that

$$F(a) = 2f(a) - q - f(2a) \leqslant 0.$$

Now

$$F(a) = f(a) + af'(a) - f(2a).$$

Since

$$f(2a) = f(a) + af'(a + \theta a), \qquad (0 < \theta < 1)$$
$$F(a) = af'(a) - af'(a + \theta a) \leqslant 0.$$

Suppose next that P_1 lies in I and P_2 in II, so that

$$x_1 \geqslant 0,\ y_1 \geqslant 0,\ x_1/p + y_1/q \leqslant \tfrac{1}{2},\ x_2 \geqslant a,\ y_2 \geqslant 0,\ y_2 + q/2 \leqslant f(x_2).$$

It suffices now to show that

$$G \equiv q(\tfrac{1}{2} - x_1/p) + f(x_2) - q/2 - f(x_1 + x_2) \leqslant 0.$$

Here

$$\partial G/\partial x_1 = -q/p - f'(x_1 + x_2) \leqslant -q/p - f'(a) = 0.$$

Hence G is a decreasing function of x_1, and, as before, of x_2, and so G attains its greatest value when $x_1 = 0$, $x_2 = a$, and then $G = 0$.

Suppose finally that P_1, P_2 lie in I. Then

$$x_1 \geqslant 0,\ y_1 \geqslant 0,\ x_2 \geqslant 0,\ y_2 \geqslant 0,\ x_1/p + y_1/q \leqslant \tfrac{1}{2},\ x_2/p + y_2/q \leqslant \tfrac{1}{2}.$$

Then by addition

$$x_1 + x_2 \geqslant 0,\ y_1 + y_2 \geqslant 0,\ (x_1 + x_2)/p + (y_1 + y_2)/q \leqslant 1,$$

and so the point $P_1 + P_2$ lies in that part of $\bar{R}$ for which $x/p + y/q \leq 1$.

Hence we have constructed a suitable S' on taking S as defined. Clearly we have for V_S, the area of S,

$$V_S = \tfrac{1}{2}pq + 4\iint dx\,dy,$$

the integral being taken over $x/p + y/q \geq \frac{1}{2}$, $y + q/2 \leq f(x)$, $y \geq 0$, $x \geq a$, or again

$$V_S = 4(V_1 + V_2),$$

where

$$V_1 = \iint dx\,dy \text{ over } 0 \leqslant x \leqslant a,\ y \geqslant 0,\ x/p + y/q \leqslant \tfrac{1}{2},$$

$$V_2 = \iint dx\,dy \text{ over } x \geqslant a,\ y \geqslant 0,\ y + q/2 \leqslant f(x).$$

2. The two-dimensional result has an immediate extension to problems of simultaneous approximation. For the simplest case, let now Λ be an $n+1$-dimensional lattice and let R be the $n+1$-dimensional region

$$|x_1| \leqq f(|x|),\ |x_2| \leqq f(|x|), \dots |x_n| \leqq f(|x|),$$

where $f(x)$ is as before. Then $\overline{R}$ is defined as

$$x \geqq 0,\ x_r \geqq 0,\ x_r \leqq f(x), \qquad (r = 1, 2, \dots n).$$

Let P be any point on the boundary of R given by

$$(x, x_1, \dots x_n) = (a, b, \dots b),$$

where $a > 0$, $b > 0$, $b = f(a)$. We define p, q as before and suppose that $q < 2b$ and so $p > 2a$.

The region $\overline{S}$ is now defined as the aggregate of

$$\text{(I)} \quad 0 \leqq x \leqq a,\ x_r \geqq 0,\ x/p + x_r/q \leqq \tfrac{1}{2} \qquad (r = 1, 2, \dots n),$$

$$\text{(II)} \qquad x \geqq a,\ x_r \geqq 0,\ x_r + \tfrac{1}{2} q \leqq f(x) \qquad (r = 1, 2, \dots n).$$

Clearly S' lies in R from the result of § 1.

Denote by V_1, V_2 the volumes of I, II. Then

$$V_1 = \int\int \dots \int dx\, dx_1 \dots dx_n \text{ over I}$$

$$= q^n \int_0^a (\tfrac{1}{2} - x/p)^n\, dx$$

$$= [q^n p/(n+1)]\,[1/2^{n+1} - (\tfrac{1}{2} - a/p)^{n+1}].$$

Also

$$V_2 = \int_a^{f(x) = \frac{1}{2} q} (f(x) - \tfrac{1}{2} q)^n\, dx.$$

Hence

$$\frac{V_S}{2^{n+1}} = \frac{q^n p}{n+1} \left\{ \frac{1}{2^{n+1}} - \left(\frac{1}{2} - \frac{a}{p} \right)^{n+1} \right\} + \int_a^{f(x) = \frac{1}{2} q} [f(x) - \tfrac{1}{2} q]^n\, dx,$$

and then

$$\triangle_R = V_S.$$

We take now the special case $f(x) = x^{-1/n}$ $(n > 1)$ and so $b = a^{-1/n}$. The tangent at (a, b) to $x_1 = f(x)$ is

$$\frac{x}{(n+1)\,a} + \frac{n x_1}{(n+1)\,b} = 1,$$

and so $p = (n+1)a$, $q = (1 + 1/n)b$, and so the condition $q < 2b$ is satisfied.

Hence

$$V_1=\left(1+\frac{1}{n}\right)^n\left[\frac{1}{2^{n+1}}-\left(\frac{n-1}{2n+2}\right)^{n+1}\right];$$

$$V_2=\int_a^{x^{-1/n}=\frac{1}{2}q}\left[x^{-1/n}-\left(\frac{n+1}{2n}\right)b\right]^n dx.$$

Put $x=b^{-n}u^{-n}$, then

$$V_2=-\int_1^{(n+1)/2n}\left(u-\frac{n+1}{2n}\right)^n n\,u^{-n-1}\,du.$$

Put $2u=(1+1/n)\,(1+w)$, then

$$V_2=n\int_0^{\frac{n-1}{n+1}}\frac{w^n}{(1+w)^{n+1}}\,dw.$$

Put $w/(1+w)=v$. Then

$$\frac{V_2}{n}=\int_0^{\frac{n-1}{2n}}\frac{v^n}{1-v}\,dv$$

$$=\frac{1}{n+1}\left\{\frac{n-1}{2n}\right\}^{n+1}+\int_0^{\frac{n-1}{2n}}\frac{v^{n+1}}{1-v}\,dv.$$

Hence

$$\triangle_R=2^{n+1}(V_1+V_2)$$

$$=(1+1/n)^n+2^{n+1}n\int_0^{\frac{n-1}{2n}}\frac{v^{n+1}}{1-v}\,dv.$$

This result is due to Blichfeldt, who gave it in a slightly different form (using the w integral for V_2 in V_1+V_2).

We can also take for $\triangle_R$ any number less than the right-hand side, e.g.

$$\triangle_R=(1+1/n)^n+\frac{2^{n+1}n}{n+2}\left(\frac{n-1}{2n}\right)^{n+2}$$

$$=(1+1/n)^n\,[1+\delta],$$

where

$$\delta=\frac{(n-1)^{n+2}}{2n\,(n+2)\,(n+1)^n}$$

BLICHFELDT gives as his estimate from the integral the following value which is better i.e. larger, except when $n = 2, 3$

$$\delta = \left\{\frac{n-1}{n+1}\right\}^{n+3}.$$

I owe to Prof. DAVENPORT a better approximation to the integral, namely,

$$J = \int_0^{\frac{n-1}{2n}} \frac{v^{n+1}}{1-v}\, dv > \frac{1}{(2\,n)^{n+1}}\, \frac{(n-1)^{n+2}\,(n+3)}{(n+2)\,(n^2+5\,n+2)}.$$

For

$$J = \left\{\frac{n-1}{2n}\right\}^{n+2} \left[\frac{1}{n+2} + \frac{1}{n+3}\left\{\frac{n-1}{2n}\right\} + \frac{1}{n+4}\left\{\frac{n-1}{2n}\right\}^2 + \dots\right].$$

Since for $r \geq 2$,

$$\frac{n+r}{n+2} \leqslant \left\{\frac{n+3}{n+2}\right\}^{r-2} = \left[1 + \frac{1}{n+2}\right]^{r-2},$$

we have

$$J > \left[\frac{n-1}{2n}\right]^{n+2} \left[\frac{1}{n+2}\right] \frac{1}{\left\{1 - \frac{n+2}{n+3}\,\frac{n-1}{2n}\right\}}$$

$$> \frac{1}{(2\,n)^{n+1}}\, \frac{(n-1)^{n+2}\,(n+3)}{(n+2)\,(n^2+5\,n+2)},$$

and so

$$\triangle_R = (1 + 1/n)^n \left[1 + \frac{(n-1)^{n+2}\,(n+3)}{(n+1)^n\,(n+2)\,(n^2+5\,n+2)}\right].$$

This is better than BLICHFELDT's result since

$$\frac{n+3}{(n+2)\,(n^2+5\,n+2)} > \frac{n-1}{(n+1)^3}$$

i.e. $n^4 + 6\,n^3 + 12\,n^2 + 10\,n + 3 > n^4 + 6\,n^3 + 5\,n^2 - 8\,n - 4$.
Hence at least one point $(x, x_1, \dots x_n) \neq (0)$ of every lattice Λ with determinant $\triangle \leq \triangle_R$, satisfies the simultaneous inequalities

$$|\, x^{1/n}\, x_r \,| \leqslant 1, \qquad (r = 1, 2, \dots n).$$

In fact, there are an infinity of solutions since a and b may be any positive numbers satisfying $a^{1/n}\, b = 1$. Thus if we take a very great and b small, all the x_r, $(r = 1, 2, \dots n)$ will be arbitrarily small from the definitions of the regions I and II. Consideration of homogeneity shows also that an infinity of points of every lattice Λ of determinant $\triangle$ satisfy the inequalities

$$|\, x^{1/n}\, x_r \,| \leqslant (\triangle/\triangle_R)^{1/n}, \qquad (r = 1, 2, \dots n),$$

From the special lattice

$$x = X, \; x_r = X_r - \theta_r X \qquad (r = 1, 2, \dots n)$$

where θ_r is real and the X's are integers, we see that there are an infinity of integer solutions of

$$| X^{1/n} (X_r - \theta_r X) | \leqslant k, \qquad (r = 1, 2, \dots n)$$

where

$$k = \left\{ \frac{n}{n+1} \right\} \Big/ \left[1 + \frac{(n+3)(n-1)^{n+2}}{(n+1)^n (n^2 + 5n + 2)(n+2)} \right]^{\frac{1}{n}}.$$

Further $X \neq 0$ for these solutions. For if $X = 0$, then on taking b sufficiently small, for every point $(x, x_1, \dots x_n)$ of the region S, $x_r = o(b)$, i.e. $X_r = 0$. $(r = 1, 2, \dots n)$.

Mathematics. — *Lattice points in some n-dimensional non-convex regions.* II. By L. J. MORDELL. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 29, 1946.)

3. We now extend Theorem I to n-dimensional regions. It will suffice to prove the result for three dimensions.

Let R be the region

$$|z| \leqslant f(|x|, |y|),$$

and then $\bar{R}$ is the first octant region defined by

$$x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0,\ z \leqslant f(x, y).$$

We suppose that for $x > 0$, $y > 0$, $f(x, y) \geq 0$ and is a steadily decreasing function of x and y, and is a differentiable function of x, y, that $\partial f/\partial x$ is a steadily increasing function of x and y separately, and similarly for $\partial f/\partial y$.

Then also $\partial f(a, b)/\partial a \leq \partial f(x, y)/\partial x$ if $x \geq a > 0$, $y \geq b > 0$, for

$$\left(\frac{\partial f(x, y)}{\partial x} - \frac{\partial f(x, b)}{\partial x}\right) + \left(\frac{\partial f(x, b)}{\partial x} - \frac{\partial f(a, b)}{\partial a}\right) \geqslant 0$$

since each bracket ≥ 0.

Write the equation to the tangent plane at (a, b, c), a first octant point on $z = f(x, y)$, namely

$$z - c = \frac{\partial f(a, b)}{\partial a}(x - a) + \frac{\partial f(a, b)}{\partial b}(y - b),$$

as

$$x/p + y/q + z/r = 1,$$

so that

$$r = f(a, b) - a\,\frac{\partial f(a, b)}{\partial a} - b\,\frac{\partial f(a, b)}{\partial b}.$$

We *postulate* that $c > \tfrac{1}{2}r$, i.e.

$$f(a, b) + a\,\frac{\partial f(a, b)}{\partial a} + b\,\frac{\partial f(a, b)}{\partial b} > 0.$$

We now define $\bar{S}$ as the aggregate of the two regions

I $\quad x \geq 0,\ y \geq 0,\ z \geq 0,\ x/p + y/q + z/r \leq \tfrac{1}{2},$

II $\quad x \geq a,\ y \geq b,\ z \geq 0,\ x/p + y/q + z/r \geq \tfrac{1}{2},\ z + r/2 \leq f(x, y),$

and S is composed of $\bar{S}$ together with the seven regions obtained from $\bar{S}$ by replacing x, y, z by $\pm x$, $\pm y$, $\pm z$.

We have to show that for any two points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ of S

$$z_1 + z_2 \leq f(x_1 + x_2, y_1 + y_2).$$

This is obvious when P_1, P_2 lie in the region I.

Suppose next that P_1, P_2 lie in the region II. It suffices to prove that

$$F \equiv f(x_1, y_1) + f(x_2, y_2) - r - f(x_1 + x_2, \; y_1 + y_2) \leq 0.$$

Now

$$\frac{\partial F}{\partial x_1} = \frac{\partial f(x_1, y_1)}{\partial x_1} - \frac{\partial f(x_1 + x_2, y_1 + y_2)}{\partial x_1} \leqslant 0.$$

Hence F is a steadily decreasing function of each of x_1, x_2, y_1, y_2, and since $x_1 \geq a$, $x_2 \geq a$, $y_1 \geq b$, $y_2 \geq b$,

$$\begin{aligned} F &\leqslant 2f(a, b) - f(a, b) + a\frac{\partial f(a, b)}{\partial a} + b\frac{\partial f(a, b)}{\partial b} - f(2a, 2b) \\ &= f(a, b) + a\frac{\partial f(a, b)}{\partial a} + b\frac{\partial f(a, b)}{\partial b} \\ &\quad - \left[f(a, b) + a\frac{\partial f(a', b')}{\partial a'} + b\frac{\partial f(a', b')}{\partial b'} \right], \end{aligned}$$

where $a' = a + \theta a \geq a$, $b' = b + \theta b \geq b$, $0 < \theta < 1$ and so $F \leq 0$.

Suppose next that P_1 lies in the region I and that P_2 lies in the region II. It suffices to prove now that

$$G = r(\tfrac{1}{2} - x_1/p - y_1/q) + f(x_2, y_2) - \tfrac{1}{2}r - f(x_1 + x_2, \; y_1 + y_2) \leq 0.$$

Here

$$\begin{aligned} \partial G/\partial x_1 &= -r/p - \partial f(x_1 + x_2, \; y_1 + y_2)/\partial x_1 \\ &= \partial f(a, b)/\partial a - \partial f(x_1 + x_2, \; y_1 + y_2)/\partial x_1 \leq 0 \end{aligned}$$

since $x_2 \geq a$, $y_2 \geq b$. Hance G is a decreasing function of each of x_1, y_1, and as before of each of x_2, x_2. Since $x_1 \geq 0$, $y_1 \geq 0$, we have $G \leq 0$.

Hence we have constructed a suitable S' by taking S as defined above. The volume of S is given by

$$V_S = (pqr)/3! + 8\,V_1,$$

where V_1 is the volume bounded by

$$x/p + y/q + z/r \geq \tfrac{1}{2}, \;\; x \geq a, \;\; y \geq b, \;\; z + r/2 \leq f(x, y).$$

Thus we have improved the crude result

$$\triangle_R = (pqr)/3! \text{ to } \triangle_R = (pqr)/3! + 8\,V_1.$$

For special forms of $f(x, y)$, this result can also be improved. Suppose, for instance that

$$f(x, y) = f(x + y),$$

where $f(X)$ is a decreasing and $f'(X)$ an increasing function of X. Then in (II) we can replace $x \geq a$, $y \geq b$ by $x + y \geq a + b$, since the proof clearly still holds. The result then takes the form

$$\triangle_R = (p^2 r)/3! + 8 \int\int\int dx\, dy\, dz$$

taken over $(x + y)/p + z/r \geq \frac{1}{2}$, $x + y \geq a + b$, $z + r/2 \leq f(x + y)$. On writing $x + y = X$, $a + b = A$, and integrating for x, y, i.e. using $\int\int dx\, dy = \int X\, dX$, we now have

$$\triangle_R = (P^2 Q)/3! + 8 \int\int X\, dX\, dY$$

taken over $X/P + Y/Q \geq \frac{1}{2}$, $X \geq A$, $Y \geq 0$, $Y + Q/2 \leq f(X)$, where $X/P + Y/Q = 1$ is the tangent at (A, B) on $Y = f(X)$. This can also be written as

$$\triangle_R = 8\,(V_1 + V_2)$$

where

$$V_1 = \int\int X\, dX\, dY \text{ over } 0 \leqslant X \leqslant A,\ Y \geqslant 0,\ X/P + Y/Q \leqslant \tfrac{1}{2},$$

$$V_2 = \int\int X\, dX\, dY \text{ over } X \geqslant A,\ Y \geqslant 0,\ Y + Q/2 \leqslant f(X).$$

4. A similar proof applies to the corresponding n-dimensional result. It suffices to state the

Theorem IV. *Let R be the n-dimensional region defined by*

$$|x_n| \leqslant f(|x_1|, \ldots, |x_{n-1}|),$$

where for $x_1 > 0, \ldots, x_{n-1} > 0$,

(*a*) *$f(x_1, \ldots x_{n-1}) \geq 0$, f is a steadily decreasing function of each variable, and is a differentiable function of all the variables.*

(*b*) $\dfrac{\partial f(x_1, \ldots x_{n-1})}{\partial x_1}, \ldots \dfrac{\partial f(x_1, \ldots x_{n-1})}{\partial x_{n-1}}$ *are all steadily increasing functions of each variable separately.*

(*c*) *Let $(\xi_1, \ldots \xi_n)$ be any point on $x_n = f(x_1, \ldots x_{n-1})$ with*

$$\xi_1 > 0, \ldots \xi_n > 0$$

with hypertangent plane

$$x_1/p_1 + \ldots + x_n/p_n = 1;$$

then we p o s t u l a t e *that $p_n \leq 2\,\xi_n$.*

Then the region R contains a point not O of every n-dimensional lattice Λ of determinant $\triangle > 0$ when

$$\triangle \leq (p_1 p_2 \ldots p_n)/n! + 2^n V.$$

where V is the volume of the region

$$x_1/p_1 + \dots x_n/p_n \geq \tfrac{1}{2},\ x_1 \geq \xi_1 \dots x_{n-1} \geq \xi_{n-1},\ x_n \geq 0,$$
$$x_n + \xi_n/2 \leq f(x_1, x_2, \dots x_{n-1}).$$

These results can be simplified when sets of the variables in

$$x_n \leq f(x_1, \dots x_{n-1})$$

occur combined only in their sums.
Thus suppose $\alpha, \beta, \dots \varkappa, \lambda$ are positive integers such that

$$\alpha + \beta + \dots + \varkappa + \lambda = n,$$

and suppose that

$$x_{n-\lambda+1} + \dots + x_n \leqslant f(x_1 + \dots + x_\alpha, x_{\alpha+1} + \dots + x_{\alpha+\beta}, \dots,$$
$$x_{\alpha+\beta+\dots+1} + \dots + x_{n-\lambda}).$$

Write

$$Y_\alpha = x_1 + \dots + x_\alpha,\ Y_\varkappa = x_{\alpha+\beta+\dots+1} + x_{n-\lambda},\ Y_\lambda = x_{n-\lambda+1} + \dots + x_n,$$
$$\eta_\alpha = \xi_1 + \dots + \xi_\alpha,\ \eta_\varkappa = \xi_{\alpha+\beta+\dots+1} + \dots,\ \eta_\lambda = \xi_{n-\lambda+1} + \dots$$

We can replace the limits of integration $x_1 \geq \xi_1, \dots x_\alpha \geq \xi_\alpha$ by $Y_\alpha \geq \eta_\alpha$, and the integral

$$\int\int\dots\int dx_1 \dots dx_\alpha \text{ by } \int (Y_\alpha^{\alpha-1}\, d\, Y_\alpha)/(\alpha-1)!.$$

Similarly when α is replaced by $\beta, \dots \varkappa$.

For the variable Y_λ we have $Y_\lambda \geq \xi_{n-\lambda+1} + \dots \xi_{n-1}$. We take

$$\xi_{n-\lambda+1} = \dots = \xi_{n-1} = 0, \text{ and so } Y_\lambda \geqslant 0.$$

We now have

$$Y_\lambda \leq f(Y_\alpha, \dots Y_\varkappa)$$

with a point $(\eta_\alpha, \dots \eta_\varkappa, \eta_\lambda)$ and the tangent plane at $(\eta_\alpha, \dots \eta_\lambda)$ making intercepts $q_\alpha, \dots q_\lambda$ (where $q_\alpha = p_1 = \dots = p_\alpha$) on the Y axis and where by hypothesis $q_\lambda < 2\,\eta_\lambda$. Then the result of Theorem IV holds for

$$\triangle \leqslant \frac{q_\alpha^\alpha \dots q_\lambda^\lambda}{\alpha!\,\beta!\dots\lambda!} + 2^n\, V,$$

where

$$V = \int\int\dots\int (Y_\alpha^{\alpha-1})/(\alpha-1)! \dots (Y_\lambda^{\lambda-1})/(\lambda-1)!\, dY_\alpha \dots dY_\lambda$$

over

$$Y_\alpha/q_\alpha + \dots Y_\lambda/q_\lambda \geqslant \tfrac{1}{2},\ \ Y_\alpha \geqslant \eta_\alpha, \dots Y_\varkappa \geqslant \eta_\varkappa,\ \ Y_\lambda \geqslant 0,$$
$$Y_\lambda + q_\lambda/2 \leqslant f(Y_\alpha, \dots Y_\varkappa).$$

Suppose in particular that the x's split into two sets so that now $\alpha \geq 1$, $\beta \geq 1$, $\alpha + \beta = n$, and let $\beta \geq \frac{1}{2}n$. We then define $\bar{R}$ by

$$x_{\alpha+1} + \dots + x_n = f(x_1 + \dots + x_\alpha).$$

Consider now the first quadrant curve of § 1, namely $Y = f(X)$ with (A, B) any point on it, and P, Q the intercepts of the tangent at (A, B) on the X, Y axes. We suppose that $Q < 2B$, that $f(X)$ is a decreasing function of X and that $f'(X)$ is an increasing function of X.

Write A for η_α, B for η_β, P for $q_\alpha = p_\alpha$, Q for $q_\beta = p_\beta$. Then

$$\Delta_R = \frac{P^\alpha Q^\beta}{\alpha!\,\beta!} + 2^n V,$$

where

$$V = \frac{1}{(\alpha-1)!\,(\beta-1)!}\int\int X^{\alpha-1}\, Y^{\beta-1}\, dX\, dY$$

taken over $X/P + Y/Q \geq \frac{1}{2}$, $Y + Q/2 \leq f(X)$, $X \geq A$, $Y \geq 0$. We can also write

$$\Delta_R \leqslant 2^n (V_1 + V_2),$$

where

$$V_1 = \frac{1}{(\alpha-1)!\,(\beta-1)!}\int\int X^{\alpha-1}\, Y^{\beta-1}\, dX\, dY,$$

taken over $0 \leq X \leq A$, $Y \geq 0$, $X/P + Y/2 \leq \frac{1}{2}$, and

$$V_2 = \frac{1}{(\alpha-1)!\,(\beta-1)!}\int\int X^{\alpha-1}\, Y^{\beta-1}\, dX\, dY$$

taken over $x \geq A$, $Y \geq 0$, $y + Q/2 \leq f(X)$.

This result gives a simple proof of Theorem II. For take

$$f(X) = X^{-\alpha/\beta}, \text{ and so } A^\alpha B^\beta = 1,\; P = \frac{\alpha+\beta}{\alpha} A,\; Q = \frac{\alpha+\beta}{\beta} B.$$

Hence

$$V_2 = \frac{1}{(\alpha-1)!\,\beta!}\int_A^{f(X)=\frac{1}{2}Q} X^{\alpha-1}\left(X^{-\alpha/\beta} - \frac{\alpha+\beta}{2\beta} B\right)^\beta dX$$

$$= \frac{1}{(\alpha-1)!\,\beta!}\int_1^{X^{-\alpha/\beta}=(\alpha+\beta)/2\beta} X^{\alpha-1}\left(X^{-\alpha/\beta} - \frac{\alpha+\beta}{2\beta}\right)^\beta dX$$

Put $X = U^{-\beta/\alpha}$, then

$$V_2 = \frac{1}{\alpha!\,(\beta-1)!}\int_{(\alpha+\beta)/2\beta}^{1} U^{-\beta-1}\left(U - \frac{\alpha+\beta}{2\beta}\right)^\beta dU$$

$$= \frac{1}{\alpha!\,(\beta-1)!}\int_1^{2\beta/(\alpha+\beta)} U^{-\beta-1}\,(U-1)^\beta\, dU.$$

Put $U = 1/(1-V)$, then

$$V_2 = \frac{1}{\alpha!\,(\beta-1)!}\int_0^{(\beta-\alpha)/2\beta} \frac{V^\beta}{1-V}\,dV.$$

Next

$$V_1 = \frac{1}{(\alpha-1)!\,\beta!}\int_0^A X^{\alpha-1}\,(\tfrac{1}{2}-X/P)^\beta\,Q^\beta\,dX$$

$$= \frac{(\alpha+\beta)^\beta}{(\alpha-1)!\,\beta^\beta\,(\beta)!}\int_0^1 X^{\alpha-1}\,\{\tfrac{1}{2}-\alpha X/(\alpha+\beta)\}^\beta\,dX.$$

Then in Theorem II, $k = 2^n(V_1+V_2)$, and the first series in $k = \Sigma_1+\Sigma_2$, say Σ_1, arises from V_1, and the second, say Σ_2, arises from V_2. For we can write

$$\frac{\Sigma_2}{2^n} = \frac{1}{n!}\beta\,\frac{n!}{\alpha!\,\beta!}\sum_{s=\beta+1}^{\infty}\frac{1}{s\beta^s}\left(\frac{\beta-\alpha}{2}\right)^s$$

$$= \frac{1}{\alpha!\,(\beta-1)!}\sum_{s=\beta+1}^{\infty}\frac{1}{s}\left(\frac{\beta-\alpha}{2\beta}\right)^s = V_2,$$

on expanding the integrand in V_2 in powers of V.

Next

$$\frac{\Sigma_1}{2^n} = \frac{1}{\beta^\beta}\sum_{s=0}^{\beta}\frac{1}{s!\,(n-s)!}\left(\frac{\beta-\alpha}{2}\right)^s\alpha^{\beta-s}.$$

Also

$$V_1 = \frac{(\alpha+\beta)^\beta}{(\alpha-1)!\,\beta^\beta\,(\beta)!}\int_0^1 (1-X)^{\alpha-1}\left(\frac{\beta-\alpha}{2}+\alpha X\right)^\beta\frac{dX}{(\alpha+\beta)^\beta}$$

$$= \frac{1}{(\alpha-1)!\,\beta^\beta\,(\beta)!}\sum_{s=0}^{\beta}\int_0^1\left(\frac{\beta-\alpha}{2}\right)^s\alpha^{\beta-s}\binom{\beta}{s}(1-X)^{\alpha-1}X^{\beta-s}\,dX$$

$$= \frac{1}{(\alpha-1)!\,\beta^\beta\,(\beta)!}\sum_{s=0}^{\beta}\left(\frac{\beta-\alpha}{2}\right)^s\alpha^{\beta-s}\frac{\beta!}{s!\,(\beta-s)!}\,\frac{(\alpha-1)!\,(\beta-s)!}{(\alpha+\beta-s)!}$$

$$= \Sigma_1/2^n.$$

This proves the result. For $\alpha = 1$, and for an $n+1$-dimensional lattice Λ, it can be written as

$$(|x_1|) + \ldots + (|x_n|)^n\,|x_{n+1}| \leqslant \Delta/\Delta_R,$$

where

$$\Delta_R = \frac{(n+1)^{n+1}}{(n+1)!\,n^n} + \frac{2^{n+1}}{(n-1)!}\int_0^{(n-1)/2n}\frac{v^{n+1}}{1-v}\,dv.$$

It is easy to generalise the result of KOKSMA and MEULENBELD. Thus let $\alpha \geq 1$, $\beta \geq 1$, $\gamma \geq 1$, $\alpha + \beta + \gamma = n$, $\gamma \geq \frac{1}{2} n$. Then an infinity of points of the lattice Λ of determinant $\triangle > 0$ satisfy

$$\left(\sum_{r=1}^{\alpha} |x_r|^{\alpha} \right) \left(\sum_{r=\alpha+1}^{\alpha+\beta} |x_s|^{\beta} \right) \left(\sum_{t=\alpha+\beta+1}^{n} |x_t|^{\gamma} \right) \leqslant \triangle / \triangle_R$$

where

$$\triangle_R = \frac{(n/\alpha)^{\alpha} (n/\beta)^{\beta} (n/\gamma)^{\gamma}}{n!} + V$$

where

$$V = \frac{1}{(\alpha-1)!\,(\beta-1)!\,(\gamma-1)!} \iiint X^{\alpha-1}\, Y^{\beta-1}\, Z^{\gamma-1}\, dX\, dY\, dZ$$

over

$$\alpha X + \beta Y + \gamma Z \geqslant \tfrac{1}{2} n, \quad X \geqslant 1, \quad Y \geqslant 1, \quad Z \geqslant 0,$$
$$(Z + n/2\gamma)^{\gamma}\; X^{\alpha}\, Y^{\beta} \leqslant 1.$$

The integral splits into two parts $V = V_1 + V_2$, say, corresponding to $\alpha X + \beta Y \geq \frac{1}{2} n$, $\alpha X + \beta Y \leq \frac{1}{2} n$. Also

$$V_1 = \frac{1}{(\alpha-1)!\,(\beta-1)!\,\gamma!} \iint X^{\alpha-1}\, Y^{\beta-1}\, (X^{-\alpha/\gamma}\, Y^{-\beta/\gamma} - n/2\gamma)^{\gamma}\, dX\, dY$$

over

$$X \geqslant 1, \quad Y \geqslant 1, \quad \alpha X + \beta Y \geqslant \tfrac{1}{2} n, \quad X^{-\alpha/\gamma}\, Y^{-\beta/\gamma} \geqslant n/2\gamma;$$

$$V_2 = \frac{1}{(\alpha-1)!\,(\beta-1)!\,\gamma!} \iint \Big[X^{\alpha-1}\, Y^{\beta-1}\, (X^{-\alpha/\gamma}\, Y^{-\beta/\gamma} - n/2\gamma)^{\gamma} - \frac{X^{\alpha-1}\, Y^{\beta-1}}{\gamma^{\gamma}} (\tfrac{1}{2} n - \alpha X - \beta Y)^{\gamma} \Big]\, dX\, dY$$

over

$$X \geqslant 1, \quad Y \geqslant 1, \quad \alpha X + \beta Y \leqslant \tfrac{1}{2} n, \quad X^{-\alpha/\gamma}\, Y^{-\beta/\gamma} \geqslant n/2\gamma.$$

It seems hardly worth while evaluating the double integrals.

5. Some interesting applications of our results now follow. Let the region R be defined by $x_n \leq f(x_1, \ldots x_{n-1})$, $x_1 \geq 0, \ldots x_n \geq 0$, where for simplicity, we suppose that $f(x_1, \ldots x_{n-1})$ for $x_1 \geq 0, \ldots x_n \geq 0$, is defined as a symmetrical non-negative function of $x_1, \ldots x_{n-1}$, no other conditions being imposed as yet upon $f(x_1, \ldots x_{n-1})$. Write

$$x_1 + \ldots + x_{n-1} = X,$$

and denote by $f(X)$ the least value of $f(x_1, \ldots x_{n-1})$ for given X with $x_1 \geq 0, \ldots x_{n-1} \geq 0$. We assume that for $X > 0$, $f(X)$ is a decreasing and

$f'(X)$ an increasing function of X. We write the tangent plane T, say, to $x_n = f(x_1, \dots x_{n-1})$ at $(x_1, \dots x_n) = (a, \dots a, b)$, in the form

$$\frac{x_1 + \dots + x_{n-1}}{p_1} + \frac{x_n}{p_n} = 1,$$

and this is also a tangent plane to $x_n = f(x_1 + \dots + x_{n-1})$, the point of contact now not being unique, since it need only satisfy

$$x_1 + \dots + x_{n-1} = (n-1)\,a,\ X_n = b.$$

The region $\bar{S}$ defined by $x_n \leq f(x_1 + \dots + x_{n-1})$, $x_1 \geq 0, \dots x_n \geq 0$ is inscribed in R since the inequality implies $x_n \leq f(x_1, \dots x_{n-1})$. Hence any improvement on the crude result for S based upon the tangent T, will also give an improvement for R based upon T.

We now proceed as in § 4 if $p_n < 2\,b$. A slight modification is required if $p_n > 2\,b$. We now take for the point of contact of T with

$$x_n = f(x_1 + \dots + x_{n-1}),$$

the point $(0, 0, \dots (n-1)\,a, b)$. Since

$$\frac{(n-1)\,a}{p_{n-1}} + \frac{b}{p_n} = 1,$$

and $p_n > 2\,b$, then $\frac{1}{2}\,p_{n-1} < (n-1)a$. Hence we have merely to interchange the role of x_{n-1}, x_n in the result of § 4.

Write $A = (n-1)a$, $B = b$, $P = p_1 = \dots = p_{n-1}$, $Q = p_n$. Then

$$\triangle_R = 2^n\,(V_1 + V_2),$$

where

$$V_1 = \int\!\!\int \frac{X^{n-2}}{(n-2)!}\,dX\,dY \text{ over } X/P + Y/Q \leqslant \tfrac{1}{2},\ Y \leqslant B,\ X \geqslant 0,$$

$$V_2 = \int\!\!\int \frac{X^{n-2}}{(n-2)!}\,dX\,dY \text{ over } Y \leqslant f(X + \tfrac{1}{2}\,P),\ Y \geqslant B,\ X \geqslant 0.$$

Results can also be found by writing

$$x_1 + \dots + x_{n-2} = X$$

and denoting by $f(x_{n-1}, X)$ the least value of $f(x_1, \dots x_{n-1})$ for given X and x_{n-1}.

For a particular case [1]) of the previous method take

$$|x_1|^\lambda + \dots + |x_n|^\lambda \leq 1, \text{ where } 0 < \lambda < 1.$$

[1]) From an Amsterdam dissertation by PIETER MULLENDER in 1945, received after writing this paper, see results have been found by noting that

$$\left(\frac{|x_1|^\lambda + \dots |x_n|^\lambda}{n}\right)^{1/\lambda} \leq \left(\frac{|x_1|^\mu + \dots |x_n|^\mu}{n}\right)^{1/\mu}$$

if $\lambda \leq \mu$, and using the results for $\mu \geq 2$ due to VAN DER CORPUT and SCHAAKE.

Here $n a^\lambda = 1$, and the tangent plane at $(a, a \ldots a)$ is

$$x_1 + \ldots + x_n = a^{1-\lambda},$$

and so

$$P = Q = a^{1-\lambda}.$$

Since

$$x_n^\lambda \leqslant 1 - x_1^\lambda - \ldots - x_{n-1}^\lambda,$$

and

$$x_1^\lambda + \ldots + x_{n-1}^\lambda \leqslant (n-1)^{1-\lambda}\ (x_1 + \ldots + x_{n-1})^\lambda,$$

the X, Y region becomes

$$Y^\lambda \leqslant 1 - (n-1)^{1-\lambda} X^\lambda.$$

The point on this is now $(A, B) = \{(n-1)a, a\}$, and the tangent at (A, B) has of course the intercepts P, Q on the X, Y axes. Since $Q = a^{1-\lambda} \geq 2B = 2a$, we have to interchange the variables, and the integration in V_1 is given by $X/P + Y/Q \leq \frac{1}{2}$, $Y \leq B$, $X \geq 0$, and in V_2 by $Y \leq f(X + \frac{1}{2}P)$, $Y \geq B$, $X \geq 0$.

Hence

$$V_1 = \frac{1}{(n-1)!}\int_0^B P^{n-1}\left(\tfrac{1}{2} - Y/Q\right)^{n-1} dY.$$

$$= \frac{a^{(n-1)(1-\lambda)}}{(n-1)!}\int_0^a \{\tfrac{1}{2} - Y/a^{1-\lambda}\}^{n-1} dY.$$

$$= \frac{a^{n(1-\lambda)}}{n!}\{1/2^n - (\tfrac{1}{2} - 1/n)^n\}.$$

Also

$$V_2 = \frac{1}{(n-2)!}\iint X^{n-2}\, dX\, dY \text{ over } Y^\lambda + (n-1)^{1-\lambda}\{X + \tfrac{1}{2}a^{1-\lambda}\}^\lambda \leqslant 1,$$
$$X \geqslant 0,\ Y \geqslant a$$

$$= \frac{1}{(n-2)!}\int_0^{(\frac{1}{2}n-1)a} X^{n-2}\left[\{1 - (n-1)^{1-\lambda}(X + \tfrac{1}{2}a^{1-\lambda})^\lambda\}^{1/\lambda} - a\right] dX,$$

the upper limit $(\frac{1}{2}n - 1)a$ being the value of X for which $X + \frac{1}{2}P = A$. The integral can be evaluated in finite terms when $1/\lambda$ is an integer. Finally, for any lattice Λ, a point $\neq 0$ exists such that

$$|x_1|^\lambda + \ldots + |x_n|^\lambda \leqslant \Delta^{\lambda/n}/[2^n(V_1 + V_2)]^{\lambda/n}.$$

For another illustration, I prove that a point of $\Lambda \neq 0$ satisfies

$$\sum_{(r)}^{n} |x_1 x_2 \ldots x_r| \leqslant (\Delta/\Delta_R)^{r/n},$$

where the notation means that the summation is extended over the $\binom{n}{r}$ sets of the $x_1 \dots x_n$ taken $r \geq 1$ at a time, and

$$\triangle_R = 2^n (V_1 + V_2),$$

where V_1, V_2 are defined below. We have now to consider

$$\sum_{(r)}^{n} x_1 x_2 \dots x_r \leqslant 1.$$

Here

$$\binom{n}{r} a^r = 1, p_1 = \dots = p_n = r \Big/ \binom{n-1}{r-1} a^{r-1} = na.$$

Now

$$x_n \leqslant \frac{1 - \sum\limits_{(r)}^{n-1} (x_1 \dots x_{r-1})}{\sum\limits_{(r-1)}^{n-1} (x_1 \dots x_{r-1})}.$$

Since

$$\sum_{(r)}^{n-1} x_1 \dots x_r \leqslant \left(\frac{x_1 + \dots + x_{n-1}}{n-1}\right)^r \binom{n-1}{r},$$

and

$$\sum_{(r-1)}^{n-1} x_1 \dots x_{r-1} \leqslant \left(\frac{x_1 + \dots + x_{n-1}}{n-1}\right)^{r-1} \binom{n-1}{r-1},$$

we now have

$$Y \leqslant \frac{1}{\left(\frac{X}{n-1}\right)^{r-1} \binom{n-1}{r-1}} - \frac{(n-r)\,X}{r\,(n-1)}.$$

Also $A = (n-1)a$, $B = a$, $P = Q = p_1 = na$.

Since $P \leq 2A$, our result is

$$V_1 = \frac{1}{(n-2)!}\int\int X^{n-2}\,dX\,dY \ \text{ over } \ X/P + Y/Q \leqslant \tfrac{1}{2},\ X \geqslant 0,\ Y \leqslant B,$$

$$V_2 = \frac{1}{(n-2)!}\int\int X^{n-2}\,dX\,dY \ \text{ over } \ Y \leqslant f(X + \tfrac{1}{2}P),\ X \geqslant 0,\ Y \geqslant B.$$

We note in V_2 that $X = A - \frac{1}{2}P$ when $Y = B$.

Now

$$\begin{aligned} V_1 &= \frac{1}{(n-1)!}\int_0^B P^{n-1} (\tfrac{1}{2} - Y/Q)^{n-1}\,dY \\ &= \frac{1}{n!} P^{n-1} Q\,[1/2^n - (\tfrac{1}{2} - B/Q)^n] \\ &= \frac{(na)^n}{n!}\,[1/2^n - (\tfrac{1}{2} - 1/n)^n]. \end{aligned}$$

and

$$2^n V_1 = \frac{(na)^n}{n!} - \frac{(n-2)^n a^n}{n!}.$$

Next

$$V_2 = \frac{1}{(n-2)!}\int_0^{(\frac{1}{2}n-1)a} X^{n-2}\left\{\frac{1}{(X+\frac{1}{2}na)^{r-1}\binom{n-1}{r-1}} - \frac{(X+\frac{1}{2}na)(n-r)}{(n-1)r} - a\right\} dX$$

$$= J - C$$

say, where the integral

$$2^n J = \frac{2^r}{(n-2)!\binom{n-1}{r-1}}\int_0^{(n-2)a} \frac{X^{n-2}}{(X+na)^{r-1}}\, dX$$

$$= \frac{2^r a^{n-r}}{(n-2)!\binom{n-1}{r-1}}\int_0^{(n-2)} \frac{X^{n-2}}{(X+n)^{r-1}}\, dX$$

$$= \frac{n\, 2^r a^n}{r(n-2)!}\int_0^{(n-2)} \frac{X^{n-2}}{(X+n)^{r-1}}\, dX.$$

which can be evaluated in finite terms.

Also

$$C = \frac{n-r}{rn!}(\tfrac{1}{2}n-1)^n a^n + \frac{(n-r)n}{2r(n-1)(n-1)!} a^n (\tfrac{1}{2}n-1)^{n-1} + \frac{a^n}{(n-1)!}(\tfrac{1}{2}n-1)^{n-1}.$$

Finally

$$2^n(V_1 - C) = \frac{a^n n^n}{n!} - \frac{n}{rn!}(n-2)^n a^n - \frac{2a^n(n-2)^{n-1}}{(n-1)!} - \frac{(n-r)\,n\,a^n(n-2)^{n-1}}{r(n-1)(n-1)!}$$

and

$$\triangle_R = 2^n J + 2^n(V_1 - C).$$

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Mathematics. — *A theorem concerning analytic continuation II.* By J. DE GROOT. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of June 29, 1946.)

In this paper we have two objects in view; in 1. we shall give a discussion of the results reached in a previous paper (A theorem concerning analytic continuation, Proc. Kon. Ned. Akad. v. Wet. **49** (1946), p. 213—222, denoted by [1]), in connection with a possible simplification of a certain proof, and the results already known about this subject. In 2. — 4. we shall give a generalisation of the theorem of continuation of [1], where the following question will be answered: *under which conditions does a one-valued function f, defined on an arbitrary closed point-set of the complex plane, present a one-valued and analytic function?*

1. In [1] the following problem (among others) was considered: in the complex plane be given a sequence of points $z_0, z_1, z_2, \ldots,$ converging to z'. Each point z_i is given a function-value w_i. Under which conditions is it possible to find a one-valued and analytic function $f(z)$ defined in a region containing z' (and therefore also almost all points z_i) such that $f(z_i) = w_i$ for *nearly* all values of i? We found a necessary and adequate condition ([1], theorem I), which implied that for the nth difference-quotient $\triangle^n$ (defined for the points $z_0, z_1, z_2, \ldots, z_n$):

$$|\triangle^n| \leqq n!\, r^n \qquad (n = 1, 2, 3, \ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where r is a suitably chosen positive constant.

The necessity of this condition was proved by means of an unequality ([1], theorem IV), which again was proved by [1], theorem III, i.e., the generalized mean value theorem for complex functions. It now appears that this unequality may be proved in a very simple way by well-known expedients [1]), while the necessity of (1) may also be proved immediately by use of the following well-known integral which represents $\triangle^n$ [2]) — Mr. POPKEN kindly drew my attention to this —

$$\triangle^n = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{\prod_{i=0}^{n} (\zeta - z_i)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

---

[1]) This unequality follows for instance from

$$\triangle^n = \lambda f^{(n)}(\lambda_0 z_0 + \ldots + \lambda_n z_n), \ (|\lambda| \leqq 1, \lambda_i \text{ real, with } \sum_0^n \lambda_i = 1),$$

comp. NÖRLUND, Differenzenrechnung.

[2]) Compare also NÖRLUND, op. cit.

Here $f(z)$ is an analytic function on and within the contour C, while the points $z_0, z_1, z_2, \ldots, z_n$ are within C. That (2) is true is immediately obvious if one applies the theorem of residues on the right side of (2). Thus we get

$$n! \sum_{i=0}^{n} \frac{f(z_i)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (z_i - z_j)}$$

and this is exactly $\triangle^n$. From (2) one now finds immediately a relation of the shape (1).

Further it may be remarked, that theorem V, following immediately from [1], theorem I, is known (comp. BENDIXSON, Acta Mathematica **9** (1887), p. 1—34). Summing up we may say that [1], theorem I is related to already known theorems and may, by means of simple expedients, be deduced from those theorems. Moreover [1], theorem III (mean value theorem for complex functions) is already known. This theorem was proved by MONTEL (comp. F. MONTEL, Sur quelques propriétés des différences divisées, Journal de Mathématiques (9me s.) **16** (1937), p. 222).

Mr. POPKEN informed me that [1], theorem I may be generalized in the following manner. While we considered a sequence of points, which converged to z', consisting of different points $z_0, z_1, z_2, \ldots$, POPKEN permits the coinciding of finitely many points z_i succeeding each other, for instance $z_0, z_1, z_1, z_1, z_2, z_2, z_3, z_4, \ldots$. To each point are given function-values w_i, to the sequence just mentioned for instance the values $w_0, w_1, w_1', w_1'', w_2, w_2', w_3, w_4, \ldots$, and we now look for a necessary and adequate condition that there exist a one-valued and analytic function, which in the points z_i has the values w_i, while moreover if a point z_i occurs n times at one stretch, the first until and including the $(n-1)th$ derivate of the function in z_i should be (respectively) $w', w'', \ldots, w^{(n-1)}$. Naturally the common shape of the difference-quotient is senseless for coinciding points z_i, but one may — in a well-known way — give a definition of the difference-quotient by means of determinants, which holds sense, when a number of points z_i coincide. As necessary and adequate condition one finds again (1).

2. The problem of continuation considered in [1] may be generalized as follows: there be given a (for the time being) bounded, closed but for the rest arbitrary set A in the complex plane. To each point $a \subset A$ is given a certain complex value $f(a)$, in other words, a one-valued function f has been defined on A. The problem now is to find a necessary and adequate condition to be imposed upon the values $f(a)$, in order that f be a one-valued analytic function on A.

A function f is, as known, analytic in a point or on a curve if it is possible to find a region containing that point or that curve — so this region is a neighbourhood of this point or curve — on which f is a one-valued analytic function. This definition may also be used when considering an arbitrary bounded closed set A instead of a point or a curve. In case A is not connected, in other words, is not a continuum, we shall ask less; then we shall only ask that there may be found finitely many disjunct regions containing A (i.e., which together form a neighbourhood of A), such that f is a one-valued and analytic function on each of those regions. We do not ask, for instance, that the function-values which f takes in one region be found by analytic continuation of the function-values of another region. We shall, however, call our f — unlike the custom — in this case, for the sake of simplicity, *analytic on that neighbourhood of* A, consisting of a finite number of regions. The problems rising by this definition we hope to discuss later. In case A *is a continuum*, however, *only one region is formed and our definition coincides with the known one.*

We consider the n^{th} *divided difference* $\Phi^n = \Phi^n\,[z_0 z_1 \ldots z_n]$ of $f(A)$ corresponding with $(n+1)$ arbitrary points $z_0, z_1, \ldots, z_n$ of A for every natural number n. Φ^n is defined by

$$\Phi^n = \frac{\Delta^n}{n!} = \sum_{k=0}^{n} \frac{f(z_k)}{\prod\limits_{\substack{j=0\\ j\neq k}}^{n} (z_k - z_j)}.$$

Now we demand that there may be found a fixed positive number r such that

$$\sqrt[n]{|\Phi^n|} \leqq r \qquad (n = 1, 2, \ldots) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We contend that condition (3), composed for every finite combination of different points of A, is a necessary and adequate condition for the given f being analytic on A (i.e., on a neighbourhood of A consisting of a finite number of regions).

Theorem I'. *A one-valued function f defined on a bounded closed set A is analytic on A only if the set of all values*

$$\sqrt[n]{|\Phi^n|} \qquad\qquad (n = 1, 2, 3, \ldots),$$

composed for every combination of n points of A, is bounded.

P r o o f. *The condition is adequate.*

We may suppose that A is an infinite point-set (in the opposite case the following proof is greatly simplified). It is possible to construct a countable set Z, which is everywhere dense in A (i.e., A is the closure of Z: $\bar{Z} = A$). Be a an arbitrary, but not isolated point of A. Then there may be found a sequence $a_0, a_1, a_2, \ldots$, consisting of points belonging to Z, which converges to a. If we take all points a_i $(i = 0, 1, 2, \ldots)$ out of Z there remains a countable set $b_0, b_1, b_2, \ldots$. We may suppose there are infinitely many points b_i. We now enumerate Z by taking alternately points a_i and b_i: $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$ and shall note this sequence in the following by $z_0, z_1, z_2, \ldots$.

We define the function $g(z)$ by

$$g(z) = f(z_0) + \sum_{n=0}^{\infty} \Phi^{n+1} \cdot (z-z_0)(z-z_1) \ldots (z-z_n) \quad . \quad . \quad . \quad (4)$$

where $\Phi^n = \Phi^n \; [z_0 z_1 \ldots z_n]$. We shall prove first that $g(z)$ is a one-valued and analytic function in a certain neighbourhood O of a and that in the points of the intersection $O \cdot A$ this function takes precisely the given f-values. In the first place it is obvious — according to the interpolation-formula of NEWTON — that in the points z_i $(i = 0, 1, 2, \ldots)$ $g(z)$ takes exactly the prescribed values $f(z_i)$.

Further be $\frac{1}{2}^d$ the diameter of A, then for the points z of a certain adequately small neighbourhood of a

$$|z - z_j| < d \qquad\qquad (j = 0, 1, 2, \ldots).$$

Since $z_{2j} = a_j$ and the points a_i are converging to a there may be found an index m for every arbitrarily small $\delta > 0$, such that

$$|z - z_{2j}| < \delta \text{ for } j > m.$$

From these two unequalities one may deduce, in connection with (3) and (4), in a simple way that there may be found an index k and two positive constants C_1 and C_2 such that for the values z which are within a δ-neighbourhood of a

$$|g(z)| < C_1 + C_2 \cdot \sum_{n=k}^{\infty} r^n \cdot \delta^{1/2 n} \cdot d^{1/2 n} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now we choose $\delta < \dfrac{1}{dr^2}$, from which follows that the series of (5) converges, so that the series of (4) is uniformly converging in the δ-neighbourhood of a. As the terms of the series of (4) are all analytic functions in that neighbourhood, $g(z)$ is regular within that neighbourhood.

Further we must prove that in the points a' of A which belong to the mentioned δ-neighbourhood, $g(z)$ takes exactly the given values $f(a')$. Be a' such a point; in the first place we suppose a' to be a limit-point of A and consider the sequence $a', z'_0, z'_1, z'_2, \ldots$, where z'_i $(i = 0, 1, 2, \ldots)$ runs through the countable set Z, while z'_{2i} is a sequence converging to a'. We consider the function

$$g_{a'}(z) = f(a') + \Phi^1_{a'} \cdot (z - a') + \sum_{n=0}^{\infty} \Phi^{n+2}_{a'} \cdot (z-a')(z-z'_0) \ldots (z-z'_n)$$

with

$$\Phi^n_{a'} = \Phi^n \, [a' \; z'_0 \, z'_1 \ldots z'_{n-1}] \, .$$

This series, like that of $g(z)$, is certainly uniformly converging in an adequately small neighbourhood of a' and is there a one-valued and analytic function. $g_{a'}(z)$ further takes the given values $f(a')$ and $f(z_i)$ in a' and $z'_i = z_j$ respectively; $g(z)$ and $g_{a'}(z)$ have the same value particularly in the sequence of points z'_{2i} $(i = 1, 2, \ldots)$ of Z, converging to a', in other words, $g(z)$ and $g_{a'}(z)$ have the same value in infinitely many points which have a limit-point belonging to their intersection, i.e., they have the same value, according to a well-known theorem of analytic continuation, in their whole intersection; so especially in a' $g(z)$ takes exactly the value $g_{a'}(a') = f(a')$.

Secondly it is possible that a' is not a limit-point but an isolated point of A. In this case a' already appears in the mentioned Z and according to the above of course $g(a') = f(a')$.

For every not-isolated point a of A it is possible, according to (4), to find a function $g(z) = g_a(z)$, which is one-valued and analytic in a certain δ_a-neighbourhood of a and which in the points of A belonging to that neighbourhood has the same values as the given function f. Apart from the open δ_a-neighbourhoods we also consider the closed $\frac{1}{2}\delta_a$-neighbourhoods (the closures of the circle-regions with center a and radius $\frac{1}{2}\delta_a$). Since A is bounded and closed it is — excepting a finite number of isolated points of A, which, for the time being, we remove out of A, while we call the remaining set again A — according to the covering-theorem of HEINE–BOREL, already covered by a finite number of such closed $\frac{1}{2}\delta_a$-neighbourhoods. Take two arbitrary closed neighbourhoods of that kind, denoted by $O = O(\frac{1}{2}\delta_{a'})$ and $O' = O'(\frac{1}{2}\delta_{a''})$. If the intersection $O \cdot O'$ of these two does contain only isolated points of A or no points of A at all, then there are none or finitely many, for the intersection is also closed. $O \cdot O'$ in this case has a certain distance from the set $A - A \cdot O \cdot O'$; $O \cdot O'$ then has an adequately small open neighbourhood $U(O \cdot O')$ which has no common point with $A - A \cdot O \cdot O'$. We now remove from O and O' the intersections of these both with $U(O \cdot O')$. Then we have left two closed sets O_1 and O'_1, while at most finitely many isolated points (viz. the set $A \cdot O \cdot O'$) have been removed from A. This new set $A - A \cdot O \cdot O'$ is again denoted by A. The closed sets O_1 and O'_1 have a vacuous intersection and therefore have a positive distance.

The second possibility is that O and O' have one or more limit-points of A for intersection. In these cases nothing is changed. By doing this for every pair of $\frac{1}{2}\delta_a$-neighbourhoods an at most finite number of isolated points is removed from A; the new set is again denoted by A. Thus we have achieved that the new set A is covered by a finite number of closed O_1-sets. Such a set of the kind O_1 which is finally formed, is not necessarily connected, but consists of an at most finite number of components. From each O_1-set we finally remove those components which contain only isolated points of A. The remaining set we again call A, and this set is finally covered by a finite number of closed sets $A_1, A_2, A_3, \ldots, A_s$ which originated from the sets of the kind O_1. These closed sets are divided into a finite number of systems, where every system will consist of those sets A_i which together form a component (i.e., a maximally connected set). Two sets A_j and A_k of such a component, having no vacuous intersection, must according to the above have a limit-point of A in this intersection.

Be these components $C_1, C_2, \ldots, C_r$. These components are, being closed sets, all at a certain positive distance from each other.

We now consider open neighbourhoods $O(A_1), O(A_2), \ldots, O(A_s)$ of the closed sets $A_1, A_2, \ldots, A_s$, satisfying the following conditions:

1°. A_i is formed from a certain corresponding closed $\frac{1}{2}\delta_a$-neighbourhood; $O(A_i)$ will belong to the corresponding open δ_a-neighbourhood.

2°. $O(A_i)$ will not contain any of the finitely many isolated points which have been removed from A.

3°. $O(A_i)$ will have a vacuous intersection with $O(A_j)$ if $A_i \cdot A_j = 0$.

It is easy to see that these conditions may be satisfied. The sum of the $O(A_i)$ $(i = 1, 2, \ldots, s)$ is divided into the components $O(C_1), O(C_2), \ldots, O(C_r)$ corresponding with $C_1, C_2, \ldots, C_r$ respectively. Apparently on every $O(A_i)$ a certain one-valued and analytic function $g(z)$ had been defined. We shall prove that therefore also on an arbitrary $O(C_k)$ a one-valued and analytic function is defined. For: if two sets $O(A_i)$ and $O(A_j)$ belonging to $O(C_k)$, have a not-vacuous intersection, $A_i \cdot A_j \neq 0$ and then according to the above there even is a limit-point $a' \subset A$ belonging to $A_i \cdot A_j$. Since Z is dense in A, a sequence of points belonging to Z converges to a', in other words, the functions $g(z)$ defined on $O(A_i)$ and $O(A_j)$ are taking the same values in a sequence of points having a limit-point a' which belongs to the intersection, and therefore in those points denote the same one-valued and analytic function. On every $O(C_i)$ there has in this way been defined a one-valued and analytic function. There still remain finitely many isolated points $a_1, a_2, \ldots, a_v$ of A; for every point a_i of this sequence we consider an open neighbourhood $O(a_i)$, which has no points in common either with $A - a_i$ or with one of the sets $O(C_i)$; to all points of $O(a_i)$ we are giving the constant given function-value $f(a_i)$. In this manner A is finally covered by a finite number of disjunct regions, on each of which a one-valued and analytic function has been defined, which in the points of A coincides with the given f. These functions together produce the required analytic function, defined on a neighbourhood of A.

The condition is necessary.

We must start from an arbitrary bounded closed set A and a neighbourhood O' of A. Within O' is a neighbourhood O of A consisting of a finite number of disjunct regions $G_1, G_2, \ldots, G_r$; the easy proof of this contention we shall not give. On O', and therefore on O, an arbitrary one-valued and analytic function f has been defined, i.e., on every G_i has been defined a one-valued and analytic function $f_i(z)$. We must prove that condition (3) is satisfied. Since A is closed it may be covered by a finite number of open circle-regions, which including their boundaries belong to O and the sum of which is divided into a finite number of disjunct regions $C_1, C_2, \ldots, C_s$ $(s \geq r)$. Every C_i must be an n_i-fold connected region, since C_i is composed from a finite number of open circle-regions.

The boundary R_i of C_i apparently consists of n_i curves $k_1, k_2, \ldots, k_{n_i}$ belonging to O (every k_j consists of a finite number of circular arcs). The integral of $f(z)$ along the curves $k_1, k_2, \ldots, k_{n_i}$, so along R_i, all described in positive direction, vanishes according to a well-known extension of the integral-theorem of CAUCHY, since $f(z)$ is analytic on the region G_i. The set A has a certain distance 2ε from the closed sum $\sum_{i=1}^{s} R_i$. Further we consider $(n+1)$ arbitrary points $z_0, z_1, \ldots, z_n$ of A. Then apparently (one applies the theorem of residues a finite number of times):

$$\Phi^n = \sum_{i=0}^{n} \frac{f(z_i)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{n} (z_i - z_j)} = \frac{1}{2\pi i} \sum_{k=1}^{s} \int\limits_{\dot{R}_k} \frac{f(\zeta)}{\prod\limits_{j=0}^{n} (\zeta - z_j)} \, d\zeta \quad . \quad . \quad (6)$$

Further

$$\left| \int\limits_{\dot{R}_k} \frac{f(z)}{\prod\limits_{j=0}^{n} (\zeta - z_j)} \, d\zeta \right| \leqq \frac{M_k \, . \, L_k}{\varepsilon^{n+1}},$$

where M_k is the maximum of $|f(z)|$ on R_k and L_k is the length of R_k. From this follows immediately

$$|\Phi^n| \leqq \frac{1}{2\pi\varepsilon^{n+1}} \, . \sum_{k=0}^{s} M_k \, . \, L_k \leqq r^n,$$

where r is a suitably chosen positive number independent of n. Since this holds true for every natural number n the proof is hereby finished.

3. *Other expressions for the necessary and adequate condition.* The necessary and adequate condition (3) may be varied in different ways. In (3) there are more than countably many conditions for the case that A is a not-countable set. This number of conditions may be made countable, however, in the following way: be Z an arbitrary countable set, everywhere dense in A. A necessary and adequate condition for f to be analytic is *the demand that* (3) *holds true for every finite combination of points of* Z *and that* f *is continuous on* A. The number of conditions (3) is now indeed at most countably infinite; but, on the other hand, we must now ask the continuity of f, which was not the case in theorem I′. But if A itself is countable this demand is not necessary of course, since we then may take $Z = A$.

Condition (3) may also be weakened in the following way: for every enumeration (α) of Z, say $z_1^\alpha, z_2^\alpha, \ldots$ there is a positive number r_α, an index i_α and a natural number n_α such that for every $n \geqq n_\alpha$

$$\sqrt[n]{|\Phi_\alpha[z_i \, z_{i+1} \ldots z_{i+n}]|} \leqq r_\alpha.$$

The numbers r_α may be different for different enumerations (α) and e.g. need not have an upper boundary. This condition may be weakened still more. The conditions we mentioned are conditions for entire A or Z. One may replace these by conditions in the points themselves; for instance: for every point of A a neighbourhood O may be found such that on $O \cdot Z$ condition (3) holds true.

4. *Further generalization of theorem I'.*

Of more importance is the question to what extent the conditions of boundedness and closedness, imposed upon A, are essential.

The condition of boundedness may be left out provided one only asks that condition (3) holds true for every bounded closed subset D of A (for different subsets D the value of r may be different); one may also impose the somewhat weaker condition that for the part of A lying within every closed circle-region with the origin for center and radius R, condition (3) holds true, while for different values R one may have different r's. The necessity and adequacy of this condition are proved by means of the method developed in 2. There is, however, one difference with the result reached in theorem I': for bounded closed A the given f may be continued analytically on a neighbourhood of A which consisted of a finite number of disjunct regions; in case boundedness is no longer conditioned the number of regions on each of which the continued f is one-valued and analytic may be infinitely great. In every bounded set of the complex plane there are, on the other hand, only finitely many regions of that kind; the regions accumulate only in the infinite.

In case one also takes into consideration the point in infinity, in other words, in case one considers a closed set on the complex sphere, one may find again a necessary and adequate condition perfectly analogous with (3). In that case one studies $f(z)$ in a neighbourhood of the point in infinity by examining the conduct of $f(1/z')$ in a neighbourhood of the point $z' = 0$, i.e., in a suitably chosen neighbourhood of $z' = 0$ condition (3) should hold true for $f(1/z')$.

The condition that A be closed plays a more essential part. If A is not closed and f is a one-valued function on A one may impose different conditions concerning the required analytic continuation, as we already explained in [1], 2. The problem is, whether one requires an analytic function which does or does not contain the limit-points of A, not belonging to A, in its region of regularity. In case one does not demand all limit-points of A to belong to the region of regularity, weaker conditions than (3) will often suffice. This problem will not be discussed here further; we only remark that for certain categories of not-closed sets one may find a necessary and adequate condition analogous with (3); for an arbitrary region e.g. by taking this as the sum of countably many closed sets. If one demands, however, continuation on a neighbourhood of the closure $\overline{A}$ of A (consisting of a finite number of regions) one apparently can use condition (3) again. Summarizing we may give the following theorem.

Theorem I''. *A one-valued function $f(A)$, defined on an arbitrary set A of the complex plane, may be continued to a one-valued and analytic function $g(z)$, defined on a certain neighbourhood $O(\overline{A})$ of the closure $\overline{A}$ of A (such that therefore $f(a) = g(a)$ for all points $a \subset A$), only if for every bounded subset A' of A there may be found a number $r_{A'}$ such that for the n^{th} divided difference Φ^n of f, composed for every combination of n points of A',*

$$\sqrt[n]{|\Phi^n|} \leqq r_{A'} \qquad (n = 1, 2, \ldots).$$

Here we may manage that the at most countably many regions which together form $O(\overline{A})$ do not accumulate anywhere in the finite (on each of those regions $g(z)$ is one-valued and analytic, but the values of $g(z)$ in one region need not necessarily form an analytic continuation of the values of $g(z)$ in another region).

As a special case we mention further:

Corollary. If A is a bounded closed point-set in the complex plane, consisting of a continuum and an arbitrary number of isolated points, and if on A is defined a one-valued function f, then f is analytic on A (except in a finite number of isolated points of A) and may be continued analytically on a *region* containing A (excepting again an at most finite number of isolated points of A), only if $f(A)$ satisfies condition (3).

Finally we remark that the generalisation of POPKEN, mentioned in 1., may be combined with our generalisation, which combination brings us to new theorems that are obvious without further discussion.

Mathematics. — *A P-adic Analogue of a Theorem of* LEBESGUE *in the Theory of Measure*. By J. POPKEN and H. TURKSTRA. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of June 29, 1946.)

1. We consider on the real axis a measurable set of points A and an arbitrary interval I [1]. The ratio

$$\frac{m(A\,I)}{m(I)}$$

is said to be the *mean density* of A in I.

Now let α be an arbitrary point on the real axis. Let I denote an interval containing α and let this interval "contract itself on α", i.e. let its length $m(I)$ tend to zero. Whenever the mean density of A in I always tends to the same limit $d = d(\alpha)$, then d is said to be the *density* of A at α.

The following fundamental theorem in the theory of density is due to LEBESGUE [2]:

If A denotes an arbitrary measurable set on the real axis, then there exists a set Z of measure zero, such that at all points outside [3] *Z the density exists, and is equal to unity at points of A and equal to zero at points outside A.*

This theorem also is of special importance in the metric theory of Diophantic approximations [4].

In this note we will prove the analogue of this theorem if we take the field of P-adic numbers in stead of the field of real numbers (theorem I). It is clear that we have to use a theory of measure in the field of all P-adic numbers. Such a theory was established by TURKSTRA in his dissertation [5].

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1) In this note we use the general symbolism of the theory of sets. Let A, B be two sets, then $A + B$, the *sum* of A and B, denotes the set of all elements in A or in B; $A - B$, the *complement* of B with respect to A, denotes the set of all elements in A but not in B; AB, the *intersection* of A and B, consists of the elements in A and also in B. Further $A \subset B$ or $B \supset A$ means that A is a sub-set of B, the case that A coincides with B not being excluded. A set is said to be *empty*, if it has no elements. The measure of a measurable set A is denoted by $m(A)$.

2) H. LEBESGUE, Sur l'intégration des fonctions discontinues, Ann. Scient. de l'École Normale Supérieure (3) 27, p. 405—407 (1910).

3) "Outside Z" means here: "belonging to the complement of Z".

4) Cf. e.g. J. F. KOKSMA, Diophantische Approximationen (Ergebnisse der Mathematik IV 4), Kap. III § 5.

5) H. TURKSTRA, Metrische bijdragen tot de theorie der Diophantische approximaties in het lichaam der P-adische getallen. Dissertation of the „Vrije Universiteit" at Amsterdam, Groningen 1936.

We shall refer to this book as "T".

See also: W. FELLER and E. TORNIER, Mass- und Inhaltstheorie des Baireschen Nullraumes, Math. Ann. **107**, 165—187 (1933).

In the next section of this note we give an extract of this theory as far as we use it in our present investigation.

2. From now on we confine ourselves to the field $K = K(P)$ of all P-adic numbers, P denoting an arbitrary prime.

A P-adic number α can be written in the form

$$\alpha = \sum_{n=-\infty}^{\infty} a_n P^n, \quad \ldots \ldots \ldots \ldots \quad (1)$$

where the coefficients a_n are integers taken from the interval $0 \leq a \leq P - 1$. such that only a finite number of the coefficients $a_{-1}, a_{-2}, a_{-3}, \ldots$ do not vanish.

Suppose $\alpha \neq 0$, and let a_{-t} be the first coefficient in (1) different from zero. Then

$$\alpha = \sum_{n=-t}^{\infty} a_n P^n \quad , \quad a_{-t} \neq 0.$$

The *P-adic value* $|\alpha|_P$ of α is defined to be

$$|\alpha|_P = P^t \qquad (|0|_P = 0).$$

Let α, β be two P-adic numbers or "points", then $|\alpha - \beta|_P$ is said to be the P-adic *distance* between α and β; we shall denote it by $\overline{\alpha\beta}$. Clearly $\overline{\alpha\beta} \geq 0$ and $\overline{\alpha\beta} = 0$ if and only if α and β coincide; also $\overline{\alpha\beta} = \overline{\beta\alpha}$ [6]. Finally the important "inequality of the triangle" is satisfied: If α, β, γ are three arbitrary P-adic points, then

$$\overline{\alpha\beta} \leqq \overline{\alpha\gamma} + \overline{\gamma\beta};$$

even a sharper inequality holds:

$$\overline{\alpha\beta} \leqq \max(\overline{\alpha\gamma}, \overline{\gamma\beta}) \text{ [7]};$$

in other words this conception of distance is "non-Archimedic", and this property is responsible for some peculiarities in the theory of P-adic point sets.

Evidently the set $K(P)$ of all P-adic points is a metric space.

Now we are in a position to define a *P-adic interval* of *order n*: Let α be an arbitrary P-adic number and let n be a fixed integer, then the set of all P-adic numbers ξ, satisfying

$$\overline{\alpha\xi} \leqq P^{-n}$$

is said to be a P-adic interval of order n [8].

Open and closed sets can be defined in the ordinary manner, but it is convenient to enlarge these definitions by the convention that an empty set

[6]) T. p. 39 (Proof of II Stelling 11).

[7]) T. p. 39 (Proof of II Stelling 11) and p. 30 (II Stelling 4).

[8]) For an equivalent definition see T. p. 72 (V Definitie 1); compare p. 74 (V Stelling 1).

is considered as open and also as closed. Then the following theorems are true [9]):

The complement of an open set is closed and the complement of a closed set is open.

The sum of a finite number or of an infinity of open sets is open.

The sum of a finite number of closed sets is closed.

The intersection of a finite number of open sets is open.

The intersection of a finite number or of an infinity of closed sets is closed.

Now we state some properties of P-adic intervals:

Theorem 1 [10]): *The set of all P-adic intervals is enumerable.*

Theorem 2 [11]): *If I_1 and I_2 denote intervals, such that they have at least one point in common; if the order of I_1 is not less than the order of I_2, then*

$$I_1 \subset I_2.$$

It follows, that *two intervals of the same order either coincide, or do not overlap.*

Let α be an arbitrary P-adic number. Now

$$\overline{\alpha\xi} \leqq P^{-n}$$

defines an interval $I^{(n)}$ of order n containing α. Hence:

Theorem 3: *Let α be an arbitrary P-adic number. For every integer n there exists one and only one interval $I^{(n)}$ of order n, enclosing α. Moreover*

$$\ldots \subset I^{(n+1)} \subset I^{(n)} \subset I^{(n-1)} \subset I^{(n-2)} \subset \ldots.$$

Let $I^{(n)}$ denote an interval of order n. Take an arbitrary P-adic number α of $I^{(n)}$. By theorem 3 there exists for every integer m one and only one interval $I^{(m)}$ of order m containing α. Moreover

$$I^{(n)} \subset I^{(n-1)} \subset I^{(n-2)} \subset \ldots.$$

Hence we have:

Theorem 4: *Let $I^{(n)}$ denote an interval of order n. For every integer $m \leqq n$ there exists one and only one interval $I^{(m)}$ of order m enclosing $I^{(n)}$. Moreover*

$$I^{(n)} \subset I^{(n-1)} \subset I^{(n-2)} \subset \ldots.$$

It follows that all intervals, containing $I^{(n)}$ as a sub-set, belong to the sequence $I^{(n)}, I^{(n-1)}, I^{(n-2)}, \ldots$.

Other fundamental properties of intervals are given by:

Theorem 5 [12]): *Any interval is an open and also a closed set.*

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[9]) T. p. 40—41.

[10]) T. p. 78 (V Stelling 3).

[11]) T. p. 79 (V Stelling 4).

[12]) T. p. 78 (V Stelling 2).

Theorem 6 [13]: *An open set* [14] *consists of a finite number or an enumerable infinity of non-overlapping intervals.*

Now the theory of measure in the field of all P-adic numbers can be developed in a manner similar to the ordinary theory of measure.

First the *measure of an interval $I^{(n)}$ of order n is defined to be the number*

$$m(I^{(n)}) = P^{-n} \text{ [15]}.$$

Then the measure of a *bounded open set* O is introduced. If O is an empty set, then by definition its measure $m(O)$ shall be equal to zero. If O is not an empty set, then by theorem 6 there exists at least one decomposition of O in a set of non-overlapping intervals $I_1, I_2, I_3, \ldots$, such that

$$O = I_1 + I_2 + I_3 + \ldots$$

Now it can be shown, that the sum

$$m(I_1) + m(I_2) + m(I_3) + \ldots$$

does not depend on the particular decomposition of O we choose; it is said to be the measure $m(O)$ of O [16].

Next we consider the so-called *bounded* sets:

Let t be an arbitrary integer; then the inequality

$$|\xi|_P \leqq P^t$$

defines the interval of order $-t$ containing zero; we shall denote this interval by K_t. All intervals K_t enclose zero, hence by theorem 3

$$\ldots \subset K_1 \subset K_0 \subset K_{-1} \subset \ldots$$

A set B is said to be *bounded* if, for a suitably chosen integer t, B is contained in the set K_t; in other words if it is possible to choose a positive number T, such, that all points β of B satisfy the inequality

$$|\beta|_P \leqq T.$$

It is easily shown, that every interval is a bounded set.

The *exterior measure* $\overline{m}(B)$ of a bounded set B is defined to be the lower bound of the measures of all bounded open sets O, which contain B [17].

B is a bounded set, hence by definition it is contained in an interval K_t. The quantity

$$\underline{m}(B) = m(K_t) - \overline{m}(K_t - B)$$

can be shown not to depend on the particular interval K_t we choose; it defines the *interior measure* of B [18]. Always $\underline{m}(B) \leqq \overline{m}(B)$ [19].

[13]) T. p. 80 (V Stelling 6, Gevolg 2).

[14]) Different from an empty set.

[15]) T. p. 86 (V Definitie 5a).

[16]) T. p. 96, V Definitie 9, but here the author uses a particular decomposition $I_1, I_2, \ldots$ of O, where $I_1, I_2, \ldots$ are the "largest" intervals of O (see p. 79 and 80, V Definitie 3 and Gevolg 2). Afterwards (p. 97, 98, V Stelling 18) it is shown, that we may substitute it by an arbitrary decomposition of O.

[17]) T. p. 101 (V Definitie 10; it is clear that in this definition O denotes a bounded open set).

[18]) T. p. 103 (V Definitie 11).

[19]) T. p. 104 (V Stelling 24).

If for a bounded set B the exterior and interior measures are equal, then B is said to be a *measurable* set with its *measure* $m(B) = \overline{m}(B) = \underline{m}(B)$ [20].

Finally the conception of measure is extended to *unbounded* sets: An arbitrary set A is said to be *measurable* if for any integer t the intersection AK_t is measurable in the above sense [21]; its *measure* is defined by

$$m(A) = \lim_{t \to \infty} m(A K_t) \text{ [22]}.$$

This limit always exists, but may be infinite. Evidently $m(A) \geqq 0$.

These extended definitions of a measurable set and of its measure are in accordance with the previous definitions with respect to open and bounded sets [23].

The following theorems are true:

Theorem 7 [24]: *The complement $K - A$ of a measurable set is measurable again.*

Theorem 8 [25]: *If A and B denote measurable sets, such that*

$$A \supset B,$$

then $m(A) \geqq m(B)$; moreover $m(A - B)$ is measurable, and

$$m(A - B) = m(A) - m(B).$$

Theorem 9 [26]: *The sum $A_1 + A_2 + A_3 + \ldots$ and the intersection $A_1 A_2 A_3 \ldots$ of a finite number or of an enumerable infinity of measurable sets $A_1, A_2, A_3, \ldots$ are measurable again. Moreover*

$$m(A_1 + A_2 + A_3 + \ldots) \leqq m(A_1) + m(A_2) + m(A_3) + \ldots,$$

and, if $A_1, A_2, A_3, \ldots$ do not overlap, even

$$m(A_1 + A_2 + A_3 + \ldots) = m(A_1) + m(A_2) + m(A_3) + \ldots$$

3. Let I be an arbitrary interval in $K(P)$. The *mean density* of a measurable set A in I is defined to be

$$\frac{m(A\,I)}{m(I)}.$$

Now let α be an arbitrary P-adic number. By theorem 3 there exists for any integer n exactly one interval $I^{(n)}$ of order n containing α. The quantity

$$\overline{d} = \overline{\lim_{n \to \infty}} \frac{m(A\,I^{(n)})}{m(I^{(n)})}$$

is said to be the *upper density* of A at the point α. Evidently $0 \leq \overline{d} \leq 1$.

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[20]) T. 108 (V Definitie 12).
[21]) T. p. 112 (V Definitie 13).
[22]) T. p. 112 (V Definitie 14).
[23]) T. p. 112 (Opmerking 1 and 2).
[24]) T. p. 115 (V Stelling 35).
[25]) T. p. 115 (V Stelling 34).
[26]) T. p. 116 (V Stelling 37), p. 117 (V Stelling 38) and p. 113 (V Stelling 33).

If

$$d = \lim_{n \to \infty} \frac{m(A\, I^{(n)})}{m(I^{(n)})}$$

exists, then d is said to be the *density* of A at the point a. If $\bar{d} = 0$ then it follows $d = 0$.

The analogue of the theorem of LEBESGUE we want to prove is:

Theorem I: *If A denotes an arbitrary measurable set in the field of all P-adic numbers, then there exists a set Z of measure zero, such that at all points outside Z the density exists, and is equal to unity at points of A and equal to zero at points outside A.*

This theorem was stated without proof in the dissertation of TURKSTRA [27]) and there it was used to obtain certain results in the theory of Diophantic approximations in the field of P-adic numbers. The authors of this note proved the theorem some years before the war, but the publication was delayed on account of several circumstances and the war.

It is easily shown, that it is sufficient to prove only the second part of theorem I:

Theorem II: *If A denotes a measurable set, then there exists a set Z of measure zero, such that at all points outside A and outside Z the density of A is equal to zero.*

For let us suppose that this last theorem has been proved, then we shall show that also theorem I is true: Let A be a measurable set. Now the complement $K - A$ of A also is measurable. Applying theorem II with $K - A$ in stead of A we obtain the following result: There exists a set Z' of measure zero, such that at all points a outside $K - A$ and outside Z' the density of $K - A$ is equal to zero, or

$$\lim_{n \to \infty} \frac{m\{(K - A)\, I^{(n)}\}}{m(I^{(n)})} = 0$$

at all points a of A outside Z'; here $I^{(n)}$ is the interval of order n containing a. Now $A\, I^{(n)}$ and $(K - A) I^{(n)}$ do not overlap, hence $m(A\, I^{(n)}) + m\{(K - A) I^{(n)}\} = m(I^{(n)})$.

It follows

$$\lim_{n \to \infty} \frac{m(A\, I^{(n)})}{m(I^{(n)})} = 1$$

at all points a of A outside Z'.

The set $Z + Z'$ evidently is of measure zero; outside $Z + Z'$ the density is equal to unity at points of A and on account of theorem II equal to zero at points outside A.

Hence theorem I is a corollary of theorem II.

In the next section of this paper we first prove the fundamental lemma 1,

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[27]) T. p. 137 (VII Stelling 1*).

then we show that the assertion of theorem II is true if we substitute a bounded open set O for the measurable set A (lemma 2), next we prove the theorem for a bounded set B (lemma 3) and lastly we establish theorem II in its general form.

4. In this section we often consider a sequence of intervals $i_1, i_2, i_3, \ldots$, satisfying certain conditions, then $i_1 + i_2 + i_3 + \ldots$ denotes the sum of these intervals. But it may happen, that there are no intervals satisfying the conditions. In this case we still write formally $i_1, i_2, i_3, \ldots$ in order to avoid the consideration of several special cases. Then $i_1 + i_2 + i_3 + \ldots$ denotes an empty set, and $m(i_1) + m(i_2) + m(i_3) + \ldots$ by definition will be equal to zero.

Now let O be a bounded open set. By theorem 6 the set O consists of a finite number or an enumerable infinity of non-overlapping intervals $I_1, I_2, I_3, \ldots$, or

$$O = I_1 + I_2 + I_3 + \ldots \quad \ldots \ldots \ldots \quad (2)$$

It follows

$$m(O) = m(I_1) + m(I_2) + m(I_3) + \ldots,$$

where the series at the right-hand side is either finite or convergent.

For every positive integer r we define the open set O_r by

$$O_r = I_r + I_{r+1} + I_{r+2} + \ldots, \quad \ldots \ldots \ldots \quad (3)$$

where O_r may be an empty set. Clearly

$$\lim_{r \to \infty} m(O_r) = 0. \quad \ldots \ldots \ldots \quad (4)$$

Lemma 1: *Let O be a bounded open set and let* (2) *be a decomposition of O in non-overlapping intervals. Let r and k denote arbitrary positive integers. Then the set* O_r, *defined in* (3), *is enclosed in an open set* $\overline{O}_{r,k} = \overline{O}_r$ *with the following two properties:*

1° $m(\overline{O}_r) \leqq kP\, m(O_r)$;

2° *if the point* a *is outside O and outside* $\overline{O}_r$, *then the upper density of O at* a *is at most* $\frac{1}{k}$.

Proof: We denote the *mean density of* O_r in an arbitrary interval i by

$$\delta(i) = \frac{m(O_r\, i)}{m(i)}.$$

Let $I = I_{r+\varrho}$ $(\varrho = 0, 1, 2, \ldots)$ be an arbitrary interval of O_r and let its order be n. If ν is an arbitrary integer $< n$, then by theorem 4 there exists exactly one interval $I^{(\nu)}$ of order ν, containing $I = I^{(n)}$ as a sub-set. Moreover

$$I = I^{(n)} \subset I^{(n-1)} \subset I^{(n-2)} \subset \ldots, \quad \ldots \ldots \ldots \quad (5)$$

and all intervals containing I as a sub-set belong to the sequence $I^{(n)}$, $I^{(n-1)}$, $I^{(n-2)}$, ...

Now $m(I^{(\nu)}) = P^{-\nu}$, hence the measure of the intervals in this sequence increases indefinitely.

By definition

$$\delta(I^{(\nu)}) = \frac{m(O_r I^{(\nu)})}{m(I^{(\nu)})}, \text{ thus } \delta(I^{(\nu)}) \leqq \frac{m(O)}{m(I^{(\nu)})}.$$

Hence the mean density of O_r in $I = I^{(n)}$ is equal to unity, but the mean density in $I^{(\nu)}$ tends to zero if ν decreases indefinitely.

Let $I^{(\mu)}$ be the *last* interval of the sequence (5) with mean density $> \frac{1}{Pk}$. It follows

$$\delta(I^{(\mu)}) > \frac{1}{kP}, \quad \ldots \ldots \quad (6)$$

and, for the next interval $I^{(\mu-1)}$ in the sequence,

$$\delta(I^{(\mu-1)}) = \frac{m(O_r I^{(\mu-1)})}{m(I^{(\mu-1)})} \leqq \frac{1}{kP}. \quad \ldots \ldots \quad (7)$$

But

$$m(I^{(\mu-1)}) = P^{-\mu+1} = P\, m(I^{(\mu)}) \quad \ldots \ldots \quad (8)$$

and $I^{(\mu)} \subset I^{(\mu-1)}$, hence

$$m(O_r I^{(\mu)}) \leqq m(O_r I^{(\mu-1)}). \quad \ldots \ldots \quad (9)$$

It follows from (7), (8) and (9)

$$\frac{m(O_r I^{(\mu)})}{P\, m(I^{(\mu)})} \leqq \frac{1}{kP}. \quad \ldots \ldots \quad (10)$$

Hence, if we denote $I^{(\mu)}$ by $\overline{I}$, we derive from (6) and (10):

An arbitrary interval $I_{r+\varrho}$ of O_r can be enclosed in an interval $\overline{I}_{r+\varrho}$, such that the mean density of O_r in $\overline{I}_{r+\varrho}$ satisfies the inequalities

$$\frac{1}{kP} < \delta(\overline{I}_{r+\varrho}) \leqq \frac{1}{k}. \quad \ldots \ldots \quad (11)$$

Evidently O_r is a sub-set of the open set

$$\overline{O}_r = \overline{I}_r + \overline{I}_{r+1} + \overline{I}_{r+2} + \ldots \quad \ldots \ldots \quad (12)$$

and we shall prove that $\overline{O}_r$ also has the other properties stated in the lemma [28]).

First we shall show that two intervals $\overline{I}_{r+\sigma}$ and $\overline{I}_{r+\tau}$ in the right-hand side of (12) either coincide or do not overlap.

Let us suppose, that $\overline{I}_{r+\sigma}$ and $\overline{I}_{r+\tau}$ at least have one point in common

[28]) If O_r is an empty set, then, by the conventions assumed in the beginning of section 4, the set $\overline{O}_r$ is empty also.

and that the order of, say, $\overline{I}_{r+\sigma}$ is not less than the order of $\overline{I}_{r+\tau}$. It follows from theorem 2

$$I_{r+\sigma} \subset \overline{I}_{r+\tau},$$

hence

$$I_{r+\sigma} \subset \overline{I}_{r+\sigma} \subset \overline{I}_{r+\tau}.$$

These intervals belong to the sequence (compare (5))

$$I_{r+\sigma} = I_{r+\sigma}^{(n)} \subset I_{r+\sigma}^{(n-1)} \subset I_{r+\sigma}^{(n-2)} \subset \ldots$$

and $\overline{I}_{r+\sigma}$ by definition is the last interval in this sequence with mean density $> \frac{1}{kP}$. But it follows from (11), that $\overline{I}_{r+\tau}$ also has a mean density $> \frac{1}{kP}$. We obtain a contradiction unless $\overline{I}_{r+\sigma}$ and $\overline{I}_{r+\tau}$ coincide.

If we denote the different intervals of (12) by $\overline{I}_{r_1}, \overline{I}_{r_2}, \overline{I}_{r_3}, \ldots$, then these intervals do not overlap and

$$\overline{O}_r = \overline{I}_{r_1} + \overline{I}_{r_2} + \overline{I}_{r_3} + \ldots$$

Hence

$$m(\overline{O}_r) = m(\overline{I}_{r_1}) + m(\overline{I}_{r_2}) + m(\overline{I}_{r_3}) + \ldots \quad (13)$$

The mean density of O_r in $\overline{I}_{r_\nu}$ ($\nu = 1, 2, 3, \ldots$) is $> \frac{1}{Pk}$, or

$$m(O_r \overline{I}_{r_\nu}) > \frac{1}{Pk} m(\overline{I}_{r_\nu});$$

we derive from (13), remembering that O_r and $\overline{O}_r$ may be empty,

$$\frac{1}{Pk} m(\overline{O}_r) = \frac{1}{Pk} m(\overline{I}_{r_1}) + \frac{1}{Pk} m(\overline{I}_{r_2}) + \ldots$$
$$\leqq m(O_r \overline{I}_{r_1}) + m(O_r \overline{I}_{r_2}) + \ldots = m(O_r \overline{O}_r) = m(O_r),$$

hence $\overline{O}_r$ has the property 1° of lemma 1.

Let a be a point outside O and outside $\overline{O}_r$ (the conditions in lemma 1, 2°). In particular a is outside $I_1 + I_2 + \ldots + I_{r-1}$; this set is closed, for every interval by theorem 5 is a closed set. Hence the complement is open. Therefore we can include a in an interval $E^{(p)}$ of order p, outside $I_1 + I_2 + \ldots + I_{r-1}$.

Let n denote an integer $\geqq p$. By theorem 3 there exists exactly one interval $E^{(n)} = E$ of order n containing a; moreover $E^{(n)} \subset E^{(p)}$. Hence E is outside $I_1 + I_2 + \ldots + I_{r-1}$, or

$$EO = EO_r. \quad (14)$$

We have to show, that the upper density of O at a is $\frac{1}{k}$ at most, or

$$\overline{\lim_{n \to \infty}} \frac{m(E^{(n)} O)}{m(E^{(n)})} \leqq \frac{1}{k}.$$

Evidently it is sufficient to prove, that the mean density of O in $E^{(n)} = E$ is $\frac{1}{k}$ at most, i.e.

$$\delta(E) = \frac{m(E\,O)}{m(E)} \leqq \frac{1}{k},$$

for every integer $n \geqq p$.

By (14)

$$\delta(E) = \frac{m(E\,O_r)}{m(E)}.$$

We may suppose $m(E\,\overline{O}_r) \neq 0$, for otherwise $m(E\,O_r) = 0$, hence $\delta(E) = 0$. It follows

$$\delta(E) \leqq \frac{m(E\,O_r)}{m(E\,\overline{O}_r)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Now

$$E\,\overline{O}_r = E\,\overline{I}_{r_1} + E\,\overline{I}_{r_2} + E\,\overline{I}_{r_3} + \ldots \quad . \quad . \quad . \quad . \quad . \quad (16)$$

If $\overline{I}_{r_\nu}$ is an arbitrary interval of the sequence $\overline{I}_{r_1}, \overline{I}_{r_2}, \overline{I}_{r_3}, \ldots$, and if E and $\overline{I}_{r_\nu}$ have at least one point in common, then by theorem 2 either $\overline{I}_{r_\nu} \subset E$ or $E \subset \overline{I}_{r_\nu}$. But this latter possibility would involve, that a was a point of $\overline{O}_r$, contrary to hypothesis, hence $\overline{I}_{r_\nu} \subset E$. Hence for an arbitrary interval $\overline{I}_{r_\nu}$ either $E\,\overline{I}_{r_\nu}$ is empty or $E\,\overline{I}_{r_\nu}$ coincides with $\overline{I}_{r_\nu}$.

Therefore we deduce from (16) the existence of a sequence $\overline{I}_{p_1}, \overline{I}_{p_2}, \overline{I}_{p_3}, \ldots$ of non-overlapping intervals, such that

$$E\,\overline{O}_r = \overline{I}_{p_1} + \overline{I}_{p_2} + \overline{I}_{p_3} + \ldots$$

Hence, taking account of $O_r \subset \overline{O}_r$,

$$E\,O_r = O_r\,\overline{I}_{p_1} + O_r\,\overline{I}_{p_2} + O_r\,\overline{I}_{p_3} + \ldots,$$

where $O_r\,\overline{I}_{p_1}, O_r\,\overline{I}_{p_2}, O_r\,\overline{I}_{p_3}, \ldots$ are non-overlapping measurable sets.

From (15) it follows therefore

$$\delta(E) \leqq \frac{m(O_r\,\overline{I}_{p_1}) + m(O_r\,\overline{I}_{p_2}) + \ldots}{m(\overline{I}_{p_1}) + m(\overline{I}_{p_2}) + \ldots}.$$

We know by (11) that the mean density of O_r in $\overline{I}_{p_\nu}$

$$\frac{m(O_r\,\overline{I}_{p_\nu})}{m(\overline{I}_{p_\nu})} \leqq \frac{1}{k} \qquad (\nu = 1, 2, \ldots).$$

It follows $\delta(E) \leqq \frac{1}{k}$ and this proves the lemma.

Lemma 2: *If O denotes an open bounded set, then there exists a set Z of measure zero, such that at all points outside O and outside Z the density of O is equal to zero.*

Proof: 1. Let k be an integer > 1, further let β be a P-adic number, such that the upper density of O at β is $> \frac{1}{k}$.

We introduce for $r = 1, 2, \ldots$ the open sets $\overline{O}_r$ considered in lemma 1. Then — applying this lemma — we deduce that outside O and outside $\overline{O}_r$ the upper density of O at every point is $\leqq \frac{1}{k}$.

It follows that β belongs to every set $O + \overline{O}_r$, hence it is contained in the intersection S_k of these sets. All sets $O + \overline{O}_r$ are open, hence S_k is measurable. Moreover

$$O \subset S_k \subset O + \overline{O}_r \qquad (r = 1, 2, \ldots),$$

hence

$$m(O) \leqq m(S_k) \leqq m(O) + m(\overline{O}_r).$$

By lemma 1

$$m(\overline{O}_r) \leqq k P\, m(O_r)$$

and by (4)

$$\lim_{r \to \infty} m(O_r) = 0;$$

it follows

$$m(S_k) = m(O).$$

Introducing the set $Z_k = S_k - O$, we obtain: Every point β with upper density $> \frac{1}{k}$ either belongs to O or it is contained in a set Z_k of measure zero $(k = 2, 3, \ldots)$.

2. The set

$$Z = Z_2 + Z_3 + Z_4 + \ldots$$

clearly is of measure zero.

We consider a point β outside O and outside Z. We shall show that the density at β is equal to zero. For otherwise the upper density $\overline{d}$ at β was positive. Now take an integer k, such that $\overline{d} > \frac{1}{k}$, then $k > 1$, and by the previous result β belongs either to O or to the set Z_k, contrary to hypothesis.

Lemma 3: *Let B be a bounded measurable set. Then there exists a set Z of measure zero, such that at all points outside B and outside Z the density is equal to zero.*

Proof: B is a bounded set; hence for a suitably chosen integer t the set K_t of all P-adic numbers ξ with

$$|\xi|_p \leqq P^t$$

contains the set B. The measure of B is equal to the exterior measure, hence it is the lower bound of the measures of all open bounded sets O which contain B.

It follows the existence of a sequence

$$O_1,\ O_2,\ O_3, \ldots$$

of bounded open sets, such that

$$B \subset O_n \text{ and } \lim_{n \to \infty} m(O_n) = m(B).$$

Now every set

$$B_n = O_n - B$$

is measurable and

$$\lim_{n \to \infty} m(B_n) = \lim_{n \to \infty} \{m(O_n) - m(B)\} = 0.$$

Hence the intersection Z_0 of these sets $B_1, B_2, B_3, \ldots,$

$$Z_0 = B_1 B_2 B_3 \ldots,$$

is of measure zero.

By lemma 2 there exists for every (bounded open) set O_n a set Z_n of measure zero, such that at all points outside O_n and outside Z_n the density of O_n is equal to zero ($n = 1, 2, \ldots$).

We will show that the set

$$Z = Z_0 + Z_1 + Z_2 + \ldots$$

of measure zero has the property stated in lemma 3.

Therefore let a denote a point outside B and outside Z, then clearly a is outside $Z_0 = B_1 B_2 B_3 \ldots$, hence it is not contained in at least one of the sets $B_1, B_2, B_3, \ldots$; say a is outside $B_n = O_n - B$. But a is not contained in B either, hence a is outside O_n. Moreover a is outside Z_n. Applying lemma 2 we find that the density of O_n at a is zero, i.e.

$$\lim_{\nu \to \infty} \frac{m(O_n I^{(\nu)})}{m(I^{(\nu)})} = 0;$$

here $I^{(\nu)}$ denotes the interval of order ν containing a. Now $B \subset O_n$, hence $m(B I^{(\nu)}) \leq m(O_n I^{(\nu)})$, hence the density of B at a

$$\lim_{\nu \to \infty} \frac{m(B I^{(\nu)})}{m(I^{(\nu)})} = 0.$$

Proof of theorem II: The set of all intervals is enumerable (theorem 1). Let $I_1, I_2, I_3, \ldots$ denote the set of all intervals of order zero. Now A is an arbitrary measurable set, hence every set

$$B_n = A I_n \qquad (n = 1, 2, \ldots)$$

is bounded and measurable; moreover

$$A = B_1 + B_2 + B_3 + \ldots$$

Applying lemma 3 on the sets $B_1, B_2, B_3, \ldots$ we derive the existence of sets $Z_1, Z_2, Z_3, \ldots$ of measure zero, such that at all points outside B_n and outside Z_n the density of B_n is equal to zero $(n = 1, 2, 3, \ldots)$. The set

$$Z_1 + Z_2 + Z_3 + \ldots = Z$$

is of measure zero. We shall show that at a point α outside Z and outside A the density of A is equal to zero.

The point α belongs to the interval

$$|\xi \alpha|_p \leqq 1$$

of order zero. This interval belongs to the sequence $I_1, I_2, I_3, \ldots$; we shall denote it by I_N.

Now α is outside B_N and also outside Z_N, hence by lemma 3 the density of B_N at α is equal to zero:

$$\lim_{\nu \to \infty} \frac{m(B_N I^{(\nu)})}{m(I^{(\nu)})} = 0,$$

where $I^{(\nu)}$ denotes the interval of order ν containing α. Both intervals $I^{(0)}$ and I_N contain α and are of order zero, hence $I^{(0)}$ coincides with I_N and all intervals $I^{(\nu)}$ with $\nu > 0$ are enclosed in I_N, hence

$$A I^{(\nu)} \subset A I_N = B_N.$$

It follows

$$A I^{(\nu)} \subset B_N I^{(\nu)};$$

clearly

$$B_N I^{(\nu)} \subset A I^{(\nu)},$$

so that $A I^{(\nu)}$ coincides with $B_N I^{(\nu)}$ for $\nu \geqq 0$, hence

$$\lim_{\nu \to \infty} \frac{m(A I^{(\nu)})}{m(I^{(\nu)})} = 0$$

and this shows that the density of A at α in fact is equal to zero.

This proves the theorem.

Mathematics. — *Non-homogeneous binary quadratic forms.* By H. DAVENPORT. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of June 29, 1946.)

1. Let α, β, γ, δ be real numbers with $\triangle = \alpha\delta - \beta\gamma \neq 0$. A famous theorem of MINKOWSKI asserts that for any real numbers λ, μ there exist integers x, y such that

$$|(\alpha x + \beta y + \lambda)(\gamma x + \delta y + \mu)| \leqslant \tfrac{1}{4}|\triangle|.$$

I shall suppose that α/β and γ/δ are irrational; it is then known that the result is true with the sign of strict inequality. If we write

$$(\alpha x + \beta y)(\gamma x + \delta y) = ax^2 + bxy + cy^2 = f(x, y),$$

we can express MINKOWSKI's theorem in the form: if $f(x, y)$ is any indefinite binary quadratic form which does not represent zero, then for any real x_0, y_0 there exist real x, y with

$$x \equiv x_0 \pmod 1, \quad y \equiv y_0 \pmod 1$$

such that [1])

$$|f(x, y)| < \tfrac{1}{4}\sqrt{d}. \qquad (1)$$

where $d = b^2 - 4ac = \triangle^2$.

Many proofs of MINKOWSKI's theorem have been given, but I believe it is still possible to add to the existing knowledge [2]). In the first place, one can easily deduce from the existing proofs slightly more than has been stated above. For any such quadratic form $f(x, y)$ there exists a number $M(f)$ satisfying

$$M(f) < \tfrac{1}{4} \qquad (2)$$

such that, instead of (1), one can satisfy

$$|f(x, y)| \leqslant M(f)\sqrt{d}.$$

I define $M(f)$ to be the lower bound of all such numbers, and the present note is concerned with the investigation of some properties of $M(f)$.

In the first place, I prove an estimate for $M(f)$ in terms of any value of f which satisfies $0 < |f| < \sqrt{d}$.

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[1]) For a positive definite quadratic form, it is easily seen that no result of this type can be valid. The best possible inequality in terms of the coefficients of the equivalent reduced form was given by DIRICHLET (*Werke*, II, 29—48).

[2]) For references to literature, see KOKSMA, *Diophantische Approximationen*. See also MORDELL, *Journal London Math. Soc.*, **16** (1941), 86—88 and **18** (1943), 218—221.

Theorem 1. *Let f_1 be any value of $|f(x, y)|$ which corresponds to coprime integral values of x, y and which satisfies*

$$0 < f_1 < \sqrt{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Then

$$M(f) \leq \tfrac{1}{4}\,\Phi\left(\frac{f_1}{\sqrt{d}}\right),$$

where

$$\Phi(t) = \begin{cases} \sqrt{1-4t^2} \ \text{for} \ 0 < t \leq \dfrac{1}{2\sqrt{2}}, \\ \dfrac{1}{4t} \ \text{for} \ \dfrac{1}{2\sqrt{2}} \leq t \leq \tfrac{1}{2}, \\ t \ \text{for} \ \tfrac{1}{2} \leq t < 1. \end{cases}$$

Since $(t) < 1$, this, incidentally, proves (2). The result is best when $|f|$ has a value f_1 which is about $\frac{1}{2}\sqrt{d}$. The existence of *some* value of $|f|$ satisfying (3) is well known from GAUSS's theory of reduction [3]).

The known results (see KOKSMA, 77—79) on non-homogeneous linear forms suggest that (2) is the best possible general inequality for $M(f)$, but this does not seem to have been proved. I give a proof in:

Theorem 2. *If $f(x, y) = x^2 + 2kxy - y^2$, where k is a positive integer, then*

$$M(f) = \frac{k}{4\sqrt{k^2+1}}.$$

Among the most interesting indefinite binary forms are those of MARKOFF's series [4]): $x^2 + xy - y^2$, $x^2 - 2y^2$, $5x^2 + 11xy - 5y^2$, ... The precise values of $M(f)$ for the first two forms are given in the following theorems.

Theorem 3. *If $f(x, y) = x^2 + xy - y^2$, then*

$$M(f) = \frac{1}{4\sqrt{5}}.$$

Theorem 4. *If $f(x, y) = x^2 - 2y^2$, then*

$$M(f) = \frac{1}{4\sqrt{2}}.$$

The second of these results is an immediate consequence of Theorem 1. For $d = 8$, and we can take $f_1 = 1$, whence $M(f) \leq \frac{1}{4\sqrt{2}}$. On the other

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[3]) See, for example, DICKSON, *Introd. to the theory of numbers*, 101.

[4]) See, for example, BACHMANN, *Die Arithmetik der quadratischen Formen*, II, Kap. 4.

hand, if $x \equiv 0 \pmod{1}$ and $y \equiv \frac{1}{2} \pmod{1}$ then obviously

$$|x^2 - 2y^2| \geqq \tfrac{1}{2} = \frac{1}{4\sqrt{2}}\sqrt{d}, \text{ so that } M(f) \geqq \frac{1}{4\sqrt{2}}.$$

The third form of MARKOFF's series presents more difficulty and here I have not yet found the exact value of $M(f)$. A result valid for all the MARKOFF forms is:

Theorem 5. *For any form of* MARKOFF*'s series,*

$$M(f) < \tfrac{1}{4}\sqrt{\tfrac{5}{6}}.$$

2. For the proof of Theorem 1 we need two lemmas [5]).

Lemma 1. *For any real β and x_0, there exists x with $x \equiv x_0$ (mod. 1) such that*

$$|x^2 - \beta^2| \leqq \begin{cases} \frac{1}{4} - \beta^2 & \text{if } \beta^2 \leqq \frac{1}{8}, \\ \beta^2 & \text{if } \frac{1}{8} \leqq \beta^2 \leqq \frac{1}{2}, \\ \sqrt{\beta^2 - \frac{1}{4}} & \text{if } \beta^2 \geqq \frac{1}{2}. \end{cases}$$

P r o o f. (1) If $\beta^2 \leqq \frac{1}{8}$, we choose x to satisfy $|x| \leqq \frac{1}{2}$, and have $-\beta^2 \leqq x^2 - \beta^2 \leqq \frac{1}{4} - \beta^2$, whence the result.

(2) If $\frac{1}{8} \leqq \beta^2 \leqq \frac{1}{2}$, we choose x to satisfy

$$\beta\sqrt{2} - 1 \leqq x \leqq \beta\sqrt{2}.$$

Since $1 - \beta\sqrt{2} \leqq \beta\sqrt{2}$, we have $x^2 \leqq 2\beta^2$, whence $|x^2 - \beta^2| \leqq \beta^2$.

(3) If $\beta^2 \geqq \frac{1}{2}$ we choose x to satisfy

$$\sqrt{\beta^2 - \tfrac{1}{4}} - \tfrac{1}{2} \leqq x \leqq \sqrt{\beta^2 - \tfrac{1}{4}} + \tfrac{1}{2}.$$

Since the number on the left is positive or zero, we have

$$\beta^2 - \sqrt{\beta^2 - \tfrac{1}{4}} \leqq x^2 \leqq \beta^2 + \sqrt{\beta^2 - \tfrac{1}{4}},$$

whence the result.

Lemma 2. *For any $\alpha > \frac{1}{4}$, and any β with $|\beta| \leqq \alpha$, and any x_0, there exists $x \equiv x_0$ (mod. 1) such that*

$$|x^2 - \beta^2| \leqq \alpha\, \Phi\left(\frac{1}{4\alpha}\right),$$

where $\Phi(t)$ is the function defined in the enunciation of Theorem 1.

P r o o f. (1) If $\alpha^2 \leqq \frac{1}{4}$, then $\beta^2 \leqq \frac{1}{4}$, and $\frac{1}{4\alpha} \geqq \frac{1}{2}$, and

$$\max\left(\tfrac{1}{4} - \beta^2, \beta^2\right) \leqq \tfrac{1}{4} = \alpha\, \Phi\left(\frac{1}{4\alpha}\right).$$

[5]) MORDELL, in *Journal London Math. Soc.*, **3** (1928), 19—22 gave a direct proof of MINKOWSKI's theorem, using an inequality which is a particular case of that proved here.

(2) If $\frac{1}{4} \leqslant \alpha^2 \leqslant \frac{1}{2}$ then $\beta^2 \leqslant \frac{1}{2}$, and $\frac{1}{2\sqrt{2}} \leqslant \frac{1}{4\alpha} \leqslant \frac{1}{2}$, and

$$\max\left(\tfrac{1}{4} - \beta^2, \beta^2\right) \leqslant \alpha^2 = \alpha\, \Phi\left(\frac{1}{4\alpha}\right).$$

(3) If $\alpha^2 \geqslant \frac{1}{2}$ then $\frac{1}{4\alpha} \leqslant \frac{1}{2\sqrt{2}}$, and $\alpha\, \Phi\left(\frac{1}{4\alpha}\right) = \sqrt{\alpha^2 - \frac{1}{4}}$.

If $\beta^2 \leq \frac{1}{8}$, we have

$$\tfrac{1}{4} - \beta^2 \leqslant \tfrac{1}{4} < \sqrt{\alpha^2 - \tfrac{1}{4}}.$$

If $\frac{1}{8} \leq \beta^2 \leq \frac{1}{2}$, we have

$$\beta^2 \leqslant \tfrac{1}{2} \leqslant \sqrt{\alpha^2 - \tfrac{1}{4}}.$$

Finally, if $\beta^2 \geq \frac{1}{2}$, we have

$$\sqrt{\beta^2 - \tfrac{1}{4}} \leqslant \sqrt{\alpha^2 - \tfrac{1}{4}}.$$

Thus Lemma 2 follows from Lemma 1.

P r o o f o f T h e o r e m 1. After an integral unimodular transformation applied to x and y, we can suppose that

$$f(x, y) = ax^2 + bxy + cy^2, \text{ where } |a| = f_1.$$

The conditions $x \equiv x_0$ (mod. 1), $y \equiv y_0$ (mod. 1) are transformed into similar conditions. Now

$$f(x, y) = a\left\{(x + \theta y)^2 - \frac{d}{4a^2}\, y^2\right\},$$

where $\theta = \frac{b}{2a}$. We choose y to satisfy $|y| \leq \frac{1}{2}$, then we choose x so that $x + \theta y$ satisfies the inequality of Lemma 2, where

$$\alpha^2 = \frac{d}{16\,a^2},\ \beta^2 = \frac{d}{4a^2}\, y^2 \leqslant \alpha^2.$$

We obtain

$$|f(x, y)| \leqslant |a|\, \frac{\sqrt{d}}{4\,|a|}\, \Phi\left(\frac{|a|}{\sqrt{d}}\right) = \tfrac{1}{4}\sqrt{d}\, \Phi\left(\frac{f_1}{\sqrt{d}}\right),$$

as required.

3. To prove Theorem 2 we note first that $M(f) \leq \frac{k}{4\sqrt{k^2+1}}$, on taking $f_1 = 1$ in Theorem 1. Hence it suffices (taking $x_0 = y_0 = \frac{1}{2}$) to prove that

$$\left|(x + \tfrac{1}{2})^2 + 2\,k\,(x + \tfrac{1}{2})\,(y + \tfrac{1}{2}) - (y + \tfrac{1}{2})^2\right| \geqslant \tfrac{1}{2}\,k$$

for all integers x, y, since the quadratic form in the theorem has

$$d = 4(k^2 + 1).$$

This is proved in the following lemma, which is of some interest in itself, though it is probably not new.

Lemma 3. *If k is any positive integer, and x, y are any odd integers, then*

$$|x^2 + 2\,kxy - y^2| \geqslant 2\,k.$$

P r o o f. The result is suggested by the fact that all the convergents to the continued fraction for $\sqrt{k^2+1} - k$ have either numerator or denominator even, so that approximations to this irrational number by fractions $\frac{x}{y}$ with x, y odd are necessarily bad. But it is easy to give a direct proof.

Suppose there exists a solution of

$$|x^2 + 2\,kxy - y^2| < 2\,k, \quad x, y \text{ odd},$$

and consider the solution for which $|y|$ is least. Without loss of generality we can suppose $y > 0$, since otherwise we change the signs of both variables. The inequality can be written

$$|(x + ky)^2 - (k^2 + 1)\,y^2| < 2\,k \quad \ldots\ldots \quad (4)$$

Put

$$|x + ky| = ky + z, \quad \ldots\ldots \quad (5)$$

then z is odd. Also $|z| < y$, for if $z \geqq y$ we get

$$(x + ky)^2 - (k^2 + 1)\,y^2 \geqslant (k+1)^2\,y^2 - (k^2+1)\,y^2 = 2\,ky \geqslant 2\,k,$$

and if $z \leqq -y$ we get

$$(x + ky)^2 - (k^2 + 1)\,y^2 \leqslant (k-1)^2\,y^2 - (k^2+1)\,y^2 = -2\,ky \leqslant -2\,k.$$

From (4) and (5),

$$|(-y)^2 + 2\,k\,(-y)\,z - z^2| < 2\,k,$$

and since $|z| < y$ this contradicts the hypothesis that $|y|$ was least.

4. **Lemma 4.** *If* $\frac{1}{\sqrt{5}} \leq y \leq \frac{1}{2}$ *then*

$$\sqrt{\tfrac{5}{4}\,y^2 + \tfrac{1}{4}} + \sqrt{\tfrac{5}{4}\,(y-1)^2 + \tfrac{1}{4}} \geqslant \tfrac{3}{2},$$

and

$$\sqrt{\tfrac{5}{4}\,y^2 - \tfrac{1}{4}} + \sqrt{\tfrac{5}{4}\,(y-1)^2 - \tfrac{1}{4}} \leqslant \tfrac{1}{2}.$$

P r o o f. On squaring both sides, the first inequality becomes

$$5y^2 + 5\,(1-y)^2 + 2 + 2\sqrt{(5y^2+1)\,(5\,(1-y)^2+1)} \geqslant 9,$$

or, after squaring again,

$$(5y^2 + 1)\,(5\,(1-y)^2 + 1) \geqslant (1 + 5y - 5y^2)^2.$$

The difference is $5(2\,y - 1)^2 \geqq 0$.

Similarly, the second inequality is

$$5y^2 + 5(1-y)^2 - 2 + 2\sqrt{(5y^2-1)(5(1-y)^2-1)} \leqq 1,$$

or

$$(5y^2-1)(5(1-y)^2-1) \leqq (5y-5y^2-1)^2,$$

on noting that $5y - y^2$ increases as y increases, so that

$$5y - 5y^2 \geqq \sqrt{5} - 1 > 1.$$

The right hand side exceeds the left hand side by $5(2y-1)^2$.

Proof of Theorem 3. We shall prove that we can find $x \equiv x_0$ (mod. 1) and $y \equiv y_0$ (mod. 1) to satisfy

$$|x^2 + xy - y^2| \leqq \tfrac{1}{4}, \quad \ldots \ldots \ldots \quad (6)$$

and since for this form $d = 5$, the result follows.

We first choose y to satisfy $|y| \leq \frac{1}{2}$. We can suppose without loss of generality that $y \geq 0$, since otherwise we can put $y = -y'$, $x = x' + y'$. Writing the inequality as

$$|(x + \tfrac{1}{2}y)^2 - \tfrac{5}{4}y^2| \leqq \tfrac{1}{4},$$

we observe first that if $\frac{5}{4}y^2 \leq \frac{1}{4}$ we can choose x so that $x + \frac{1}{2}y$ satisfies the inequality of Lemma 1, and this suffices for our purpose, since

$$\max(\tfrac{1}{4} - \tfrac{5}{4}y^2, \tfrac{5}{4}y^2) \leqq \tfrac{1}{4}.$$

Hence we can suppose $\frac{1}{\sqrt{5}} \leq y \leq \frac{1}{2}$. Consider the two intervals

$$\sqrt{\tfrac{5}{4}y^2 - \tfrac{1}{4}} - \tfrac{1}{2}y \leqq x \leqq \sqrt{\tfrac{5}{4}y^2 + \tfrac{1}{4}} - \tfrac{1}{2}y,$$

$$-\sqrt{\tfrac{5}{4}(y-1)^2 + \tfrac{1}{4}} - \tfrac{1}{2}(y-1) \leqq x \leqq -\sqrt{\tfrac{5}{4}(y-1)^2 - \tfrac{1}{4}} - \tfrac{1}{2}(y-1),$$

or say

$$\lambda_1 \leqq x \leqq \lambda_2,$$
$$\lambda_3 \leqq x \leqq \lambda_4.$$

By Lemma 4, we have

$$\lambda_1 \leqq \lambda_4, \quad \lambda_2 \geqq \lambda_3 + 1.$$

Hence these two intervals cover the whole of the interval

$$\lambda_3 \leqq x \leqq \lambda_3 + 1$$

and so we can find a value of $x \equiv x_0$ (mod. 1) in one of them. In the first interval,

$$\tfrac{5}{4}y^2 - \tfrac{1}{4} \leqq (x + \tfrac{1}{2}y)^2 \leqq \tfrac{5}{4}y^2 + \tfrac{1}{4}.$$

In the second interval,

$$\tfrac{5}{4}(y-1)^2 - \tfrac{1}{4} \leqq (x + \tfrac{1}{2}(y-1))^2 \leqq \tfrac{5}{4}(y-1)^2 + \tfrac{1}{4}.$$

Thus in the former case the pair x, y and in the latter case the pair $x, y-1$ satisfy (6).

That (6) is the best possible inequality is obvious on taking $x \equiv \frac{1}{2}$ (mod. 1) and $y \equiv \frac{1}{2}$ (mod. 1).

5. Finally, Theorem 5 is a simple deduction from Theorem 1. Any form of MARKOFF's series has [6]) minimum Q and discriminant $d = 9\,Q^2 - 4$, where Q is one of the MARKOFF numbers

1, 2, 5, 13, 29, 34, 89, ...

Hence we can take $f_1 = Q$ and have

$$\frac{f_1}{\sqrt{d}} = \frac{1}{\sqrt{9 - \frac{4}{Q^2}}},$$

so that

$$\tfrac{1}{3} < \frac{f_1}{\sqrt{d}} \leq \frac{1}{\sqrt{5}}.$$

By Theorem 1, since $\Phi(t)$ decreases as t increases for $t < \frac{1}{2}$,

$$M(f) < \tfrac{1}{4}\sqrt{1 - \tfrac{4}{9}},$$

whence the result.

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6) BACHMANN, loc. cit. 123.

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Mathematics. — *The product of n homogeneous linear forms.* By H. DAVENPORT. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of June 29, 1946.)

1. Let $L_1, \ldots, L_n$ be n homogeneous linear forms in n variables $u_1, \ldots, u_n$, with real coefficients and determinant 1. A well known theorem of MINKOWSKI [1]) asserts that there exist integral values of $u_1, \ldots, u_n$, not all zero, for which

$$|L_1| + \ldots + |L_n| \leqslant (n!)^{\frac{1}{n}},$$

and so, by the inequality of the arithmetic and geometric means,

$$|L_1 \ldots L_n|^{\frac{1}{n}} \leqslant \frac{(n!)^{\frac{1}{n}}}{n}.$$

The number on the right is approximately $1/e$ when n is large. A stronger inequality, for large n, may be deduced from BLICHFELDT's theorem [2]) on the minimum of a positive definite quadratic form in n variables. Let γ_n be the greatest value of this minimum for all such quadratic forms of determinant 1. Then there exist integral values of $u_1, \ldots, u_n$, not all zero, for which

$$L_1^2 + \ldots + L_n^2 \leqslant \gamma_n,$$

and so

$$|L_1 \ldots L_n|^{\frac{1}{n}} \leqslant \left(\frac{\gamma_n}{n}\right)^{\frac{1}{2}}. \qquad (1)$$

BLICHFELDT's theorem is that

$$\gamma_n \leqslant \frac{2}{\pi}\left\{\Gamma\left(2+\frac{n}{2}\right)\right\}^{\frac{2}{n}} \backsim \frac{n}{\pi e} \quad \text{as } n \to \infty,$$

so that the number on the right of (1) can be taken to be $\frac{1}{\sqrt{\pi e}}$ approximately, when n is large.

I obtain a further improvement in the following

Theorem. *If $L_1, \ldots, L_n$ are homogeneous linear forms in $u_1, \ldots, u_n$ with*

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[1]) See, for example, HARDY and WRIGHT, *Introd. to the theory of numbers*, second edition (1945), Theorem 450.

[2]) *Trans. American Math. Soc.*, 15 (1914), 227—235. See also REMAK, *Math. Zeitschr.*, 26 (1927), 694—699 and BLICHFELDT, *Math. Annalen*, 101 (1929), 605—608.

real coefficients and determinant 1, *then the lower bound* M *of* $|L_1 \dots L_n|^{\frac{1}{n}}$ *for integral values of* $u_1, \dots, u_n$, *not all zero, satisfies*

$$M \leqslant \left(\frac{\gamma_n}{n\, \varkappa^{1-\frac{1}{n}}} \right)^{\frac{1}{2}},$$

where $\varkappa > 1.4$ *is a certain absolute constant.*

For large n, we have, therefore,

$$M \leqslant \frac{1}{\sqrt{\varkappa \pi e}}$$

approximately. It is not easy to determine $\varkappa$ accurately, but the numerical evidence strongly suggests that it is about 1.473.

The definition of $\varkappa$ is as follows. For any $\alpha > 0$, the equation

$$\xi^{-\alpha} + (\xi - 1)^{-\alpha} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

defines a unique number $\xi > 1$, which is obviously a continuous function of α. We define

$$\Phi(\alpha) = \frac{\alpha \xi + \xi^{-\alpha}}{\alpha + 1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is also a continuous function of α, and we define $\varkappa$ to be the minimum of $\Phi(\alpha)$. In order not to interrupt the main argument, we postpone the investigation of $\varkappa$ till Lemma 3, and take for granted the result of that Lemma, namely that $1.4 < \varkappa < 1.5$.

2. **Lemma 1.** *Suppose* $0 < \varepsilon < \frac{1}{4}$, *and let* $x_1, \dots, x_n$ *be real numbers satisfying*

$$|x_1 \dots x_n| > 1 - \varepsilon,\ |(x_1 - 1) \dots (x_n - 1)| > 1 - \varepsilon,\ |(x_1 + 1) \dots (x_n + 1)| > 1 - \varepsilon. \quad (4)$$

Then

$$x_1^2 + \dots + x_n^2 > (1 - \varepsilon)^2 \varkappa n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Proof. By (4), we have

$$x_1^2 \dots x_n^2 > (1 - \varepsilon)^2, \quad |(x_1^2 - 1) \dots (x_n^2 - 1)| > (1 - \varepsilon)^2 \quad . \quad . \quad . \quad (6)$$

It is plainly impossible that $x_1^2, \dots, x_n^2$ should all be less than 1, for then either $x_1^2 \leq \frac{1}{2}$ or $1 - x_1^2 \leq \frac{1}{2}$, and one at least of the products in (6) would be less than $\frac{1}{2}$. We can also suppose that not all of $x_1^2, \dots, x_n^2$ are greater than 1, for this would imply

$$\frac{x_1^2 + \dots + x_n^2}{n} - 1 \geqslant \{(x_1^2 - 1) \dots (x_n^2 - 1)\}^{\frac{1}{n}} > (1 - \varepsilon)^{\frac{2}{n}},$$

whence

$$x_1^2 + \dots + x_n^2 > \{1 + (1 - \varepsilon)^{\frac{2}{n}}\} n,$$

which is a stronger inequality than (5).

We can therefore suppose, without loss of generality, that

$$x_1^2 > 1, \dots, x_p^2 > 1, \quad x_{p+1}^2 < 1, \dots, x_n^2 < 1,$$

where $1 \leq p \leq n-1$. Let

$$\xi = \frac{x_1^2 + \dots + x_p^2}{p}, \quad \eta = \frac{x_{p+1}^2 + \dots + x_n^2}{n-p},$$

so that $\xi > 1 > \eta > 0$. By the inequality of the arithmetic and geometric means, applied to (6), we have

$$\xi^p \, \eta^{n-p} > (1-\varepsilon)^2,$$
$$(\xi-1)^p \, (1-\eta)^{n-p} > (1-\varepsilon)^2.$$

Put $\xi = (1-\varepsilon)^2 \xi_0$, then $\xi - 1 < (1-\varepsilon)^2 (\xi_0 - 1)$, and we have

$$\xi_0^p \, \eta^{n-p} > 1,$$
$$(\xi_0-1)^p \, (1-\eta)^{n-p} > 1.$$

If we write $\alpha = \frac{p}{n-p}$, these become

$$\xi_0^\alpha \, \eta > 1,$$
$$(\xi_0-1)^\alpha \, (1-\eta) > 1,$$

and consequently ξ_0 satisfies

$$\xi_0^{-\alpha} + (\xi_0-1)^{-\alpha} < 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Now

$$x_1^2 + \dots + x_n^2 = p\,\xi + (n-p)\,\eta > (1-\varepsilon)^2 \{ p\,\xi_0 + (n-p)\,\eta \}$$
$$> (1-\varepsilon)^2 \, n \, \frac{(\alpha\,\xi_0 + \xi_0^{-\alpha})}{(\alpha + 1)}.$$

For fixed α, the expression $\alpha\xi_0 + \xi_0^{-\alpha}$ decreases as ξ_0 decreases, provided $\xi_0 > 1$, since its derivative with respect to ξ_0 is $\alpha(1 - \xi_0^{-\alpha-1}) > 0$. Hence the last inequality remains valid if ξ_0 is defined as a function of α by equality in (7). It follows from (2) and (3) that

$$x_1^2 + \dots + x_n^2 > (1-\varepsilon)^2 \, n \, \Phi(\alpha) \geqslant (1-\varepsilon)^2 \, n \, \varkappa,$$

which proves the Lemma.

3. Our next lemma concerns the successive minima of a positive definite quadratic form $Q(u_1, \dots, u_n)$. These were defined by MINKOWSKI as follows. The first minimum S_1 is the minimum of Q for all integral values of $u_1, \dots, u_n$, not all zero. After a suitable integral unimodular transformation of the variables, we can suppose that $S_1 = Q(1, 0, \dots, 0)$. Now S_2 is defined as the minimum of Q for all integral values of $u_1, \dots, u_n$ such that

not all of $u_2, \ldots, u_n$ are zero. After a further integral unimodular transformation of the variables $u_2, \ldots, u_n$ we can suppose that

$$S_2 = Q\,(u_1^{(2)}, u_2^{(2)}, 0, \ldots, 0).$$

We define S_3 to be the minimum of Q when $u_3, \ldots, u_n$ are not all zero, and make a similar integral unimodular transformation of $u_3, \ldots, u_n$. Continuing in this way, we obtain n numbers $S_1, \ldots, S_n$ with

$$0 < S_1 \leq S_2 \leq \ldots \leq S_n,$$

such that $Q(u_1, \ldots, u_n) \geq S_r$ unless $u_r = u_{r+1} = \ldots = u_n = 0$.

Lemma 2. *For any positive definite quadratic form Q of determinant D, we have*

$$S_1 \ldots S_n \leqq \gamma_n^n D.$$

This result is due to MINKOWSKI (Geometrie der Zahlen, § 51), but I give the proof here as it is very simple. It is an interesting unsolved problem whether the corresponding result for an arbitrary convex body is true or not.

P r o o f. Without loss of generality we can suppose $D = 1$. We can express Q in the form

$$Q = X_1^2 + \ldots + X_n^2,$$

where X_r is a linear form in $u_r, \ldots, u_n$ and the determinant of $X_1, \ldots, X_n$ is 1. Define the quadratic form Q' by

$$Q' = \frac{X_1^2}{S_1} + \frac{X_2^2}{S_2} + \ldots + \frac{X_n^2}{S_n},$$

then Q' is a positive definite form of determinant $(S_1 \ldots S_n)^{-1}$. For all integral $u_1, \ldots, u_n$ for which $u_n \neq 0$, we have

$$Q' \geqslant \frac{Q}{S_n} \geqslant 1.$$

Generally, for any integral $u_1, \ldots u_n$ for which $u_r \neq 0$, $u_{r+1} = \ldots = u_n = 0$ we have

$$Q' = \frac{X_1^2}{S_1} + \ldots + \frac{X_r^2}{S_r} \geqslant \frac{X_1^2 + \ldots + X_r^2}{S_r} = \frac{Q}{S_r} \geqslant 1.$$

Hence $Q' \geq 1$ for all integral values of $u_1, \ldots, u_n$, not all zero. Thus $(S_1 \ldots S_n)^{\frac{1}{n}} Q'$ is a quadratic form of determinant 1, whose minimum is at least $(S_1 \ldots S_n)^{\frac{1}{n}}$, and the result is a consequence of the definition of γ_n.

4. P r o o f o f t h e T h e o r e m. Let M denote the lower bound of $|L_1 \ldots L_n|^{\frac{1}{n}}$ for integral values, not all zero, of $u_1, \ldots, u_n$. If $M = 0$, there

is nothing to prove, hence we can suppose $M > 0$. For an arbitrarily small positive number ε there exist values $L_1^*, \dots, L_n^*$ of $L_1, \dots, L_n$ corresponding to integral values $u_1^*, \dots, u_n^*$ of $u_1, \dots, u_n$ for which

$$M^n \leqslant |L_1^* \dots L_n^*| < \frac{M^n}{1-\varepsilon} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Put $x_1 = \dfrac{L_1}{L_1^*}, \dots, x_n = \dfrac{L_n}{L_n^*}$; then $x_1, \dots, x_n$ are linear forms in $u_1, \dots, u_n$ of determinant $\dfrac{1}{L_1^* \dots L_n^*}$. For all integral values of $u_1, \dots, u_n$ other than $0, \dots, 0$ we have

$$|x_1 \dots x_n| = \left| \frac{L_1 \dots L_n}{L_1^* \dots L_n^*} \right| \geqslant \frac{M^n}{|L_1^* \dots L_n^*|} > 1-\varepsilon.$$

Hence, for all integral values of $u_1, \dots, u_n$ other than $u_1^*, \dots, u_n^*$ and $-u_1^*, \dots, -u_n^*$ we have

$$|(x_1-1) \dots (x_n-1)| > 1-\varepsilon \text{ and } |(x_1+1) \dots (x_n+1)| > 1-\varepsilon.$$

By Lemma 1, apart from these exceptions

$$x_1^2 + \dots + x_n^2 > (1-\varepsilon)^2 \varkappa n.$$

Hence $Q = x_1^2 + \dots + x_n^2$ has successive minima which satisfy

$$S_1 = n, \quad S_r \geqslant (1-\varepsilon)^2 \varkappa n \quad (r = 2, \dots, n),$$

and the determinant of Q is $(L_1^* \dots L_n^*)^{-2}$. By Lemma 2,

$$n (1-\varepsilon)^{2(n-1)} \varkappa^{n-1} n^{n-1} \leqslant \gamma_n^n (L_1^* \dots L_n^*)^{-2}.$$

Finally, by (8), this implies

$$M^{2n} \leqslant \frac{\gamma_n^n}{n^n \varkappa^{n-1} (1-\varepsilon)^{2(n-1)}},$$

and since ε is arbitrarily small, this proves the Theorem.

5. We now come to the investigation of the value of $\varkappa$.

Lemma 3. $1.4 < \varkappa < 1.5$.

Proof. In the first place, $\Phi(1) = 1.5$. For when $\alpha = 1$, the solution of (2) is $\xi = \frac{1}{2}(3 + \sqrt{5})$, and

$$\Phi(1) = \tfrac{1}{2}(\xi + \xi^{-1}) = \tfrac{3}{2}.$$

Next we prove that if $\alpha > 3$ or $\alpha < \frac{1}{2}$ then $\Phi(\alpha) > \frac{3}{2}$, from which it follows that $\Phi(\alpha)$ attains its minimum somewhere in the interval $\frac{1}{2} \leq \alpha \leq 3$.

First, if $\alpha > 3$, we have

$$\Phi(\alpha) = \frac{\alpha \xi + \xi^{-\alpha}}{\alpha + 1} > \frac{\alpha \xi}{\alpha + 1} > \tfrac{3}{4} \xi > \tfrac{3}{2}.$$

since $\xi > 2$ by (2). Secondly, if $\alpha < \frac{1}{2}$ we note that, by (2),

$$(\xi - 1)^{-\alpha} > \tfrac{1}{2} \text{ and } \xi^{-\alpha} < \tfrac{1}{2},$$

so that

$$2^{\frac{1}{\alpha}} < \xi < 2^{\frac{1}{\alpha}} + 1 .$$

Hence

$$\Phi(\alpha) > \frac{\alpha\, 2^{\frac{1}{\alpha}} + (2^{\frac{1}{\alpha}} + 1)^{-\alpha}}{\alpha + 1} > \tfrac{2}{3}\{\alpha\, 2^{\frac{1}{\alpha}} + (2^{\frac{1}{\alpha}} + 1)^{-\alpha}\}.$$

Now

$$\alpha\, 2^{\frac{1}{\alpha}} + (2^{\frac{1}{\alpha}} + 1)^{-\alpha} > \alpha\, 2^{\frac{1}{\alpha}} + (2^{\frac{1}{\alpha}} + \tfrac{1}{4}\, 2^{\frac{1}{\alpha}})^{-\alpha} = \alpha\, 2^{\frac{1}{\alpha}} + \tfrac{1}{2}\,(\tfrac{4}{5})^{\alpha}.$$

The last expression increases as α decreases, since its derivative is

$$2^{\frac{1}{\alpha}} - \frac{1}{\alpha}\,(\log\, 2)\, 2^{\frac{1}{\alpha}} - \tfrac{1}{2}\,(\log \tfrac{5}{4})\,(\tfrac{4}{5})^{\alpha},$$

which is negative since $\alpha < \frac{1}{2} < \log 2$. Hence

$$\Phi(\alpha) > \tfrac{2}{3}\{\tfrac{1}{2} \cdot 4 + \tfrac{1}{2}\,(\tfrac{4}{5})^{\frac{1}{2}}\} = \tfrac{2}{3}\left(2 + \frac{1}{\sqrt{5}}\right) > \tfrac{3}{2}.$$

We have now to consider the values of $\Phi(\alpha)$ for $\frac{1}{2} \leq \alpha \leq 3$. The following table [3] gives the values of ξ, $\xi^{-\alpha}$, $\Phi(\alpha)$ at intervals of 0.1.

α	ξ	$\xi^{-\alpha}$	$\Phi(\alpha)$	α	ξ	$\xi^{-\alpha}$	$\Phi(\alpha)$
0.5	4.546	0.468	1.828	1.8	2.171	0.247	1.484
0.6	3.736	453	1.684	1.9	2.150	233	1.489
0.7	3.268	436	1.602	2.0	2.132	219	1.494
0.8	2.969	418	1.552	2.1	2.116	207	1.500
0.9	2.765	400	1.520	2.2	2.103	194	1.507
1.0	2.618	381	1.500	2.3	2.091	183	1.513
1.1	2.508	363	1.487	2.4	2.081	172	1.520
1.2	2.423	345	1.479	2.5	2.073	161	1.526
1.3	2.357	327	1.475	2.6	2.065	151	1.533
1.4	2.304	310	1.473	2.7	2.058	142	1.540
1.5	2.261	294	1.474	2.8	2.052	133	1.547
1.6	2.225	277	1.476	2.9	2.047	125	1.554
1.7	2.196	262	1.480	3.0	2.042	117	1.561

From this it is plain that $\varkappa < 1.474$, and it is very plausible that

$$\varkappa = 1.473 \text{ approximately.}$$

We proceed to prove rigorously that $\varkappa > 1.4$.

[3] The meaning of any entry, e.g. 4.546, in the table is that $4.546 \leq \xi < 4.547$. This deviation from normal practice is advisable for the purpose we have in view.

First we prove that ξ and $\xi^{-\alpha}$ both decrease steadily as α increases. By (2),

$$\{\xi^{-\alpha}\log\xi+(\xi-1)^{-\alpha}\log(\xi-1)\}+\alpha\{\xi^{-\alpha-1}+(\xi-1)^{-\alpha-1}\}\frac{d\xi}{d\alpha}=0. \quad (9)$$

and since $\xi>2$ it follows that $\dfrac{d\xi}{d\alpha}<0$. Also

$$\frac{d}{d\alpha}(\xi^{-\alpha})=-\xi^{-\alpha}\log\xi-\alpha\,\xi^{-\alpha-1}\frac{d\xi}{d\alpha},$$

and to prove that this is negative, we have to prove that

$$\frac{d\xi}{d\alpha}>-\frac{\xi\log\xi}{\alpha}.$$

By (9), this reduces to proving

$$(\xi-1)^{-\alpha}\log(\xi-1)-\xi(\log\xi)(\xi-1)^{-\alpha-1}<0,$$

which is obvious since $(\xi-1)\log(\xi-1)<\xi\log\xi$ for $\xi>2$.

It follows that for $\alpha_1\leq\alpha\leq\alpha_2$, we have

$$\Phi(\alpha)=\frac{\alpha}{\alpha+1}\xi+\frac{1}{\alpha+1}\xi^{-\alpha}\geqslant\frac{\alpha_1}{\alpha_1+1}\xi_2+\frac{1}{\alpha_2+1}\xi_2^{-\alpha_2},$$

where $\xi_2=\xi(\alpha_2)$. In this way we obtain a lower bound for $\Phi(\alpha)$ in each of the intervals $0.5\leq\alpha\leq0.6, \ldots, 2.9\leq\alpha\leq3.0$. The least of these lower bounds is the one which comes from $1.1\leq\alpha\leq1.2$, which is

$$\Phi(\alpha)\geqslant\frac{1.1}{2.1}(2.423)+\frac{1}{2.2}(0.345)>1.425.$$

Hence $\varkappa>1.425>1.4$. This proves Lemma 3.

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Mathematics. — *Einige Sätze über mehrfach negativ-konvergente Reihen in der intuitionistischen Mathematik.* By J. G. DIJKMAN. (Communicated by Dr. A. HEYTING.)

(Communicated at the meeting of June 29, 1946.)

§ 1. *Einleitung.*

Von Prof. L. E. J. BROUWER [1]) ist der Begriff „negative Konvergenz" einer Reihe eingeführt worden.

Deuten wir die partielle Summe der ersten n Glieder der unendlichen Reihe $\sum_{p=1}^{\infty} u_p$ an mit s_n, also $s_n = \sum_{p=1}^{n} u_p$, so nennen wir die Reihe negativ-konvergent nach α, falls es für jedes $\varepsilon > 0$ unmöglich ist, dasz es eine monoton wachsende Folge natürlicher Zahlen n_k gibt derart, dasz $| s_{n_k} - \alpha | > \varepsilon$ für jedes k.

Dieser Begriff ist von M. J. BELINFANTE [2]) erweitert worden auf mehrfach negativ-konvergente Reihen.

Die Reihe $\sum_{p=1}^{\infty} u_p$ nennen wir mehrfach negativ-konvergent nach $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_N$, falls es für jedes $\varepsilon > 0$ unmöglich ist, dasz es monoton wachsende Folgen n_k^i natürlicher Zahlen gibt derart, dasz $| s_{n_k^i} - \alpha_i | > \varepsilon$ für jedes k und für jedes i mit $1 \leq i \leq N$.

Ueberdies ist es möglich, dasz eine Reihe mehrfach negativ-konvergent ist nach einer ω-Folge $\{ a_i \}$.

Die Reihe ist ω-fach negativ-konvergent nach der ω-Folge $\{ \alpha_i \}$ falls es für jedes $\varepsilon > 0$ unmöglich ist, dasz es monoton wachsende Folgen n_k^i natürlicher Zahlen gibt derart, dasz $| s_{n_k^i} - \alpha_i | > \varepsilon$ für jedes i und k.

Diese Konvergenzbegriffe sind auch für Folgen eingeführt worden.

Sie lauten:

Eine Folge $\{ a_i \}$ heisst positiv-konvergent nach a falls es sich zu jedem $\varepsilon > 0$ eine Zahl $n(\varepsilon)$ bestimmen läszt mit der Eigenschaft: $| a - a_i | < \varepsilon$ für jedes $i > n(\varepsilon)$.

Die Folge $\{ a_i \}$ heisst negativ-konvergent nach a falls es für jedes $\varepsilon > 0$ unmöglich ist, dasz es eine monoton wachsende Folge n_k natürlicher Zahlen gibt derart, dasz $| a_{n_k} - a | > \varepsilon$ für jedes k.

Ebenso wie für Reihen kann der Begriff „negative Konvergenz" auch für Folgen auf „mehrfach negative Konvergenz" erweitert worden.

Betrachten wir die partiellen Summen s_n als eine Folge $\{ a_i \}$ so können

[1]) L. E. J. BROUWER: Ueber die Bedeutung des Satzes vom ausgeschlossenen Dritten, Journ. für die reine und angew. Math., Band 154, Seite 1—7.

[2]) M. J. BELINFANTE: Ueber eine besondere Klasse von non-oszillierenden Reihen, Proc. Kon. Akad. v. Wetensch., Amsterdam, Vol. XXXIII, No. 10 (1930).

wir auch mit den Konvergenzbegriffen für Folgen anfangen und diese Begriffe auf die Reihen anwenden.

Im Ueberstehenden bedeuten „monoton wachsende Folgen" immer „Folgen, die monoton wachsen falls k wächst".

§ 2. *Zusammenhang der einfachen negativen Konvergenz mit der mehrfachen negativen Konvergenz.*

Satz 1.

Ist eine Reihe negativ-konvergent nach a, so ist die Reihe auch mehrfach negativ-konvergent nach jeder Folge $\{ a_i \}$, die negativ-konvergent ist nach a.

Beweis. Die Folge s_n ist negativ-konvergent nach a, d.h. es ist für jedes $\varepsilon' > 0$ unmöglich, dasz es eine monoton wachsende Folge n_p gäbe derart, dasz $|s_{n_p} - a| > \varepsilon'$ für jedes p. Ueberdies ist für jedes $\delta > 0$ das Bestehen einer monoton wachsenden Folge i mit $| a - a_i | > \delta$ für jedes i unmöglich.

Denken wir uns jetzt, dasz wir für ein bestimmtes $\varepsilon > 0$ monoton wachsende Folgen hätten derart dasz $| s_{n_k^i} - a_i | > \varepsilon$ für jedes i und k.

Immer gilt: $| s_{n_k^i} - a_i | \leq | s_{n_k^i} - a | + | a - a_i |$.

Wir hätten also monoton wachsende Folgen n_k^i, für die für jedes i und k gilt: $\varepsilon < | s_{n_k^i} - a | + | a - a_i |$.

Aus $\varepsilon < | s_{n_k^i} - a | + | a - a_i |$ folgt für jedes i und k:

$$\text{entweder } | s_{n_k^i} - a | > \tfrac{1}{2}\varepsilon \text{ oder } | a - a_i | > \tfrac{1}{2}\varepsilon.$$

Wäre $| a - a_{i_1} | < \frac{1}{2} \varepsilon$ für ein bestimmtes i_1, so wäre $| s_{n_k^{i_1}} - a | > \frac{1}{2} \varepsilon$ für jedes k.

Das ist aber im Widerspruch mit der Voraussetzung, dasz die Reihe negativ-konvergent ist nach a.

Für jedes i ist es also unmöglich, dasz $| a - a_i | < \frac{1}{2}\varepsilon$ ist. In diesem Fall sind wir aber sicher davon dasz für jedes i gilt $| a - a_i | > \frac{1}{3} \varepsilon$, im Widerspruch mit der Voraussetzung, dasz die Folge $\{ a_i \}$ negativ-konvergent ist nach a.

Für jedes $\varepsilon > 0$ haben wir also die Unmöglichkeit des Bestehens monoton wachsender Folgen n_k^i mit der Eigenschaft $| s_{n_k^i} - a_i | > \varepsilon$ für jedes i und k gezeigt, d.h. die Reihe ist negativ-konvergent nach der Folge $\{ a_i \}$.

Ist die Menge $\{ a_1, a_2, \ldots \}$ nicht in negativem Sinne abgeschlossen und ist die Reihe positiv-konvergent nach a, so folgt aus diesem Satze, dasz es möglich ist, dasz die Reihe mehrfach negativ-konvergent ist nach dieser Punktmenge, während es unmöglich ist, dasz die Reihe negativ-konvergent ist nach einem der Punkte dieser Menge.

Die ω-fache negative Konvergenz nach der Menge $\{ a_i \}$ stimmt also keineswegs mit dem Sachverhalt überein, den man so andeuten kann, dasz die Reihe nach einem Punkt der Menge $\{ a_i \}$ konvergieren musz, ohne dasz bekannt ist nach welchem Punkt [1]).

Satz 2.

Ist eine Reihe negativ-konvergent nach $a_1, a_2 \dots a_N, a$ und betrachten wir die Folge $\{ a_i \}$, die negativ-konvergent sei nach a, während $a_1, a_2, \dots, a_N$ Glieder der Folge $\{ a_i \}$ sind, so ist die Reihe mehrfach negativ-konvergent nach der Folge $\{ a_i \}$.

B e w e i s. Denken wir uns, dasz wir für ein bestimmtes $\varepsilon > 0$ monoton wachsende Folgen n_k^i hätten derart, dasz $| s_{n_k^i} - a_i | > \varepsilon$ für jedes i und k.

Evident ist dann:

$$| s_{n_k^i} - a_i | > \tfrac{1}{2}\varepsilon \text{ für } 1 \leqslant i \leqslant N \text{ und jedes } k. \quad . \quad . \quad . \quad (1)$$

Weiter ist $\varepsilon < | s_{n_k^i} - a_i | \leq | s_{n_k^i} - a | + | a - a_i |$ für jedes i und k. Für jedes i und k gilt also mindestens eine der folgenden Ungleichungen:

$$\text{entweder } | s_{n_k^i} - a | > \tfrac{1}{2}\varepsilon \text{ oder } | a - a_i | > \tfrac{1}{2}\varepsilon.$$

Für jedes i ist es aber unmöglich, dasz gilt $| a - a_i | < \frac{1}{2}\varepsilon$, denn in diesem Falle wäre $|s_{n_k^i} - a | > \frac{1}{2}\varepsilon$ für jedes k. Hieraus folgt in Verbindung mit (1), dasz die Reihe nicht mehrfach negativ-konvergent wäre nach

$$a_1, a_2, \dots a_N, a$$

im Widerspruch mit der Voraussetzung.

Aber wenn es für jedes i unmöglich ist, dasz $| a - a_i | < \frac{1}{2}\varepsilon$ so ist für jedes i unbedingt $| a - a_i | > \frac{1}{3}\varepsilon$ im Widerspruch mit der Voraussetzung, dasz die Folge $\{ a_i \}$ negativ-konvergent ist nach a.

§ 3. *Erweiterung der ω-fachen negativen Konvergenz auf ω^2-fache negative Konvergenz.*

Satz.

Ist eine Reihe ω-fach negativ-konvergent nach der ω-Folge $\{ a_i \}$ und ist $\{ \beta_i \}$ eine Folge, die negativ-konvergent ist nach a_1, so ist die Reihe mehrfach negativ-konvergent nach der Punktmenge $(\beta_1, \beta_2, \dots, a_2, a_3, \dots)$.

B e w e i s. Denken wir uns, dasz wir für ein bestimmtes $\varepsilon > 0$ monoton wachsende Folgen n_k^i und m_μ^λ konstruiert hätten derart, dasz:

$$| s_{n_k^i} - a_i | > \varepsilon \text{ für jedes } k \text{ und } i \geqq 2.$$
$$| s_{m_\mu^\lambda} - \beta_\lambda | > \varepsilon \text{ für jedes } \lambda \text{ und } \mu.$$

Als Sonderfall ist dann evident:

$$| s_{n_k^i} - a_i | > \tfrac{1}{2}\varepsilon \text{ für jedes } k \text{ und } i \geqslant 2 \quad . \quad . \quad . \quad . \quad (1)$$

[1]) Diese Bemerkung verdanke ich Herrn Dr. A. HEYTING, der mich auch übrigens bei der Abfassung dieser Arbeit unterstützt hat.

Für jedes λ und μ ist aber: $\varepsilon < | s_{m_\mu^\lambda} - \beta_\lambda | \leq | s_{m_\mu^\lambda} - \alpha_1 | + | \alpha_1 - \beta_\lambda |$.

Also: $\varepsilon < | s_{m_\mu^\lambda} - \alpha_1 | + | \alpha_1 - \beta_\lambda |$ für jedes λ und μ.

Für jedes λ und μ gilt also mindestens eine der folgenden Ungleichungen:

$$\text{entweder } | s_{m_\mu^\lambda} - \alpha_1 | > \tfrac{1}{2}\varepsilon \text{ oder } | \alpha_1 - \beta_\lambda | > \tfrac{1}{2}\varepsilon.$$

Für jedes λ ist es nun unmöglich, dasz $| \alpha_1 - \beta_\lambda | < \frac{1}{2}\varepsilon$ ist, denn in diesem Falle wäre $| s_{m_\mu^\lambda} - \alpha_1 | > \frac{1}{2}\varepsilon$ für jedes μ und dies gibt in Verbindung mit (1), dasz die Reihe nicht ω-fach negativ-konvergent ist nach der Folge $\{ \alpha_i \}$, im Widerspruch mit der Voraussetzung.

Für jedes λ ist es also unmöglich, dasz $| \alpha_1 - \beta_\lambda | < \frac{1}{2}\varepsilon$ ist, aber dann ist für jedes λ unbedingt $| \alpha_1 - \beta_\lambda | > \frac{1}{2}\varepsilon$, im Widerspruch mit der Voraussetzung, dasz die Folge $\{ \beta_\lambda \}$ negativ-konvergent ist nach α_1, womit wir gezeigt haben, dasz die Reihe mehrfach negativ-konvergent ist nach der Punktmenge $(\beta_1, \beta_2, \ldots, \alpha_2, \alpha_3 \ldots)$.

Wir haben also gezeigt, dasz wir α_1 ersetzen dürfen durch eine Folge, aber dann ist es evident, dasz wir jedes α_i ersetzen dürfen durch Folgen, die b.z.w. negativ-konvergent sind nach α_i.

Dies gibt eine mehrfach negative Konvergenz nach einer Menge vom Typus ω^2.

§ 4. *Reduktion ω-facher negativer Konvergenz.*

Wir beweisen jetzt den Satz:

Sei die Folge $\{\alpha_i\}$ positiv-konvergent nach α.

Sei eine Reihe mit partiellen Summen s_n ω-fach negativ-konvergent nach der Folge $\{ \alpha_i \}$, während wir für ein geeignetes $\varepsilon > 0$ eine monoton wachsende Folge n_k natürlicher Zahlen bestimmen können derart dasz $| s_{n_k} - \alpha | > \varepsilon$ für jedes k.

So gilt:

Wir können ein N bestimmen derart, dasz die Reihe negativ-konvergent ist nach $(\alpha_1, \alpha_2, \ldots \alpha_N)$.

B e w e i s. Die Reihe ist mehrfach negativ-konvergent nach der Folge $\{ \alpha_i \}$ und diese Folge ist positiv-konvergent nach α. Nach Wahl einer beliebigen Zahl $\eta > 0$ können wir also eine Zahl $N(\eta)$ bestimmen derart, dasz $| \alpha - \alpha_i | < \eta$ für jedes $i > N(\eta)$. Wähle $\eta < \varepsilon$.

Aus: $| s_{n_k} - \alpha | > \varepsilon$ folgt: $\varepsilon < | s_{n_k} - \alpha_i | + | \alpha_i - \alpha |$ für jedes i und k.

Es ist also $\varepsilon < | s_{n_k} - \alpha_i | + \eta$ für $i > N(\eta)$ und jedes k oder anders gesagt:

$$| s_{n_k} - \alpha_i | > \varepsilon - \eta \text{ für jedes } i > N(\eta) \text{ und jedes } k \quad . \quad . \quad . \quad . \quad (1)$$

Die Reihe ist aber mehrfach negativ-konvergent nach der Folge $\{ \alpha_i \}$.

Wählen wir: $\delta = \varepsilon - \eta$ so ist es also unmöglich, dasz $| s_{n_k^i} - \alpha_i | > \delta$ für jedes i und k.

Dies gibt in Verbindung mit (1): Es ist für $\delta = \varepsilon - \eta$ unmöglich, dasz $| s_{n_k} - \alpha_i | > \delta$ für: $1 \leq i \leq N(\eta)$ und jedes k.

Sei jetzt $\eta_1 \circ\rangle 0$ und es durchlaufe η eine monoton wachsende Folge (ηs), welche Folge positiv-konvergent sei nach ε.

$N(\eta)$ durchläuft dann eine Folge $N(\eta_\lambda)$ mit $N(\eta_\lambda) \leq N(\eta_{\lambda+1})$.

Dann durchläuft $\delta_\lambda = \varepsilon - \eta_\lambda$ das Intervall $0 < \delta \leq \varepsilon - \eta_1$.

Hieraus folgt die Unmöglichkeit, dasz $| s_{n_k} - a_i | > \delta_\lambda$ sei für $1 \leq i \leq N(\eta_\lambda)$ und jedes k. Es ist also auch unmöglich, dasz $| s_{n_k} - a_i | > \delta_\lambda$ ist für $1 \leq i \leq N(\eta_1) = N$ und jedes k und λ, d.h. es ist für jedes $\delta > 0$ unmöglich, dasz $| s_{n_k} - a_i | > \delta$ für jedes k und $1 \leq i \leq N$.

Einfach zeigt man jetzt, dasz dies auch für jede monoton wachsende Folge n_k^i natürlicher Zahlen der Fall ist.

Anatomy. — *Tegengestelde ontwikkelingstendenties in 's menschen gebit.* By TH. E. DE JONGE. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of May 25, 1946.)

Eerste mededeeling: *Multipele hyperodontie in boven- en onderkaak*

(met 1 plaat)

Van de vele en velerlei ontwikkelingspotenties, die sluimerend voortleven in 's menschen gebit en die de studie zijner morphologische variaties tot zulk een aantrekkelijk en vruchtbaar object stempelen, vormt de hyperodontie ontegenzeggelijk nog steeds een der meest bekende en belangwekkende problemen.

Zóó talrijk echter zijn de casuïstische mededeelingen, waarmede in schier eindelooze reeks de literatuur der laatste decennia ons als het ware overstroomde, dat wij bezwaarlijk nog van eene zeldzame anomalie kunnen spreken. Toch is hare frequentie bij de verschillende tandgroepen nog zéér uiteenloopend: bekend is b.v. de betrekkelijke veelvuldigheid van overschrijding van het normale aantal van snijtanden in de bovenkaak — ervaringsfeit, dat in volkomen overeenstemming is met en zijne natuurlijke verklaring vindt in de verschillende aanlegpotenties in het gebied der praecanine gebitselementen. Eene welsprekende bevestiging daarvan leveren ons de cijfers, waarin STAFNE (1) de uitkomsten van zijne breed opgezette statistische studie over dit onderwerp samenvatte en welke wij in onderstaande tabel weergeven:

Overzicht van de verdeeling der overtollige tanden in boven- en onderkaak volgens STAFNE.

	Incisivi I	Incisivi II	Cuspidati	Praemolares	Molares		Totaal aantal
					Paramolares	Distomolares	
Maxilla	227 [1])	19 [1])	2	9	58	131	446
Mandibula	10	0	1	33	0	10	54
Totaal aantal	237	19	3	42	58	141	500

1) STAFNE maakt blijkens zijne indeeling geen onderscheid tusschen den overtolligen mesiodens en de verhoudingsgewijs véél zeldzamer gevallen van echte tandverdubbeling: dit verklaart zijne sterk uiteenloopende cijfers voor medialen en lateralen incisivus in de bovenkaak. Met reden mogen wij echter aannemen, dat deze divergentie in werkelijkheid nog aanzienlijker is dan in de door STAFNE gevonden waarden tot uitdrukking komt: deze toch onderzocht levend materiaal en nu leert de ervaring dat juist de mesiodentes niet zoo heel zelden op grond van aesthetische overwegingen (doorbraak tusschen de beide mediale snijtanden) dan wel, omdat zij functioneele bezwaren opleveren (orale dystopie), vroegtijdig verwijderd worden [BOLK (6)].

Nemen wij bovendien in aanmerking, dat deze onderzoeker zijne 500 gevallen bij 441 individuen registreerde, dan ligt de conclusie voor de hand, dat zich in éénzelfde gebit slechts bij uitzondering meer dan één overtollig element manifesteeren zal: ook dan echter leert de ervaring, dat deze goeddeels wederom op de fronttanden der bovenkaak betrekking hebben, bij welke wij immers niet slechts bilaterale ontwikkeling van den mesiodens kennen, doch daarnaast tevens rekening dienen te houden met de mogelijkheid van bilaterale resp. bilateraal-symmetrische tandverdubbeling ingevolge schizodontie [BOLK (2), BENNEJEANT (3), DE JONGE (4), MÖLLER (5) e.a.].

Stellen de bovengegeven beschouwingen de zeldzaamheid van multipele hyperodontie in het licht, ze motiveeren tevens de beschrijving van het geval, waaraan wij in het onderstaande enkele woorden willen wijden.

* * *

Het betreft hier een vijfentwintigjarigen man, bij wien, met uitzondering van zijn dentaal systeem, geenerlei anatomische of andere orgaanafwijkingen waarneembaar waren. Zijne gebitsstructuur draagt echter een dermate hypertrophisch karakter, dat wij, de cuspidati buiten beschouwing latende, bij alle overige tandgroepen de manifeste kenmerken ervan aantreffen (zie afb. 1).

De mediale bovenkaakssnijtanden onderscheiden zich reeds bij den eersten oogopslag door vorm en aantal: de langgerekte bouw van den rechtschen, welks kroon medio-distaal niet minder dan 12 mm meet, vormt een fraai voorbeeld van beginnende schizodontie, terwijl het vrij sterk naar mediaal gerichte orale tuberculum dentis zich door zóó krachtige differentiatie kenmerkt, dat het tot het niveau van de margo incisalis reikt. Deze beide bijzonderheden vereenvoudigen in niet onbelangrijke mate de diagnosestelling der linksche elementen. Immers: de mogelijkheid van den aanleg van eenen mesiodens zouden wij niet zonder meer van de hand willen wijzen — al dient erkend, dat de kroon van dezen slechts bij zéér hooge uitzondering den normalen snijtandvorm zal aannemen. Op grond van vergelijking der symmetrieverhoudingen echter — daarbij vragen wij in het bijzonder de aandacht voor de eveneens mediale localisatie van het linksche kroontuberculum — ligt het veeleer voor de hand, den medialen incisivus ter linkerzijde tezamen met diens distalen buurman te homologiseeren met zijnen sterk verbreeden synergeet rechts. Anders omschreven: deze laatste vertegenwoordigt eene vormphase, waaruit zich op ongedwongen wijze de toestand ter andere zijde van de mediaanlijn afleiden en verklaren laat.

Een scherp contrast met de configuratie der drie centrale snijtanden vormen de beide in entosteem doorgebroken laterale, bij welke met name het atrophisch karakter van den linkschen onmiddellijk in het oog springt. Wij vinden deze ongetwijfeld opmerkelijke coïncidentie — welke onwillekeurig de gedachte wakker roept aan de vicariëerende ontwikkeling, die wij bij andere organen kennen — vaker en hebben er bij vroegere gelegenheid (4) reeds een markant voorbeeld van beschreven [2]).

In deze gevallen van eene correlatieve ontwikkeling te spreken, zoude wellicht te ver gaan; toch willen wij de principiëele mogelijkheid ervan stellig niet a priori verwerpen: bekend immers is — en BOLK (6) heeft er in zijne „*Schets der ontwikkelingsgeschiedenis van het menschelijk gebit*" [3]) reeds nadrukkelijk de aandacht op gevestigd — dat agenesie der zijdelingsche snijtanden niet zelden eene compensatie schijnt te zoeken in verbreeding der mediale. Weliswaar hebben sommigen in deze verbreede kroonvormen eene coalescentie willen zien van medialen met lateralen incisivus, doch, afgezien nog van andere bezwaren, verzet ook hun zéér uiteenloopend tijdstip van aanleg en calcificatie zich tegen deze interpretatie.

De onderste incisivi vallen ten hoogste door de forsche ontwikkeling hunner orale kroonvlakte op doch geven overigens tot geenerlei bijzondere beschouwingen aanleiding.

Van de bicuspidati in de bovenkaak vragen de linksche onze bijzondere aandacht: dat zij alle drie normaal van vorm en grootte zijn, maakt de beantwoording der vraag, welk element overtollig is, zeker niet eenvoudiger. Toch zijn wij geneigd, den middelsten als zoodanig te determineeren. Twee overwegingen leidden tot deze conclusie. De eerste is deze, dat de kroon van den meest distalen praemolaris, óók en met name in de ontvouwing zijner fijnere structuurdétails, zóó duidelijk de kenmerken van een tweeden praemolaris draagt, dat wij diens identiteit boven elken twijfel verheven achten. Bovendien echter staat de middelste praemolaris in de voornaamste afmetingen zijner kroon ten achter bij de beide andere.

Nög veel bezwaarlijker blijkt de differentieele diagnose bij de rechtsche onderpraemolares: weliswaar bezit de voorste hunner de kenmerkende vormeigenschappen van een eersten praemolaris (caniniforme differentiatie zijner mesiale kauwvlakzone, smalle orale krooncuspis, die in hare zwakke, bijkans rudimentaire vormontwikkeling eene opvallende tegenstelling vormt met den domineerenden buccalen knobbel), zoodat wij dezen *eo ipso* buiten beschouwing kunnen laten — zijne beide distale naamgenooten daarentegen zijn zóó gelijkvormig, dat ook vergelijking der symmetrieverhoudingen ons hier geen stap verder brengt. Eene redeneering per analogiam tenslotte achten wij een zóó zwakke bewijsvoering, dat wij ons van eene beslissing in deze onthouden.

Van de molares, wier kronen in boven- nòch onderkaak krachtig ontwikkeld zijn, leveren de bovenste geen stof tot bijzondere opmerkingen. Niet aldus in de onderkaak, waar zich bij alle lateraal van de mesiobuccale krooncuspis eene verhevenheid afteekent, welke wij vroeger reeds als *mesiobuccale randprominentie* omschreven hebben (7 en 8) en die zich bij den eersten molaris rechts tot een echt overtollig tuberculum geëvolueerd heeft.

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[2]) vd. pag. 182 en afb. 3.

[3]) ib. pag. 191.

Ter linkerzijde ontbreekt de eerste molaris. Met reden mogen wij echter aannemen, dat ook deze eene anterolaterale randprominentie bezeten heeft: niet slechts de aanwezigheid dezer prominentie bij zijn beide distale buren wijst in deze richting doch in gelijke mate de bilaterale symmetrie, die wij vroeger reeds als bijna specifiek kenmerk in de vormontwikkeling van onzen eersten molaris beschreven hebben (9).

De röntgenstatus van onzen casus bracht ten slotte geenerlei verdere bijzonderheden aan het licht: nòch afwijkingen in de overigens niet zoo bijzonder krachtig gestructureerde wortelformatie, nòch overtollige in het kaakbeen of elders geïmpacteerde elementen.

Beschouwen wij tenslotte boven- en onderkaak in haar geheel, dan kunnen wij ons moeilijk aan den indruk onttrekken, dat bij beide de hypertrophische ontwikkelingstendenz harer elementen in zekeren zin doorkruist wordt door eene — zij het aanmerkelijk zwakkere — regressieve tendenz: terwijl in het bijzonder bij de beide linksche mediale bovensnijtanden eene nauwelijks merkbare convergentie der beide zijdelingsche kroonvlakken a.h.w. hare natuurlijke voortzetting vindt in de afgeronde incisale randhoeken, heeft deze op bijkans classieke wijze haren stempel gedrukt op de structuur der laterale bovenkaaksincisivi; even onmiskenbaar echter is haar invloed op de vormontwikkeling van alle molares. In onze tweede mededeeling komen wij daar nader op terug.

De relatie tusschen boven- en onderkaak moge dan al niet normaal te noemen zijn, alle hyperodontie ten spijt is er niettemin eene zeer bevredigende gebitsharmose verkregen. Twee factoren hebben daar op niet te onderschatten wijze toe bijgedragen: eenerzijds de orale dystopie der laterale bovenincisivi, die hun eigenlijke plaats in de tandrij reeds door de mediale snijtanden ingenomen vonden, anderzijds het vroegtijdig verloren gaan der beide voorste molares ter linkerzijde.

* * *

Van de in de literatuur neergelegde gevallen komen er eigenlijk slechts drie als vergelijkingsobject in aanmerking: de meest bekende daarvan is ongetwijfeld nog steeds de classieke casus van LANGER (10), die in een negerschedel, behalve vier vierde molares, in de bovenkaak twee en in de onderkaak tenslotte nog één overtolligen praemolaris aantrof.

Als tweeden vermelden wij CHRIST (11), in wiens geval in de bovenkaak twee vierde molares en twee mesiodentes tot doorbraak gekomen waren, terwijl bovendien in de onderkaak een overtollige praemolaris aanwezig bleek.

Bij beide derhalve óók in dien zin eene multipliciteit, dat de overtollige elementen gelijk in ons geval over verschillende tandgroepen verspreid liggen.

Uit de latere jaren is ons — zoover onze literatuurkennis onder de huidige omstandigheden reikt — slechts één soortgelijk maar dan ook bijzonder belangwekkend geval bekend. Het werd in 1944 door SIMEK in de *Ceska Stomatologie* (12) beschreven en ons nadien met groote welwillendheid door Professor KLIMES voor nauwkeuriger onderzoek ter beschikking gesteld. Daarop vooruitloopende bepalen wij ons voor heden tot deze bijzonderheid, dat het normale aantal van tanden hier met niet minder dan acht elementen overschreden bleek.

In de onderkaak was beiderzijds een overtollige bicuspidatus aangelegd. In de bovenkaak bleek bovendien ook *in alle overige groepen van tanden het normale aantal overschreden:* naast bilaterale snijtandverdubbeling en verdubbeling van den linkschen cuspidatus bilaterale ontwikkeling van een derden praemolaris en tenslotte één vierde molaris!

Gelijk gezegd — verdere gevallen zijn ons uit de jongere literatuur niet bekend [4]): zoo hebben, om ons slechts tot enkele voorbeelden te bepalen, de mededeelingen van ERNA GREINER (13) [5]) en de recente publicaties van DUYZINGS (14) en MARTHA DE BOER (15) uitsluitend op de praemaxillaire gebitselementen betrekking: òòk in quantitatieven zin blijven deze echter nog zóó verre ten achter bij den door ons beschreven casus, dat zij op wel zéér sprekende wijze diens waarlijk unieke zeldzaamheid in het daglicht stellen!

Samenvatting.

Beschrijving en afbeelding eener gebitsformatie, welker hypertrophische ontwikkelingstendenz in boven- en onderkaak zich op verschillende wijze ontplooide. Terwijl van de drie centrale bovenincisivi de breede rechtsche in den vorm van een tweelingstand doorgebroken is (*odontopagus partim discissus* volgens de nomenclatuur ISSEL–HERBST), bevinden zich ter linkerzijde twee middelste snijtanden.

In het gebied der praemolares is in de bovenkaak links, in de onderkaak rechts een overtollig element aanwezig, terwijl bij alle ondermolares een mesiobuccale randprominentie tot ontwikkeling gekomen blijkt.

De cuspidati zijn alle vier normaal, van de zich in staat van reductie bevindende laterale snijtanden in de bovenkaak is de rudimentaire linksche vrijwel geheel aplastisch.

Résumé.

Description et réproduction d'une formation de la denture, dont la tendance à un développement hypertrophique s'est réalisée de diverses manières aux mâchoires supérieure et inférieure. Tandis que des trois

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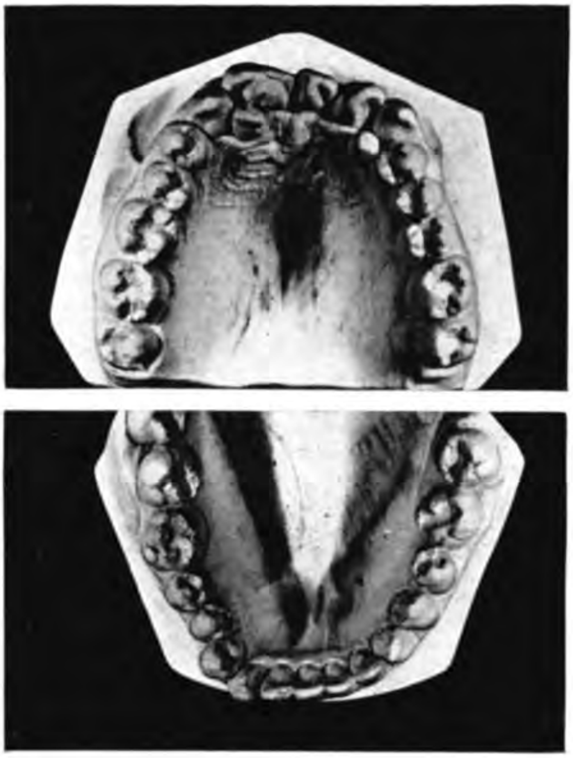
[4]) Weliswaar beschreef HERBST (16 en 17) een geval van verdubbeling van alle fronttanden en voorste bicuspidati in boven- en onderkaak: zijne formale genese is echter — gelijk die van enkele andere dergelijke dysmorphieën [KÖNIG (23)] — ongetwijfeld reeds tot het gebied der tweelingspathologie terug te brengen.

[5]) Deze beschreef een kinderschedeltje, in welks melkgebit zich rechts twee laterale bovensnijtanden bevonden, terwijl links zoowel de mediale snijtand boven als de laterale in de onderkaak in den vorm van een tweelingstand doorgebroken waren.

TH. E. DE JONGE: *Tegengestelde ontwikkelingstendenties in 's menschen gebit.*



incisives centrales supérieures l'incisive large droite a percé sous forme de dents jumelles (*odontopagus partim discissus* selon la nomenclature ISSEL–HERBST), il y a au côté gauche deux incisives médianes.

Dans le domaine des prémolaires on voit à gauche dans la mâchoire supérieure, et à droite dans la mâchoire inférieure un élément supplémentaire, tandis qu'une proéminence du bord mésiobuccal s'est développée sur toutes les molaires inférieures.

Les quatre cuspidées sont normales; parmi les incisives latérales de la mâchoire supérieure, qui sont en état de réduction, la rudimentaire gauche est presque complètement aplastique.

Zusammenfassung.

Es wird eine Gebissformation beschrieben und abgebildet, deren hypertrophische Entwicklungstendenz in Ober- und Unterkiefer sich in verschiedener Weise entfaltete. Während von den drei zentralen oberen Schneidezähnen der breite rechte in Form eines Zwillingszahnes durchgebrochen ist (*odontopagus partim discissus* nach der ISSEL–HERBSTschen Nomenklatur), befinden sich an der linken Seite zwei mittlere Schneidezähne.

In dem Gebiet der Prämolaren ist im Oberkiefer links, im Unterkiefer rechts ein überzähliges Element vorhanden, während bei allen unteren Molaren eine mesiobukkale Randprominenz zur Entwicklung gekommen ist.

Alle vier cuspidati sind normal; von den in Reduktion befindlichen lateralen Schneidezähnen im Oberkiefer ist der rudimentäre linke nahezu ganz aplastisch.

Summary.

Description and illustration of a teeth-formation of which the hypertrophical tendency of development in upper- and lower jaw expressed itself in different ways. Whereas of the three central upper incisors the large right one forced its way in the form of a twin tooth (*odontopagus partim discissus*, according to the nomenclature ISSEL–HERBST) there are two middle incisors at the left.

In the area of the premolars we find a superfluous element in the upper jaw at the left and in the lower jaw at the right, whereas all lower molars have obtained a mesiobuccal prominence.

The four canines are all normal; of the lateral incisors in the upper jaw, which are all in state of reduction, the rudimentary left one is nearly completely aplastic.

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Eene uitvoerige literatuurlijst bevindt zich aan het einde onzer tweede mededeeling.

Anatomy. — *Tegengestelde ontwikkelingstendenties in 's menschen gebit.* By TH. E. DE JONGE. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of May 25, 1946.)

Tweede mededeeling: *Regressieve vormontwikkeling in boven- en onderkaak*

(met 1 plaat)

In onze eerste mededeeling over dit onderwerp hebben wij de gebitsstructuur van een jongeman beschreven, welker morphologische progressiviteit vergezeld ging van eene nauwelijks nog waarneembare regressieve tendenz. De gelukkige omstandigheid nu, dat wij bovendien in de gelegenheid waren, de gebitten der overige gezinsleden in ons onderzoek te betrekken, geeft ons aanleiding, ten tweeden male de aandacht voor dit onderwerp te vragen.

In aanmerking voor ons onderzoek kwamen in de eerste plaats de beide zusters. Terwijl bij een harer de toestand *volkomen normaal* bleek, leverde het gebit der andere — morphologisch veel minder progressief ontwikkeld dan onze reeds besproken casus — als vergelijkingsobject onder meer dan één opzicht eene welsprekende toelichting op onze reeds eerder gegevene beschouwingen (afb. 1).

Dit geldt in de eerste plaats voor de molares, wier kronen in de onderkaak alle [1]) een mesiobuccaal randtuberculum dragen, welks bilateraalsymmetrisch karakter bij den eersten molaris de juistheid van ons reeds vroeger geleverd betoog in een helder daglicht stelt: „Ter linkerzijde", aldus in onze eerste mededeeling, „ontbreekt de eerste molaris. Met reden mogen wij echter aannemen, dat ook deze eene anterolaterale randprominentie bezeten heeft: niet slechts de aanwezigheid dezer prominentie bij zijne beide distale buren wijst in deze richting doch in gelijke mate de bilaterale symmetrie, die wij vroeger reeds als bijna specifiek kenmerk in de vormontwikkeling van onzen eersten molaris beschreven hebben" [2]).

Tot soortgelijke conclusie leidt nadere beschouwing der praemolares. Morphotisch al even weinig progredient ontwikkeld als de molares, blijkt hier slechts in de bovenkaak links een overtollig element aanwezig, welks eruptie tusschen de beide normale biscuspidati nochtans op ongedwongen wijze de juistheid bevestigt van onze in onze eerste mededeeling hoofdzakelijk ex juvantibus gestelde diagnose: „Toch zijn wij geneigd", aldus luidde onze conclusie daar, „den middelsten als zoodanig [3]) te determineeren. Twee overwegingen leidden tot deze conclusie. De eerste is deze.

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[1]) De ontbrekende tweede en derde molaris links zijn blijkens de anamnese vroeger reeds geëxtraheerd.

[2]) L.c. pag. 837.

[3]) scil. als overtollig element.

dat de kroon van den meest distalen praemolaris, óók en met name in de ontvouwing zijner fijnere structuurdétails, zóó duidelijk de kenmerken van een tweeden praemolaris draagt, dat wij diens identiteit boven elken twijfel verheven achten. Bovendien echter staat de middelste praemolaris in de voornaamste afmetingen zijner kroon ten achter bij de beide andere" [4]).

Bij de voortanden tenslotte geenerlei aanduiding meer van progressiever vormdifferentiatie. Dat geldt met name voor de bovenkaak, waar niet slechts de beide laterale incisivi ontbreken, doch de configuratie der mediale onmiddellijk het beeld der HUTCHINSONsche vormanomalie in het geheugen roept.

Men weet, dat het meest in het oog springend kenmerk dezer difformiteit, welke, voor het eerst door HUTCHINSON (18) beschreven, door dezen *slechts onder bepaalde voorwaarden* als pathognomonisch voor het beeld der connatale lues beschouwd werd — reeds in eene vroegere mededeeling wezen wij daarop (19) — bestaat in eene incisaalwaartsche convergentie van de approximale vlakken der snijtandskronen. Identiek daarmede is de nadien door MOON (20) beschreven convergentie der vier opstaande kroonvlakken bij de molares, waarop in de latere jaren PFLUEGER en anderen (21 en 22) opnieuw de aandacht vestigden. PFLUEGER sprak in dit verband van eene „Knospenform", in welke aan de ontwikkeling der bloemen ontleende omschrijving hij a.h.w. tot uitdrukking wilde brengen, dat de aanleg der kroon niet meer haren vollen wasdom vermocht te bereiken doch min of meer in „knopvorm" verkalkte. Remming derhalve der normale vormontwikkeling, in ons geval echter niet alleen bij de snijtanden doch *in gelijke mate bij de weinig gedifferentieerde kronen van alle postcanine gebitselementen*.

Ook in onze eerste mededeeling hebben wij de beteekenis dezer regressieve tendenz als vormgevend element in de gebitsontwikkeling onder oogen gezien en er nadrukkelijk op gewezen, dat zoowel bij boven- als onderkaak „de hypertrophische ontwikkelingstendenz harer elementen in zekeren zin doorkruist wordt door eene — zij het aanmerkelijk zwakkere — regressieve tendenz: terwijl in het bijzonder bij de beide linksche mediale bovensnijtanden eene nauwelijks merkbare convergentie der beide zijdelingsche kroonvlakken a.h.w. hare natuurlijke voortzetting vindt in de afgeronde incisale randhoeken, heeft deze op bijkans classieke wijze haren stempel gedrukt op de structuur der laterale bovenkaaksincisivi; even onmiskenbaar echter is haar invloed op de vormontwikkeling van alle molares" [5]).

Hier een véél zwakkere progressieve tendenz: daarnaast echter een regressieve tendenz, uitgegroeid tot een zóó domineerenden factor, dat óók descriptief-anatomisch de beide beschreven gebitsformaties scherp met elkander contrasteeren.

* * *

[4]) L.c. pag. 836.

[5]) L.c. pag. 837.

Over het gebit van den jongeren broeder (afb. 2) kunnen wij na het voorafgaande kort zijn. Het vertoont in vorm en aantal zóó groote overeenkomst met dat zijner zuster, dat uitvoeriger beschrijving gevoegelijk achterwege kan blijven: dezelfde marginale randprominenties bij alle nog aanwezige ondermolares, eenzelfde overtollige bovenpraemolaris links. Eenzelfde morphologisch-regressieve tendenz tenslotte heeft ook hier haren stempel gedrukt op de vormontplooiing van het geheel.

Eéne marquante bijzonderheid niettemin onderscheidt dit gebit van de beide andere: de bilaterale agenesie van den lateralen bovensnijtand heeft hier in zekeren zin haar natuurlijk complement gevonden in eene — ook röntgenoscopisch bevestigde — agenesie van den rechtschen medialen onderincisivus en consecutieve persistentie van diens voorganger in het melkgebit (24).

Samenvatting.

In onze eerste mededeeling over dit onderwerp beschreven wij het gebit van een jongeman, waarin multipele hyperodontie het meest sprekende kenmerk vormde eener morphologisch-progressieve ontwikkelingstendenz, welke nochtans doorkruist werd door eene — zij het véél zwakkere — regressieve tendenz.

Onze tweede mededeeling was gewijd aan de gebitsformatie van diens zuster en broeder. Terwijl bij dezen het morphologisch-progressief karakter veel minder op den voorgrond trad, was de retrogressieve tendenz daarentegen uitgegroeid tot een vormgevend element van overheerschenden invloed.

Résumé.

Notre première communication sur ce sujet décrit la denture d'un jeune homme, chez qui l'hyperodontie multipele formait la caractérique la plus frappante d'une tendance au développement morphologico-progressif, entrecroisée toutefois d'une tendance régressive (bien qu'en beaucoup plus faible mesure).

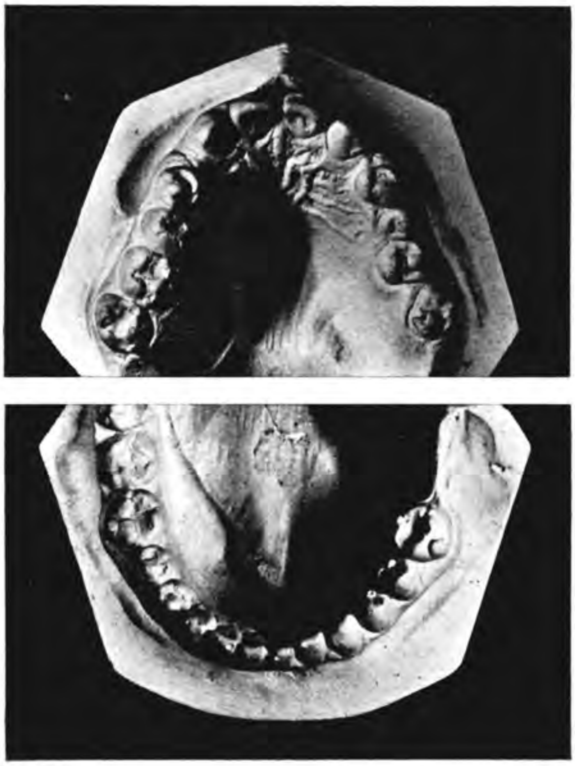
Notre seconde communication concerne la formation de la denture de la soeur et du frère de ce jeune homme. Tandis que, chez eux, le caractère morphologico-progressif se montre beaucoup moins en premier plan, la tendance rétrogressive par contre s'est développée en un élément formateur d'influence prédominante.

Zusammenfassung.

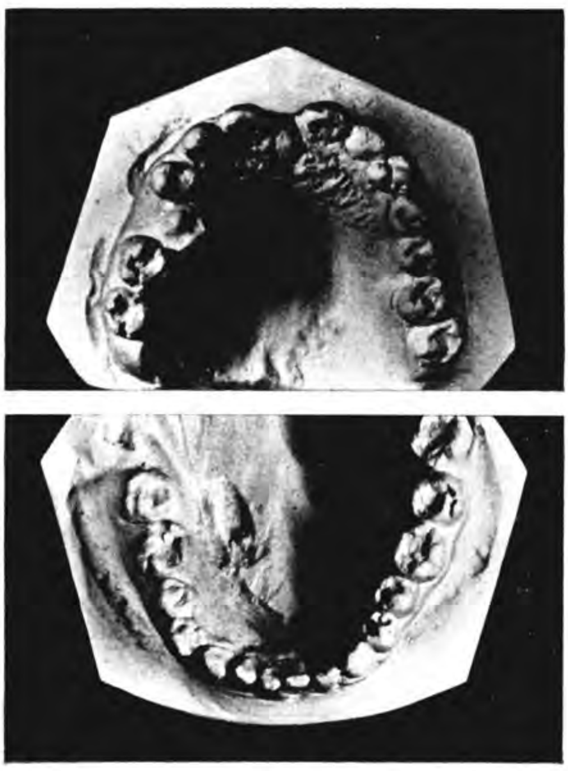
In unserer ersten Mitteilung über diesen Gegenstand beschrieben wir das Gebiss eines jungen Mannes. In diesem Gebiss war die multiple Hyperodontie das beredteste Kennzeichen einer morphologisch-progressiven Entwicklungstendenz, welche aber durch eine — wenn auch viel schwächere — regressive Tendenz durchkreuzt wurde.

Unsere zweite Mitteilung betraf die Gebissformation der Schwester und des Bruders des obengenannten jungen Mannes. Während bei diesen der

TH. E. DE JONGE: *Tegengestelde ontwikkelingstendenzen in 's menschen gebit.*



Afb. 1.



Afb. 2.

morphologisch-progressive Charakter viel weniger in den Vordergrund trat, war dagegen die retrogressive Tendenz zu einem formgebenden Element dominierenden Einflusses ausgewaschsen.

Summary.

In our first publication on this subject we described the teeth of a young man, with which multiple hyperodontia formed the most striking sign of a morphologic-progressive tendency of development, being crossed however by a — much feebler though — regressive tendency.

Our second publication concerns the teeth-formation of the young man's sister and brother. The retrogressive tendency increased to a formgiving element of dominating influence, whereas with them the morphologic-progressive character was much less obvious.

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Comparative Physiology. — *The presence of α- and β-amylase in the saliva of man and in the digestive juice of Helix pomatia. II. Polarimetric determinations.* By L. ANKER and H. J. VONK. (From the Laboratory of Comparative Physiology University of Utrecht.) (Communicated by Prof. A. DE KLEYN.)

(Communicated at the meeting of June 29, 1946.)

In our previous communication (6) we mentioned that two types of amylase may be distinguished which behave differently in the following points: 1° the limit unto which they are able to split up the starch-molecule, 2° the rapidity with which the jodine-test of the unattacked starch disappears during the reaction, 3° the stereoisomeric form of the maltose which is set free in the reaction by both of the enzymes.

The last mentioned fact has been discovered by KUHN in 1924. Maltose may appear in two forms (called α- and β) the first of which causes a stronger positive rotation than the second. These forms are unstable: when they are present together in aqueous solution they are converted into each other, so that an equilibrium-mixture is formed, which contains 36 % α-maltose and 64 % β-maltose. This transition (called mutarotation) is strongly accelerated by an excess of OH-ions (e.g. addition of soda). If originally an excess of α-maltose is present, the mutarotation may be called negative, in the reverse case the mutarotation is positive.

KUHN observed in the amylolytic cleavage of starch by amylase of different origin in some cases a negative in other a positive mutarotation. In the first case maltose was set free in the α-form, in the second in the β-form. He therefore distinguished two types of amylases: an α- and a β-amylase. Further research showed that the α-amylase was identical with OHLSSON's dextrinogen-amylase, the β-amylase with OHLSSON's saccharogen amylase. The names α- and β-amylase are mostly used nowadays.

Whereas in the vegetable kingdom both types of amylases occur as well together as separately, the animal amylases are — as far as investigated — nearly always of the α-type. Only relatively small amounts of β-amylase occurring together with the α-amylase have been observed by PURR in the pancreatic juice of the pig and by ANKER and VONK in the saliva of man and in the digestive juice of Helix pomatia.

The most conclusive evidence for recognising an amylase as belonging to the α- or to the β-type is undoubtedly the primary appearance of maltose in the α- resp. in the β-form. As far as we know this criterion has never been applied to the action of saliva on starch and glycogen [1]), so that we

[1]) These experiments have been performed by KUHN for the amylases of malt, taka and pancreas. (Cf. also SAMEC (3) p. 132—134.)

thought it important to follow the course of this reaction polarimetrically. The same experiments were carried out with the digestive juice of Helix pomatia (as far as the material allowed) which has neither been investigated with the polarimetric method. Simultaneously the course of these reactions has been followed by the determination of the change in reduction when tested with an alkaline copper-solution in order to compare the results of these methods. KUHN has applied the same procedure in his researches on the cleavage of starch by the α-amylase of malt and of the pancreas. In using these methods also an impression can be obtained of the limit unto which the hydrolysis may proceed.

I. *The cleavage of starch by saliva.*

a. Method and calculation.

The reaction-mixture contained 250 cm³ of a 1 % solution of soluble starch (KAHLBAUM), buffered by KH_2PO_4 and Na_2HPO_4 at a p_H of 6.8 to which 3 to 4 cm³ of filtered saliva were added. The saliva had been collected after a thorough cleaning of the mouth (like in our previous communication), this time to prevent that small amounts of reducing sugars arising from rests of food would disturb the result of the determinations. The reaction took place in an ERLENMEYER flask of 300 cm³ at a constant temperature of 25° C.

At fixed times three portions were taken simultaneously from the mixture. Of the first portion the rotation was determined immediately. To the second portion $2n$ soda was added [2]) to stop the reaction and to cause a rapid course of the mutarotation. The rotation of this sample was determined after 15 min. After correction of the observed value for the dilution by the addition of soda, it can be compared with the directly measured value of the first sample. The comparison of these two values gives an answer to the question whether a negative or a positive mutarotation is present, viz. whether the amylase for the largest part consists of the α- or the β-type. In the third portion the reduction was determined by means of the method of LUFF, as indicated by SCHOORL (4).

The values obtained by the polarimeter and the determination of the reduction were drawn up in curves showing the relation of cleavage and time. In order to compare the curves obtained for the polarimetrical observations and these of the reduction with each other, and to calculate the limit unto which the cleavage proceeds, some constants of the starch-solutions had to be determined previously. Firstly the percentage of amylose, viz. of digestible substance. This value was determined by boiling the starch-solution after adding hydrochloric acid (until the solution contained 2.5 % HCl) on a waterbath for 3 hours, neutralizing and determining the reduction. By bringing this reduction in relation to the weight of the starch which the solution contained, a percentage of 86.5 for the amount of

[2]) An amount of 5 cm³ 2n soda for a sample of 20 cm³.

amylose in the dry substance was found. Secondly the reductive power of the starch-solution itself had to be determined. It varied in the various experiments from 0.89 to 1.20 cm^3 of 0.1035 n thio-sulfate per 10 cm^3 solution.

The reduction-value which was observed in the samples of the reaction-mixture taken at different times after the beginning of the experiment. consists of the reduction caused by the amount of maltose, already formed at this time and the reduction power of the unaltered starch still present. As for a definite amount of starch which is split up a certain amount of reductive power has disappeared, which is replaced by the reduction of the maltose which has been formed, the amount of maltose present at a certain time can be calculated in mg.

In order to compare this amount with that which had been found polarimetrically, the rotation was calculated which is equivalent to the amount of maltose found by means of reduction-determinations, assuming that the maltose is present in the state of an equilibrium-mixture of α- and β-maltose. To this rotation had to be added the rotation-value of the unhydrolysed starch. For this purpose the specific rotation of the dissolved starch was determined before the beginning of each set of experiments.

For the specific rotation of the equilibrium-mixture of maltose the value $a_D = 137.9^\circ$ was taken, which is generally used in the literature on starch-chemistry. As substrate we used Amylum solubile of SCHERING–KAHLBAUM. The length of the polarimeter-tube was 2 dm. Pipettes had been carefully calibrated.

Assuming that the course of the reaction follows indeed this simple scheme, the values of the curves determined by reduction and by polarisation (after taking away the mutarotation by means of soda) should coincide. This is not strictly the case. At the beginning of the reaction a fairly strong deviation is seen, which diminishes with the proceeding of the reaction. Similar deviations have been observed by KUHN and SAMEC for the cleavage of starch by the α-amylase of malt and pancreas. They must be ascribed to the fact that the reaction-mixtures contain besides starch and maltose also dextrines, the reduction as well as the rotation of which are unknown. Consequently they cannot be taken into account in the calculations. Most of these dextrines are gradually split up into maltose during the reaction, as is known, by the action of the α-amylase. So it is clear that the divergences between the polarimetric values and those of the reduction disappear in the course of the reaction, as may be seen from our results.

b. Results. Fig. 1 shows the results of an experiment in which the cleavage of starch by saliva has been studied. Curve *a* represents the direct polarimetric observations, curve *b* shows the polarimetric values which are obtained after the mutarotation has finished. The values of curve *c* have been obtained by calculating the rotation-values, which an equilibrium mixture of maltose would show, from the determined reduction values.

In comparing curves *a* and *b* it may be seen that (with the exception of slight deviations at the beginning and at the end of an experiment) there is a distinct negative mutarotation, so that we may conclude that the majority of the saliva-amylase consists of the α-type. The nearly coinciding of the curves *b* (rotation of the equilibrium-mixture) and *c* (rotation of equilibrium-mixture calculated from the reduction), is a good control for the exactitude of the polarimetric as well as the reductometric determinations.

The general shape of our curves agrees with those published by KUHN for the α-amylase of malt-extract. We repeated the experiment shown in fig. 1 seven times with different samples of saliva and with approximately the same results. In two of these experiments a sharp deviation was observed in the beginning of the reaction. This deviation has been represented in fig. 1 by the dotted line. It is also present in KUHN's curves. It is absent in the curves of fig. 1, probably because of the rapid course of the reaction in this experiment, where the saliva proved to be very active. In the two cases where this anomaly was observed by us, the amylase-activity was apparently low. In the rapid course of the experiment represented in fig. 1 the phenomenon took probably place between two observations. The exact causes of this phenomenon are unknown.

Although the curves *b* and *c* in fig. 1 show a very good agreement, in some of our experiments (which are not represented here for lack of room)

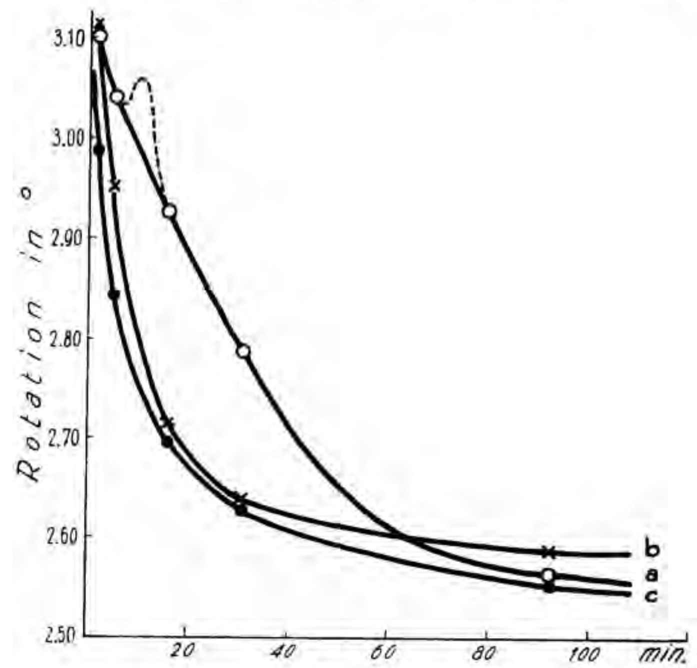

Fig. 1.
Action of saliva on starch. Explication in text.

the curve *c* is running somewhat more beneath the curve *b*. After SAMEC these curves meet if the starch is totally hydrolysed. Both KUHN and SAMEC mention deviations of those curves to both sides, without trying to give any explanation of this phenomenon. We think it possible that the deviations of KUHN and SAMEC are due to the fact that they probably did not determine the specific rotation of the dissolved starch for each experiment separately. This value which we determined for each experiment appeared to be somewhat variable according to the way in which the solution had been prepared. The same is true for the reduction of the starch-solution itself.

Without going into theoretical considerations concerning the mechanism of the enzymatic reaction or on the structure of starch we may make the following remark on the limit unto which the starch is hydrolysed by both types of amylase. Conclusions drawn from finding a certain cleavage-limit are of small value, if the nature of the substrate is not exactly known. Starch occurs chiefly in two forms which are split up by both types of amylase to a different degree. (For extensive data in this respect see the work of SAMEC (3)). It depends on the relative amounts of these forms in the substrate to which degree the latter is hydrolysed by a certain amylase. Some substrates consist already partly of cleavage-products of starch, e.g. the so-called LINTNER-starch. Our substrate was hydrolysed by amylase for about 80 %. This agrees with the values formerly found by VONK and BRAAK in this laboratory for the same substrate.

II. *The cleavage of glycogen by saliva.*

The amount of glycogen at our disposal being very limited we chiefly

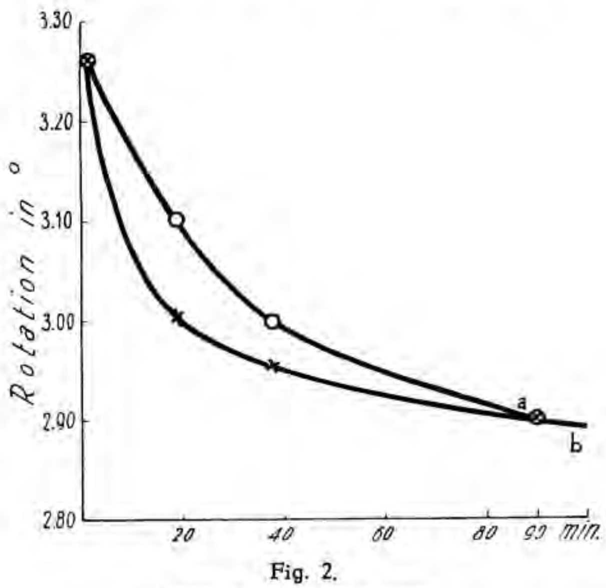

Fig. 2.
Action of saliva on glycogen.

restricted our investigation to the difference in rotation which could be observed before and after the mutarotation. Here too we found a negative mutarotation which confirms the fact already found by other investigators, that the type of maltose which is set free, does not depend on the character of the substrate, but is determined by the enzyme by which the sugar is set free.

Moreover it must be observed that a certain amount of saliva hydrolyses a definite amount of starch more rapidly than the same amount of saliva hydrolyses the same amount of glycogen. From some determinations of reduction-values we got the impression that the percentage of the saccharification in the case of glycogen is lower than in that of the starch.

We repeated the experiments with glycogen five times with the same results. One of these experiments is represented in fig. 2.

III. *The cleavage of starch by the digestive juice of Helix pomatia.*

Owing to the great amount of snails which is necessary for the collection of a few cm^3 juice from the crop of the snail we could only perform two experiments, one of which is represented in fig. 3. In both experiments we

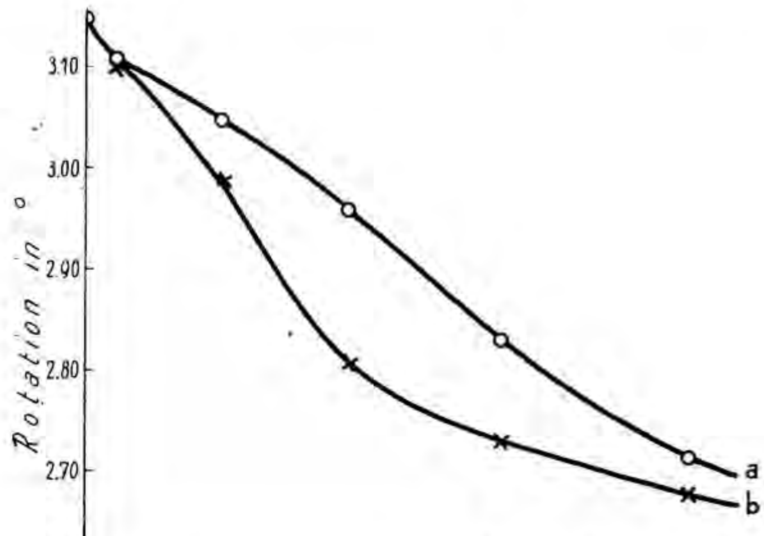

Fig. 3.
Action of digestive juice of Helix pomatia on starch.

found a strongly negative mutarotation, so that by far the largest amount of the amylase of Helix must belong to the α-type. Such in accordance with the fact that we found in our previous communication by means of the diffusion-method only a small amount of β-amylase. It is remarkable, that the amylase of the juice of Helix which contains a lot of enzymes (in rather large amounts), which are capable of hydrolysing β-hexosides, belongs for the greater part to the α-type.

Summary.

1. In the cleavage of starch- and glycogen by the saliva of man and also in the cleavage of starch by the digestive juice of Helix the maltose is set free in the α-form. In both of these digestive juices the amylase must therefore consist for the largest part of α-amylase. This agrees with the results of our previous communication (obtained with diffusion methods) where only very small amounts of β-amylase together with large amounts of α-amylase were found.

2. The results of the polarimetric determinations (after the mutarotation) nearly agree with the determinations by means of sugar-titration.

3. Starch-solutions (of Amylum solubile KAHLBAUM) are hydrolysed by human saliva for about 80 %.

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CONTENTS

Geophysics. — *Deep-focus and intermediate earthquakes in the East Indies.* By F. A. VENING MEINESZ.

(Communicated at the meeting of June 29, 1946.)

Since a long time already it is known that earthquake centres can occur at great depth far below the rigid crust. As it is well-known the centres occur generally in three groups: the shallow centres at depths less than 60 km which obviously occur in the rigid crust and mostly in tectonically active areas, the intermediate shocks at depths of 60—300 km, usually between 100 and 250 km, and the deep shocks at depths between 300 and 700 km. It is likewise well-known that in many cases these centres are more or less located in inclined planes cutting the surface in belts of strong tectonic activity. Most of the deep earthquakes occur in areas round the Pacific and, with only a few exceptions, the planes are dipping below the surrounding continental areas at an angle between 40° and 50°. Many attempts have already been made to explain these deep shocks in layers where we must accept a plastic behaviour of the materials.

A few geologists, e.g. ROBERT SCHWINNER [1]) and J. H. F. UMBGROVE [2]), have accepted the simple viewpoint that these planes are old shear planes, along which shear occurs because of the compression by tectonic forces. There is no doubt that this explanation is too simple and cannot be admitted in this shape. In the first place shear stresses of considerable amount in the plastic layers below the rigid crust must give rise to plastic deformation and not to sudden break unless the stresses increase in such a way that the high viscosity of these layers does not allow a plastic adjustment before the strength-limit is reached; it is, however, to be made clear how such an increase could take place and in any case currents must go along with it which no doubt must disturb the plane. In the second place it would be difficult to explain that the centres do not seem to occur continuously along the shear plane, but that they appear to fall apart in three groups: the shallow, the intermediate and the deep shocks which, each in itself, do not seem very clearly to follow the plane. In the third place, if such a preponderant shear plane is present, it seems difficult to explain that because of the compression the crust can bulge downwards in the tectonic belt in the way as is made probable by the narrow belts of negative anomalies

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[1]) "Der Begriff der Konvektionsströmung in der Mechanik der Erde", page 144, Gerl. Beitr. z. Geophysik, 58, 1941.

[2]) "Eilandbogen", Tijdschr. K. Nederl. Aardrijkskundig Genootschap, Amsterdam, 2e Reeks, Dl. LXIII No. 2, Mrt. 1946.

found in several of these areas [3]). If a shear plane is present we must assume that one part of the crust must be pushed over the other part and in that case an entirely different distribution of gravity anomalies must result.

Geophysicists, as e.g. B. GUTENBERG, S. W. VISSER, the writer and others, have in several publications pointed out that it is likely that the deep shocks are connected with current systems in the plastic layers below the crust and that the occurring of sudden breaks must be caused by increases of stress to values above the strength-limit arising in such a way that a plastic adjustment is impossible. In his papers on "Seismicity of the Earth", 1941 [4]) GUTENBERG is no doubt right in remarking that the shocks have to be expected in those parts of the subcrustal currents where the stresses are maximum (p. 113) and not necessarily at the level of maximum flow. The question, however, is what the character of these subcrustal currents is: how they are connected with the tectonic deformation of the crust at the surface and how it can be explained that the shocks more or less occur in the inclined planes mentioned above. Also it is necessary to show how these great stresses leading to the earthquakes can come about. In this paper the writer will make an attempt at an explanation for an area with many deep-focus and intermediate earthquakes, viz. the East-Indian Archipelago, where we have the advantage that besides the seismic data much is known about the geology and the gravity field. We shall begin by studying the south-eastern part.

In this investigation we shall suppose that in the subcrustal plastic layers convection-currents are present which must give rise to a system of stresses and that a relatively sudden deformation in the Earths crust brings about a quickly increasing other system of stresses which is superimposed on the first and occasions break in those areas where the total stress exceeds the limit of strength. It is, however, not impossible, although the facts do not seem to point this way, that the convection-current in itself leads already in some points to stresses reaching the strength-limit without a second trigger-effect being present.

It is probable that at least in the Banda Sea area, in the SE part of the Archipelago, a convection current is indeed going on, rising below the outer island-arc and sinking below the basin. Already in 1932 the writer has made this supposition for explaining the positive anomalies in the basin and the sinking down of this area. The outer island-arc is situated in the belt

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[3]) F. A. VENING MEINESZ, Maritime Gravity Survey in the East Indies; tentative interpretation of the results, Proc. Kon. Akad. v. Wetensch., Amsterdam, 33, 57 (1930).

F. A. VENING MEINESZ, Gravity Expeditions at Sea, Vol. II, Waltman Delft, 1934, page 119.

PH. H. KUENEN, The negative isostatic anomalies in the East Indies, Leidsche Geol. Med. 8, 1936, pages 169—214.

[4]) Special Papers, Geol. Soc. of America, No. 34.

of strong negative anomalies where the crust is supposed to have buckled downwards because of the tectonic compressing forces, thus forming a down-bulging root of crustal material below the crust of a cross-section of some 2.000 km^2. This root being lighter than the surrounding subcrustal material, explains the large deficiency of gravity. As we must assume that the crust is richer in radioactive materials than the subcrustal layers, the concentration of crustal material in the belt must gradually bring about a rise of temperature which may be expected to start a convection current in the subcrustal layers which are already near instability because of the negative temperature gradient in the outer layers brought about by the cooling of the Earth. The writer has pointed out elsewhere that this convection current rising below the belt and sinking below the Banda basin explains the positive anomalies as well as the sinking of this last area in a recent geological perior, i.e. some 4 or 5 million years ago. It likewise makes it clear that this sinking follows with a great time-lag on the last great folding period in the tectonic belt, which itself may be estimated to have occurred some 10 million years ago; in this period a large part if not the whole of the crustal root must have come into being. The long duration of the time-lag may be explained by the low value of the temperature conduction in the Earth because of which the above mentioned heating up of the subcrustal layers in the neighbourhood of the downfolded root must have been extremely slow.

Besides this convection-current going on in the substratum we must assume that at the surface a rising has recently set in in the tectonic belt. The island of Timor e.g. shows clear evidence of such a rising since the pliocene. We can easily explain it. The convection-current must give a decrease of the compression of the crust in the area around the belt because it rises under this belt and flows off laterally; it is even possible that this leads to tension in the crust. In any case we can understand that this makes it possible for the belt to detach itself somewhat from the adjoining areas and to rise in connection with its tendency to readjust the isostatic equilibrium. At the same time the adjoining areas which were drawn upward because of this tendency of the belt must sink back and no doubt are thus forming the basins which are found on both sides of the tectonic belt. The writer thinks that these rather quick vertical movements of the belt and the basins are causing the second field of stresses in the substratum which, added to the field brought about by the convection-current, gives rise to the intermediate and deep-focus shocks.

There is still another possibility for explaining the rising of the belt and the sinking of the adjoining basins in the present period. They may possibly also be caused by an increase in the lateral compression of the crust which might give a new wave-formation in the crust. This wave-formation would be favoured by the upwardbend already present because of the rising tendency of the tectonic belt and the wave-formation would give increased rising there. This mechanism has been investigated by the writer in 1940 and he may refer for it to that paper [5]. It leads to about the same results for the substratum as mentioned above.

We have now to investigate the fields of stress caused by the convection current which rises under the belt and sinks under the basin. For this investigation we may use the formulas for slow steady convection derived by the writer in "Equations for elastic solids in spherical coordinates, etc." [6]. For the dimensions of our current we can neglect the Earth's curvature and so we may use the formulas 25*a*, 30, 31, 32 and 33 (p. 445—477) derived for a plane fluid layer. The Z axis is chosen contrary to the direction of gravity, the X axis at right angles to the belt and the Y axis parallel to it.

For the quantity K (formulas 12, 13 and 15 ibid.) which determines the distribution in the XY plane of the vertical component $w = w_0 K$ of the speed and of the mean pressure $p = p_0 K$ $(p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z))$, w_0 and p_0 only being dependent on the z coordinate, we choose

$$K = \cos fx$$

in which

$$f = \frac{\pi}{l}$$

and l the distance between the axis of the rising and sinking colums (fig. 1).

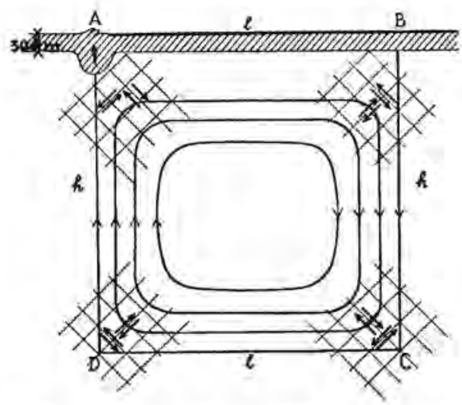

Fig. 1.
Schematic representation of the supposed Convection-current and accompanying shearstresses in SE part of the East Indies, A = tectonic belt, to the right of A: Banda basin.

By means of 31 and 32 ibid. we obtain for the stresses

$$\left.\begin{aligned} \frac{\sigma_x}{2\eta} &= -\left(\frac{p_0}{2\eta} + \frac{\partial w_0}{\partial z}\right)\cos fx \qquad & \frac{\tau_y}{2\eta} &= -\frac{1}{z} f\left(w_0 + \frac{1}{f^2}\frac{\partial^2 w_0}{\partial z^2}\right)\sin fx \\ \frac{\sigma_z}{2\eta} &= -\left(\frac{p_0}{2\eta} - \frac{\partial w_0}{\partial z}\right)\cos fx \qquad & \frac{p_0}{\eta} &= \frac{1}{f^2}\frac{\partial^3 w_0}{\partial z^3} - \frac{\partial w_0}{\partial z} \end{aligned}\right\} . \quad (1)$$

We see that for $x = 0$ and for $x = l$, i.e. for the axis of the rising and

[5]) F. A. VENING MEINESZ, The Earth's crust deformation in the East Indies, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **43**, 3 (1940).

[6]) Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48** (1945).

the sinking columns, we have a maximum shear in planes enclosing an angle of 45° with the X and Z axis (see fig. 1) and given by

$$\tau_{45} = 2\,\eta\,\frac{\partial w_0}{\partial z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

while for $x = \frac{1}{2}\,l$, i.e. for a vertical cross-section half-way between those two axis we have a shear in vertical and horizontal planes given by

$$\tau_y = \eta f\left(w_0 + \frac{1}{f^2}\,\frac{\partial^2 w_0}{\partial z^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2b)$$

For a further study of these shear-stresses we have to derive w_0 as a function of z.

By using equations (25) and (30) ibid and by introducing the equation for temperature conduction we obtain the following differential equation for stable convection which has already been derived by Lord RAYLEIGH in 1916 [7]).

$$\frac{\partial^6 w_0}{\partial z^6} - 3f^2\,\frac{\partial^4 w_0}{\partial z^4} + 3f^4\,\frac{\partial^2 w_0}{\partial z^2} - f^6\,w_0 + \frac{\varrho\,\beta\,\alpha\,g}{\eta\,\mu}\,f^2\,w_0 = 0. \quad . \quad (3)$$

where ϱ = density
β = temperature gradient
α = coeff. of thermal expansion
μ = coeff. of thermal conductivity.

The solution is simple when at the upper as well as the lower boundary-plane the temperature is constant and $w_0 = 0$ as well as $\tau_y = 0$; this last condition means $\partial^2 w_0/\partial_z{}^2 = 0$. This solution is given by Lord RAYLEIGH

$$w_0 = -\,a \sin pz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

with $p = \dfrac{\pi}{h}$ (4a) (h = thickness of the layer).

The minimum value of h for which convection is possible is given by (3):

$$\frac{l}{h} = \sqrt{q-1}$$

with

$$q = \sqrt[3]{\frac{\varrho\,\beta\,\alpha\,g}{\eta\,\mu\,f^4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

Fig. 2a gives the values of w_0 between $z = 0$ and $z = -h$. If our boundary-condition $\tau_y = 0$ has to be replaced by the condition that the speed-component in the boundary-plane is zero, which comes to the same as $\dfrac{\partial w_0}{\partial z} = 0$ our solution consists of three periodic terms and w_0 is represented

[7]) Lord RAYLEIGH, O.M., F.R.S., "On convection-currents in a horizontal layer of fluid, when the higher temperature is on the under side", Phil. Magaz. Dec. 1916.

by fig. 2*b*. We shall not enlarge on the rather complicated formulas for this case.

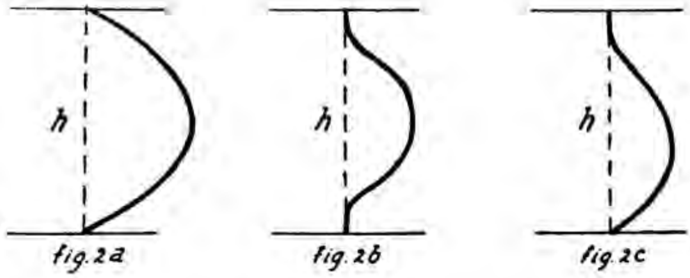

Fig. 2*a*, *b*, *c*.
Graphs of w_0 (prop. to vertical comp. of velocity).

Because of the presence of the crust on top of the plastic layer the upper boundary condition for our problem is probably more or less coinciding with this last case. At the lower boundary we may expect the first case to be nearer the truth because the layer below is likely to be also mobile and so we cannot suppose large values of shear-stresses to be transmitted by it. It is difficult, however, to be sure about the exact conditions at this boundary. Taking all considerations into account the writer thinks that the curve of fig. 2c cannot be far amiss as a representation of the actual values for w_0.

Applying this to formula 2*a* we see that the maximum value for τ_{45} coincides with that for $\partial w_0/\partial z$ and, therefore, is found at two places, viz. somewhat below the upper boundary of the current and at, or near, the lower boundary. The areas where they occur are indicated in fig. 1. For τ_y we find the maximum somewhere in the middle part of the current and because of the negative value of $\partial^2 w_0/\partial z^2$ its value is in general smaller than the maximum of τ_{45}. If we adopt the case of fig. 2*a* with formula 4 we obtain

$$\tau_y\,(\max) = \eta\, f\, a \left(\frac{p^2}{f^2} - 1\right) = \pi\,\eta\,\frac{a}{h}\left(\frac{l}{h} - \frac{h}{l}\right) \quad . \quad . \quad . \quad (5a)$$

while for this case

$$\tau_{45}\,(\max) = -\,2\,\eta\, p\, a = -\,2\,\pi\,\eta\,\frac{a}{h} . \quad . \quad . \quad . \quad . \quad . \quad (5b)$$

If $l = h$ we see that τ_y even disappears in the middle of the current. But also for other ratios of l/h we obtain a smaller value of τ_y as long as

$$\frac{l}{h} < 1 + \sqrt{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and we can prove that greater values of this ratio seldom occur. We may conclude that usually the largest values of shear are found in the rising and sinking columns at the levels mentioned above and working in planes under 45° with the vertical. By examining fig. 1 we see at once that a rapid rising of the tectonic crustal belt A and a sinking of the basin between this belt and B must give a relatively quick increase of these stresses in the areas A and C of maximum shear and so we can understand earthquakes to originate there along planes given by the arrows in the sense of AC. For the dimensions adopted in this figure they would especially occur at depths around 120 km and 500 km, i.e. at the depths and in the locations which are usual in the Banda-sea area for which this figure has been drawn.

Before, however, continuing our discussions we must examine for a moment the distribution of the currents in the horizontal sense. According to the adopted shape of the function K other symmetrical currents must be present on the other sides of AD as well as BC and this pattern is repeated periodically to both sides. In the case of the East Indies we, however, have supposed another surface condition for the temperature, i.e. the presence of a belt A of a higher temperature and so the location of the rising column AD is given by that belt. We might, therefore, expect that the current system is mainly restricted to two revolving currents to both sides of AD. This, however, is not borne out by the facts; we have only on one side deep earthquakes. The writer thinks that the curvature of the belt is the cause of this lack of symmetry. It brings about a decrease of internal friction on the concave side and an increase on the convex side and this must give rise to currents of different speed on both sides, viz. stronger on the inside and weaker on the outside. This seems in harmony with the presence of a deeper and wider basin on the inside than on the outside. It can likewise make it clear that only below the inside basin the shear-stresses can reach a value leading to intermediate and deep shocks. This difference must be particularly strong in the eastern part of the Banda basin where the loop of the tectonic belt is semi-circular and its radius smallest. It is interesting to see that here the Banda-sea has its greatest depth.

In this eastern area of the Banda-sea the horizontal dimension l is limited to 300—400 km by the presence under the island of Ceram of a continuation, though weaker than elsewhere, of the tectonic belt which probably prevents the downward current to come as far to the north as further to the west. As according to a deeper study of the formulas for convection the ratio l/h cannot be much less than the unity the depth of the deep shocks is likewise limited in this area; we only find values of 300—400 km there. Somewhat further to the west where the horizontal dimensions are larger we see higher values of the depth, viz. 500 km south of the island of Buru and 600—700 km south of Celebes. As it is well known these last values are the greatest yet found for deep shocks and the writer thinks that this is in good agreement with our supposition that they are caused by convection-currents; according to GUTENBERG [8]) the cooling has not perceptibly

[8]) GUTENBERG, "The cooling of the Earth" in: Internal Constitution of the Earth, page 160.

advanced below a depth of 800 km and so strong currents of this type may be expected to be limited to a layer of that thickness.

Examining the formulas 1 and fig. 1 we see that the shear-stress in the whole areas indicated in that figure near A and C, in planes parallel to those where the maximum stress works, is only slightly less than the maximum value and so we can understand that intermediate and deep earthquakes can originate in these entire areas. This may explain the scattering of the deep shocks south of Celebes and in most other areas of the map. As we have already remarked in the beginning of this paper this occurs in nearly all areas where we find deep shocks; they do not make the impression to be located in the inclined plane which connects the area of the deep shocks with those of the intermediate and the shallow ones.

If we accept the explanation of the intermediate and the deep shocks as it is given here, the question may be raised whether the surface effect of the rising of the belt A is indeed needed as a trigger effect or if they might already occur because of the presence of the convection-currents and their accompanying stress-field alone. Although the writer does not feel sure whether the latter type of shocks does not from time to time occur, he thinks that in general a surface effect must play a part because otherwise it seems difficult to understand why usually no shocks do occur in the area indicated near D. Examining, however, the intermediate shocks in the Banda-sea area we see e.g. a shock at a depth of 100 km at a latitude of 6° S.L. which might perhaps be attributed to the area near B in fig. 1.

The assumption of a convection-current sinking below the Banda-basin and rising below the tectonic belt may also explain the fact that the rising of this belt is accompanied on the inside of the curve by two basins separated by the inner Banda-arc. The basin between the belt and this last arc as well as the basin on the outside of the belt may be assumed to be caused by the sinking back of the crust originally drawn upwards by the tendency towards isostatic readjustment of the belt itself. The inner Banda-basin is perhaps partly caused by the sinking above a sinking column due to the lower temperature of this column but no doubt further by the fact that the part of the convection-current between A and B is weaker than the other parts because of the rising of the belt and the stowing away, therefore, of matter below A; this difference of the currents from A to B and from B to C must result in a sinking of the area above the current from B to C. It is, I think, only in this way that we can explain that a rising of the belt can bring about a sinking at such a large distance away from it. The sinking mentioned first for this basin, i.e. that connected with its being above the sinking column of the convection-current is only occurring when this current comes into being or, in general, when it increases; if the current is stable no further sinking takes place and if it disappears the basin must rise again. The sinking of the basin on the outside of the belt may be entirely due to the cause mentioned above for this basin, i.e. the release of the crust originally lifted up by the rising tendency of the belt when this detaches itself from its surroundings or it may also partly be caused by the sinking above a weak convection-current on this side of the belt.

Examining our solution we may realize that our explanation of the intermediate and deep shocks is not affected by a reversal of the currents and also of the crustal movements to which a trigger effect is attributed. The reversal will only change the sign of all the stresses and so the shear-movements accompanying the shocks will take place in an opposite sense. It seems possible that these reversed currents may occur during the formation of the downward bulge in A at the lower boundary of the crust. During that time this forming bulge must press away the lower plastic layer which is already unstable because of the cooling of the Earth and so it might be possible that this would start a subcrustal convection-current contrary to that given in fig. 1; such a current would also be favoured by the movement of the crust towards A needed for the formation of the crustal bulge. Once started it will accelerate and this may help the bulge to come into being. We may also refer here to the experiments made by GRIGGS who imitated such currents by revolving rolls for bringing the crustal deformation about.

Admitting the forming of a convection-current, the downward movement of the bulge may also give the trigger-effect for bringing about the intermediate and deep shocks. The shear would then again take place in the direction AC but contrary to the arrows of fig. 1.

L. P. G. KONING has made an investigation about the direction of the shear-plane and the sense of the movement for one of the deep shocks south of Celebes [9]) and he finds both about in the way as represented by fig. 1. So, if we may accept this result for all the shocks in the SE part of the East Indies we come to the conclusion that in that area we have to accept the original supposition. For other areas, however, the reversed supposition may fit the facts better and it seems likely that in that case we have to assume that the tectonic deformation of the crust is still going on.

We shall now shortly examine the other parts of the East Indies. For the NE part where we have a second tectonic belt from E. Celebes to the east of the Philippines the facts are similar to those we have dealt with. There is, however, a difference with regard to the elevation of the tectonic belt. The part in the Molucca Sea between E. Celebes and the small islands of Tifore and Majoe (W of Ternate) shows a depth of more than 2000 m. There is again a volcanic inner arc to the NW of the belt, viz. the N arm of Celebes and the Sangihe Is, and NW of this arc we find the Sulu Sea with deep shocks in the center while the intermediate ones are again found near and over the inner arc. There seems every reason for a similar interpretation as for the SE part of the Archipelago, i.e. a convection-current

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[9]) L. P. G. KONING, Over het mechanisme in den haard van diepe aardbevingen. Van Campen, Amsterdam, 1941.

rising below the belt and sinking below the Sulu basin. There is, however, less evidence of a rising in the tectonic belt but it is not impossible it has occurred and that thus a trigger-effect can again be accounted for. Future investigations may shed light on this point.

The deep shocks north of Java are the most difficult to explain. As there is no deep basin in that area the question arises whether we can here accept the existence of a convection-current. We notice that the deep basins north of the smaller Sunda Is. stop N of the middle of Sumbawa, i.e. opposite the spot where the elevation of the tectonic belt changes from above sea-level to 2500 m below it and so it makes the impression that the formation of the basin is especially connected with the rising of the tectonic belt. At the same time the N—S gravity profiles from the Java Sea via Java and the tectonic belt to the undisturbed Indian Ocean give an indication of the presence of a convection-current; the area north of the south-coast of Java shows a positive mean anomaly and that to the south of it a negative one. This is similar to the gravity profiles in the SE of the Archipelago where we have a positive mean anomaly over the Banda-basin and a negative one over the area around the outer island arc and where the change of sign likewise occurs over or slightly to the south of the inner arc. If we should be permitted to interpret these results over the Java area in the same way as we can do it for the eastern area, i.e. as an indication of the presence of a convection-current rising under the area of the negative mean anomaly and sinking under that of the positive one we have to assume that either the sinking of the basin over the sinking current only occurs if the tectonic belt over the ascending current is rising or that the sinking of the oil-geosyncline in the Northern part of Java takes the place here of the basin-sinking elsewhere. As for this last supposition it must, however, be remarked that the oil-geosyncline only coincides with the southern part of the sinking current and not with the whole of it.

If we accept a convection-current over the Java area we may no doubt explain the intermediate and deep shocks in this area in the same way as in the eastern part of the Archipelago. The only remaining question then is whether there is somewhere a trigger effect caused by the rising of a part of the crust. We do not know if such a movement is actually going on in the tectonic belt; this is no doubt possible. If this is not the case we might perhaps look for such an effect in a rising somewhere else.

It is interesting to notice that no deep shocks are found in the Sumatra area. The writer thinks that the tectonic belt west of that island has not witnessed any down-bulge of the crust but that we have here mainly a longitudinal shear movement accompanied by a slighter creeping of the Sumatra block along an inclined plane over het block to the west of it [10]). Such a process does not give the excessive accumulations of crustal matter

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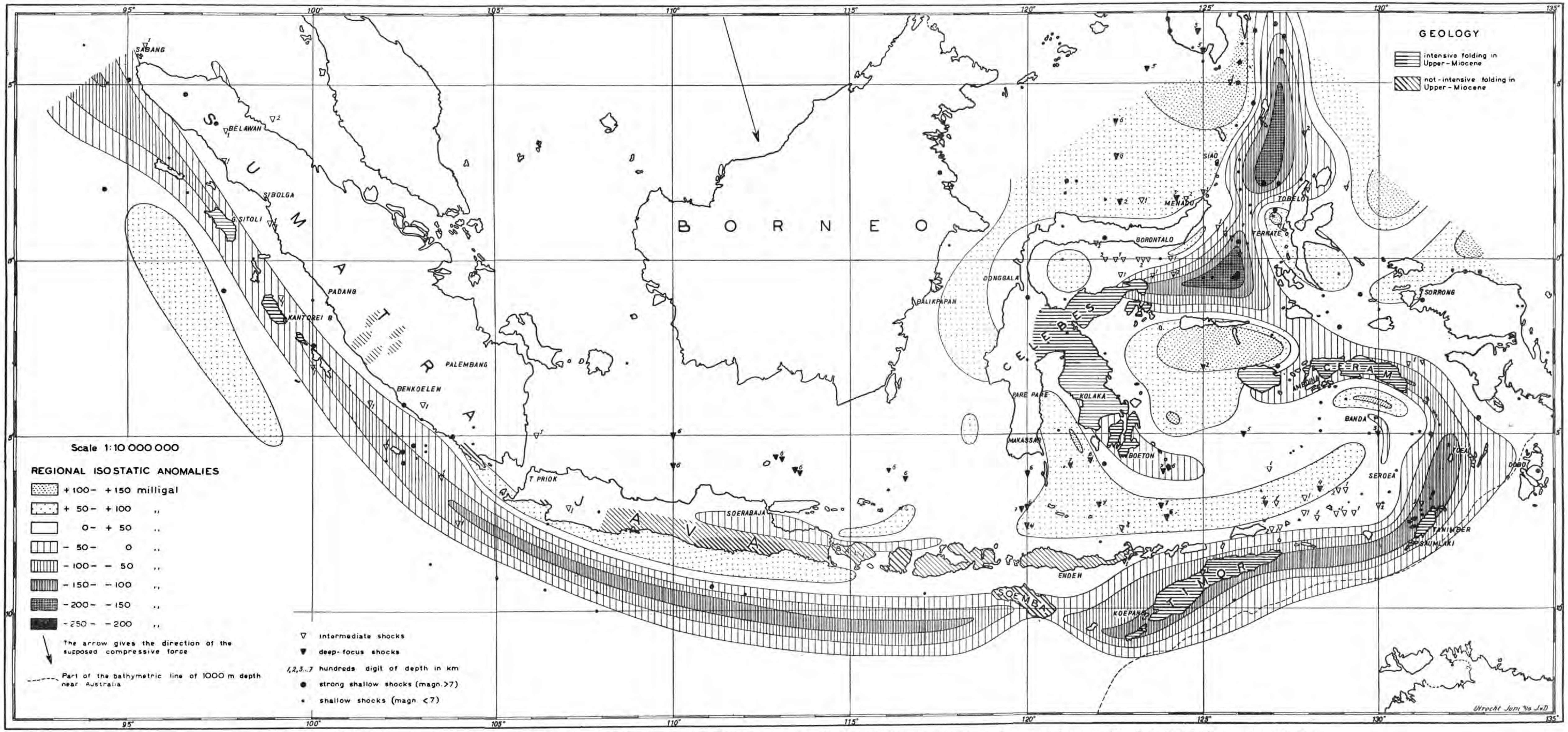
10) F. A. VENING MEINESZ, Gravity Expeditions at Sea, Vol. II, page 122, Waltman, Delft 1934.



East Indies, Gravity: Regional Anomalies, $l = 60$ km., $R = 174.3$ km. $T = 30$ km, *Geology* after J. H. F. UMBGROVE, Tijdschr. K.N.A.G. 62, 1935. *Seismicity* after B. GUTENBERG, Seismicity of the Earth, Bull. G.S.A. 56, p. 603—668, 1945.

which are brought about by the buckling in the tectonic belt elsewhere and so we can understand that the temperature distribution has not been sufficiently disturbed here for bringing about convection-currents. This would be in harmony with our supposition that only those currents can bring about the deep shocks.

If we may adhere to this hypothesis, the location of the deep shocks is a most valuable help for the study of those deep currents and of their distribution over the world. It is hardly necessary to stress the importance of this conclusion as these deep currents are probably the cause of most if not all of the tectonic deformations of the crust and of many other phenomena.

Mathematics. — *Over parametervoorstellingen met toepassing op* CAYLEY'*s formules voor de voorstelling van den orthogonalen determinant.* By W. VAN DER WOUDE.

(Communicated at the meeting of September 21, 1946.)

Laat een irreducibele m-dimensionale variëteit V_m in een n-dimensionale ruimte door een parametervoorstelling

$$x_i = q_i\,(t_1, t_2, \ldots t_m)\ \ (i = 1, 2 \ldots)$$

gegeven zijn. Hiermee wordt dan aangeduid, dat elk stelsel waarden t_a ($a = 1, \ldots, m$) een punt van V_m bepaalt — uitgezonderd misschien, wanneer voor zoo'n stelsel niet alle x_i's bepaald zijn — en dat een algemeen punt van V_m door een stelsel t_a bepaald kan worden. Het is dan echter mogelijk, dat sommige punten van V_m niet direct, d.w.z. door het invullen van waarden voor t_a, bepaald zijn.

Zoo kan b.v. de cirkel

$$x^2 + y^2 = 1$$

door

$$x = \frac{2\,t}{1+t^2},\ y = \frac{1-t^2}{1+t^2} \quad \ldots\ldots\ldots \quad (2)$$

worden voorgesteld. Het punt $(0, -1)$ van den cirkel wordt echter uit deze parametervoorstelling slechts door een limietovergang $(t \to \infty)$ gevonden.

Men kan in dit voorbeeld natuurlijk het gebrek opheffen door invoering van homogene coördinaten en parameters, dus door den cirkel voor te stellen door

$$x^2 + y^2 = z^2$$

en (2) te vervangen door

$$x = 2\,st,\ y = s^2 - t^2,\ z = s^2 + t^2.$$

Het is nu verder bekend, dat ook hierna, d.w.z. na zoo'n homogeen maken, soms nog punten der variëteit niet direct uit de parametervoorstelling worden gevonden, ook hoe ze dan toch wel gevonden worden.

Echter is het soms weinig opvallend, dat er als het ware punten aan de parametervoorstelling ontsnappen, waardoor haar gebruik dan eigenaardige gevaren meebrengt.

Het is mijn doel in het volgende eerst een paar eenvoudige m.i. karakteristieke voorbeelden te geven en daarna een toepassing op een vraagstuk van schijnbaar geheel anderen aard, n.l. op de uitdrukking der elementen van een orthogonalen determinant van den n^{den} graad, met de waarde $+1$, in $\frac{1}{2}n(n-1)$ parameters.

Het is n.l. CAYLEY [1]) gelukt deze termen door een dergelijke parametervoorstelling uit te drukken. Er zijn echter gevallen, waarin deze voorstelling niet mogelijk is.

In KOWALEWSKI's Einführung (zie voorafgaande voetnoot) wordt aangetoond, dat er voor $n = 2$ slechts één uitzondering is, die echter door homogeen maken der parameters dadelijk verdwijnt.

Het vraagstuk van deze uitzonderingsdeterminanten [2]) werd kortgeleden nog besproken in de dissertatie van Dr. LELYVELD [3]) en wel in de voorrede. Hij bewijst, dat er voor $n = 2$ en $n = 3$ geen uitzonderingsdeterminanten (in deze beteekenis) bestaan; d.w.z. dat de termen van elken orthogonalen determinant, met waarde $+1$, in de gevallen $n = 2$ en $n = 3$ in 2, resp. 4, homogene parameters kunnen worden uitgedrukt.

Verder bewijst hij, dat voor $n = 4$, ook bij gebruik van homogene parameters, uitzonderingsdeterminanten voorkomen. Deze geeft hij aan in het reëele gebied, voor zoover ze symmetrisch zijn.

In het volgende hoop ik nog aan te toonen, dat men op grond van de voorafgaande eenvoudige opmerking over parametervoorstellingen de eerste resultaten, — geen uitzonderingen voor $n = 2$ en $n = 3$ — dadelijk kan inzien en daarna voor $n = 4$ de uitzonderingsdeterminanten in het reëele gebied alle kan aangeven.

§ 1. Als eerste voorbeeld kies ik een variëteit V in de projectieve vierdimensionale ruimte R_4 voorgesteld door

$$\left.\begin{aligned} x_1 &= t_5\, t_2 \\ x_2 &= t_1\, t_3 \\ x_3 &= t_2\, t_4 \\ x_4 &= t_3\, t_5 \\ x_5 &= t_4\, t_1 \end{aligned}\right\} \qquad (1,1)$$

met
$$\Sigma\, t_i = 0. \qquad (1, 1^a)$$

Bepaalt men de doorsnede van V met $x_1 = 0$, dan vindt men a) $t_5 = 0$, *b*) $t_2 = 0$.

Toch zou men dwalen door de beide vlakken

$$x_1 = x_4 = 0 \text{ en } x_1 = x_3 = 0$$

als de volledige doorsnede van V met $x_1 = 0$ te beschouwen.

Immers stelt (1) voor de variëteit

$$x_1\, x_2\, x_3 + x_2\, x_3\, x_4 + x_3\, x_4\, x_5 + x_4\, x_5\, x_1 + x_5\, x_1\, x_2 = 0$$

de bekende derdegraadsvariëteit in R_4 van SEGRE.

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[1]) CAYLEY. Sur quelques propriétés des déterminants orthogonales Crelle's Journal 52. Zie verder: G. KOWALEWSKI, Einführung in die Determinantentheorie; Leipzig, Veit & Co., 1909.

[2]) Ik noem „uitzonderingsdeterminant" elken determinant, die uit de gegeven parametervoorstelling slechts door een limietovergang wordt gevonden; dat stemt niet geheel overeen met de definitie van Dr. LELYVELD.

[3]) Afbeeldingen van bewegingen om een punt in R_2, R_3, R_4. Proefschrift Utrecht 1943.

Tot die doorsnede behoort dus ook het vlak

$$x_1 = x_2 + x_5 = 0$$

§ 2. Beschouwen wij deze kwestie iets algemeener.

Laat een variëteit V voorgesteld worden door

$$x_i = q_i(t_1, t_2, \ldots t_\lambda) \ (i = 1, \ldots n) \ldots (2, 1)$$

aangenomen wordt, dat voor zekere waarden $t_\alpha^{(0)}$ ($\alpha = 1, \ldots, \lambda$ elke) x_i gelijk nul is.

Wij beschouwen dan t_α als functie van een parameter u, zoodat

$$t_\alpha^{(0)} = t_\alpha(u_0)$$

Hierdoor zijn dan de x_i functies van u, waarbij

$$x_i(u_0) = 0 \ (i = 1, \ldots n)$$

Wij laten nu u een stelsel waarden doorloopen met u_0 als limiet, dan beschrijft het punt x een kromme op V, waarop ook het limietpunt van deze kromme. Elk punt van V, dat niet dadelijk uit (2, 1) gevonden wordt, d.w.z. dat niet gevonden wordt door eenvoudig bepaalde waarden aan t_α te geven, kan men als limietpunt der parametervoorstelling (2, 1) vinden [4]).

Passen wij deze methode op ons voorbeeld (1, 1) toe.

Het is duidelijk, dat hier voor $x_i = 0$ noodig is, dat drie opeenvolgende (modulo 5) t's gelijk nul zijn.

Wij beschouwen dus functies $t_\alpha(u)$ waarvoor

$$t_5^{(0)} = t_1^{(0)} = t_2^{(0)} = 0 \ (t_\alpha^{(0)} = t_\alpha(u_0))$$

waaruit volgt

$$t_3^{(0)} + t_4^{(0)} = 0$$

Daar het geen zin heeft elke t_α gelijk nul te nemen, kunnen wij aannemen, dat $t_3^{(0)}$ en $t_4^{(0)}$ ongelijk nul zijn. Nu kan men uit t_α, t_β, t_γ — hiermee worden voor het oogenblik t_5, t_1, t_2 bedoeld, echter afgezien van de volgorde — er zeker één b.v. t_α nemen, zoodat $\lim\limits_{u \to 0} \frac{t_\beta}{t_\alpha}$ en $\lim\limits_{u \to 0} \frac{t_\gamma}{t_\alpha}$ eindig zijn.

Laat b.v. t_1 met t_α bedoeld zijn dan stellen wij

$$t_5 = \lambda t_1, \ t_2 = \mu t_1.$$

Voor het limietpunt, bepaald door $u \to u_0$, vinden wij

$$x_1 = 0, \ x_2 = t_3^{(0)}, \ x_3 = -\lambda^{(0)} t_3^{(0)},$$

$$x_4 = \mu^{(0)} t_3^{(0)}, \ x = -t_3^{(0)}.$$

Aldus worden als limietpunten gevonden, de punten van het vlak

$$x_1 = x_2 + x_5 = 0,$$

4) Zie B. L. VAN DER WAARDEN: Zur Algebraischen Geometrie, III, Math. Ann. p. 695.

en als meetkundige plaats der limietpunten de vlakken

$$x_i = x_{i-1} + x_{i+1} = 0 \; (i = 1, \dots 5, \text{mod. } 5).$$

§ 3. Nog meer eigenaardig is misschien het volgende voorbeeld. Een variëteit wordt, weer in de projectieve R_4, voorgesteld door

$$\left.\begin{aligned} x_1 &= t_1\, t_3\, t_4 \\ x_2 &= t_2\, t_4\, t_5 \\ x_3 &= t_3\, t_5\, t_1 \\ x_4 &= t_4\, t_1\, t_2 \\ x_5 &= t_5\, t_2\, t_3 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3,1)$$

met $\Sigma\, t_a = 0$ $(a = 1, \dots\dots 5)$ ...... $(3, 1^a)$.

Als doorsnede van deze variëteit met de lineaire ruimte $x_1 = 0$ vindt men voorloopig slechts drie rechten, dan als meetkundige plaats der limietpunten het tweedegraadsoppervlak

$$x_1 = x_2\, x_3 + x_3\, x_4 + x_4\, x_5 = 0.$$

Inderdaad is (3) een voorstelling der variëteit

$$x_1\, x_2 + x_2\, x_3 + x_3\, x_4 + x_4\, x_5 + x_5\, x_1 = 0.$$

§ 4. Voor de toepassing van het voorafgaande citeer ik in deze § en de volgende, gedeeltelijk zonder bewijs, KOWALEWSKI.

Laat $\| a_{ik} \|$ een orthogonale matrix zijn, waarbij

$$| a_{ik} | = +1.$$

Door de hoofdtermen met 1 te vermeerderen ontstaat de determinant $| a^+_{ik} |$, zoodat dus

$$a^+_{ik} = a_{ik} \; (i \neq k), \; a^+_{ii} = a_{ii} + 1.$$

Laat in dezen determinant b_{ik} de cofactor (het algebraïsch complement) zijn van a^+_{ik}. Dan is (zie Einführung, p. 169)

$$\left.\begin{aligned} b_{11} = b_{22} = \qquad\quad &= b_{nn} = \tfrac{1}{2}\, | a^+_{ik} | \\ b_{rs} &= - b_{sr} \; (r \neq s) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (4,1)$$

Is nu verder c_{ik} het algebraïsch complement van b_{ik} in $| b'_k |$, dan is bekend — mits $| a^+_{ik} | \neq 0$ —

$$a^+_{ik} = \frac{c_{ik}}{| a^+_{ik} |^{n-2}},$$

dus

$$a_{rs} = \frac{c_{rs}}{| a^+_{ik} |^{n-2}} \; (r \neq s), \; a_{rr} = -1 + \frac{c_{rr}}{| a^+_{ik} |^{n-2}}$$

Vermenigvuldigen wij de tellers van beide breuken met 2 b_{11} en de noemers met (a^+_{ik}), dan vinden wij

$$\left.\begin{aligned} a_{rs} &= \frac{2\,b_{11}\,c_{rs}}{|\,b_{ik}\,|}\;(r \neq s) \\ a_{rr} &= \frac{2\,b_{11}\,c_{rr}}{|\,b_{ik}\,|} - 1 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4,2)$$

Daar c_{rs} en $|\,b_{ik}\,|$ polynomia van de graden $(n-1)$ en n in de b_{ik} zijn, is hierdoor elke a_{rs} homogeen en van den graad nul in $\frac{1}{2}n\,(n-1)+1$ parameters uitgedrukt; door het opgeven der homogeniteit kan dit aantal met 1 worden verminderd.

Dit geldt voor elke orthogonale matrix $\|\,a_{ik}\,\|$ mits $|\,a^+_{ik}\,| \neq 0$ [5])

§ 5. Omgekeerd: Is $\|\,b_{ik}\,\|$ een scheeve niet singuliere matrix, d.w.z. $b_{rs} = -b_{sr}$ $(r \neq s)$, en is $|\,b_{ik}\,| \neq 0$, terwijl bovendien alle hoofdelementen onderling gelijk zijn, dan wordt door (4, 2), waarin c_{rs} het algebraïsch complement van b_{rs} in $\|\,b_{ik}\,\|$ voorstelt, een orthogonale matrix bepaald met $|\,a_{ik}\,| = +1$ (zie Einführung p. 173 e.v.).

Opmerking. Het valt op, dat het verband tusschen de matrices $\|\,a^+_{ik}\,\|$ en $\|\,b_{ik}\,\|$ in § 4 nauwer was dan tusschen de matrices, die hier in § 5 met dezelfde symbolen werden aangeduid. Immers in § 4 was b_{ik} het algebraïsch complement van a^+_{ik} in $|\,a^+_{ik}|$; in § 5 is dat in het algemeen niet het geval.

Dat het niet noodig is het verband tusschen $\|\,a_{ik}\,\|$ en $\|\,b_{ik}\,\|$ nauwer te leggen, dan in § 5 geschiedde, kan aldus worden verklaard.

In § 4 leidden wij de formule (4, 2) af, om uitgaande van $\|\,b_{ik}\,\|$, $\|\,a_{ik}\,\|$ terug te vinden. Nu zijn de tweede leden van (4, 2) homogeen en van den graad nul in b_{ik}. Dat beteekent, dat na vermenigvuldiging van alle b_{ik} met een zelfden factor toch (4, 2) dezelfde matrix $\|\,a_{ik}\,\|$ blijft aangeven. In § 5 was het mogelijk geweest aldus, d.w.z. door alle elementen van $\|\,b_{ik}\,\|$ met een zelfden factor te vermenigvuldigen, hetzelfde verband als in § 4 te leggen tusschen $\|\,b_{ik}\,\|$ en $\|\,a_{ik}\,\|$. Uit de gemaakte opmerking blijkt, dat dan toch dezelfde matrix $\|\,a_{ik}\,\|$ gevonden zou zijn. Dus was die vermenigvuldiging overbodig.

Opmerking. Zijn alle termen van den determinant $|\,b_{ik}\,|$ reëel, dan zijn ook die van den determinant $|\,a_{ik}\,|$ alle reëel, daar a_{rs} een rationale functie der b's is. Zijn omgekeerd alle termen van $|\,a_{ik}\,|$ reëel, dan zijn ook die van

[5]) De orthogonale matrices, die na vermeerdering van de termen der hoofddiagonaal met +1, een determinant leveren, die gelijk nul is, worden ook aangehaald door LIPSCHITZ, echter uit een geheel ander oogpunt. (R. LIPSCHITZ, Untersuchungen über die Summen von Quadraten; M. Cohen und Sohn, Bonn, 1886; een uitvoerig overzicht in Bull. d. Sc. Math., 2e Serie, T. x (1886, p. 163).) LIPSCHITZ behandelt de transformaties, die dezesommen invariant laten. De matrices, die bovengenoemde eigenschappen bezitten, leveren hierbij aanvankelijk moeilijkheid op. Hij bewijst dan dat men door de teekens van een even aantal kolommen van zoo'n matrix om te keeren steeds een nieuwe matrix kan doen ontstaan, die bruikbaar is.

$|b_{ik}|$ alle reëel of althans al hunne verhoudingen zijn dat, want de termen van $|b_{ik}|$ worden gevonden als cofactoren van die van $|a^+_{ik}|$ of zijn daarmee evenredig. Deze opmerking is later (§ 8) van belang.

§ 6. Uitgaande van een determinant

$$|b_{ik}| = \begin{vmatrix} b_{11}\, b_{12} \ldots b_{1n} \\ b_{21}\, b_{22} \ldots b_{2n} \\ b_{n1}\, b_{n2} \ldots b_{nn} \end{vmatrix} \neq 0,$$

waarin $b_{11} = b_{22} = \ldots = b_{nn}$ en $b_{rs} = -b_{sr}$ $(r \neq s)$, zal men dus door (4, 2) elken orthogonalen determinant $|a_{ik}|$ kunnen vinden, mits

$$|a^+_{ik}| = \begin{vmatrix} a_{11}+1 & a_{12} & a_{1n} \\ a_{21} & a_{22}+1 & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn}+1 \end{vmatrix} \neq 0$$

De determinant a_{ik}, die niet aldus gevonden wordt, noemde ik reeds een uitzonderingsdeterminant. Voor dezen geldt $|a^+_{ik}| = 0$; het is echter niet zeker, dat een determinant $|a_{ik}|$, waarbij $|a^+_{ik}| = 0$, een uitzonderingsdeterminant is.

Nemen wij eerst $n = 2$. Uit

$$\begin{vmatrix} \lambda & \mu \\ -\mu & \lambda \end{vmatrix} = \lambda^2 + \mu^2$$

vinden wij door (4, 2) de eenvoudige voorstelling der orthogonale matrix.

$$\left\| \begin{matrix} \dfrac{\lambda^2-\mu^2}{\lambda^2+\mu^2} & \dfrac{2\lambda\mu}{\lambda^2+\mu^2} \\ -\dfrac{2\lambda\mu}{\lambda^2+\mu^2} & \dfrac{\lambda^2-\mu^2}{\lambda^2+\mu^2} \end{matrix} \right\| \quad \ldots\ldots \quad (6,1)$$

Voor $n = 3$ vinden wij uit

$$\begin{vmatrix} \lambda & \mu & \nu \\ -\mu & \lambda & \varrho \\ -\nu & -\varrho & \lambda \end{vmatrix} = \lambda(\lambda^2 + \mu^2 + \nu^2 + \varrho^2)$$

de elementen der orthogonale matrix uitgedrukt in vier homogene parameters op een wijze, die gewoonlijk naar EULER wordt genoemd.

$$\left\| \begin{matrix} \dfrac{\lambda^2+\varrho^2-\mu^2-\nu^2}{S} & \dfrac{2(\mu\lambda-\nu\varrho)}{S} & \dfrac{2(\mu\varrho+\lambda\nu)}{S} \\ \dfrac{-2(\mu\lambda+\nu\varrho)}{S} & \dfrac{\lambda^2+\nu^2-\mu^2-\varrho^2}{S} & \dfrac{2(\lambda\varrho-\mu\nu)}{S} \\ \dfrac{2(\mu\varrho-\lambda\nu)}{S} & \dfrac{-2(\lambda\varrho+\mu\nu)}{S} & \dfrac{\lambda^2+\mu^2-\nu^2-\varrho^2}{S} \\ & S = \lambda^2+\mu^2+\nu^2+\varrho^2 & \end{matrix} \right\| \quad . \quad (6,2)$$

In de projectieve negendimensionale ruimte R_9 noemen wij negen der coördinaten a_{ik} $(i, k = 1, 2, 3)$ en de tiende a_0. Wij beschouwen de variëteit V, wier vergelijkingen zijn

$$\sum_{k=1}^{3} a_{ik}\, a_{jk} \left\langle \begin{array}{l} a_0^2\ (i=j) \\ 0\ \ (i \neq j) \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6,3)$$

Zij wordt ook voorgesteld door

$$\left. \begin{array}{c} a_{11} = \lambda^2 + \varrho^2 - \mu^2 - \nu^2,\ a_{12} = 2(\mu\lambda - \nu\varrho), \ldots\ a_{33} = \lambda^2 + \mu^2 - \nu^2 - \varrho^2 \\ a_0 = \lambda^2 + \mu^2 + \nu^2 + \varrho^2 \end{array} \right\} \quad (6,4)$$

Met elk punt, gegeven door deze parametervoorstelling, correspondeert een orthogonale matrix $\| \mathfrak{a}_{ik} \|$ met $| \mathfrak{a}_{ik} | = +1$, gegeven door

$$\mathfrak{a}_{ik} = \frac{a_{ik}}{a_0}$$

behalve met die punten, waarvoor $a_0 = 0$, zonder dat alle a_{ik} gelijk nul zijn.

De vraag is nu, of (6, 4) alle punten van V geeft. Mogelijke limietpunten der parametervoorstelling kunnen wij vinden door elke a_{ik} gelijk nul te stellen. Hieruit volgt echter $\lambda = \mu = \nu = \varrho = 0$, d.w.z. de parametervoorstelling (6, 4) heeft geen limietpunten.

Daar nu (6, 4) elk punt der variëteit (6, 3) dadelijk geeft, worden ook alle orthogonale matrices $\| \mathfrak{a}_{ik} \|$ gevonden door de parametervoorstelling (6, 2).

Het is duidelijk, dat op dezelfde wijze wordt ingezien, dat de orthogonale matrices voor $n = 2$ uit (6, 1) worden aangegeven zonder uitzondering.

§ 7. Voor $n = 4$ gaan wij uit van de scheeve matrix

$$\left\| \begin{array}{cccc} \lambda & \mu & \nu & \pi \\ -\mu & \lambda & \varrho & \sigma \\ -\nu & -\varrho & \lambda & \tau \\ -\pi & -\sigma & -\tau & \lambda \end{array} \right\| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7,1)$$

Haar determinant is gelijk aan

$$N = \lambda^4 + (\mu^2 + \nu^2 + \pi^2 + \varrho^2 + \sigma^2 + \tau^2)\, \lambda^2 + \triangle^2,$$

$$\triangle = \mu\tau - \nu\sigma + \pi\varrho.$$

Wij veronderstellen

$$N \neq 0.$$

Dan vinden wij uit (7, 1) door toepassing van (4, 2) de orthogonale matrix $\| \mathfrak{a}_{ik} \|$ met $| \mathfrak{a}_{ik} | = +1$, waarvan hieronder de elementen volgen [6]):

[6]) Zie het proefschrift van Dr. LELYVELD.

(verder aan te geven door (7, 2)

$$a_{11}=\frac{\lambda^4-(\mu^2+\nu^2+\pi^2-\varrho^2-\sigma^2-\tau^2)\,\lambda^2-\triangle^2}{N}$$

$$a_{22}=\frac{\lambda^4-(\mu^2-\nu^2-\pi^2+\varrho^2+\sigma^2-\tau^2)\,\lambda^2-\triangle^2}{N}$$

$$a_{33}=\frac{\lambda^4-(-\mu^2+\nu^2-\pi^2+\varrho^2-\sigma^2+\tau^2)\,\lambda^2-\triangle^2}{N}$$

$$a_{44}=\frac{\lambda^4-(-\mu^2-\nu^2+\pi^2-\varrho^2+\sigma^2+\tau^2)\,\lambda^2-\triangle^2}{N}$$

$$a_{12}=\frac{2\,\lambda\,\{\mu\,\lambda^2-(\nu\,\varrho+\pi\,\sigma)\,\lambda+\tau\,\triangle\}}{N}$$

$$a_{13}=\frac{2\,\lambda\,\{\nu\,\lambda^2+(\mu\,\varrho-\pi\,\tau)\,\lambda-\sigma\,\triangle\}}{N}$$

$$a_{14}=\frac{2\,\lambda\,\{\pi\,\lambda^2+(\mu\,\sigma+\nu\,\tau)\,\lambda+\varrho\,\triangle\}}{N}$$

$$a_{21}=\frac{-2\,\lambda\,\{\mu\,\lambda^2+(\nu\,\varrho+\pi\,\sigma)\,\lambda+\tau\,\triangle\}}{N}$$

$$a_{23}=\frac{2\,\lambda\,\{\varrho\,\lambda^2-(\mu\,\nu+\sigma\,\tau)\,\lambda+\pi\,\triangle\}}{N}$$

$$a_{24}=\frac{2\,\lambda\,\{\sigma\,\lambda^2-(\mu\,\pi-\varrho\,\tau)\,\lambda-\nu\,\triangle\}}{N}$$

$$a_{31}=\frac{-2\,\lambda\,\{\nu\,\lambda^2-(\mu\,\varrho-\pi\,\tau)\,\lambda-\sigma\,\triangle\}}{N}$$

$$a_{32}=\frac{-2\,\lambda\,\{\varrho\,\lambda^2+(\mu\,\nu+\sigma\,\tau)\,\lambda+\pi\,\triangle\}}{N}$$

$$a_{34}=\frac{2\,\lambda\,\{\tau\,\lambda^2-(\nu\,\pi+\varrho\,\sigma)\,\lambda+\mu\,\triangle\}}{N}$$

$$a_{41}=\frac{-2\,\lambda\,\{\pi\,\lambda^2-(\mu\,\sigma+\nu\,\tau)\,\lambda+\varrho\,\triangle\}}{N}$$

$$a_{42}=\frac{-2\,\lambda\,\{\sigma\,\lambda^2+(\mu\,\pi-\varrho\,\tau)\,\lambda-\nu\,\triangle\}}{N}$$

$$a_{43}=\frac{-2\,\lambda\,\{\tau\,\lambda^2+(\nu\,\pi+\varrho\,\sigma)\,\lambda+\mu\,\triangle\}}{N}$$

Hierin is

$$\triangle=\mu\,\tau-\nu\,\sigma+\pi\,\varrho$$

$$N=\lambda^4+(\mu^2+\nu^2+\pi^2+\varrho^2+\sigma^2+\tau^2)^2\,\lambda^2+\triangle^2$$

In de projectieve R_{16} noemen wij zestien der coördinaten a_{ik} $(i, k = 1, \ldots, 4)$ en de zeventiende a_0. Wij beschouwen de variëteit V, wier vergelijkingen zijn

$$\sum_{k=1}^{4} a_{ik}\, a_{jk} \begin{cases} a_0^2\ (i=j) \\ 0\ \ (i \neq j) \end{cases}$$

Zij wordt ook voorgesteld door a_{ik} gelijk te stellen aan den teller van de breuk, die in (7, 2) gelijk aan a_{ik} is, dus

$$\left.\begin{array}{c} a_{11} = \lambda^4 - (\mu^2 + \nu^2 + \pi^2 - \varrho^2 - \sigma^2 - \tau^2)\,\lambda^2 - \Delta^2 \\ \text{enz. met} \\ a_0 = N \end{array}\right\} \quad . \quad . \quad . \quad (7,3)$$

Evenals in § 6 correspondeert met elk punt, door deze parametervoorstelling direct gegeven, een orthogonale matrix $\| a_{ik} \|$ met $| a_{ik} | = +1$, behalve met die punten, waarvoor $a_0 = 0$.

De vraag is nogmaals of (7, 3) alle punten van V geeft. Om dit te onderzoeken stellen wij weer alle a's gelijk nul, waarbij wij uit $a_{ik} = 0$ $(i \neq k)$ in de eerste plaats afleiden

$$a_{ik} \pm a_{ki} = 0.$$

Aan al deze vergelijkingen en aan

$$a_{ii} = 0$$

wordt dan voldaan door te stellen

I

$$\lambda = 0 \ , \ \Delta = 0$$

of II

$$\begin{array}{ll} \mu\,\lambda^2 + \tau\,\Delta = 0 \ , & \nu\,\varrho + \pi\,\sigma = 0, \\ \nu\,\lambda^2 - \sigma\,\Delta = 0 \ , & \mu\,\varrho - \pi\,\tau = 0, \\ \pi\,\lambda^2 + \varrho\,\Delta = 0 \ , & \mu\,\sigma + \nu\,\tau = 0, \\ \varrho\,\lambda^2 + \pi\,\Delta = 0 \ , & \mu\,\nu + \sigma\,\tau = 0, \\ \sigma\,\lambda^2 - \nu\,\Delta = 0 \ , & \mu\,\pi - \varrho\,\tau = 0, \\ \tau\,\lambda^2 + \mu\,\Delta = 0 \ , & \nu\,\pi + \varrho\,\sigma = 0. \end{array}$$

In geval I hebben wij dus aan (7, 3) toe te voegen

$$\lambda \to 0 \ , \ \Delta \to 0.$$

(als tevoren beschouwen wij $\lambda, \mu, \ldots, \tau$ als functies van een parameter u).

Wij stellen $\Delta = \lambda\omega$ (7, 4) en gaan de verhouding der a's na voor $\lambda \to 0$, waarbij lim ω eindig wordt verondersteld.

Hierdoor vinden wij de uitzonderingsdeterminanten met de elementen

$$a_{11} = \frac{-\mu^2 - \nu^2 - \pi^2 + \varrho^2 + \sigma^2 + \tau^2 - \omega^2}{N_0}$$

$$a_{22} = \frac{-\mu^2 + \nu^2 + \pi^2 - \varrho^2 - \sigma^2 + \tau^2 - \omega^2}{N_0}$$

$$a_{33} = \frac{\mu^2 - \nu^2 + \pi^2 - \varrho^2 + \sigma^2 - \tau^2 - \omega^2}{N_0}$$

$$a_{44} = \frac{\mu^2 + \nu^2 - \pi^2 + \varrho^2 - \sigma^2 - \tau^2 - \omega^2}{N_0}$$

$$a_{12} = \frac{-2(\nu\varrho + \pi\sigma - \tau\omega)}{N_0} \qquad a_{21} = \frac{-2(\nu\varrho + \pi\sigma + \tau\omega)}{N_0}$$

$$a_{13} = \frac{2(\mu\varrho - \pi\tau - \sigma\omega)}{N_0} \qquad a_{31} = \frac{2(\mu\varrho - \pi\tau + \sigma\omega)}{N_0}$$

$$a_{14} + \frac{2(\mu\sigma + \nu\tau + \varrho\omega)}{N_0} \qquad a_{41} = \frac{2(\mu\sigma + \nu\tau - \varrho\omega)}{N_0}$$

$$a_{23} = \frac{-2(\mu\nu + \sigma\tau - \pi\omega)}{N_0} \qquad a_{32} = \frac{-2(\mu\nu + \sigma\tau + \pi\omega)}{N_0}$$

$$a_{24} = \frac{-2(\mu\pi - \varrho\tau + \nu\omega)}{N_0} \qquad a_{42} = \frac{-2(\mu\pi - \varrho\tau - \nu\omega)}{N_0}$$

$$a_{34} = \frac{-2(\nu\pi + \varrho\sigma - \mu\omega)}{N_0} \qquad a_{43} = \frac{-2(\nu\pi + \varrho\sigma + \mu\omega)}{N_0}$$

$$N_0 = \mu^2 + \nu^2 + \pi^2 + \varrho^2 + \sigma^2 + \tau^2 + \omega^2$$

$$(\mu\tau - \nu\sigma + \pi\varrho = 0)$$

Hierbij werd lim ω eindig verondersteld. Ter aanvulling kan men nog (7, 4) vervangen door

$$\lambda = \omega^+ \Delta$$

en onderstellen

$$\lim . \omega^+ = 0.$$

Men vindt dan geen nieuwe uitzonderingsdeterminanten.

§ 8. In geval II vinden wij, waneer wij de oplossing

$$\lambda = \mu = \nu = \pi = \varrho = \sigma = \tau = 0$$

niet meetellen

$$\left.\begin{array}{c} \mu = \tau,\ \nu = -\sigma,\ \pi = \varrho \\ (\text{of } \mu = -\tau,\ \nu = \sigma,\ \pi = -\varrho) \\ \lambda^2 + \mu^2 + \nu^2 + \varrho^2 = 0 \end{array}\right\} \quad \ldots \ldots \quad (8,1)$$

Wij moeten dus μ en τ tot eenzelfde limiet laten naderen, eveneens ν en σ, ook π en ϱ, verder nog stellen

$$\lambda^2 + \mu^2 + \nu^2 + \varrho^2 = 0;$$

dan vinden wij de verdere limietpunten van V bij deze parametervoorstelling.

Nu rijst de vraag: *wanneer wij ons willen beperken tot de bestaanbare limietpunten, mogen wij dan concludeeren: hieruit worden geen limietpunten gevonden, daar aan* (8, 1) *geen reëele waarden van* $\lambda, \ldots \tau$ *voldoen, behalve*

$$\lambda = \mu = \ldots \tau = 0 ?$$

Het is duidelijk, dat deze conclusie bij een willekeurige variëteit onjuist zou zijn, daar deze zeer wel bestaanbare punten kan bezitten, die alleen langs complexen weg bereikbaar zijn (b.v. een isoleerpunt van een algebraïsche kromme). Wij zullen hierna bewijzen, dat dit gevaar bij onze variëteit V niet te duchten is. Maar een tweede vraag doet zich terzelfder tijd voor.

Is het mogelijk, dat op onze variëteit V reëele punten voorkomen, die niet door uitsluitend reëele parameterwaarden $\lambda, \ldots \tau$ kunnen worden aangegeven?

Op deze laatste vraag antwoorden wij het eerst. Een reëel punt op V wordt aangegeven door reëele coördinaten a_{ik}. Nu is reeds bewezen — zie de opmerking aan het slot van § 5 —, dat de hierbij behoorende parameterwaarden reëel zijn of dat tenminste hunne verhoudingen dat zijn, daar ze gelijk zijn aan, of evenredig met, de cofactoren van a_{ik}.

Nu blijft dus nog te antwoorden op de eerste vraag: kan elk reëel punt van V als limietpunt gevonden worden?

Er is een correspondentie één aan één tusschen elk reëel punt van V en een orthogonale determinant met reëele termen (de coördinaten van dat punt) en daardoor met den stand van een reëel rechthoekig assenstelsel in de vierdimensionale Euclidische ruimte. Nu kan elke stand van dit assenstelsel uit elken anderen langs reëelen weg bereikt worden; dus geldt ook voor de punten van V, dat elk langs reëelen weg bereikt kan worden.

Ons uitgangspunt was: er worden reëele punten gezocht door te stellen

$$\mu - \tau \to 0, \quad \nu + \sigma \to 0, \quad \pi - \varrho \to 0$$
$$\lambda^2 + \mu^2 + \nu^2 + \varrho^2 \to 0.$$

Nu is het duidelijk, dat aan den laatsten eisch slechts voldaan kan worden door van $\lambda, \mu, \nu, \varrho$ minstens één langs complexen weg tot zijn limiet te laten naderen, daar het geen zin heeft alle homogene voorkomende parameters tot nul te laten naderen. Maar daaruit blijkt dan, dat er behalve de reeds gevondene bij deze parametervoorstelling geen limietpunten op V zijn.

§ 9. Een enkel woord over de meetkundige beteekenis van de matrices $\| a_{ik} \|$, die de uitzonderingsdeterminanten $| a_{ik} |$ leveren. Daarvoor geldt $| \mathbf{a}^{+}_{ik} | = 0$, d.w.z. de karakteristieke vergelijking heeft een wortel -1. Bij een orthogonale transformatie zijn de wortels van deze vergelijking invariant.

Nu kunnen wij elke beweging in R_4 door een orthogonale transformatie herleiden tot twee draaiinge om onderling absoluut-loodrechte vlakken. Het is nu dadelijk zichtbaar, dat een wortel van de hierbij behoorende karakteristieke vergelijking gelijk is aan -1, dan en slechts dan, als de hoek van een dezer draaiingen gelijk π is. Hierdoor is dus elke matrix $\| a_{ik} \|$, waarbij $| a^{+}_{ik} \| = 0$, gekarakteriseerd. Wij zagen, dat zoo'n matrix niet *steeds* een uitzonderingsdeterminant levert; op de vraag, wanneer dat wel of niet gebeurt, gaan wij thans niet in.

Mathematics. — *On the fundamental theorem of algebra.* (Second communication)[1]). By J. G. VAN DER CORPUT.

(Communicated at the meeting of September 21, 1946.)

§ 2. *Proof of the transitivity.*

Definition. A polynomial is called simple if it is relatively prime to its derivative.

Lemma 1. *Every divisor $D(X)$ of a simple polynomial $F(X)$ is simple,* for a common divisor of $D(X)$ and $\frac{dD}{dX}$ would occur both in $F(X) = D(X)\,U(X)$ and in $\frac{dF}{dX} = U\frac{dD}{dX} + D\frac{dU}{dX}$, which is impossible.

Lemma 2. *The product of simple relatively prime polynomials is simple.*

It is sufficient to give a proof for the product of two polynomials. Be $F(X) = U(X)\,V(X)$. Then $\frac{dF}{dX} = U\frac{dV}{dX} + V\frac{dU}{dX}$ is relatively prime both to $U(X)$ and $V(X)$, hence also to $U(X)\,V(X)$.

Lemma 3. *Two polynomials, the product of which is simple, are simple and relatively prime.*

In fact, a common divisor of $U(X)$ and $\frac{dU}{dX}$ (or $V(x)$) occurs both in $F(X) = U(X)\,V(X)$ and in $\frac{dF}{dX} = U\frac{dV}{dX} + V\frac{dU}{dX}$, which is impossible, if $U(X)\,V(X)$ is simple.

Lemma 4. *Any polynomial $F(X)$ may be written in the form*

$$F(X) = F_1^{\alpha_1} \dots F_\lambda^{\alpha_\lambda},$$

where the exponents are positive and $F_1(X), \dots, F_\lambda(X)$ are simple, relatively prime polynomials.

Proof. The theorem is obvious if $F(X)$ is simple. If not, put $F(X) = D(X)\,F^*(X)$, where D is the greatest common divisor of $F(X)$ and $\frac{dF}{dX}$. In this case the degree of both $D(X)$ and $F^*(X)$ is less than the degree of $F(X)$. We may suppose, that for these polynomials the

[1]) Compare Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 722—732 (1946) and Indagationes Mathematicae 4 (1946).

proof is already given. Consequently we may write

$$F = F_1^{\alpha_1} \dots F_\lambda^{\alpha_\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the exponents α_i are natural numbers and $F_1 \dots F_\lambda$ are simple polynomials. We may even assume that these polynomials are relatively prime. For, if for instance $F_1 = U V$ and $F_2 = U W$, then U, V and W, being divisors of simple polynomials, are simple and

$$F = U^{\alpha_1 + \alpha_2} V^{\alpha_1} W^{\alpha_2} F_3^{\alpha_3} \dots F_\lambda^{\alpha_\lambda}.$$

Continuing in this manner we finally find $F(X)$ written in a form, similar to (6) with simple, relatively prime polynomials.

Lemma 5. *If a polynomial F is written in the form (6), where $F_1, \dots, F_\lambda$ are simple and relatively prime and the exponents are positive, then $F_1(X) \dots F_\lambda(X)$ is the characteristic divisor of $F(X)$.*

Proof. $\dfrac{dF}{dX}$ is the sum of λ terms; all but one of these terms are divisible by $F_1^{\alpha_1}$. The exceptional term possesses the form $\alpha_1 \dfrac{F}{F_1}$ and is divisible by $F_1^{\alpha_1 - 1}$ but has further no common factor with F_1. The derivative $\dfrac{dF}{dX}$ is therefore divisible by $F_1^{\alpha_1 - 1}$ and has no further common factor with F_1. A similar argument is valid for the polynomials $F_2, \dots, F_\lambda$, so that $G = F_1^{\alpha_1 - 1} \dots F_\lambda^{\alpha_\lambda - 1}$ is the greatest common divisor of F and $\dfrac{dF}{dX}$. Hence it follows, that $F_1 \dots F_\lambda$ is equal to the characteristic divisor.

Corollaries. The characteristic divisor $F^*(X)$ of a polynomial $F(X)$ is simple, being the product of simple, relatively prime polynomials.

$F(X)$ divides $(F^*(X))^\mu$, where μ denotes the degree of $F(X)$, since each $\alpha_\nu \leqq \mu$.

If $D(X)$ is a divisor of $F(X)$, then $D^*(X)$ is a divisor of $F^*(X)$. In fact, we may write $D(X) = F_1^{\beta_1} \dots F_\lambda^{\beta_\lambda}$ where $0 \leqq \beta_\nu \leqq \alpha_\nu$, so that the characteristic divisor $\prod\limits_{\beta_\varrho > 0} F_\varrho$ of $D(X)$ is a divisor of the characteristic divisor $\prod\limits_{\varrho=1}^{\lambda} F_\varrho$ of $F(X)$.

Lemma 6. *If U is the greatest common divisor of F and G, then U^* is the greatest common divisor of F^* and G^*.*

Proof. U is a divisor of both F and G. By corollary of lemma 5 the polynomial U^* is a divisor of both F^* and G^*, hence also of their greatest common divisor.

The greatest common divisor D of F^* and G^* is a divisor of U. Hence the characteristic divisor D^* of D is a divisor of U^*. But D is a divisor of the simple polynomial $F^*(X)$, so that it is also simple, consequently identical with its own characteristic divisor. Hence D is a divisor of U^*. Consequently U^* and D are identical.

Lemma 7. *If q denotes a positive element of Ω, then every interval can be divided in a finite number of subintervals, each with length $\leqq q$.*

In fact, Ω being Archimedeanly ordered, a natural number ϱ exists, such that $\varrho > \frac{b-a}{q}$, where a and b denote the endpoints of the interval. The points $a + \frac{\sigma(b-a)}{q}$ where σ runs through $0, 1, \ldots \varrho$, divide the given interval in subintervals, each with length $\leqq q$.

Lemma 8. *To any two relatively prime polynomials $F(X)$ and $G(X)$ corresponds a positive element $p = p(F, G) \leqq e$ of Ω, satisfying the inequality*

$$|F(X)| + |G(X)| > p$$

for all elements X of Ω.

Proof. We can restrict ourselves to an interval Γ outside which $|F(X)| > e$. Since $F(X)$ and $G(X)$ are relatively prime, the unit element of Ω can be written in the form

$$e = U(X)\,F(X) + V(X)\,G(X),$$

where $U(X)$ and $V(X)$ denote appropiate polynomials. The absolute value of $U(X)$ and $V(X)$ in Γ is less than $\frac{e}{p}$, where p is a suitable positive element of Ω. Then we have for all X of Ω in Γ

$$e < \frac{e}{p}\,(|F(X)| + |G(X)|),$$

which implies the required result.

Lemma 9. *To any two relatively prime polynomials $F(X)$ and $G(X)$ corresponds a positive element $q = q(F, G)$ of Ω, such that $G(X)$ is definite in every interval with length $\leqq q$, where $F(X)$ changes sign, and $F(X)$ is definite in every interval with length $\leqq q$, where $G(X)$ changes sign* ("definite" means: always positive or always negative).

Proof. We may restrict ourselves to a bounded interval $\triangle$. Consider the elements $p(F^*, G)$ and $p(F, G^*)$ of lemma 8. Be p_1 the smaller of) these two elements. Ω contains a positive element $m \geqq e$, such that each derivative of $F^*(X) = H(X)$, of $F(X)$, of $G^*(X)$ and of $G(X$ has in $\triangle$ an absolute value $\leqq m$. To prove that the element $q = \frac{p_1}{4m}$ possesses the required property we consider a subinterval $\triangle_1$ of $\triangle$ with length $\leqq q$, where $F(X)$ changes sign. For any two elements x and

$x+h$ of $\triangle_1$ we have $|h| \leqq q \leqq e$, hence, if μ denotes the degree of F,

$$\left.\begin{aligned} |H(x+h)-H(x)| &= |h| \left| \frac{H'(x)}{1!} + \ldots + \frac{h^{\mu-1} H^{(\mu)}(x)}{\mu!} \right| \\ &\leqq |h| \, m \left(\frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{\mu!} \right) < 2 |h| \, m \leqq 2 q m = \frac{p_1}{2} \end{aligned}\right\} . \quad (7)$$

Similarly

$$|G(x+h) - G(x)| < \frac{p_1}{2} . \quad \ldots \ldots \quad (8)$$

Since $F(X)$ changes sign in $\triangle_1$, this interval contains two elements u and v with $F^*(v) \leqq 0 \leqq F^*(u)$.

Hence from (7) we infer

$$0 \leqq F^*(u) \leqq F^*(u) - F^*(v) < \frac{p_1}{2}.$$

From the preceeding lemma it follows then $|G(u)| > \frac{p_1}{2}$. From (8) we find for each element x of $\triangle_1$

$$G(x) > G(u) - \frac{p_1}{2} > 0, \text{ if } G(u) > \frac{p_1}{2},$$

and

$$G(x) < G(u) + \frac{p_1}{2} < 0, \text{ if } G(u) < -\frac{p_1}{2}.$$

Hence $G(X)$ is definite in $\triangle_1$.

The second part of the theorem follows by exchanging F and G.

Definition. Two polynomials $F(X)$ and $G(X)$ are said to be equivalent in an interval Φ, if two polynomials P and Q, both definite in Φ, exist, such that $F^* P = G^* Q$.

Lemma 10. *If a polynomial $F(X)$ changes sign in an interval Φ, then every multiple $G(X)$ of $F(X)$ changes also sign in Φ. If further is given, that $G(X)$ changes sign at most once in Φ, then $F(X)$ and $G(X)$ are equivalent in Φ* (and therefore both change sign exactly once in Φ).

Proof. G^* is a multiple of F^*. Be $G^*(X) = F^*(X)\, U(X)$. Since $F^*(X)$ changes sign in Φ, by lemma 9 and 3 we can divide Φ in subintervals, each with length $\leqq q$, where q denotes the element $q(F^*, U)$ of lemma 9. In at least one of these subintervals, say Φ_1, the polynomial $F^*(X)$ changes sign. In virtue of the choice of q the polynomial $U(X)$ is definite in Φ_1, hence $G^*(X) = F^*(X)\, U(X)$ changes sign in Φ_1.

Suppose that $G(X)$ changes sign only once in Φ. If $U(X)$ were not definite in Φ, in one of the constructed subintervals, say Φ_2, the polynomial $U(X)$ would change sign and there $F^*(X)$ would be definite.

Hence $G^*(X) = F^*(X)\, U(X)$ would change sign in Φ_2. Since $U(X)$ is definite in Φ_1, but not in Φ_2, the intervals Φ_1 and Φ_2 would be different and $G^*(X)$ would change sign in Φ more than once. This not being the case, we find $U(X)$ to be definite in Φ and therefore $F(X)$ and $G(X)$ to be equivalent in Φ.

Remark. From $(\Gamma, C) = (\triangle, D)$ it follows, that (C, D) changes sign at most once in $(\Gamma, \triangle)$, for (C, D) changes sign there and has a multiple $D(X)$, which changes sign at most once in $(\Gamma, \triangle)$.

Lemma 11. *If $F(X)$ and $G(X)$ are equivalent in Φ, they are also equivalent to their greatest common divisor in Φ.*

Proof. Put $F^* = U^* P$ and $G^* = U^* Q$, where U^* is the greatest common divisor of F^* and G^*, so that P and Q are relatively prime. By lemma 6 the polynomial U^* is the characteristic divisor of the greatest common divisor U of F and G. If $P(X)$ is not definite in Φ, then a subinterval of Φ exists, in which $P(X)$ changes sign and which has a length $< q(U^*, P)$ and $< q(P, Q)$, where q denotes the element of Ω, introduced in lemma 9. By this lemma U^* and Q are definite in this subinterval; hence also G^*, while F^* changes sign in that subinterval. This is impossible, since $F(X)$ and $G(X)$ are equivalent in Φ.

Proof of the transitivity. From $(\Gamma, C) = (\triangle, D)$ and $(\triangle, D) = (\Lambda, L)$ we must deduce $(\Gamma, C) = (\Lambda, L)$. The greatest common divisor (C, D) of C and D changes sign in $(\Gamma, \triangle)$ and possesses a multiple $D(X)$, which changes sign at most once in that interval. From lemma 10 these polynomials are equivalent in $(\Gamma, \triangle)$ and both change sign there. Similarly (D, L) and D are equivalent in $(\triangle, \Lambda)$ and both change sign there. By lemma 11 the polynomial $D(X)$ is in $(\Gamma, \triangle, \Lambda)$ equivalent to the greatest common divisor (C, D, L) of (C, D) and (D, L). In $(\Gamma, \triangle, \Lambda)$ the polynomial $D(X)$ changes sign, for otherwise $D(X)$ would change sign both in $(\Gamma, \triangle)$ and $(\triangle, \Lambda)$ outside $(\Gamma, \triangle, \Lambda)$, consequently more than once in $\triangle$, which is impossible. Hence the polynomial (C, D, L) which is equivalent with $D(X)$, changes sign in $(\Gamma, \triangle, \Lambda)$, and therefore also its multiple (C, L) by lemma 10. Then (C, L) changes sign also in (Γ, Λ).

§ 3. *Definition of sum and product.*

Consider two polynomials

$$F(X) = f_0 + f_1 X + \ldots + f_\mu X^\mu \text{ and } G(X) = g_0 + g_1 X + \ldots + g_\nu X^\nu,$$

where $f_\mu = g_\nu = e$. The products $\Pi(X - Y_\varrho - Z_\sigma)$ and $\Pi(X - Y_\varrho Z_\sigma)$, where ϱ runs through $1, \ldots, \mu$ and where σ runs through $1, \ldots, \nu$, and where X, Y_ϱ and Z_σ denote indeterminates, may be written as integral rational functions of X, of the elementary symmetric functions of $Y_1, \ldots, Y_\mu$ and of the elementary symmetric functions of $Z_1, \ldots, Z_\nu$. If we replace the elementary symmetric functions ΣY_1, $\Sigma Y_1 Y_2, \ldots, Y_1 Y_2 \ldots Y_\mu$, ΣZ_1, $\Sigma Z_1 Z_2, \ldots, Z_1 Z_2 \ldots Z_\nu$ respectively by $-f_{\mu-1}, f_{\mu-2}, \ldots, (-1)^\mu f_0, -g_{\nu-1}, g_{\nu-2}, \ldots, (-1)^\nu g_0$, these products become polynomials in X, which I denote by $F(X) + G(X)$ and $F(X) \times G(X)$.

If we put $G(X) = G_1(X)\, G_2(X)$, the left side of the identity

$$\prod_{\varrho=1}^{\mu} G(X-Y_\varrho) = \prod_{\varrho=1}^{\mu} G_1(X-Y_\varrho) \cdot \prod_{\varrho=1}^{\mu} G_2(X-Y_\varrho)$$

becomes $F(X) + G(X)$, hence

$$F(X) + G(X) = (F(X) + G_1(X))\ (F(X) + G_2(X)).$$

Herefrom it follows: If $G_1(X)$ is a divisor of $G(X)$, then $F(X) + G_1(X)$ is a divisor of $F(X) + G(X)$. In the same way we get: If also $F_1(X)$ is a divisor of $F(X)$, then $F_1(X) + G_1(X)$ is a divisor of $F(X) + G_1(X)$, hence of $F(X) + G(X)$. In particular:

Lemma 12. *If u is a root of $F(X)$ and v is a root of $G(X)$, then $(X-u) + (X-v) = X-u-v$ is a divisor of $F(X) + G(X)$, hence $u+v$ is a root of $F(X) + G(X)$.*

Lemma 13. *A polynomial $F(X)$, the derivative of which is always $\geqq 0$ in an interval Φ, satisfies the inequality $F(u) \leqq F(v)$ for all elements u and v of Φ with $u \leqq v$.*

Proof. The second and higher derivatives of $F(X)$ in Φ are absolutely less than a suitably chosen element m of Ω. Divide the interval with endpoints u and v into σ equal parts, each of length $l = \frac{v-u}{\sigma} \leqq e$. For the endpoints a and b ($a < b$) of such a part we have

$$F(b) - F(a) = \frac{b-a}{1!} F'(a) + \frac{(b-a)^2}{2!} F''(a) + \ldots + \frac{(b-a)^\mu F^{(\mu)}(a)}{\mu!},$$

where μ denotes the degree of F. From $F'(a) \geqq 0$ it follows

$$F(b) - F(a) \geqq -m\, l^2 \left(\frac{1}{2!} + \ldots + \frac{1}{\mu!}\right) \geqq -m\, l^2,$$

and adding we obtain

$$F(v) - F(u) \geqq -m\, l^2\, \sigma = -\frac{m\,(v-u)^2}{\sigma}.$$

Since the number σ may be taken arbitrary large, we find $F(v) - F(u) \geqq 0$, for otherwise the number σ could be taken as large as to contradict the inequality.

Remark. From this lemma it follows immediately: If a polynomial has a definite derivative in an interval, the polynomial changes sign there at most once.

Lemma 14. *If the polynomial $C(X)$ changes sign in Γ and the polynomial $D(X)$ in $\triangle$, then the polynomial $C(X) + D(X)$ changes sign in $\Gamma + \triangle$.*

Proof. Suppose first that C and D are simple. Put $C(X) + D(X) = F(X)$. If a and b denote the endpoints of $\Gamma + \triangle$, we may assume without loss of generality $F^*(a) \neq 0$ and $F^*(b) \neq 0$, for otherwise the lemma is evident. Ω contains a positive element m, such that the second and higher derivatives of $F^*(X)$ are all absolutely $\leqq m$ in $\Gamma + \triangle$. Choose in Ω a positive element l, satisfying the inequalities

$$l \leqq e\,;\; l \leqq \frac{p}{4m}\,;\; l \leqq \frac{2}{p} F^*(a)\,;\; l \leqq \frac{2}{p} F^*(b). \quad . \quad . \quad . \quad . \quad (9)$$

(where p denotes the element $p\left(F^*, \dfrac{dF^*}{dX}\right)$, introduced in lemma 8), such that in the interval with endpoints a and $a + l$ the inequality $|F^*(x)| > \frac{1}{2} |F^*(a)|$ is valid and in the interval with endpoints $b - l$ and b similarly $|F^*(x)| > \frac{1}{2} |F^*(b)|$. From lemma 5, corollary we know $(F^*)^\mu = F\,G$, where μ denotes the degree of F and G is a suitable polynomial. Ω contains a positive element g, such that in $\Gamma + \triangle$ the polynomial G possesses an absolute value $\leqq g$.

If s and t are arbitrary elements of Ω, then

$$H(X, s, t) = \{(C(X) - s) + (D(X) - t)\} - \{C(X) + D(X)\}$$

is a polynomial in X, s and t. In each term of H either a factor s or a factor t occurs, for $H(X, 0, 0)$ is identically equal to 0. Hence a positive element k of Ω exists, such that from $|s| < k$ and $|t| < k$ it follows

$$|H(w, s, t)| < \frac{1}{g} \left(\frac{p\,l}{4}\right)^{\mu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

for all elements w of $\Gamma + \triangle$. Finally we choose the positive element h of Ω such that in every subinterval of Γ with length $\leqq h$, the oscillation of $C(X)$ is less than k, and that also in each subinterval of $\triangle$ with length $\leqq h$, the oscillation of $D(X)$ is less than k. Divide Γ and $\triangle$ into subintervals, each of length $\leqq h$. In at least one of these subintervals of Γ, say Γ_1, the polynomial $C(X)$ changes sign and in at least one of the subintervals of $\triangle$, say $\triangle_1$, the polynomial $D(X)$ changes sign.

Then the interval Γ_1 contains two elements u and u_1 with $C(u_1) \leqq 0 \leqq C(u)$. Since the oscillation of $C(X)$ in Γ_1 is less than k, it follows

$$0 \leqq C(u) \leqq C(u) - C(u_1) < k.$$

Similarly $\triangle_1$ contains a point v with $0 \leqq D(v) < k$. Hence inequality (10) is valid for $w = u + v$, $s = C(u)$ and $t = D(v)$. Then u is a root of $C(X) - s$ and v is a root of $D(X) - t$, hence $w = u + v$ is a root of $(C(X) - s) + (D(X) - t)$ by lemma 12. Therefore

$$|F(w)| = |C(w) + D(w)| = |H(w, s, t)| < \frac{1}{g} \left(\frac{p\,l}{4}\right)^{\mu}.$$

Hence

$$|F^*(w)|^\mu = |F(w)\,G(w)| < \left(\frac{p\,l}{4}\right)^\mu.$$

Consequently

$$|F^*(w)| < \frac{p\,l}{4} < \tfrac{1}{2}\,p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

From the definition of $p = p\left(F^*, \dfrac{dF^*}{dX}\right)$ it follows

$$\left|\frac{dF^*(w)}{dw}\right| > \tfrac{1}{2}\,p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From (11) and (9) we infer

$$|F^*(w)| < \tfrac{1}{2}\,|F^*(a)| \quad \text{and} \quad |F^*(w)| < \tfrac{1}{2}\,|F^*(b)|.$$

Therefore it is impossible that w lies either in the interval with endpoints a and $a+l$ or in the interval with endpoints $b-l$ and b. As w lies in the interval with endpoints a and b, it lies in the interval with endpoints $a+l$ and $b-l$. The interval $\Gamma + \triangle$ contains consequently the elements $w-l$ and $w+l$.

Since the second and higher derivatives of F^* are absolutely $\leqq m$ in $\Gamma + \triangle$ and since $l \leqq e$, the Taylor development gives

$$\left|F^*(w \mp l) - F^*(w) \pm l\,\frac{dF^*(w)}{dw}\right| \leqq m\,l^2\left(\frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{\mu!}\right) < m\,l^2 \leqq \tfrac{1}{4}\,p\,l,$$

hence by (11) we get

$$\left|F^*(w \pm l) \mp l\,\frac{dF^*(w)}{dw}\right| < \tfrac{1}{2}\,p\,l.$$

From (12) we infer that $F^*(w-l)$ and $F^*(w+l)$ have different sign; consequently $F(X)$ changes sign in $\Gamma + \triangle$.

Suppose now that C and D are not both simple. Then C^* and D^*, and therefore $C^* + D^*$ change sign resp. in Γ, $\triangle$ and $\Gamma + \triangle$; consequently the multiple $C + D$ of $C^* + D^*$ changes sign also in $\Gamma + \triangle$.

Now we pass to the definition of the sum of $\gamma = (\Gamma, C)$ and $\delta = (\triangle, D)$. Put $F = C + D$. A subinterval Γ' of Γ and a subinterval $\triangle'$ of $\triangle$ can be found, both with length $\leqq \tfrac{1}{2}\,q$, where q denotes the element $q\left(F^*, \dfrac{dF^*}{dX}\right)$, introduced in lemma 9, such that C changes sign in Γ' and D in $\triangle'$. In the interval $\Gamma' + \triangle'$ with length $\leqq q$ the polynomial F changes sign by lemma 14. By lemma 9 the derivative $\dfrac{dF^*}{dX}$ is definite throughout that interval; therefore F^* changes sign there at most once. So we have proved the existence of subintervals Γ' of Γ and $\triangle'$ of $\triangle$, such that C changes sign once in Γ', D in $\triangle'$ and $C + D$ in $\Gamma' + \triangle'$. Now we may put

$$\gamma + \delta = (\Gamma' + \triangle',\ C + D),$$

if we show that the couple $(\Gamma' + \triangle',\ C + D)$ is uniquely determined. Suppose

$$(\Gamma, C) = (\Gamma_1, C_1) \text{ and } (\triangle, D) = (\triangle_1, D_1).$$

We have to prove

$$(\Gamma' + \triangle',\ C + D) = (\Gamma_1' + \triangle_1',\ C_1 + D_1),$$

where Γ', $\triangle'$, Γ_1' and $\triangle_1'$ are subintervals respectively of Γ, $\triangle$, Γ_1 and $\triangle_1$, such that C changes sign only once in Γ', D in $\triangle'$, C_1 in Γ_1', D_1 in $\triangle_1'$, $F = C + D$ in $\Phi = \Gamma' + \triangle'$ and $F_1 = C_1 + D_1$ in $\Phi_1 = \Gamma_1' + \triangle_1'$. We must prove that the greatest common divisor L of $F = C + D$ and $F_1 = C_1 + D_1$ changes sign in the common part Λ of Φ and Φ_1. Since $S = (C, C_1) + (D, D_1)$ is a divisor both of $F = C + D$ and $F_1 = C_1 + D_1$, the polynomial S is also a divisor of their greatest common divisor L. By assumption the couples $(\Gamma, C) = (\Gamma' C)$ and $(\Gamma_1, C_1) = (\Gamma_1', C_1)$ are equal, so that (C, C_1) changes sign in (Γ', Γ_1') and similarly (D, D_1) in $(\triangle', \triangle_1')$. By lemma 14 S changes sign in $\Sigma = (\Gamma', \Gamma_1') + (\triangle', \triangle_1')$. By lemma 10 the multiple L of S changes sign in Σ, hence certainly in Λ, which contains Σ: in fact each point w of Σ may be written in the form $u + v$, where u lies both in Γ' and Γ_1', and v lies both in $\triangle'$ and $\triangle_1'$, hence w lies both in $\Gamma' + \triangle'$ and $\Gamma_1' + \triangle_1'$, consequently also in their common part Λ. This establishes the proof.

In a similar way we define the product of two couples.

Chemistry. — *De spreiding van enkele gliadine- en gluteline-praeparaten.* By E. GORTER and G. J. ELINGS.

(Communicated at the meeting of May 25, 1946.)

De bedoeling van dit onderzoek was om bij verschillende pH te bepalen hoe groot het oppervlak was van een monomoleculaire laag gespreid eiwit. Hierbij verdient dadelijk te worden opgemerkt, dat kleine cijfers niet beteekenen, dat de monomoleculaire laag veel dikker is, maar de kleine cijfers beteekenen, dat slechts een deel van het eiwit spreidt en het andere in oplossing gaat.

Techniek. De vloeistoffen, waarop de spreiding van het eiwit werd nagegaan, waren:

tusschen pH 1 en 3.3	HCl oplossing
,, ,, 3.6 ,, 6	azijnzuur-acetaatbuffers 0.0033 m
,, ,, 6 ,, 8	veronal-natrium HCl buffers volgens MICHAELIS 0.0033 m

Deze lage ionen-concentraties zijn gekozen, omdat grootere hoeveelheden electrolyt de neiging tot spreiden versterken. Daarom zijn ook tweewaardige en driewaardige ionen niet te gebruiken, daar deze ook de spreiding zeer sterk bevorderen. De metingen geschiedden bij ongeveer 19° C. De pH-waarden werden geschat met een colorimetrische methode, waarbij als indicatoren de nitrophenolreeksen werden gebruikt.

Spreiding van de eiwitten werd verkregen door van een oplossing, die ongeveer 5 mg/cc bevatte, uit een capillair-pipet 5 mm^3 op het oppervlak te brengen. De metingen geschiedden na één minuut wachten. Door draaiing aan den torsiekop van de balans, werd de laag gespreid eiwit geleidelijk onder toenemenden druk geplaatst, waarbij de grootte van het oppervlak, die bij elken druk behoorde, werd genoteerd. Met de meeste eiwitten kan er nu, wanneer men de oppervlakken uitzet tegen den druk, een reeks punten verkrijgen, waarvan een groot aantal in een rechte lijn gelegen zijn. Door deze lijn door te trekken naar de nullijn, verkrijgt men een getal voor het oppervlak bij een druk 0. Bij deze eiwitten was deze methode niet toe te passen, omdat de punten op een kromme liggen. Wij hebben ons uit deze moeilijkheid gered door telkens de oppervlakken te meten bij een druk van 6.6 $\frac{\text{dynes}}{\text{cm}}$; 13.2 $\frac{\text{dynes}}{\text{cm}}$; 19.6 $\frac{\text{dynes}}{\text{cm}}$; 26.4 $\frac{\text{dynes}}{\text{cm}}$,

voor wat de gliadine betreft, terwijl de getallen waren

$$5.5\,\frac{\text{dynes}}{\text{cm}};\ 11.0\,\frac{\text{dynes}}{\text{cm}};\ 23.0\,\frac{\text{dynes}}{\text{cm}};\ 33.0\,\frac{\text{dynes}}{\text{cm}},$$

voor de gluteline.

Gliadine-praeparaten.

Het eerste praeparaat is bereid volgens DILL en ALSBERG [1]), het tweede op soortgelijke wijze, terwijl het derde praeparaat uit Manitoba-tarwe is bereid volgens de methode van BLISH en SANDSTEDT [2]). Het laatste praeparaat is met oplossingen van zuur en alcali van geringere sterkte dan de beide andere praeparaten in aanraking geweest.

Gluteline-praeparaten.

Het eerste gluteline is bereid door dr. P. C. BLOKKER [3]). Het tweede gluteline-praeparaat is bereid uit de volgens BLISH en SANDSTEDT gedroogde gluten, door behandelen met 0.07 m azijnzuur, waardoor gliadine wordt opgelost en gluteline achterblijft. Deze gluteline is daarna bij 50° C in vacuo gedroogd. Het derde gluteline-praeparaat werd uit het tweede bereid door dit in zeer slappe loog op te lossen. Hierna werd de oplossing gefiltreerd en daarna werd het filtraat weer aangezuurd met azijnzuur tot een concentratie van 0.07 m. De gluteline slaat dan weer neer, terwijl de gliadine in oplossing blijft. Er is dus een grootere waarschijnlijkheid, dat de aldus bereide gluteline vrij is van gliadine [4]).

Beschrijving der resultaten.

De krommen, die aangeven de grootte van de spreiding bij de verschillende pH, verschillen voor de drie gliadine-praeparaten onderling slechts weinig (zie figuur 1). Ze komen grosso modo overeen met de resultaten van BLOKKER. De verschillen van onze metingen met gliadine vallen bijna binnen de waarnemingsfouten, die voor dit te oplosbare eiwit vrij groot zijn.

Voor de gluteline-praeparaten geldt, dat de beide eerste geheel dezelfde resultaten opleveren. Ook het derde praeparaat volgt met zijn pH kromme precies den loop van de beide andere praeparaten tusschen pH 1 en het iso-electrische punt. Naar den alcalischen kant van het iso-electrische punt evenwel, wordt de spreiding veel minder. Er bestaat tusschen het derde en de beide andere praeparaten hetzelfde verschil als wij hebben gevonden bij de spreiding van insuline en protamine-insuline, waarvan het laatste zich gedraagt als het eerste en tweede praeparaat gluteline (GORTER en MAASKANT [5]).

Het ligt voor de hand om de verschillen in beide gevallen toe te schrijven aan de vervanging van een aantal vrije COOH-groepen van het eiwit door NH_2-groepen of andere positieve groepen. Men ziet hetzelfde bij het

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[1]) DILL en ALSBERG, J. biol. Chem., **65**, 279 (1925).

[2]) BLISH en SANDSTEDT, Cereal Chem., **3**, 144 (1926).

[3]) P. C. BLOKKER, Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 228 (1942).

[4]) Deze praeparaten zijn grootendeels voor ons bereid in de laboratoria van WESSANEN's Koninklijke Fabrieken N.V. te Wormerveer, die ons ook de bovenstaande gegevens omtrent de bereidingswijze verstrekten. Wij danken de directeuren zeer voor hunne medewerking.

[5]) E. GORTER en L. MAASKANT, The spreading of protamine insulinate. Proc. Kon. Akad. v. Wetensch., Amsterdam, **40**, 71 (1937).

geconjugeerde eiwit pepsine-spermidine. GORTER en VAN ORMONDT [6]).

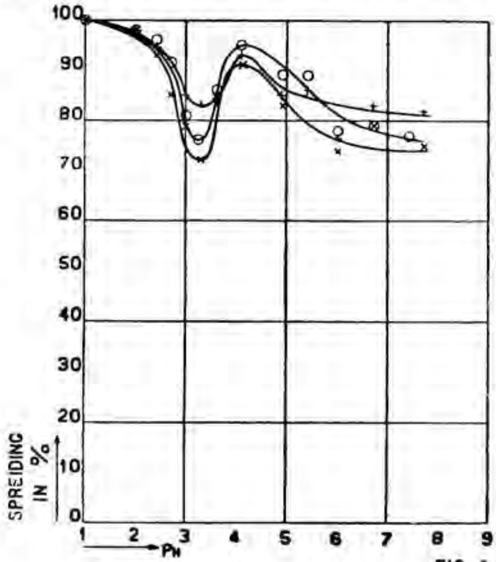

FIG. 1

Gliadine + Eerste praeparaat bereid volgens DILL en ALSBERG.
O Tweede praeparaat idem,
× Derde praeparaat bereid uit Manitoba volgens BLISH en SANDSTEDT.

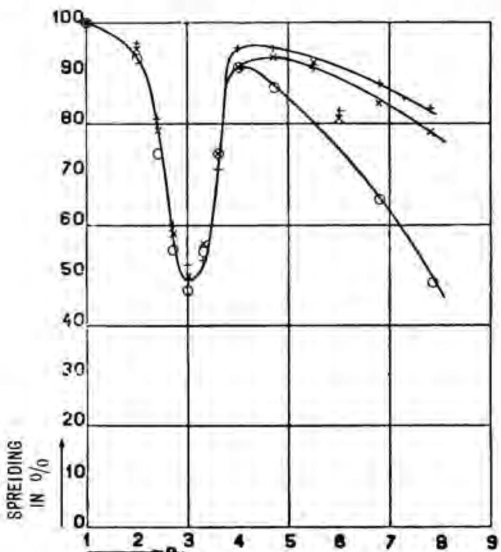

FIG. 2

Gluteline + Eerste praeparaat.
× Tweede praeparaat (uitgewasschen met 0.07 m. azijnzuur).
O Derde praeparaat. Opgelost in slappe loog en weer neergeslagen met 0.07 m. azijnzuur. Waarschijnlijk is dit praeparaat beter vrij van gliadine.

[6]) E. GORTER, H. VAN ORMONDT, TH. M. MEYER, The spreading of complex proteins, Biochem. J., 29, 38 (1935).

Hierbij is duidelijk, wat er gebeurt: één der NH_2-groepjes van het spermidine, resp. protamine bindt zich aan de CO-groep van het eiwit en de andere NH_2-groep van spermidine (of protamine) blijft vrij.

Er is tusschen het gliadine en het gluteline wel verschil, terwijl zij hetzelfde maximum vertoonen bij het iso-electrische punt (pH 4). Het voornaamste verschil is, dat het minimum aan den zuren kant van het iso-electrische punt dieper ligt bij het gliadine (resp. 50 en 80 %) van de maximale spreiding. De verklaring hiervan is, dat er in gluteline meer vrije NH_2-groepen voorkomen dan in het gliadine. Dat dit de verklaring is, kan men afleiden uit de resultaten van de spreiding van ovalbumine — tartrazine [6]).

Tartrazine is een kleurstof, die 2 sterk negatieve SO_3-groepen heeft.

Conclusies. De resultaten worden meegedeeld met de spreiding van gliadine en gluteline op verdunde bufferoplossingen met een pH tusschen 1 en 8. De verschillen zijn niet zeer groot. De overeenkomst betreft vooral, dat de maximumspreiding bij het iso-electrische punt ongeveer bij dezelfde pH ligt (ongeveer 4.3 ± 0.3). Terwijl ook de ligging van het minimum aan den zuren kant van het iso-electrische punt ongeveer op hetzelfde punt ligt (gliadine 3.3 ± 0.3; gluteline 3.0 ± 0.2). Ook de grootte van de spreiding bij het iso-electrische punt aan den alcalischen kant daarvan verschilt niet zeer belangrijk. Het grootste verschil is, dat gliadine een kleiner minimum heeft aan den zuren kant van het iso-electrische punt dan gluteline, resp. 50 en 80 % van de maximale spreiding.

Summary.

We have determined the size of the surface occupied by a monomolecular layer of the proteins gliadin and glutelin at various pH of the liquid in the tray. As has already been observed by P. C. BLOKKER [1]), gliadin is not easy to study by the spreading because it has a too great solubility. Therefore the points obtained for the size of the surface at various pressures do not lie in a straight line. We have tried to overcome this difficulty by measuring the size of the surface at 4 fixed pressures, n.l. 6.6, 13.2, 19.6 and 26.4 dynes/cM. The curves obtained by plotting the surface against the pH of the solution in the tray show some points of correspondance but differ in other respects. Both proteins have a maximum spreading at the same isoelectric point. They also behave as most proteins in showing on both sides of the isoelectric point a minimum. They differ by the depth of the minimum principally on the acid side of the isoelectric point.

The possible cause of the special form of these area-pH curves are discussed and compared with the spreading curves of known conjugated proteins e.g. ovalbumin-tartrazin, insulin-protamin and pepsin-spermidin.

Winschoten, 1944.

Chemistry. — *De spreiding van het seroglycoïd en het crystalbumine van* HEWITT. By E. GORTER and G. J. ELINGS.

(Communicated at the meeting of May 25, 1946.)

Onlangs hebben GORTER en HERMANS [1]) de resultaten meegedeeld van een onderzoek naar de spreiding van een complex eiwit, opgebouwd uit albumine en 40 % lipoïden: het lipo-proteïne van MACHEBOEUF. Daarbij bleek, dat dit complex bestaat uit eiwit met 39.6 % lipoïden en 13.1 % cholesterine van de droge stof. De spreiding onderscheidde zich weinig van die van een eiwit zonder lipoïden.

Wij hebben nu onderzocht, hoe de spreiding is van de glyco-proteïnen. Wij kozen daarvoor het seroglycoïd van HEWITT [2]).

Bereiding. Het seroglycoïd werd door ons bereid uit runderbloed of uit dat van een nuchter kalf. Na stolling van het bloed wordt het serum afgegoten. Aan het serum wordt een gelijke hoeveelheid verzadigde ammoniumsulfaatoplossing, die op pH 7.0 was gebracht, toegevoegd om de globulinen neer te slaan. Deze worden door centrifugeeren verwijderd. De albuminen, die in oplossing zijn, worden nu gefractionneerd neergeslagen met azijnzuur, totdat de pH 4.4 is. Het neerslag bevat het crystalbumine. Dit wordt weer telkens opgelost in water en neergeslagen. De aldus gevormde achtereenvolgende neerslagen met ammonium-sulfaat en azijnzuur zijn B_1—B_9 genoemd. De oplossingen, die bij het praecipiteeren telkens in oplossing blijven, bevatten voornamelijk seroglycoïd: dit is *b* genoemd. In vele gevallen werd nog een verdere fractionneering toegepast door zooveel verzadigd ammonium-sulfaat van een pH 4.4 toe te voegen aan de oplossing, die dezelfde pH had tot een eindconcentratie van 65 % en later van 75 % bereikt was. De hierbij ontstane neerslagen werden genoemd C en de oplossingen, die overbleven en die het seroglycoïd bevatten, *c*.

Tenslotte werd nog nagegaan of een verdere fractionneering met trichloorazijnzuur 20 % mogelijk was. Hierbij bleek, dat wanneer men zooveel toevoegde, dat de eindconcentratie 2 % trichloorazijnzuur bedroeg, alle eiwitten, dus ook het seroglycoïd, uit de vloeistof was neergeslagen. Voegde men evenwel eerst een kleinere hoeveelheid trichloorazijnzuur tot een eindconcentratie van 0.5 % bereikt was toe, dan gelukte het om het serumglycoïd neer te slaan. De oplossing van dit trichloorazijnzuurneerslag van seroglycoïd is *D* genoemd.

(Alle neerslagen zijn steeds aangegeven met *A.B.C.D.*, alle eiwitoplossingen met *a.b.c.d.*)

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[1]) Proc. Ned. Akad. v. Wetensch., Amsterdam, **45**, 804 (1942).

[2]) HEWITT, L. F., Biochem. J., **30** II, 2230 (1936), **31** I, 360 (1936), **31** II, 1535 (1937).

Gehalte aan trisacchariden. In verschillende fracties werd het gehalte aan koolhydraten bepaald. Dit is volgens HEWITT een galactose-mannose-glucosamine. Daartoe werd gebruik gemaakt van de reactie van SØRENSEN en HAUGAARD [3]).

Wij hebben daarbij gevonden een koolhydraatgehalte, uitgedrukt in glucose, zooals de tabel aangeeft.

Serum op 50 % verzadiging gebracht met ammonium-sulfaat bij pH 7.0.

<table>
<tr><th rowspan="2">Globulinen
A</th><th colspan="4">Resteerende oplossing *a*</th></tr>
<tr><th>Met azijnzuur</th><th colspan="3">Op pH 4,4 brengen</th></tr>
<tr><td rowspan="3"></td><td>B_1 2,7 % trisaccharid
B_3 1,6 % trisaccharid
B_6 0,7 % trisaccharid
B_8 0,5 % trisaccharid</td><td colspan="3">Resteerende oplossing *b*
Ammonium-sulfaat oplossing pH 4,4
Toevoegen tot de eindconcentratie
65 % of 75 % bedraagt</td></tr>
<tr><td rowspan="2">B_9 0,[illegible] % trisaccharid
Crystalbumine</td><td rowspan="2">Neerslag *C*</td><td colspan="2">Resteerende oplossing *c* bevat 5—6,5—7 % trisaccharide, met 20 % CCl_3CO_2H brengen op 0,5 % eindconcentratie</td></tr>
<tr><td>Neerslag *D*
Seroglycoïd</td><td>Rest *d* met 2 % trichloor-azijnzuur slaat alle eiwit neer
6,5 % trisaccharide</td></tr>
</table>

Uit bovenstaand schema kan men zoowel de wijze van bereiding als de resultaten van de fractionneering, beoordeeld naar het trisaccharide-gehalte, aflezen.

Bepaling van trisaccharide.

Benoodigde reagentia: *a.* 60 vol. % H_2SO_4.
b. 1.6 % orcinol in 30 vol. % H_2SO_4.
c. een versch bereide oplossing van gelijke deelen galactose en mannose in water.

Uitvoering: 1 cc van de te onderzoeken vloeistof die 0,02 tot 0,2 mg/cc trisaccharide bevat wordt verwarmd met 2,5 cc van de 1.6 % orcinol-oplossing in 15 cc van 60 vol. % H_2SO_4; dit geschiedt in een kookbuis van 18 cm bij 2½ cm, die geplaatst wordt in een waterbad van 80° gedurende 20 minuten. Na afloop van dezen tijd wordt de buis in koud water gedompeld. Hierbij ziet men een bruine kleur optreden, die lichtgevoelig is, zoodat alle bewerkingen in het donker moe-

[3]) SØRENSEN, MARGR. en HAUGAARD, G., Bioch. Zeitschr., **260**, 247 (1933).

ten worden uitgevoerd. Men bepaalt dan colorimetrisch de sterkte van de oplossing door de kleur te vergelijken met die van een standaardoplossing van mannose en galactose, die dezelfde bewerking ondergaat.

Wanneer men nu op de gebruikelijke wijze [4]) deze eiwitoplossingen tracht te spreiden op verdunde buffer-oplossingen van verschillende pH, dan vindt men verschillen, die in de onderstaande figuren zijn aangegeven.

Als bufferoplossingen werden gebruikt:

tusschen pH 1 en 3.3 verdunde HCl-oplossing,
tusschen pH 3.6 en 5.6 azijnzuur-acetaatmengsels 0.0033 mol.
Boven pH 5.6 de veronal-HCl-buffer van MICHAELIS van dezelfde molaire concentratie.

Van elk der eiwit-oplossingen werd eerst de grootte van de spreiding bij pH 1 bepaald. Deze diende als methode om het eiwitgehalte van de oplossing te bepalen. Daarna werd een meting gedaan van de eiwit-oplossingen

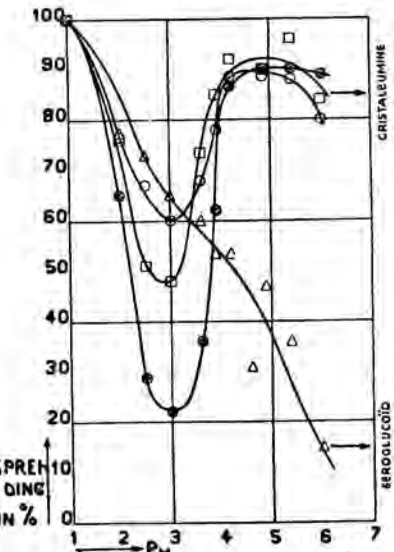

FIG. 1

Over de pH-spreidingscurven van twee eiwitfractie's uit serum („Hewitt").

B_1 ● 2.7 % Trisaccharide B_6 □ 0.7 % Trisaccharide
B_3 ○ 1.6 % Trisaccharide C △ 6.5 % Trisaccharide

op de verschillende buffers. Ook werd een drukoppervlakte-kromme opgeteekend, waaruit door extrapolatie naar een druk 0 de grootte van de spreiding werd bepaald. Na het opbrengen van het eiwit werd steeds 1 minuut gewacht.

Men ziet dat de B_1 fracties (een mengsel van crystalbumine en sero-

[4]) GORTER, E. e.a., Proc. Kon. Akad. v. Wetensch., Amsterdam, **37**, 788 (1934); **29**, 371 (1926).

glycoïd met een trisaccharid gehalte van 2.7 % en dat waarschijnlijk nog verontreinigd is met andere eiwitten) zich gedragen als het albumine uit

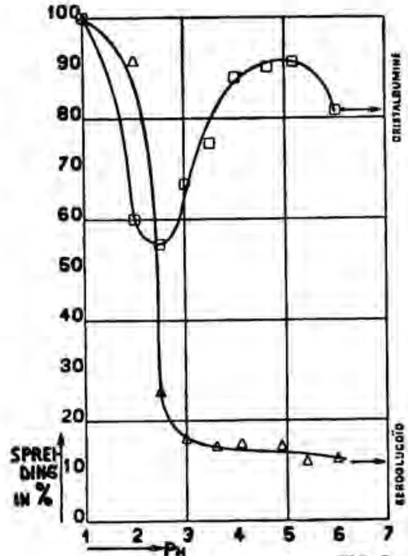

FIG. 2.

Over de pH-spreidingscurven van twee eiwitfractie's uit serum („Hewitt").

B_6 □ 0.8 % Trisaccharide D_1 △ 7 % Trisaccharide

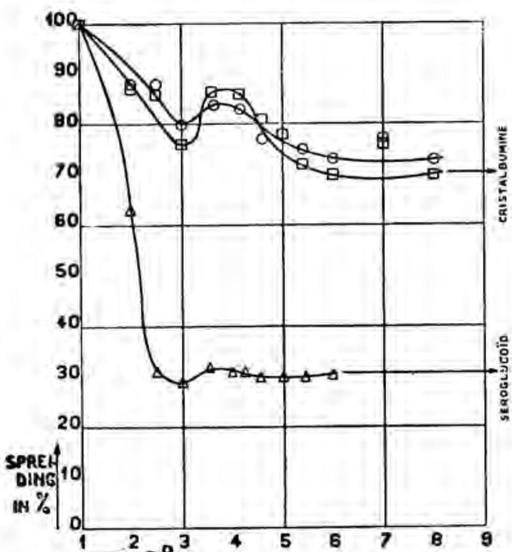

FIG. 3

Over de pH-spreidingscurven van twee eiwitfractie's uit serum („Hewitt").

B_8 ○ 0.5 % Trisaccharide B_9 □ 0.5 % Trisaccharide

C_1 △ 5 % Trisaccharide

serum. De kromme waarin de grootte van de spreiding is uitgezet tegen de pH vertoont de typische eigenschappen van dit eiwit: maximale spreiding bij pH 1 en bij het iso-electrische punt (± pH 5) en minimum bij pH 3. Scherp daarvan onderscheiden is de kromme voor het seroglycoïd *c*.

De kromme voor het *seroglycoïd* is in de beide laatste proeven gekenmerkt, doordat reeds bij pH 2.5 een sterke vermindering van de spreiding

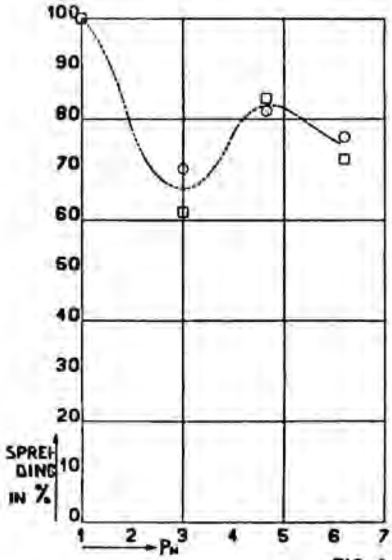

FIG. 4

Over de pH-spreidingscurven van twee eiwitfractie's uit serum („Hewitt").

B_6 ○ Suikerarme fractie van serum-albumine.

B_6 □ na toevoegen van CCl_3 CO_2H tot 7 % na 30 min. affiltreeren en weer oplossen in AQ met wat NH_4OH ter neutralisatie

te verkrijgen is bij meer alkalische reactie van de vloeistof. Het ziet er naar uit alsof de aanwezigheid van trisaccharid in het eiwitmolecuul de oplossing hiervan sterk verhoogt.

Het blijkt dat B_3 met een trisaccharid gehalte van 1.6 % juist is wat men verwachten kan, als men aanneemt, dat dit preparaat nog een mengsel is van B_6 en *c*.

Mogelijk is, dat bij de lage pH het koolhydraat van het eiwit wordt gesplitst. Het is begrijpelijk, dat het invoeren van zoovele polaire groepen als het trisaccharid bevat in het eiwitmolecuul de oplosbaarheid doet toenemen en de neiging tot spreiden vermindert.

Er is eenige reden om aan te nemen, dat de koppeling van dit trisaccharid aan het eiwit geschiedt tusschen de aminogroep van het glucosamine en de COOH-groepen van het eiwit. Dit zou inderdaad de gebrekkige spreiding aan den alkalischen kant van het iso-electrische punt kunnen verklaren [5]. Hiermee is niet in tegenspraak wat S. PRZYLECHI en J. CICHOCHEU [6]) hebben

[5]) GORTER, E., VAN ORMONDT, H., MEYER, TH., Complex proteïns. Biochem. J. **29**, 38 (1935).

[6]) Biochem. Z. **299**, 92 (1938).

ben gevonden, die ertoe besluiten, dat de koppeling niet kan geschieden door middel van de OH-groepen van de suiker onder normale physiologische omstandigheden.

We hebben nog overwogen of de fractie *c*, die seroglycoïd bevat, mis-

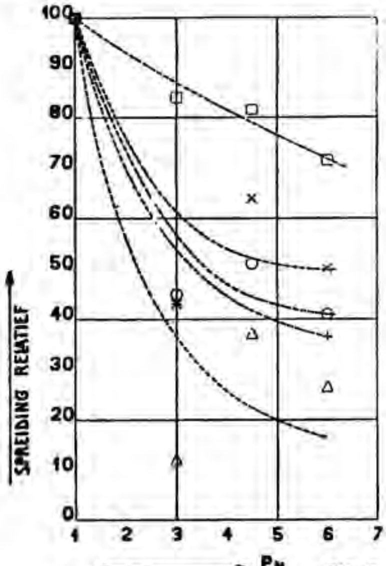

FIG. 5

Bij pH 1 m²/cc 0.55 O glasvocht.
0.41 + glasvocht met 2½ dln. alcohol neergeslagen en weer opgelost.
0.34 × glasvocht met 10 dln. aceton neergeslagen en weer opgelost.
0.93 □ Hoornvlies in water met spoor KOH opgelost, helder gecentrifugeerd.
27 △ lenzen in water met spoor KOH opgelost, helder afgecentrifugeerd.

schien daarom slecht spreidt bij een pH > 3.3, omdat het bij de bewerking gedenatureerd is. Men [7]) heeft gevonden, dat gedenatureerae albumine niet spreidt, maar dat het, nadat het op pH 2.4 is gebracht, deze eigenschap weer terugkrijgt. Tegen deze opvatting pleit, dat toevoegen van zuur aan de crystalbumine-oplossing niet de eigenschap om te spreiden herstelt. Bovendien kan men een oplossing van albumine of caseïne niet denatureeren en omzetten in een niet-spreidende stof door dezelfde bewerking als het seroglycoïd heeft ondergaan; neerslaan met half verzadigd ammoniumsulfaat, filtreeren, gevolgd door toevoegen van azijnzuur tot pH 4.4 en wederom toevoegen van 2 % trichloorazijnzuur, waarbij men behalve de eerste maal telkens het neerslag weer oplost en neerslaat. Ook het crystalbumine blijkt de bewerking goed te doorstaan; als men een oplossing daarvan een half uur lang wegzet met 7 % trichloorazijnzuur verandert de spreiding nauwelijks.

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[7]) GORTER, E. en VAN ORMONDT, H., Biochem., J. 29, 48 (1935).

Als tweede voorbeeld van een glycoproteïne hebben wij het mucine van het glasvocht onderzocht.

Het glasvocht was afkomstig van kalfsoogen. De cornea wordt weggeknipt, waarbij het kamervocht afloopt, de lens wordt verwijderd, dan het oog verder opengeknipt, en het glasvocht verwijderd door druk, waarbij men zorgt, dat geen vocht uit de spieren van het oog of bloed in het glasvocht terechtkomt. Het wordt geknipt en gefiltreerd door watten. Het filtraat is een water-heldere vloeistof.

Deze vrij visqueuze eiwitoplossing vertoont op een $^1/_{10}$ N HCl-oplossing pH 1 een duidelijke spreiding, waaruit men kan schatten, dat zij 0.05 % eiwit bevat.

Wanneer men dit glasvocht nu spreidt bij verschillende pH, dan vindt men een soortgelijke kromme als bij het seroglycoïd, als men de grootte van de spreiding uitzet tegen de pH.

Als men het mucoproteïne uit het glasvocht met 2.5 vol. alcohol neerslaat en dit neerslag weer in water oplost, krijgt men eenzelfde resultaat! Ook het neerslag, dat ontstaat door 1 dl glasvocht in 10 dl aceton te brengen, kan men in water weer oplossen. En deze oplossing spreidt ongeveer als het oorspronkelijke glasvocht (zie fig. 4).

Dit zijn slechts zeer voorloopige en onvolledige metingen.

Conclusie. Het seroglycoïd van HEWITT, dat als een mucoproteïn dat ± 6 % van een trisaccharid (galactose — mannose — glucosamine) is op te vatten, vertoont bij het onderzoek in monomoleculaire lagen een spreiding die slechts bij een pH < 3 gemakkelijk is te verkrijgen. Op een vloeistof met een hoogere pH is de oplosbaarheid van het seroglycoïd te groot.

Deze geringe spreiding en verhoogde oplosbaarheid worden toegeschreven aan de vervanging van de COOH-groepen van het eiwit door de OH-groepen van het trisaccharid, waarbij dan de aminogroep van het glucosamine zich hecht aan de COOH-groepen van het eiwit.

De spreiding van het crystalbumine is daarentegen zeer veel beter en is ongeveer te verklaren door van de spreiding van het serum albumine, die van het seroglycoïd af te trekken, m.a.w. de kromme van de spreiding van serumalbumine ligt tusschen die van het crystalbumine en het seroglycoïd in.

Summary.

In a previous paper one of us has published the results of the spreading of a conjugated protein known as MACHEBOEUF's lipoprotein which contains 39.6 % total lipoïds. The curve indicating the spreading of this lipoprotein at different pH was not very different from that of other proteins.

Now we shall communicate the results of the spreading of seroglycoïd and crystalbumin, isolated following Hewitt's technique.

After precipitation of the globulins, the albumin fraction was separated in two parts: one soluble fraction containing much sugar and the other far less soluble fraction, which contains only small amounts of sugar. For the determination of the content of trisaccharid of the protein fraction SØRENSEN and HAUGAARD [3]) colorimetric orcinol method was used. The

two fractions were called after Hewitt: one the most soluble fraction is *seroglycoid* (with 7 % carbohydrate), and the other crystalbumine, (which contains only 0.5 % trisaccharid). A mixture of both proteins has the same properties as serum-albumin.

In order to find out the difference in spreading tendency between the two proteins, one has to examine this spreading at various pH. It appears that there is a very great difference in so far that crystalbumin shows the properties of a protein, in which only a few free COO- and $NH^{3\cdot}$ groups are present, whereas the spreading of seroglycoid with a low minimum at a pH 3.0 and only very little spreading tendency at a higher pH above 3.0 behaves like a protein in which the $NH^{3\cdot}$ groups have disappeared and are replaced by acid groups. It is possible that the seroglycoid is more soluble than the crystalbumin fraction owing to the presence in this conjugated protein molecule of several OH groups.

If now we examine the spreading of a natural glycoprotein similarr esults are obtained.

We have used the humor aqueus from the eyes of a calf. This liquid can be examined in a fresh state. It shows the same type of spreading as the seroglycoid. After precipitation of the protein with acetone or with 2.5 vol alcohol a precipitate is formed that is easily soluble in water and has retained the same spreading qualities as the less pure product (fig. 5). The spreading of these glycoproteins is not due to denaturation.

Winschoten, 1944.

Mathematics. — *La structure des pseudo-évaluations d'un anneau élémentaire.* By F. LOONSTRA. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of September 21, 1946.)

§ 1. Supposons que R est un anneau commutatif quelconque ayant un élément unité. Une fonction $w(a)$ des éléments a de R, qui est toujours réelle et non-négative, sera dite une pseudo-évaluation de R, si $w(a)$ satisfait aux conditions suivantes:

$$w(a-b) \leqq w(a) + w(b) \quad ; \quad w(a \,.\, b) \leqq w(a) \,.\, w(b);$$
$$w(0) = 0 \quad ; \quad w(1) > 0.$$

Il faut mentionner ici les contributions de K. MAHLER (Acta Mathematica 66, 79—119, 1936; 67, 51—80, 283—328, 1936; Proceedings Koninklijke Akademie van Wetenschappen, Amsterdam Vol. XXXIX, N⁰. 1, 1936).

Pour chaque anneau il existe une pseudo-évaluation triviale

$$w_0(a) = \begin{cases} 0 \text{ pour } a = 0 \\ 1 \quad ,, \quad a \neq 0. \end{cases}$$

Il se trouve qu'il est important de considérer la fonction $0(a) = 0$, chaque a de R ayant une pseudo-évaluation égale à zéro.

Une suite infinie $\{a_i\}$ d'éléments d'un anneau R, possédant une pseudo-évaluation $w(a)$ s'appelle w-suite fondamentale, si à tout nombre ε positif arbitrairement petit, on peut faire correspondre un nombre positif entier $p(\varepsilon)$, tel que $w(a_m - a_n) \leqq \varepsilon$, m et $n \geqq p(\varepsilon)$.

Une propriété importante déduite par M. MAHLER est la suivante:

Supposons $w_1(a), w_2(a), \ldots, w_n(a)$ un nombre fini des pseudo-évaluations de R, alors

$$w^{\Sigma}(a) = w_1(a) + w_2(a) + \ldots + w_n(a)$$

aussi bien que

$$w^*_{\Sigma}(a) = \max(w_1(a), w_2(a), \ldots, w_n(a))$$

est une pseudo-évaluation de R.

Si $w_1(a)$ et $w_2(a)$ présentent deux pseudo-évaluations de R, M. MAHLER définit que $w(a)$ contient $w_1(a)$, de cette manière: $w_1(a) \subset w_2(a)$ ou $w_1 \subset w_2$, si pour chaque suite infinie $\{a_i\}$ d'éléments de R

$$\lim_{n\to\infty} w_2(a_n) = 0 \text{ entraîne } \lim_{n\to\infty} w_1(a_n) = 0.$$

Si $w_1 \subset w_2$ et $w_2 \subset w_1$, les pseudo-évaluations w_1 et w_2 seront dites équivalentes, $w_1 \approx w_2$.

§ 2. Après ces préparations, indiquées par M. MAHLER, nous appelons l'attention en particulier sur l'ensemble de toutes les pseudo-évaluations d'un anneau élémentaire, c'est à dire d'un anneau commutatif, ayant la propriété que chaque pseudo-évaluation $w(a)$ ne contient qu'un nombre fini des pseudo-évaluations non-équivalentes. Nous appellerons, suivant M. O. ORE, structure un ensemble S d'éléments $a, b, c, \dots$ lorsque les trois conditions suivantes seront remplies:

1°. L'ensemble S contient des couples d'éléments, a, b, liés entre eux par une relation $a \subset b$ („a précède b") telle que:

$a \subset a$; $a \subset b$ et $b \subset a$ entraîne $a \approx b$ et réciproquement;

$a \subset b$ et $b \subset c$ entraîne $a \subset c$.

2°. A tout couple d'éléments, a, b de l'ensemble S correspond un élément $a \cap b$ de S („produit de a et de b") tel que:

$a \cap b \subset a$, $a \cap b \subset b$; $c \subset a$ et $c \subset b$ entraîne $c \subset a \cap b$.

3°. A tout couple d'éléments a, b de l'ensemble S correspond un élément $a + b$ de S („somme de a et de b") tel que:

$a \subset a + b$, $b \subset a + b$; $a \subset c$ et $b \subset c$ entraîne $a + b \subset c$.

Une structure S satisfaisant à l'égalité suivante

$$a \cap c + b \cap c \approx a \cap (b + c)$$

s'appelle une structure distributive.

§ 3. Supposons R un anneau fourni de pseudo-évaluations.

3.1 L'équivalence des pseudo-évaluations est réflexible, symétrique et transitive.

3.2 Si l'on a $w \subset w'$, on a aussi pour tout w'': $w \subset w' + w''$.

En effet, si $\{a_i\}$ est une w'-suite fondamentale avec $\lim_{i \to \infty} w'(a_i) = 0$, elle est aussi une w-suite fondamentale avec $\lim_{i \to \infty} w(a_i) = 0$; alors, une suite $\{a_i\}$, étant $(w' + w'')$-suite fondamentale avec $\lim_{i \to \infty} (w' + w'')(a_i) = 0$ est également une w'-et de même une w''-suite fondamentale avec $\lim_{i \to \infty} w'(a_i) = 0$ et $\lim_{i \to \infty} w''(a_i) = 0$. En vertu de la supposition elle est aussi une w-suite fondamentale avec $\lim_{i \to \infty} w(a_i) = 0$. Il s'ensuit encore $w_1 \subset w_1 + w_2$ pour tout w_1 et w_2.

3.3 Si l'on a $w_1 \subset w$ et $w_2 \subset w$, on a aussi $w_1 + w_2 \subset w$.

En vertu de la supposition une suite $\{a_i\}$, étant une w-suite fondamentale avec $\lim_{i \to \infty} w(a_i) = 0$ est aussi une w_1-suite fondamentale et une w_2-suite fondamentale, ensuite une $(w_1 + w_2)$-suite fondamentale, alors on a $w_1 + w_2 \subset w$.

3.4 Si w_1 et w_2 sont deux pseudo-évaluations de R, $w \approx w_1 + w_2$ est la „plus petite" pseudo-évaluation, contenant w_1 et w_2.

Démonstration: Soit $w \approx w_1 + w_2$, et puis w' une pseudo-évaluation, de sorte que $w_1 \subset w'$, $w_2 \subset w'$ il s'ensuit que $w \subset w'$; c'est une conséquence de 3.3.

En outre, si $w_1 \subset w$ et $w_2 \subset w$, on a aussi $w_1 + w_2 \subset w$.

Une application suivante résulte de 3.4:

3.5 Si $w_1 \subset w'$ et $w_2 \subset w''$, on a aussi $w_1 + w_2 \subset w' + w''$.

3.6 Supposons R_e un anneau élémentaire, il y a une pseudo-évaluation $w(a) = w_1(a) \cap w_2(a)$ ayant la propriété suivante: $w(a) \subset w_1(a)$, $w(a) \subset w_2(a)$, de sorte que $w' \subset w_1$, $w' \subset w_2$ entraîne $w' \subset w$.

Démonstration: Nous nous servons de ce qui a été établi:

$$\overline{w} \subset w_1, \overline{w} \subset w_2 \text{ et } \overline{\overline{w}} \subset w_1, \overline{\overline{w}} \subset w_2 \text{ entraîne } \overline{w} + \overline{\overline{w}} \subset w_1, \overline{w} + \overline{\overline{w}} \subset w_2.$$

Soit $\overline{w}$ une pseudo-évaluation de sorte que $\overline{w} \subset w_1$, $\overline{w} \subset w_2$, la pseudo-évaluation $w = \Sigma \overline{w}$ remplit la condition demandée, $\overline{w}$ parcourant toutes les pseudo-évaluations avec $\overline{w} \subset w_1$, $\overline{w} \subset w_2$. En effet: il n'y a qu'un nombre fini de ces pseudo-évaluations $\overline{w}(a)$ et la somme d'un tel ensemble est aussi une pseudo-évaluation de R. En vertu de la construction de w on a, si $w' \subset w_1$, $w' \subset w_2$, $w' \subset w$.

De 3.1, 3.4 et 3.6 on conclut que l'ensemble S_w de toutes les pseudo-évaluations d'un anneau élémentaire est une structure, si l'on appelle élément de structure une classe des pseudo-évaluations equivalent entre elles. Nous démontrerons que S_w est une structure distributive; c'est pourquoi nous donnons la lemme suivante:

3.7 Pour tout w_1, w_2 et w_3 on a: $w_1 \cap w_2 \subset w_1 \cap (w_2 + w_3)$.

En effet, soit $w_1 \cap w_2 = w$, il en résulte $w \subset w_1$ et $w \subset w_2$; en vertu de 3.2 il résulte $w \subset w_2 + w_3$ et alors $w \subset w_1 \cap (w_2 + w_3)$.

3.8 La structure S_w des pseudo-évaluations d'un anneau élémentaire R_e est une structure distributive.

Démonstration:

1°. On a $w_1 \cap w_2 \subset w_1 \cap (w_2 + w_3)$ et $w_1 \cap w_3 \subset w_1 \cap (w_2 + w_3)$ et de là

$$w_1 \cap w_2 + w_1 \cap w_3 \subset w_1 \cap (w_2 + w_3) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

2°. Soit $w_1 \cap w_2 = w'$, $w_1 \cap w_3 = w''$, on a $w' \subset w_1$, $w' \subset w_2$ et $w'' \subset w_1$, $w'' \subset w_3$, et $\overline{w} \subset w_1$, $\overline{w} \subset w_2$ entraîne $\overline{w} \subset w'$; de même pour $\overline{\overline{w}} \subset w_1$, $\overline{\overline{w}} \subset w_3$ on a $\overline{\overline{w}} \subset w''$. D'après

$$\left.\begin{matrix} w' \subset w_1 \\ w' \subset w_2 \end{matrix}\right\} \text{ on a } \left.\begin{matrix} w' \subset w_1 \\ w' \subset w_2 + w_3 \end{matrix}\right\}$$

et d'après

$$\left.\begin{matrix} w'' \subset w_1 \\ w'' \subset w_3 \end{matrix}\right\} \text{ on a } \left.\begin{matrix} w'' \subset w_1 \\ w'' \subset w_2 + w_3 \end{matrix}\right\}.$$

Supposons $w' + w'' = w$, on obtient

$$w \subset w_1 + w_1 \approx w_1, \; w \subset w_2 + w_3 + w_2 + w_3 \approx w_2 + w_3.$$

Si cependant $\overline{w} \subset w_1$, $\overline{w} \subset w_2$, on a $\overline{w} \subset w'$ et de $\overline{\overline{w}} \subset w_1$, $\overline{\overline{w}} \subset w_3$ il s'ensuit $\overline{\overline{w}} \subset w''$, alors on a $\overline{w} + \overline{\overline{w}} \subset w' + w''$ ou $\overline{w} + \overline{\overline{w}} \subset w$.

Maintenant $w_1 \cap (w_2 + w_3)$ est une pseudo-évaluation w^* avec $w^* \subset w_1$, $w^* \subset w_2 + w_3$, alors $w^* \subset w$, c'est à dire

$$w_1 \cap (w_2 + w_3) \subset w_1 \cap w_2 + w_1 \cap w_3 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

En vertu des inégalités (1) en (2) on obtient

$$w_1 \cap (w_2 + w_3) \approx w_1 \cap w_2 + w_1 \cap w_3,$$

la structure S_w d'un anneau élémentaire est distributive.

M. MAHLER appelle les pseudo-évaluations $w_1(a)$, $w_2(a), \ldots, w_n(a)$ indépendantes, si on peut faire correspondre à tout ensemble de n nombres $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$ de R une suite $a_1, a_2, \ldots, a_n, \ldots$ d'éléments de R, possédant les propriétés suivantes:

$$\lim_{m \to \infty} w_h(a^{(h)} - a_m) = 0 \; (h = 1, 2, \ldots, n).$$

En vertu de cette définition nous démontrons:

3,9 Le produit $w(a) = w_1(a) \cap w_2(a)$ des pseudo-évaluations indépendantes $w_1(a)$ et $w_2(a)$ c'est la pseudo-évaluation $0(a) = 0$ pour tout a de R.

Démonstration: Si $a^{(1)}$ et $a^{(2)}$ sont deux éléments quelconques de R, il y a d'après la supposition une suite $\{a_i\}$ d'éléments de R avec

$$\lim_{i \to \infty} w_1(a^{(1)} - a_i) = 0, \; \lim_{i \to \infty} w_2(a^{(2)} - a_i) = 0.$$

D'après $w \subset w_1$ et $w \subset w_2$, on a donc

$$\lim_{i \to \infty} w(a^{(1)} - a_i) = 0 \text{ et } \lim_{i \to \infty} w(a^{(2)} - a_i) = 0$$

alors

$$w(a^{(1)} - a_i - a^{(2)} + a_i) = 0 \text{ ou } w(a^{(1)} - a^{(2)}) = 0.$$

On a

$$w(a^{(1)}) \leqq w(a^{(2)}) + w(a^{(1)} - a^{(2)}) \text{ alors } w(a^{(1)}) \leqq w(a^{(2)})$$

et de même $w(a^{(2)}) \leqq w(a^{(1)})$, et on en obtient $w(a^{(1)}) = w(a^{(2)})$ pour deux éléments quelconques $a^{(1)}$ et $a^{(2)}$ de R; parce que $w(0) = 0$, tous les éléments de R satisfont la propriété $w(a) = 0$.

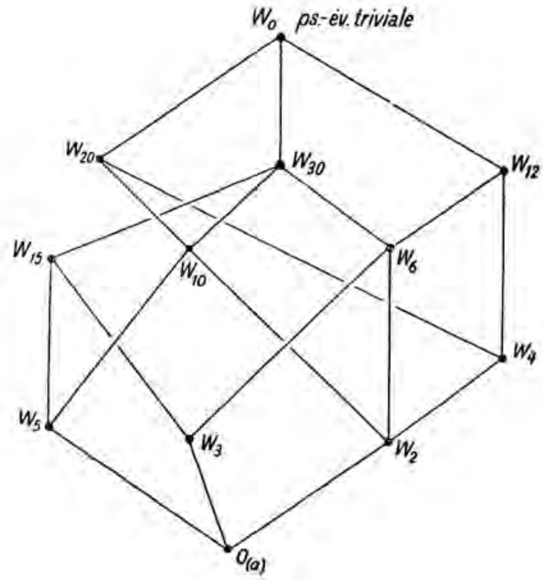
§ 4. Pour avoir un exemple simple nous prendrons comme exemple un anneau R contenant un nombre fini d'éléments. Toutes les pseudo-évaluations de cet anneau R sont d'un type

$$w_{\mathfrak{r}}(a) = \begin{cases} 0 & \text{pour } a \equiv 0 \pmod{\mathfrak{r}} \\ 1 & \text{,, } \; a \not\equiv 0 \pmod{\mathfrak{r}} \end{cases}$$

dans lequel $\mathfrak{r}$ présente un idéal quelconque de R.

L'énumération de tous les ideaux d'un anneau fini nous fournit toutes les pseudo-évaluations de R; R est un anneau élémentaire.

Comme exemple d'une pareille structure S_w nous donnons les pseudo-évaluations de l'anneau R des classes résiduaires (mod 60). Si l'idéal $\mathfrak{r}_1$ correspond à la pseudo-évaluation w_1, l'idéal $\mathfrak{r}_2 \rightarrow w_1$, alors $\mathfrak{r}_1 . \mathfrak{r}_2 \rightarrow w_{12} = w_1 + w_2$



et $\mathfrak{r}_1 + \mathfrak{r}_2 \rightarrow w_1 \cap w_2$. L'ensemble des idéaux $\mathfrak{r}$ de R est également une structure S, qu'on dérive de S_w en échangeant la somme contre le produit, c'est à dire en renversant S_w. Finalement il est intéressant de noter ici le caractère particulier d'un anneau élémentaire.

4.1 Un anneau R, ayant la propriété que toute pseudo-évaluation de R (qui n'équivaut pas à la pseudo-évaluation triviale ou à 0(a)) équivaut à une somme fini des évaluations $w_{i_1}, w_{i_2}, \ldots, w_{i_n}$ non-équivalentes, déterminées univoque, est élémentaire.

Démonstration: Chaque pseudo-évaluation w de R est une somme finie d'évaluations $w \approx w_{i_1} + w_{i_2} + \ldots + w_{i_n}$; maintenant w ne contient que des évalutions-équivalant à $w_{i_1}, w_{i_2}, \ldots$ ou w_{i_n}. En effet, si l'évaluation $w_i \subset w$ on aurait

$$w_i \subset w_{i_1} + w_{i_2} + \ldots + w_{i_n}$$

et puis

$$w_{i_1} + w_{i_2} + \ldots + w_{i_n} \approx w_i + w_{i_1} + w_{i_2} + \ldots + w_{i_n},$$

alors

$$w \approx w_i + w_{i_1} + w_{i_2} + \ldots + w_{i_n},$$

c'est à dire w serait de deux manières différentes équivalent à une somme finie d'évaluations non-équivalentes et ce n'est pas possible. Alors, seulement $w_{i_1} \subset w, w_{i_2} \subset w, \ldots, w_{i_n} \subset w$; alors chaque pseudo-évaluation

$w' \subset w$ ne peut contenir qu'un nombre fini de pseudo-évaluations $w_{i_1}, w_{i_2}, \ldots, w_{i_n}$. Le nombre de pseudo-évaluations $w' \subset w$ est alors fini, par conséquent R est élémentaire.

Dans son article „Über Pseudobewertungen" II, Acta Mathematica 67, 51—80, 1936, M. MAHLER démontre que chaque pseudo-évaluation d'un corps, étant une extension finie algébrique équivaut à une somme finie d'évaluations de K; K est alors élémentaire.

Mathematics. — *The existence of 1- and 2-dimensional subspaces of a compact metric space.* By A. VAN HEEMERT. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of September 21, 1946.)

According to the definition of dimension of BROUWER–MENGER–URYSOHN, it is evident that each metric separable space of finite dimension d contains a subspace of dimension d' for any of $d' < d$. If, on the other hand, the dimension of R is infinite, we cannot conclude anything directly from the definition we just mentioned about the existence of subspaces of finite dimension $d' > 0$ [1]. As far as I know, in this case there has never been obtained any results concerning this question. In the following we will prove the existence of subspaces of dimension $d' = 1, 2$ if we still suppose that R is compact. In doing this we will suppose that the theory of producing a space as the limit space of a R_n-adic sequence of topological spaces is known [2]. In particular we make use of the well-known theorem, that a compact metric space is the limit space of a R_n-adic sequence of polytopes P_n, $n = 1, 2, \ldots$, so that this sequence is normal and irreducible and the successive mappings f_n^{n+1} are mappings of P_{n+1} on P_n. If the dimension of R is infinite, $\dim R = \infty$, it is also known that the elements of this sequence have the property that for sufficient large values of n $\dim P_n$ is larger than any natural number [3].

For terms and notations compare the publications we just quoted ([2]).

1. **Theorem.** If T is a simplex of any dimension, $\mathfrak{R}(T)$ its boundary and $fR \subset T$ is a reducible map of the compact metric space R in T and if we denote the boundary of T by $\mathfrak{R}(T)$, then there exists an allowed modification („zulässige Abänderung") g of f so that

1° $gp = fp$ for each $p \subset f^{-1}\mathfrak{R}(T)$,

2° for some inner point a of T we have $a \not\subset gR$,

3° if for some $p \subset R$ we have $fp \not\subset \mathfrak{R}(T)$, then we have also

$$gp \not\subset \mathfrak{R}(T).$$ [4]

Proof. Since f is reducible, there exists an allowed modification h of f so that some inner point a of T is not covered by hR ($hR \not\supset a!$). First of

[1]) The equivalent definition of dimension of ALEXANDROFF yields no better results.

[2]) This theory has been developed by H. FREUDENTHAL, Entwicklungen von Räumen und ihren Gruppen, Compositio Mathematica **4** (1937), 145—234, especially Kap. IV. Compare also A. VAN HEEMERT, De R_n-adische ontwikkeling etc., Thesis (Groningen, 1943).

[3]) H. FREUDENTHAL, l.c. Kap. VII.

[4]) This theorem is a strengthening of a property which has been proved by H. FREUDENTHAL, l.c., Hilfssatz III, S. 185, formulated under „Hinreichend". The proof we give here is essentially the same as that of FREUDENTHAL.

all we suppose for a moment that h has the property that from $fp \not\subset \mathfrak{R}(T)$, for any p, it follows that $hp \not\subset \mathfrak{R}(T)$. Afterwards we will free ourselves from this supposition.

Now let

$$f^{-1}\,\mathfrak{R}(T) = M \subset R;$$

then M is closed and consequently M is compact. Consider any point $p \subset M$. Then

$$a \not\subset \overline{fp, hp}$$

(by $\overline{fp, hp}$ we understand the rectilinear segment which joins fp with hp), for hp lies in the same boundary-simplex as fp and owing to $fp \subset \mathfrak{R}(T)$, we have $\overline{fp, hp} \subset \mathfrak{R}(T)$. Consequently there exists a neighbourhood $U(p)$ of p so that for each point $q \subset U(p)$

$$a \not\subset \overline{fq, hq}.$$

If we take for each $p \subset M$ such a neighbourhood, then they form together a neighbourhood M^* of M, so that for each point $q \subset M^*$ again $a \not\subset \overline{fq, hq}$.

Now, if we denote for each point $p \subset R$

$$\varrho_p = \varrho(p, M) \quad , \quad \bar{\varrho}_p = \varrho(p, R - M^*)$$

the following relations are true:

$$\begin{aligned} &\varrho_p = 0,\ \bar{\varrho}_p \neq 0 \quad \text{for} \quad p \subset M \\ &\varrho_p \neq 0,\ \bar{\varrho}_p = 0 \quad \text{for} \quad p \subset R - M^* \\ &\varrho_p \neq 0,\ \bar{\varrho}_p \neq 0 \quad \text{for} \quad p \subset M^* - M. \end{aligned}$$

This follows from the fact that M and $R - M^*$ are closed.

Define the mapping $gR \subset T$ as follows: For $p \subset R$, gp divides the segment $\overline{fp, hp}$ in the ratio $\varrho_p : \bar{\varrho}_p$. Then g coincides with f for the points of M and g coincides with h for the points of $R - M^*$, while as a consequence of the definition of M^*

$$gp \neq a \text{ for each } p \subset M^* - M$$

so that even

$$gp \neq a \text{ for each } p \subset R.$$

From the foregoing it is easily seen, that we have

α) $gp = fp$ if $p \subset M$,

β) for $p \subset R$, $fp \not\subset \mathfrak{R}(T)$ it follows that $gp \not\subset \mathfrak{R}(T)$,

γ) g is a continuous mapping.

To complete the proof of our theorem we have only to show, that we can free ourselves from the restricting hypothesis with respect to h which we have made at the beginning. Now if h is an allowed modification of f and if this hypothesis is not true, than we define an other allowed modification h' of f in the following way: $h'p$ is derived from hp by multiplying hp with

$$\frac{1+\varrho_p}{1+2\varrho_p}$$

with respect to a. h' is easily seen to be continuous; it is even an allowed modification of f (not of h!), for

$$h'p = hp \text{ if } p \subset M,$$

while $h'R \not\supset a$. For this new allowed modification h' of f our hypothesis is true. This completes the proof.

Remark. We make use of the just proved theorem in the following way: It is evidently possible to find an allowed modification g' of f so that

$$g'p = fp \text{ if } fp \subset \Re(T)$$

while for any $p \subset R$ we have

$$g'p \subset \Re(T).$$

Therefore we only have to project every point gp of the original allowed modification g from a on $\Re(T)$. We don't use further 3° of our theorem. This point has only been stated since it easily can be deduced therefrom, that this *characteristic of a reducible mapping* is true:

$fR \subset T$ is reducible if and only if it is possible to find a continuous mapping $gR \subset T$ which is a continuous deformation of f with the following property: the images of all points belonging to the interior of T remain in the interior of T and the images of all points belonging to $\Re(T)$ remain fixed, so that there is at least one point a of T which is not covered by gR.

2. **Theorem.** If the polytope P of dimension $n \geqq 2$ is normally and irreducibly mapped on the simplex S of dimension d, $d = 1, 2$, and if by ${}^dP$ we understand the sub-polytope consisting of all d-dimensional simplices of P, then f is also an irreducible mapping of ${}^dP$ on S.

Proof. We consider the cases $d = 1$ and $d = 2$ separately. In both cases there is at least one component of P which is irreducibly mapped on S by f. Therefore we may suppose that P is connected.

A. $d = 1$. The proof is elementary in this case. ${}^1P$ is a connected polytope and as f is normal, there is for any vertex of S at least one vertex of ${}^1P$ which is mapped on it by f. But every allowed modification g of f maps the vertices of ${}^1P$ in the same vertex of S as f did and because ${}^1P$ is connected we have

$$g\,{}^1P = S.$$

Consequently f is an irreducible mapping ${}^1P$ on S.

Remark. The supposition that $fP = S$ is normal is essential: It is easy to define an irreducible mapping of a 2-dimensional simplex P^* on a 1-dimensional simplex S so that ${}^1P^*$ is mapped on a single inner point of S. Evidently this mapping is not normal and ${}^1P^*$ is mapped reducibly in S by f.

B. $d = 2$. In this case we will have to use a wellknown theorem of combinatorial topology:

If for $n \geqq 2$ a continuous mapping of the boundary of a $(n+1)$-dimensional simplex T^{n+1} in the boundary $\Re(S)$ of a 2-dimensional simplex S is given, it is possible to extend this mapping to a continuous mapping of T^{n+1} in $\Re(S)$ [5].

The proof of our theorem can now be given indirectly as follows: Let

$$f\,{}^2P \subset S$$

be reducible. According to 1 there exists an allowed modification g of f, defined in the points of ${}^2P$, so that

$$1^\circ \ g\,{}^2P \subset \Re(S),$$
$$2^\circ \ fq = gq \text{ for } q \subset {}^2P\,.\,(f^{-1}\,\Re(S)).$$

By means of the extension-theorem which has just been mentioned, we extend this mapping, which is defined in the points of ${}^2P$, to a continuous mapping of P in $\Re(S)$ which is an allowed modification of the original mapping $fP = S$. This new mapping will be denoted by g.

Assume that g has already been defined for a certain value d, $2 \leqq d \leqq n$,

$$g\,{}^dP \subset \Re(S),$$

where g is an allowed modification of

$$f\,{}^dP \subset S,$$

so that $fp = gp$ for all points of ${}^dP$ which are mapped in $\Re(S)$ by f. This g exists for $d = 2$ according to our assumptions. Now from the extension-theorem it follows, that g can be extended continuously about each $(d+1)$-dimensional simplex T^{d+1} of P. Here we take for the extended mapping the original mapping f, if already $fT^{d+1} \subset \Re(S)$. With that, all cases have been mentioned: If a normal mapping of a simplex T^{d+1} in an other one S is given so that an inner point of T^{d+1} is not mapped in $\Re(S)$, than no inner point of T^{d+1} is mapped in $\Re(S)$, if some inner point of T^{d+1} is mapped in $\Re(S)$, then the entire simplex T^{d+1} is mapped in $\Re(S)$. This remark (which is true for every dimension of S) shows that the extension of the mappings is always possible in such a way that the mappings of points of T^{d+1} which are mapped in $\Re(S)$ is not changed by passing to the allowed modification.

The mappings of all simplices of dimension $d+1$ which are defined in this way link together to a single continuous mapping of ${}^{d+1}P$ in $\Re(S)$; for it is an extension of the mapping $g^dP \subset \Re(S)$. For this new mapping

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[5]) P. ALEXANDROFF and H. HOPF, Topologie I (Berlin, 1935), S. 516, Satz II.

$g\,^{d+1}P \subset \Re(S)$ we have again that $fp = gp$ for all points p which have already been mapped in $\Re(S)$ bij f.

Herefrom by complete induction the continuous mapping

$$gP \subset \Re(S)$$

has been defined which coincides with f for all points of P that have already been mapped in $\Re(S)$ by f. But then g is an allowed modification of f and consequently f would have been reducible in contradiction with our assumptions. Therefore the mapping

$$f\,^2P \subset S$$

is irreducible.

Corollary. If a normal and irreducible mapping is given

$$fP = Q,$$

where P is any polytope and Q is any polytope of dimension d, $d = 1, 2$, then we have

$$f\,^dP = Q$$

and this mapping is (normal and) irreducible.

3. **Theorem.** Every compact metric space R of any dimension (finite or infinite) contains a subspace of dimension d, $d = 1, 2$, if $\dim R \geqq d$.

Proof. Let R be the R_n-adic limit space of the R_n-adic sequence of polytopes P_n of dimension $\geqq d$ [6])

$$P_1 \leftarrow P_2 \leftarrow \dots \qquad (f_n^{n+1}\, P_{n+1} = P_n),$$

where all mappings f_n^{n+1}, $n = 1, 2, \dots$, are normal and irreducible. Let

$$P_2' = (f_1^2)^{-1}\,{}^dP_1.$$

According to 2, Corollary, we have

$$f_1^2\,{}^dP_2' = {}^dP_1$$

and this is a normal and irreducible mapping. For

$$f_1^2\,P_2' = {}^dP_1$$

is normal and irreducible.

Evidently ${}^dP'_2$ is not empty. Let further

$$P_3' = (f_2^3)^{-1}\,{}^dP_2'.$$

Again we have that

$$f_2^3\,{}^dP_3' = {}^dP_2'$$

6) H. FREUDENTHAL, l.c., Hauptsatz I and S. 229, Satz 4.

is a normal and irreducible mapping. Continuing in this way we find for each n a ${}^dP'_n$ and if we write ${}^dP'_1$ for ${}^dP_1$, then the R_n-adic sequence

$$ {}^dP'_1 \leftarrow {}^dP'_2 \leftarrow \ldots \qquad (f_n^{n+1}\, {}^dP'_{n+1} = {}^dP'_n) $$

is normal and irreducible, while all elements of this sequence are not empty and of dimension d. But then

$$ R' = \lim_{n} {}^dP'_n \subset R $$

is a (compact) subspace of R of dimension d. This completes the proof [7].

[7]) H. FREUDENTHAL, l.c., S. 229, Satz 4.

Mathematics. — *Sur la méthode de* LAGUERRE *pour l'approximation des racines de certaines équations algébriques et sur la critique d'*HERMITE. By E. BODEWIG. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 21, 1946.)

Pour les équations algébriques ayant toutes leurs racines réelles LAGUERRE a fourni une méthode élégante [1]) qui nous permet, en partant d'une valeur arbitraire réelle x, d'approximer les deux racines „voisines" de x simultanément. (L'expression „voisine" est claire, si x est situé entre deux racines consécutives. Dans le cas où x est situé en dehors de l'intervalle le plus petit qui contient toutes les racines de l'équation, les deux racines „voisines" de x seront par définition la plus petite et la plus grande racine de l'équation.) La méthode de LAGUERRE est toujours convergente ce qui n'est pas toujours le cas avec celle de NEWTON.

Malheureusement LAGUERRE n'a mentionné le point principal de sa démonstration qu'en passant [2]). Il donne seulement le résultat. C'est HERMITE qui a attaqué ce point et ajouté un mémoire propre aux oeuvres de LAGUERRE [3]). Il regrette [4]) „que l'éminent géomètre ne se soit pas étendu davantage sur ce point essentiel et qu'on n'ait pas suffisamment la trace des idées qui l'ont conduit à la découverte d'un résultat important dont il a fait des applications nombreuses et extrêmement remarquables". Après cette remarque il donne une démonstration indépendante, sans savoir la voie par laquelle LAGUERRE a obtenu son résultat.

En réalité le cours des idées de LAGUERRE doit avoir été simple, et nous croyons l'avoir retrouvé ci-dessous.

Une autre lacune dans la démonstration de LAGUERRE est qu'il prouve seulement qu'on s'approche des racines voisines, mais pas que son procès tend vers les racines. Cette démonstration n'est pas difficile naturellement.

Ensuite nous prouverons une autre propriété de la méthode. Dans son mémoire LAGUERRE a démontré que sa méthode donne une approximation meilleure que celle de NEWTON. Nous éclaircirons *le caractère de convergence* de la méthode en introduisant une nouvelle notion qui me semble être d'une grande importance dans l'analyse numérique.

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[1]) Sur une méthode pour obtenir par approximation les racines d'une équation algébrique qui a toutes ses racines réelles. Nouvelles Annales Math., 2e série, XIX, 1880. — Oeuvres de LAGUERRE, publiés par HERMITE, POINCARÉ et ROUCHÉ, Paris, Gauthier—Villars, 1898, —vol. I, p. 87—103.

[2]) Oeuvres, I, p. 93.

[3]) Sur un mémoire de LAGUERRE concernant les équations algébriques. Oeuvres de LAGUERRE, vol. I, p. 461—468.

[4]) L.c., p. 463.

Définition. La suite

$$x_1, x_2, \ldots, x_n, \ldots \text{ avec la limite } X$$

converge en degré k, si

$$\frac{x_n - X}{(x_{n-1} - X)^k} \rightarrow c \neq 0.$$

Au moyen d'un résultat d'HERMITE [5]) nous prouverons que la méthode de LAGUERRE converge en 3e degré en général, tandis que la méthode de NEWTON ne converge qu'en 2e degré (si elle converge du tout).

Nous ajouterons une démonstration indépendante et directe en complétant le résultat de la manière suivante: *La méthode de* LAGUERRE *converge cubiquement* (ou, en cas exceptionels, en degré supérieure) *au cas d'une racine simple et linéairement au cas d'une racine multiple.* Pour avoir un procédé qui converge toujours cubiquement il faut modifier la méthode légèrement.

Enfin nous ferons une simple application d'un résultat de LAGUERRE–HERMITE en démontrant la *proposition* curieuse:

Soit $f(x) = 0$ une équation algébrique ayant toutes ses racines réelles. En partant d'un point arbitraire x, la méthode de NEWTON ne converge pas nécessairement. Mais n fois la correction Newtonienne $h = -f(x)/f'(x)$ détermine toujours un intervalle $(x, x + nh)$ dans lequel se trouve au moins une racine de $f(x) = 0$.

Pous être clair sans devoir renvoyer aux mémoires cités j'expliquerai brèvement toute la méthode de nouveau. Les développements n'y deviendront pas beaucoup plus longs.

I. Supposons que l'équation

$$f(X) = X^n + \binom{n}{1} a_{n-1} X^{n-1} + \binom{n}{2} a_{n-2} X^{n-2} + \ldots + a_0 = 0 \quad (1)$$

a toutes ses racines X_i réelles, de sorte que

$$X_1 \leqq X_2 \leqq \ldots \leqq X_n,$$

et posons avec HERMITE

$$S_u = \left(\frac{u - X_1}{x - X_1}\right)^2 + \ldots + \left(\frac{u - X_n}{x - X_n}\right)^2, \quad x = \text{arbitraire et réel, mais fixe.} \quad (2)$$

Alors l'équation $S_u = 0$ n'a pas de racines réelles u. Mais considérons l'équation quadratique en X:

$$F(X) \equiv S_u - \left(\frac{u - X}{x - X}\right)^2 = 0, \quad \ldots \ldots \quad (3)$$

[5]) L.c., p. 465—468.

où u, x sont fixes, $x \neq X_i$ et où, en cas que $x > X_n$, le nombre u n'est égal à une des deux valeurs pour lesquelles $S_u = 1$. Cette équation a toujours deux racines réelles X', X'', dépendantes de x et u. Car à cause de

$$F(x) < 0, \quad F(X_i) > 0, \quad i = 1, \ldots, n$$

la parabole $y = (x - X)^2 . F(X)$ est située au-dessus de l'axe des abscisses pour tous les n points avec les abscisses X_i et au-dessous de l'axe des abscisses pour le point avec l'abscisse x. Ainsi nous avons deux cas: Si x est situé entre deux racines consécutives X_m, X_{m+1}, l'intervalle $X'X''$ ne contient aucune racine de l'équation. De l'autre côté, si x est situé à la gauche de la plus petite racine ou à la droite de la plus grande racine de $f = 0$, la parabole contient toutes les racines X_i dans son intérieur. Au premier cas, X' et X'' sont plus proches de X_m et X_{m+1} que ne l'était x. Au second cas, X' et X sont plus proches de X_1 et X_n que ne l'était x.

Il reste le cas $u = \infty$ qui est traité comme il suit. L'équation $u^{-2}F(X) = 0$ devient pour $u = \infty$:

$$1/(x - X)^2 = \sum_i 1/(x - X_i)^2.$$

Mais

$$\Sigma 1/(x - X_i) = f'(x)/f(x)$$

différentié et inséré ci-dessus donnera

$$1/(x - X)^2 = \sum_i 1/(x - X_i)^2 = (f'^2 - ff'')/f^2, \quad \ldots \quad (4)$$

donc

$$1/(x - X_i)^2 < (f'^2 - ff'')/f^2,$$

d'où

$$(x - X_i)^2 > f^2/(f'^2 - ff''), \quad (i = 1, \ldots, n).$$

C'est à dire:

Aucune racine de $f = 0$ n'est située dans le cercle avec le rayon $|f/\sqrt{f'^2 - ff''}|$ autour de x.

Pour simplifier la manière d'exprimer nous imaginons que la ligne droite soit fermée à l'infini. Alors, si $X_{n+1} \equiv X_1$, toute valeur $x \neq X_i$ est située entre deux racines consécutives X_m et X_{m+1}. Alors dans le précédent, pour chaque u et chaque x, nous avons déterminé deux valeurs X' et X'' qui sont plus proches des racines voisines de x que ne l'était x. Pour $u = \infty$ ces X', X'' ont été calculés définitivement ci-dessus. Pour chaque autre valeur de u il faut les calculer de cas en cas.

La question se pose naturellement de trouver *l'intervalle $X'X''$ le plus avantageux possible pour toutes les u* (x étant fixe). Car alors l'approximation des deux racines sera la plus grande. C'est à dire, il faut choisir une valeur u' telle que X' soit situé aussi éloigné que possible à la gauche de x et une valeur u'' telle que X'' soit situé aussi éloigné que possible à la droite de x. Et il faut combiner u' et u'' pour obtenir l'intervalle (extr X', extr X'').

Insérant

$$\left(\frac{u-X_i}{x-X_i}\right)^2 = \left(\frac{u-x}{x-X_i}\right)^2 + 2\,\frac{u-x}{x-X_i} + 1$$

en (2) et usant (4), l'équation (3) devient

$$(f'^2 - ff'')M^2 + 2ff'M + nf^2 - \left(\frac{Mf}{x-X} + f\right)^2 = 0, \quad \text{où} \quad M = u - x. \quad (5)$$

Au lieu de u, maintenant la différence M doit être choisie telle que l'approximation soit aussi grande que possible.

C'est à ce point de la démonstration de LAGUERRE que se rapporte *la critique d'*HERMITE. Car LAGUERRE dit brèvement que la valeur la meilleure de M est fournie par une telle valeur de X pour laquelle le discriminant de (5) disparaît. Probablement le cours de ses idées était comme il suit.

L'équation (5) est quadratique en x et quadratique en M. C'est à dire, prescrivant x et X on aura deux valeurs pour M. Laissant maintenant X croître à partir de la valeur donnée x, on obtient pour chaque point passé X deux quantités $M : M_1, M_2$. Chacune de cettes quantités, insérées en (5), fournira une paire de valeurs dont l'une est la valeur donnée X. Ainsi nous aurons les deux paires (Y_1, X), (Y_2, X). Mais désignant les racines de $f = 0$ situées immédiatement à la gauche resp. à la droite de x par X_m resp. X_{m+1}, X ne pourra pas croître au delà de X_{m+1}. Car alors M et u seront complexes, puisque chaque M réel fournirait, comme nous l'avons *vu*, un intervalle où il n'y a pas de racines de $f = 0$, tandis que dans notre cas il y aurait la racine X_{m+1}. Ainsi X ne peut aller que jusqu'à un certain point P situé devant X_{m+1}. Ce P doit avoir la propriété que M_1, M_2, appartenants à P, fourniront deux paires de racines (Q, P), (R, P) de (5) de sorte qu'aucun des intervalles QP, RP ne contiendra une racine de $f = 0$. C'est à dire, Q et R sont situés à la droite de la racine X_m. Chacune valeur de X située à la droite de P fournira donc deux valeurs complexes M, ou pour une telle X le radicand apparaissant quand l'équation (5) est résolue par rapport à M, sera négatif. La valeur extrême P de X sera donc telle que les deux M appartenants à P coincident: $M_1 = M_2$. C'est à dire, le discriminant de (5) par rapport à M sera égal à zéro:

$$D(X, x) \equiv [(n-2)f'^2 - (n-1)ff''](x-X)^2 + 2ff'(x-X) - nf^2 = 0, \quad (6)$$

donc

$$X - x = -nf/(f' \pm \sqrt{H}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

où

$$H = (n-1)^2 f'^2 - n(n-1)ff''$$

est le HESSIEN de f.

(7) fournit donc deux valeurs X', X'' de X pour chacune desquelles les deux M coincident. Evidemment X' sera la limite à la gauche, X'' celle à la droite de x. En effet, partant de x à la gauche nous aurons le même discriminant D qu'en (6). Ainsi nous avons *le résultat de* LAGUERRE:

(7) fournit les points extrêmes de tous les intervalles obtenus pour toutes les valeurs de u de $-\infty$ à $+\infty$.

Il faut donc faire *distinguer l'intervalle extrême* ($X'X''$) *fourni par une seule M et l'intervalle* (*extr* X', *extr* X'') fourni par (7) à l'aide de deux M différents, à savoir par $M'_1 = M'_2 \equiv M'$ et $M''_1 = M''_2 \equiv M''$, où $M' \neq M''$. Ces deux valeurs de M seront obtenues en résolvant l'équation (5) ou brèvement

$$A(X) \,.\, M^2 + 2B(X) \,.\, M + C(X) = 0,$$

ce qui fournit

$$M_{1,2} \equiv M = -B(X)/A(X),$$

puisque le discriminant est 0. Insérant les deux valeurs de X résultantes de (7) on aura les deux M: M' et M''.

Examinons encore la relation des deux intervalles pour *le cas* $x = \infty$. Nous partons de nouveau de (3) dont le premier membre reste positif pour $x \to \infty$ et pour tous les $X = X_i$ et est négatif pour $X \to x$. La parabole (3) renferme donc toutes les racines X_i. Ses points X d'intersection avec l'axe des abscisses sont donc donnés par

$$(n-1)\,u^2 + 2\,(na_{n-1} + X)\,u + (n^2\, a^2_{n-1} - n\,(n-1)\,a_{n-2} - X^2) = 0.$$

Ici comme en (5), les valeurs extrêmes de X seront fournies quand le discriminant de l'équation disparaît:

$$D \equiv X^2 + 2\,a_{n-1}\,X + (n-1)^2\,a_{n-2} - n\,(n-2)\,a^2_{n-1} = 0,$$

donc

$$X = -\,a_{n-1} \pm (n-1)\,.\sqrt{a^2_{n-1} - a_{n-2}}.$$

De l'autre côté, en déterminant la valeur de u qui fait l'intervalle ($X'X''$) aussi petit que possible, nous aurons à cause de

$$X'' - X' = 2\sqrt{R},\ \text{où}\ R = nu^2 + 2\,na_{n-1} + n^2\,a^2_{n-1} - (n^2-n)\,a_{n-2}:$$

$$R' = 0\ \text{ou}\ u = -\,a_{n-1},$$

c'est à dire

$$R = n\,(n-1)\,(a^2_{n-1} - a_{n-2}).$$

Ainsi nous avons

$$(\text{extr}\ X',\ \text{extr}\ X'') = 2\,(n-1)\,.\sqrt{a^2_{n-1} - a_{n-2}},\ \text{tandis que}$$

$$\text{extr}\ (X'\,X'') = 2\sqrt{n\,(n-1)}\,.\sqrt{a^2_{n-1} - a_{n-2}}.$$

Le deuxième intervalle est donc moins effectif que le premier. Le quotient des deux est $\sqrt{n/(n-1)}$, donc approximativement égal à $1 + \frac{1}{2n}$.

Des développements précédents il s'ensuit que X', X'' sont toujours réelles. Au moyen de (7) nous avons donc une démonstration indirecte de la proposition connue que *le* HESSIEN *d'une fonction algébrique ayant toutes ses racines réelles est toujours non-négatif:*

$$H(x) \geqq 0 \text{ pour tous les } x. \qquad (8)$$

Une estimation plus précise sera donnée par J. G. VAN DER CORPUT dans la note suivante.

Il reste à considérer *le cas que toutes les racines de $f = 0$ sont identiques.* Alors pour que la méthode ne soit pas complètement en défaut, il faudra que les deux points dans lesquels la parabole (3) coupe l'axe des abscisses, coincident. En effet, pour la puissance $f = (x-a)^n$ d'une fonction linéaire, le HESSIEN disparaît identiquement de sorte qu'en (7) les deux racines coincident:

$$X = x - nf/f' = a. \qquad (9)$$

C'est à dire, dans ce cas la méthode de LAGUERRE fournit directement la seule racine par un seul procès.

Presque le même s'ensuit de (3). Car si $X_1 = \ldots = X_n = a$, on aura

$$F(X) = n\left(\frac{u-a}{x-a}\right)^2 - \left(\frac{u-X}{x-X}\right)^2$$

et la parabole

$$n(u-a)^2(x-X)^2 - (x-a)^2(u-X)^2,$$

devient pour $u = a$ une double ligne passante par le point $(a, 0)$.

Enfin, quand $f = 0$ est une équation linéaire ou quadratique, on obtient facilement que les racines de l'équation sont fournies par (7) dans une seule opération.

En somme, *la méthode de* LAGUERRE *n'est jamais en défaut, elle donne toujours une approximation X', X'' des deux racines „voisines".*

II. Dans ce qui précède nous avons démontré que, en désignant par X_m la racine „voisine" de x au côté „gauche" de x (la ligne droite étant fermée à l'infini) et en partant de x et répétant la méthode d'approximation de LAGUERRE, nous aurons une suite infinie, monotone et bornée $x, x_1, x_2, \ldots$, où

$$x_{k+1} = x_k - \frac{nf(x_k)}{f'(x_k) \pm \sqrt{H(x_k)}}. \qquad (10)$$

Ici le signe $\pm$ doit être choisi tel que x_1 est situé „à la gauche" de x, c'est à dire tel que $x_1 < x$ quand $X_1 < x$ et tel que $x_1 > x$ quand $X_1 > x$.

Apparamment on a alors

$$\lim x_k = X_m. \qquad (11)$$

Car à cause de $x_k \geqq X_m$ on a aussi lim $x_k \geqq X_m$. Mais lim x_k est une racine de $f = 0$, autrement il serait impossible que lim $(x_{k+1} - x_k) = 0$. Et parcequ'il n'y a pas de racine entre x et X_m, nous avons lim $x_k = X_m$.

En choisissant l'autre signe de $\sqrt{H(x)}$, nous aurons une infinité de valeurs convergente vers X_{m+1}.

III. Considérons maintenant *la manière de convergence* de la méthode. Interrompant le développement de TAYLOR

$$0 = f(X) = f(x) + (X-x) f'(x) + \tfrac{1}{2}(X-x)^2 f''(x) + \dots \quad . \quad (12)$$

après les termes quadratiques, l'erreur de la valeur résultante de X est tout au plus de l'ordre de grandeur de $(X-x)^3$. C'est à dire, la convergence du procès est au moins du 3e degré.

Maintenant dans son mémoire cité [6]) HERMITE démontre que l'équation (6) est la meilleure de sa sorte, c'est à dire meilleure que chaque autre équation dont les coefficients sont plus généralement

$$(af'^2 - bff'')(x-X)^2 - 2cff'(x-X) - df^2 = 0.$$

Il est vrai qu'il borne les coefficients par les relations

$$a - b + c = 0, \quad a + 2c - d \geqq 0, \text{ [7])}$$

mais cela suffit pour nous, car ces relations sont satisfaites par (12) puisque $a = 0$, $b = c = \frac{1}{2}$, $d = 1$. Et parce que l'équation (6) est meilleure que l'équation (12), le procédé basé sur (6) converge en 3e degré au moins.

Mais pour savoir *le caractère précis de la convergence*, il faut développer $x_{k+1} - X$ par des puissances de $x_k - X = \delta$ et chercher le coefficient de la puissance la plus petite du développement.

Or au moyen de (10) nous obtenons, en omettant l'argument x_k dans tout ce qui suit:

$$x_{k+1} - X = \delta - nf/(f' + \sqrt{H}) = T/U(f' + \sqrt{H}),$$

où

$$T = (\delta f' - nf)^2 - \delta^2 H, \quad U = \delta f' - nf - \delta\sqrt{H}.$$

Nous supposons que la racine X qu'il faut approximer ait la multiplicité m de sorte que

$$f(x) = L^m . g(x), \text{ où } L = x - X \text{ et } g(X) \neq 0.$$

Alors

$$f = f(X+\delta) = \frac{1}{m!}\delta^m f^{(m)}(X) + \frac{1}{(m+1)!}\delta^{m+1} f^{(m+1)}(X) + \dots$$

$$f' = f'(X+\delta) = \frac{1}{(m-1)!}\delta^{m-1} f^{(m)}(X) + \frac{1}{m!}\delta^m f^{(m+1)}(X) + \dots$$

$$f'' = f''(X+\delta) = \frac{1}{(m-2)!}\delta^{m-2} f^{(m)}(X) + \frac{1}{(m-1)!}\delta^{m-1} f^{(m+1)}(X) + \dots\dots$$

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[6]) L.c., p. 467—468.

[7]) Dans la relation dernière HERMITE omet le signe d'égalité quoique sa démonstration reste valide dans ce cas.

Maintenant

$$\frac{1}{m!} f^{(m)}(x) = g(x) + \textbf{puissances ascendantes de } L$$

$$\frac{1}{(m+1)!} f^{(m+1)}(x) = g'(x) + \textbf{puissances ascendantes de } L$$

$$\frac{1}{(m+2)!} f^{(m+2)}(x) = \tfrac{1}{2} g''(x) + \textbf{puissances ascendantes de } L$$

d'où

$$f \;= \delta^m g(X) + \delta^{m+1} g'(X) + \tfrac{1}{2}\delta^{m+2} g''(X) + \dots$$

$$f' = m\,\delta^{m-1} g(X) + (m+1)\,\delta^m g'(X) + \frac{m+2}{2}\,\delta^{m+1} g''(X) + \dots$$

$$f'' = m(m-1)\,\delta^{m-2} g(X) + m(m+1)\,\delta^{m-1} g'(X) + \binom{m+2}{2}\delta^m g''(X) + \dots$$

$$H = m(n-m)(n-1)\,\delta^{2m-2} g^2(X) + 2m(n-m-1)(n-1)\,\delta^{2m-1} g(X)\,g'(X)$$
$$+ (n-1)[(m+1)(n-m-1)g'^2(X) + (mn-m^2-2m-n)\,g(X)g''(X)]\,\delta^{2m} + \dots$$

$$(\delta f' - nf)^2 = (n-m)^2 g^2(X)\,\delta^{2m} + 2(n-m)(n-m-1)\,g(X)\,g'(X)\,\delta^{2m+1} +$$
$$+ [(n-m-1)^2 g'^2(X) + (n-m)(n-m-2)\,g(X)\,g''(X)]\,\delta^{2m+2} + \dots$$

$$T = -a\,\delta^{2m} - b\,\delta^{2m+1} - c\,\delta^{2m+2} + \dots, \text{ où}$$

$$a = n(m-1)(n-m)\,g^2(X)$$

$$b = 2n(n-m-1)(m-1)\,g(X)\,g'(X)$$

$$c = mn(n-m-1)\,g'^2(X) + n(mn-m^2-m-2n+3)\,g(X)\,g''(X)$$

$$\sqrt{H} = \sqrt{P}\,g(X)\,\delta^{m-1} + \dots, \text{ où } P = m(n-m)(n-1)$$

$$f' + \sqrt{H} = (m + \sqrt{P})\,g(X)\,\delta^{m-1} + \dots$$

$$U = -(n-m+\sqrt{P})\,g(X)\,\delta^m + \dots$$

Ainsi en omettant les puissances plus élevées de δ nous aurons

$$x_{k+1} - X = +\delta\,\frac{a + b\delta + c\delta^2 + \dots}{n[m(n-m) + \sqrt{P}]\,g^2(X) + \dots} \quad . \quad . \quad . \quad (13)$$

Ici $a \neq 0$, quand $m \neq 1$, tandis que pour $m = 1$: $a = b = 0$. Ainsi nous avons *deux cas*:

1. $m = 1$. Alors le développement (13) commence par la puissance δ^3. *La convergence est au moins du 3e degré.*

2. $m > 1$. Alors le développement (13) commence par la puissance δ. *La convergence est linéaire.*

C'est à dire, seulement pour une racine simple la méthode converge au 3e degré (au moins). Pour une racine multiple elle ne converge que linéairement.

Nous avons donc un phénomène semblable que dans *la méthode de* NEWTON qui converge quadratiquement dans les environs d'une racine simple, tandis qu'elle ne converge que linéairement dans les environs d'une racine de multiplicité $m > 1$. Pour obtenir dans le dernier cas une convergence quadratique il faut remplacer la formule NEWTONienne usuelle

$$x_{k+1} = x_k - f(x_k)/f'(x_k) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

par

$$x_{k+1} = x_k - m \cdot f(x_k)/f'(x_k). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14')$$

(La formule dernière peut être employée avantageusement aussi dans le cas que m racines sont situées très proches les uns des autres.)

Semblablement en changeant la formule (10) en

$$x_{k+1} = x_k - \frac{n f(x_k)}{f'(x_k) \pm \sqrt{\lambda H(x_k)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (10')$$

où λ est un facteur numérique on obtient un développement analogue à (13). Maintenant on peut choisir λ tel que de nouveau

$$a = b = 0,$$

savoir

$$\lambda = (n-m)/m\,(n-1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10'')$$

car λ n'est autre que le quotient des deux premiers termes correspondants de $(\delta f' - nf)^2$ et H. Il va sans dire que (10′) n'est qu'une des généralisations de (10) qui fournissent une approximation cubique.

Il y a encore un autre point de correspondence des deux méthodes. Si $m = n$, c'est à dire si toutes les racines de $f = 0$ sont identiques, la formule (14′) fournit directement la racine en question. Et le même est le cas avec la formule (10) ou (7) comme nous l'avons vu. — De l'autre côté, dans ce cas $m = n$ la formule (10′, 10″) devient la formule générale (14′) de NEWTON.

IV. En résumé nous avons le *théorème*:

Soit (1) une équation ayant toutes ses racines réelles: $X_1 \leqq X_2 \leqq \ldots \leqq X_n$. Imaginons que la ligne droite soit fermée de sorte que, si $X_{n+1} \equiv X_1$, chaque valeur arbitraire et réelle $x \neq \mathbf{X}_i$ sera située entre deux racines X_m et X_{m+1}. Partant maintenant du point $x \equiv x_0$ formons les deux suites (10). Alors quelque soit x, l'une des suites va converger monotonement vers $\mathbf{X}_m$ et l'autre vers X_{m+1}. La convergence est du 3e degré (au moins) ou du 1er degré, selon que la racine à approximer est simple ou multiple [8]).

[8]) Dans le dernier cas la méthode n'est pas essentiellement supérieure à la méthode de NEWTON qui ne converge aussi que linéairement et qui a l'avantage d'être plus simple (Il va sans dire que la méthode de NEWTON doit converger, c'est à dire que l'approximation de x est déjà satisfaisante).

Pour avoir une convergence toujours cubique, il faut remplacer la formule originale (7) ou (10) par la formule généralisée (10', 10"), où la multiplicité m de la racine doit être connue naturellement [9]. Quand la multiplicité m exacte n'est pas connue, mais que l'on sait au moins que la racine n'est pas simple, on peut poser $m = 2$ dans les formules (10, 10") pour obtenir une convergence plus rapide.

Si toutes les racines de l'équation sont identiques, la formule (7) fournit la racine exacte par une seule opération.

Si l'équation est quadratique, ses deux racines seront fournies par (7) par une seule opération.

L'avantage principal de la méthode ne consiste pas en ce qu'elle converge cubiquement — ce qu'elle a commun avec d'autres méthodes —, mais en ce qu'elle converge toujours.

V. Si nous polarisons la forme S_u en (2) par rapport à u, nous obtenons

$$S_{uv} = \frac{(u-X_1)(v-X_1)}{(x-X_1)^2} + \ldots + \frac{(u-X_n)(v-X_n)}{(x-X_n)^2} \quad (u, v \text{ réelles}). \qquad (15)$$

Décomposons

$$\frac{(u-X_i)(v-X_i)}{(x-X_i)^2} = \frac{(u-x)(v-x)}{(x-X_i)^2} + \frac{u+v-2x}{x-X_i} + 1.$$

Alors usant les deux relations (4) et la règle avant (4), nous aurons

$$f^2(x) \,.\, S_{uv} = (u-x)(v-x)(f'^2 - ff'') + (u+v-2x)ff' + nf^2.$$

A cause de (15) S_{uv} ne peut pas disparaître si tous les numérateurs $(u-X_i)(v-X_i)$ ont le même signe. Si $S_{uv} = 0$, une partie des X_i se trouve donc dans l'intervalle (u, v) et les autres en dehors de (u, v). C'est la proposition suivante de LAGUERRE [10]).

Pour x réel et arbitraire les nombres u et v qui satisfont à la relation:

$$(u-x)(v-x)(f'^2 - ff'') + (u+v-2x)ff' + nf^2 = 0, \quad . \quad . \qquad (16)$$

et dont l'un est arbitraire, „séparent" les racines de l'équation $f = 0$, qui sont supposées toutes réelles.

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[9]) Nous supposons ici que le procédé basé sur (10', 10") est pareillement convergent.

[10]) Oeuvres, I, p. 101. — HERMITE, Oeuvres de LAGUERRE, I, p. 461—2).

Posons maintenant $v = x$; la relation (16) devient

$$(u - x) f' + n f = 0, \text{ donc } u = x - \frac{nf}{f'}$$

et dans l'intervalle (u, x) il y a au moins une racine de $f = 0$. C'est la proposition énoncée sur p. 912 ci-dessus.

Prenant $x = 0$ on a encore:

D a n s l ' i n t e r v a l l e $(0; -a_0/a_1)$ i l y a a u m o i n s u n e r a c i n e d e $f = 0$.

Mathematics. — *Sur l'approximation de* LAGUERRE *des racines d'une équation qui a toutes ses racines réelles.* By J. G. VAN DER CORPUT.

(Communicated at the meeting of September 21, 1946.)

Dans la note intéressante précédente M. BODEWIG démontre que la méthode de LAGUERRE, appliquée à un polynôme, qui a toutes ses racines réelles, donne une approximation des racines qui est au moins du troisième degré pour une racine simple et du premier degré pour une racine multiple. Le cours des idées de LAGUERRE, développé par M. BODEWIG, m'a permis de déduire une méthode qui donne une approximation seulement des racines dont la multiplicité est $\geqq q$ (où q désigne un nombre naturel, qui est tout au plus égal au degré du polynôme) de telle façon que cette approximation est au moins du troisième degré pour les racines à multiplicité q et du premier degré pour les racines dont la multiplicité est supérieure à q.

Considérons un polynôme $f(x)$ du $n^{\text{ième}}$ degré dont toutes les racines et tous les coëfficients sont réels et qui possède au moins une racine dont la multiplicité est $\geqq q$, où $q \leqq n$. Dans ce cas le HESSien

$$H(x) = (n-1)^2 f'^2(x) - n(n-1) f(x) f''(x)$$

est $\geqq 0$ pour chaque nombre réel x. Posons pour chaque nombre réel x (fini ou infini), qui ne coincide pas avec une racine de $f(x)$,

$$u(x) = x + \frac{n f(x)}{-f'(x) \pm \sqrt{\lambda H(x)}} \text{ et } v(x) = x + \frac{n f(x)}{-f'(x) \mp \sqrt{\lambda H(x)}},$$

où $\lambda = \dfrac{n-q}{(n-1)\,q}$; nous employons les signes supérieurs ou inférieurs selon que $f(x)$ est positif ou négatif. Par $u(\infty)$ et $v(\infty)$ je désigne les limites auxquelles $u(x)$ et $v(x)$ tendent si x croît indéfiniment.

Pour le polynôme

$$f(x) = x^n + a\,x^{n-1} + b\,x^{n-2} + \dots$$

nous obtenons

$$u(\infty) = -\frac{a}{n} - \frac{1}{qn}\sqrt{q(n-q)\{(n-1)\,a^2 - 2\,n\,b\}}$$

et

$$v(\infty) = -\frac{a}{n} + \frac{1}{qn}\sqrt{q(n-q)\{(n-1)\,a^2 - 2\,n\,b\}}$$

Si $f'(x) = \sqrt{\lambda H(x)}$, on peut choisir $u(x)$ égal à $+\infty$ ou $-\infty$; si $f'(x) = -\sqrt{\lambda H(x)}$, on peut choisir $v(x)$ égal à $+\infty$ ou $-\infty$.

J'obtiens le résultat suivant:

Considérons un polynôme $f(x)$ du $n^{\text{ième}}$ degré qui a toutes ses racines et tous ses coëfficients réels et qui possède au moins une racine dont la multiplicité est $\geqq q$. Soit x un nombre réel, fini ou infini, qui ne coincide pas avec une racine de $f(x)$. Si l'intervalle ouvert dont les points extrêmes sont $u(x)$ et $v(x)$ contient x, il ne contient aucune racine de $f(x)$ à multiplicité $\geqq q$; si l'intervalle fermé dont les points extrêmes sont $u(x)$ et $v(x)$ ne contient pas x, il contient toutes les racines de $f(x)$, dont la multiplicité est $\geqq q$.

Afin d'approximer les racines dont la multiplicité est $\geqq q$, je choisis un point réel quelconque x_1 (fini ou infini) qui ne coincide pas avec une racine de $f(x)$. Je pose constamment $x_{h+1} = u(x_h)$ $(h = 1, 2, \ldots)$ ou constamment $x_{h+1} = v(x_h)$ $(h = 1, 2, \ldots)$ et je suppose qu'aucun des points $x_2, x_3, \ldots$ ne coincide avec une racine de $f(x)$. Dans ces conditions je démontrerai:

Considérons un polynôme dont toutes les racines et tous les coëfficients sont réels et qui possède au moins une racine dont la multiplicité est $\geqq q$. Les points $x_1, x_2, \ldots$, construits ci-dessus, tendent vers une racine ξ de $f(x)$ à multiplicité $\geqq q$. Cette approximation est lineaire ou au moins du troisième degré selon que m est supérieur ou égal à q, notamment

$$\lim_{h\to\infty} \frac{x_{h+1}-\xi}{x_h-\xi} = 1 - \frac{n\,q}{m\,q + \sqrt{q\,m\,(n-q)\,(n-m)}} \quad \text{pour } m > q \quad . \qquad (1)$$

et

$$\left.\begin{aligned} &\lim_{h\to\infty} \frac{x_{h+1}-\xi}{(x_h-\xi)^3} = \\ &\frac{(m+2)\,(n-m-1)\,f^{(m+1)}(\xi)\,f^{(m+1)}(\xi) - 2\,(m+1)\,(n-m)\,f^{(m)}(\xi)\,f^{(m+2)}(\xi)}{2\,(n-m)\,m\,(m+1)^2\,(m+2)\,f^{(m)}(\xi)\,f^{(m)}(\xi)} \\ &\hspace{20em} \text{pour } m = q. \end{aligned}\right\} . \qquad (2)$$

I. *Considérons d'abord le cas, où nous avons constamment*

$$x_{h+1} = u(x_h) \quad (h = 1, 2, \ldots).$$

Si $f(x)$ possède au moins une racine à multiplicité $\geqq q$ située à droite de x_1, les points $x_1, x_2, \ldots$ forment une suite constamment croissante qui tend vers la racine à multiplicité $\geqq q$ située à droite de x_1, aussi près de x_1 que possible.

Si $f(x)$ ne possède aucune racine à multiplicité $\geqq q$ située à droite de x_1, il y existe un seul nombre naturel k tel que x_k soit situé à droite de x_{k+1}, que $x_1, \ldots, x_k$ forment une suite constamment croissante et que les points $x_{k+1}, x_{k+2}, \ldots$ forment une suite constamment croissante qui tend vers la racine à multiplicité $\geqq q$ située aussi éloignée de x_1 que possible.

II. *Dans le cas ou l'on prend constamment* $x_{h+1} = v(x_h)$ *le résultat énoncé sous I reste valable, si l'on remplace les mots „droite" et „croissante" par „gauche" et „décroissante".*

Dans le cas particulier où $f(x)$ ne possède qu'une racine (simple ou multiple), le théorème n'est pas applicable, car alors on a

$$f(x) = a(x-\xi)^n \text{ et } H(x) = 0,$$

de sorte que $u(x)$ et $v(x)$ coincident avec ξ pour chaque nombre x. Dans ce cas on obtient pour chaque choix de x_1, que x_2 coincide avec la racine ξ de $f(x)$, contrairement à l'hypothèse qu'aucun des nombres $x_1, x_2, \ldots$ ne coincide avec une racine de ce polynôme.

Maintenant les démonstrations.

Considérons un nombre réel x (fini ou infini) qui ne coincide pas avec une racine de $f(x)$. Pour chaque racine X de $f(x)$ à multiplicité $m \geqq q$ et pour chaque nombre réel t on obtient

$$\sum_{h=1}^{n}\left(t+\frac{1}{X_h-x}\right)^2 \geqq m\left(t+\frac{1}{X-x}\right)^2 \geqq q\left(t+\frac{1}{X-x}\right)^2,$$

où $X_1, \ldots, X_n$ désignent les racines de $f(x)$.

Des identités

$$\sum_{h=1}^{n}\frac{1}{X_h-x} = -\frac{f'(x)}{f(x)} \text{ et } \sum_{h=1}^{n}\frac{1}{(X_h-x)^2} = \frac{f'(x)f'(x)-f(x)f''(x)}{f^2(x)}$$

on déduit immédiatement

$$n t^2 - 2t\frac{f'(x)}{f(x)} + \frac{f'(x)f'(x)-f(x)f''(x)}{f^2(x)} \geqq q t^2 + \frac{2qt}{X-x} + \frac{q}{(X-x)^2}.$$

Si l'on transmet tout au membre gauche, on obtient un polynôme quadratique, qui est $\geqq 0$ pour chaque nombre réel t, de sorte que le discriminant de cette forme est $\geqq 0$. Par conséquent

$$(n-q)\left(\frac{f'(x)f'(x)-f(x)f''(x)}{f^2(x)} - \frac{q}{(X-x)^2}\right) \geqq \left(\frac{f'(x)}{f(x)} + \frac{q}{X-x}\right)^2.$$

Ainsi nous obtenons

$$\lambda H(x) \geqq \left(\frac{n f(x)}{X-x} + f'(x)\right)^2,$$

d'où il suit que $H(x) \geqq 0$ et que

$$-f'(x) - \sqrt{\lambda H(x)} \leqq \frac{n f(x)}{X-x} \leqq -f'(x) + \sqrt{\lambda H(x)}. \quad . \quad (3)$$

Distinguons trois cas:

1°. Soit $f'(x) \neq \pm \sqrt{\lambda H(x)}$.

Si l'intervalle ouvert dont les points extrêmes sont $u(x)$ et $v(x)$ contient x, les deux nombres $\frac{n f(x)}{-f'(x) \pm \sqrt{\lambda H(x)}}$, donc aussi les deux nombres $-f'(x) \pm \sqrt{\lambda H(x)}$, ont des signes différents. Par conséquent le membre à gauche de (3) est négatif et le membre à droite positif. Il est donc exclu que les inégalités

$$\frac{1}{-f'(x) - \sqrt{\lambda H(x)}} > \frac{X-x}{n f(x)} > \frac{1}{-f'(x) + \sqrt{\lambda H(x)}} \quad . \quad . \quad (4)$$

sont valables, et pareillement que les inégalités

$$\frac{1}{-f'(x) - \sqrt{\lambda H(x)}} < \frac{X-x}{n f(x)} < \frac{1}{-f'(x) + \sqrt{\lambda H(x)}} \quad . \quad . \quad (5)$$

sont valables, d'où il suit que X n'est pas situé dans l'intervalle ouvert dont les points extrêmes sont $u(x)$ et $v(x)$.

Si l'intervalle fermé dont les points extrêmes sont $u(x)$ et $v(x)$ ne contient pas x, les deux nombres $\frac{n f(x)}{-f'(x) \pm \sqrt{\lambda H(x)}}$, donc aussi les deux nombres $-f'(x) \pm \sqrt{\lambda H(x)}$ ont le même signe. Alors (4) ou (5) sont valables avec $\geqq$ au lieu de $>$ et avec $\leqq$ au lieu de $<$. Dans ce cas X est situé dans l'intervalle fermé, dont les points extrêmes sont $u(x)$ et $v(x)$.

2°. Soit $f'(x) = -\sqrt{\lambda H(x)}$.

Alors (3) devient

$$0 \leqq \frac{n f(x)}{X-x} \leqq -f'(x) + \sqrt{\lambda H(x)}.$$

Le dernier membre est >0. En effet, sinon on aurait $f'(x) = H(x) = 0$, de sorte que x serait fini et la dernière inégalité impliquerait $f(x) = 0$, contrairement à l'hypothèse que x ne coincide pas avec une racine de $f(x)$. Ainsi nous obtenons

$$x < u(x) \leqq X \text{ ou } x > u(x) \geqq X,$$

selon que $f(x)$ est positif ou négatif.

Si l'interval ouvert avec les points extrêmes $u(x)$ et $v(x)$ (par définition $v(x)$ est ici égal à $+\infty$ ou $-\infty$) contient x, il ne contient pas X et si l'interval fermé avec les points extrêmes $u(x)$ et $v(x)$ ne contient pas x, il contient X.

3°. Soit $f'(x) = \sqrt{\lambda H(x)}$.

Dans ce cas on trouve

$$-f'(x) - \sqrt{\lambda H(x)} \leqq \frac{n f(x)}{X-x} \leqq 0.$$

d'où il suit
$$x > v(x) \geqq X \text{ ou } x < v(x) \leqq X,$$
selon que $f(x)$ est positif ou négatif.

Le raisonnement est ici le même que dans le cas 2°.

Dans chacun de ces trois cas l'énoncé du premier théorème est valable, de sorte que cette proposition est démontrée.

De ce théorème résultent les deux résultats suivants:

Si $x < u(x)$, le polynôme $f(x)$ ne possède aucune racine à multiplicité $\geqq q$ située entre x et $u(x)$.

En effet, si $v(x) < x$ le polynôme $f(x)$ ne possède aucune racine à multiplicité $\geqq q$ dans l'intervalle dont $v(x)$ et $u(x)$ sont les points extrêmes. Si $x < u(x) \leqq v(x)$, chaque racine à multiplicité $\geqq q$ est située dans l'intervalle, dont $u(x)$ et $v(x)$ sont les points extrêmes. Finalement le cas $x < v(x) < u(x)$ est exclu, puisque dans ce cas on aurait

$$-f'(x) \pm \sqrt{\lambda H(x)} > 0; \frac{n f(x)}{-f'(x) + \sqrt{\lambda H(x)}} < \frac{n f(x)}{-f'(x) - \sqrt{\lambda H(x)}}, \quad \text{si } f(x) > 0;$$

$$-f'(x) \pm \sqrt{\lambda H(x)} < 0; \frac{n f(x)}{-f'(x) - \sqrt{\lambda H(x)}} < \frac{n f(x)}{-f'(x) + \sqrt{\lambda H(x)}}, \quad \text{si } f(x) < 0,$$

donc $u(x) < v(x)$.

Si $u(x) < x$, le point $v(x)$ est situé entre $u(x)$ et x, tandis que l'intervalle fermé, dont les points extrêmes sont $u(x)$ et $v(x)$, contient toutes les racines de $f(x)$ à multiplicité $\geqq q$.

En effet dans ce cas on a

$$-f'(x) + \sqrt{\lambda H(x)} < 0 \text{ et } 0 < \frac{n f(x)}{f'(x) + \sqrt{\lambda H(x)}} < \frac{n f(x)}{f'(x) - \sqrt{\lambda H(x)}}, \quad \text{si } f(x) > 0,$$

$$-f'(x) - \sqrt{\lambda H(x)} > 0 \text{ et } 0 < \frac{-n f(x)}{-f'(x) + \sqrt{\lambda H(x)}} < \frac{-n f(x)}{-f'(x) - \sqrt{\lambda H(x)}}, \quad \text{si } f(x) < 0,$$

d'où il suit $u(x) < v(x) < x$. L'intervalle dont les points extrêmes sont $v(x)$ et $u(x)$ ne contient pas x, donc il contient toutes les racines à multiplicité $\geqq q$.

Dans la démonstration du second théorème je puis admettre que x_{h+1} est constamment égal à $u(x_h)$: dans le cas où $x_{h+1} = v(x_h)$ la démonstration est analogue.

De ce qui précède il suit: Si $x_h < x_{h+1}$, le polynôme $f(x)$ ne possède aucune racine à multiplicité $\geqq q$ située entre x_h et x_{h+1}; si $x_h > x_{h+1}$, l'interval fermé, dont les points extrêmes sont x_h et x_{h+1} contient toutes les racines à multiplicité $\geqq q$.

Il y existe au plus un nombre naturel k tel que $x_{k+1} < x_k$. En effet, soit k le plus petit nombre naturel avec cette propriété. Chaque racine à multiplicité $\geqq q$ est située entre x_{k+1} et x_k. Si l'on aurait $x_{k+2} < x_{k+1}$, chaque racine à multiplicité $\geqq q$ serait située entre x_{k+2} et x_{k+1}, ce qui est impossible. Donc $x_{k+2} > x_{k+1}$. Chaque racine à multiplicité $\geqq q$ est située entre x_{k+2} et x_k et en même temps à gauche de x_k, de sorte que $x_{k+1} < x_{k+2} < x_k$. On obtient de la même manière $x_{k+2} < x_{k+3} < x_k$ etc. Ainsi on obtient pour $h = k+1,\ k+2, \ldots$

$$x_h < x_{h+1} < x_k$$

et toutes les racines à multiplicité $\geqq q$ sont situées entre x_h et x_k.

Il y a donc seulement deux possibilités: ou bien les nombres $x_1, x_2, \ldots$ forment une suite toujours croissante, ou bien il y existe un nombre naturel k tel que $x_{k+1} < x_k$. Dans le premier cas $f(x)$ ne possède aucune racine à multiplicité $\geqq q$ située entre x_1 et x_h, où h désigne un nombre naturel quelconque; dans le second cas chaque racine à multiplicité $\geqq q$ est située entre x_h et x_k $(h = k+1,\ k+2, \ldots)$.

Dans tous les deux cas il y existe un nombre entier $k \geqq 0$, tel que les nombres $x_{k+1}, x_{k+2}, \ldots$ forment une suite toujours croissante. Il est exclu, que x_h croît indéfiniment, puisque, si x est suffisamment grand, le nombre $\sqrt{\lambda H(x)}$ est tout au plus du même ordre de grandeur que x^{n-2}, de sorte que les deux nombres $\dfrac{n f(x)}{-f'(x) \pm \sqrt{\lambda H(x)}}$ sont négatifs, d'où il suit que $u(x)$ est inférieur à x.

De cette manière nous avons démontré, que les nombres x_h $(h = 1, 2, \ldots)$ tendent vers une limite finie ξ. Puisque $x_{h+1} - x_h$ tend vers zéro, cela est aussi le cas avec $f(x_h)$, de sorte que ξ est une racine de $f(x)$. Si m désigne la multiplicité de cette racine et si nous posons $x - \xi = \varrho$ et $f(x) = \varrho^m g(x)$, le facteur $g(x)$ est un polynôme tel que $g(\xi) \neq 0$. Alors on a

$$f'(x) = m\,\varrho^{m-1} g(x) + \varrho^m g'(x),$$

$$f''(x) = m(m-1)\,\varrho^{m-2} g(x) + 2\,m\,\varrho^{m-1} g'(x) + \varrho^m g''(x),$$

et

$$\lambda H(x) = \varrho^{2m-2} g^2(x)\,(A - 2B\varrho + C\varrho^2),$$

où

$$A = \frac{(n-q)\,m\,(n-m)}{q}\ ;\ B = \frac{(n-q)\,m}{q}\,\frac{g'(x)}{g(x)}\ ;$$

$$C = \frac{n-q}{q}\,\frac{(n-1)\,g'^2(x) - n\,g(x)\,g''(x)}{g^2(x)}.$$

et

$$x_{h+1} = u(x_h) = x + \frac{n\,\varrho^m g(x)}{-m\,\varrho^{m-1} g(x) - \varrho^m g'(x) \pm \sqrt{\lambda H(x)}}.$$

où x désigne x_h et où j'emploie le signe supérieur ou inférieur selon que $\varrho^m g(x)$ est positif ou negatif. Le dernier terme dans le dénominateur est donc, en vertu de $\varrho < 0$, égal à

$$-\varrho^{m-1} g(x) \sqrt{A-2B\varrho + C\varrho^2}.$$

Ainsi nous obtenons la rélation

$$\lim_{h\to\infty} \frac{x_{h+1}-\xi}{x_h-\xi} = 1 + \frac{n}{-m-\sqrt{A}},$$

qui donne (1). Par conséquent l'approximation est au moins linéaire. Elle est exactement du premier degré, excepté si la limite trouvée est égale à zéro. Ceci est seulement le cas, si $\sqrt{A} = n-m$, donc $q=m$, d'où il suit qu' alors $x_{h+1}-\xi$ est approximativement égal à

$$\varrho + \frac{n\varrho^m g(x)}{-m\varrho^{m-1} g(x) - \varrho^m g'(x) - (n-m)\varrho^{m-1} g(x)\left\{1 - \frac{B}{A}\varrho + D\varrho^2\right\}}$$

où

$$D = \frac{AC-B^2}{2A^2} = \frac{n(n-m-1)g'^2(x) - n(n-m)g(x)g''(x)}{2m(n-m)^2 g^2(x)}.$$

Le résultat trouvé est donc égal à

$$\frac{-(n-m)D\varrho^3 g(x)}{-ng(x) - (n-m)D\varrho^2 g(x)},$$

par conséquent

$$\lim_{h\to\infty} \frac{x_{h+1}-\xi}{(x_h-\xi)^3} = \frac{n-m}{n} \lim_{x\to\xi} D = \frac{(n-m-1)g'^2(\xi) - (n-m)g(\xi)g''(\xi)}{2m(n-m)g^2(\xi)},$$

de sorte que nous obtenons (2) en vertu de

$$\varrho^m g(x) = f(x) = \frac{\varrho^m f^{(m)}(\xi)}{m!} + \frac{\varrho^{m+1} f^{(m+1)}(\xi)}{(m+1)!} + \dots,$$

d'où il suit

$$g(\xi) = \frac{f^{(m)}(\xi)}{m!};\ g'(\xi) = \frac{f^{(m+1)}(\xi)}{(m+1)!} \text{ et } g''(\xi) = 2\frac{f^{(m+2)}(\xi)}{(m+2)!}.$$

Si h est suffisamment grand, $x_{h+1}-\xi$ et $x_h-\xi$ sont négatifs, de sorte que la limite figurant dans (1) est $\geqq 0$, donc

$$\sqrt{qm(n-q)(n-m)} \leqq (n-m)q,$$

d'où il suit, en vertu de $m \neq n$, que $m \leqq q$.

Ainsi nous obtenons que seules les racines à multiplicité $\leqq q$ peuvent être approximées de cette manière.

Finalement nous devons distinguer deux cas différents:

1°. Supposons qu'au moins une racine de $f(x)$ à multiplicité $\geqq q$ soit située à droite de x_1.

Alors x_h tend vers une racine ξ à multiplicité $\geqq q$, située à droite de x_1 et puisque $f(x)$ ne possède aucune racine à multiplicité $\geqq q$ entre x_1 et x_h, la limite ξ est la racine à multiplicité $\geqq q$, située à droite de x, aussi près de x_1 que possible.

2°. Supposons qu'aucune racine à multiplicité $\geqq q$ ne soit située à droite de x_1. La limite ξ des nombres x_h est une racine à multiplicité $\geqq q$ de sorte que ξ soit situé à gauche de x_1. Dans ce cas ξ est la racine à multiplicité $\geqq q$, située aussi à gauche que possible, puisque, si h est suffissamment grand, x_h est situé à gauche de toutes les racines à multiplicité $\geqq q$.

Ainsi le théorème est complètement démontré.

Mathematics. — *A note on a theorem of* BLICHFELDT. By C. A. ROGERS. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 21, 1946.)

1. Suppose L is an n-dimensional lattice, with determinant $\triangle > 0$, and S is any set of points with inner volume $V(S)$. BLICHFELDT [1]) has proved that if k is a positive integer and

$$V(S) > k \triangle,$$

then S contains a set of $(k+1)$ points, all mutually congruent with respect to L. A direct consequence of this result, with $k=1$, is the fundamental theorem of MINKOWSKI that, if K is a convex body, symmetrical about the origin O and containing no lattice point other than O, then

$$V(K) \leqslant 2^n \triangle.$$

If λK is used to denote the set of points obtained by multiplying by λ all the coordinates of each of the points of K, and if λ_1 is defined to be the lower bound of the values of λ for which the body λK contains a lattice point other than O, then this theorem of MINKOWSKI asserts that

$$\lambda_1^n V(K) \leqslant 2^n \triangle.$$

MINKOWSKI [2]) generalized his theorem by associating with the convex body K a series of "successive minima" $\lambda_1, \lambda_2, \ldots, \lambda_n$, satisfying $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, and by proving that

$$\lambda_1 \lambda_2 \ldots \lambda_n V(K) \leqslant 2^n \triangle.$$

This suggests that in order to generalize BLICHFELDT's result we should define numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ in a similar way. Suppose S is any closed bounded set of points with inner volume $V(S) > 0$. As $V(S) > 0$, there is a cube which is contained in S. So, if λ is sufficiently large, λS contains a set of $(n+1)$ points, $A_0, A_1, \ldots, A_n$, such that the vectors $A_0A_1, A_0A_2, \ldots, A_0A_n$ are a set of n linearly independent lattice vectors. As S is bounded, if λ is sufficiently small, λS does not contain any pair of points, congruent with respect to L. Thus, as S is closed, there is a positive least value, λ_1 say, of λ for which the set λS contains two points, P_1 and Q_1 say, congruent with respect to L. Suppose λ_ν, P_ν and Q_ν have been defined for $\nu = 1, 2, \ldots, r$, where $r < n$. Then, if λ is sufficiently large, at least one of the vectors $A_0A_1, A_0A_2, \ldots, A_0A_n$ is a lattice vector linearly

1) BLICHFELDT, *Trans. Amer. Math. Soc.*, **15**, 227—235 (1914).

2) MINKOWSKI, *Geometrie der Zahlen*, (Berlin, 1910), Kapitel 5. For a simplified proof see DAVENPORT, *Quart. J. of Math.*, **10**, 119—121 (1939).

independent of the vectors $P_1Q_1, P_2Q_2, \ldots, P_rQ_r$. So, as S is closed, there is a least value, λ_{r+1} say, of λ for which the set λS contains two points, P_{r+1} and Q_{r+1} say, such that the vector $P_{r+1}Q_{r+1}$ is a lattice vector linearly independent of the vectors $P_1Q_1, P_2Q_2, \ldots, P_rQ_r$. The numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ given by this definition (which will be called definition (I)) satisfy the inequalities $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

If K is a convex body, symmetrical about the origin O, and $S = \frac{1}{2}K$, then the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ of definition (I) are identical with the "successive minima" $\lambda_1, \lambda_2, \ldots, \lambda_n$ defined by MINKOWSKI. Thus, with the notation of definition (I), the theorems of BLICHFELDT and MINKOWSKI give the inequalities:

(1) If S is any closed bounded set with inner volume $V(S) > 0$, then

$$\lambda_1^n V(S) \leqslant \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(2) If S is any closed convex body, symmetrical about the origin O, with volume $V(S)$, then

$$\lambda_1 \lambda_2 \ldots \lambda_n V(S) \leqslant \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

An examination of DAVENPORT's proof of MINKOWSKI's generalized inequality shows that this proof in fact establishes the result (2) for all closed convex bodies, without use of the hypothesis of symmetry.

In this note I shall give an example to show that the inequality (2) is not in general true, even for the simplest types of non-convex bodies. I shall then prove the weaker result of a positive character that, if S is any closed bounded set with inner volume $V > 0$, then

$$(\lambda_2^n/k) V(S) \leqslant \Delta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where k is the unique integer satisfying

$$\frac{\lambda_2}{\lambda_1} \leqslant k < \frac{\lambda_2}{\lambda_1} + 1.$$

Finally the definition of $\lambda_1, \lambda_2, \ldots, \lambda_n$ will be reconsidered.

2. Take L to be the two-dimensional lattice of points with integral coordinates. Take S to be the set of points inside and on the polygon with vertices at the points A, B, C, D, E, F, G, H, I and J with coordinates $(\frac{1}{2}, \frac{1}{2})$, $(\varepsilon, 1-\eta)$, $(-\varepsilon, \eta)$, $(-1+\varepsilon, 1-\eta)$, $(-1, \frac{1}{2})$, $(-\frac{1}{2}, -\frac{1}{2})$, $(-\varepsilon, -1+\eta)$, $(\varepsilon, -\eta)$, $(1-\varepsilon, -1+\eta)$ and $(1, -\frac{1}{2})$, where

$$0 < \varepsilon < \eta < \tfrac{1}{2}.$$

By use of the figure it is easy to verify that

(1) $\lambda_1 = \frac{2}{3}$, the segments $\frac{2}{3}AJ$ and $\frac{2}{3}EF$ being congruent with respect to L,

(2) $\lambda_2 = 1$, each point of the polygonal line $ABCDE$ being

congruent with respect to L to one or two points of the polygonal line $JIHGF$, and

(3) $V(S)$ is equal to the sum of the area of the parallelogram $AEFJ$ and twice the area of the triangle ABM, where M is the mid-point of BC, so that

$$V(S) = \tfrac{3}{2} \cdot 1 + \tfrac{1}{2}\left(\tfrac{1}{2} - \eta\right) = \tfrac{7}{4} - \tfrac{1}{2}\eta.$$

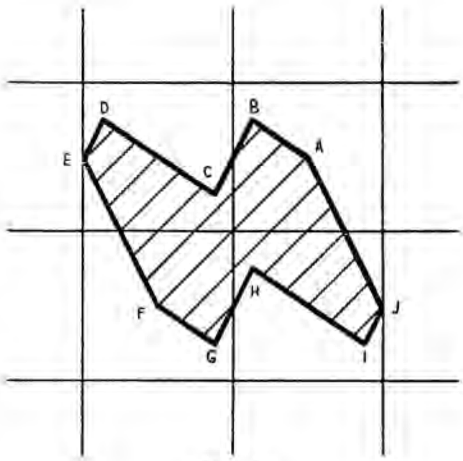

Fig. 1.

Thus

$$\lambda_1 \lambda_2 V(S) = \tfrac{7}{6} - \tfrac{1}{3}\eta > 1 = \Delta.$$

This shows that the inequality (2) is not valid even for this simple non-convex body.

§ 3. **Lemma.** *Let L be an n-dimensional lattice, with determinant $\Delta > 0$, and let S be a closed bounded set of points, with inner volume $V(S)$, satisfying*

$$V(S) > k\,\Delta,$$

where k is a positive integer. If $k > 1/\lambda_1$, where λ_1 is given by definition (1), and $R_0, R_1, \ldots, R_k$ are $(k+1)$ points of S all mutually congruent with respect to L, then these points $R_0, R_1, \ldots, R_k$ cannot all lie on one line.

Proof. Suppose L, S and k satisfy the hypotheses and the points $R_0, R_1, \ldots, R_k$ of S are all mutually congruent with respect to L; that such a set of points always exists follows by BLICHFELDT's theorem. Suppose $R_0, R_1, \ldots, R_k$ lie on a line, and let l be the length of the shortest lattice vector parallel to this line. Then the distance between the two extreme points (R and R' say) of the set $R_0, R_1, \ldots, R_k$ must be ml, where $m \geq k$ is an integer. The set $\frac{1}{m}S$ will thus contain the points $\frac{1}{m}R$ and $\frac{1}{m}R'$, which will be congruent with respect to L. As

$$\frac{1}{m} \leqslant \frac{1}{k} < \lambda_1,$$

this is contrary to definition (I), and so the lemma is proved.

Theorem. *If L is any n-dimensional lattice with determinant $\triangle > 0$, and S is any closed bounded set of points with inner volume $V(S) > 0$, then*

$$(\lambda_2^n/k)\ V(S) \leqslant \triangle. \qquad (4)$$

where k is the unique integer satisfying

$$\frac{\lambda_2}{\lambda_1} \leqslant k < \frac{\lambda_2}{\lambda_1} + 1,$$

and λ_1 and λ_2 are given by definition (I).

Proof. Suppose, if possible, that

$$(\lambda_2^n/k)\ V(S) = \triangle\,(1+\delta)^{2n}$$

where $\delta > 0$. Then

$$V(\lambda_2(1+\delta)^{-1} S) > k\,\triangle.$$

Now the first minimum for the body $S' = \lambda_2(1+\delta)^{-1}S$ is

$$\lambda'_1 = \lambda_1\ (1+\delta)/\lambda_2,$$

so that

$$\frac{1}{\lambda'_1} = \frac{\lambda_2}{\lambda_1(1+\delta)} < \frac{\lambda_2}{\lambda_1} \leqslant k.$$

Thus, using BLICHFELDT's theorem and the lemma, S' contains a set of points $R_0, R_1, \ldots, R_k$, all congruent with respect to L but not all on the same straight line. This is, however, contrary to the definition of λ_2, and so (4) is established.

4. Suppose S is any closed bounded set of points with inner volume $V(S) > 0$, and consider the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ defined in section 1. For certain sets S, it is possible to find values of λ greater than λ_n, for which the set λS does not contain even a single pair of points congruent with respect to L. For such sets definition (I) is rather artificial; it is only really natural when S has the property:

> (α) if P and Q are any two points of S, then, for every $\lambda > 1$, there exist points P_λ and Q_λ in λS such that $P_\lambda Q_\lambda$ is equal and parallel to PQ.

Quite a large class of sets have this property, in particular, all convex bodies and, more generally, all star bodies. A closed set S is a star body, if it contains an inner point C such that, if P is any point of S, then all the points on the segment CP are points of S. When S does not have the property (a) it is, perhaps, more natural to define μ_r, for $r = 1, 2, \ldots, n$, as the least value of μ such that, for all $\lambda \geq \mu$, the set λS contains points $P_1, Q_1, P_2, Q_2, \ldots, P_r, Q_r$, the vectors $P_1Q_1, P_2Q_2, \ldots, P_rQ_r$ being a set of r linearly independent lattice vectors. The existence of the μ's follows, as in definition (I), from the assumptions that S is closed and bounded and has a positive inner volume. This definition, which will be called definition (II), clearly implies that

$$0 < \mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n,$$
$$\lambda_\nu \leqslant \mu_\nu \text{ for } \nu = 1, 2, \ldots, n,$$

and, if S has the property (a), then the numbers $\mu_1, \mu_2, \ldots, \mu_n$ reduce to the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$.

With this notation the theorem of BLICHFELDT, with $k = 1$, is equivalent to the inequality

$$\mu_1^n V(S) \leqslant \Delta, \qquad (5)$$

and the theorem of the last section can be generalized slightly to yield the curious mixed inequality which follows.

Theorem. *If L is an n-dimensional lattice, with determinant $\Delta > 0$, and S is a closed bounded set of points with inner volume $V(S) > 0$, then*

$$(\mu_2^n/k)\, V(S) \leqslant \Delta \,. \qquad (6)$$

where k is the unique integer satisfying

$$\frac{\mu_2}{\lambda_1} \leqslant k < \frac{\mu_2}{\lambda_1} + 1,$$

and λ_1 and μ_2 are given by definitions (I) and (II).

Proof. Suppose, if possible, that

$$(\mu_2^n/k)\, V(S) = \Delta\,(1+\varrho)^n$$

where $\varrho > 0$. Then, using the definition of μ_2, it is possible to choose δ, such that $0 < \delta < \varrho$ and, if the set $S' = \mu_2(1+\delta)^{-1} S$ contains a set of points $R_0, R_1, \ldots, R_k$, all congruent with respect to L, then these points all lie on the same straight line.

But, as above,

$$V(S') > k\,\Delta$$

and

$$\frac{1}{\lambda_1'} = \frac{\mu_2}{\lambda_1(1+\delta)} < \frac{\mu_2}{\lambda_1} \leqslant k.$$

Thus, using BLICHFELDT's theorem and the lemma, S' contains a set of points $R_0, R_1, \ldots, R_k$, all congruent with respect to L but not all on the same straight line. This contradiction proves (6).

I am grateful to Professors DAVENPORT and VAN DER CORPUT, who have encouraged me to write this note and have advised me during its preparation.

University College, London.

Note added on the 4th of July, 1946.

Since I wrote this note I have seen a paper by V. JARNIK in *Věstník Královské Ceské Společnosti Nauk,* (1941). In this paper JARNIK gives a definition equivalent to definition (I), and proves that

$$\lambda_1 \lambda_2 \dots \lambda_n V(S) \leqslant 2^{n-1} \triangle \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

He also states that Mr. VL. KNICHAL has shown by an example that it is not possible to replace the 2^{n-1} in this inequality by unity I hope to publish shortly a proof of a sharper inequality of the type (7). C. A. R.

Mathematics. — *On the G-function.* VI. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of September 21, 1946.)

§ 17. Further investigation of the expansions of § 16.

I may recall that the function $G_{p,q}^{m,n}(z)$ satisfies the homogeneous linear differential equation (34) and that, if $q > p$ and the conditions (36), (37) and (38) are satisfied, a system of fundamental solutions of this equation valid in the neighbourhood of $z = \infty$ is formed by the p functions (39) together with the $q - p$ functions (40) [48].

In this § I will express the function $G_{p,q}^{m,n}(z)$ $(q > p)$ as a linear combination of these fundamental solutions; the expressions in question appear to be special cases of the expansion formulae (145), (148), (149), (150) and (152). My results can be stated as follows:

Theorem 11. Assumptions: *m, n, p and q are integers with*

$$1 \leqq n \leqq p < q,\ 2 \leqq m \leqq q \text{ and } m + n \geqq q + 1; \quad . \quad . \quad (153)$$

the number z satisfies the inequality

$$-(m + n - \tfrac{1}{2} p - \tfrac{1}{2} q)\pi < \arg z < (m + n - \tfrac{1}{2} p - \tfrac{1}{2} q)\pi; \quad . \quad (154)$$

the numbers $a_1, \ldots, a_n$, *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); λ *is an arbitrary integer which satisfies the inequalities* [49]

$$0 \leqq \lambda \leqq m + n - q - 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

$$(m + n + \tfrac{1}{2} p - \tfrac{3}{2} q - 2)\pi - \arg z < 2\lambda\pi < (m + n - \tfrac{1}{2} p - \tfrac{1}{2} q)\pi - \arg z. \quad (156)$$

Assertion: *The function* $G_{p,q}^{m,n}(z)$ *can by means of* (145) *be expressed in terms of fundamental solutions valid near* $z = \infty$.

Theorem 12 A. Assumptions: *m, n, p and q are integers which fulfil the conditions* (153); *the number z satisfies the inequality*

$$(m + n + \tfrac{1}{2} p - \tfrac{3}{2} q - 2)\pi < \arg z < (m + n - p + \varepsilon)\pi; \quad . \quad (157)$$

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[48]) Comp. also the Remark at the end of § 4.

[49]) By (154) we have

$$0 < (m + n - \tfrac{1}{2} p - \tfrac{1}{2} q)\pi - \arg z,$$
$$(m + n + \tfrac{1}{2} p - \tfrac{3}{2} q - 2)\pi - \arg z < (2m + 2n - 2q - 2)\pi;$$

from these relations it follows that there exists at least one integer λ satisfying (155) and (156).

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequalities*[50])

$$r \geqq 0, \quad \ldots \ldots \ldots \quad (158)$$

$$(\tfrac{1}{2}p + \tfrac{1}{2}q - m - n)\pi + \arg z < 2r\pi < (\tfrac{3}{2}q - \tfrac{1}{2}p - m - n + 2)\pi + \arg z. \quad (159)$$

Assertion: *The function* $G^{m,n}_{p,q}(z)$ *can by means of* (148) *be expressed in terms of fundamental solutions valid near* $z = \infty$.

Theorem 12 B. Assumptions: *m, n, p and q are integers which fulfil the conditions* (153); *the number z satisfies the inequality*

$$-(m + n - p + \varepsilon)\pi < \arg z < -(m + n + \tfrac{1}{2}p - \tfrac{3}{2}q - 2)\pi;$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequalities*

$$r \geqq 0,$$

$$(\tfrac{1}{2}p + \tfrac{1}{2}q - m - n)\pi - \arg z < 2r\pi < (\tfrac{3}{2}q - \tfrac{1}{2}p - m - n + 2)\pi - \arg z.$$

Assertion: *The function* $G^{m,n}_{p,q}(z)$ *can by means of* (149) *be expressed in terms of fundamental solutions valid near* $z = \infty$.

Theorem 13 A. Assumptions: *m, n, p and q are integers with*

$$0 \leqq n \leqq p < q, \; 1 \leqq m \leqq q \quad \ldots \ldots \quad (160)$$

and

$$\tfrac{3}{4}p + \tfrac{1}{4}q - \tfrac{1}{2}\varepsilon < m + n \leqq q + 1; \quad \ldots \ldots \quad (161)$$

the number z satisfies the inequality

$$-(m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi < \arg z < (m + n - p + \varepsilon)\pi; \quad . \quad (162)$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequalities*[51])

$$r \geqq q - m - n + 1, \quad \ldots \ldots \ldots \quad (163)$$

$$(\tfrac{1}{2}p + \tfrac{1}{2}q - m - n)\pi + \arg z < 2r\pi < (\tfrac{3}{2}q - \tfrac{1}{2}p - m - n + 2)\pi + \arg z. \quad (164)$$

Assertion: *The function* $G^{m,n}_{p,q}(z)$ *can by means of* (148) *be expressed in terms of fundamental solutions valid near* $z = \infty$.

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50) From (157) it follows

$$(\tfrac{3}{2}q - \tfrac{1}{2}p - m - n + 2)\pi + \arg z > 0;$$

hence there exists at least one integer r satisfying (158) and (159).

51) Because of (162) we have

$$(2q - 2m - 2n + 2)\pi < (\tfrac{3}{2}q - \tfrac{1}{2}p - m - n + 2)\pi + \arg z;$$

from this it appears that there exists at least one integer r which satisfies (163) and (164).

Theorem 13 B. Assumptions: *m, n, p and q are integers which fulfil the conditions* (160) *and* (161); *the number z satisfies the inequality*

$$-(m+n-p+\varepsilon)\pi < \arg z < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi;$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequalities*

$$r \geqq q-m-n+1,$$

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi-\arg z < 2r\pi < (\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi-\arg z.$$

Assertion: *The function* $G^{m,n}_{p,q}(z)$ *can by means of* (149) *be expressed in terms of fundamental solutions valid near* $z=\infty$.

Theorem 14. Assumptions: *m, n, p and q are integers with*

$$0 \leqq n \leqq p < q,\ 1 \leqq m \leqq q$$

and

$$p+1 \leqq m+n \leqq \tfrac{3}{4}q+\tfrac{1}{4}p-\tfrac{1}{2}\varepsilon+1; \quad \ldots \quad (165)$$

the number z satisfies the inequality

$$-(m+n-p+\varepsilon)\pi < \arg z < (m+n-p+\varepsilon)\pi; \quad \ldots \quad (166)$$

the numbers $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (20); *r is an arbitrary integer which satisfies the inequalities*

$$0 \leqq r \leqq q-m-n+1, \quad \ldots \quad (167)$$

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi+\arg z < 2r\pi < (\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z. \quad (168)$$

Assertion: *The function* $G^{m,n}_{p,q}(z)$ *can by means of* (150) *be expressed in terms of fundamental solutions valid near* $z=\infty$.

Theorem 15. Assumptions: *m, n, p and q are integers with*

$$0 \leqq n \leqq p < q \text{ and } 0 \leqq m \leqq q;$$

λ *is an arbitrary integer; the number z satisfies the inequality*

$$(m+n-p+\varepsilon+2\lambda-2)\pi \leqq \arg z < (m+n-p+\varepsilon+2\lambda)\pi; \quad (169)$$

the numbers $a_1, \ldots, a_p$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (38); μ *is an arbitrary integer which satisfies the condition*

$$(m+n-\tfrac{3}{2}p+\tfrac{1}{2}q+2\lambda-2)\pi-\arg z < 2\mu\pi < (m+n-\tfrac{5}{2}p+\tfrac{3}{2}q+2\lambda)\pi-\arg z. \quad (170)$$

Assertion: *The function* $G^{m,n}_{p,q}(z)$ *can by means of* (152) *be expressed in terms of fundamental solutions valid near* $z=\infty$.

If λ runs through the sequence of all positive and negative integers (zero included), we find by means of theorem 15 for all values of $\arg z$ an expression of $G^{m,n}_{p,q}(z)$ $(q>p)$ in terms of fundamental solutions valid near $z=\infty$.

If $m+n \geqq p+1$ and at the same time $|\arg z| < (m+n-p+\varepsilon)\pi$, we may obtain in a simpler way such an expression by means of the theorems 11, 12, 13 and 14.

For instance:

If $m+n \geqq q+1$ and $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, we may use theorem 11.

If $m+n \geqq q+1$ and $(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi \leqq \arg z < (m+n-p+\varepsilon)\pi$, we may use the theorems 12 A, B.

If $\frac{1}{2}p+\frac{1}{2}q < m+n \leqq q+1$ and $|\arg z| < (m+n-p+\varepsilon)\pi$, we may use the theorems 13 A, B.

If $p+1 \leqq m+n \leqq \frac{1}{2}p+\frac{1}{2}q$ and $|\arg z| < (m+n-p+\varepsilon)\pi$, we may use theorem 14.

Proof of theorem 11. From (156) it follows

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi < \arg z+(q-m-n+2\lambda+1)\pi < (\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi.$$

Hence condition (36) (with $-\lambda$ instead of λ) holds for the functions $G^{q,1}_{p,q}$ on the right of (145) and so these functions are fundamental solutions.

Proof of theorem 12 A. By (159) we have

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi < \arg z+(q-m-n-2r+1)\pi < (\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi.$$

The functions $G^{q,1}_{p,q}$ on the right-hand side of (148) are therefore fundamental solutions.

From (159) and (157) it follows

$$2r\pi < (\tfrac{3}{2}q-\tfrac{3}{2}p+\varepsilon+2)\pi,$$

consequently

$$r < q-p+1.$$

Hence the number of the functions $G^{q,0}_{p,q}$ occurring on the right of (148) is at most equal to $q-p$. These functions satisfy the condition (37) (with $\psi=s$); for we have by (157)

$$\arg z+(q-m-n)\pi < (q-p+\varepsilon)\pi$$

and by (159)

$$-(q-p+\varepsilon)\pi < -(\tfrac{1}{2}q-\tfrac{1}{2}p)\pi < \arg z+(q-m-n-2r+2)\pi.$$

The functions $G^{q,0}_{p,q}$ in (148) are therefore also fundamental solutions.

Proof of theorem 12 B. Similar to that of 12 A.

Proof of theorem 13 A. The inequality (162) has a meaning; for it follows from (161) that

$$-(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi < (m+n-p+\varepsilon)\pi.$$

For the rest the proof is similar to that of theorem 12 A.

Proof of theorem 13 B. Similar to that of 13 A.

Proof of theorem 14. From (166) it follows

$$(\tfrac{3}{2}q+\tfrac{1}{2}p-2m-2n-\varepsilon+2)\pi<(\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z \quad (171)$$

and

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi+\arg z<(\tfrac{1}{2}q-\tfrac{1}{2}p+\varepsilon)\pi. \quad (172)$$

Because of (165) we have

$$0\leqq\tfrac{3}{2}q+\tfrac{1}{2}p-2m-2n-\varepsilon+2 \quad (173)$$

and

$$\tfrac{1}{2}q-\tfrac{1}{2}p+\varepsilon\leqq 2+2q-2m-2n. \quad (174)$$

By combining (171) and (173) we obtain

$$0<(\tfrac{3}{2}q-\tfrac{1}{2}p-m-n+2)\pi+\arg z; \quad (175)$$

similarly by combining (172) and (174)

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi+\arg z<(2+2q-2m-2n)\pi. \quad (176)$$

From (175) and (176) it appears that there exists at least one integer r satisfying (167) and (168).

Now on the right-hand side of (150) there occur $q-m-n+1$ functions of the type $G_{p,q}^{q,0}(\zeta)$, the values of $\arg\zeta$ being

$$\arg z+(m+n-q)\pi, \arg z+(m+n-q+2)\pi, \ldots, \arg z+(q-m-n)\pi. \quad (177)$$

But by (165) we have $q-m-n+1\leqq q-p$. Hence the number of these functions $G_{p,q}^{q,0}(\zeta)$ is at most equal to $q-p$. It is easily seen that these functions are fundamental solutions. For it follows from (166) that the values (177) lie between $-(q-p+\varepsilon)\pi$ and $(q-p+\varepsilon)\pi$. The condition (37) is therefore satisfied and so the functions $G_{p,q}^{q,0}(\zeta)$ on the right-hand side of (150) are fundamental solutions. The functions $G_{p,q}^{q,1}$ are also fundamental solutions. This may be established in the same manner as in the proof of theorem 12 A.

Proof of theorem 15. From (170) and (169) it follows

$$(\tfrac{1}{2}q-\tfrac{1}{2}p-\varepsilon-2)\pi<2\mu\pi<(\tfrac{3}{2}q-\tfrac{3}{2}p-\varepsilon+2)\pi,$$

hence

$$-1<\mu<q-p+1;$$

the condition (151) of theorem 10 is therefore satisfied.

The functions $G_{p,q}^{q,1}$ on the right-hand side of (152) are fundamental solutions, since we have by (170)

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi<\arg z+(2p-q-m-n-2\lambda+2\mu+1)\pi<(\tfrac{1}{2}q-\tfrac{1}{2}p+1)\pi.$$

We will still show that the $q-p$ functions $G_{p,q}^{q,0}(\zeta)$ on the right-hand side of (152) are fundamental solutions. Now the values of $\arg\zeta$ are

$$\arg z + (2p-q-m-n-2\lambda+2)\pi,\ \arg z + (2p-q-m-n-2\lambda+4)\pi,\ \ldots,\ \arg z + (q-m-n-2\lambda)\pi,$$

and these values lie because of (169) between $-(q-p+\varepsilon)\pi$ and $(q-p+\varepsilon)\pi$. The functions $G_{p,q}^{q,0}(\zeta)$ satisfy therefore the condition (37) and so they are fundamental solutions.

With this the theorem has been proved.

§ 18. The asymptotic expansion of the function $G_{p,q}^{m,n}(z)$ $(q>p)$.

We are now able, for all values of m, n, p, q and $\arg z$, to investigate the behaviour of $G_{p,q}^{m,n}(z)$ $(q>p)$ as $|z|\to\infty$. For the theorems of § 17 in connection with the expansion formulae (145), (148), (149), (150) and (152) enable us to express the function $G_{p,q}^{m,n}(z)$ linearly in terms of functions $G_{p,q}^{q,1}$ and $G_{p,q}^{q,0}$ of which the asymptotic expansions can immediately be deduced from the theorems A and C of § 2.

In order to get an asymptotic expansion of $G_{p,q}^{m,n}(z)$ for $|z|\to\infty$ I investigate all the functions $G_{p,q}^{q,0}$ and $G_{p,q}^{q,1}$ on the right-hand side of one of the mentioned expansion formulae and I determine the dominant[52]) or the dominants among them[53]). Unless the coefficients of all the dominant functions vanish, we need only take account of the asymptotic expansions of these dominants and may neglect the others. Now these coefficients are functions of the parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ and these functions are in general not zero. Such a function is only zero if the parameters $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ satisfy a certain equation. Since these parameters are mutually independent, there exists in general no relation between them.

---

[52]) I say that $\phi(z)$ is dominant compared with $\psi(z)$ if the leading term of the asymptotic expansion of $\psi(z)$ is of an order less than the error term of the asymptotic expansion of $\phi(z)$.

For instance: If $\phi_1(z),\ldots,\phi_6(z)$ possess the asymptotic expansions

$$\phi_1(z)\backsim e^z\left(a_{1,0}+\frac{a_{1,1}}{z}+\ldots\right),\quad \phi_2(z)\backsim z^5\left(a_{2,0}+\frac{a_{2,1}}{z}+\ldots\right),$$

$$\phi_3(z)\backsim z^{-1}\left(a_{3,0}+\frac{a_{3,1}}{z}+\ldots\right),\quad \phi_4(z)\backsim e^{iz}z^{-2}\left(a_{4,0}+\frac{a_{4,1}}{z}+\ldots\right),$$

$$\phi_5(z)\backsim e^{-z}\left(a_{5,0}+\frac{a_{5,1}}{z}+\ldots\right),\quad \phi_6(z)\backsim e^{-2z}\left(a_{6,0}+\frac{a_{6,1}}{z}+\ldots\right)$$

and z is positive, then $\phi_1(z)$ is dominant compared with $\phi_2(z),\ldots,\phi_6(z)$; $\phi_2(z)$, $\phi_3(z)$ and $\phi_4(z)$ are dominant compared with $\phi_5(z)$ and $\phi_6(z)$; $\phi_5(z)$ is dominant compared with $\phi_6(z)$; but $\phi_2(z)$ is not dominant compared with $\phi_3(z)$ and $\phi_4(z)$. Among the functions $\phi_2(z),\ldots,\phi_6(z)$ there are three dominants, viz. $\phi_2(z)$, $\phi_3(z)$ and $\phi_4(z)$.

[53]) In many cases there is only one dominant function, viz. a function $G_{p,q}^{q,0}$.

If there is only one dominant function, I will suppose in this § that the coefficient of this function is not zero; if there are two or more than two dominant functions, I assume that at least one of them possesses a coefficient which is not zero. So in formula (195) it is tacitly supposed that the coefficient $D_{p,q}^{m,n}(\lambda)$ does not vanish; in formula (196) that at most one of the coefficients $D_{p,q}^{m,n}(\lambda)$ and $D_{p,q}^{m,n}(\lambda-1)$ vanishes.

If the coefficients of all the dominant functions are zero, it is necessary to make a closer investigation, with which I will not occupy myself.

Except in some simple cases the asymptotic behaviour of the function $G_{p,p+1}^{m,n}(z)$ is quite different from that of the function $G_{p,q}^{m,n}(z)$ with $q \geqq p+2$ (comp. the theorems 20 and 21).

Substantially the results run as follows:

1. If $n \geqq 1$ and $m+n > \frac{1}{2}p + \frac{1}{2}q$, then $G_{p,q}^{m,n}(z)$ has for large values of $|z|$ with $|\arg z| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ an asymptotic expansion of algebraic order (This is the case of theorem B of § 2).

2. If $m > \frac{1}{2}p + \frac{1}{2}q$, then $G_{p,q}^{m,0}(z)$ has for large values of $|z|$ with $|\arg z| < (m-\frac{1}{2}p-\frac{1}{2}q)\pi$ an asymptotic expansion which is exponentially zero.

3. If $q \geqq p+2$ and $m+n > \frac{1}{2}p + \frac{1}{2}q$, then $G_{p,q}^{m,n}(z)$ has for large values of $|z|$ with $|\arg z| > (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ an asymptotic expansion which is exponentially infinite.

4. If $q \geqq p+2$ and $m+n \leqq \frac{1}{2}p + \frac{1}{2}q$, then $G_{p,q}^{m,n}(z)$ has for large values of $|z|$ an asymptotic expansion which is exponentially infinite [54]).

5. If $m+n \geqq p+1$ and λ is either an arbitrary integer $\geqq 0$ or an arbitrary integer $\leqq p-m-n$, then $G_{p,p+1}^{m,n}(z)$ has for large values of $|z|$ with

$$(m+n-p+2\lambda-\tfrac{1}{2})\pi < \arg z < (m+n-p+2\lambda+\tfrac{1}{2})\pi \quad . \quad (178)$$

an asymptotic expansion which is exponentially infinite.

6. If λ is an arbitrary integer, then $G_{p,p+1}^{m,n}(z)$ has for large values of $|z|$ with

$$(m+n-p+2\lambda-\tfrac{3}{2})\pi < \arg z < (m+n-p+2\lambda-\tfrac{1}{2})\pi$$

an asymptotic expansion of algebraic order.

7. If $m+n \leqq p$ and λ is an arbitrary integer, then $G_{p,p+1}^{m,n}(z)$ has for large values of $|z|$ in the sector (178) an asymptotic expansion which is exponentially infinite.

I will now state my results. The simplest case is afforded by

Theorem 16. Assumptions: *m, n, p and q are integers with*

$$1 \leqq n \leqq p < q,\ 1 \leqq m \leqq q \text{ and } m+n > \tfrac{1}{2}p + \tfrac{1}{2}q;$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ satisfy the conditions (1) *and* (20).

[54]) There are certain special values of arg z for which 3. and 4. are not true; comp. assertion 4 of theorem 18 and assertion 3 of theorem 20.

Assertion: *The function* $G_{p,q}^{m,n}(z)$ *possesses for large values of* $|z|$ *with*

$$-(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi < \arg z < (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi \quad . \qquad (179)$$

the asymptotic expansion [55])

$$G_{p,q}^{m,n}(z) \backsim \sum_{t=1}^{n} e^{(m+n-q-1)\pi i a_t} \triangle^{m,n}_{q}(t) E_{p,q}(z e^{(q-m-n+1)\pi i} \| a_t). \qquad (180)$$

Remark. This theorem is equivalent to theorem B of § 2.

Proof: We may distinguish two cases:

First case: $m+n \geqq q+1$ [56]). We apply theorem 11. The asymptotic expansions of the functions $G_{p,q}^{q,1}$ on the right of (145) can be deduced from theorem A and (15) (with $\gamma=\lambda$): the result is

$$G_{p,q}^{q,1}(z e^{(q-m-n+2\lambda+1)\pi i} \| a_t) \backsim e^{2\lambda\pi i a_t} E_{p,q}(z e^{(q-m-n+1)\pi i} \| a_t).$$

From this relation and (145) follows (180).

Second case: $\frac{1}{2}p+\frac{1}{2}q < m+n \leqq q+1$. We use theorem 13 A. The asymptotic behaviour of the functions $G_{p,q}^{q,0}(z e^{(q-m-n-2s)\pi i})$ on the right of (148) can be determined by means of theorem C. Now it follows from (179)

$$\arg(z e^{(q-m-n)\pi i}) < (\tfrac{1}{2}q-\tfrac{1}{2}p)\pi$$

and from (164)

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p)\pi < \arg(z e^{(q-m-n-2r+2)\pi i}).$$

Hence we have for $s=0,1,\ldots,r-1$

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p)\pi < \arg(z e^{(q-m-n-2s)\pi i}) < (\tfrac{1}{2}q-\tfrac{1}{2}p)\pi.$$

From this relation and (26) and (25) it appears that the functions $G_{p,q}^{q,0}(z e^{(q-m-n-2s)\pi i})$ on the right of (148) tend exponentially to zero as $|z| \to \infty$.

The functions $G_{p,q}^{q,1}$ in (148) yield by means of theorem A and (15) the same asymptotic expansions of algebraic order as in the first case. With this the theorem is established.

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[55]) Comp. footnote [12]).

[56]) If $m+n \geqq q+1$, then $m \geqq 2$, since $n < q$.

Physics. — *The distribution of energy in continuous X-ray spectra corresponding to different forms of high tension, and the influence of filtering.* Introduction and section I. By R. H. DE WAARD (*X*-ray department of the Medical University Clinic, Utrecht). (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of September 21, 1946.)

Introduction.

Let an X-ray tube be acted upon by a constant high tension V. Experiments carried out by KULENKAMPFF [1]) suggest that the distribution of energy in the continuous spectrum of the radiation then excited is given by a formula

$$\overline{E}_\nu = K\, i\, (\nu_0 - \nu) \quad \ldots \ldots \ldots \quad (1)$$

where K is a constant characteristic of the tube, i the tube-current, and ν_0 the maximum frequency given by DUANE and HUNT's famous quantum relation. If V is expressed in kilovolts we have

$$\nu_0 = \gamma\, V \quad \ldots \ldots \ldots \quad (2)$$

where

$$\gamma = 2.43 \times 10^{17} \quad \ldots \ldots \ldots \quad (3)$$

Formulae (1), (2) and (3) also result from a theory given by H. A. KRAMERS in 1923 [2]).

We see from the formulae that to any value of V corresponds a definite relation between the energy $\overline{E}_\nu$ and the frequency ν. Fig. 1 gives a graphic representation of the relations corresponding to the high tensions $V = 70$ kilovolts and $V = 50$ kilovolts. In this figure the spectral energy $\overline{E}_\nu$ is plotted against the frequency ν. If we plot the energy against the wavelength instead of the frequency, i.e. $\overline{E}_\lambda$ against λ, the straight lines are transformed into the curves of fig. 1*b*, which are very similar to the experimental curves obtained by several investigators. As an example of the latter curves we give those of fig. 2 showing the results of measurements carried out by BOUWERS on a tube which was fed by a transformer at peak voltages of 70 and 50 kilovolts [3]).

It is not surprising that the agreement of experimental curves as shown in fig. 2 with the curves of fig. 1*b* is only qualitative. The obvious reason is that the latter curves may be assumed to refer to constant tension and

[1]) Annalen der Physik, 69, 548, 1922.

[2]) See M. et L. DE BROGLIE, Physique des Rayons X et Gamma, Paris 1928, pp. 78—79.

[3]) A. BOUWERS, Physica en Techniek der Röntgenstralen, Deventer 1927, pp. 17, 79.

to the radiation excited in an X-ray tube whereas experimental data have been chiefly obtained with certain forms of variable tension and apply to the emitted radiation which has first traversed some thickness of the anodic material and thereafter the window of the tube. From (1), (2) and

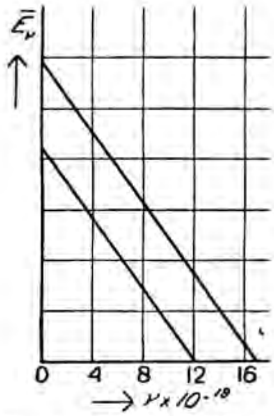

Fig. 1*a*. Theoretical $\overline{E}_\nu$, ν-relations resulting from formulae (1), (2) and (3) for constant tensions $V = 70$ kV and $V = 50$ kV.

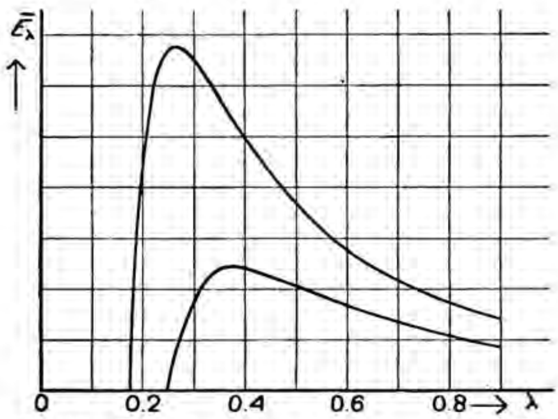

Fig. 1*b*. $\overline{E}_\lambda$, λ-relations corresponding to the $\overline{E}_\nu$, ν-relations of fig. 1*a*.

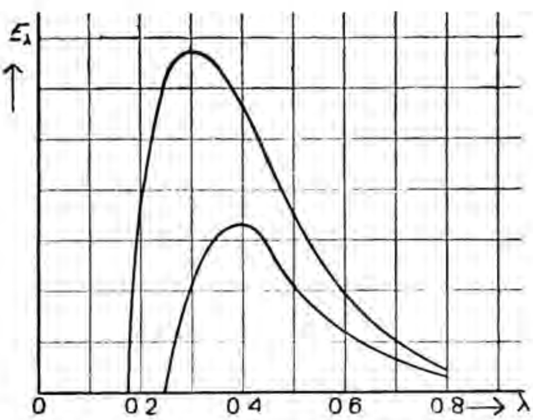

Fig. 2. Experimental E_λ, λ-curves obtained by BOUWERS for peak voltages $V_p = 70$ kV (upper curve) and $V_p = 50$ kV (lower curve).

(3) formulae referring to current experimental conditions can be easily derived, however, and with such formulae experimental curves can be well compared.

Formulae of the sort in question are obtained in section I of the present paper. They are concerned with the distribution of energy in the continuous spectra of filtered X-radiation and apply not only to constant tension but also to three simple forms of variable tension. In section II these results are applied to some special cases and tested by comparison with experimental data. The outcome is that the distribution of spectral energy in the radiation leaving the window of an X-ray tube can be satisfactorily described by means of the formulae referring to filtered radiation. If, therefore, we introduce the idea of an intrinsic tube filter, these formulae enable us to account for this distribution.

I. *Deduction of fundamental formulae.*

The formulae which will be deduced in this section refer to four current forms of high tension. The main results are theoretical E_λ,λ-relations giving the distribution of energy in the continuous spectra of filtered X-radiation. These relations can be largely characterized by two wavelengths, viz. the minimum wavelength λ_p and the wavelength λ_m of maximum intensity, and it will appear that these wavelengths can be determined in a very simple way.

A. Constant high tension.

Minimum wavelength.

To the maximum frequency $\nu_0 = \gamma V$ corresponds a minimum wavelength λ_0 given by the formula

$$\lambda_0 = \frac{12.3}{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in ÅNGSTRÖM-units if V is expressed in kilovolts.

Expression for $\overline{E}_\lambda$.

The expression for $\overline{E}_\lambda$ corresponding to that for $\overline{E}_\nu$ given by (1) is

$$\overline{E}_\lambda = L\,i\left(\frac{1}{\lambda_0} - \frac{1}{\lambda}\right)\frac{1}{\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where L is just as K a constant characteristic of the X-ray tube. The value of this constant is different according as λ_0 and λ are expressed in cm, or in ÅNGSTRÖM-units.

Influence of filtering. Expression for E_λ. Wavelength of maximum intensity.

We imagine that the radiation excited in an X-ray tube falls perpendicularly on a sheet of matter of thickness d, and we will assume the coefficient of enfeeblement μ of the matter composing the sheet to be given by the well-known formula

$$\mu = C\,\lambda^3 + \sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where C and σ are constants. In this formula the term $C\lambda^3$ accounts for the influence of absorption, and the term σ for that of scattering. Now let $E_\lambda d\lambda$ be the energy of transmitted radiation in the range of wavelengths between λ and $\lambda + d\lambda$; we then have

$$E_\lambda = \overline{E}_\lambda\, e^{-\mu d} = L\,i\left(\frac{1}{\lambda_0} - \frac{1}{\lambda}\right)\frac{1}{\lambda^2}\, e^{-(f\lambda^3 + g)} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where

$$f = Cd \text{ and } g = \sigma d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If a series of filters is traversed characterized by the sets of constants $(C_1\sigma_1d_1)$, $(C_2\sigma_2d_2)$ ... the latter formulae are to be replaced by

$$f = C_1\,d_1 + C_2\,d_2 + \ldots \text{ and } g = \sigma_1, d_1 + \sigma_2\,d_2 + \ldots \quad . \quad (8a)$$

In order to illustrate formula (7) we will apply it to the case of a 70 kV X-radiation falling on aluminium filters. We then have $\lambda_0 = 0.176$ Å whilst $C = 37$ and $\sigma = 0.38$ if λ and d are expressed in ÅNGSTRÖM-units and centimeters respectively. The E_λ,λ-curves corresponding to filters of thickness $d = 0$, $d = 1$ mm., and $d = 3$ mm. are shown in fig. 3.

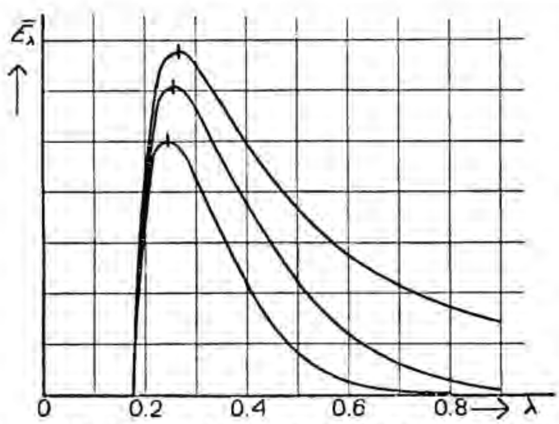

Fig. 3. Theoretical E_λ, λ-curves for constant tension $V = 70$ kV and aluminium filters of thickness 0, 1.0 mm., and 3.0 mm.

We see from this figure that the transmitted radiation is more homogeneous according as it has passed a thicker filter, and that the inhomogeneousness can be characterized by the minimum wavelength λ_0 together with the wavelength λ_m to which corresponds the maximum of the expression for E_λ. Now λ_0 is given by formula (4), and we will show that λ_m, too, can be found by a simple method. If

$$\frac{\lambda_0}{\lambda_m} = \cos \varphi_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

it can be shown that

$$f\lambda_0^3 = \cos^3 \varphi_m \, \frac{\cos \varphi_m - \frac{2}{3}}{1 - \cos \varphi_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and the relation between $f\lambda_0^3$ and $\cos \varphi_m$ given by this equation is graphically represented by curve A in fig. 4. With the help of this figure λ_m can be determined in every given case. Indeed, if f and V are known the value of λ_0 can be found from formula (4) and thereafter the product $f\lambda_0^3$ calculated. The corresponding value of $\cos \varphi_m$ can then be taken from the figure and from this value and λ_0 the value of λ_m can be easily derived.

The physical meaning of the angle φ_m follows at once from fig. 5. This figure shows that the angle is closely connected with the inhomogene-

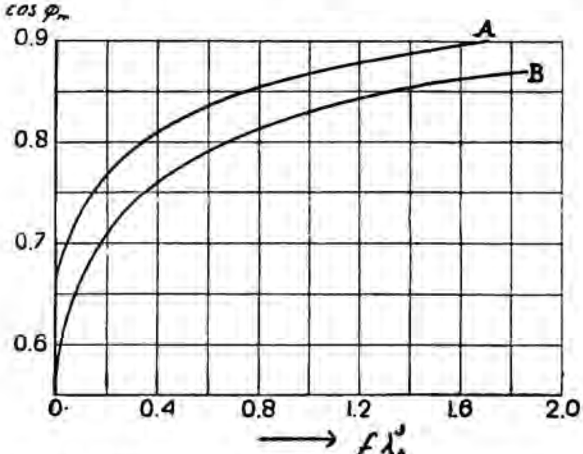

Fig. 4. Curves giving the relation between $\cos\varphi_m$ and $f\lambda_p^3$: A for constant tension (λ_p is then the same wavelength as λ_0), B for single phase alternating tension.

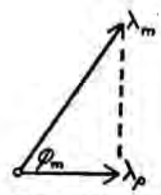

Fig. 5. Physical interpretation of the angle φ_m: in case of constant tension the wavelength λ_p is the same as λ_0.

ousness of the radiation in question. We will therefore call it the angle of inhomogeneousness and its cosine the cosine of inhomogeneousness.

B. High tension produced by single phase alternating current apparatus.

Distribution of spectral energy.

Let an X-ray tube be acted upon by a high tension with sine variation such as can be supplied by a transformer without valve tubes or with one, two or four. In every half period in which this tension works (fig. 6) the

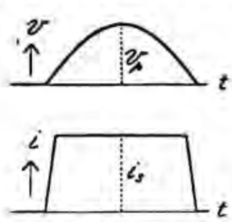

Fig. 6. Courses of high tension V and tube current i in case of single phase alternating tension.

tube current will almost continuously have its saturation value i_s; only during the short time in which the tension is very low the tube current will be considerably smaller. If τ is the length of a half period the course of the high tension V can be expressed by a formula

$$V = V_p \cos \frac{\pi}{\tau} t$$

where V_p is the peak-value of the tension. We will now suppose that during any short time dt the distribution of spectral energy over the various frequencies ν is characterized by the formula

$$\overline{E}_\nu = K\,i\,(\gamma V - \nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which can be derived from (1) and (2), V being the momentary value of the high tension. If, then, $\overline{E}_\nu^{\sim} d\nu$ is the spectral energy emitted by the tube per second in the range of frequencies between ν and $\nu + d\nu$ we have

$$\overline{E}_\nu^{\sim} = \frac{2}{\pi} K\,i\left(\nu_p \sqrt{1-\left(\frac{\nu}{\nu_p}\right)^2} - \nu \text{ arc cos } \frac{\nu}{\nu_p}\right). \quad . \quad . \quad . \quad (12)$$

where

$$\nu_p = \gamma\, V_p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

is the maximum frequency corresponding to the peak value V_p of the high tension.

It was assumed in the deduction of formula (12) that $i = i_s$ as long as X-rays are emitted by the tube. Moreover, it had to be observed that in the half period between the moments $t = -\frac{\tau}{2}$ and $t = \frac{\tau}{2}$ radiation of a given frequency ν is only emitted in the interval of time in which $\gamma V > \nu$, that is when

$$-\frac{\tau}{\pi} \text{ arc cos } \frac{\nu}{\nu_p} < t < \frac{\tau}{\pi} \text{ arc cos } \frac{\nu}{\nu_p}$$

A useful simplification of (12) can be obtained by the introduction of the angle φ given by the equation

$$\cos\varphi = \frac{\nu}{\nu_p} = \frac{\lambda_p}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If, then, we introduce the function

$$B_\varphi^{\sim} = \frac{2}{\pi}(\sin\varphi - \varphi\cos\varphi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

the relation (12) can be written

$$\overline{E}_\nu^{\sim} = K\,i\,\nu_p\,B_\varphi^{\sim} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The essentials of this relation are given by the curve of fig. 7 in which the values of $B_\varphi^{\sim}$ are plotted against those of $\cos\varphi$.

For the sake of comparison the figure also shows the straight line representing the function

$$B_\varphi^{=} = 1 - \cos\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which corresponds to the case of constant tension treated in IA. If in this case we write $\overline{E}_\nu^{=}$ for $\overline{E}_\nu$ and ν_p for the maximum frequency ν_0 we have obviously

$$\overline{E}_\nu^{=} = K\, i\, \nu_p\, B_\varphi^{=}. \qquad (18)$$

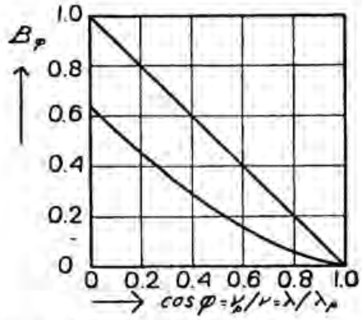

Fig. 7. B_φ, cos φ-relations: straight line for constant tension, curve for single phase alternating tension.

Expression for $\overline{E}_\lambda^{-}$.

The expression for $\overline{E}_\lambda^{-}$ corresponding to that for $\overline{E}_\nu^{-}$ given by (13) is easily found to be

$$\overline{E}_\lambda^{-} = \frac{L\, i}{\lambda_p} B_\varphi^{-} \frac{1}{\lambda^2} \qquad (19)$$

where λ_p is the minimum wavelength corresponding to the peak-value of the high tension:

$$\lambda_p = \frac{12.3}{V_p}. \qquad (20)$$

Expression for E_λ^{-}. Wavelength of maximum intensity.

The distribution of spectral energy in transmitted X-radiation is, in the case under consideration, given by the expression

$$E_\lambda^{-} = \overline{E}_\lambda^{-}\, e^{-(f\lambda^3 + g)} \qquad (21)$$

and the cosine of inhomogeneousness $\cos\varphi_m = \frac{\lambda_p}{\lambda_m}$ can be shown to satisfy the equation

$$f\lambda_p^3 = \cos^3\varphi_m \frac{\varphi_m \cot\varphi_m - \frac{2}{3}}{1 - \varphi_m \cot\varphi_m} \qquad (22)$$

to which corresponds curve B of fig. 4. With the help of this curve λ_m can be determined whenever f and $\lambda_p = \frac{12.3}{V_p}$ are known.

C. High tension produced by three phase alternating current apparatus.

The form of tension supplied by a three phase generator with three transformers and six valve-tubes is schematically represented in fig. 8. It is clear that the tube current will, in this case, have continually the saturation value is i_s:

$$i = i_s.$$

If τ is the length of a half period of the alternating tension the duration of a single high tension tip (such as that between a and b in fig. 8) will

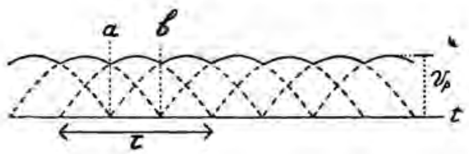

Fig. 8. Course of high tension V in case of three phase alternating tension.

be $\frac{\tau}{3}$, and in the course of such a tip the tension can be represented by the formula

$$V = V_p \cos \frac{\pi}{\tau} t$$

where

$$-\tfrac{1}{6}\tau < t < \tfrac{1}{6}\tau.$$

Now let $\overline{E}_\lambda^\nabla \, d\lambda$ be the energy emitted every second in the range of wave-lengths between λ and $\lambda + d\lambda$. If, then, we put as in (14)

$$\cos \varphi = \frac{\nu}{\nu_p} = \frac{\lambda_p}{\lambda}$$

and in a similar way as in (19)

$$\overline{E}_\lambda^\nabla = \frac{L\,i}{\lambda_p} B_\varphi^\nabla \frac{1}{\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

we can prove that

$$B_\varphi^\nabla = \frac{6}{\pi} (\sin \varphi - \varphi \cos \varphi) \text{ if } \tfrac{1}{2}\sqrt{3} < \cos \varphi = \frac{\nu}{\nu_p} \quad . \quad . \quad (24a)$$

and

$$B_\varphi^\nabla = \frac{3}{\pi} - \cos \varphi \text{ if } \cos \varphi = \frac{\nu}{\nu_p} < \tfrac{1}{2}\sqrt{3}. \quad . \quad . \quad . \quad (24b)$$

In fig. 9 the relation between B_φ^∇ and $\cos \varphi$ is graphically represented. The angle of inhomogeneousness φ_m to which corresponds the maximum value of the expression

$$E_\lambda^\nabla = \overline{E}_\lambda^\nabla \, e^{-(f\lambda^3 + g)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

giving the distribution of spectral energy in transmitted X-radiation can

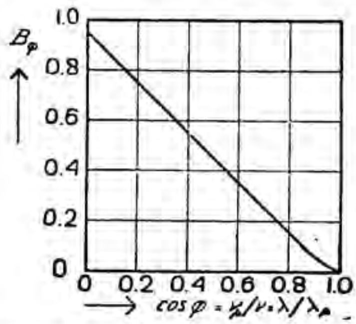

Fig. 9. B_φ, cos φ-relation for three phase alternating tension.

be found to satisfy the equations

$$f\lambda_p^3 = \cos^3\varphi_m \frac{\varphi_m \cot\varphi_m - \frac{1}{3}}{1-\varphi_m \cot\varphi_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26a)$$

and

$$f\lambda_p^3 = \cos^3\varphi_m \frac{\frac{\pi}{3}\cos\varphi_m - \frac{2}{3}}{1-\frac{\pi}{3}\cos\varphi_m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26b)$$

the first being valid when the resulting value of cos φ_m exceeds $\frac{1}{2}\sqrt{3} = 0.866$, the second when the resulting value of cos φ_m is smaller than 0.866; they are illustrated by the curve of fig. 10. We see from this

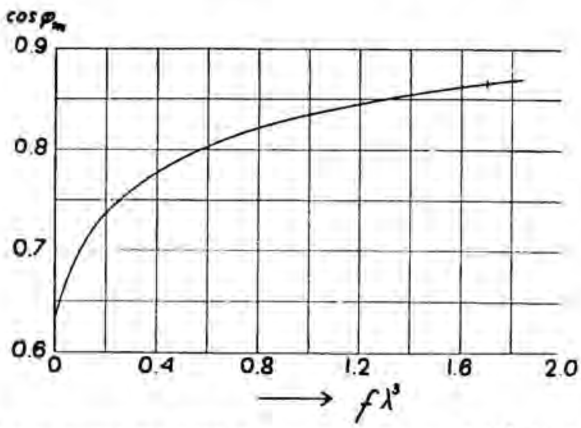

Fig. 10. Curve giving the relation between cos φ_m and $f\lambda_p^3$ for three phase alternating tension.

figure that to any value of $f\lambda_p^3$ corresponds only one value of cos φ_m. There is, therefore, no ambiguity in the definition of the angle of inhomogeneousness.

D. VILLARD'*s type of pulsating high tension.*

Form of the high tension curve.

Let us suppose that the electric energy consumed by an X-ray tube is small in comparison to the power of the apparatus in VILLARD-arrangement by which it is fed. The high tension acting on the tube then has the form schematically represented in fig. 11, and its course in a single period can

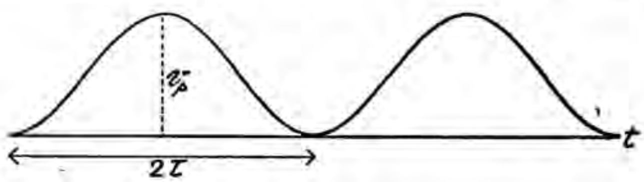

Fig. 11. Course of pulsating high tension of VILLARD's type.

be represented by the formula

$$V = \frac{V_p}{2}\left(1 + \cos\frac{\pi}{\tau}\,t\right)$$

where τ is again a half period. The tube current i will almost continually have its saturation value i_s and we will therefore base our calculations on the assumption that i is invariably $= i_s$.

Distribution of spectral energy.

Let $\overline{E}_{\lambda}^{\frown} d\lambda$ be the energy emitted every second in the range of wavelengths between λ and $\lambda + d\lambda$. If, then, we put as in (19)

$$\overline{E}_{\lambda}^{\frown} = \frac{Li}{\lambda_p} B_{\varphi}^{\frown} \frac{1}{\lambda^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

we can prove that

$$B_{\varphi}^{\frown} = \frac{1}{2\pi}(\sin\psi - \psi\cos\psi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where

$$\cos\psi = 2\cos\varphi - 1 \text{ and } \cos\varphi = \frac{\nu}{\nu_p} = \frac{\lambda_p}{\lambda} \quad . \quad . \quad . \quad . \quad (29)$$

In fig. 12 the relation between $B_{\varphi}^{\frown}$ and $\cos\varphi$ is graphically represented together with that between $B_{\varphi}^{=}$ and $\cos\varphi$.

If φ_m is the angle of inhomogeneousness to which corresponds the maximum value of the expression

$$E_{\lambda}^{\frown} = \overline{E}_{\lambda}^{\frown} e^{-(f\lambda^3 + g)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

giving the distribution of spectral energy in transmitted X-radiation and if

$$\cos\psi_m = 2\cos\varphi_m - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

the angle ψ_m can be found to satisfy the equation

$$f\lambda_p^3 = \frac{(\cos\psi_m + 1)^3}{24(\sin\psi_m - \psi_m\cos\psi_m)}\{\psi_m(1 + 3\cos\psi_m) - 2\sin\psi_m\}. \quad (32)$$

and the corresponding relation between $f\lambda_p^3$ and

$$\cos\varphi_m = \frac{1 + \cos\psi_m}{2}$$

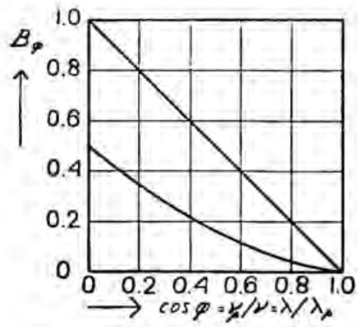

Fig. 12. B_φ, cos φ-relations: straight line for constant tension, curve for pulsating tension of VILLARD's type.

is graphically illustrated by the curve of fig. 13. With the help of this figure it is possible to determine cos φ_m and

$$\lambda_m = \frac{\lambda_p}{\cos\varphi_m}$$

in every case where C, d and $\lambda_p = \dfrac{12.3}{V_p}$ are known.

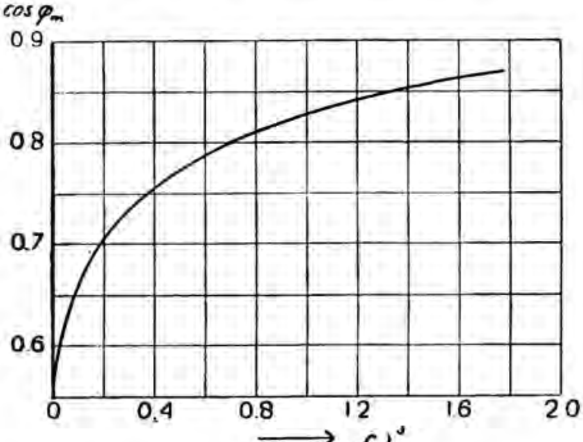

Fig. 13. Curve giving the relation between cos φ_m and $f\lambda_p^3$ for pulsating tension of VILLARD's type.

Physics. — *The intensity of scattered X-radiation in medical radiography*. I. By R. H. DE WAARD. (*X*-ray department of the Medical University Clinic, Utrecht.) (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of September 21, 1946.)

1. *On the importance of scattering of X-rays in medical radiography.*

The simplest arrangement used in medical radiography is schematically represented in fig. 1. The *X*-rays by which the image is formed start from the target *F* of a tube and reach the film *f* after having passed through the object *Ob*. Apart from these rays, however, the film is reached by scattered radiation which has its origin in every part of the object which is itself exposed to *X*-rays.

Scattered radiation obviously causes a fog on the film by which the contrasts of the image are lessened, and in many cases satisfactory results can be only obtained when special measures are taken to reduce it. The

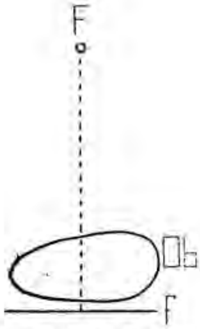

Fig. 1. Scheme of the simplest arrangement used in medical radiography. *F* focus of *X*-ray tube, Ob object, f film.

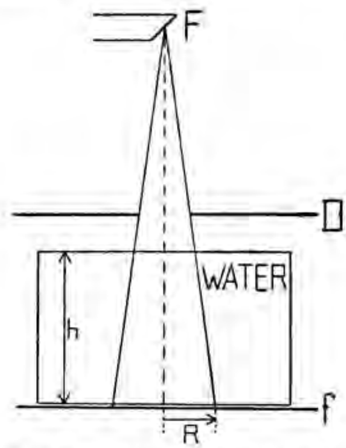

Fig. 2. Arrangement used by WILSEY in his measurements. *F* focus of *X*-ray tube, *D* circular diaphragm, f film, *R* radius of image.

simplest of these measures is the use of plane, conical or cylindrical diaphragms; the effect of these instruments is due to the restriction they impose on the quantity of direct radiation which passes through the object and by which scattering is brought about. They have, however, the drawback of not allowing larger parts of the object to be represented in a single image, and it is for this reason that in many instances preference is given to the use of grids which are to be placed between the object and the film.

At the present moment our knowledge of the quantity of scattered radiation leaving the object in the direction of the film is chiefly based on experiments carried out with water-sheets. As in most tissues X-rays behave in very much the same way as in water such experiments may give a reliable impression of what happens in radiographic practice. The oldest were made by WILSEY [1]) many years ago, and the sort of cases he investigated is represented in fig. 2. The distance of the target to the film was 50 cm, that between the water-sheet and the film 0.15 cm, and the thicknesses h of the sheets used were 5, 10, 15, 20 and 25 cm. Between the target and the sheets circular diaphragms could be placed of such sizes that the diameters of their images varied from 1.5 to 50 cm. The measurements concerned the ratio p of scattered to total radiation reaching the film when these different diaphragms were used, and some of the main results are represented by the curves of fig. 3. From these curves were deduced those of fig. 4 which give the ratio s of scattered to direct radia-

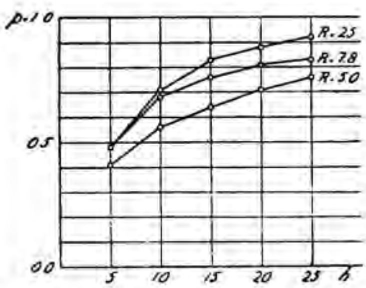

Fig. 3. Some results of WILSEY's measurements: relations between p and h corresponding to images of different sizes.

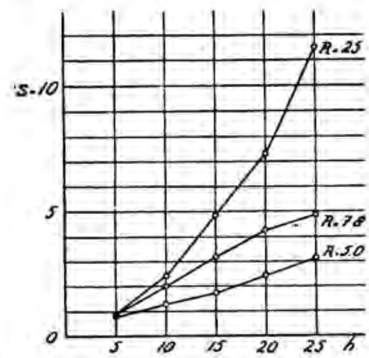

Fig. 4. Relations between s and h derived from the experimental data of fig. 3.

tion under the same conditions. As the curves refer to one single condition of tension only the information they give is rather incomplete. They do show, however, that with the use of large diaphragms a water-sheet of no more than 10 cm thickness may give rise to such an amount of scattered radiation that it reaches the film with more than double the intensity of the direct radiation, and it is therefore obvious that in many practical cases scattering may impair the contrasts to an extent as to make it necessary to reduce its influence by the measures mentioned above.

In the present paper the quantities of scattered radiation will be calculated which leave water-sheets of various thicknesses when incident beams of different sizes and different penetrating power are applied. For the sake of simplicity the calculations will be confined to cases in which the incident beams consist of parallel X-rays and have circular cross-sections.

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1) Amer. Jrn. of Roent., VIII, 1921, p. 328.

2. *Fundamentals of X-ray scattering;* J. J. THOMSON's *theory.*

When a beam of X-rays AB passes with intensity $D(v)$ through an element v of a waterfilled space the amount of energy it will lose there per second may be written

$$\mu D(v)\, v.$$

A part

$$\alpha D(v)\, v$$

of this energy is absorbed by the medium whilst the other part

$$\sigma D(v)\, v$$

is transformed into scattered radiation.

In medical radiography image formation is due to X-rays whose wavelengths are between 0.1 and 0.6 Å. For these rays the values of α and σ in water are approximately

$$\alpha = 2.5\,\lambda^3 \quad \text{and} \quad \sigma = 0.18$$

and we have, consequently,

$$\mu = 2.5\,\lambda^3 + 0.18. \qquad (1)$$

The mathematical developments to be given in this paper will be based on J. J. THOMSON's classical theory of X-ray scattering. According to this theory the quantity of radiation scattered by v per steradian and per second in a direction which makes an angle θ with that of the beam AB (fig. 5) is expressed by the formula

$$\frac{3}{16\,\pi}\,\sigma D(v)\, v\,(1 + \cos^2\theta). \qquad (2)$$

In the range of wavelengths used in medical radiography this consequence is in satisfactory agreement with facts. Another consequence, however, viz. that the wavelength of scattered radiation is the same as that of the original beam, was shown by COMPTON to be incorrect; a part of the radiation due to scattering has wavelengths λ' which are somewhat larger, the difference varying between 0 and 0.0484 Å. In § 11 the COMPTON effect will be taken into account in the interpretation of the results of our calculations.

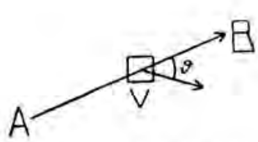

Fig. 5. Beam AB of X-rays passing through a volume v where scattering occurs with different angles θ of deviation.

3. *Infinite beam falling perpendicularly on a watersheet: calculation of intensity of secondary radiation.*

Consider a plane sheet of water extending to infinity in all horizontal directions, and let its surface be exposed to an equally infinite homogeneous beam of vertical X-rays with one single wavelength λ. It is then comparatively easy to calculate the intensity with which secondary radiation (scattered radiations of higher orders being *not* included) will leave the sheet in a downward direction. Let D be the intensity with which the incident radiation reaches the bottom of the sheet; in a plane a distance x higher inside the sheet the intensity will then amount to

$$D(x) = D e^{\mu x}.$$

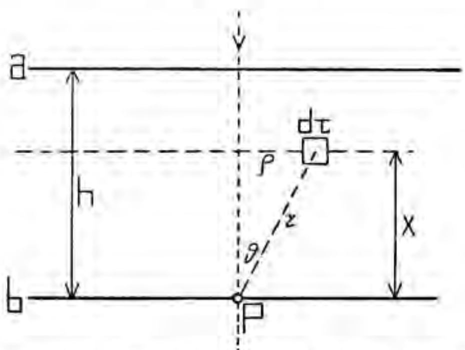

Fig. 6. Horizontal watersheet exposed to vertical X-rays forming a beam of infinite extent. *a* surface, *b* bottom of the watersheet.

Secondary radiation arriving in a point P of the bottom (fig. 6) obviously has its origin in every part of the sheet. The intensity in such a point will therefore consist of contributions due to different levels. Now the contribution due to a lamella with thickness dx situated in a level x will be proportional to dx and to the local intensity $D(x)$ of direct radiation; it may therefore be written

$$\Phi D(x)\, dx$$

where Φ is a quantity depending on μ and x. A calculation shows that this quantity is a function of the product $u = \mu x$; denoting it by $\Phi(u)$ we have

$$\Phi(u) = \frac{3\sigma}{8} \left\{ \left(1 + \frac{u^2}{2}\right) H(u) + \tfrac{1}{2}(1-u)\, e^{-u} \right\} \quad \ldots \ldots \quad (3)$$

where

$$H(u) = \int_u^\infty \frac{e^{-z}}{z}\, dz. \quad \ldots \ldots \ldots \ldots \quad (4)$$

The values of $H(u)$ can for different values if u be found in a table of the exponential integral; formula (3) then provides the corresponding values of $\Phi(u)$. The resulting relation between $\Phi(u)$ and u is graphically represented by the small circles in fig. 7.

Formula (3) is obtained as follows. If $d\tau$ is an element of the lamella

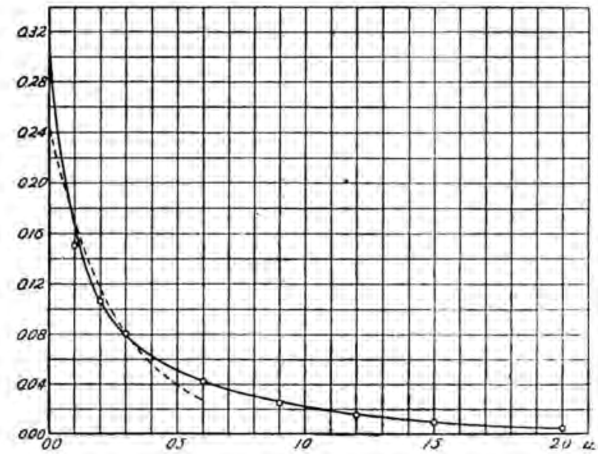

Fig. 7. Graphic representation of the functions $\Psi_1(u) = 0.242\, e^{-3.63}$ (dotted curve) and $\Psi_2(u) = 0.2\, e^{-10u} + 0.108\, e^{-1.53\, u}$ (full curve). The small circles show the course of the function $\Phi(u)$.

considered the secondary radiation originating there will arrive in P with an intensity given by the expression

$$\frac{3}{16\,\pi}\,\sigma\, D\,(x)\, d\tau\,(1 + \cos^2\theta)\,\frac{e^{-\mu r}}{r^2}$$

in which the symbols θ and r have the meanings indicated in fig. 6. Integrating this expression over all elements contained in the lamella we find

$$\frac{3}{16\,\pi}\,\sigma\, D\,(x)\, dx \int_{\varrho=0}^{\varrho=\infty} (1 + \cos^2\theta)\,\frac{e^{-\mu r}}{r^2}\; 2\,\pi\,\varrho\; d\varrho$$

and this expression can be transformed into

$$\frac{3\sigma}{8}\, D\,(x)\, dx \int_{z=\mu x}^{z=\infty} \left(1 + \frac{(\mu\, x)^2}{z^2}\right) e^{-z}\,\frac{dz}{z}.$$

Hence

$$\Phi\,(u) = \frac{3\sigma}{8} \int_{z=u}^{z=\infty} \left(1 + \frac{u^2}{z^2}\right) e^{-z}\,\frac{dz}{z}$$

and as

$$\int_{u}^{\infty} \frac{e^{-z}}{z^3}\, dz = -\left[\frac{e^{-z}}{2\, z^2}\,(1 - z)\right]_{z=u}^{z=\infty} + \tfrac{1}{2}\, H\,(u)$$

this formula for $\Phi(u)$ is equivalent to (3).

The intensity of secondary radiation in P is now given by the expression

$$\int_0^h \Phi(\mu x)\, D(x)\, dx \quad \ldots\ldots\ldots\ldots (5)$$

where h is the thickness of the watersheet.

4. *Approximations of the function* $\Phi(u)$.

We see from fig. 7 that the course of the function $\Phi(u)$ is to a certain extent analoguous to that of functions of the form

$$k\, e^{-\gamma u} \quad \ldots\ldots\ldots\ldots (6)$$

where k and γ are positive numbers. There is, however, an essential difference: $\Phi(u)$ tends to infinity when u tends to zero. Nevertheless $\Phi(u)$ can in any region between 0 and u be reasonably approximated by a function of the form (6) or by the sum of a number of such functions.

When the region in question is small one function is sufficient for many purposes. If, for instance, $\Phi(u)$ is to be represented in the region between $u = 0$ and $u = 0.3$ we can obtain applicable values for k and γ by solving the equations

$$k\, e^{-0.3\gamma} = \Phi(0.3) = 0.0814$$

and

$$\int_0^{0.3} k\, e^{-\gamma u}\, du = \frac{k}{\gamma}\left(1 - e^{-0.3\gamma}\right) = \int_0^{0.3} \Phi(u)\, du = 0.0443$$

the result being

$$k = 0.242 \qquad \gamma = 3.63.$$

The corresponding relation between

$$\Psi_1(u) = 0.242\, e^{-3.63u} \quad \ldots\ldots\ldots (7_1)$$

and u is shown by the dotted curve of fig. 7.

An approximation of $\Phi(u)$ valid in a much greater region is given by the function

$$\Psi_2(u) = 0.200\, e^{-10u} + 0.108\, e^{-1.53u} \quad \ldots\ldots (7_2)$$

which is graphically represented by the full curve. The constants figuring in this function have been so adjusted that

$$\Phi(0.6) = \Psi_2(0.6) \quad \text{and} \quad \int_0^{0.6} \Phi(u)\, du = \int_0^{0.6} \Psi_2(u)\, du;$$

moreover, the curves make it clear that a number between 1.5 and ∞ exists by which the upper limit 0.6 can be replaced without the validity of the latter formula being affected. The function $\Psi_2(u)$ therefore gives a good approximation of $\Phi(u)$ in a region which extends from 0 to over 2.0.

The substitution of $\Phi(u)$ by one of the approximating functions $\Psi_1(u)$ or $\Psi_2(u)$ considerably simplifies the evaluation of the expression (5) for the intensity of secondary radiation in P. More important, however, is that it enables us to calculate the total intensity of all scattered radiation leaving the bottom of the sheet, that is to say of scattered radiations of all orders and not of secondary radiation only.

5. *Integral equation for the intensity of scattered radiation.*

It is obvious that inside the watersheet of fig. 6 scattered radiation will propagate in all possible upward and downward directions. We will now denote by $S(x)$ the total intensity of scattered radiation in points in the level x. This intensity consists of contributions due to different directions, and the steeper a direction is the larger its contribution will be. It is therefore natural to assume that the amount of scattered radiation originating in an element $d\tau$ of the sheet will be distributed over the various directions in a way which is approximately that given by THOMSON's formula (2). If this is true $S(x)$ will be the sum of contributions

$$\{D(y) + S(y)\}\, \Phi\{\mu(y-x)\}\, dy$$

due to lamellae of thickness dy in which the space between the levels x and h can be subdivided, and of contributions

$$\{D(y) + S(y)\}\, \Phi\{\mu(x-y)\}\, dy$$

corresponding to lamellae between the levels 0 and x. Hence

$$S(x) = \int_x^h \{D e^{\mu y} + S(y)\}\, \Phi\{\mu(y-x)\}\, dy + \int_0^x \{D e^{\mu y} + S(y)\}\, \Phi\{\mu(x-y)\}\, dy \quad (8)$$

and from this integral equation the function $S(x)$ can be derived.

Integral equations of the sort (8) also occur in the theory of scattering of light in gaseous media. An equation very similar to (8) was obtained by L. V. KING in 1912 [2]). In his discussion of the equation he gives an approximate solution for thin sheets. It will be shown in the following sections that by substituting for Φ expressions as given by (7_1) and (7_2) more general solutions can be found in a comparatively simple way.

6. *Solution of the integral equation for thin sheets (first approximation).*

If in the integral equation (8) we substitute for Φ the approximating function Ψ_1 we find

$$S(x) = k \int_x^h \{D e^{\mu y} + S(y)\}\, e^{-\gamma\mu(y-x)}\, dy + k \int_0^x \{D e^{\mu y} + S(y)\}\, e^{-\gamma\mu(x-y)}\, dy. \quad (9)$$

From the latter equation we can deduce the following differential equation for $S(x)$:

$$\frac{d^2 S}{dx^2} + \{2k(\gamma\mu) - (\gamma\mu)^2\} = -2k(\gamma\mu)\, D e^{\mu x}. \quad . \quad . \quad . \quad (10)$$

[2]) Absorption of light in gaseous media. Phil. Trans. London. A 211, 1912, p. 375.

The way in which this deduction can be performed may be briefly indicated. If we put

$$D\, e^{\mu y} + S(y) = \varphi(y)$$

we may write for the equation (9)

$$S(x) = k\, e^{\gamma\mu x} \int_x^h \varphi(y)\, e^{-\gamma\mu y}\, dy + k e^{-\gamma\mu x} \int_0^x \varphi(y)\, e^{\gamma\mu y}\, dy.$$

Multiplication by $e^{-\gamma\mu x}$ and subsequent differentiation now gives

$$\frac{d}{dx}\{e^{-\gamma\mu x}\, S(x)\} = -k\,\varphi(x)\, e^{-\gamma\mu x} + k \frac{d}{dx}\left\{ e^{-2\gamma\mu x} \int_0^x \varphi(y)\, e^{\gamma\mu y}\, dy \right\}$$

and this equation can be transformed into

$$\left(-\gamma\mu + \frac{d}{dx}\right) S(x) = -k\, e^{-\gamma\mu x}\, 2\gamma\mu \int_0^x \varphi(y)\, e^{\gamma\mu y}\, dy.$$

In this way one of the two integrals in the right hand member of (9) is removed. The other integral can be removed in a similar way, viz. by multiplication with $e^{\gamma\mu x}$ and subsequent differentiation, and the result is exactly the equation (10).

The solution of the differential equation (10) may be written

$$S(x) = D\,(F e^{\mu x} + G_1\, e^{\xi x} + G_2\, e^{-\xi x}) \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where

$$F = \frac{m}{1-m} \quad \text{with} \quad m = \frac{2k(\gamma\mu)}{(\gamma\mu)^2 - \mu^2} \quad . \quad . \quad . \quad . \quad . \quad (11a)$$

and

$$\xi = \sqrt{(\gamma\mu)^2 - 2k(\gamma\mu)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11b)$$

The constants G_1 and G_2 can be determined by substitution in (9) of the expression for $S(x)$ resulting from (11), (11a) and (11b); by equating corresponding coefficients we then find the equations

$$\begin{aligned} \varepsilon_1 (1+F)\, e^{\mu h} + \alpha_1\, G_1\, e^{\xi h} + \alpha_2\, G_2^{-\xi h} &= 0 \\ \varepsilon_2 (1+F) \quad + \alpha_2\, G_1 \quad + \alpha_1\, G_2 \quad &= 0 \end{aligned}$$

where

$$\begin{aligned} \varepsilon_1 &= (\mu\gamma - \mu)^{-1} \qquad & \alpha_1 &= (\mu\gamma - \xi)^{-1} \\ \varepsilon_2 &= (\mu\gamma + \mu)^{-1} \qquad & \alpha_2 &= (\mu\gamma + \xi)^{-1}. \end{aligned}$$

From these equations G_1 and G_2 can be calculated. For the purpose of the present paper, however, only their sum

$$G = G_1 + G_2$$

is of interest, and this quantity is given by the expression

$$G = (1 + F)\,\frac{(\varepsilon_1 a_2 - \varepsilon_1 a_1)\, e^{\mu h} - \varepsilon_2 a_1 e^{\xi h} + \varepsilon_2 a_2 e^{-\xi h}}{a_1^2 e^{\xi h} - a_2^2 e^{-\xi h}}$$

Now let the film be in immediate contact with the watersheet [3]. The intensity S of scattered radiation to which it is exposed is then obviously $S(0)$. Hence

$$S = S(0) = (F + G)\,D = D\left\{\frac{m}{1-m} + \frac{1}{1-m}\,\frac{\varepsilon_1 (a_2 - a_1)\, e^{\mu h} - \varepsilon_2 a_1 e^{\xi h} + \varepsilon_2 a_2 e^{-\xi h}}{a_1^2 e^{\xi h} - a_2^2 e^{-\xi h}}\right\}.$$

This expression for S is of the form

$$S = D\,\frac{A\, e^{ah} + B + C_1 e^{-ch}}{1 - C_2 e^{-ch}} \quad \ldots\ldots \quad (12)$$

where the quantities A, a, B, C_1, C_2, c are independent of h. These quantities can be conveniently calculated by means of the following set of formulae:

$$\left.\begin{array}{l}
k = 0.242;\quad \gamma = 3.63;\quad m = \dfrac{2k(\gamma\mu)}{(\gamma\mu)^2 - \mu^2};\quad \xi = \sqrt{(\gamma\mu)^2 - 2k(\gamma\mu)} \\
\qquad\qquad e_1 = \mu\gamma - \mu \qquad a_1 = \mu\gamma - \xi \\
\qquad\qquad e_2 = \mu\gamma + \mu \qquad a_2 = \mu\gamma + \xi \\
A = \dfrac{1}{1-m}\,\dfrac{a_1}{e_1}\left(-1 + \dfrac{a_1}{a_2}\right);\quad B = \dfrac{m}{1-m} - \dfrac{1}{1-m}\,\dfrac{a_1}{e_2};\ a = \mu - \xi \\
C_1 = \dfrac{-m}{1-m}\left(\dfrac{a_1}{a_2}\right)^2 + \dfrac{1}{1-m}\,\dfrac{a_1}{e_2}\,\dfrac{a_1}{a_2};\quad C_2 = \left(\dfrac{a_1}{a_2}\right)^2;\ c = 2\xi
\end{array}\right\} \quad . \quad (13)$$

Applying these formulae to the cases $\mu = 0.20$ and $\mu = 0.30$ we find

$$S = D\,\frac{-1.53\, e^{-0.219 h} + 1.40 + 0.13\, e^{-0.838 h}}{1 - 0.07\, e^{-0.838 h}} \quad \text{for } \mu = 0.20$$

and

$$S = D\,\frac{-0.578\, e^{-0.512 h} + 0.542 + 0.036\, e^{-1.624 h}}{1 - 0.02\, e^{-1.624}} \quad \text{for } \mu = 0.30 \qquad (14)$$

The relations between $\dfrac{S}{D}$ and h given by these formulae are graphically represented in fig. 8.

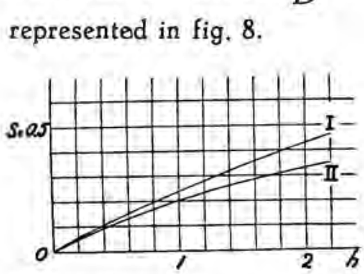

Fig. 8. Theoretical s,h-curves valid for small values of h ($h < 3$ cm.) and corresponding to infinite incident beams of X-rays: I for $\mu = 0.20$, II for $\mu = 0.30$.

[3]) In the cases under consideration the resulting formulae for S are valid for any distance between the watersheet and the film. The formulae to be obtained in § 9, however, which apply to cases with limited incident beams, only hold if film and watersheet are in immediate contact.

7. *Solution of the integral equation for thicker sheets (second approximation).*

We will now consider the integral equation which is obtained when in (8) the function $\Phi(u)$ is replaced by an approximating function of the form

$$\Psi_2(u) = k_1 e^{-\gamma_1 u} + k_2 e^{-\gamma_2 u}.$$

This new equation can be solved in very much the same way as the equation (9); the resulting expression for S is, however, much more complicated. For the sake of completeness we will first give this solution in full.

Starting from the data

$$k_1 = 0.200 \qquad \gamma_1 = 10 \qquad k_2 = 0.108 \qquad \gamma_2 = 1.53$$

we can for any given value of μ calculate in succession the following quantities:

$$\left.\begin{aligned} m &= \frac{2k_1(\mu\gamma_1)}{(\mu\gamma_1)^2-\mu^2} + \frac{2k_2(\mu\gamma_2)}{(\mu\gamma_2)^2-\mu^2} \\ p &= (\mu\gamma_1)^2 + (\mu\gamma_2)^2 - 2k_1(\mu\gamma_1) - 2k_2(\mu\gamma_2) \\ q &= (\mu\gamma_1)^2(\mu\gamma_2)^2\left(1 - \frac{2k_1}{\mu\gamma_1} - \frac{2k_2}{\mu\gamma_2}\right) \\ \xi &= \sqrt{\tfrac{1}{2}p - \sqrt{\tfrac{1}{4}p^2 - q}} \\ \eta &= \sqrt{\tfrac{1}{2}p + \sqrt{\tfrac{1}{4}p^2 - q}} \end{aligned}\right\} \quad . \quad . \quad . \quad (15)$$

$$\left.\begin{aligned} \varepsilon_1 &= (\mu\gamma_1 - \mu)^{-1} & \alpha_1 &= (\mu\gamma_1 - \xi)^{-1} & \beta_1 &= (\mu\gamma_1 - \eta)^{-1} \\ \varepsilon_2 &= (\mu\gamma_2 - \mu)^{-1} & \alpha_2 &= (\mu\gamma_2 - \xi)^{-1} & \beta_2 &= (\mu\gamma_2 - \eta)^{-1} \\ \varepsilon_3 &= (\mu\gamma_1 + \mu)^{-1} & \alpha_3 &= (\mu\gamma_1 + \xi)^{-1} & \beta_3 &= (\mu\gamma_1 + \eta)^{-1} \\ \varepsilon_4 &= (\mu\gamma_2 + \mu)^{-1} & \alpha_4 &= (\mu\gamma_2 + \xi)^{-1} & \beta_4 &= (\mu\gamma_2 + \eta)^{-1} \end{aligned}\right\} . \quad (16)$$

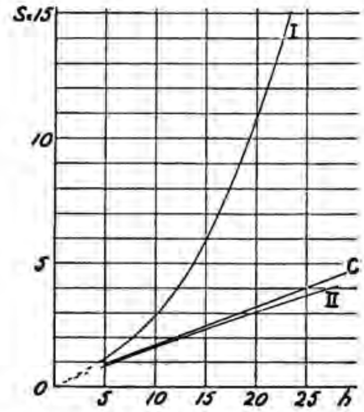

Fig. 9. Theoretical s, h-curves corresponding to infinite incident beams of X-rays and valid for greater values of h: I for $\mu = 0.20$, II for $\mu = 0.30$.

$$
\left.\begin{aligned}
M^{\mu\xi} &= \begin{vmatrix} \varepsilon_1 & \alpha_1 \\ \varepsilon_2 & \alpha_2 \end{vmatrix} \left\{ \begin{vmatrix} \alpha_1 & \beta_3 \\ \alpha_2 & \beta_4 \end{vmatrix} + \begin{vmatrix} \beta_3 & \beta_1 \\ \beta_4 & \beta_2 \end{vmatrix} + \begin{vmatrix} \beta_1 & \alpha_1 \\ \beta_2 & \alpha_2 \end{vmatrix} \right\} \\
M^{\mu}_{\xi} &= \begin{vmatrix} \varepsilon_1 & \alpha_3 \\ \varepsilon_2 & \alpha_4 \end{vmatrix} \left\{ \begin{vmatrix} \alpha_3 & \beta_1 \\ \alpha_4 & \beta_2 \end{vmatrix} + \begin{vmatrix} \beta_1 & \beta_3 \\ \beta_2 & \beta_4 \end{vmatrix} + \begin{vmatrix} \beta_3 & \alpha_3 \\ \beta_4 & \alpha_4 \end{vmatrix} \right\} \\
M^{\mu\eta} &= \begin{vmatrix} \varepsilon_1 & \beta_1 \\ \varepsilon_2 & \beta_2 \end{vmatrix} \left\{ \begin{vmatrix} \beta_1 & \alpha_3 \\ \beta_2 & \alpha_4 \end{vmatrix} + \begin{vmatrix} \alpha_3 & \alpha_1 \\ \alpha_4 & \alpha_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \right\} \\
M^{\mu}_{\eta} &= \begin{vmatrix} \varepsilon_1 & \beta_3 \\ \varepsilon_2 & \beta_4 \end{vmatrix} \left\{ \begin{vmatrix} \beta_3 & \alpha_1 \\ \beta_4 & \alpha_2 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{vmatrix} + \begin{vmatrix} \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{vmatrix} \right\} \\
M^{\xi\eta} &= \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \left\{ \begin{vmatrix} \alpha_1 & \varepsilon_3 \\ \alpha_2 & \varepsilon_4 \end{vmatrix} + \begin{vmatrix} \varepsilon_3 & \beta_1 \\ \varepsilon_4 & \beta_2 \end{vmatrix} \right\}; \quad M^{\xi}_{\eta} = -\begin{vmatrix} \alpha_1 & \beta_3 \\ \alpha_2 & \beta_4 \end{vmatrix} \left\{ \begin{vmatrix} \alpha_1 & \varepsilon_3 \\ \alpha_2 & \varepsilon_4 \end{vmatrix} + \begin{vmatrix} \varepsilon_3 & \beta_3 \\ \varepsilon_4 & \beta_4 \end{vmatrix} \right\} \\
M^{\eta}_{\xi} &= -\begin{vmatrix} \alpha_3 & \beta_1 \\ \alpha_4 & \beta_2 \end{vmatrix} \left\{ \begin{vmatrix} \alpha_3 & \varepsilon_3 \\ \alpha_4 & \varepsilon_4 \end{vmatrix} + \begin{vmatrix} \varepsilon_3 & \beta_1 \\ \varepsilon_4 & \beta_2 \end{vmatrix} \right\}; \quad M_{\xi\eta} = \begin{vmatrix} \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{vmatrix} \left\{ \begin{vmatrix} \alpha_3 & \varepsilon_3 \\ \alpha_4 & \varepsilon_4 \end{vmatrix} + \begin{vmatrix} \varepsilon_3 & \beta_3 \\ \varepsilon_4 & \beta_4 \end{vmatrix} \right\} \\
M_0 &= \begin{vmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{vmatrix} \left\{ \begin{vmatrix} \beta_1 & \varepsilon_3 \\ \beta_2 & \varepsilon_4 \end{vmatrix} + \begin{vmatrix} \varepsilon_3 & \beta_3 \\ \varepsilon_4 & \beta_4 \end{vmatrix} \right\} + \begin{vmatrix} \beta_1 & \beta_3 \\ \beta_2 & \beta_4 \end{vmatrix} \left\{ \begin{vmatrix} \alpha_1 & \varepsilon_3 \\ \alpha_2 & \varepsilon_4 \end{vmatrix} + \begin{vmatrix} \varepsilon_3 & \alpha_3 \\ \varepsilon_4 & \alpha_4 \end{vmatrix} \right\} \\
N^{\xi\eta} &= -\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}^2; \quad N^{\xi}_{\eta} = \begin{vmatrix} \alpha_1 & \beta_3 \\ \alpha_2 & \beta_4 \end{vmatrix}^2 \\
N^{\eta}_{\xi} &= \begin{vmatrix} \alpha_3 & \beta_1 \\ \alpha_4 & \beta_2 \end{vmatrix}^2; \quad N_{\xi\eta} = -\begin{vmatrix} \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \end{vmatrix}^2 \\
N_0 &= -2 \begin{vmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{vmatrix} \cdot \begin{vmatrix} \beta_1 & \beta_3 \\ \beta_2 & \beta_4 \end{vmatrix}
\end{aligned}\right\} \quad . \quad . \quad (17)
$$

If, then, we put

$$
\left.\begin{aligned}
\triangle = M^{\mu\xi} e^{(\mu+\xi)h} + M^{\mu}_{\xi} e^{(\mu-\xi)h} + M^{\mu\eta} e^{(\mu+\eta)h} + M^{\mu}_{\eta} e^{(\mu-\eta)h} + \\
+ M^{\xi\eta} e^{(\lambda+\eta)h} + M^{\eta}_{\xi} e^{(-\xi+\eta)h} + M^{\xi}_{\eta} e^{(\xi-\eta)h} + M_{\xi\eta} e^{(-\xi-\eta)h} + M_0
\end{aligned}\right\} \quad (18a)
$$

and

$$
\triangle_0 = N^{\xi\eta} e^{(\xi+\eta)h} + N^{\eta}_{\xi} e^{(-\xi+\eta)h} + N^{\xi}_{\eta} e^{(\xi-\eta)h} + N_{\xi\eta} e^{(-\xi-\eta)h} + N_0 \quad (18b)
$$

the quantity S is given by the formula

$$
S = D \frac{\triangle + m \triangle_0}{(1-m) \triangle_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)
$$

The expressions for $\triangle$ and $\triangle_0$ consisting of 9 and 5 terms respectively this result is complicated enough. Numerical calculations show, however, that in cases of watersheets thicker than 3 cm most of these terms are so small that they may be neglected, and so the formula for S assumes the same simple form as in the case treated in § 6, viz.

$$
S = D \frac{A e^{ah} + B + C_1 e^{-ch}}{1 - C_2 e^{-ch}}.
$$

The constants A, a, B, C_1, C_2, c can be approximated by means of the formulae

$$\left.\begin{array}{lll} e_1 = \mu\gamma_1 - \mu & a_1 = \mu\gamma_1 - \xi & b_1 = \mu\gamma_1 - \eta \\ e_2 = \mu\gamma_2 - \mu & a_2 = \mu\gamma_2 - \xi & b_2 = \mu\gamma_2 - \eta \\ e_3 = \mu\gamma_1 + \mu & a_3 = \mu\gamma_1 + \xi & - \\ e_4 = \mu\gamma_2 + \mu & a_4 = \mu\gamma_2 + \xi & - \end{array}\right\} \quad . \quad . \quad (19)$$

$$\left.\begin{array}{l} A = \dfrac{-1}{1-m}\left\{1 + \dfrac{b_1}{b_2}\left(2\,\dfrac{a_2}{a_1} - \dfrac{e_2}{e_1}\right)\right\}\left\{\dfrac{a_2\,b_2}{e_2}\left(\dfrac{a_2}{a_1\,a_4} - \dfrac{1}{a_3}\right) + \dfrac{a_2}{e_2}\left(1 - \dfrac{a_2}{a_4}\right)\right\} \\ a = \mu - \xi \\ B = \dfrac{m}{1-m} - \dfrac{1}{1-m}\left(1 + \dfrac{a_2\,b_1}{a_1\,b_2}\right)\left\{\dfrac{b_1}{e_3}\left(1 - \dfrac{a_2}{b_2}\right) - \dfrac{a_2\,b_1}{e_4\,a_1} + \dfrac{a_2}{e_4}\right\} \\ C_1 = -\dfrac{m}{1-m}\left(\dfrac{a_2}{a_4}\right)^2 + \dfrac{1}{1-m}\,\dfrac{a_2\,a_2}{a_4\,e_4} \\ C_2 = \left(\dfrac{a_2}{a_4}\right)^2 \\ c = 2\xi \end{array}\right\} \quad (20)$$

where m, ξ and η are the quantities given by the set of formulae (15). Applying these formulae to the cases $\mu = 0.20$ and $\mu = 0.30$ we find the following expressions for S:

$$S = D\,\frac{1.99\,e^{0.095\,h} - 2.26 + 0.34\,e^{-0.210\,h}}{1 - 0.24\,e^{-0.210\,h}} \quad \text{for } \mu = 0.20.$$

and

$$S = D\,\frac{-16.2\,e^{-0.010\,h} + 16.2 + 0.05\,e^{-0.620\,h}}{1 - 0.04\,e^{-0.620\,h}} \quad \text{for } \mu = 0.30.$$

The corresponding relations between $\dfrac{S}{D}$ and h are graphically represented by the fully drawn curves I and II of fig. 9; the dotted parts by which these curves are completed are derived from data given by fig. 8.

Botany. — *The effect of phenoxycompounds on waterplants.* By G. L. FUNKE. (From the Botanical Institute, Government University, Ghent.) (Communicated by Prof. W. H. ARISZ.)

(Communicated at the meeting of September 21, 1946.)

It is only a short time since phenoxycompounds have become known as plant growth regulators, but their remarkable and manifold effects have created quite a sensation among plant physiologists. They are applied as weed killers (7, 8, 11, 13), for inducing parthenocarpy (5, 6, 10, 15), different correlative and formative effects (1, 12, 14, 15, 16), inhibiting of seed germination (4), etc.

ZIMMERMAN and HITCHCOCK (15*b*) describe very peculiar modifications of leaves due to a treatment with several phenoxycompounds. They reminded me of the effect of α-naphtylacetic acid and indol compounds on the leaves of waterlilies which I observed in former experiments (2 *e*). In Nymphaeaceae we see a succession of different leaf forms, a more or less gradual transition from the primary sagittate leaves, which remain submerged under all circumstances, towards the secondary, round, floating leaves. In my experiments the secundary floating leaves became sagittate under influence of the hormones above mentioned (in a concentration of 3 mg/L) and therefore I called it a reversal to the primary form. I was anxious to see whether phenoxycompounds have a similar effect and therefore I made some experiments with 4-chlorophenoxyacetic acid and 2,4-dichlorophenoxyacetic acid. The plant material was grown in aquaria of 40 cm deep which were kept in hothouses. The apparatuses of my laboratory having been neglected in my absence from Ghent during the war and nobody being available to put them in working order, I was dependent on the temperature of these hothouses (20°—24°). The summer, unfortunately, was very cool, and consequently I could not keep the water at optimal temperatures. The results, therefore, cannot be accurately compared with those obtained with other growth regulators, but nevertheless may be considered sufficiently interesting to justify this report.

Nymphaea odorata. Young specimens were transplanted from a shallow pond into aquariums, each specimen separately. After 8 or 10 days the plants had recovered and had formed new leaves; then the substances were added. In the solutions of 2,4-dichlorophenoxyacetic acid the reaction became visible after 1, 2 or 3 days; the petioles showed a renewal of growth with the consequence that they soon meandered on the surface. The reaction was far from being as explosive as with α-naphtylacetic acid (2 *f, g, h, i, j, k, l, m*), but a constant elongation was maintained during a more or less large number of days. The maximum growth per day did not exceed 25 cm, whereas in solutions of α-naphtylacetic acid it can be 50, 60, 70 and even 90 cm per day. But here especially we should not forget that my plants had not the occasion to grow at a temperature of 30°. I could confirm the observations in my former experiments, viz. that only those leaves reacted strongly which were neither very young nor quite adult; leaves which at the beginning of the experiment had reached about half the height of the aquarium, therefore with petioles about 20 cm long and with blades still folded, showed the best reaction. Three experiments gave identical results; in table 1 a selection is given of the ultimate lengths reached in the different concentrations.

TABLE 1.
Ultimate lengths of the petioles of Nymphaea odorata in solutions of phenoxycompounds.

Subst.	Conc. in mg/L	Ultimate lengths in cm	Reached after-days	Condition of plants at end of exp.
Control		45— 52— 55	36	good
2,4-dichl.	$\frac{1}{4}$	178—185—209	29	good
	$\frac{1}{2}$	223—238—262	29	some decaying
	1	198—221—225	20	decaying
	2	185—196—258	17	decaying
4-chlor.	$\frac{1}{2}$	65— 75— 90	36	good
	1	65— 80—100	36	good
	2	111—113—165	36	good

The response in the solutions of 2,4-dichlorophenoxyacetic acid 1 and 2 mg/L was rapid, but came soon to an end because blades and petioles began to decay; not over their whole surface or length, but in definite places. The optimal concentration for longitudinal growth must be somewhat below $\frac{1}{2}$ mg/L, which is higher than for α-naphtylacetic acid where it is between $\frac{1}{8}$ and $\frac{1}{4}$ mg/L for most Nymphaeaceae. The response to 4-chlorophenoxyacetic acid is much slower and its effect is much less; up till now I have not had the occasion to determine its optimal concentration.

The young leaves which at the beginning of the experiment had not yet unfolded their blades, kept them in the rolled-up position; the youngest leaves with outspread blades showed a hyponastic rolling-up; the older ones did not react any more. Otherwise no formative effects of the substances could be observed. Higher concentrations are perhaps needed for phenomena of reversal to the primitive leaf form, but they would probably inhibit any development owing to their noxious effect.

Nymphaea terminalis — N. Devoniensis. The specimens were taken from material in the Victoria basin; the experiments were made twice; the results are summarized in table 2.

We see again that, at least as far as N. terminalis is concerned, 4-chlorophenoxyacetic acid has hardly any influence in a concentration in which 2,4-dichlorophenoxyacetic acid is very effective. The specimen of N. Devoniensis in 2,4-dichlorophenoxyacetic acid 1 mg/L of 293 cm in-

TABLE 2.
Ultimate lengths of the petioles of Nymphaea Devoniensis and N. terminalis in solutions of phenoxycompounds.

Subst.	Conc. in mg/L	Ultimate lengths in cm	
		N. Devoniensis	N. terminalis
Control		45— 50— 65	45— 60 —70
2,4-dichl.	$\frac{1}{4}$	65— 70— 71	81— 87— 90
	$\frac{1}{2}$	80— 82— 90	140—168—186
	1	125—150—192—293	166—195—203
4-chlor.	$\frac{1}{2}$	70— 72— 75	78— 82— 90

dicates that a very strong reaction is possible, but during these experiments it was shown in this case only.

The reaction of the leafblades was very feeble; only those which had not yet reached the surface at the beginning of the experiment kept their blades in rolled-up position.

Limnanthemum nymphaeoides. The material was taken from a very shallow pond outdoors. The petioles did not measure more than 10—12 cm; when planted in the aquaria of 40 cm, these petioles started a new growth immediately and after 24—36 hours the blades were floating again. This proves that the plants were in excellent condition (comp. 2 *d, g*: 3). The experiments were made twice, the results can be seen in table 3.

TABLE 3.
Ultimate lengths of the petioles of Limnanthemum nymphaeoides in solutions of phenoxycompounds.

Subst.	Conc. in mg/L	Ultimate lengths in cm	Reached after-days	Condition of plants at end of exper.
Control		50— 57— 60	16	good
2,4-dichl.	$\frac{1}{2}$	83— 83—132	10	poor
	1	149—152—180	11	poor
	2	85— 90—100	5	decaying
4-chlor.	1	85—100—119	16	poor

The concentration 2 mg/L of 2,4-dichlorophenoxyacetic acid is distinctly noxious; after no more than 5 days the experiment had to be discontinued because the petioles were hardly measurable owing to their strong coiling; they broke easily and were partly decaying. 1 mg/L and even $\frac{1}{2}$ mg/L are not favorable either. The same phenomena were to be seen in the solution of 4-chlorophenoxyacetic acid, although to a minor degree. The effect of the latter substance is again less than that of 2,4-dichlorophenoxyacetic acid, but the difference of effect between these two acids is not so great in this case as for the three species of Nymphaea.

Although both phenoxycompounds appear to be noxious to Limnanthemum, its leafblades showed no abnormalities save the usual hyponastic bending.

Some dozens of petioles of every species were examined as to their cell-lengths. These measurements confirm completely the outcome of the thousands of measurements made in foregoing years, viz. that in the three species of Nymphaea the definite length is reached partly by cell elongation and partly by the formation of new cells, and that in Limnanthemum the elongation is mainly the result of cell division only. In the noxious concentrations of the acids cells of the Nymphaeae were considerably longer than in the optimal concentrations; this is also according to what I have found earlier.

Discussion. ZIMMERMAN and HITCHCOCK have stated the interesting fact that the different phenoxycompounds vary in their effect on different plant species. For example, 4-chlorophenoxyacetic acid modifies the leaves of tobacco(15 *b*, pag. 330); 2,4-dichlorophenoxyacetic acid does not modify tobacco leaves, but greatly modifies the leaves of tomato (l.c., pag. 338). As far as my experiments go we see that 2,4-dichlorophenoxyacetic acid has a much stronger effect than 4-chlorophenoxyacetic acid on all species examined. In another paper (9) HITCHCOCK and ZIMMERMAN report that of several phenoxy- and other compounds tested, only 4-chlorophenoxyacetic acid caused distinct responses in Kalanchoe daigremontiana. I have treated plants of Kalanchoe thyrsiflora with lanoline preparations of both my substances around the stem (8 mg/g). The leaves above the treated region responded by a strong epinasty, and afterwards the stems decayed at their bases. And again these phenomena were much more distinct in the plants treated with 2,4-dichlorophenoxyacetic acid than in those treated with 4-chlorophenoxyacetic acid.

The noxious effect of the phenoxycompounds on Limnanthemum and in higher concentrations on Nymphaea goes parallel to what they do to weeds. But this effect is by no means particular to these substances. In my former experiments I have invariably stated that even the concentrations which may be called optimal as far as the elongation goes, are noxious to the plant because they shorten its life cycle when the treatment is not discontinued in time. It remains to be seen why substances such as α-naphtylacetic acid have a similar and probably much stronger effect on waterplants than 2,4-dichlorophenoxyacetic acid, whilst on higher land plants they are inactive, at least when they are applied in the way in which phenoxycompounds are used.

Summary. 2,4-dichlorophenoxyacetic acid, in concentrations of $\frac{1}{4}$, $\frac{1}{2}$, 1 and 2 mg/L, causes a very strong excess growth in the petioles of Nymphaea odorata, N. terminalis, N. Devoniensis and Limnanthemum nymphaeoides. When immersed in solutions of this substance the petioles can

attain a manifold of their original length within one or a few weeks. The optimal concentration seems to be about ½ mg/L. The daily growth can attain 25 cm, and probably much more at optimal temperatures. 4-chlorophenoxyacetic acid is much less efficient and it makes its influence felt only after a considerable lapse of time.

I have great pleasure in thanking the Direction of the N.V. Amsterdamsche Chininefabriek, Amsterdam, who kindly provided me with the phenoxycompounds.

Botanical Institute.

Ghent, September 1946.

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Comparative Physiology. — *Digestion in the stomach of birds. I. The acidity in the stomach of young chickens.* By H. J. VONK, GRIETJE BRINK and N. POSTMA. (From the laboratory of Comparative Physiology, University of Utrecht.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of September 21, 1946.)

In carnivore vertebrates the *outer layer* of the stomach contents is (chiefly in the fundus) imbibed with gastric juice and in this way obtains a p_H of 2.0 to 2.8, favourable for digestion by pepsin (the optimum of pepsin varies from 1.75—2.2 in diverse animals). The superficial layer of the stomach contents is disintegrated in this way and the products of this disintegration are carried to the pyloric part by means of the gentle movements of the fundus. In the pyloric region only pepsin and no HCl is secreted. This part shows strong movements which knead the half-digested food before passing it on to the duodenum. The layer of the stomach contents which assumes a p_H favourable for the digestion is not more than 1 or 2 mm thick. At a distance greater than 2 mm from the surface, the p_H rises rapidly. The diffusion of the pepsin into the protein is very slow because of the large molecules of both.

In herbivore vertebrates the conditions are different. Because the contents of their stomach can bind far less hydrochloric acid, this diffuses rapidly through these contents. After some hours parts of the contents of a rabbits stomach, situated some cm below the surface, have already attained a p_H of 2.0 or less. The pepsin also spreads rapidly through the stomach contents. After 24 hours it can be demonstrated in the centre of the food (VONK and V. D. KROGT, 7). In the herbivore stomach the bulk of the food is not digested. Only the 1 or 2 procent of protein which it contains can be attacked by pepsin.

In omnivorous animals the conditions must be intermediate between those of carnivores and herbivores, but they have not yet been investigated adequately.

The digestion in the stomach of birds is again of a total different type. At least this is the case for those birds where the glandular and muscular stomach (gizzard) are well differentiated. The glandular stomach is to be compared with the fundus of mammals (secretion of digestive juice), the muscular stomach (gizzard) with the pyloric part of the mammalian stomach. The surface of the gizzard is covered with a sheat of coiline, a keratine-like protein, which is secreted by and adheres strongly to the wall of this stomach. During the passage through the glandular stomach the food is mixed with gastric juice. In the muscular stomach the contents are kneaded heavily; this part of the stomach can develop enormous pressures as was already known to RÉAUMUR and SPALLANZANI. The mechanical function is reinforced by small stones which are regularly taken up by the animals.

In view of these conditions it is to be expected that the p_H in the muscular stomach of birds will be more or less uniform throughout its contents. This supposition has been confirmed by the work of MENNEGA (1). For a review on this and other work on this topic see VONK (2). The p_H-values which she has found for the stomach contents of adult herons, (3.50—5.89) are surprisingly high, so high that it seems doubtful whether pepsin can act at all at these p_H-values. Nevertheless fish and meat are digested rapidly in the stômach of the heron. How this is possible is still an unsolved problem.

In young birds the p_H of the stomach-contents has never been investigated. MAC LAUGHLIN (3) found in the gizzard of adult birds 3.39 but his technique is not very exact.

The present investigation [1]) has been undertaken with the purpose to detect whether the feeding of young chickens with food, containing different percentages of protein, would lead to differences in the p_H of the stomach-contents. It has been carried out parallel with researches of other investigators who studied for practical purposes the effect of these different kinds of food on the growth of chickens (determining weight, development of feather-cover, crest and gills, general exterior etc.). These results will be published elsewhere.

The experiments were made as soon as possible after birth and 23 and 53 days after birth. A control after 100 days had to be omitted, the chickens of the investigated group having been afflicted after the third series of observations with coccidiosis.

We received the first group of chickens as soon as possible after birth. The secretion of the stomach was stimulated [2]) by feeding with food [3]) of low protein content which could not bind much of the acid secretion. Directly after feeding the animals were killed, opened and the p_H of the stomach contents determined by means of a glass electrode. A Cambridge potentiometer (Electrometer Valve Potentiometer) was used for these determinations, together with a small glass-electrode of the type used by MAC INNES and DOLE. For the description of this arrangement the dissertation of MENNEGA (1), the literature cited there and the article of VAN DEN BROEK (6) may be consulted. For each determination the electrode was checked against a standard buffer.

In this way an idea of the acidity of the stomach-contents was obtained. In some animals of the first group the p_H of the stomach was measured in

[1]) We are indebted to the "Kennemer Chemische Industrie", IJmuiden, who put the greater part of the means for this investigation at our disposal.

[2]) The secretion of the stomach of the chicken has been investigated by FRIEDMANN (4), but he has not measured the p_H of the gastric juice or of the contents.

[3]) This food consisted chiefly of barley-grit.

hunger. The results of the total first series are shown in Table I, which contains also at the bottom peculiarities about the time of feeding, the contents of the stomach etc.

Average p_H-values were calculated for several groups of results. In doing so it must be born in mind that for determining the average of the p_H-values (these being logarithms) the corresponding hydrogenion-concentrations must be calculated. From these $[H^+]$-values the average and the standard-deviation can be calculated in the usual way. (The standard-deviation $\sigma = \pm \sqrt{\frac{\Sigma d^2}{n(n-1)}}$, where d is the difference between each single value and the calculated average and n the number of observations.) Finally from the average of the hydrogenion concentrations the corresponding p_H can be calculated. For comparing the results of different groups of observations, the error of the difference of these groups [4]) has to be calculated from the mean-errors of each series of determinations according to the formula $\sigma_{diff} = \pm \sqrt{\sigma_1^2 + \sigma_2^2}$, σ_1 and σ_2 being the mean-errors of each group. According to the rules of these calculations a difference is considered to be essential (significant) if it exceeds 2 to 3 times its error. In view of the relatively small amount of observations in this case a difference can only be taken as essential if it exceeds 3 times its error. All these calculations have to be performed on the hydrogenion concentrations and not on the p_H-values.

By means of these calculations the following results have been obtained for Table I:

Average of all the glandular stomachs:

	p_H 2.66 (+ 0.38 or —0.20);	$[H^+] = 0.002\ 191 \pm 0.001\ 284$
id. gizzards:	p_H 2.86 ($\pm$ 0.08)	; $[H^+] = 0.001\ 392 \pm 0.000\ 253$
Difference		$= 0.000\ 799 \pm 0.001\ 309$

The difference of 0.000799 between the glandular stomachs and the gizzards is not essential, being in fact even smaller than its error.

Moreover the averages of all the gizzards and of all the glandular stomachs of fed animals were compared with each other:

Average gizzards of all fed animals

	p_H 2.77 $\pm$ 0.06;	$H = 0.001\ 708 \pm 0.000\ 235$
id. gl. stomachs	p_H 2.89 $\pm$ 0.01;	$H = 0.001\ 293 \pm 0.000\ 036$
Difference		$= 0.000\ 415 \pm 0.000\ 238$

This difference is neither significant.

The results for the animals in hunger (X, XI, XII) were:

Average gizzards	p_H 2.77 (+ 0.11 or —0.10);	$H = 0.001\ 706 \pm 0.000\ 392$
,, gl.stom.	p_H 4.25 (+ 0.19 or —0.13);	$H = 0.000\ 056 \pm 0.000\ 019$
Difference		$= 0.001\ 650 \pm 0.000\ 392$

4) This difference must be calculated again from the hydrogenion concentrations.

This difference should be considered as significant, but in view of the small amount of observations (especially for the glandular stomach) and of the results for the observations on all the stomachs and those of the fed animals, we cannot consider it to be of real importance.

TABLE I.

p_H in the first stages of life (3—7 days). Born May 1; arrived May 3.

	May 4		May 5							May 8		
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
Crop, coecum	4.45	4.20	4.80	4.70	4.10	4.70	4.00	5.00	4.10	5.85	4.85	4.85
„ transition	—	4.90	5.80	4.75	4.50	5.10	5.00	5.25	4.60	6.25	—	5.10
Gland. stomach	3.70	4.45	1.85	2.45	2.10	4.80	4.50	5.50	3.65	4.25	4.65	4.05
Musc. stomach l. (gizzard)	3.00	3.50	2.30	2.75	—	3.35	2.90	3.60	3.00	3.05	3.00	2.50
Musc. stomach r. (gizzard)	3.60	3.50	2.65	3.30	—	3.05	2.40	3.60	3.05	3.05	2.60	2.75
Beginn. duodenum	5.20	5.70	6.00	5.75	6.15	5.70	5.45	6.05	6.05	6.80	6.55	6.85
Fresh food mixed with some water	5.60											

Arrangement and particulars of the experiments in Table I.

Feeding: 3 May. All animals 14.30 water; 14.45—15.00 and 18—18.30 food.
4 May. All animals: 8.30—10.00, 11.30—12.00, 13.30—14.30 food; then I and II killed and measured, the others food at 15.30—16.30.

Observations: I crop moderately filled; gland. stom. empty; gizzard moderately filled;
II crop strongly filled; gland. stom. distally filled; gizzard moderately filled.

If the glandular stomach is not mentioned, it was empty. The contents of the gizzard are generally dry, consisting of chaff, more or less sand, some meal and broken grain.

5 May. IV and V no food; the others fed at 9.00—9.15.
III killed and measured at 9.15—9.45. Crop and gizzard well filled.
IV and V k. and measured at 10.15. IV crop empty, gizzard much sand; V crop filled, gizzard as IV.

10.45—11.15 the others fed; 11.15 VI k. and m.; crop and gizzard moderately filled.
13.00—13.30 the others fed; 13.30 VII k. and m.; crop and gizzard well filled.
14.30—15.00 the others fed; 15.00 VIII k. and m.; crop and gizzard well filled.
16.00—16.30 the others fed; 16.30 IX k. and m.; crop well filled; gizzard nearly empty.

May 6 and 7. Animals X, XI and XII until 7 May 20.00. After that animals without any straw, only with water.

May 8. 10.15. X, XI and XII killed and measured. Crop and gland. stomach empty, some contents in gizzard.

I, II, III, VI, VII, VIII, IX are fed animals.

X, XI, XII are animals in hunger.

IV and V had no food during 14 hours but could pick up pieces of straw and some remnants of previous feeding.

The chief result of the total series of observations of Table I is the low acidity of the stomach contents (both for gizzard and glandular stomach) in fed animals as well as for the total of all animals. This acidity (averages varying between p_H 2.66 and 2.89) is far lower than that of the content of the gizzard and muscular stomach of all the adult birds formerly measured by MENNEGA. This result demonstrates that at least for young chickens a strongly acid gastric juice is secreted quite comparable in acidity to that of mammals.

We now proceed to the measurements carried out on animals which had been fed after birth on the farm with food containing different percentages of protein. These three kinds of food are indicated as —30, —25 and —20; —30 having the highest, —20 the lowest content of protein. The composition of these foods may be seen from the following list:

	Food — 30	Food — 25	Food — 20
Maize-flour	4.—	4.—	5.—
Barley-flour	10.—	11.—	13.—
Oat-meal	7.—	8.5	10.—
Bran	7.—	8.5	10.—
Fish-meal	3.—	3.—	3.—
Animal-meal	6.—	6.—	6.—
Yeast	1.—	1.—	1.—
Minerals	1.—	1.—	1.—
Grass-meal	4.—	4.—	4.—
Oat-grits	34.—	36.—	39.—
Milk powder	5.—	5.—	5.—
Meat-meal	18.—	12.—	3.—
Vitamine-mixture	0.05	0.05	0.05

The results of the measurements 24—25 days after birth are given in Table II.

The comparison of several averages showed the following results:

Average of all the gizzards: p_H 2.77 ± 0.06; $[H^+] = 0.001\,708 \pm 0.000\,235$
,, ,, ,, gland.stom.: p_H 4.06 ± 0.16; $[H^+] = 0.000\,094 \pm 0.000\,032$

Difference $= 0.001\,614 \pm 0.000\,237$

This difference exceeding more than 6 times its error, is essential and therefore the contents of the gizzards are far more acid than those of the glandular stomachs.

Average of all the gizz. (food —30):
p_H 2.49 ± 0.06 ; $[H^+] = 0.003\,228 \pm 0.000\,445$
Average of all the gizz. (food —20):
p_H 3.08 (+ 0.08 or —0.07); $[H^+] = 0.000\,828 \pm 0.000\,141$

Difference $= 0.002\,400 \pm 0.000\,467$

This difference is essential.

Average of all the gizz. (food —30)

p_H 2.49 ± 0.06 ; [H+] = 0.003 228 ± 0.000 445

Average of all the gizz. (food —25)

p_H 3.00 (+ 0.10, or —0.08); [H+] = 0.001 001 ± 0.000 199

Difference = 0.002 227 ± 0.000 487

This difference is essential.

Average of all the gizz. (food —25)

p_H 3.00 (+ 0.10, or —0.08); [H+] = 0.001 001 ± 0.000 199

Average of all the gizz. (food —20)

p_H 3.08 (+ 0.08, or —0.07; [H+] = 0.000 828 ± 0.000 141

Difference = 0.000 173 ± 0.000 244

This difference is smaller than its error.

TABLE II.

p_H 24—25 days after birth, after feeding with food with different percentages of protein.

Data	May 25.10.00 h					May 26.10.00 h					May 26.14.00 h	
No. food	—30	—30	—30	—25	—20	—30	—25	—20	—20	—20	—25	—25
No. animal	I	II	III	I	I	IV	II	II	III	IV	III	IV
Crop, cran.	5.29	5.32	5.44	5.46	5.76	3.86	5.30	5.15	5.56	5.08	5.61	5.68
„ coecum	4.77	5.44	5.46	5.60	5.37	3.62	5.14	4.79	5.56	4.85	4.78	5.49
„ caudal	5.15	5.45	5.32	5.44	5.48	5.20	5.00	5.25	5.15	4.94	5.51	5.57
Glandular stomach	4.48	4.16	3.39	3.88	4.31	3.80	4.54	4.00	4.28	4.60	4.46	4.98
Condition	e	e	f	f	e	fl	f	e	fl	f	e	e
Musc. stomach (gizzard)												
conditon	m.fl	s.fl	mo.fl	mo.fl	s.fl	n.fl	s.fl	mo.fl	mo.fl	n.fl	s.fl	v.m.fl
Left	2.46	2.43	2.47	2.89	3.40	2.14	3.38	3.45	2.86	3.25	2.71	2.62
Dorsal	2.63	2.74	2.88	3.20	3.68	2.18	3.12	2.94	3.00	2.90	3.09	3.68
Right	2.82	2.33	2.93	3.14	3.75	2.39	3.12	3.14	2.73	3.12	2.51	3.01
Ventral	2.46	2.77	2.65	3.50	—	2.52	3.35	3.37	2.76	3.38	4.04	2.94
Food		5.68							5.70		5.64	

n.fl = no fluid, s.fl = small amount of fluid, mo.fl = moderate amount of fluid, m.fl = much Fluid, v.m.fl. = very much fluid, e = empty, f = full, fl. = fluid.

Arrangement and particulars of the experiments in Table II.

Feeding: 24 May all animals from 17—21 h.

25 May all animals from 8.30—10.00; then killed and measured food —30 nos I, II and III, food —25 no I, food —20 no I; the others fed until 21.00.

26 May all remaining animals from 8.30—10.00; then killed and measured food —30 IV, —25 II and —20 II, III and IV.

The two remaining animals fed until 14.00, then killed and measured (—25 III and IV).

The p_H of the three kinds of food is practically the same.

In the objects —25 III and IV which have been fed during 5.30 hours before the measurements were taken, the deviations of the p_H-values per animal are much larger, than for the animals which were killed after 1.30 hour of feeding. It is therefore advisable that the feeding be continued not too long a time before observation takes place.

It seems therefore that the food —30, richer in protein, causes a decidedly higher acidity in the gizzard than the food poorer in protein. This is the more striking because, if on the three kinds of food an equal amount of gastric juice of the same acidity were secreted, the food richer in protein would show the higher p_H. The contrary is the case and therefore either the food richer in protein must cause the secretion of a larger quantity of stomach-juice, or of a stomach-juice of higher acidity than that secreted on the food with less protein. Both these factors could also coöperate. The difference between the foods —30 and —25 (—25 with a medium amount of protein) is also real. The difference between food —25 and —20 is not essential, but nevertheless there is a difference in the same direction as that between —30 and —25. As to the possibility of pepsin-action the result is that for the food richer in protein the conditions (average p_H 2.49) are very good and for the others (3.00 and 3.08) tolerable.

TABLE III.

p_H 52—53 days after birth, after feeding with food with different percentages of protein.

Data	June 22.9.30 h					June 23.9.30 h						
No. food No. animal	—30 V	—25 V	—25 VI	—25 VII	—20 V	—30 VI	—30 VII	—30 VIII	—25 VIII	—20 VI	—20 VII	—20 VIII
Crop, cran.	4.54	4.08	4.41	4.48	5.26	5.14	5.06	5.15	4.11	4.81	5.33	4.50
„ coecum	4.72	4.42	4.36	4.86	5.33	4.09	5.10	4.96	4.60	5.67	5.66	5.00
„ caudal	4.79	4.29	4.45	4.72	5.10	4.62	4.81	5.24	4.42	5.35	5.53	5.24
„ transition ("Strasse")	4.46	4.72	4.56	4.62	5.41	4.40	—	5.20	—	5.38	5.66	5.34
glandular stomach												
Condition	f	e	fl	e	f	p	p	p	fl	fl	fl	p
Cranial	3.44	4.48	4.84	4.42	4.55	4.28	5.25	4.80	4.86	4.68	4.53	4.48
Caudal	3.93	3.70	4.75	4.29	3.89	4.22	4.62	4.65	4.58	4.53	4.30	4.80
Musc. stomach (gizzard)												
Condition	m.fl	n.fl	mo.fl	s.fl	s.fl	m.fl	m.fl	m.fl	m,fl	n.fl	n.fl	mo.fl
Left	—	2.40	3.46	3.88	3.54	3.58	3.38	2.94	3.05	3.22	3.40	3.44
Dorsal	3.61	—	3.90	3.57	3.45	3.80	3.40	3 40	2.96	3.72	3.46	3.67
Right	—	2.75	3.80	3.08	3.30	3.84	3.18	2.74	2.93	3.52	3.36	3.78
Ventral	3.34	2.52	3.56	3.48	3.42	3.92	3.44	3.00	2.64	3.19	3.52	3.68

n.fl = no fluid, s.fl = small amount of fluid, mo.fl = moderate amount of fluid, m.fl = much fluid, e = empty, f = full, fl = fluid, p = pulpy.

Arrangement and particulars of the experiments in Table III.

Feeding: 21 June from 16—22 h.
22 June from 8.00—9.30, then the animals indicated in the table under 22 June killed and measured.
The remaining chickens were fed until 22.00 h.
23 June fed from 8.00—9.30; then the animals under June 23 killed and measured.

The same experiments were performed 52 and 53 days after birth. These results are shown in Table III. The comparison of several averages showed the following results:

Average of all the gizzards
p_H 3.18 (+ 0.08 or —0.07): $[H^+]$ = 0.000 667 ± 0.000 117
Average of all the gl.stom.
p_H 4.24 (+ 0.14 or —0.11): $[H^+]$ = 0.000 058 ± 0.000 016
Difference = 0.000 609 ± 0.000 118

This difference is essential and therefore like in Table II the average acidity of the gizzard is decidedly lower than that of the glandular stomach. However the acidity in the gizzards of this Table (III) is lower than that of the gizzards in Table II. We return to this point later on.

The comparison of the gizzards containing different kinds of food does not show the same regular results as that in the experiments of Table II. If we take as essential a difference between two averages which exceeds 3 times the error of this difference, we must conclude that there are no real differences between the acidity of the gizzards containing the three different kinds of food. This may be seen from the following calculations:

Average of all the gizz. (—25)
p_H 2.95 (+ 0.14 or —0.10): $[H^+]$ = 0.001 1208 ± 0.000 3051
Average of all the gizz. (—30)
p_H 3.30 (+ 0.09 or —0.08): $[H^+]$ = 0.000 5039 ± 0.000 1011
Difference = 0.000 6169 ± 0.000 3214

Average of all the gizz. (—25)
p_H 2.95 (+ 0.14 or —0.10): $[H^+]$ = 0.001 1208 ± 0.000 3051
Average of all the gizz. (—20)
p_H 3.45 ± 0.04 : $[H^+]$ = 0.000 3564 ± 0.000 0348
Difference = 0.000 7644 ± 0.000 0314

Average of all the gizz. (—30)
p_H 3.30 (+ 0.09 or —0.08): $[H^+]$ = 0.000 5039 ± 0.000 1011
Average of all the gizz. (—20)
p_H 3.45 ± 0.04 : $[H^+]$ = 0.000 3564 ± 0.000 0348
Difference = 0.000 1475 ± 0.000 1069

In so far as any significance can be attributed to these differences, we see that the greatest difference (greater than 2 times its error) exists between the animals fed with food —25 and those fed with food —20. However the gizzards of animals with food —25 are (quite contrary to the results of Table II) more acid (though not significantly) than those of the animals with food —30. The most justified conclusion we may draw from these results, is that in this stage there are no real differences between the results with the different kinds of food. This is in agreement with the fact that the average acidity of all gizzards of the animals of Table III is significantly less than that of the animals of Table II, as may be seen from the following figures:

Average of all the gizz. Table II

p_H 2.77 ± 0.06 ; $[H^+] = 0.001\,708 \pm 0.000\,235$

Average of all the gizz. Table III

p_H 3.18 (+ 0.08 or —0.07); $[H^+] = 0.000\,667 \pm 0.000\,117$

Difference $= 0.001\,041 \pm 0.000\,263$

This difference exceeds about 4 times its error. It is the more reliable because the number of the determinations (47 for all the gizz. Table II and 45 for all the gizz. Table III) is much larger than that of the determinations in the experiments with different kinds of food.

Finally we had at our disposal 11 adult cocks for our experiments. These animals were fed with different kinds of food. Six of them were fed with mixed chickenfood (containing meat-meal, barley-grit-and meal etc.) as used in chicken feeding. One was fed with raw minced potatoes, two were fed with earthworms and some meat, two with mixed food and some earthworms. We propose to deal with these experiments extensively in a following communication. We may however compare the averages obtained in these experiments with those of the young animals in the present paper. As an average of the determinations on 8 animals fed with mixed food with or without worms (in total 72 determinations) we obtained a p_H of 2.96 (+ 0.05 or —0.04) corresponding with a hydrogenion concentration of 0.00108 ± 0.000114. The difference of this value with the average of the animals of Table II was 0.000628 ± 0.000261. In view of the great amount of observations (72 for the adults and 47 for Table II) we may consider this difference as real.

If we include in the observations of the adults all the fed animals viz. also one animal fed with minced potatoes and two animals fed with meat and worms we obtain a somewhat higher average of 0.000867 ± 0.0000983 corresponding to a p_H of 3.06 (+ 0.05 or —0.04), because especially the animals fed with meat showed higher p_H-values. The difference with the average of Table II is then more pronounced, whereas a real difference with the results of Table III does not exist [5]).

We may therefore come to the final conclusion that the acidity in the investigated adult animals appears to be somewhat lower than that of the animals in Table III but that this difference (if all the adult fed animals are included) is not real. Only in the first stages of life (2 to 23 days after birth) a decidedly lower acidity of the stomach contents is found. After 52 days the conditions do not differ from those of adult animals.

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[5]) For the difference between the values of Table III and of the 8 adults fed with mixed food we get 0.000413 ± 0.000163, which can be considered as essential (in view of the rather large amount of observations).

Discussion. We may now try to answer the question, whether the acidity in the stomach of the chickens is sufficient to enable a good action of pepsin. This answer can only be a prelimary one, as the optimum of extracts of the glandular stomach of the chickens has never been determined. We intend to publish the results of this determination in our next article. There is however no reason to expect that this optimum will be situated far beyond the range of p_H 1.75—2.40 which has been found until now for different animals (VONK, 8, MENNEGA, 1). As may be seen from the average of Table II for the young animals the p_H of the stomach contents (2.77) is near the probable optimum.

For the older animals (Table III: 3.18, adults 3.06) the p_H of the contents will probably show a fair deviation from the optimum. From former results of one of us (VONK 8) it may be concluded that at p_H 3.0 the action of pepsin is about 12 times slower than at its optimum and at p_H 3.2 about 16 times. However it has been demonstrated in the same paper that stirring of the digestive mixture has a strong accelerating influence on the peptic action. Now in the gizzard of the chicken very strong movements take place which develop an enormous pressure. Our preliminary conclusion therefore is, that the conditions in the chicken's stomach regarding p_H and movement are sufficient to enable a good peptic digestion. Further research on the accelerating action of strong pressures on the action of pepsin is desirable.

The rapid digestion in the stomach of the adult heron remains to be explained as the p_H-values found by MENNEGA are higher and the movements (as far as known) negligeable in comparison with those of the grain consuming birds.

Summary.

The p_H of the stomach (gizzard) contents of young chickens were determined in order to see whether there is a difference in these p_H-values when food with different percentages of protein was given. Such difference has only been found about three weeks after birth, the food with the highest protein content showing the lowest p_H-values. About 7 weeks after birth these differences have disappeared.

The contents of the gizzard always show a higher acidity than that of the glandular stomach.

In young animals (23 days) the contents of the gizzard show an average p_H of 2.77, in animals 52 days after birth the average p_H of the gizzard has risen to p_H 3.18. In adult cocks we found an average p_H of the gizzard of 3.06. This figure does not differ essentially from the p_H (3.18) after 52 days.

The p_H in young chickens (until 3 weeks) enables a strong action of pepsin. In adult animals this action must be slower because of the higher p_H. Probably this is compensated by the promotion of the digestion through the strong movements of the gizzard.

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CONTENTS

Mathematics. — *On the fundamental theorem of algebra.* (Third communication)[1]. By J. G. VAN DER CORPUT.

(Communicated at the meeting of October 26, 1946.)

§ 4. *Proof that the couples form a field.*

It is obvious that in the set of the couples the addition and multiplication are commutative. For a proof of the associative law we remark that each three intervals Γ_1, $\triangle_1$ and Λ_1 satisfy the relation

$$(\Gamma_1 + \triangle_1) + \Lambda_1 = \Gamma_1 + (\triangle_1 + \Lambda_1),$$

and that each three polynomials C, D and L satisfy the relation

$$(C + D) + L = C + (D + L).$$

Similar formulae are valid for $\times$ instead of $+$. Take three arbitrary couples $\gamma = (\Gamma, C)$, $\delta = (\triangle, D)$ and $\lambda = (\Lambda, L)$. If the subintervals Γ_1, $\triangle_1$ and Λ_1 of Γ, $\triangle$ and Λ, where C, D and L respectively change sign, are chosen small enough, then both $(\gamma + \delta) + \lambda$ and $\gamma + (\delta + \lambda)$ are equal to $(\Gamma_1 + \triangle_1 + \Lambda_1,\ C + D + L)$. Similarly

$$(\gamma\,\delta)\,\lambda = \gamma\,(\delta\,\lambda) = (\Gamma_1 \times \triangle_1 \times \Lambda_1,\ C \times D \times L).$$

This proves the associative law of addition and multiplication.

Consider $1 + \mu + \nu + \varkappa$ indeterminates X, $Y_\varrho\,(1 \leqq \varrho \leqq \mu)$, $Z_\sigma\,(1 \leqq \sigma \leqq \nu)$, $U_\tau\,(1 \leqq \tau \leqq \varkappa)$. The product

$$\prod_{\varrho,\sigma,\tau} (X - Y_\varrho (Z_\sigma + U_\tau)) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

is a divisor of

$$\prod_{\varrho,\sigma,\tau,\zeta} (X - Y_\varrho Z_\sigma - Y_\zeta U_\tau), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $1 \leqq \zeta \leqq \mu$. If these products are developped and every elementary symmetric function of $Y_1, \ldots, Y_\mu$ is replaced by the corresponding coefficient (multiplied by ± 1) of $C(X)$, every elementary symmetric function of $Z_1, \ldots, Z_\nu$ is replaced by the corresponding coefficient (multiplied by ± 1) of $D(X)$ and every elementary symmetric function of $U_1, \ldots, U_\varkappa$ is replaced by the corresponding coefficient (multiplied by ± 1) of $L(X)$, then (13) becomes $A = C \times (D + L)$ and (14) becomes $B = (C \times D) + (C \times L)$. Hence A is a divisor of B. If the subintervals Γ_1 of Γ, $\triangle_1$ of $\triangle$ and Λ_1 of Λ, where respectively C, D and L change sign, are chosen small enough, then

$$\gamma\,(\delta + \lambda) = (\Gamma_1 \times (\triangle_1 + \Lambda_1), A) \text{ and } \gamma\delta + \gamma\lambda = (\Gamma_1 \times \triangle_1 + \Gamma_1 \times \Lambda_1, B).$$

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[1]) Compare Proceedings Academy Amsterdam **49** (1946), p. 722—732 and 878—886); Indagationes Mathematicae **7** (1946).

Here

$$\Gamma_1 \times (\Delta_1 + \Lambda_1) = \Gamma_1 \times \Delta_1 + \Gamma_1 \times \Lambda_1$$

and the greatest common divisor A of A and B changes sign in this interval. Hence $\gamma(\delta + \lambda) = \gamma\delta + \gamma\lambda$, which proves the distributive law.

All couples (Γ, C), where Γ contains the element 0 of Ω, and where $C(0) = 0$, are equal, for they are equal to (Π_0, X), where Π_0 denotes the interval, which only consists of the element 0. In fact, the greatest common divisor X of $C(X)$ and X changes sign in the common part Π_0 of Γ and Π_0. This couple (Π_0, X) is the element 0 in the set of couples, i.e. a couple (Δ, D) remains unchanged, if (Π_0, X) is added, for

$$(\Pi_0, X) + (\Delta, D) = (\Delta, X + D) = (\Delta, D).$$

In order to prove that in the set of couples subtraction is always possible, it is sufficient to show, that an arbitrary couple (Γ, C) being given, a couple (Λ, L) exists, such that

$$(\Gamma, C) + (\Lambda, L) = (\Pi_0, X).$$

I denote by $L(X)$ the polynomial $(-1)^{\mu} C(-X)$, where μ denotes the degree of $C(X) = c_0 + c_1 X + \ldots + c_{\mu} X^{\mu}$ $(c_{\mu} = e)$. To find $C + L$ we have to develop the product $\Pi(X - Y_{\varrho} - Z_{\sigma})$, where ϱ and σ run through $1, \ldots, \mu$, and to replace ΣY_1, ΣZ_1, $\Sigma Y_1 Y_2$, $\Sigma Z_1 Z_2, \ldots$ by $-c_{\mu-1}$, $c_{\mu-1}$, $c_{\mu-2}$, $c_{\mu-2}, \ldots$ We get the same result if in $\Pi(X - Y_{\varrho} + Z'_{\sigma})$ we replace ΣY_1, $\Sigma Z'_1$, $\Sigma Y_1 Y_2$, $\Sigma Z'_1 Z'_2, \ldots$ respectively by $-c_{\mu-1}$, $-c_{\mu-1}$, $c_{\mu-2}$, $c_{\mu-2}, \ldots$ We get also the same result if in $\Pi(X - Y_{\varrho} + Y_{\sigma})$ we replace ΣY_1, $\Sigma Y_1 Y_2, \ldots$ by $-c_{\mu-1}, c_{\mu-2}, \ldots$ This product contains then at least one factor X, so that $C(X) + L(X)$ vanishes in the point 0. If Λ is the set of elements $-x$, where x runs through the interval Γ, then $L(X)$ changes sign in Λ and $\Gamma + \Lambda$ contains the element 0. Hence $(\Gamma + \Lambda, C + L) = (\Pi_0, X)$, which proves the possibility of subtraction.

The couples therefore form a ring. All couples (Γ, C), where Γ contains the unit element e of Ω and $C(e) = 0$, are equal, viz. equal to $(\Pi_1, X - e)$, where Π_1 denotes the interval which consists only of the element e. In fact, the greatest common divisor $X - e$ of $C(X)$ and $X - e$ changes sign in the common part Π_1 of Γ and Π_1. This couple $(\Pi_1, X - e)$ is the unit element of the ring, i.e. a couple (Δ, D) remains unchanged, if multiplied by $(\Pi_1, X - e)$. In fact

$$(\Pi_1, X - e)(\Delta, D) = (\Delta, (X - e) \times D) = (\Delta, D).$$

In order to prove that the set of couples is a field, it is sufficient to show that to every couple $(\Gamma, C) \neq (\Pi_0, X)$ corresponds a couple (Λ, L), such that

$$(\Gamma, C)(\Lambda, L) = (\Pi_1, X - e) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

We have to consider several cases:

1. Γ does not contain the element 0 of Ω and $C(0) \neq 0$, hence $c_0 \neq 0$.

Put

$$L(X)=\frac{1}{c_0}(c_\mu+c_{\mu-1}X+\ldots+c_0X^\mu).$$

To find $C\times L$ we have to develop the product $\Pi(X-Y_\varrho Z_\sigma)$, where ϱ and σ run through $1,\ldots,\mu$ and then to replace

$$\Sigma Y_1,\ \Sigma Y_1 Y_2,\ldots,\Sigma Z_1,\ \Sigma Z_1 Z_2,\ldots,Z_1 Z_2\ldots Z_\mu$$

respectively by $-c_{\mu-1}, c_{\mu-2},\ldots,-\frac{c_1}{c_0},\frac{c_2}{c_0},\ldots,\frac{(-1)^\mu c_\mu}{c_0}$. We get the same result, if we develop $\frac{(-1)^\mu c_\mu^\mu}{c_0^\mu}\Pi(U_\sigma X-Y_\varrho)$, and replace

$$\Sigma Y_1,\ \Sigma Y_1 Y_2,\ldots,\Sigma U_1,\ \Sigma U_1 U_2,\ldots,U_1 U_2\ldots U_\mu$$

respectively by $-c_{\mu-1}, c_{\mu-2},\ldots,-\frac{c_{\mu-1}}{c_\mu},\frac{c_{\mu-2}}{c_\mu},\ldots,\frac{(-1)^\mu c_0}{c_\mu}$. We get also the same result, if we develop

$$P=\frac{(-1)^\mu c_\mu^\mu}{c_0^\mu}\Pi(Y_\sigma X-Y_\varrho)$$

and replace $\Sigma Y_1,\ \Sigma Y_1 Y_2,\ldots$ by $-c_{\mu-1}, c_{\mu-2},\ldots$. Since P contains the factor $X-e$, which is found by putting $\sigma=\varrho=1$, the polynomial $C(X)\times L(X)$ is divisible by $X-e$, so that e is a root of this polynomial. If Λ is the set of points $\frac{1}{x}$, where x runs through the interval Γ, then $L(X)$ changes sign in Λ and $\Gamma\times\Lambda$ contains the point e. Consequently

$$(\Gamma\times\Lambda,\ C\times L)=(\Pi_1,\ X-e).$$

2. Γ contains the element 0 of Ω and $C(0)\neq 0$. The element 0 of Ω divides Γ into two subintervals Γ' and Γ''. We may put $(\Gamma,C)=(\Gamma',C)$. Since $C(0)\neq 0$, the interval Γ' contains a subinterval Γ_1, which does not contain the element 0 of Ω, such that C changes sign in Γ_1. Then $(\Gamma,C)=(\Gamma_1,C)$ and by 1. a couple (Λ,L) exists with (15).

3. Γ does not contain the element 0 of Ω and $C(0)=0$. Then a natural number ϱ exists, such that $C(X)=X^\varrho F(X)$, where $F(0)\neq 0$. Since X^ϱ is definite in Γ, the polynomial $F(X)$ changes sign in Γ, whence $(\Gamma,C)=(\Gamma,F)$ and by 1. a couple (Λ,L) exists, satisfying (15).

4. Γ contains the element 0 of Ω and $C(0)=0$. This case does not occur, since then (Γ,C) would be the element 0 of the set of couples.

Now I proceed to show that (Γ,C) is a root of $C(X)$. Be

$$A(X)=a_0+\ldots+a_\nu X^\nu \qquad (\nu\geqq 0)$$

a polynomial with coefficients belonging to Ω. If $\gamma_{\tau\sigma}$ $(1\leqq\sigma\leqq\tau\leqq\nu)$ run through an interval Γ, then

$$\sum_{\varrho=0}^{\nu} a_\varrho\gamma_{\varrho 1}\gamma_{\varrho 2}\ldots\gamma_{\varrho\varrho}$$

runs through an interval, which I call $A(\Gamma)$. I show first the existence of a polynomial $P(X)$, such that the couple $\gamma=(\Gamma, C)$ satisfies the relation

$$A(\gamma)=(A(\Gamma_1), P) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

for every sufficiently small subinterval Γ_1 of Γ, in which $C(X)$ changes sign.

Relation (16) is obvious in the case $\nu=0$, for then $P(X)=X-a_0$ has the required property. In fact, in that case $A(\Gamma_1)$ consists, whatever be the subinterval Γ_1 of Γ, only of the element a_0, so that $(A(\Gamma_1), X-a_0)$ is by convention the element a_0 itself. Now I suppose $\nu \geqq 1$ and assume the theorem already proved with $\nu-1$ instead of ν. Put

$$G(X)=a_0+\ldots+a_{\nu-1}X^{\nu-1}; \quad H(X)=a_\nu X^{\nu-1}, \quad K(X)=X,$$

whence

$$A(\gamma) = G(\gamma)+H(\gamma)K(\gamma)$$

and

$$A(\Gamma)=G(\Gamma)\boldsymbol{+}(H(\Gamma)\times K(\Gamma)).$$

By our assumption two polynomials $Q(X)$ and $R(X)$ exist, such that for every sufficiently small subinterval Γ_1 of Γ, where $C(X)$ changes sign, the relations

$$G(\gamma)=(G(\Gamma_1), Q) \text{ and } H(\gamma)=(H(\Gamma_1), R)$$

are valid. We know already

$$K(\gamma)=\gamma=(\Gamma_1, C)=(K(\Gamma_1), C).$$

From the definition of multiplication it follows

$$H(\gamma)K(\gamma)=(H(\Gamma_1)\times K(\Gamma_1), R\times C)$$

and from the definition of addition

$$G(\gamma)+H(\gamma)K(\gamma)=(G(\Gamma_1)\boldsymbol{+}(H(\Gamma_1)\times K(\Gamma_1)), Q\boldsymbol{+}(R\times C)).$$

This proves (16) by putting $P=Q\boldsymbol{+}(R\times C)$.

This relation implies in particular, that $P(X)$ changes sign once in the interval $A(\Gamma_1)$.

Let us now consider the special case $A(X)=C^*(X)$ By assumption $C(X)$ changes sign in Γ_1, so that this interval contains two elements u and v such that $C^*(v)\leqq 0 \leqq C^*(u)$. From the definition of $C^*(\Gamma_1)$ it follows, that this interval contains the points $C^*(u)$ and $C^*(v)$, consequently also the element 0. If $P(0)$ were $\neq 0$, we might choose the subinterval Γ_1 of Γ so small, that $P(X)$ would not change sign in $C^*(\Gamma_1)$. This being impossible, we have $P(0)=0$. Hence $C^*(\gamma)=(C^*(\Gamma_1), P)$ is the element 0 of Ω, so that γ is a root of $C^*(X)$, consequently also of $C(X)$.

§ 5. *Proof that Ω' is algebraically closed.*

In the introduction we stated that Ω' contains all elements, which are real algebraic with respect to Ω'. To prove this, it is sufficient to show that an arbitrary polynomial $F'(X)$ being given, the coefficients of which are real algebraic with respect to Ω', a polynomial $F(X)$ can be constructed, which is not identically equal to 0, such that the coefficients of $F(X)$ belong to Ω and that $F'(X)$ is a divisor of $F(X)$. In the proof I employ the following lemma.

Lemma 15. *If a is algebraic with respect to a commutative field Ω and if b is algebraic with respect to Ω (a), then b is algebraic with respect to Ω.*

Proof. Let a satisfy an equation of degree ν with coefficients belonging to Ω. We do not know whether this equation is reducible or not. But we know that b satisfies an algebraic equation, the coefficients of which all belong to $\Omega(a)$. Each of these coefficients may be written as a polynomial in a of degree $< \nu$, the coefficients of which belong to Ω. Hence the equation may be written in the form

$$F_0(X) + a\, F_1(X) + \ldots + a^{\nu-1} F_{\nu-1}(X) = 0,$$

where the left hand side of the equation is not identically equal to 0. Since a satisfies an equation of degree ν, we may write for $\lambda = 0, 1, \ldots, \nu - 1$:

$$a^{\lambda}(F_0(X) + \ldots + a^{\nu-1} F_{\nu-1}(X)) \doteq F_{\lambda 0} + a\, F_{\lambda 1} + \ldots + a^{\nu-1} F_{\lambda, \nu-1}. \quad (17)$$

The left hand sides of each of these ν relations takes the value 0, if X is replaced by b, hence b is also a root of

$$G(X) = \begin{vmatrix} F_{00} \ldots F_{0,n-1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ F_{n-1,0} \ldots F_{n-1,n-1} \end{vmatrix}.$$

The coefficients of this polynomial $G(X)$ belong to Ω. If this polynomial is not identically equal to 0, the proof is established. If $G(X)$ is identically equal to 0, then the ν polynomials (17) are linearly dependent, i. e. ν polynomials $H_0(X), \ldots, H_{\nu-1}(X)$ exist with coefficients belonging to Ω, such that the ν polynomials are not all identically equal to 0 and satisfy the identity

$$\left(\sum_{\lambda=0}^{\nu-1} a^{\lambda} H_{\lambda}(X)\right)(F_0(X) + \ldots + a^{\nu-1} F_{\nu-1}(X)) = 0.$$

Since the latter factor does not vanish identically, the first factor is identically equal to 0. Hence ν numbers $k_0, \ldots, k_{\nu-1}$, not all 0, exist with $\sum_{\lambda=0}^{\nu-1} k_{\lambda} a^{\lambda} = 0$. These numbers k_{λ} all belong to Ω, so that a satisfies an algebraic equation of degree $< \nu$. We may repeat our argument untill finally we find a polynomial with root b with coefficients belonging to Ω.

Lemma 16. *If Ω is a commutative field and all coefficients of a polynomial $F'(X) = c_0 + \ldots + c_\nu X^\nu$ are algebraic with respect to Ω, then a polynomial $F(X)$, not identically vanishing, can be constructed, such that the coefficients of $F(X)$ belong to Ω and that $F'(X)$ a divisor of $F(X)$.*

Proof. Put $F'(X) = b$; then b depends on X and b is algebraic with respect to $\Omega(X, c_0, \ldots, c_\nu)$. Here c_ν is algebraic with respect to Ω, hence also with respect to $\Omega(X, c_0, \ldots, c_{\nu-1})$. By lemma 15 we know b to be algebraic with respect to $\Omega(X, c_0, \ldots, c_{\nu-1})$. Repeating this argument we find finally b to be algebraic with respect to $\Omega(X)$. So we obtain a number of polynomials $F_0(X), \ldots, F_\lambda(X)$, where $F_0(X)$ does not vanish identically, such that

$$F_0(X) + F_1(X)\,b + \ldots + F_\lambda(X)\,b^\lambda = 0 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The coefficients of the polynomials $F_0(X), \ldots, F_\lambda(X)$ all belong to Ω. From (18) it follows that b is a divisor of $F_0(X)$, hence $F_0(X)$ possesses the required property.

§ 6. *Intuitionistic proof of the fundamental theorem.*

Consider a polynomial

$$A(X) = a_0 + \ldots + a_\mu X^\mu,$$

the coefficients of which belong to $\Omega'(i)$ and do not vanish simultaneously. If $z = x + iy$ and $z' = x' + iy'$ are elements of $\Omega(i)$ and u is a positive element of Ω, I denote by $|z| \leqq \frac{1}{u}|z'|$ the inequality $x^2 + y^2 \leqq \frac{x'^2 + y'^2}{u^2}$. I will show:

Lemma 17. *If there exists an integer $\lambda \geqq 0$ and $\leqq \mu$ such that*

$$|a_\lambda| \geqq \frac{|a_\varrho|}{u}, \qquad (\varrho = 0, \ldots, \mu) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where u denotes a positive element of Ω then $A(X)$ possesses at least λ roots, each of which has an absolute value less than an appropriate positive element of Ω, depending only on μ and u.

If $\lambda = 0$, there is nothing to prove, so that I may suppose $\lambda \geqq 1$. I may assume that the proof has already been established, if λ is replaced by $\lambda - 1$ and moreover, if $x_1, \ldots, x_\mu$ denote the roots of $A(X)$, arranged in order of not decreasing absolute value, we may assume that $x_1, \ldots, x_{\lambda-1}$ have the required property. It is sufficient to show that x_λ has the same property.

If $a_0 = 0$, then $\frac{A(X)}{X}$ possesses by induction at least $\lambda - 1$ roots, each having the required property, so that $A(X)$ possesses at least λ roots with this property. I may therefore suppose $a_0 \neq 0$. I distinguish two cases:

1°. Be

$$\left|\frac{a_{\lambda-1}}{a_0}\right| \leqq \frac{e}{2\,|x_1 \ldots x_{\lambda-1}|}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The expression

$$(-1)^{\lambda-1}\frac{a_{\lambda-1}}{a_0} - \frac{e}{x_1 \ldots x_{\lambda-1}}$$

equals a sum of $\binom{\mu}{\lambda-1}-1$ terms, each absolutely $\leqq \frac{e}{|x_1 \ldots x_{\lambda-2}\,x_\lambda|}$, so that the absolute value of this expression is less than

$$\frac{\binom{\mu}{\lambda-1} e}{|x_1 \ldots x_{\lambda-2}\,x_\lambda|}.$$

Hence

$$\frac{e}{|x_1 \ldots x_{\lambda-1}|} < \frac{2\binom{\mu}{\lambda-1} e}{|x_1 \ldots x_{\lambda-2}\,x_\lambda|},$$

so that

$$|x_\lambda| < 2\binom{\mu}{\lambda-1}|x_{\lambda-1}|$$

has the required property.

2°. Suppose, that (20) is not true. By (19)

$$|a_\lambda| \geqq \frac{|a_{\lambda-1}|}{u},$$

hence

$$\left|\Sigma\frac{e}{x_1 \ldots x_\lambda}\right| = \left|\frac{a_\lambda}{a_0}\right| \geqq \frac{1}{u}\left|\frac{a_{\lambda-1}}{a_0}\right| > \frac{e}{2u\,|x_1 \ldots x_{\lambda-1}|}.$$

The left hand side is at most equal to

$$\frac{\binom{\mu}{\lambda} e}{|x_1 \ldots x_\lambda|} < \frac{2^\mu e}{|x_1 \ldots x_{\lambda-1}|\,|x_\lambda|}.$$

Hence we obtain

$$|x_\lambda| < 2^{\mu+1} u,$$

so that x_λ possesses the required property.

This result, applied on $X^\mu A\left(\frac{1}{X}\right)$ instead of $A(X)$ and with $\mu-\lambda$ instead of λ, gives:

Lemma 18. *If* $0 \leqq \sigma \leqq \mu$ *and*

$$|a_\sigma| \geqq \frac{|a_\varrho|}{u} \qquad (\varrho = 0, \ldots, \mu), \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

then $A(X)$ *possesses at least* $\mu-\sigma$ *roots, the reciprocal values of which are absolutely less than an appropriate positive element of* Ω, *depending only on* μ *and* u.

It is possible that some of these roots have an infinite value; the number of such roots is $\mu-\zeta$, where ζ is the greatest integer such that $a_\zeta \neq 0$.

Let us now consider a natural number μ and two positive elements u and v of Ω. The notation $p \ll q$, where p and q are elements of $\Omega'(i)$, denotes, that Ω contains a positive element t, depending only on μ, u and v, such that $|p| \leqq t\,|q|$.

Consider a polynomial $A(X)$, written in the form

$$A(X) = a'(X-x_1)\dots(X-x_\lambda)(e-x_{\lambda+1}X)\dots(e-x_\mu X), \quad (22)$$

where $0 \leqq \lambda \leqq \mu$, and a' denotes an element $\neq 0$ of $\Omega'(i)$, whereas the elements $x_1, \dots, x_\mu$ of $\Omega'(i)$ have an absolute value $\leqq v$. [1])

Each coefficient of this polynomial $A(X)$ is $\ll a'$. Moreover we find successively that each of the polynomials

$$A_1(X) = \frac{A(X)}{X-x_1}, \; \dots, \; A_\lambda(X) = \frac{A_{\lambda-1}(X)}{X-x_\lambda},$$

$$A_{\lambda+1}(X) = \frac{A_\lambda(X)}{e-x_{\lambda+1}X}, \; \dots, \; A_\mu(X) = \frac{A_{\mu-1}(X)}{e-x_\mu X}$$

possesses the property, that each coefficient is $\ll a$, where a is the coefficient of $A(x)$ such that $|a_\varrho| \leqq |a|$ for $\varrho = 0, \dots, \mu$. The last polynomial $A_\mu(X)$ is equal to the constant a', which is therefore also $\ll a$. So we have found $a \ll a' \ll a$.

Consider a second polynomial $B(X)$, which possesses at least λ roots with absolute value $\leqq v$ and at least $\mu-\lambda$ roots, the reciprocal values of which are absolutely $\leqq v$. Moreover I assume that each coefficient of the polynomial $B(X)-A(X)$ is $\ll a\,r^{\mu !}$, where r denotes a positive element $\leqq e$ of Ω'.

I assert that $B(X)$ may be written in the form

$$B(X) = b'(X-y_1)\dots(X-y_\lambda)(e-y_{\lambda+1}X)\dots(e-y_\mu X),$$

where

$$y_\varrho - x_\varrho \ll r \;\; (\varrho = 1, \dots, \mu) \text{ and } b'-a' \ll a\,r. \quad (23)$$

To give a proof of this assertion, I consider an arbitrary root y of $B(X)$; this root may have an infinite value. I distinguish two cases, according to whether $|y| \leqq e$ or $> e$.

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[1]) If the polynomial $A(X)$ satisfies the inequalities $|a_\sigma|$ and $|a_\tau| \geqq \frac{|a_\varrho|}{u}$ for $\varrho = 0, \dots, \mu$, where $\sigma \leqq \lambda \leqq \tau$, it is possible to obtain an element v, depending only on μ and u, such that $A(X)$ may be written in the form (22). In fact, if $x'_1, \dots, x'_\mu$ are the roots of $A(X)$, arranged in order of not decreasing absolute value and if v is suitably chosen, the roots $x'_1, \dots, x'_\lambda$ have by the lemma 17, applied with τ in stead of λ, an absolute value $\leqq v$, whereas $\frac{e}{x'_{\lambda+1}}, \dots, \frac{e}{x'_\mu}$ have by lemma 18 also an absolute value $\leqq v$.

1°. Be $|y| \leqq e$. Each coefficient of the polynomial $B(X) - A(X)$ is $\ll a r^{\mu!}$, consequently

$$A(y) = A(y) - B(y) \ll a r^{\mu!} \ll a' r^{\mu!},$$

hence

$$(y - x_1) \dots (y - x_\lambda)(e - x_{\lambda+1} y) \dots (e - x_\mu y) \ll r^{\mu!},$$

so that at least one of the μ factors is $\ll r^{(\mu-1)!}$.

If at least one of the factors

$$y - x_\varrho \ll r^{(\mu-1)!} \; (1 \leqq \varrho \leqq \lambda),$$

then I may suppose, without loss of generality

$$y - x_1 \ll r^{(\mu-1)!}.$$

In that case I put $y_1 = y$, so that y_1 is a root of the polynomial $B(X)$, such that

$$y_1 - x_1 \ll r^{(\mu-1)!} \leqq r. \quad \dots \dots \dots \quad (24)$$

Now I must consider the case where at least one of the factors

$$e - x_\varrho y \ll r^{(\mu-1)!} \; (\lambda + 1 \leqq \varrho \leqq \mu).$$

In that case I may assume, without loss of generality

$$e - x_{\lambda+1} y \ll r^{(\mu-1)!}.$$

If we have

$$|e - x_{\lambda+1} y| \geqq \tfrac{1}{2} e, \quad \dots \dots \dots \quad (25)$$

we have $e \ll r$. By assumption the roots $y_1, \dots, y_\lambda, y'_{\lambda+1}, \dots, y'_\mu$ of $B(X)$ may be arranged such that y_ϱ $(\varrho \leqq \lambda)$ and $y_\varrho = \frac{e}{y'_\varrho}$ $(\varrho > \lambda)$ are $\ll e \, (1 \leqq \varrho \leqq \mu)$. Hence

$$y_\varrho - x_\varrho \ll e \ll r.$$

Moreover it follows from our above argument, that $b' \ll b$, when b is the coefficient of $B(X)$, such that $|b_\varrho| \leqq |b|$ for $\varrho = 0, \dots, \mu$. Since each coefficient of $B(X) - A(X)$ is $\ll a r^{\mu!} \ll a$, we obtain

$$b' \ll b \ll a, \text{ hence } b' - a' \ll a \ll a r.$$

Hence it appears that the assertion is obvious, if (25) is true. Consequently I may suppose, that this inequality is not valid. Hence

$$|e - x_{\lambda+1} y| < \frac{e}{2}.$$

Therefore

$$\frac{e}{y} \ll x_{\lambda+1} \ll e.$$

In this case I put $y_{\lambda+1} = \frac{e}{y}$, so that $y_{\lambda+1}$ possesses an absolute value $\ll e$ and is equal to the reciprocal value of a root of $B(X)$. By

$$y_{\lambda+1} - x_{\lambda+1} = y_{\lambda+1}(e - x_{\lambda+1} y) \ll e - x_{\lambda+1} y$$

we find

$$y_{\lambda+1} - x_{\lambda+1} \ll r^{(\mu-1)!} \leqq r \quad \dots \dots \dots \quad (26)$$

2°. Be $|y| > e$. Similarly we find, by putting $y_1 = \frac{e}{y}$ or $y_{\lambda+1} = y$, that (24) or (26) are valid again.

Next I have to distinguish two cases:

I. Be y_1 a root of $B(X)$, satisfying (24). Put

$$C(X) = \frac{A(X)}{X-x_1} \text{ and } D(X) = \frac{B(X)}{X-y_1}.$$

Each coefficient of $D(X)-C(X)$ may be written as a polynomial of x_1, of y_1-x_1, of the coefficients of $A(X)$ and of the coefficients of $B(X)-A(X)$. Then this coefficient is a sum of terms, each of which is divisible by y_1-x_1 or by at least one coefficient of $B(X)-A(X)$; in fact $D(X)-C(X)$ equals identically 0, if $y_1-x_1=0$ and simultaneously each coefficient of $B(X)-A(X)$ vanishes. Since each coefficient of $B(X)-A(X)$ is $\ll r^{\mu!} \leqq r^{(\mu-1)!}$, we obtain by (24), that each coefficient of $D(X)-C(X)$ is $\ll r^{(\mu-1)!}$. By induction $D(X)$ may be written in the form

$$D(X) = b'(X-y_2)\dots(X-y_\lambda)(e-y_{\lambda+1}X)\dots(e-y_\mu X),$$

where y_ϱ $(2 \leqq \varrho \leqq \mu)$ and b' satisfy (23). So we obtain the required form for the polynomial $B(X)$.

II. Be $\frac{e}{y_{\lambda+1}}$ a root of $B(X)$, satisfying (26). By a similar argument, in which $C(X)$ and $D(X)$ are replaced respectively by

$$\frac{A(X)}{e-x_{\lambda+1}X} \text{ and } \frac{B(X)}{e-y_{\lambda+1}X},$$

we obtain the same result.

This gives a proof of our assertion, and therefore also of the lemma:

Be $0 \leqq \sigma \leqq \lambda \leqq \tau \leqq \mu$. If the polynomials $A(X) = a_0 + \dots + a_\mu X^\mu$ and $B(X) = b_0 + \dots + b_\mu X^\mu$, the coefficients of which belong to $\Omega'(i)$, satisfy for $\varrho = 0, \dots, \mu$ the inequalities

$$|a_\tau| \text{ and } |a_\tau| \geqq \frac{|a_\varrho|}{u}; \quad |b_\tau| \text{ and } |b_\tau| \geqq \frac{|b_\varrho|}{u}; \quad |b_\varrho - a_\varrho| \leqq |a_\sigma| r^{\mu!},$$

then the assertion of the lemma, stated at the end of the introduction, is valid. I do not use this lemma itself, in order to avoid the definition of $\sum_{\varrho=0}^{\lambda} |a_\varrho|$, etc. The argument, given in the introduction, shows that from this lemma follows immediately a simple intuitionistic proof of the fundamental theorem of algebra.

Mathematics. — *Determinants and quadratic forms.* (First communication.) By J. G. VAN DER CORPUT and H. J. A. DUPARC.

(Communicated at the meeting of October 26, 1946.)

In his paper "Over eenige determinanten" [1]) J. G. VAN DER CORPUT deduces three theorems, which enable him to evaluate a great number of determinants. We propose to give simpler proofs for the first and second of these three propositions and to generalise his third theorem.

We make use of the following lemma, where $|e_{rs}|$ denotes the determinant of n columns and n rows, such that e_{rs} is the element in the rth row and sth column.

If $f_s(x)$ and $g_s(x)$ $(1 \leq s \leq n)$ are polynomials of x of degree $\leq n-1$, then we have

$$|f_s(x_r)|\,|g_s(y_r)| = |g_s(x_r)|\,|f_s(y_r)|. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In fact, putting

$$g_s(x) = a_{1s} + a_{2s}\,x + \ldots + a_{ns}\,x^{n-1},$$

we find by the multiplication theorem of determinants

$$|g_s(y_r)| = |a_{1s} + a_{2s}\,y_r + \ldots + a_{ns}\,y_r^{n-1}| = |a_{rs}|\,|y_r^{s-1}|$$

and similarly

$$|g_s(x_r)| = |a_{rs}|\,|x_r^{s-1}|,$$

hence

$$|x_r^{s-1}|\,|g_s(y_r)| = |y_r^{s-1}|\,|g_s(x_r)|.$$

In the same manner we obtain

$$|y_r^{s-1}|\,|f_s(x_r)| = |x_r^{s-1}|\,|f_s(y_r).$$

Consequently

$$|x_r^{s-1}|\,|y_r^{s-1}|\,|g_s(y_r)|\,|f_s(x_r)| = |y_r^{s-1}|\,|x_r^{s-1}|\,|g_s(x_r)|\,|f_s(y_r)|.$$

This identity implies (1).

VAN DER CORPUT's first and second theorem assert:

If $3n-2$ numbers $a_1, a_2, \ldots, a_{n-1}, x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_n$ can be found such that

$$e_{r,s+1} = a_s\,(y_r - x_s)\,e_{rs} \qquad (1 \leqq r \leqq n,\ 1 \leqq s \leqq n-1), \quad . \quad . \quad (2)$$

[1]) J. G. VAN DER CORPUT, Over eenige determinanten. Verh. Kon. Akad. v. Wetensch., Amsterdam, 14, N^0. 3, 44 (1930).

Compare J. POPKEN, Ueber einige Determinanten. Proc. Kon. Akad. v. Wetensch., 35, 542—546 (1932).

and if $\varphi_s(y)$ is a polynomial of y of degree $\leq n-s$, then

$$|e_{rs}\,\varphi_s(y_r)| = |e_{rs}|\,\varphi_1(x_1)\dots\varphi_n(x_n), \quad \dots \quad (3)$$

where

$$|e_{rs}| = e_{11}\dots e_{n1}\,a_1^{n-1}\,a_2^{n-2}\dots a_{n-1}^1 \prod_{1\leq\sigma<\varrho\leq n}(y_\varrho - y_\sigma) \quad \dots \quad (4)$$

To deduce this result we put

$$g_s(x) = a_1\,a_2\dots a_{s-1}(x-x_1)\dots(x-x_{s-1})\,\varphi_s(x) \quad (s=1,\dots,n).$$

In the determinant $|g_s(x_r)|$ each element above the principal diagonal vanishes. Thus this determinant is equal to the product of the elements in the principal diagonal and therefore equal to

$$a_1^{n-1}\dots a_{n-1}^1\,\varphi_1(x_1)\dots\varphi_n(x_n) \prod_{1\leq\sigma<\varrho\leq n}(x_\varrho - x_\sigma).$$

Putting $f_s(x) = x^{s-1}$, we find by our lemma

$$|g_s(y_r)| = a_1^{n-1}\dots a_{n-1}^1\,|y_r^{s-1}|\,\varphi_1(x_1)\dots\varphi_n(x_n).$$

Putting particularly $\varphi_s(x) = 1$, we find (3) and (4) in virtue of

$$e_{rs}\,\varphi_s(y_r) = e_{r1}\,g_s(y_r).$$

Instead of VAN DER CORPUT's third theorem we prove two propositions, the second of which involves his third theorem as a particular case.

Let A_{hk} denote the minor of the element in the h^{th} row and k^{th} column in the determinant $|e_{rs}\varphi_s(y_r)|$. Let the relations (2) be valid, be $1\leq h\leq n$ and *$1\leq k\leq n$; be $\varphi_s(x)$ $(1\leq s<k)$ a polynomial in x of degree $\leq k-s-1$ and $\varphi_s(x)$ $(s\leq k\leq n)$ a polynomial in x of degree $\leq n-s$. Then*

$$A_{hk} = \frac{|e_{rs}\,\varphi_s(y_r)|\;p(y_h)}{e_{h1}\,a_1\dots a_{k-1}\;\varphi_k(x_k)\;q'(y_h)},$$

where $q(y) = (y-y_1)\dots(y-y_n)$ and the polynomial $p(y)$ denotes the coefficient of $\frac{1}{x}$ in the LAURENT *expansion of*

$$\frac{q(x)}{(x-y)(x-x_1)\dots(x-x_k)}.$$

To give a proof we deduce a more general result in which the conditions, imposed on the degrees of the polynomials $\varphi_s(x)$ are replaced by the weaker condition, that the degree of $\varphi_s(x)$ $(1\leq s\leq n)$ is $\leq n-s$. We suppose therefore for the present, that the polynomials $\varphi_s(x)$ $(1\leq s\leq n)$ are arbitrary polynomials of degree $\leq n-s$.

If we put

$$\psi_s(y) = (y-x_1)\dots(y-x_{s-1})\,\varphi_s(y), \quad \dots \quad (5)$$

we find

$$\psi_s(x_r) = 0 \quad \text{for} \quad r < s, \quad \ldots \quad (6)$$

and

$$|e_{rs}\,\varphi_s(y_r)| = \sum_{s=1}^{n} A_{ns}\, e_{ns}\, \varphi_s(y_n) = \sum_{s=1}^{n} A_{ns}\, e_{n1}\, a_1 \ldots a_{s-1}\, \psi_s(y_n). \quad (7)$$

Since A_{nk} does not involve the indeterminate y_n, this indeterminate does not occur by (3) and (4) in

$$B_k = A_{nk} \frac{e_{n1}\, a_1 \ldots a_{k-1}\, (y_n - y_1) \ldots (y_n - y_{n-1})}{|e_{rs}\,\varphi_s(y_r)|} = A_{nk} \frac{e_{n1}\, a_1 \ldots a_{k-1}\, q'(y_n)}{|e_{rs}\,\varphi_s(y_r)|}.$$

Consequently by (7)

$$(y_n - y_1) \ldots (y_n - y_{n-1}) = \sum_{s=1}^{n} B_s\, \psi_s(y_n).$$

Putting y_n successively equal to $x_1, x_2, \ldots, x_k$, we obtain

$$(x_r - y_1) \ldots (x_r - y_{n-1}) = \sum_{s=1}^{k} B_s\, \psi_s(x_r) \qquad (r = 1, \ldots, k)$$

in virtue of (6). Hence

$$B_k = \begin{vmatrix} \psi_1(x_1) \ldots \ldots \psi_{k-1}(x_1) & (x_1 - y_1) \ldots (x_1 - y_{n-1}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \psi_1(x_k) \ldots \ldots \psi_{k-1}(x_k) & (x_k - y_1) \ldots (x_k - y_{n-1}) \end{vmatrix} : \begin{vmatrix} \psi_1(x_1) \ldots \psi_k(x_1) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \psi_1(x_k) \ldots \psi_k(x_k) \end{vmatrix}.$$

In the last determinant each element above the principal diagonal vanishes in virtue of (6), so that this determinant is equal to $\psi_1(x_1) \ldots \psi_k(x_k)$, hence

$$B_k = \frac{1}{\psi_1(x_1) \ldots \psi_k(x_k)} \begin{vmatrix} \psi_1(x_1) \ldots \psi_{k-1}(x_1) \dfrac{q(x_1)}{x_1 - y_n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \psi_1(x_k) \ldots \psi_{k-1}(x_k) \dfrac{q(x_k)}{x_k - y_n} \end{vmatrix}.$$

In this manner we find a value for A_{nk} and similarly we obtain for $h = 1, \ldots, n;\ k = 1, \ldots, n$

$$A_{hk} = \frac{|e_{rs}\,\varphi_s(y_r)|\, D}{e_{h1}\, a_1 \ldots a_{k-1}\, \psi_1(x_1) \ldots \psi_k(x_k)\, q'(y_h)}, \quad \ldots \quad (8)$$

where

$$D = \begin{vmatrix} \psi_1(x_1) \ldots \psi_{k-1}(x_1) \dfrac{q(x_1)}{x_1 - y_h} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \psi_1(x_k) \ldots \psi_{k-1}(x_k) \dfrac{q(x_k)}{x_k - y_h} \end{vmatrix} \quad \ldots \quad (9)$$

We have deduced this result under the condition that the degree of the polynomials $\varphi_s(y)$ is at most $n-s$ $(1 \leqq s \leqq n)$. This formula may be used with advantage for small values of k, since in that case the determinant D can be calculated easily.

In order to calculate D for all values of k we impose on the polynomials $\varphi_s(y)$ the stronger condition that their degree is at most $k-s-1$ for $1 \leqq s < k$ and at most $n-s$ for $k \leqq s \leqq n$.

If D_r denotes the minor in D of the element in the rth row and last column, we obtain by (9) and (5)

$$D_1 = (-1)^{k-1} \begin{vmatrix} \psi_1(x_2) \dots \psi_{k-1}(x_2) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \psi_1(x_k) \dots \psi_{k-1}(x_k) \end{vmatrix} =$$

$$= (-1)^{k-1} \begin{vmatrix} \varphi_1(x_2) & (x_2-x_1)\varphi_2(x_2) \dots \dots (x_2-x_1)\dots(x_2-x_{k-1})\varphi_{k-1}(x_2) \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \varphi_1(x_k) & (x_k-x_1)\varphi_2(x_k) \dots \dots (x_k-x_1)\dots(x_k-x_{k-1})\varphi_{k-1}(x_k) \end{vmatrix}.$$

Since the degree of the polynomials $\varphi_s(x)$, occurring in this determinant D_1, is at most $k-s-1$, we find by (3) and (4) (applied with $k-1$ instead of n, with $x_2, x_3, \dots$ instead of $y_1, y_2, \dots$, with $e_{r1} = 1$ and $a_s = 1$)

$$D_1 = (-1)^{k-1} \varphi_1(x_1) \dots \varphi_{k-1}(x_{k-1}) \prod_{2 \leqq \sigma < \varrho \leqq k} (x_\varrho - x_\sigma)$$

$$= \frac{\varphi_1(x_1) \dots \varphi_k(x_k) \prod_{1 \leqq \sigma < \varrho \leqq k} (x_\varrho - x_\sigma)}{(x_1-x_2) \dots (x_1-x_k)\, \varphi_k(x_k)}$$

$$= \frac{\psi_1(x_1) \dots \psi_k(x_k)}{(x_1-x_2) \dots (x_1-x_k)\, \varphi_k(x_k)}$$

by (5). Similarly we obtain

$$D_r = \frac{\psi_1(x_1) \dots \psi_k(x_k)}{\varphi_k(x_k) \prod_{\substack{\sigma=1 \\ \sigma \neq r}}^{k} (x_r - x_\sigma)},$$

so that the determinant D in (8) is equal to

$$D = \sum_{r=1}^{k} D_r \frac{q(x_r)}{x_r - y_h} = \frac{\psi_1(x_1) \dots \psi_k(x_k)}{\varphi_k(x_k)} \sum_{r=1}^{k} \frac{q(x_r)}{(x_r - y_h) \prod_{\substack{\sigma=1 \\ \sigma \neq r}}^{k} (x_r - x_\sigma)}.$$

Hence

$$A_{hk} = \frac{|e_{rs}\varphi_s(y_r)|}{e_{h1} a_1 \dots a_{k-1} \varphi_k(x_k) q'(y_h)} \sum_{r=1}^{k} \frac{q(x_r)}{(x_r - y_h) \prod_{\substack{\sigma=1 \\ \sigma \neq r}}^{k} (x_r - x_\sigma)}.$$

The term behind the sign Σ is equal to the residu of the function $\dfrac{q(x)}{(x-y_h)\prod\limits_{\sigma=1}^{k}(x-x_\sigma)}$, taken at the point $x=x_r$. The only singular points of this function are $x_1, \ldots, x_k$ and ∞, so that the sum of the residues, taken at the points $x_1, \ldots, x_k$ is equal to the opposite value of the residu, taken at ∞. This last residu has the value $-p(y_h)$ by definition of the polynomial $p(y)$. Thus

$$A_{hk}=\frac{|e_{rs}\,\varphi_s(y_r)|\,p(y_h)}{e_{h1}\,a_1\ldots a_{k-1}\,\varphi_k(x_k)\,q'(y_h)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Theorem. *Let* $|\,e_{rs};\,e_r\,|_k$ *be the determinant, obtained by replacing the elements in the* kth *column of the determinant* $|\,e_{rs}\,|$ *respectively by* $e_1, \ldots, e_n$. *Be the relations* (2) *valid, be* $1\leq k\leq n$, *be* $\varphi_s(x)$ $(1\leq s<k)$ *a polynomial of degree* $\leq k-s-1$ *and be* $\varphi_s(x)$ $(k\leq s\leq n)$ *a polynomial of degree* $\leq n-s$; *be* $f(y)$ *a function, which is analytical with the exception of a finite number of points at most and which is regular in the points* $y_1, \ldots, y_n$. *Then*

$$|\,e_{rs}\,\varphi_s(y_r);\;e_{r1}\,f(y_r)\,|_k=-\frac{|\,e_{rs}\,\varphi_s(y_r)\,|}{a_1\ldots a_{k-1}\,\varphi_k(x_k)}\,S\,\frac{f(y)\,p(y)}{q(y)},$$

where $p(y)$ *and* $q(y)$ *have the same meaning as in the preceding lemma and where* S *denotes the sum of the residues taken at the singular points of* $f(y)$ *and at infinity.*

By putting $k=n$ and therefore $p(y)=1$, we obtain VAN DER CORPUT's third theorem.

Proof. By developing $|\,e_{rs}\,\varphi_s(y_r);\;e_{r1}\,f(y_r)\,|_k$ by the elements of the kth column, we obtain

$$|\,e_{rs}\,\varphi_s(y_r);\;e_{r1}\,f(y_r)\,|_k=\sum_{r=1}^{n}e_{r1}\,f(y_r)\,A_{rk},$$

hence by the preceding lemma

$$|\,e_{rs}\,\varphi_s(y_r);\;e_{r1}\,f(y_r)\,|_k=\frac{|\,e_{rs}\,\varphi_s(y_r)\,|}{a_1\ldots a_{k-1}\,\varphi_k(x_k')}\sum_{r=1}^{n}\frac{p(y_r)\,f(y_r)}{q'(y_r)}.$$

The term behind Σ is the residu of the function $\dfrac{p(y)\,f(y)}{q(y)}$, taken at the point y_r, so that $-\sum\limits_{r=1}^{n}\dfrac{p(y_r)\,f(y_r)}{q'(y_r)}$ is equal to the sum of the residues of this function, taken at all other singular points of $\dfrac{p(y)\,f(y)}{q(y)}$ i.e. at all singular points of $f(y)$ and at infinity. Hence

$$|\,e_{rs}\,\varphi_s(y_r);\;e_{r1}\,f(y_r)\,|_k=-\frac{|\,e_{rs}\,\varphi_s(y_r)\,|}{a_1\ldots a_{k-1}\,\varphi_k(x_k)}\,S\,\frac{f(y)\,p(y)}{q(y)},$$

which is the required result.

Lemma. *A positive definite quadratic form $Q(u_1, \dots, u_n)$ with determinant D is* $\geqq \frac{D u_k^2}{A_k}$ *for $k = 1, \dots, n$, where A_k is the minor of the element in the kth row and kth colum of D, provided that $A_k \neq 0$. Here* $\frac{D}{A_k}$ *is the best possible factor independant of $u_1, \dots, u_n$.*

Proof. Let the indeterminates $u_1, \dots, u_n$ be ordered such, that u_k is the last of them. Then $Q - \frac{D u_k^2}{A_k}$ becomes a quadratic form, all principal determinants of which are $\geqq 0$; the last one is exactly equal to 0. Hence the quadratic form $Q - \frac{D u_k^2}{A_k}$ is positive definite. If $\frac{D}{A_k}$ is replaced by a greater number, then $Q - \frac{D u_k^2}{A_k}$ becomes a quadratic form, the determinant of which is negative, so that this quadratic form is no longer positive definite. Herefrom it follows, that $\frac{D}{A_k}$ is the best possible factor.

We now proceed to give some applications on some special quadratic forms, which we know to be positive definite [2]).

For $\alpha > -2$ we get

$$Q_1 = \sum_{r=1}^{n} \sum_{s=1}^{n} \Gamma(\alpha + r + s)\, u_r u_s \geqq \frac{D}{A_k} u_k^2.$$

Here

$$x_r = -r;\; y_r = \alpha + r;\; e_{r1} = \Gamma(\alpha + r + 1);\; a_r = 1;\; \varphi_r(y) = 1.$$

By (8) and (9) we find

$$Q_1 \geqq \frac{\Gamma(n)\,\Gamma(\alpha+2)\,\Gamma(\alpha+3)}{\Gamma(\alpha+n+2)} u_1^2.$$

Moreover we find

$$\frac{A_2}{D} = \frac{\dfrac{(x_2-y_1)\dots(x_2-y_n)}{x_2-y_2} - \dfrac{(x_1-y_1)\dots(x_1-y_n)}{x_1-y_2}}{(x_2-x_1)\,\Gamma(\alpha+3)\,q'(y_2)}$$
$$= \frac{\Gamma(\alpha+n+2)\,\{(n-1)\,\alpha + 3\,n - 2\}}{\Gamma(n-1)\,\Gamma(\alpha+3)\,\Gamma(\alpha+5)}.$$

Hence

$$Q_1 \geqq \frac{\Gamma(n-1)\,\Gamma(\alpha+3)\,\Gamma(\alpha+5)}{\Gamma(\alpha+n+2)\,\{(n-1)\,\alpha + 3n - 2\}} u_2^2.$$

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[2]) Confer J. G. VAN DER CORPUT and J. POPKEN, Ueber den kleinsten Wert einiger quadratischen Formen, Proc. Kon. Akad. v. Wetensch., Amsterdam, **34** (1931), quoted as. C. P.

Further

$$\frac{A_3}{D}=(-1)^{n-1}\frac{\dfrac{q(x_1)}{x_1-y_3}-\dfrac{2q(x_2)}{x_2-y_3}+\dfrac{q(x_3)}{x_3-y_3}}{4\,\Gamma(n-2)\,\Gamma(a+4)}$$

$$=\frac{\Gamma(a+n+2)\,\{(a+4)(a+5)\,n^2-(a+4)(3a+11)n+2(a+2)(a+3)\}}{4\,\Gamma(n-2)\,\Gamma(a+4)\,\Gamma(a+7)},$$

hence

$$Q_1\geqq\frac{4\,\Gamma(n-2)\,\Gamma(a+4)\,\Gamma(a+7)}{\Gamma(a+n+2)\,\{(a+4)(a+5)n^2-(a+4)(3a+11)n+2(a+2)(a+3)\}}\,u_3^2.$$

By (10), applied with $h=k=n$ and $p(y)=1$, we obtain the result, occurring in CP., p. 624

$$Q_1\geqq\Gamma(n)\,\Gamma(a+n+1)\,u_n^2.$$

We apply the same formula with $h=k=n-1$ and

$$p(y)=y+x_1+\ldots+x_{n-1}-y_1-\ldots-y_n,$$

therefore

$$p(y_{n-1})=-(n-1)\,a-n^2+n-1,$$

whereas

$$q'(y_{n-1})=-\Gamma(n-1),$$

hence

$$\frac{A_{n-1}}{D}=\frac{(n-1)\,a+n^2-n+1}{\Gamma(n-1)\,\Gamma(a+n)}\text{ and }Q_1\geqq\frac{\Gamma(n-1)\,\Gamma(a+n)}{(n-1)\,a+n^2-n+1}\,u_{n-1}^2.$$

In the same manner we obtain

$$Q_1\geqq\frac{4\,\Gamma(n-2)\,\Gamma(a+n-1)}{(n-1)(n-2)\,a^2+(n-2)(2n^2-3n+5)a+n^4-4n^3+9n^2-14n+12}\,u_{n-2}^2.$$

Replacing a by $2a+1$ and u_r by $\dfrac{u_r}{\Gamma(a+r+1)}$ we obtain for $a>-\dfrac{3}{2}$:

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\binom{2a+r+s}{a+r}u_r\,u_s$$

is

$$\geqq\frac{\binom{2a+2}{a+1}}{\binom{2a+n+2}{2a+3}}\,u_1^2$$

and

$$\geqq\frac{\binom{2a+3}{a+1}}{\binom{2a+n+2}{2a+4}}\,\frac{2a+5}{a+2}\,\frac{u_2^2}{2(n-1)\,a+4n-3}$$

and

$$\geqq \frac{\binom{2a+6}{a+3}}{\binom{2a+n+2}{2a+5}} \frac{4a+14}{2a+5} \frac{u_3^2}{2(n-1)(n-2)a^2+(n-2)(11n-7)a+15n^2-35n+12}$$

and

$$\geqq \frac{\binom{2a+2n-4}{a+n-2}}{\binom{2a+2n-4}{2a+n-1}} \frac{4u_{n-2}^2}{4(n-1)(n-2)a^2+2(n-2)(2n^2-n+3)a+(n-1)(n^3-n^2+2n-4)}$$

and

$$\geqq \frac{\binom{2a+n}{a+1}}{\binom{a+n-1}{a+1}} \frac{u_{n-1}^2}{2(n-1)a+n^2}$$

and

$$\geqq \frac{\binom{2a+2n}{a+n}}{\binom{2a+2n}{2a+n+1}} u_n^2.$$

Zoology. — *The influence of high concentrations of $CaCl_2$ on maturation in the egg of Limnaea stagnalis.* By CHR. P. RAVEN and J. C. A. MIGHORST. (From the Zoological Laboratory, University of Utrecht.)

(Communicated at the meeting of October 26, 1946.)

RAVEN and KLOMP (1946) showed that the cleavage of the eggs of *Limnaea stagnalis* in distilled water and solutions of urea and sucrose is abnormal; the blastomeres, after their separation by the cleavage furrows, do not join and flatten against each other, but remain nearly spherical; the vitelline membrane withdraws from the egg surface, beginning at the cleavage furrows, and loosely surrounds the blastomeres at last. As the addition of a small quantity of $CaCl_2$ to the medium leads to normal cleavage, it was concluded that the observed abnormalities are due to the lack of $Ca^{\cdot\cdot}$-ions in the surrounding fluid, affecting the properties of the vitelline membrane.

O. HUDIG (1946) studied the influence of weak $CaCl_2$-solutions, varying between 0.005 % and 0.2 %, on the development of the *Limnaea* egg. Her experiments lead to the conclusion that both the properties of the vitelline membrane and of the egg cortex proper are altered in the absence of $Ca^{\cdot\cdot}$-ions. The vitelline membrane shows a swelling in distilled water; in consequence of this, the first polar body is formed inside the membrane. In weak solutions of $CaCl_2$, the swelling of the membrane is prevented and the first polar body lies in most cases outside the membrane, like in normal eggs in the egg capsule fluid. Furthermore, in concentrations varying from 0.005 % to 0.04 % $CaCl_2$ the cleavage is normal, the blastomeres flatten against each other and a cleavage cavity is formed.

In view of the important part played by $Ca^{\cdot\cdot}$-ions in the phenomena of maturation and activation in other eggs, and their known influence on the properties of the egg cortex, it seemed desirable to study the influence of higher concentrations of $CaCl_2$ on the egg of *Limnaea*. These experiments, however, meet with a difficulty which thwarts the analysis. Whereas it is possible, in marine eggs, to study the influence of certain ions by varying the composition of the medium without changing its isotonicity or pH with respect to sea water, here a rise in the concentration of $CaCl_2$ involves an augmentation of osmotic pressure too. Most of the solutions studied were hypertonic to the eggs. It is not easy to decide, therefore, whether an observed effect is due to the influence of $Ca^{\cdot\cdot}$-ions or to hypertonicity. As a matter of fact, however, a comparison with experiments performed with other eggs gives sufficient evidence by which the two influences can be distinguished. Of course, further experiments will be needed to make sure of the conclusions arrived at.

Material and methods.

Egg-masses were obtained in the usual way by stimulation of the snails with *Hydrocharis* (RAVEN and BRETSCHNEIDER 1942). The eggs were liberated immediately after oviposition by pricking the egg capsules on a dry glass plate under a binocular microscope, washed in distilled water in order to remove the adhering egg capsule fluid, and then transferred to the solution to be studied.

The $CaCl_2$-solutions were made from crystallized calcium chloride containing about 25 % of water; the percentages given are computed to net weights of $CaCl_2$. Solutions of 0.2 % to 3 % have been used. The eggs were exposed to the solutions at room temperatures of $\pm$ 20° C.

Behaviour of the eggs in various concentrations of $CaCl_2$.

In the higher concentrations studied, the eggs shrink considerably in consequence of the marked hypertonicity of the medium; at the same time, they become less transparent. Soon, cytolysis sets in, preceded by a renewed swelling of the eggs, even beyond their original volume. E.g. in 3 % solutions, after 20 minutes all eggs show cytolysis; in weaker solutions, cytolysis occurs less regularly and after a greater lapse of time. The behaviour of the eggs of different egg-masses varies widely; evidently, differences in susceptibility of individual eggs and between different egg-masses play an important part in their reactions.

In 3 % and 1.5 % $CaCl_2$ solutions, no polar bodies are formed. In a 1.1 % solution, some eggs extrude the first polar body; it is very small and spindle-shaped. In 0.75 % solutions, a larger percentage of the eggs form the first polar body; exceptionally, also a second polar body may appear. No further development takes place, however. In 0.5 % solutions, the differences in susceptibility between the egg-masses are very evident; whereas in some batches nearly all eggs extrude both polar bodies, in other egg-masses only one polar body is formed in most eggs. In 0.4 % $CaCl_2$, nearly all eggs develop to the 2-cell stage, whereas in 0.2 % solutions a morula stage may be attained.

Interesting results were obtained by transferring the eggs to distilled water after a temporary treatment with $CaCl_2$; of course, they have to be washed thoroughly in order to remove the adhering fluid. After 5 minutes in $CaCl_2$ 0.56 %, most eggs develop in distilled water in the same way as the controls transferred immediately to distilled water; they show the abnormal cleavage which is characteristic of this medium, and development stops at the 4-cell stage. Some eggs, however, can develop further into a rather normal morula. After a stay of 30 minutes in 0.56 % $CaCl_2$, all eggs develop into a normal morula. Eggs, subjected for 10 minutes to 1.1 % $CaCl_2$, can develop into a morula after return to distilled water; after a stay of 30 minutes in 1.1 % $CaCl_2$, both polar bodies are formed and a normal 4-cell stage is reached in distilled water. It seems, therefore, that the egg can accumulate $Ca^{\cdot\cdot}$-ions from a concentrated solution, and form, in this way, a reserve which influences the course of further development.

In higher concentrations of $CaCl_2$, small clear spots appear in the vitelline membrane; after some hours, in these places small vesicles have been formed on the surface of the egg. Still later, the vitelline membrane is lifted locally from the egg surface, forming great blebs. Both polar bodies may lie within such blebs; this forms an argument in favour of the view considered by Miss HUDIG that the position of the first polar body outside the membrane is only apparent, whilst it is surrounded smoothly by the membrane in reality.

Influence of $CaCl_2$ upon the maturation.

In order to study the influence of $CaCl_2$ upon the maturation, a number of eggs were subjected to a solution of 0.5 % $CaCl_2$. This solution was chosen because it is the lowest concentration in which a large percentage of eggs extruded only one polar body. Furthermore, it is only slightly hypertonic to the eggs so that we may expect that the observed effects will largely be due to the chemical composition of the medium.

Eggs from 3 different egg-masses *A*, *B* and *C* have been studied. They were decapsulated immediately after oviposition and transferred to 0.5 % $CaCl_2$. Samples of these eggs were fixed at 3 different moments: 1°. 15 minutes after the controls (normal eggs of the same egg-mass left within the capsules) had formed the second polar body (*A* 1, *B* 1, *C* 1); 2°. 35 minutes after the formation of the second polar body in the controls (*A* 2, *B* 2, *C* 2); and 3°. when the controls showed the first cleavage (*A* 3, *B* 3, *C* 3).

The difference in behaviour between different egg-masses manifested itself also in this case. In most eggs of *A*, the maturation proceeded in a normal way. Only in 2 of the 15 eggs sectioned, a marked delay in the maturation divisions, surpassing the normal range of variation, could be observed; one egg of sample *A* 2 showed a second maturation spindle in metaphase, 35 minutes after the controls had formed the second polar body; whereas in an egg of sample *A* 3, at the time of first cleavage in the controls, the second polar body was just being formed.

In the eggs of egg-mass C, the disturbance of the maturation divisions was more pronounced. 35 Minutes after the formation of the second polar body in the controls, the eggs in the $CaCl_2$ solution had not yet formed their second polar body. They show signs of depolarization, like those of *B* described below. At the time of first cleavage in the controls, the eggs of C 3 have extruded both polar bodies; in some cases, however, these are very big; perhaps, they may be considered as "giant polar bodies". At the animal pole of the eggs, 2 pronuclei have been formed. As the number of eggs of this batch is rather low, however, it will not be discussed in detail.

In the eggs of egg-mass *B*, the effect of the treatment is most evident. In only 1 out of 17 eggs of this batch, a second polar body has been formed; in the other eggs, its formation has been suppressed. The first maturation division is not interfered with; the first polar body is extruded in a normal way. As a matter of fact, in some of the sectioned eggs no first polar body has been observed in the sections; as they agree in all other respects with the other eggs of their sample, however, it is evident that this polar body has been lost during the preparation of the sections. After the completion of the first maturation division, a second maturation spindle is formed. In normal eggs, this spindle places itself in the egg axis; its outer end comes into contact with the egg cortex at the animal pole. In the eggs in the $CaCl_2$ solution, however, the second maturation spindle sinks into the interior of the egg and places itself perpendicular to the egg axis. In this position, the nuclear division takes place, be it with great delay as compared with the controls; ultimately, it leads to the formation of 2 submerged groups of karyomeres. A cell division, following the submerged division of the nucleus, has never been observed.

Some examples, illustrating this course of events, will be given:

B 1—9 (fig. 1). Fixed 15 minutes after formation of 2d polar body in controls. At animal pole one polar body. In interior of egg 2d maturation spindle in meta-to early anaphase, perpendicular to egg axis; it is situated at a considerable distance from the animal pole, near the centre of the egg, but still in the animal half. No spermaster can be found in the egg.

B 1—Va (fig. 2). Same sample as B 1—9. One polar body. 2d Maturation spindle in late anaphase, with rather large asters, at some distance from animal pole and perpendicular to egg axis. No spermaster.

B 2—IIIa (fig. 3). Fixed 35 minutes after formation of 2d polar body in controls. First polar body in other section. 2d Maturation spindle near centre of egg, perpendicular to egg axis. It is in late anaphase; the asters have grown very large. No spermaster or male pronucleus.

B 2—9 (fig. 4). Same sample as B 2—IIIa. First polar body in other section. Two groups of karyomeres in animal half of egg (one of these shown in section); they have a somewhat abnormal appearance, each containing a swollen chromosome in a clear vacuolar space. No spermaster or male pronucleus.

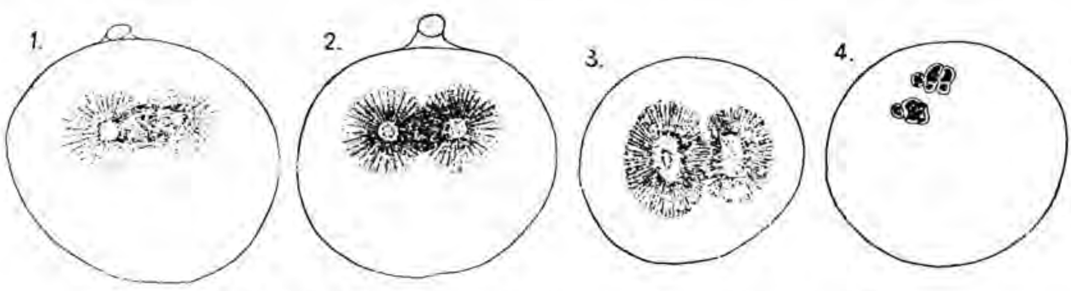

Fig. 1—4. Depolarization of second maturation spindle in eggs of *Limnaea stagnalis* in 0.5 % $CaCl_2$.

1. B 1—9. Early anaphase. 2. B 1—Va. Late anaphase.
3. B 2—IIIa. Late anaphase. Enlargement of asters.
4. B 2—9. Karyomere nucleus.

The two last-mentioned eggs show the great disparity in stage of development between different eggs fixed at the same moment; this is also clear in sample *B* 3, where most eggs have reached the karyomere stage, but one still shows a 2d maturation spindle in metaphase.

Discussion.

1. The effects of the treatment of the eggs with concentrated solutions of $CaCl_2$ are, in part, of a rather general nature. The eggs shrink by loss of water; in the higher concentrations, this shrinkage may be followed by cytolysis. The development comes to a standstill; the stronger the solution employed, the earlier development stops. All these effects may be attributed to the hypertonicity of the medium; in hypertonic solutions of urea and sucrose the same phenomena have been observed.

2. In $Ca^{\cdot\cdot}$-free media, the cleavage of the eggs of *Limnaea* is abnormal. Addition of a small amount of $CaCl_2$ to the medium leads to normal cleavage. In solutions of $CaCl_2$ of increasing concentrations, as long as they permit development, cleavage has a normal course. When the eggs are subjected, immediately after oviposition, for a short time to still higher concentrations of $CaCl_2$, they may afterwards, upon return to distilled water, cleave normally. Evidently, they accumulate $Ca^{\cdot\cdot}$-ions from the surrounding solution, forming a reserve which suffices to warrant normal cleavage.

3. In a solution of 0.5 % $CaCl_2$, in some egg-masses development is blocked during maturation. The first polar body is formed normally, but the second maturation spindle looses its contact with the animal pole, sinks into the interior of the egg and assumes a position at right angles to the egg axis. The nuclear division may take place and leads to the formation of 2 groups of karyomeres; it is not followed by a cell division.

This aberration of development clearly belongs to a class of phenomena, known as "depolarization": the displacement of the maturation spindle expresses the fact that the forces working in the direction of the polar axis have been diminished in strength.

This notion of "depolarization" has first been used by DALCQ (1925) in his study on the influence of the composition of the medium on maturation and activation in *Asterias glacialis*. The following phenomena are brought under this heading: The first maturation spindle looses its connection with the egg cortex (in other cases it does not even ascend to the animal pole, but retains its original position in the interior of the egg); its centrosomes enlarge, ultimately they divide and a pluripolar spindle is formed. The second maturation spindle may place itself perpendicular to the egg axis; it grows considerably in length; a temporary furrow may indent the egg from the animal pole in the plane of the spindle equator. The maturation may come to a standstill in the metaphase of the first or second maturation division, or it may be completed but giant polar bodies are formed. Finally, the sperm nuclei may imitate the condition of the maturation nucleus ("mise à l'unisson").

This depolarization of the *Asterias* egg may arise spontaneously or it may be brought about by a variety of methods: isotonic solution of pure $CaCl_2$ or other isotonic media with excess of $CaCl_2$; addition of sucrose or of CO_2 to the sea water; diluted sea water. It is made probable that also in these latter cases the $Ca^{\cdot\cdot}$-ions are the real cause of the phenomenon.

PASTEELS (1930) obtained depolarization in the eggs of *Barnea candida* by treatment with $CaCl_2$-solutions; $MgCl_2$ had a similar, but weaker influence; also by dilution of the sea water depolarization could be brought about. The effect is different in unfertilized and fertilized eggs. In unfertilized eggs, the first maturation division remains blocked in metaphase; the spindle poles may divide so that a polycentric spindle is formed. When fertilized eggs are subjected to the treatment, the reaction of the eggs may differ: the first maturation division may take place in the interior of the egg and lead to the formation of 2 independent second maturation spindles or one pluripolar spindle. In other cases, only the second maturation spindle sinks into the interior of the egg and remains blocked in metaphase. A marked enlargement of spindle and asters may occur secondarily, when the division is blocked in metaphase.

In later experiments (PASTEELS 1938), in which unfertilized *Barnea* eggs were subjected to increasing doses of $CaCl_2$, it was found that depolarization can be induced in two ways. With very low concentrations of $CaCl_2$, the eggs are insufficiently activated; the maturation divisions begin, but are not completed normally, the spindle sinks into the interior of the egg, no polar bodies are formed. Increasing doses of $CaCl_2$ lead to complete activation and expulsion of 1 or 2 polar bodies. But with still higher concentrations of $CaCl_2$ a new phase of depolarization, first of the second, then of the first maturation division occurs. This is explained by a secondary paralysis of the maturation divisions by a too high concentration of $CaCl_2$.

TYLER (1931) reports on polar body suppression in eggs of *Urechis caupo* poorly activated by sub- or supra-optimum exposure to diluted sea water. In these eggs, the polar spindle becomes centrally located and places itself perpendicular to the polar axis; it is converted into a cleavage spindle and division of the egg occurs (TYLER 1932). The failure of the spindle to approach the animal pole appears to be correlated with a retardation in the surface changes of the egg.

Similar results were obtained by TYLER and BAUER (1937) in *Urechis caupo* by short exposure of the eggs to ammoniacal sea water. On the other hand, by over-exposure to this activating agent "submerged" maturation divisions occurred: the first maturation division takes place in the interior of the egg and gives rise to the formation of two nuclei; two spindles appear in the egg at the time of the second polar division. When the over-exposure is not too great the two spindles come to the surface, where each gives rise to a polar body. Slightly longer treatment results in only one of the spindles attaining the surface position, where it extrudes a polar body. Still longer exposure causes both to divide submerged within the egg. The polar bodies can be suppressed in fertilized eggs or in eggs optimally activated by hypotonic solution by means of subsequent exposure to ammoniacal sea water (TYLER 1941).

Finally, LINDAHL (1941) reports on depolarization phenomena in eggs of *Srongylocentrotus droebackiensis* exposed to supra-optimum temperature. The maturation spindle grows in length, its asters enlarge; furthermore, it may remain in the interior of the egg and give rise to the formation of giant polar bodies. LINDAHL relates the effect of the treatment to changes in the viscosity of the egg protoplasm.

Probably, the different agents bringing about depolarization in the eggs of various animals have something in common, which is the true cause of the phenomenon; whether the $Ca^{\cdot\cdot}$-ions (DALCQ) or the changes in the viscosity (LINDAHL) or still another agent are to be considered as such, cannot be decided at this moment.

4. In the depolarized eggs of *Limnaea*, the ascent of the spermaster and sperm nucleus to the animal pole, which is accomplished in normal eggs during and in the first hour after the maturation divisions, does not occur. Hence, it seems probable that the same factors which are, in normal eggs, responsible for the fixation of the maturation spindle at the animal pole, attract the spermaster and sperm nucleus to this pole, too. Whether the accumulation of the animal pole plasm is due to the same attractive force, could not yet be made out. In our preparations of depolarized eggs, this plasm is not clearly recognizable, in consequence of inappropriate staining; further experiments are needed to answer this question.

5. Treatment with $CaCl_2$-solutions gives us a means to influence the polar gradient-field of the egg. The further development of depolarized eggs, after return to a medium more suitable to development, is still to be studied. A comparison of the influence of $CaCl_2$-solutions with that of LiCl, which also acts upon the polar gradient-field (RAVEN 1942), may yield interesting results.

Summary.

1. Eggs of *Limnaea stagnalis* were treated with solutions of $CaCl_2$, varying between 0.2 % and 3 %, immediately after oviposition.

2. The development of the egg soon comes to a standstill; the stronger the solution employed, the earlier development stops. This effect may be attributed to the hypertonicity of the medium.

3. Cleavage is normal in these solutions. In a concentrated solution of $CaCl_2$, the eggs accumulate $Ca^{\cdot\cdot}$-ions, forming a reserve which suffices to warrant normal cleavage after their return to distilled water.

4. In a solution of 0.5 % $CaCl_2$, in some egg-masses a depolarization

of the eggs occurs. The second maturation spindle looses its contact with the animal pole, becomes centrally located and assumes a position at right angles to the polar axis.

5. The ascent of the spermaster and sperm nucleus to the animal pole is likewise suppressed in these depolarized eggs.

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Physics. — *The distribution of energy in continuous X-ray spectra corresponding to different forms of high tension, and the influence of filtering.* Section II. By R. H. DE WAARD (*X*-ray department of the Medical University Clinic, Utrecht) *). (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of September 21, 1946.)

II. *Application to special cases and comparison with experimental data.*

Discussion of the curves of fig. 1.

The formulae obtained in section I will now be compared with the experimental data contained in fig. 2. The curves represented in this figure bear on experiments carried out by BOUWERS with an *X*-ray tube fed by single phase alternating tension with peak-values of 70 and 50 kV respectively. As to these curves it is important to notice that — as the author was told by Dr. BOUWERS personally — they are drawn to somewhat different scales of the ordinates. This implies that the proportion of the ordinates of the two curves is to be considered as perfectly arbitrary. Therefore, if these curves are to be compared with theoretical ones, any of the latter may also be drawn to an arbitrary scale of ordinates. In this paper the scales will be chosen in such a way as to give corresponding curves maxima of equal height. Now, according to the theory, the distribution of spectral energy in the unfiltered radiation corresponding to single phase alternating tensions with peak-values of 70 and 50 kilovolts is given by the formulae

$$\overline{E_\lambda} = \frac{Li}{\lambda_p}\,\frac{2}{\pi}\,(\sin\varphi - \varphi\cos\varphi)\,\frac{1}{\lambda^2} \text{ and } \cos\varphi = \frac{\lambda_p}{\lambda}$$

where

$$\lambda_p = \frac{12.3}{70} = 0.176\text{ Å and } \lambda_p = \frac{12.3}{50} = 0.247\text{ Å}$$

respectively. The resulting relations between $\overline{E_\lambda}$ and λ are graphically represented by the curves of fig. 14 and we see that the courses of

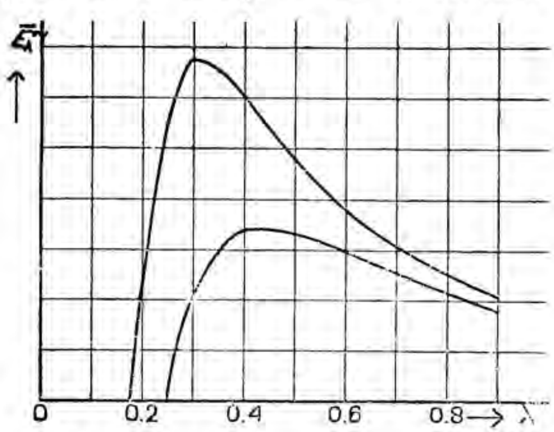

Fig. 14. Theoretical $\overline{E_\lambda}$, λ-curves for peak voltages $V_p = 70$ kV (upper curve) and $V_p = 50$ kV (lower curve).

*) Introduction and Section I in Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, p. 944 (1946).

corresponding curves are rather different. The curves of fig. 1 can, however, be remarkably well represented by formulae of the form (21) referring to filtered radiation. Let us first consider the case of a peak-voltage of 70 kV where $\lambda_p = 0.176$ Å. If we substitute in (19) 0.176 for λ_p and in (21) 3.54 for f we obtain the relation between E_λ^- and λ which is graphically represented by the upper curve of fig. 15 and we see that this

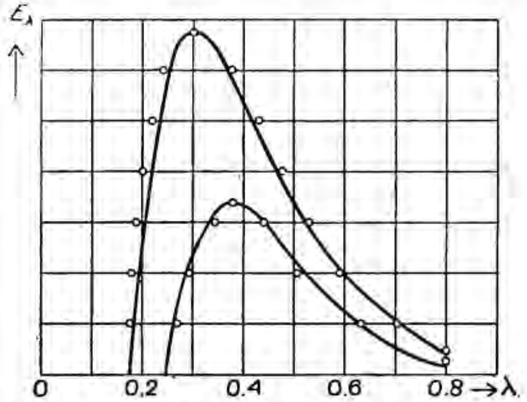

Fig. 15. Theoretical E_λ^-, λ-curves. Upper curve: $V_p = 70$ kV, $f_i = 3.54$. Lower curve: $V_p = 50$ kV, $f_i = 5.83$. The circles indicate the courses of BOUWERS' experimental curves given in fig. 2.

curve is, indeed, hardly different from the corresponding one in fig. 2. The value of the quantity g is of no importance here: it is independent of λ and so its effect is entirely eliminated by our agreement on the scale of ordinates to be chosen.

The wavelength λ_m of maximum intensity can be easily calculated. We have

$$f\lambda_p^3 = 0.0193$$

and curve B of fig. 4 shows that the corresponding value of $\cos\varphi_m = \frac{\lambda_p}{\lambda_m}$ is 0.60; hence

$$\frac{\lambda_m}{\lambda_p} = 1.67$$

a value which is in satisfactory agreement with that deduced by BOUWERS from the experimental curve, i.e. 1.65.

In the case of a peak-value of 50 kV we have $\lambda_p = 0.246$ Å, and if we substitute in (19) 0.246 for λ_p and in (21) 5.83 for f we arrive at the relation between E_λ^- and λ which is graphically represented by the lower curve of fig. 15. This curve, too, hardly differs from the corresponding one in fig. 2. The value of $\frac{\lambda_m}{\lambda_p}$, however, is smaller than in the preceding case; we have

$$f\lambda_p^3 = 0.088$$

and consequently

$$\cos \varphi_m = 0.66 \text{ and } \frac{\lambda_m}{\lambda_p} = \frac{1}{\cos \varphi_m} = 1.52$$

Obviously the quantities $f = 3.54$ and $f = 5.83$ will, among other things, account for the influence of the window of the tube on the composition of the radiation emitted. This influence, however, can be shown to be of comparatively little importance. In the case in question the windows were aluminium foils of 0.005 cm thickness, and as $C = 37$ the corresponding value of Cd is 0.158 and so, indeed, rather small when compared with 3.54 or 5.83.

The result obtained makes it clear that even the difference of the latter quantities cannot be due to properties of the windows themselves.

However, these windows often bear a thin coat of tungsten on their inner surface, and we will see that the presence or absence of such a coat can easily explain the difference in question. According to a formula given by W. H. and W. L. BRAGG [4]) we have $C_{wo} = 6350$; the difference can, therefore, be accounted for by a tungsten thickness of $\frac{5.83 - 3.54}{6350} =$ $= 0.00036$ cm, and we may expect that in practice coats of such a thickness will, indeed, often occur. There is, therefore, no reason why the difference should be considered as a feature of essential importance.

As to the values of the quantity g corresponding to the tensions to which the curves of fig. 1 refer there is evidence that they, too, are not essentially different. We have seen that such evidence cannot be contained in fig. 2 itself. The experimental curves by which KULENKAMPFF was led to the fundamental assumption (1), however, show a marked parallelism in the region of small wavelengths where the influence of the factor $e^{-f\lambda^3}$ is unimportant, and this parallelism is inconsistent with any important difference between the factors e^{-g} and consequently between the values of the quantity g. We are therefore led to the conclusion that these values are approximately equal. If this conclusion is correct the formulae obtained enable us to calculate the actual proportion of the height of the maxima of E_{λ}^{-}, λ-curves corresponding to different tensions. The g's being approximately equal they can be simply omitted in the calculations in question. It may be observed in this connection that a tungsten thickness of 0.00036 cm by which the difference of the quantities $f = 3.54$ and $f = 5.83$ can be accounted for will not bring about any sensible difference of the g's either.

We have seen that one phenomenon accounted for by the quantities $f = 3.54$ and $f = 5.83$ is the absorption in the window and in a tungsten

[4]) *X*-rays and crystal structure, London 1924, p. 42. The formula in question is $C = \frac{C^* N^4}{A} \varrho$, where N is the atomic number $(= 74)$, A the atomic weight $(= 182.6)$, ϱ the specific weigt $(= 19.1)$, and $C = 0.00188$ in the range of wavelengths between the K- and L-lines ($\lambda = 0.178$ Å and $\lambda = 1.02$ Å respectively).

coat on its inner surface. Another phenomenon is the internal absorption in the anode. The electrons falling on the anode of an X-ray tube penetrate in it to a certain depth, and so the X-radiation leaving the anode will have its origin some distance under the focal area. This distance must be expected to be somewhat smaller than the range of electrons propagating in tungsten, and an estimate of it can be derived from experimental data on the range of electrons in aluminium [5]); in case of tensions of about 60 kV the range in tungsten is of the order of 0.0003 cm. The radiation excited in an X-ray tube will therefore have to traverse a tungsten-thickness of something like 0.0002 cm before leaving the anode.

Now a calculation shows that the total tungsten thicknesses corresponding to the values 3.54 and 5.83 of f are of the order of 0.0005—0.001 cm. As the internal absorption accounts for about 0.0002 cm an additional thickness of 0.0003—0.0008 cm is therefore required. The simplest possible assumption is that this additional thickness is altogether due to the window of the tube and to tungsten coats covering its inner surface. This assumption is quite consistent with a difference of 0.00036 cm in thickness of tungsten coats as found above, and it can therefore be considered as reasonable.

Intrinsic tube filter.

We have seen that experimental evidence suggests that the composition of the radiation emitted by an X-ray tube is that given by the theory when applied to the case of a definite filter. This filter will be called the intrinsic tube filter. If f and g are its characteristic quantities the radiation emitted is characterized by the formula

$$E_\lambda^- = \frac{L\,i}{\lambda_p}\,\frac{2}{\pi}\,(\sin\varphi - \varphi\cos\varphi)\,\frac{1}{\lambda^2}\,e^{-(f_i\lambda^3 + g_i)}$$

where L is a constant characteristic of the tube. Now, as this is also true for the constant g_i the same will hold for the product

$$L\,e^{-g_i};$$

when denoting this product Λ we can simplify the formula for E_λ^- to

$$E_\lambda^- = \frac{\Lambda\,i}{\lambda_p}\,\frac{2}{\pi}\,(\sin\varphi - \varphi\cos\varphi)\,\frac{1}{\lambda^2}\,e^{-f_i\lambda^3}.$$

Consequently we will obtain the formula characterizing the radiation after it has traversed an external filter (f_e, g_e) by substituting in (21)

$$f_i + f_e \text{ for } f \text{ and } g_e \text{ for } g.$$

Obviously the same substitutions can be performed in the formulae applying to other forms of tension. It follows from what has been said

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[5]) RASETTI, Elements of nuclear physics, London and Glasgow 1937, fig. 15. From the curve for aluminium approximate data for tungsten can be derived by multiplication of the ordinates with the ratio of the quantities $\frac{N\varrho}{A}$ (See p. 1013, footnote).

that an essential physical phenomenon accounted for by the introduction of the intrinsic filter might be the internal absorption of X-radiation in the anode

A few applications.

In conclusion we will apply our formulae to the following special case: X-radiation due to an apparatus for single phase alternating tension traverses a 20 cm thick watersheet, the peak-value of the high tension being 80 kV. We then have

$$\lambda_p = \frac{12.3}{80} = 0.154 \text{ Å}.$$

Now, as $C_{\text{water}} = 2.5$ we have for the sheet of water

$$(Cd)_{\text{water}} = f_e = 2.5 \times 20 = 50$$

and we will assume that for the intrinsic tube filter

$$(Cd)_{\text{intr. filter}} = f_i = 4.$$

We then have $\lambda_p = 0.154$ Å and $f = f_i + f_e = 4 + 50 = 54$ and the wavelength for which the resulting expression for $E_\lambda^{\sim}$ assumes its maximum value can be easily found with the help of fig. 4. We have $f\lambda_p^3 = 0.197$ and consequently $\cos \varphi_m = \dfrac{0.154}{\lambda_m} = 0.71$ and

$$\lambda_m = 0.22 \text{ Å}.$$

In case of constant tension the result would have been

$$\lambda_m = 0.20 \text{ Å}$$

and the value $\lambda_m = 0.22$ Å would have corresponded to a 3.8 cm thick watersheet.

Summary.

In section I the distribution of energy in the continuous X-ray spectra corresponding to different forms of high tension is derived from a simple assumption suggested by experimental curves given by KULENKAMPFF, and in addition the influence of filtering is calculated. It appears that any of the theoretical distributions obtained can be largely characterized by two wavelengths which can be determined in a simple way, viz. the minimum wavelength λ_p and the wavelength λ_m of maximum intensity.

In section II it is shown that the distribution of energy in the continuous spectrum of the radiation leaving the window of an X-ray tube is in good agreement with the theoretical formulae referring to filtered radiation. The filter in question will be called the intrinsic tube filter. The introduction of this filter makes it possible to account for the distribution of energy in continuous X-ray spectra, both before and after filtering, by the formulae and curves obtained in section I.

Physics. — *The intensity of scattered X-radiation in medical radiography.* II. By R. H. DE WAARD. (*X*-ray department of the Medical University Clinic, Utrecht) *). (Communicated by Prof. H. R. KRUYT.)

(Communicated at the meeting of September 21, 1946.)

8. *Discussion of critical case.*

The curves I and II of fig. 9 are of different types, and the difference has some importance from a physical point of view. When h increases the slope of curve I increases as well whereas the slope of curve II decreases and tends to zero. Moreover, in the case of curve II, the quantity $\frac{S}{D}$ tends to a limit which is never surpassed. It is now clear that in case I a given increase of h will be most effective when h is large whereas in case II the effect of such an increase will decrease with increasing h.

From formulae (12) and (19) we see that in the cases I and II we have $a > 0$ and $a < 0$ respectively. The case corresponding to $a = 0$ is an intermediary between I and II; in this "critical" case we have $\mu = \xi$ and in view of (15)

$$\mu = \frac{2\,k_1\,\gamma_1}{\gamma_1^2-1} + \frac{2\,k_2\,\gamma_2}{\gamma_2^2-1}.$$

Now, if

$$\frac{2\,k_1\,\gamma_1}{\gamma_1^2-1} + \frac{2\,k_2\,\gamma_2}{\gamma_2^2-1} = \mu_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

it is easy to see that

$\mu < \mu_c$ in the cases I
$\mu = \mu_c$ in the critical case, and
$\mu > \mu_c$ in the cases II.

Let us now consider the critical case in some detail. From (15) and (18) we see that in this case we have $m = 1$ and that the general solution of the integral equation does not hold. This solution must, in fact, be replaced by a solution of the form

$$S = D\,\frac{A_c\,h + B + C_1\,e^{-ch}}{1 - C_2\,e^{-ch}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where the quantities A_c, B, C_1, C_2, c can be calculated by means of the following set of formulae

$$\left.\begin{aligned} \mu_c &= \frac{2\,k_1\,\gamma_1}{\gamma_1^2-1} + \frac{2\,k_2\,\gamma_2}{\gamma_2^2-1} \\ p &= (\mu_c\,\gamma_1)^2 + (\mu_c\,\gamma_2)^2 - 2\,k_1\,(\mu_c\,\gamma_1) - 2\,k_2\,(\mu_c\,\gamma_2) \\ q &= (\mu_c\,\gamma_1)^2\,(\mu_c\,\gamma_2)^2\left(1 - \frac{2\,k_1}{\mu_c\,\gamma_1} - \frac{2\,k_2}{\mu_c\,\gamma_2}\right) \\ \eta &= \sqrt{\tfrac{1}{2}\,p + \sqrt{\tfrac{1}{4}\,p^2 - q}} \end{aligned}\right\} \quad . \quad . \quad . \quad (23)$$

*) Part I in Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 955 (1946).

$$\left.\begin{array}{ll} e_1 = \mu_c\,\gamma_1 - \mu_c & b_1 = \mu_c\gamma_1 - \eta \\ e_2 = \mu_c\,\gamma_2 - \mu_c & b_2 = \mu_c\gamma_2 - \eta \\ e_3 = \mu_c\,\gamma_1 + \mu_c & - \\ e_4 = \mu_c\,\gamma_2 + \mu_c & - \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$\left.\begin{array}{l} G_c = \dfrac{-\mu_c^2}{\dfrac{2\,k_1\,\gamma_1}{(\gamma_1^2-1)^2} + \dfrac{2\,k_2\,\gamma_2}{(\gamma_2^2-1)^2}} \\ A_c = G_c\left(1 + \dfrac{b_1\,e_2}{b_2\,e_1}\right)\left(\dfrac{e_2}{e_4} + \dfrac{b_1}{e_3} - \dfrac{b_1\,e_2}{e_1\,e_4} - 1\right) \\ B\ = G_c\left(1 + 2\,\dfrac{b_1\,e_2}{b_2\,e_1}\right)\left\{\dfrac{1}{e_4} + \dfrac{b_1}{e_2\,e_3} - \dfrac{b_1}{e_1\,e_4} - \dfrac{1}{e_2} + \right. \\ \qquad\qquad \left. + \dfrac{e_2}{e_4^2}\left(1 - \dfrac{b_1\,e_2}{b_2\,e_1}\right)\right\} - 1 \\ C_1 = \left(\dfrac{e_2}{e_4}\right)^2 \qquad C_2 = \left(\dfrac{e_1}{e_4}\right)^2 \qquad c = 2\,\mu_c \end{array}\right\} \quad . \quad . \quad (25)$$

Substituting in these formulae

$$k_1 = 0.200 \qquad \gamma_1 = 10 \qquad k_2 = 0.108 \qquad \gamma_2 = 1.53$$

we find

$$\mu_c = 0.287 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and

$$S = D\,\frac{0.161\,h + 0.015 + 0.04\,e^{-0.574\,h}}{1 - 0.04\,e^{-0.574\,h}}.$$

The corresponding relation between $\frac{S}{D}$ and h is represented in fig. 9 by the "critical" curve C. This curve is very nearly a straight line passing through the origin. It divides the $\frac{S}{D}$, h-plane in two parts, one containing the curves of type I and the other the curves of type II.

9. *Incident beams with limited (circular) cross-sections.*

In the cases treated in the preceding sections the cross-sections of the incident beams of X-rays were infinite, and the intensities of direct and scattered radiation were therefore constant throughout any horizontal plane. In the present section we will consider cases in which the situation is different. The incident beams will as before be homogeneous and consist of vertical X-rays with one single wavelength λ; their cross-sections, however, will be circular and they will therefore occupy cylindriform areas.

In any cross-section scattered radiation will then obviously be most intense in the centre; towards the boundary it will show some decrease and a small part of it will penetrate beyond that. The situation is therefore rather complicated. However, it may be reasonably approximated by a simple scheme on which mathematical deductions can be based. This scheme can be summarized as follows:

1. No scattered radiation is present beyond the area reached by direct radiation.

2. Inside the area reached by direct radiation scattered radiation has constant intensity throughout any horizontal cross-section, and its distribution over various directions is in accordance with THOMSON's formula (2).

3. $S(x)$ satisfies an integral equation similar to (8), viz.

$$\left. \begin{aligned} S(x) = \int_x^h \{D e^{\mu y} + S(y)\}\, \Phi_{\mu R}\{\mu(y-x)\}\, dy \,+ \\ + \int_0^x \{D e^{\mu y} + S(y)\}\, \Phi_{\mu R}\{\mu(x-y)\}\, dy \end{aligned} \right\} \quad . \quad . \quad (27)$$

4. The definition of the function $\Phi_{\mu R}$ appearing in this equation can just as that of Φ be based on a calculation of secondary radiation.

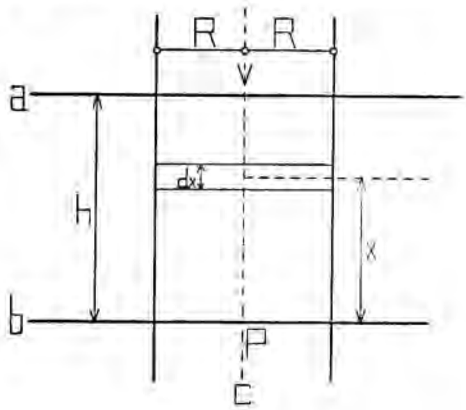

Fig. 10. Horizontal watersheet exposed to vertical X-rays forming a beam with circular cross-sections. a surface, b bottom of the watersheet, c central ray, R radius of cross-section of the incident beam.

Let in fig. 10 a be the surface and b the bottom of the watersheet, c the axis of the incident beam and R the radius of its cross-section. We now imagine a thin lamella of the sheet situated at a distance x above b, and consider the contribution of secondary radiation originating in this lamella (i.e. not of scattered radiations of higher orders!) to the intensity with which secondary radiation leaves the water in P. As this contribution will be proportional to $D(x)$ and to the thickness dx of the lamella it may be written

$$\Phi_{\mu R}\, D(x)\, dx$$

and in this expression the definition of the function $\Phi_{\mu R}$ is implied. In a similar way as in § 3 we find

$$\Phi_{\mu R}(\mu x) = \frac{3\sigma}{16\pi} \int_{\vartheta=0}^{\vartheta=\operatorname{arc\,tg}\frac{R}{x}} (1+\cos^2\vartheta)\,\frac{e^{-\mu r}}{r^2}\, 2\pi\varrho\, d\varrho$$

and consequently

$$\Phi_{\mu R}(u) = \frac{3\sigma}{8}\left[\left(1+\frac{u^2}{2}\right) H(z) + \frac{u^2}{2}\,\frac{1-z}{z^2}\, e^{-z}\right]_{z=\sqrt{u^2+(\mu R)^2}}^{z=u} \quad . \tag{28}$$

where $H(u)$ is the function defined by equation (4). Formula (28) shows that the function $\Phi_{\mu R}$ contains μR as a parameter and so justifies the use of the subscript.

Now the relations between $\Phi_{\mu R}$ and $u = \mu x$ can just as the relation between Φ and u be approximated by equations of the form

$$\Phi_{\mu R}(u) = k_1\, e^{-\gamma_1 u} + k_2\, e^{-\gamma_2 u}.$$

The integral equation (27) can therefore be solved in the same way as (8) and the resulting formulae for S will have the same structure.

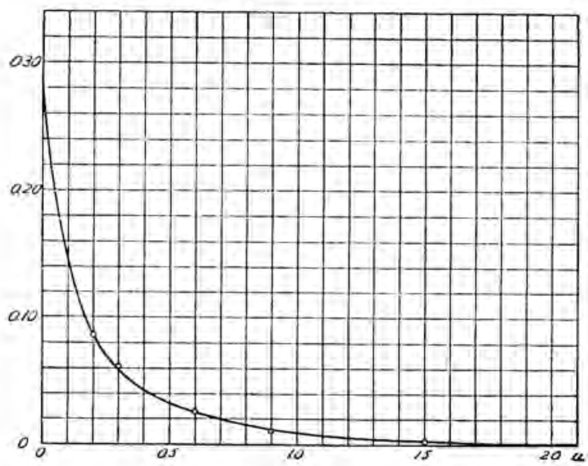

Fig. 11. Curve representing the function $0.2\,e^{-10u} + 0.088\,e^{-2.06u}$; the small circles show the course of the function $\Phi_{0.8}(u)$.

10. *Numerical results.*

Fig. 11 shows the degree of approximation with which functions like $\Phi_{0.8}(u)$ can be represented by expressions of the form

$$k_1\, e^{-\gamma_1 u} + k_2\, e^{-\gamma_2 u}.$$

The sets of coefficients k_1, γ_1, k_2, γ_2 corresponding to the functions $\Phi_{0.4}$,

$\Phi_{0.55}$, $\Phi_{0.8}$, $\Phi_{1.3}$, $\Phi_{2.1}$ are given in table I, and they are such that the condition

$$\int_0^{0.6} \Phi_{\mu R}(u)\, du = k_1 \int_0^{0.6} e^{-\gamma_1 u}\, du + k_2 \int_0^{0.6} e^{-\gamma_2 u}\, du$$

is satisfied. The table shows that for all important values of μR we have

$$k_1 = 0.200 \quad \text{and} \quad \gamma_1 = 10 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

and as to k_2 and γ_2 it provides the data from which the curves of fig. 12 were derived.

TABLE I.

μR	0.40	0.55	0.80	1.30	2.10	∞
k_1	0.20	0.20	0.20	0.20	0.20	0.20
γ_1	10.0	10.0	10.0	10.0	10.0	10.0
k_2	0.058	0.074	0.088	0.099	0.105	0.108
γ_2	2.74	2.39	2.06	1.74	1.59	1.53

These curves and the formulae (29) enable us to determine the values of k_1, γ_1, k_2, γ_2 corresponding to any given set of values of μ and R.

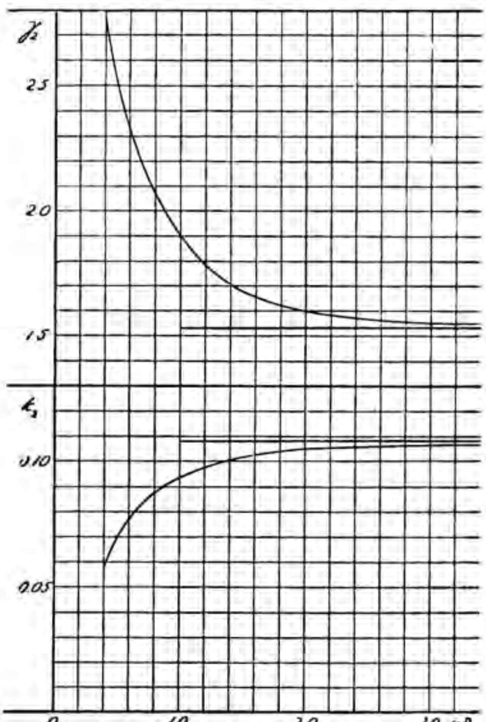

Fig. 12. Graphs showing the dependence of k_2 and γ_2 on μR.

By means of formulae (15), (19) and (20) we can then calculate the quantities A, a, B, C_1, C_2, c figuring in formula (12) and so obtain a numerical expression for $s = \frac{S}{D}$. In this way were obtained the following results which apply to $\mu = 0.20$, i.e. to the wavelength $\lambda = 0.20$ Å (see formula (1)).

$$\left.\begin{aligned} s &= \frac{1.99\,e^{0.95\,h} - 2.26 + 0.34\,e^{-0.21\,h}}{1 - 0.24\,e^{-0.21\,h}} \text{ for } R = \infty \\ s &= \frac{2.94\,e^{0.065\,h} - 3.10 + 0.24\,e^{-0.27\,h}}{1 - 0.16\,e^{-0.27\,h}} \text{ for } R = 10 \text{ cm.} \\ s &= -\,4.76\,e^{-0.035\,h} + 4.74 \text{ for } R = 5 \text{ cm.} \\ s &= -\,0.55\,e^{-0.208\,h} + 0.65 \text{ for } R = 2.5 \text{ cm.} \end{aligned}\right\} \quad . \quad . \quad (30)$$

These results are graphically represented in fig. 13a, and we see that the curve given here are in good qualitative agreement with those of fig. 4 as deduced from WILSEY's measurements. Obviously the curves for $R = \infty$

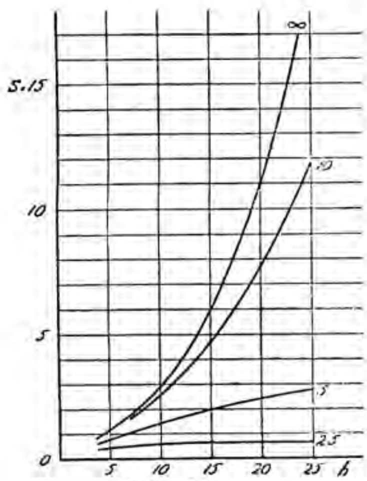

Fig. 13a.

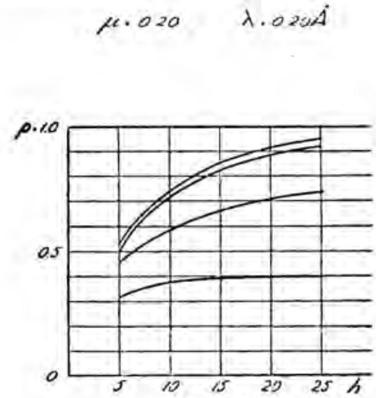

Fig. 13b.

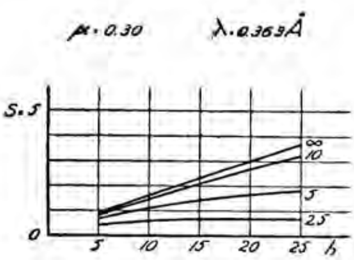

Fig. 14a.

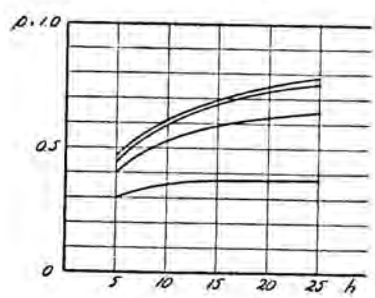

Fig. 14b.

Figs. 13a, b and 14a, b. Theoretical s, h- and p, h-curves corresponding to two different values of μ and λ and to four different sizes of the incident beam ($R = 2.5$ cm, $R = 5$ cm, $R = 10$ cm, and $R = \infty$).

and $R = 10$ cm are of type I and those for $R = 5$ cm and $R = 2.5$ cm of type II. The p, h-curves corresponding to the s, h-curves of fig. 13*a* are given in fig. 13*b*.

The s, h-curves and p, h-curves of fig. 14*a*, *b* were obtained in the same way as those of fig. 13*a*, *b*; they apply to $\mu = 0.30$, i.e. to the wavelength $\lambda = 0.363$ Å. The values of the constants A, a, B, C_1, C_2, c occurring in the s, h-formulae represented in figs. 13*a* and 14*a* are given in the tables II and III.

TABLE II [1]).

$\mu = 0.20$	A	a	B	C_1	C_2	c
$R = \infty$	1.99	0.095	−2.26	0.34	0.24	0.21
$R = 10$ cm	2.94	0.065	−3.10	0.24	0.16	0.27
$R = 5$ cm	−4.76	−0.035	4.74	[0.06]	[0.06]	[0.47]
$R = 2.5$ cm	−0.55	−0.208	0.65	[0.02]	[0.01]	[0.82]

TABLE III [1]).

$\mu = 0.30$	A	a	B	C_1	C_2	c
$R = \infty$	−16.2	−0.010	16.2	[0.06]	[0.04]	[0.62]
$R = 10$ cm	−8.82	−0.018	8.84	[0.05]	[0.04]	[0.64]
$R = 5$ cm	−2.18	−0.068	2.23	[0.03]	[0.02]	[0.74]
$R = 2.5$ cm	−0.53	−0.228	0.60	[0.01]	[0.01]	[1.06]

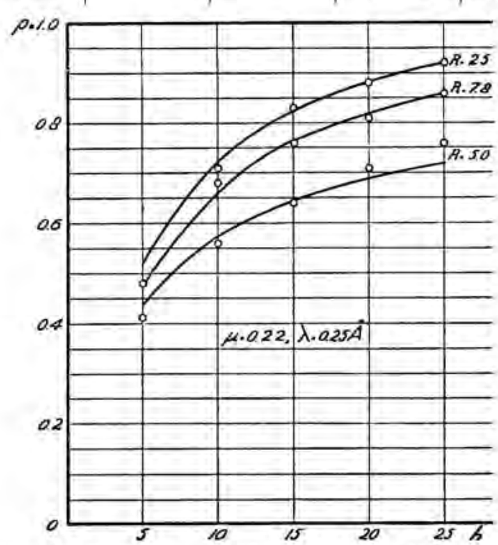

Fig. 15. Theoretical p, h-curves corresponding to $\mu = 0.22$ ($\lambda = 0.25$ Å) and to images of the sizes to which the experimental data of fig. 3 refer. The small circles give a copy of the latter data.

[1]) The numbers in [ ] have no practical importance and are given for the sake of completeness. In the expressions for $s = \dfrac{S}{D}$ they only figure in terms which are negligibly small.

11. *Comparison with* WILSEY'*s measurements.*

A quantitative test of the mathematical developments given in this paper can be obtained by applying them to conditions similar to those of WILSEY's measurements and comparing the results. A difficulty is that theory and experiment bear upon somewhat different geometrical situations: the mathematical deductions are based on the assumption that the part of the watersheet reached by direct radiation has the form of a circular cylinder, and in the arrangements used by WILSEY this assumption is often far from correct. On the whole, however, the experimental results may be expected to agree with calculations relating to images of the same sizes and to a suitable value of μ.

Now let us consider the p, h-curves of fig. 15. These curves were obtained by calculation and they bear upon images of the same sizes as the experimental data given in fig. 3. To facilitate comparison these data are copied in fig. 15 and we see that they are in close agreement with the curves. This agreement could be obtained by putting $\mu = 0.22$ in the calculations; according to (1) the wavelength corresponding to this value of μ is 0.252 Å and the question arises whether this is in accordance with experimental evidence.

An estimate of the wavelengths of the X-rays with which WILSEY carried out his experiments can be derived from the fact that in a spark gap between dull points arranged parallel to the tube the critical distance between the points was 12.5 cm. According to SPIEGLER and FERNAU [2]) the corresponding peak-voltage V_p is something like 85 or 90 kV, and for the radiation transmitted by a given watersheet the wavelength λ_m of maximum intensity can be determined by means of a set of formulae and curves given by the author in a preceding paper [3]). For watersheets of 5—25 cm we find $\lambda_m = 0{,}23 - 0.20$ Å, and when we compare these wavelengths with that of 0.252 Å mentioned above we see that there is a difference of 0.02 — 0.05 Å. Such a difference now can be fully explained by the COMPTON effect. This effect causes an enlargement of the mean wavelengths both of scattered and of total radiation, and so the results of calculations applying to a given wavelength must be expected to correspond to a smaller mean wavelength of transmitted direct radiation. As the increase of wavelength brought about by the COMPTON effect in a single process of scattering is 0.024 Å on the average (see § 2) an increase between 0.02 and 0.05 Å can be due to a very small number of these processes. Now scattered radiation leaving the watersheet is partly due to repeated scattering, and so the calculations given in this paper may be considered to be in satisfactory agreement with WILSEY's measurements.

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[2]) G. SPIEGLER u. H. W. FERNAU, Taschenbuch der medizinischen Röntgen- und Radium-Technik, Wien 1930, S. 174.

[3]) R. H. DE WAARD, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 1011 (1946).

Summary.

The contrasts in pictures obtained in medical radiography greatly depend on the proportion of scattered to direct radiation reaching the film during the exposure. It is therefore of interest to study the proportion of scattered to direct radiation leaving the object in the direction of the film. Measurements of this proportion were carried out by WILSEY on water-sheets: he found that in cases as occurring in practice it can have values as large as 10. In the present paper are given calculations from which can be derived the dependence of the proportion on the thickness of the water-sheet and on the wavelength and the size of the incident beam. The results of these calculations are in good agreement with WILSEY's measurements.

ERRATUM.

In the expression for A given in part I of this paper (Vol. **49**, p. 966, formulae (20)) the fraction $\frac{a_2 b_2}{e_2}$ is erroneous and should be replaced by $\frac{a_2 b_1}{e_2}$.

Mathematics. — *Concerning the homotopy groups of the components of the mapping space Y^{S^p}.* By SZE-TSEN HU. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of October 26, 1946.)

1. Introduction.

Let Y be a connected compact absolute neighbourhood retract, [8, p. 58]. Let us denote by G^p the mapping space Y^{S^p}, which consists of the totality of the mappings of a p-sphere S^p into Y. Let $x_0 \in S^p$, $y_0 \in Y$ be given points, and denote by F^p the closed subset of G^p, which consists of the totality of the mappings $f \in G^p$ with $f(x_0) = y_0$. Let $\pi^p(Y)$ denote the p-th homotopy group of Y with x_0, y_0 as base points. Let F^p_α be the component of F^p which consists of the totality of the representatives of the element $\alpha \in \pi^p(Y)$. Since Y is arcwise connected, each component of G^p contains at least one component of F^p. Let G^p_α be the component of G^p which contains F^p_α.

The fundamental group of the component G^p_0 was first studied by M. ABE, [1]; the higher homotopy groups of G^p_0 were determined by the author in terms of those of Y, [6, § 10], during the early months of 1946. At that time, practically nothing was known concerning the homotopy properties of the component G^p_α, $\alpha \neq 0$. Most recently, it appears the work of G. W. WHITEHEAD, [10], in which an example has been given to show that G^p_0 and G^p_α are in general of the different homotopy types if $\alpha \neq 0$. In the present note, two isomorphisms will be given in § 2 regarding the structures of the homotopy groups of the component G^p_α, which indicate the close relation between WHITEHEAD products, [11], and the homotopy groups of G^p_α. They have been used to determine the homotopy groups of G^p_α in terms of those of Y for a certain number of special cases, of which the most interesting one is $Y = S^2$.

2. General theorems.

For each pair of elements $\alpha \in \pi^p(Y)$, $\beta \in \pi^q(Y)$, let us denote by $[\alpha, \beta] \in \pi^{p+q-1}(Y)$ the WHITEHEAD product of α and β, [11]. For $q > 1$ and a given $\alpha \in \pi^p(Y)$, the transformation $\beta \rightarrow [\alpha, \beta]$ is a homomorphism of $\pi^q(Y)$ into $\pi^{p+q-1}(Y)$, denoted by ϱ_α. Let K^q_α and J^{p+q-1}_α denote the kernel and the image of ϱ_α respectively. Choose $a \in F^p_\alpha$ as the base point for all the homotopy groups of F^p_α and G^p_α. There is a natural homomorphism $\mu\colon \pi^q(F^p_\alpha) \rightarrow \pi^q(G^p_\alpha)$ induced by the injection mapping

$F_\alpha^p \to G_\alpha^p$. Let P_α^q be the image of $\pi^q(F_\alpha^p)$ under μ. For the higher homotopy groups of G_α^p, the following theorem is a direct consequence of the theorems of G. W. WHITEHEAD, [10, (2. 4) and (3. 2)].

Theorem 2. 1. *For each $q > 1$ and $a \in \pi^p(Y)$, we have*

(2. 11) $$\pi^q(G_\alpha^p)/P_\alpha^q \approx K_\alpha^q,$$

(2. 12) $$\pi^{p+q}(Y)/J_\alpha^{p+q} \approx P_\alpha^q.$$

According to S. EILENBERG, [2], the fundamental group $\pi^1(Y)$ is a group of operators for the group $\pi^p(Y)$ with the unit element of $\pi^1(Y)$ as unit operator. Let Q_α^1 denote the subgroup of $\pi^1(Y)$, wich consists of the totality of the elements $\omega \in \pi^1(Y)$ such that $\omega(a) = a$. For the fundamental group $\pi^1(G_\alpha^p)$, we shall prove the following theorem.

Theorem 2. 2. *For each $a \in \pi^p(Y)$ we have*

(2. 21) $$\pi^1(G_\alpha^p)/P_\alpha^1 \approx Q_\alpha^1,$$

(2. 22) $$\pi^{p+1}(Y)/J_\alpha^{p+1} \approx P_\alpha^1.$$

[Proof] Let τ denote the projection of G_α^p into Y, defined by $\tau(f) = f(x_0)$ for each $f \in G_\alpha^p$. Then τ is a fibre mapping, [10, (2. 1)], and $\tau(F_\alpha^p) = y_0$. τ induces a homomorphism of $\pi^1(G_\alpha^p)$ into $\pi^1(Y)$ still denoted by τ.

Let I denote the closed interval (0, 1) of real numbers. Let $\xi \in \pi^1(G_\alpha^p)$ be represented by a mapping $\Phi: I \to G_\alpha^p$ such that $\Phi(0) = a = \Phi(1)$. Let $\psi = \tau\Phi$, then $\psi(0) = y_0 = \psi(1)$. ψ represents an element $\omega \in \pi^1(Y)$, and $\omega = \tau\xi$. Φ defines a homotopy $f_t: S^p \to Y$ by means of the relation $\Phi(t) = f_t$ for each $t \in I$. Since $f_0 = a = f_1$ and $f_t(x_0) = \psi(t)$ for each $t \in I$, it follows that $\omega(a) = a$. Hence $\omega \in Q_\alpha^1$. Conversely, suppose $\omega \in Q_\alpha^1$ be an arbitrary element, represented by a mapping $\psi: I \to Y$ with $\psi(0) = y_0 = \psi(1)$ From the Covering Homotopy Theorem, [7], it follows that there exists a mapping $\Phi: I \to G_\alpha^p$ such that $\tau\Phi = \psi$ and $\Phi(0) = a$. Φ defines a homotopy $f_t: S^p \to Y$ by the relation $\Phi(t) = f_t$ for each $t \in I$. Since $f_0 = a$ and $f_t(x_0) = \psi(t)$, f_1 represents the element $\omega(a) \in \pi^p(Y)$. Hence $f_1 \in F_\alpha^p$, for $\omega \in Q_\alpha^1$. From the arcwise connectedness, it follows that there exists a homotopy $\Phi_t: I \to G_\alpha^p$, $0 \leqslant t \leqslant 1$, such that $\Phi_0 = \Phi$ $\Phi_t(0) = a$, $\Phi_t(1) \in F_\alpha^p$, and $\Phi_1(1) = a$. Let $\psi_t = \tau\Phi_t$; then $\psi_t: I \to Y$ $0 \leqslant t \leqslant 1$, is a homotopy such that $\psi_0 = \psi$ and $\psi_t(0) = y_0 = \psi_t(1)$ for each $t \in I$. Hence ψ_1 is also a representative of ω. Φ_1 represents an element $\xi \in \pi^1(G_\alpha^p)$ and $\tau\xi = \omega$. Hence, we have proved that the image of $\pi^1(G_\alpha^p)$ under the homomorphism τ is Q_α^1.

It is trivial that P_α^1 is contained in the kernel of τ. Conversely, suppose $\xi \in \pi^1(G_\alpha^p)$ be an arbitrary element of the kernel of τ, represented by a mapping $\Phi: S^1 \to G_\alpha^p$ with $\Phi(z_0) = a$, z_0 being a given point of S^1. Then the mapping $\psi = \tau \Phi$ represents the unit element of $\pi^1(Y)$; hence there exists a homotopy $\psi_t : S^1 \rightarrow Y$, $0 \leqq t \leqq 1$, such that $\psi_0 = \psi$, $\psi_1(S^1) = y_0$, and $\psi_t(z_0) = y_0$ for each $0 \leqq t \leqq 1$. From the Covering Homotopy Theorem, it follows that there exists a homotopy $\Phi_t : S^1 \rightarrow G^p_\alpha$ such that $\Phi_0 = \Phi$, $\tau \Phi_t = \psi_t$ and $\Phi_t(z_0) = a$ for each $0 \leqq t \leqq 1$. Since $\Phi(S^1) \subset F^p_\alpha$, we obtain $\xi \in P^1_\alpha$. Hence (2. 21) follows.

The isomorphism (2. 22) can be proved as (2. 12). Q. E. D.

For the use of the sequel, we mention the results of the author, [6, § 10], and M. ABE, [1], for the component G^p_0.

(2. 3) *If* $q > 1$, $\pi^q(G^p_0)$ *is isomorphic with the direct sum of* $\pi^{p+q}(Y)$ *and* $\pi^q(Y)$.

(2. 4) *If* Y *is* $(p+1)$*-simple*, [2], $\pi^1(G^p_0)$ *is isomorphic with the direct product of* $\pi^{p+1}(Y)$ *and* $\pi^1(Y)$.

Theorem 2. 5. *If* $\alpha + \beta = 0$, *then the components* G^p_α *and* G^p_β *are homeomorphic.*

[Proof] Let $\theta : S^p \rightarrow S^p$ be a homeomorphism of S^p which reverses orientation and has x_0 as a fixed point. Then, a homeomorphism h of G^p_α onto G^p_β is given by $h(f) = f\theta$ for each $f \in G^p_\alpha$. Q. E. D.

3. Spaces with continuous multiplication.

Theorem 3. 1. *If* Y *admits a continuous multiplication with a two-sided identity* e *(e.g., if* Y *is a topological group), then* G^p_0 *and* G^p_α *are of the same homotopy type for each* $\alpha \in \pi^p(Y)$.

[Proof] According to G. W. WHITEHEAD, [10, p. 464], it remains to prove that there exists a mapping $\lambda : Y \rightarrow G^p_\alpha$ such that $\tau\lambda$ is the identity, where τ denotes the projection of G^p_α onto Y defined by $\tau(f) = f(x_0)$ for each $f \in G^p_\alpha$. For each $y \in Y$, let $\lambda_0 y \in G^p_0$ be the constant mapping of S^p into y. Hence λ_0 is a mapping of Y into G^p_0 such that $\tau\lambda_0$ is the identity. From the arcwise connectedness of Y, it follows that there exists a mapping $\Phi \in G^p_\alpha$ such that $\Phi(x_0) = e$. Define $\lambda : Y \rightarrow G^p$ by

$$\lambda y(x) = \Phi(x) \cdot \lambda_0 y(x), \qquad (y \in Y,\ x \in S^p).$$

Since $\lambda e = \Phi \in G^p_\alpha$ and Y is arcwise connected, it follows that $\lambda(Y) \subset G^p_\alpha$. Further, $\tau(\lambda y) = e \, . \, y = y$; hence $\tau\lambda$ is the identity. The proof has been completed. Q. E. D.

Corollary 3. 2. *If* $Y = S^r$, $(r = 1, 3, 7)$, *then for each* $\alpha \in \pi^p(S^r)$ *and each* $q \geqq 1$ *we have*

$$\pi^q(G^p_\alpha) \approx \pi^{p+q}(S^r) + \pi^q(S^r).$$

4. The sphere S^r.

In the present paragraph, let Y be the r-sphere S^r. Since the cases $r=1, 3, 7$ have been solved in (3. 2) and the case $r=2$ will be treated in § 5, we may suppose that $r>3$.

If $p<r$ or $p=r+2$, then the space G^p is connected and our proplem reduces to (2. 3) and (2. 4). It remains to investigate the case $p \geqq r$ and $p \neq r+2$.

Theorem 4. 1. *For each* $\alpha \in \pi^p(S^r)$, *we have*

(4. 11) $\pi^q(G^p_\alpha) \approx \pi^{p+q}(S^r)$, $(q<r-1)$;

(4. 12) $\pi^{r-1}(G^p_\alpha) \approx \pi^{p+r-1}(S^r)/J^{p+r-1}_\alpha$;

(4. 13) $\pi^{r+1}(G^p_\alpha)$ *has a subgroup* $P^{p+r+1}_\alpha \approx \pi^{p+r+1}(S^r)$;

(4. 14) $\pi^{r+2}(G^p_\alpha) \approx \pi^{p+r+2}(S^r)/J^{p+r+2}_\alpha$.

[Proof] These are immediate consequences of the general theorems (2. 1), (2. 2), and the facts $\pi^{r+2}(S^r)=0$ and $\pi^q(S^r)=0$ if $q<r$.

Lemma 4. 2. *If r is any positive even integer and α, β are arbitrary elements of $\pi^r(S^r)$ both different from zero, then $[\alpha, \beta] \in \pi^{2r-1}(S^r)$ is also different from zero.*

[Proof] Suppose, $[\alpha, \beta]=0$. Then by a corollary of G. W. WHITEHEAD, (10, p. 467), there exists a mapping $f: S^r \times S^r \rightarrow S^r$ of the type (α, β). It follows from a theorem of H. HOPF, [5, p. 431], that there exists an element of $\pi^{2r+1}(S^{r+1})$ with HOPF invariant ab, where a and b are the degrees of α and β. Since $r+1$ is odd, it follows that $ab=0$. Hence, at least one of the elements α, β must be zero. Q. E. D.

Following ALLEXANDROFF—HOPF, we shall denote by $\mathfrak{G}_0=\mathfrak{G}$ the infinite cyclic group, and by $\mathfrak{G}_m$ the finite cyclic group of the order m.

Theorem 4. 3. *If r is even and $\alpha \in \pi^r(S^r)$ is different from zero, then $K^r_\alpha=0$ and $J^{2r-1}_\alpha=\mathfrak{G}$.*

[Proof] This is a consequence of Lemma 4. 2.

Theorem 4. 4. *If r is odd, then*

(4. 41) $K^r_\alpha=\mathfrak{G}$ *for each* $\alpha \in \pi^r(S^r)$;

(4. 42) $J^{2r-1}_\alpha=\mathfrak{G}_2$ *if* $\pi^{2r+1}(S^{r+1})$ *has no element with Hopf invariant 1 and α is of odd degree, and* $J^{2r-1}_\alpha=0$ *otherwise.*

[Proof] If $\pi^{2r+1}(S^{r+1})$ has an element of HOPF invariant 1, then by Theorem (3. 12) of G. W. WHITEHEAD, [10], we have $[\alpha, \beta]=0$ for each α, $\beta \in \pi^r(S^r)$; hence $K^r_\alpha=\mathfrak{G}$ and $J^{2r-1}_\alpha=0$. If $\pi^{2r+1}(S^{r+1})$ has no element with HOPF invariant 1, then $[\alpha, \beta]=0$ if and only if at least one of the elements α, β is of even degree. If α is of even degree, then $[\alpha, \beta]=0$ for each $\beta \in \pi^r(S^r)$; hence $K^r_\alpha=\mathfrak{G}$ and $J^{2r-1}_\alpha=0$. If α is of odd degree, then K^r_α consists of the elements of even degree; hence $K^r_\alpha=\mathfrak{G}$, $J^{2r-1}_\alpha=\mathfrak{G}_2$. Q. E. D.

H. FREUDENTHAL, [4], announced without proof the existence of the elements of $\pi^{2r+1}(S^{r+1})$ with HOPF invariant 1 for every odd r. See also G. W. WHITEHAED, [9].

From (4.3) and (4.4), the following two theorems can be deduced easily.

Theorem 4.5. *If r is even, then for each $\alpha \in \pi^r(S^r)$ different from zero, we have*

(4.51) $\pi^{r-1}(G^r_\alpha) \approx \pi^{2r-1}(S^r)/J^{2r-1}_\alpha$, $J^{2r-1}_\alpha \approx \mathfrak{G}$;

(4.52) $\pi^r(G^r_\alpha) \approx \pi^{2r}(S^r)/J^{2r}_\alpha$.

Theorem 4.6. *If r is odd, then for each $\alpha \in \pi^r(S^r)$, we have*

(4.61) $\pi^{r-1}(G^r_\alpha) \approx \pi^{2r-1}(S^r)/J^{2r-1}_\alpha$, *where* $J^{2r-1}_\alpha=0$ *or* $\mathfrak{G}_2$ *as described in* (4.42);

(4.62) $\pi^r(G^r_\alpha)/P^r_\alpha \approx \mathfrak{G}$.

5. The sphere S^2.

Throughout this paragraph, let Y be the 2-sphere S^2, and let $\iota \in \pi^2(S^2)$ denote the element represented by the identity of S^2.

Lemma 5.1. *The generator γ of the group $\pi^3(S^2)$ can be so chosen that $[\iota, \iota]=2\gamma$.*

[Proof] Let γ^* be an arbitrary generator of $\pi^3(S^2)$; then we have $[\iota, \iota]=\delta m \gamma^*$, where $\delta=\pm 1$ and $m>0$. Let D be the subgroup of $\pi^3(S^2)$ generated by $[\iota, \iota]$, then we have $\pi^3(S^2)/D$ is isomorphic with $\mathfrak{G}_m$. On the other hand, let E denote the Einhängung operation of H. FREUDENTHAL, [3]; then E is a homomorphism of $\pi^3(S^2)$ onto $\pi^4(S^2)$. By a result of G. W. WHITEHEAD, [10, p. 470], the kernel of E is the subgroup D; hence $\pi^3(S^2)/D$ is isomorphic with $\mathfrak{G}_2$. Then it follows that $m=2$. Choosing $\gamma=\delta\gamma^*$ as new generator of $\pi^3(S^2)$, we have $[\iota, \iota]=2\gamma$. Q. E. D.

Lemma 5.2. $[\gamma, \iota]=0$.

[Proof] This is contained in the second example of G. W. WHITEHEAD, [10, p. 474].

Theorem 5. 3. *If $Y=S^2$ and $\alpha \in \pi^2(S^2)$ is different from zero, then*

(5. 31) $\pi^1(G^2_\alpha)=\mathfrak{G}_{2m}$, *where $m>0$ is determined by $\alpha=\pm m\iota$;*

(5. 32) $\pi^2(G^2_\alpha)=\mathfrak{G}_2$;

(5. 33) $\pi^3(G^2_\alpha)=\mathfrak{G}$;

(5. 34) $\pi^4(G^2_\alpha)/P^4_\alpha \approx \mathfrak{G}_2$, $P^4_\alpha \approx \pi^6(S^2)$;

(5. 35) $\pi^5(G^2_\alpha) \approx \pi^7(S^2)/J^7_\alpha$.

[Proof] (5. 31) Since $Q^1_\alpha \subset \pi^1(S^2)=0$, it follows from (2. 2) that $\pi^1(G^2_\alpha) \approx \pi^3(S^2)/J^3_\alpha$. Since $\alpha=\pm m\iota$, we have $[\alpha, \iota]=\pm 2m\gamma$; therefore, J^3_α is the subgroup of $\pi^3(S^2)$ generated by $2m\gamma$. Hence $\pi^3(S^2)/J^3_\alpha \approx \mathfrak{G}_{2m}$.

(5. 32) $K^2_\alpha=0$, by (5. 1); $J^4_\alpha=0$, by (5. 2). Then, by (2. 1),

$$\pi^2(G^2_\alpha) \approx \pi^4(S^2) \approx \mathfrak{G}_2.$$

(5. 33) Since $J^4_\alpha=0$, we get $K^3_\alpha=\pi^3(S^2)$; since $\pi^5(S^2)=0$, we get $P^3_\alpha=0$. Hence $\pi^3(G^2_\alpha) \approx \pi^3(S^2) \approx \mathfrak{G}$.

(5. 34) Since $\pi^5(S^2)=0$, we have $K^4_\alpha=\pi^4(S^2)$ and $J^6_\alpha=0$.

Then (5. 34) follows from (2. 1).

(5. 35) follows from $K^5_\alpha \subset \pi^5(S^2)=0$. Q. E. D.

It would be worthwhile to mention here that

$\pi^1(G^2_0)=\mathfrak{G}$, $\pi^2(G^2_0)=\mathfrak{G}+\mathfrak{G}_2$, $\pi^3(G^2_0)=\mathfrak{G}$,

$\pi^4(G^2_0) \approx \pi^6(S^2)+\mathfrak{G}_2$, $\pi^5(G^2_0) \approx \pi^7(S^2)$.

(5. 31) yields the complete solution of the classification of the homotopy types of the components of G^2 as stated in the

Theorem 5. 4. *If $Y=S^2$, the two different components G^2_α, G^2_β of G^2 are of the same homotopy type, if and only if α and β are negative to each other.*

[Proof] If $\alpha+\beta=0$, then G^2_α and G^2_β are of the same homotopy type by (2. 5). Conversely, if G^2_α and G^2_β are of the same homotopy type, then by (5. 31), the absolute values of the degrees of α and β are equal. Hence $\alpha+\beta=0$, for they are supposed to different. Q. E. D.

The following two theorems can be proved by the similar methods as used in the proof of (5. 3).

Theorem 5. 5. *If $Y=S^2$ and $\alpha \in \pi^3(S^2)$ be arbitrary element then we have*

(5. 51) $\pi^1(G^3_\alpha)=\mathfrak{G}_2$;

(5. 52) $\pi^2(G^3_\alpha)=\mathfrak{G}$;

(5. 53) $\pi^3(G^3_\alpha)/P^3_\alpha \approx \mathfrak{G}$;

(5. 54) $\pi^4(G^3_\alpha)$ has a subgroup $P^4_\alpha \approx \pi^7(S^2)$;

(5. 55) $\pi^5(G^3_\alpha) \approx \pi^8(S^2)/J^8_\alpha$.

Theorem 5. 6. *If $Y = S^2$ and $\alpha \in \pi^4(S^2)$ be arbitrary element, then we have*

(5. 61) $\pi^1(G^4_\alpha) = 0$;

(5. 62) $\pi^2(G^4_\alpha)/P^2_\alpha \approx \mathfrak{G}$;

(5. 63) $\pi^4(G^4_\alpha)$ *has a subgroup* $P^4_\alpha \approx \pi^8(S^2)$:

(5. 64) $\pi^5(G^4_\alpha) \approx \pi^9(S^2)/J^9_\alpha$.

For $Y = S^2$, G^5 is connected; hence our problem has been solved in (2. 3) and (2. 4). For $p > 5$, I know no more than the general theorems in § 2.

6. Aspherical spaces.

Y is said to be aspherical, if $\pi^p(Y) = 0$ for each $p > 1$. Then Y is p-simple for each $p > 1$.

Since G^p, $p > 1$, is connected, we deduce from (2. 3) and (2. 4) the following theorem.

Theorem 6. 1. *If Y is aspherical and $p > 1$, then G^p is also aspherical and $\pi^1(G^p) \approx \pi^1(Y)$.*

From (2. 1), (2. 2) and the 2-symplicity, we deduce the following

Theorem 6. 2. *If Y is aspherical and $\alpha \in \pi^1(Y)$, then G^1_α is also aspherical and $\pi^1(G^1_\alpha) \approx \pi^1(Y)$.*

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Mathematics. — *On a generalisation of the formula of* HILLE *and* HARDY *in the theory of Laguerre polynomials.* By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 26, 1946.)

1. The Laguerre polynomials $L_n^{(\alpha)}(x)$ can be defined by means of a generating function

$$\frac{1}{(1-t)^{\alpha+1}} e^{-\frac{xt}{1-t}} = \sum_{n=0}^{\infty} t^n L_n^{(\alpha)}(x) \qquad |t|<1 \quad . \quad . \quad . \quad . \quad (1)$$

or by

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{\alpha+n}) = \sum_{r=0}^{n} \binom{n+\alpha}{n-r} \frac{(-x)^r}{r!} \quad . \quad . \quad (2)$$

For these polynomials the following theorem holds

$$\left.\begin{aligned} &\sum_{n=0}^{\infty} t^n \frac{n!\, e^{-\frac{1}{2}(x+y)} (xy)^{\frac{1}{2}\alpha}}{\Gamma(n+\alpha+1)} L_n^{(\alpha)}(x) L_n^{(\alpha)}(y) = \\ &= \frac{t^{-\frac{1}{2}\alpha}}{1-t} \exp\left(-\frac{1}{2}(x+y)\frac{1+t}{1-t}\right) \cdot I_\alpha\left(\frac{2\sqrt{xyt}}{1-t}\right) \quad (|t|<1) \end{aligned}\right\} \quad (3)$$

where I_α is the "Bessel function of imaginary argument". This formula is often called the Hille–Hardy formula, but besides those of HILLE [1]) and HARDY [2]) the names of WIGERT, BATEMAN and MYLLER LEBEDEW have been associated with the discovery of the theorem [3]). HARDY obtained his result by an application of Mellin's inversion formula; the proof of Hille involved the use of infinite integrals containing Bessel functions. A simple proof for (3) has been given by WATSON [4]) by means of generalized hypergeometric functions.

If we write

$$\varphi_n^{(\alpha)}(x) = \left[\frac{e^{-x} x^{\alpha} n!}{\Gamma(n+\alpha+1)}\right]^{\frac{1}{2}} L_n^{(\alpha)}(x). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the left member of (3) is seen to be

$$\Sigma t^n \varphi_n^{(\alpha)}(x) \varphi_n^{(\alpha)}(y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now in recent years series of the form

$$\Sigma c_n t^n \varphi_n^{(\alpha)}(x) \cdot \varphi_n^{(\alpha)}(y) \qquad (|t| \leqq 1) \quad . \quad . \quad . \quad (6)$$

have been investigated by WATSON and by ERDELYI. WATSON [5]) has

[1]) HILLE, Proc. Nat. Acad. of Sci., **12**, 261—265, 265—269, 348—352 (1926).

[2]) HARDY, Journal London Math. Soc. **7**, 138—139 (1932).

[3]) For the history of the formula see WATSON, Journal London Math. Soc., **8**, 190 (1933), ERDELYI, Compositio Math. **6**, 336—347 (1939), BATEMAN, Zb. Mathem. **21**, 24 (1940).

[4]) WATSON, l.c. 189—192.

WATSON, Sitzungsber. Ak. Wiss. Wien, **147**, 151—159 (1938).

shown that for $t=1$, $c_n = \dfrac{1}{n+1}$ (6) can be expressed by incomplete Γ-functions, thus generalizing a formula which had been given by R. NEUMANN and by KOSCHMIEDER for $\alpha = 0$. ERDELYI has published several papers on the subject [6]), showing finally that the results obtained by WATSON and by himself can be considered as special cases of a general theorem concerning bilinear series of confluent hypergeometric functions.

The aim of the present note is far more unpretending and it tries to show only that for $c_n = \dfrac{\Gamma(n+\alpha+k+1)}{\Gamma(n+\alpha+1)}$ where k is an integer and for $|t|<1$, the series (6) can be written as a sum of k Bessel functions which coefficients are expressions containing Laguerre polynomials of the argument $\dfrac{(x+y)\,t}{1-t}$.

2. We prove the following generalisation of (3)

$$\left.\begin{aligned}&\sum_{n=0}^{\infty} t^n \frac{n!\, e^{-\frac{1}{2}(x+y)}\,\Gamma(n+\alpha+k+1)\,(xy)^{\frac{1}{2}\alpha}}{\Gamma^2(n+\alpha+1)} L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y) = \\ &= \frac{t^{-\frac{1}{2}\alpha}}{(1-t)^{k+1}} \cdot \exp\left(-\frac{1}{2}(x+y)\frac{1+t}{1-t}\right) \cdot k! \sum_{p=0}^{k} \left[\left[\sum_{m=0}^{k-p} \binom{p}{m} \frac{t^m}{(1+t)^m} L_{k-p-m}^{(\alpha+p+m)}\left(\frac{(x+y)\,t}{1-t}\right)\right] \times \right. \\ &\qquad \left. \times \frac{1}{p!}(xyt)^{\frac{p}{2}} \left(\frac{1+t}{1-t}\right)^p I_{\alpha+p}\left(\frac{2\sqrt{xyt}}{1-t}\right)\right] \qquad (|t|<1,\ k \text{ an integer})\end{aligned}\right\} \quad (7)$$

Since $\dbinom{p}{m} = 0$ for $m>p$ the second Σ on the right has $q+1$ terms, where $q = \min\,[k-p, p]$. For $k=0$ we have the Hille–Hardy theorem. The proof is extremely elementary for (7) can be derived from (3) by multiplying with $t^{\alpha+k}$, differentiating k times with respect to t and dividing by t^{α}. The only difficulty arises from the arrangement of the righthand member. Once the formula discovered, the proof can best be given by induction. We assume that (7) is valid for k, multiply by $t^{\alpha+k+1}$, differentiate with respect to t and divide by $t^{\alpha+k}$. We obtain then the left member of (7) for $k+1$. For the reduction of the right member we make use of the following relations

$$\frac{d\,I_\beta(z)}{dz} = \frac{\beta}{z} I_\beta(z) + I_{\beta+1}(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$z\,\frac{d\,L_n^{(\alpha)}(z)}{dz} = (n+1)\,L_{n+1}^{(\alpha)}(z) - (\alpha+n+1-z)\,L_n^{(\alpha)}(z) \quad . \quad . \quad (9)$$

$$L_n^{(\alpha)}(z) + L_{n-1}^{(\alpha+1)}(z) = L_n^{(\alpha+1)}(z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

[6]) ERDELYI, Rend. Acc. Lincei. **24**, 347—350 (1936); S.—B. Akad. Wiss. Wien IIa, **147**, 513—520 (1938); id. **148**, 38—39 (1939); Compositio Math. **6**, 336—347 (1939); id. **7**, 340—352 (1939).

For $I_\beta\left(\frac{2\sqrt{xyt}}{1-t}\right)$ and $L_n^{(\alpha)}\left(\frac{(x+y)t}{1-t}\right)$ we obtain accordingly

$$\frac{d}{dt}I_\beta=\frac{\beta(1+t)}{2t(1-t)}I_\beta+\sqrt{xyt}\,\frac{1+t}{t(1-t)^2}I_{\beta+1}\,. \quad . \quad . \quad . \qquad (8a)$$

$$\frac{d}{dt}L_n^{(\alpha)}=\frac{n+1}{t(1-t)}L_{n+1}^{(\alpha)}-\frac{\alpha+n+1-u}{t(1-t)}L_n^{(\alpha)}\,. \quad . \quad . \quad . \qquad (9a)$$

where

$$u=\frac{(x+y)t}{1-t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

From ($8a$) it is obvious that in the right member we obtain an expression

$$\frac{t^{-1/2\alpha}}{(1-t)^{k+2}}\exp\left(-\tfrac{1}{2}(x+y)\frac{1+t}{1-t}\right)\cdot k!\sum_{p=0}^{k+1}A_p(x,y,t)\,I_{\alpha+p} \quad . \qquad (12)$$

and the functions $A_p(x,y,t)$ have to be found. If we differentiate

$$\frac{t^{\frac{1}{2}\alpha+k+1}}{(1-t)^{k+1}}\cdot\exp\left(-\tfrac{1}{2}(x+y)\frac{1+t}{1-t}\right)$$

with respect to t and divide by $t^{\alpha+k}$, we obtain

$$\frac{t^{-1/2\alpha}}{(-t)^{k+2}}\cdot\exp\left(-\tfrac{1}{2}(x+y)\frac{1+t}{1-t}\right)\cdot(\tfrac{1}{2}\alpha+k+1-\tfrac{1}{2}\alpha t-u).$$

Again, differentiating $t^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)^p$ we have

$$p\cdot t^{\frac{p}{2}-1}\left(\frac{1+t}{1-t}\right)^{p-1}\cdot\frac{1+4t-t^2}{2(1-t)^2}.$$

Thus if

$$\sum_{m=0}^{k-p}\binom{p}{m}\frac{t^m}{(1+t)^m}L_{k-p-m}^{(\alpha+p+m)}\left(\frac{(x+y)t}{1-t}\right)=S_p(t) \quad . \quad . \quad . \qquad (13)$$

it follows that for $0<p<k+1$

$$\left.\begin{aligned}
A_p=&(\tfrac{1}{2}\alpha+k+1-\tfrac{1}{2}\alpha t-u)\cdot\frac{1}{p!}(xyt)^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)^p S_p\\
&+\frac{pt(1-t)}{p!}(xy)^{\frac{p}{2}}\cdot t^{\frac{p}{2}-1}\left(\frac{1+t}{1-t}\right)^{p-1}\cdot\frac{(1+4t-t^2)}{2(1-t)^2}S_p\\
&+t(1-t)\cdot\frac{1}{p!}(xyt)^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)^p\frac{d}{dt}S_p(t)\\
&+t(1-t)\frac{1}{p!}(xyt)^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)S_p\cdot\frac{(\alpha+p)(1+t)}{2t(1-t)}\\
&+t(1-t)\cdot\frac{1}{(p-1)!}(xyt)^{\frac{p-1}{2}}\left(\frac{1+t}{1-t}\right)^{p-1}\cdot S_{p-1}\cdot(xyt)^{1/2}\frac{1+t}{t(1-t)^2}\\
&=\frac{1}{p!}(xyt)^{\frac{p}{2}}\left(\frac{1+t}{1-t}\right)^p B_p(xyt)
\end{aligned}\right\} \qquad (14)$$

where

$$\left.\begin{aligned}B_p = & \left[\tfrac{1}{2}\alpha + k + 1 - \tfrac{1}{2}\alpha t - u + \frac{p}{2}\cdot\frac{1+4t-t^2}{1+t} + \frac{\alpha+p}{2}(1+t)\right] S_p \\ & + t(1-t)\frac{d}{dt} S_p + p S_{p-1} \\ & = \left(\alpha + k + 1 + p - u + \frac{2pt}{1+t}\right) S_p + t(1-t)\frac{d}{dt} S_p + p S_{p-1}\end{aligned}\right\} \quad (15)$$

Now

$$\left.\begin{aligned}\frac{d}{dt} S_p = & \sum_{m=0}^{k-p} \binom{p}{m} \left[\frac{m\, t^{m-1}}{(1+t)^{m+1}} L_{k-p-m}^{(\alpha+p+m)} + \right. \\ & \left. + \frac{t^{m-1}}{(1-t)(1+t)^m} \{k-p-m+1) L_{k-p-m+1}^{(\alpha+p+m)} - (\alpha+k+1-u) L_{k-p-m}^{(\alpha+p+m)}\}\right]\end{aligned}\right\}. \quad (16)$$

Thus we obtain

$$\left.\begin{aligned}B_p = & \sum_{m=0}^{k-p} \binom{p}{m} \frac{t^m}{(1+t)^m} \left[\left(\alpha + k + 1 + p - u + \frac{2pt}{1+t}\right) L_{k-p-m}^{(\alpha+p+m)} + \frac{m(1-t)}{1+t} L_{k-p-m}^{(\alpha+p+m)} + \right. \\ & \left. + (k-p-m+1) L_{k-p-m+1}^{(\alpha+p+m)} - (\alpha+1-u) L_{k-p-m}^{(\alpha+p+m)} + (p-m) L_{k-p+1-m}^{(\alpha+p-1+m)}\right] \\ = & \sum_{m=0}^{k-p+1} \binom{p}{m} \frac{t^m}{(1+t)^m} \left[(\alpha+k+1+p-u) L_{k-p-m}^{(\alpha+p+m)} + \frac{2mp}{p-m+1} L_{k-p-m+1}^{(\alpha+p+m-1)} + \right. \\ & + m L_{k-p-m}^{(\alpha+p+m)} - \frac{2m(m-1)}{p-m+1} L_{k-p-m+1}^{(\alpha+p+m-1)} + (k-p-m+1) L_{k-p-m+1}^{(\alpha+p+m)} - \\ & \left. - (\alpha+k+1-u) L_{k-p-m}^{(\alpha+p+m)} + (p-m) L_{k-p+1-m}^{(\alpha+p-1+m)}\right] \\ = & \sum_{m=0}^{k-p+1} \binom{p}{m} \frac{t^m}{(1+t)^m} [(p+m) L_{k-p-m}^{(\alpha+p+m)} + (p+m) L_{k-p+1-m}^{(\alpha+p-1+m)} + (k-p-m+1) L_{k-p-m+1}^{(\alpha+p+m)}] \\ = & \sum_{m=0}^{k+1-p} \binom{p}{m} \frac{t^m}{(1+t)^m} [(p+m) L_{k+1-p-m}^{(\alpha+p+m)} + (k-p-m+1) L_{k+1-p-m}^{(\alpha+p+m)}] \\ = & (k+1) \sum_{m=0}^{k+1-p} \binom{p}{m} \frac{t^m}{(1+t)^m} L_{k+1-p-m}^{(\alpha+p+m)}\end{aligned}\right\} \quad (17)$$

It follows that for $0 < p < k+1$

$$A_p = (k+1)(xyt)^{\frac{p}{2}} \left(\frac{1+t}{1-t}\right)^p \sum_{m=0}^{k+1-p} \binom{p}{m} \frac{t^m}{(1+t)^m} L_{k+1-p-m}^{(\alpha+p+m)} \quad . \quad (18)$$

and the same result is valid for $p = 0$ and for $p = k + 1$. Thus from (12) we take the conclusion that (3) is due for $k + 1$.

3. From the special cases derivable from (7) we quote only the case $y \to 0$. We obtain then

$$\sum_{n=0}^{\infty} \binom{n+\alpha+k}{k} t^n L_n^{(\alpha)}(x) = \frac{1}{(1-t)^{k+1+\alpha}} \exp\left(-\frac{xt}{1-t}\right) \cdot L_k^{(\alpha)}\left(\frac{xt}{1-t}\right) \quad (19)$$

a formula we have given elsewhere [7]) and which can be used for the evaluation of some definite integrals involving products of Laguerre polynomials.

[7]) BOTTEMA, Een betrekking voor de polynomen van LAGUERRE en VAN HERMITE, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 65—71 (1946).

Mathematics. — *On the zeros of a polynomial and of its derivative.* By
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1. Considering polynomials $f(z)$ with real coefficients, we observe that, very roughly speaking, there exists a tendency for the zeros of its derivative $f'(z)$ to lie closer to the real axis, than those of $f(z)$. This is illustrated by the following well-known facts:

A. The number of imaginary [1]) zeros of $f'(z)$ does not exceed that of $f(z)$. This is a consequence of ROLLE's theorem.

B. If the zeros of $f(z)$ lie in a strip $|\operatorname{Im} z| \leq a$, the same can be said concerning the zeros of $f'(z)$. This is a special case of a famous theorem of GAUSS, expressing that the zeros of $f'(z)$ all lie in any convex domain, containing all the zeros of $f(z)$.

C. JENSEN's *Circles Theorem.* [2]) If $a_\nu \pm i b_\nu$ denote the imaginary zeros of $f(z)$, then any imaginary zero of $f'(z)$ lies inside at least one of the circles $|z - a_\nu| \leq |b_\nu|$.

In this paper we prove another theorem, illustrating the same tendency.

Theorem 1. *Let the real* [3]) *polynomial $f(z)$ of degree n $(n > 1)$ have the zeros* [4]) $\alpha_1, \ldots, \alpha_n$, *and let* $\beta_1, \ldots, \beta_{n-1}$ *be those of $f'(z)$. Then we have*

$$\frac{1}{n-1} \sum_{\nu=1}^{n-1} |\operatorname{Im} \beta_\nu| \leqslant \frac{1}{n} \sum_{\nu=1}^{n} |\operatorname{Im} \alpha_\nu|. \qquad (1)$$

There is equality if, and only if, all the zeros of $f(z)$ are real.

In section **2**, a generalization of Theorem 1 is stated and proved. Section **3** contains some applications of Theorems 1 and 2.

We do not know, whether Theorem 1 remains valid if the condition about the reality of the coefficients of $f(z)$ is omitted. The cases $n = 2, 3$ are easy to deal with, but the general case seems to be difficult. We are, however, able to prove it if all the zeros of $f(z)$ are purely imaginary, but not necessarily conjugated by pairs. This will be carried out in section **4**, which is independent of sections **1**, **2**, **3**. As a corrolary, we will obtain

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[1]) We call a complex number α $(\alpha = \operatorname{Re}\alpha + i \operatorname{Im}\alpha)$ imaginary, if $\operatorname{Im}\alpha \neq 0$; and purely imaginary, if $\operatorname{Re}\alpha = 0$.

[2]) Acta Math., **36**, 181—195 (1913).

[3]) Throughout the paper, a polynomial or rational function of z is called a real function, if it is real for real z.

[4]) A double zero is counted twice, etc.

a simple and more general inequality, containing an arbitrary convex function (Theorem 7).

2. The results concerning the zeros of $f'(z)$ and $f(z)$ can also be expressed in terms of zeros and poles of the function $f'(z)/f(z)$. We extend our considerations to rational functions of the more general type

$$\left.\begin{array}{l}\varphi(z)=az+b+\sum_{j=1}^{k}\frac{s_j}{z-a_j}+\sum_{j=1}^{l}\left(\frac{t_j}{z-\varrho_j}+\frac{t_j}{z-\bar{\varrho}_j}\right),\\ a\leqslant 0,\ b,\ a_1,\ldots,a_k \text{ real};\ s_1\geqslant 0,\ldots,s_k\geqslant 0;\ t_1\geqslant 0,\ldots,t_l\geqslant 0,\ \varrho_1,\ldots,\varrho_l \\ \hfill\text{imaginary}\end{array}\right\}. \quad (2)$$

Henceforth, we consider the point $z=\infty$ as a possible pole or zero of rational functions. For instance, if $a\neq 0$, $z=\infty$ represents a simple pole of (2); we call $-a$ its residue [5]). If $a=b=0$, $z=\infty$ is a zero of $\varphi(z)$. Moreover, $z=\infty$ is considered as a point on the real axis; thus we always take Im z to be zero, if z represents the point at infinity.

We can now describe a function of the type (2) as a real rational function, all of whose poles are simple, with positive residues. In analogy to the behaviour of $f'(z)/f(z)$, referred to above, we observe the tendency for the zeros of $\varphi(z)$ to lie closer to the real axis than the poles of $\varphi(z)$. Properties analogous to A, B, C hold true, and also their proofs are analogous [6]).

Our generalization of theorem 1 reads:

Theorem 2. *Let $\varphi(z)$ be of the type (2), and let $a_1,\ldots,a_n$ denote the poles of $\varphi(z)$, and $\beta_1,\ldots,\beta_n$ its zeros. Then we have*

$$\sum_{\nu=1}^{n}|\text{Im}\,\beta_\nu|\leqslant\sum_{\nu=1}^{n}|\text{Im}\,a_\nu|.$$

If, moreover, $a=b=0$, we have

$$\sum_{\nu=1}^{n}|\text{Im}\,\beta_\nu|\leqslant\sum_{\nu=1}^{n}|\text{Im}\,a_\nu|-2\,(\sum_1^k s_j+2\sum_1^l t_j)^{-1}\sum_1^l t_j\,|\text{Im}\,\varrho_j|.$$

In both cases, there is equality if, and only if, the poles of $\varphi(z)$ all lie on the real axis.

Theorem 1 follows immediately from the second part of Theorem 2. For, on taking $\varphi(z)=f'(z)/f(z)$, we obtain $a=b=0$, $t_j=1$, $\Sigma s_j+2\Sigma t_j=n$.

Our proof of Theorem 2 is relatively simple when $\varphi(z)$ has neither real poles, nor real zeros. In the general case, however, we need an auxiliary function $\psi(z)$, to be constructed in Lemma 2.

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5) With this (rather unusual) convention, the sign of the residue of a pole on the real axis remains invariant with respect to transformations $z=(az'+b)/(cz'+d)$ $(ad-bc>0;\ a,\ b,\ c,\ d$ real).

6) The analogue of JENSEN's Circles Theorem was proved by J. v. SZ. NAGY, Jhrsber. D. Math. Ver. **31**, 238—251 (1922).

Lemma 1. *If*

$$f(z) = \prod_{\nu=1}^{n} (z - \alpha_\nu),\ g(z) = \prod_{\nu=1}^{n} (z - \beta_\nu),$$

then

$$\text{V.P.} \int_{-\infty}^{\infty} \log \left| \frac{g(z)}{f(z)} \right| dz = \pi \left\{ \sum_{1}^{n} |\operatorname{Im} \beta_\nu| - \sum_{1}^{n} |\operatorname{Im} \alpha_\nu| \right\},$$

where V.P. *means 'valeur principale': V.P.* $\int_{-\infty}^{\infty} = \lim_{A \to +\infty} \int_{-A}^{A}$.

Proof. It is easily verified, that, for α arbitrary and $A \to +\infty$

$$\int_{-A}^{A} \log |z - \alpha|\, dz = 2 (A \log A - A) + \pi |\operatorname{Im} \alpha| + O(A^{-1}).$$

Now our lemma is evident.

Lemma 2. *Let $\varphi(z)$ be a real rational function (not identically zero) of the type* (2). *Then we can construct a real rational function $\psi(z)$, all of whose poles are simple and real, with negative residues, such that*

a) $0 \leq \varphi(z)\, \psi(z) < \infty$ *for real* z,

and b) $\varphi(z)\, \psi(z) = 1$ *for* $z = \infty$.

Proof. A point a on the real axis is called a *positive change of sign* for $\varphi(z)$, if the behaviour of the sign of $\varphi(z)$ in the neighbourhood of a is the same, as it is for a real function, which has a simple pole at $z = a$ with positive residue. So if a is finite, this means $(z - a)\, \varphi(z) > 0$ for $0 < |z - a| < \varepsilon$; if $a = \infty$ it means $z\, \varphi(z) < 0$ for $0 < |z^{-1}| < \varepsilon$. Negative changes of sign are defined accordingly. Any change of sign represents a pole or zero of odd order of $\varphi(z)$.

Now construct a real rational function $\psi(z)$, which has its (simple) zeros in the positive changes of sign for $\varphi(z)$, and its (simple) poles in the negative ones. Moreover, we take care, that $\varphi(z)\ \psi(z) \geq 0$ for all real values of z. It follows, that the poles of $\psi(z)$ are negative changes of sign for $\psi(z)$; hence all the residues of $\psi(z)$ are negative.

Any pole or zero of $\psi(z)$ is a pole or zero of $\varphi(z)$. The residues of $\varphi(z)$ being positive, any real pole of $\varphi(z)$ is a positive change of sign for $\varphi(z)$, and hence it is a zero of $\psi(z)$. So if a is a pole of one of the functions $\varphi(z)$ or $\psi(z)$, it is a zero of the other one. $\varphi(z)$ and $\psi(z)$ having simple zeros only, we now conclude, that the product $\varphi(z)\ \psi(z)$ *has no poles on the real axis.*

We shall prove, that $\varphi(z)\ \psi(z)$ has no zero at $z = \infty$. If $z = \infty$ is a pole of $\varphi(z)$, it will be a zero of $\psi(z)$. Both pole and zero are simple, and hence $0 \neq \varphi(\infty)\ \psi(\infty) \neq \infty$. If, in the second place, $0 \neq \varphi(\infty) \neq \infty$, we have also $0 \neq \psi(\infty) \neq \infty$, whence it follows $0 \neq \varphi(\infty)\ \psi(\infty) \neq \infty$. If, lastly, $\varphi(\infty) = 0$, we represent $\varphi(z)$ in the form (2), with $a = b = 0$.

From $s_j \geq 0$, $t_j \geq 0$, we infer, that $z = \infty$ is a simple zero, and moreover, that it is a negative change of sign for φ. Hence $z = \infty$ is a pole of $\psi(z)$, and again $0 \neq \varphi(\infty)\,\psi(\infty) \neq \infty$.

Now having obtained $\varphi(\infty)\,\psi(\infty) = c > 0$, we take $\psi_1(z) = c^{-1}\,\psi(z)$, which is easily seen to satisfy the conditions of our lemma.

Proof of Theorem 2. Let $\varphi(z)$ be given by (2), and let $\psi(z)$ satisfy the conditions of Lemma 2. Then we have, by Lemma 1:

$$\text{V. P.} \int_{-\infty}^{\infty} \log |\varphi(z)\,\psi(z)|\, dz = \pi \{ \sum_1^n |\operatorname{Im} \beta_\nu| - \sum_1^n |\operatorname{Im} \alpha_\nu| \}, \quad . \quad . \quad (3)$$

where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are the poles and zeros of $\varphi(z)$, respectively. For, the zeros and poles of $\psi(z)$ are real, and do not contribute to the right-hand side of (3).

It follows from the inequality

$$\log u \leqslant u - 1 \qquad (u > 0), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that, for z real,

$$\log |\varphi(z)\,\psi(z)| = \log \varphi(z)\,\psi(z) \leqslant \varphi(z)\,\psi(z) - 1. \quad . \quad . \quad . \quad (5)$$

In the upper half plane, the function $\varphi(z)\,\psi(z) - 1$ has the poles ϱ_j (we may, of course, suppose $\operatorname{Im} \varrho_j > 0 > \operatorname{Im} \bar{\varrho}_j$), with residues $t_j\,\psi(\varrho_j)$. At $z = \infty$ this function behaves like $c_1 z^{-1} + c_2 z^{-2} + \ldots$, with c_1 real. Now contour integration shows that

$$\text{V. P.} \int_{-\infty}^{\infty} \{ \varphi(z)\,\psi(z) - 1 \}\, dz = -2\pi \sum_{j=1}^{l} \operatorname{Im} \{ t_j\,\psi(\varrho_j) \}. \quad . \quad . \quad (6)$$

By Lemma 2, $\psi(z)$ is a function of the type

$$\psi(z) = A\,z + B - \frac{R_1}{z - d_1} - \ldots - \frac{R_p}{z - d_p} \quad . \quad . \quad . \quad . \quad (7)$$

where B, $d_1, \ldots, d_p$ are real, and $A \geq 0$, $R_1 \geq 0$, $\ldots$, $R_p \geq 0$. This implies $\operatorname{Im} \psi(\varrho_j) \geq A \operatorname{Im} \varrho_j$, hence

$$-\operatorname{Im} \{ t_j\,\psi(\varrho_j) \} \leqslant 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and even

$$-\operatorname{Im} \{ t_j\,\psi(\varrho_j) \} \leqslant -A\,t_j \operatorname{Im} \varrho_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The first part of Theorem 2 follows from (3), (5), (6) and (8). Now take $a = b = 0$. In this case, we have

$$\lim_{z \to \infty} z\,\varphi(z) = s_1 + \ldots + s_k + 2\,(t_1 + \ldots + t_l).$$

According to $\varphi(\infty)\,\psi(\infty) = 1$, we infer from (7), that

$$A = [s_1 + \ldots + s_k + 2\,(t_1 + \ldots + t_l)]^{-1}.$$

Now using (9) instead of (8), the second part of Theorem 2 is readily proved.

In (4), the sign $=$ only holds for $u = 1$. Hence, in Theorem 2, equality occurs only, if $\varphi(z)\,\psi(z) = 1$ identically in z. This means, that $\varphi(z)$ has no imaginary poles.

3. An interesting specialization of Theorem 1 is obtained by taking all the zeros of $f(z)$ to lie on the imaginary axis. This leads to:

Theorem 3. *Let $F(y)$ be a real polynomial, whose zeros $\gamma_1, \dots, \gamma_n$ $(n > 0)$ are all ≥ 0, but not all $= 0$, and let $\delta_1, \dots, \delta_{n-1}$ be the zeros of $F'(y)$. Then we have the inequality*

$$\sum_1^{n-1} \delta_\nu^{\frac{1}{2}} < \frac{n - \frac{1}{2}}{n} \sum_1^n \gamma_\nu^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Proof. The real polynomial $f(z) = F(-z^2)$ has the 2n zeros $\pm i\gamma_1^{\frac{1}{2}}, \dots, \pm i\gamma_n^{\frac{1}{2}}$. Its derivative $f'(z) = -2zF'(-z^2)$ having the zeros $0, \pm i\delta_1^{\frac{1}{2}}, \dots, \pm i\delta_{n-1}^{\frac{1}{2}}$. Theorem 3 follows by applying Theorem 1 to $f(z)$.

An inequality in the opposite direction was given by KAKEYA [7]) for general exponents. He proved, under the assumptions of Theorem 3, putting

$$D_p = \frac{1}{n} \sum_1^n \gamma_\nu^p - \frac{1}{n-1} \sum_1^{n-1} \delta_\nu^p,$$

that $D_p \geq 0$ for $p \geq 1$ or $p \leq 0$, and $D_p \leq 0$ for $0 \leq p \leq 1$, with equality only in three cases: $p = 0$, $p = 1$, and $\gamma_1 = \dots = \gamma_n$ (p arbitrary). For $p = \frac{1}{2}$, KAKEYA's result reads

$$\sum_1^{n-1} \delta_\nu^{\frac{1}{2}} \geqslant \frac{n-1}{n} \sum_1^n \gamma_\nu^{\frac{1}{2}}.$$

In section 4, KAKEYA's inequality for D_p will appear to be a special case of Theorem 7.

We give two more applications of Theorem 2.

Theorem 4. *Let the real polynomial $f(z) = a_0 + a_1 z + \dots + a_n z^n$ have the zeros $\alpha_1, \dots, \alpha_n$, and let $P(y)$ be a polynomial, all of whose zeros are ≤ 0. Furthermore, let $\beta_1, \dots, \beta_n$ be the zeros of*

$$g(z) = a_0\, P(0) + a_1\, P(1)\, z + \dots + a_n\, P(n)\, z^n.$$

We then have

$$\sum_{\nu=1}^n |\operatorname{Im} \beta_\nu| \leqslant \frac{P(n-1)}{P(n)} \sum_{\nu=1}^n |\operatorname{Im} \alpha_\nu|.$$

Theorem 1 follows from this one by taking $P(y) = y$.

Proof. It is sufficient to prove the case $P(y) = y + a$ $(a \geq 0)$, the

[7]) Proc. Phys. Math. Soc. Japan (3) **15**, 149—154 (1933). The case, that p is a natural number, was considered before by H. E. BRAY, Am. J. Math. **53**, 864—872 (1931). KAKEYA uses BRAY's result.

general case being obtained by iteration. Now $g(z) = zf'(z) + af(z)$; hence $\beta_1, \ldots, \beta_n$ and ∞ are the zeros of the function

$$\varphi(z) = \frac{f'(z)}{f(z)} + \frac{a}{z},$$

whose poles are $a_1, \ldots, a_n, 0$. Applying the second part of Theorem 2, we obtain $\Sigma s_j + 2\Sigma t_j = n + a$, $t_2 = \ldots = t_k = 1$, and, consequently

$$\sum_1^n |\operatorname{Im} \beta_\nu| \leqslant \left(1 - \frac{1}{n+a}\right) \sum_1^n |\operatorname{Im} a_\nu| = \frac{P(n-1)}{P(n)} \sum_1^n |\operatorname{Im} a_\nu|.$$

The following theorem is obtained by a similar iteration process.

Theorem 5. *Let $a_1, \ldots, a_n$ be the zeros of the real polynomial $f(z)$, and suppose that $g(y) = b_0 + b_1 y + \ldots + b_m y^m$ has real zeros only. If, furthermore, $\beta_1, \ldots, \beta_n$ are the zeros of $b_0 f(z) + b_1 f'(z) + \ldots + b_m f^{(m)}(z)$, then we have*

$$\sum_1^n |\operatorname{Im} \beta_\nu| \leqslant \sum_1^n |\operatorname{Im} a_\nu|. \quad \ldots \ldots \quad (11)$$

Proof. It is sufficient to consider the linear function $g(y) = b_0 + y$. Taking $\varphi(z) = b_0 + f'(z)/f(z)$ and applying the first part of Theorem 2, we immediately obtain (11).

4. We do not know whether the inequality (1) holds true for polynomials with complex coefficients. We can however prove it, if all the zeros of $f(z)$ are assumed to lie on the imaginary axis. Introducing a rotation $z = ix$, our result reads

Theorem 6. *Let the polynomial $f(x)$ of degree $n > 1$ have the zeros $a_1, \ldots, a_n$, and let $\beta_1, \ldots, \beta_{n-1}$ denote the roots of $f'(x) = 0$. Supposing $a_1, \ldots, a_n$ to be real (which implies the reality of $\beta_1, \ldots, \beta_{n-1}$), we have*

$$\frac{1}{n-1} \sum_1^{n-1} |\beta_\nu| \leqslant \frac{1}{n} \sum_1^n |a_\nu|,$$

with equality only if

$$a) \quad a_1 \leqslant 0, a_2 \leqslant 0, \ldots, a_n \leqslant 0,$$

or if

$$b) \quad a_1 \geqslant 0, a_2 \geqslant 0, \ldots, a_n \geqslant 0.$$

Proof. Since we have

$$\frac{1}{n-1} \sum_1^{n-1} \beta_\nu = \frac{1}{n} \sum_1^n a_\nu, \quad \ldots \ldots \quad (12)$$

(both sides equalling $-a_{n-1}/na_n$, if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots$) it is sufficient to prove

$$D(f) = \frac{1}{n} \sum_1^n \varphi(a_\nu) - \frac{1}{n-1} \sum_1^{n-1} \varphi(\beta_\nu) \geqslant 0. \quad \ldots \quad (13)$$

where

$$\varphi(x) = \tfrac{1}{2}(x + |x|) = \begin{cases} x \text{ if } x \geqslant 0, \\ 0 \text{ if } x \leqslant 0. \end{cases}$$

In case *a*) we have also $\beta_1 \leq 0, \ldots, \beta_{n-1} \leq 0$, and consequently $D = 0$. The same holds true in case *b*). Now suppose

$$0 < k < n,\ a_1 \geq 0, \ldots, a_k \geq 0,\ a_{k+1} < 0, \ldots, a_n < 0.$$

Suppose furthermore, that at least one β_l is positive, and not equal to a multiple root of $f(x) = 0$, for otherwise (13) is trivial. Put

$$f_\varrho(x) = \prod_1^k (x - a_\nu) \prod_{k+1}^n (x - \varrho a_\nu), \qquad f(x) = f_1(x).$$

The zeros $\beta_1(\varrho), \ldots, \beta_{n-1}(\varrho)$ of $f'_\varrho(x)$ are continuous functions of ϱ for $0 \leq \varrho \leq 1$, and differentiable at least for $0 < \varrho \leq 1$. Let $\beta_1, \ldots, \beta_j$ $(j \geq 1)$ be the positive zeros of $f'(x)$, and suppose $h \geq 1$, $f(\beta_1) \neq 0, \ldots, f(\beta_h) \neq 0$, $f(\beta_{h+1}) = \ldots = f(\beta_j) = 0$. The zeros $\beta_1(\varrho), \ldots, \beta_h(\varrho)$ are increasing, if ϱ decreases from 1 to 0. For, by differentiation of the relations

$$\sum_{\nu=1}^k \frac{1}{\beta_l - a_\nu} + \sum_{\nu=k+1}^n \frac{1}{\beta_l - \varrho a_\nu} = 0$$

with respect to ϱ, we obtain

$$\frac{d\beta_l}{d\varrho}\left(\sum_{\nu=1}^k \frac{1}{(\beta_l - a_\nu)^2} + \sum_{\nu=k+1}^n \frac{1}{(\beta_l - \varrho a_\nu)^2}\right) = \sum_{\nu=k+1}^n \frac{a_\nu}{(\beta_l - \varrho a_\nu)^2},$$

whence it follows $\dfrac{d\beta_i}{d\varrho} < 0$ $(i = 1, \ldots, h;\ 0 < \varrho \leq 1)$. Furthermore

$$\beta_{h+1}(\varrho), \ldots, \beta_j(\varrho)$$

are constant. Hence

$$\sum_{i=1}^j \beta_i(1) < \sum_{i=1}^j \beta_i(0) \leqslant \sum_{\nu=1}^{n-1} \varphi(\beta_\nu(0)). \quad \ldots \ldots \quad (14)$$

Now considering the expressions $D\{f_1(x)\}$ and $D\{f_0(x)\}$, we observe the contributions of the a's to be equal for both. So by (14), we have

$$D\{f(x)\} > D\{f_0(x)\}. \quad \ldots \ldots \ldots \quad (15)$$

The polynomial $f_0(x)$ belonging to case *b*), we have $D\{f_0(x)\} = 0$. Now (15) proves our theorem.

Theorem 7. *Let the polynomial $f(x)$ of degree $n > 1$ have the n real zeros $a_1 \leq a_2 \leq \ldots \leq a_n$, and let $\beta_1, \ldots, \beta_{n-1}$ denote the zeros of $f'(x)$. Let the function $\psi(x)$ be convex in the interval $a_1 \leq x \leq a_n$. Putting*

$$D(\psi;\ f) = \frac{1}{n}\sum_1^n \psi(a_\nu) - \frac{1}{n-1}\sum_1^{n-1} \psi(\beta_\nu),$$

we have $D(\psi; f) \geq 0$. $D(\psi; f) = 0$ *holds only if* $\psi(x)$ *is linear for* $a_1 \leq x \leq a_n$ [8]).

Proof. Let $\gamma_1 < \gamma_2 < \ldots < \gamma_m$ denote the set $a_1, \ldots, a_n, \beta_1, \ldots, \beta_{n-1}$, arranged in ascending order (each number of this set is taken only once). It follows that $\gamma_1 = a_1$, $\gamma_m = a_n$. We can evidently construct a convex function $\psi^*(x)$, satisfying $\psi^*(\gamma_i) = \psi(\gamma_i)$ $(i = 1, \ldots, m)$, which is linear and continuous upon the intervals

$$x \leqslant \gamma_2, \gamma_2 \leqslant x \leqslant \gamma_3, \ldots, \gamma_{m-2} \leqslant x \leqslant \gamma_{m-1}, \gamma_{m-1} \leqslant x.$$

This function can be represented as

$$\psi^*(x) = \sum_{i=2}^{m-1} \lambda_i |x - \gamma_i| + C\,x, \quad \lambda_i \geqslant 0. \quad . \quad . \quad . \quad . \quad (16)$$

According to Theorem 6, we have

$$D\{|x-\gamma_i|;\, f(x)\} = D\{|x|;\, f(x+\gamma_i)\} \geqslant 0,$$

and by (12) $D(x; f) = 0$. Hence

$$D(\psi; f) = D(\psi^*, f) = \sum_2^{m-1} \lambda_i\, D\{|x-\gamma_i|; f\} + C\, D(x; f) \geqslant 0 \;. \quad (17)$$

Now suppose $a_1 < a_n$, and suppose that $\psi(x)$ is not linear upon the interval $a_1 \leq x \leq a_n$. The open interval $a_1 < x < a_n$ containing at least one root of $f'(x) = 0$, we observe that $m > 2$, and that $\psi^*(x)$ is not linear upon $a_1 \leq x \leq a_n$. Hence at least one term on the right-hand side of (16), say the term with $i = k$, is non-linear, and consequently $\lambda_k > 0$. By theorem 6 it follows, that $\lambda_k D(|x - \gamma_k|; f) > 0$, the roots of $f(x)$ occurring on both sides of γ_k. Now (17) yields $D(\psi; f) > 0$, which proves our theorem.

KAKEYA's result, mentioned in section 3, is contained in Theorem 6. This follows from the convexity of y^p for $y > 0$, $p \geq 1$ or $p \leq 0$, and of $-y^p$ for $y > 0$, $0 \leq p \leq 1$.

We are indebted to Mrs. T. VAN AARDENNE-EHRENFEST for some valuable remarks.

8) This includes the case $a_1 = a_2 = \ldots = a_n$.

Eindhoven, September 1946.

Mathematics. — *Sur les espaces linéaires normés.* I. By A. F. MONNA.
(Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 26, 1946.)

I. *Classification des espaces.*

§ 1. Soit E un espace linéaire dont les coefficients appartiennent à un corps K. Préalablement on ne suppose de K qu'il soit commutatif; on ne suppose pas que les éléments de K soient des nombres réels. Soit E un espace métrique et, en désignant par θ l'élément unique de E tel que $x + \theta = \theta + x = x$ pour tout x en E, supposons que la distance (x, y) de x et y en E vérifie la relation

$$(x, y) = (x - y, \theta) = \|x - y\|. \quad \quad (1)$$

(x, y) est réel.

Comme on sait, on appelle un pareil espace E *normé* si l'on a pour tout x en E et a en K

$$\|a x\| = |a| \cdot \|x\|. \quad \quad (2)$$

Cependant la relation (2) n'a que de sens que si l'on impose à K une condition de plus: il faut qu'une valuation sur K soit donnée. Remarquons que la relation $\| ax \| = a \,.\, \| x \|$ n'a aucun sens, même si K est le corps des nombres réels (en effet $\| ax \|$ et $\| x \|$ sont tous les deux $\geqq 0$).

Dans tout ce qui suit nous supposons donc que sur le corps K est donnée une valuation, dont les valeurs sont des nombres réels.

Les espaces que nous allons considérer satisfassent tous à (1) et (2).

Soit désigné par N_K l'ensemble des valeurs $|a|$ des éléments a de K; soit N_E l'ensemble des normes $\| x \|$ des éléments x de E.

N_K se compose d'un groupe multiplicatif de nombres > 0 et du nombre 0. En effet si α et β appartiennent à N_K, ils existent des éléments a et b de K avec $|a| = \alpha$, $|b| = \beta$; alors $|ab| = \alpha\beta$ et, si $\beta \neq 0$, $\left|\frac{a}{b}\right| = \frac{\alpha}{\beta}$ de sorte que $\alpha\beta$ et $\frac{\alpha}{\beta}$ appartiennent à N_K. A ce propos on ne peut rien dire de N_E.

Pour autant que je sais on n'a considéré que le cas où on a

$$N_E \subset N_K. \quad \quad (3)$$

Cette condition implique qu'on peut normer chaque vecteur de E: étant donné le vecteur x, on a $\| x \| \in N_K$ et il existe donc un élément a de K tel que $|a| = \| x \|$. On a alors

$$\left\|\frac{x}{a}\right\| = \frac{\|x\|}{|a|} = 1.$$

Sans cette condition ceci n'est pas toujours possible.

Dans ce qui suit on ne suppose pas (3), *à moins que cela est expressément indiqué.*

On peut distinguer deux cas:

A. La valuation de K est archimédienne.

En vertu d'un théorème connu, K est alors isomorphe à un corps à valuation absolue composé de nombres réels ou complexes.

Les espaces, résultants du cas où K est identique au corps des nombres réels ou complexes, ont été déjà amplement traités. Pour autant que je sais on n'a pas étudié le cas où K est un vrai sous-corps des dits corps. Il donne lieu à des difficultés auxquelles nous viendrons plus tard.

B. La valuation de K est non-archimédienne.

Supposons qu'il n'existe aucun couple x et y de E avec x et $y \neq \theta$ tel que

$$\|x+y\| \leqq \max(\|x\|, \|y\|).$$

Alors on aurait pour tout x et $y \neq \theta$

$$\max(\|x\|, \|y\|) < \|x+y\| \leqq \|x\| + \|y\|.$$

En particulier

$$\|x\| = \max(\|x\|, \|x\|) < \|2x\| = |2| \cdot \|x\|,$$

donc puisque $\|x\| \neq 0$,

$$1 < |2|.$$

Plus général $1 < |n|$. Alors la valuation de K serait archimédienne: contradiction.

La supposition B conduit donc à deux cas:

1. pour *tout* x et y de E on a

$$\|x+y\| \leqq \max(\|x\|, \|y\|).$$

2. ils existent des couples x et y de E avec x, $y \neq \theta$ tels que

$$\|x+y\| \leqq \max(\|x\|, \|y\|).$$

Les espaces du cas $B1$ seront appelés *totalement-non-archimédiens*. Au cas $B2$ nous parlons des espaces *semi-non-archimédiens*.

Remarques. 1. Si pour tout x, y de E on a $\|x+y\| \leq \max(\|x\|, \|y\|)$, la valuation de K est non-archimédienne. En effet, on a

$$\|nx\| = |n| \, . \, \|x\| \leq \|x\| \text{ donc } |n| \leq 1.$$

Ceci justifie le nom totalement-non-archimédien.

2. Dans les cas $B1$ et $B2$ on peut immédiatement indiquer des couples qui vérifient l'inégalité triangulaire rigoureuse, p. ex. le couple x, x:

$$\|x+x\| = \|2x\| = |2| \cdot \|x\| \leqq \|x\|.$$

3. Même au cas A ils existent de tels couples; par exemple les vecteurs x et y tels que

$$\|x+y\|=\|y\|, \qquad \|x\| \geqq \|y\|,$$

donc dans un espace euclidien les triangles à côtés égaux et dont la troisième côté est plus petite.

4. Supposons a) E lui-même est un corps b) $\| xy \| = \| x \| \cdot \| y \|$ c) la valuation de K est non-archimédienne. Alors E est totalement-non-archimédien. On peut suivre une marche de démonstration de V. D. WAERDEN [1]).

5. Il serait désirable d'obtenir des conditions nécessaires et suffisantes pour discerner entre les cas $B1$ et $B2$ et entre A et $B2$.

6. Pour les espaces du type A voir le livre connu de BANACH [2]).

7. J'ai donné un exemple d'un espace semi-non-archimédien dans l'article $M1$: Over een lineaire P-adische ruimte, Proc. Nederl. akad. v. Wetensch. 52, 74—82 (1943).

J'ai commencé à étudier les espaces totalement-non-archimédiens pour lesquels $N_E \subset N_K$ dans les articles:

$M2$: Over niet-archimedische lineaire ruimten, idem 52, 308—321 (1943),

$M3$: Lineaire functionaalvergelijkingen in niet-archimedische Banachruimten, idem 52, 654—661 (1943),

$M4$: Over geordende groepen en lineaire ruimten, idem 53, 178—182 (1944).

Un résultat de $M2$ sera corrigé ci-après [3]).

Dans ce qui suit nous traitons principalement les espaces totalement-non-archimédiens, cependant sans la condition supplémentaire (3).

II. *Propriétés topologiques des espaces linéaires totalement-non-archimédiens.*

§ 2.

Théorème 1. *Chaque espace linéaire totalement-non-archimédien est de dimension* 0.

Démonstration. Il suit de l'inégalité triangulaire rigoureuse que chaque ε-voisinage fermé d'un point arbitraire x_0 de E: $\| x - x_0 \| \leqq \varepsilon$ est ouvert, d'où résulte le théorème.

Dans ce qui suit nous étudions les propriétés des espaces totalement-non-archimédiens localement compacts. *Nous supposons que la valuation de K est non-triviale.* Il sera mentionné expressément si la démonstration d'un théorème reste vraie si la valuation est triviale. Plus tard nous ferons quelques remarques à propos du cas de la valuation triviale.

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[1]) B. L. V. D. WAERDEN, Moderne algebra I, p. 248 (Berlin 1937).

[2]) S. BANACH, Théorie des opérations linéaires (Warszawa 1932).

[3]) Dans $M\,2$ et $M\,3$ je suis parti d'un groupe ordonné multiplicatif abélien P, complété d'un élément 0, vérifiant d'ailleurs quelques autres axiomes. Cependant j'ai montré dans $M\,4$ que ceci ne présente pas une généralisation par rapport au cas où N_K consiste de nombres réels.

Théorème 2. *L'espace linéaire totalement-non-archimédien*[4]) *E est supposé localement compact. On a les propriétés suivantes:*

1. *l'ensemble N_E a 0 comme seul point d'accumulation;*
2. *le corps K est localement-totalement-borné;*
3. *E est complet;*
4. *soit $C \in N_E$; alors E ne contient qu'un nombre fini d'éléments $e_i (1 \leqq i \leqq n)$ tels que*

$$\| e_i \| = C \, (i = 1, \ldots, n), \qquad \| e_i - e_j \| = C \, (i \neq j).$$

Démonstration. 1. D'abord on a: si $\{ x_n \}$ est une suite convergente, alors ou $\lim\limits_{n \to \infty} \| x_n \| = 0$ ou $\| x_n \| =$ const. pour presque tout n. En effet, si $\lim x_n = x$ et $\lim \| x_n \| \neq 0$, on a

$$\| x_n \| = \| x_n - x + x \| = \max (\| x - x_n \|, \| x \|) = \| x \|$$

pour toutes les valeurs de $n > N$ telles que $\| x - x_n \| < \| x \|$.

Puisque E est localement compact il existe une sphère compacte B: $\| x \| \leqq \varepsilon$. Soit B_1 la sphère $\| x \| \leqq \varepsilon_1$. La valuation de K n'étant pas triviale, il existe un élément a de K tel que $| a | \leqq \varepsilon / \varepsilon_1$. Pour tout ax avec $x \in B_1$ on a $\| ax \| \leqq \dfrac{\varepsilon}{\varepsilon_1} \varepsilon_1 = \varepsilon$, donc $ax \in B$. B_1 est donc compact et, plus général, il en résulte que chaque ensemble fermé borné est compact.

Soit ensuite V un ensemble compact et donc borné de E, N_V l'ensemble des normes $\| x \|$ pour $x \in V$. Alors ou $0 \in \overline{N}_V$ ou bien N_V est fini. En effet, supposons que $\alpha \neq 0$ est un point d'accumulation de N_V, $\alpha_i \to \alpha$, $\alpha_i \in N_V$, $\alpha_i \neq \alpha$ et $\{ x_i \}$ une suite de points de V avec $\| x_i \| = \alpha_i \to \alpha$, alors en vertu de la compacité de V il existerait une suite partielle $\{ x_{i'} \}$ avec la limite x telle que $\| x_{i'} \| = \alpha_{i'} \to \alpha = \| x \|$, en contradiction avec la propriété du premier alinéa.

Il résulte de ce qui précède que seulement 0 peut être un point d'accumulation de N_E; 0 est certainement point d'accumulation puisque N_K et donc N_E contiennent des nombres arbitrairement petits.

2. Il y a équivalence métrique (au facteur $\| x_0 \|$ près) entre le corps K et l'ensemble ax_0 pour a dans K et x_0 fixé. Comme sous-ensemble de E ce dernier ensemble est localement totalement borné; donc K possède cette propriété.

3. Parmi les espaces métriques on peut caractériser les espaces complets par la propriété suivante: toute suite $B_1 \supset B_2 \supset \ldots\ldots$ de sphères dont les

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[4]) Supposons que l'espace E est archimédien (le cas A de la partie I) et localement compact. On montre, comme au cas totalement-non-archimédien, que E est complet; nous verrons (§ 5), qu'on peut alors supposer, sans restreindre la généralité, que K est complet. En ce cas K est un corps de nombres réels ou complexes, et en nous bornant au cas de nombres réels, K est identique au corps des nombres réels. Ce cas a été déjà traité par BANACH l.c. 2).

diamètres convergent vers 0 possède une intersection $B_1 B_2 \ldots\ldots$ non vide. En utilisant cette propriété, la propriété 3 devient triviale lorsqu'on remarque que chaque sphère B est compacte (on fait usage du fait que la valuation est non triviale; voir 1).

4. Supposons qu'il existe une infinité dénombrable d'éléments e_i tels que $\| e_i \| = C$, $\| e_i - e_j \| = C$ $(i \neq j)$. Soient U_ε le voisinage $\| x \| \leqq \varepsilon$ de θ et a un élément de K avec $0 < | a | < \frac{\varepsilon}{C}$. Un tel élément existe puisque la valuation de K est non triviale.

L'ensemble V des éléments ae_1, $ae_2, \ldots$ est contenu dans U_ε puisque

$$\| a e_i \| = | a | \cdot \| e_i \| \leqq \frac{\varepsilon}{C} C = \varepsilon.$$

De plus

$$\| a e_i - a e_j \| = | a | \cdot \| e_i - e_j \| = | a | \cdot C.$$

Il s'ensuit que V n'admet pas de point d'accumulation, en contradiction avec la compacité locale de E.

Le théorème est ainsi démontré.

Conséquences. 1. Il suit de la propriété 1 que la valuation de K est discrète. Autrement N_K serait partout dense. En considérant les points ax de E (x fixé, a parcourt K), on voit que N_E serait alors partout dense aussi.

2. On tire de la propriété 4 qu'il n'existe qu'un nombre fini d'éléments a_i de K tels que $| a_i | = 1$, $| a_i - a_j | = 1$ $(i \neq j)$. C'est aussi une conséquence de 2. En effet, le plus petit corps complet contenant K est localement compact et pour ces corps la propriété est déjà connue [5]). Il suffit alors de remarquer que l'ensemble N_E ne change pas si l'on passe de K au dit corps complet (comparer le début de la démonstration de la propriété 1).

4. La propriété 1 du théorème 2 s'étend aux espaces totalement-non-archimédiens localement-totalement-bornés car N_E ne change pas si l'on passe de E à l'espace complet le plus petit contenant E, qui est localement compact donc vérifie 1°. J'ai montré ailleurs que N_E est dénombrable pour les espaces totalement-non-archimédiens séparables [6]).

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[5]) F. LOONSTRA, Im Kleinen kompakte nichtarchimedisch bewertete Körper. Proc. Ned. Akad. v. Wetensch., Amsterdam, 45, 665—668 (1942).

J. DE GROOT und F. LOONSTRA, Topologische Eigenschaften bewerteter Körper. Idem p. 658—664.

[6]) Sur l'approximation de fonctions abstraites, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, 49, 404—408 (1946).

Remarquons qu'un espace, dont N_E a 0 comme seul point d'accumulation, est semi-non-archimédien ou totalement-non-archimédien.

§ 3. Le but de ce paragraphe sera de montrer une inversion du théorème 2. Rappelons que la valuation de K est discrète si N_E admet 0 comme seul point d'accumulation. Alors N_K consiste des nombres

$$\varrho^n \ (n = 0, \pm 1, \pm 2, \ldots)$$

si ϱ est un nombre > 1. Dans tout ce qui suit ϱ a partout cette signification. Sans mention contraire la valuation de K est supposée non-triviale.

Théorème 3. *Soit E un espace linéaire totalement-non-archimédien dont N_E n'admet pas d'autre point d'accumulation que* 0. *Soit L un sous-espace linéaire fermé de E; $L \neq E$. Soit $C \in N_E$; $C \neq 0$. Alors $E - L$ contient un élément x_0 tel que*

$$C \leqq \| x_0 \| \leqq C\varrho$$

$$\| x_0 - x \| \geqq \| x_0 \| \text{ pour tout } x \in L.$$

Démonstration. Soient $y \in E - L$ et d la plus petite distance de y à L, c. à d. la borne inférieure des nombres $\| y - x \|$ pour $x \in L$; $d \neq 0$ puisque L est fermé. N_E n'admettant pas d'autre point d'accumulation que 0, il existe en L un point x' où la plus petite distance est atteinte. On a donc

$$\| y - x' \| \leqq \| y - x \| \text{ pour tout } x \in L.$$

Posons $y - x' = z$. Alors $\| z \| = d$ et $\| z + x' - x \| \geqq d$. Puisque $x' - x \in L$, donc

$$\| z - x \| \geqq d \text{ pour } x \in L.$$

Soit alors a un élément de K tel que

$$\frac{C}{d} \leqq |a| \leqq \frac{C}{d}\varrho.$$

Un pareil élément existe. En effet, soit $\alpha \in N_K$ le plus petit des nombres $\geqq \frac{C}{d}$ de N_K. On a $\alpha \geqq \frac{C}{d}$ et $\frac{\alpha}{\varrho} < \frac{C}{d}$, donc $\alpha < \frac{C}{d}\varrho$. On voit que chaque élément a avec $|a| = \alpha$ satisfait. Posons alors $x_0 = az$. On a

$$\| x_0 \| = |a| \cdot \| z \|,$$

donc

$$C \leqq \| x_0 \| \leqq C\varrho.$$

De plus $\| az - ax \| \geqq |a| \, d = |a| \, \| z \| = x_0$ et puisque $ax \in L$,

$$\| x_0 - x \| \geqq \| x_0 \|.$$

Remarques. 1. Si l'on suppose $N_E \subset N_K$ on peut déterminer le vecteur x_0 tel que $\| x_0 \| = C$, puisqu'alors $\frac{C}{d} \in N_K$.

2. Comparer ce théorème avec le théorème correspondant de RIESZ (voir BANACH, l.c. 2) p. 83).

Théorème 4. *Soit E un espace linéaire totalement-non-archimédien dont N_E n'admet autre point d'accumulation que 0. Soit K complet.*
Alors pour tout r entier l'espace linéaire, formé des points

$$L_r \ldots x = a_1 x_1 + \ldots + a_r x_r$$

où a_i parcourt K et où les x_i sont des points fixés, est fermé.

Démonstration. Puisque K est complet, l'espace L_1 est fermé. Supposons que L_{r-1} est fermé et montrons que L_r est fermé. L_r consiste des points

$$x = y + a x_r$$

où $y \in L_{r-1}$, $a \in K$, x_r pas dans L_{r-1}. Soit $L_r^{(1)}$ la variété $y + x_r$ $(y \in L_{r-1})$; $L_r^{(1)}$ est fermé. Soit z un point d'accumulation de $L_r^{(1)}$. Il existe une suite $y_n + x_r \to z$ $(n \to \infty)$. Donc $y_n \to z - x_r$. Puisque $y_n \in L_{r-1}$ et L_{r-1} est fermé, on a $z - x_r \in L_{r-1}$, donc $z - x_r = y$ ou $z = y + x_r$, q.e.d. θ n'est pas dans $L_r^{(1)}$ et a donc une plus petite distance $\neq 0$ à $L_r^{(1)}$. Puisque N_E n'admet pas d'autres points d'accumulation que 0, il y a des points dans $L_r^{(1)}$ où cette plus petite distance à θ est atteinte. Soit $\bar{x}_r$ un tel point. Alors

$$\bar{x}_r = y_0 + x_r \qquad (y_0 \in L_{r-1})$$

$$\|\bar{x}_r\| \leqq \|y + x_r\| \text{ pour tout } y \in L_{r-1}, \text{ donc } y + x_r \in L_r^{(1)}.$$

$L_r^{(1)}$ est identique à l'ensemble des points $y + \bar{x}_r$ $(y \in L_{r-1})$. En effet, si $u = y + x_r$, $u \in L_r^{(1)}$, on a $u = y + \bar{x}_r - y_0 = y - y_0 + \bar{x}_r$. Or, y et y_0 sont dans L_{r-1}, donc $y - y_0$ dans L_{r-1}, de sorte que u appartient à $y + \bar{x}_r$. Même démonstration pour la propriété inverse. On montre de la même façon que L_r est identique à l'ensemble $y + a\bar{x}_r$. On a maintenant

$$\|\bar{x}_r\| \leqq \|y + \bar{x}_r\|$$

et

$$\|a\bar{x}_r\| \leqq \|ay + a\bar{x}_r\|$$

pour chaque y en L_{r-1} et a en K. Puisque ay parcourt tout L_{r-1}, on a

$$\|a\bar{x}_r\| \leqq \|y + a\bar{x}_r\|.$$

De l'inégalité

$$\|y + a\bar{x}_r\| \leqq \max(\|y\|, \|a\bar{x}_r\|)$$

on obtient

$$\|y + a\bar{x}_r\| = \max(\|y\|, \|a\bar{x}_r\|),$$

valable pour tout y en L_{r-1} et a en K.

Soit alors z un point d'accumulation de L_r. Il existe donc une suite $y_n + a_n\bar{x}_r \to z$ c. à d.

$$\|y_n - y_m + (a_n - a_m)\bar{x}_r\| \leqq \varepsilon \text{ pour } n \text{ et } m > N(\varepsilon).$$

On a en vertu de ce qui précède

$$\| y_n - y_m + (a_n - a_m)\bar{x}_r \| = \max (\| y_n - y_m \|, \| (a_n - a_m)\bar{x}_r \|)$$

donc

$$\| (a_n - a_m)\bar{x}_r \| \leqq \varepsilon \text{ pour } n \text{ et } m > N.$$

$\{ a_n \}$ est donc une suite fondamentale et, K étant complet, elle a une limite a. Alors

$$\| y_n - (z - a\bar{x}_r) \| = \| y_n - z + a\bar{x}_r - a_n\bar{x}_r + a_n\bar{x}_r \| \leqq$$

$$\leqq \max (\| y_n + a_n\bar{x}_r - z \|, |a - a_n| \, \| \bar{x}_r \|) \leqq \varepsilon$$

de sorte que lim y_n existe et vaut $z - a\bar{x}_r$. Puisque y_n est dans L_{r-1} et L_{r-1} est fermé, cette limite y est dans L_{r-1}; donc $z = y + a\bar{x}_r$. q.e.d.

Théorème 5. *Mêmes suppositions que dans le théorème 4. Supposons de plus que pour tout $C \in N_E$ il n'existe en E qu'un nombre fini d'éléments e_i tels que*

$$\left. \begin{array}{l} C \leqq \| e_i \| \leqq C\varrho \\ C \leqq \| e_i - e_j \| \leqq C\varrho \quad (i \neq j). \end{array} \right\} \quad . \quad . \quad . \quad . \quad (4)$$

Alors il existe en E un nombre fini de points $x_1, \ldots, x_r$ tels que chaque x de E peut s'écrire dans la forme

$$x = a_1 x_1 + \ldots + a_r x_r$$

où a_i parcourt K $(i = 1, \ldots, r)$.

Démonstration. Supposons que le théorème n'est pas vrai. Soit $C \in N_E$; $C \neq 0$. Supposons qu'on a déjà déterminé r points $x_1, \ldots, x_r$ linéairement indépendants de E tels que

$$\begin{array}{ll} C \leqq \| x_i \| \leqq C\varrho & (i = 1, \ldots, r) \\ C \leqq \| x_i - x_j \| \leqq C\varrho & (i \neq j) \end{array}$$

Soit L_r l'espace linéaire des points

$$x = a_1 x_1 + \ldots + a_r x_r \quad (a_i \in K).$$

En vertu du théorème précédent L_r est fermé et le théorème 5 étant supposé faux $L_r \neq E$. Le théorème 3 assure donc l'existence en $E - L_r$ d'un point x_{r+1} tel que

$$C \leqq \| x_{r+1} \| \leqq C\varrho$$

$$\| x_{r+1} - x \| \geqq \| x_{r+1} \| \text{ pour tout } x \in L_r.$$

En particulier pour $i = 1, \ldots, r$

$$\| x_{r+1} - x_i \| \geqq \| x_{r+1} \| \geqq C.$$

Or

$$\| x_{r+1} - x_i \| \leqq \max (\| x_{r+1} \|, \| x_i \|) \leqq C\varrho.$$

Donc

$$C \leqq \| x_{r+1} - x_i \| \leqq C\varrho.$$

A partir du système $\{x_i\}$ $(i = 1, \ldots, r)$ à propriétés (4) nous avons ainsi obtenu un système de $r+1$ points avec les propriétés (4). D'après la supposition $L_r \neq E$ pour chaque valeur de r. On obtient donc une suite infinie $x_1, x_2, \ldots$ avec les propriétés (4); contradiction.

A côté de l'espace du théorème précédent considérons maintenant l'espace K^r dont les éléments sont les suites $y = (a_1, a_2, \ldots, a_r)$ où $a_i \in K$ et où $(a_1, a_2, \ldots, a_r) \neq (b_1, \ldots, b_r)$ s'il y a une valeur de i telle que $a_i \neq b_i$. En supposant que la valuation de K est discrète, ce qui est vrai au théorème 5, cet espace devient un espace linéaire normé totalement-non-archimédien si l'on pose

$$\| y \| = \max_{1 \leqq i \leqq r} |a_i|.$$

Bien entendu, l'addition et la multiplication par les constantes de K sont définies de la manière ordinaire.

Considérons encore l'espace E du théorème 5 et *supposons de plus que cet espace est complet*. L'espace K^r précédent sera celui où K est le corps des coefficients de E. En faisant correspondre le point

$$x = a_1 x_1 + \ldots + a_r x_r$$

de l'espace E (en vertu du théorème 5 chaque point de E admet une telle représentation unique) au point $(a_1, \ldots, a_r)$ de K^r, on obtient une représentation biunivoque, additive et homogène de E sur K^r tout entier. Il suit de $\| x \| \leqq \max \| a_i x_i \|$ que cette transformation de K^r en E est continue. E et K, et donc K^r, étant complets, il suit du théorème 14 de la troisième partie, dont la démonstration sera indépendante des présents théorèmes, que cette transformation est alors bicontinue (donc une homéomorphie). Remarquons maintenant que K, et donc K^r, est localement compact (voir la note 5). On en tire que E est localement compact. Nous avons ainsi obtenu le théorème suivant:

Théorème 6. *Soit E un espace linéaire totalement-non-archimédien satisfaisant à:*

1. *N_E admet le point 0 comme seul point d'accumulation;*
2. *le corps K des coefficients est complet;*
3. *E est complet;*
4. *pour tout $C \in N_E$ il n'existe en E qu'un nombre fini d'éléments e_i tels que*

$$C \leqq \| e_i \| \leqq C\varrho$$
$$C \leqq \| e_i - e_j \| \leqq C\varrho \quad (i \neq j).$$

Alors E est localement compact.

Ce théorème est l'inversion du théorème 2. Bien entendu, la condition 4 est plus grave que la propriété 4 de ce théorème. Cependant, cette condition plus grave est aussi nécessaire pour les espaces localement compacts (même démonstration qu'auparavant). Le théorème 5 s'applique donc aux espaces localement compacts de sorte que les points d'un pareil espace puissent s'écrire dans la forme

$$x = a_1 x_1 + \dots + a_r x_r.$$

Nous avons déjà remarqué que K est localement compact. On trouve les formes possibles de ces corps l.c. 4). A un point près — l'image du point à l'infini au moyen duquel K peut être compacticé — le corps K est homéomorphe à l'ensemble parfait non-dense de CANTOR. L'espace E lui-même est, à un point près, homéomorphe au produit topologique d'un nombre r de tels ensembles de CANTOR: c'est une caractérisation topologique des espaces totalement-non-archimédiens localement compacts.

Il est possible d'introduire dans les espaces totalement-non-archimédiens localement compacts une métrique telle que la condition $N_E \subset N_K$ du § 1 est vérifiée. Si, en effet $x = a_1 x_1 + \dots + a_r x_r$, il suffit de poser

$$\|x\| = \max |a_i|.$$

On voit que les conditions auxquelles une norme doit satisfaire, sont vérifiées (remarquer que la représentation de x est unique). On a même $N_E \equiv N_K$.

Une métrique satisfaisant à $N_E \subset N_K$ ayant été introduite en E, la remarque 1 après le théorème 3 et la démonstration du théorème 5 font voir qu'on peut trouver r points $x_1, \dots, x_r$ en E satisfaisant à

$$\|x_i\| = C \qquad (i = 1, \dots, r;\ C \in N_E;\ C \neq 0),$$
$$\|x_i - x_j\| = C \quad (i \neq j),$$

tel que chaque point x de E peut s'écrire d'une façon unique dans la forme

$$x = a_1 x_1 + \dots + a_r x_r.$$

Cela nous permet de définir en E une notion d'*indice*. Rappelons d'abord la notion d'indice du corps K. En vertu de la valuation non-archimédienne de K l'ensemble des éléments $|a| < 1$ de K et celui des éléments $|a| \leqq 1$ constituent un groupe additif. Les éléments $|a| \leqq 1$ se partagent en un nombre de classes modulo $|a| < 1$. La supposition que K soit localement compact entraîne que ce nombre est fini. En effet, si ce nombre était infini et si $a_1, a_2, \dots$ désignaient les représentants d'une infinité de classes différentes entre eux et différentes de la classe-zéro, on aurait

$$|a_i| = 1,\ |a_i - a_j| = 1 \qquad (i \neq j)$$

en contradiction avec la compacité locale. Soit K_0 ce système fini de classes. K_0 est un corps fini puisque de $|a| = 1$, $|b| = 1$ il résulte $|ab| = 1$, $\left|\frac{a}{b}\right| = 1$.

En partant d'un nombre $C \in N_K$ on trouve de la même façon un système fini de classes dans le groupe $|a| \leqq C$. On trouve le même nombre de classes puisque de l'existence d'un système a_i $(i = 1, \ldots, n)$ avec $|a_i| = 1$, $|a_i - a_j| = 1$ $(i \neq j)$ il résulte celle d'un système $\bar{a}_i$ $(i = 1, \ldots, n)$ avec $|\bar{a}_i| = C$, $|a_i - a_j| = C$ $(i \neq j)$ pour chaque $C \in N_K$ et inversement. Ce nombre est appelé l'*indice du corps K*.

Dans un espace localement compact on trouve par la même voie, grâce aux théorèmes précédents, un nombre fini de classes en partant d'un nombre $C \in N_E$; les ensembles des points $\| x \| \leqq C$ et $\| x \| < C$ constituent des groupes additifs. Cependant la méthode précédente ne s'applique plus pour montrer que ce nombre ne dépend pas du nombre C, puisque si $C' \neq C$, $C' \in N_E$, on n'est pas sûr qu'il existe en K un élément a tel que $|a| = C'/C$. L'indépendance résulte alors du théorème 5, modifié pour le cas $N_E \subset N_K$ suivant la remarque précédente. Ce nombre est appelé *l'indice de l'espace*. L'indice de E vaut au moins celui de K. Je n'ai pas réussi à établir une relation entre l'indice de K, l'indice E et le nombre r du théorème 5.

Mathematics. — *Sur les espaces linéaires normés.* II. By A. F. MONNA.
(Communicated by Prof. W. VAN DER WOUDE.)

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§ 4. Au §§ 2 et 3 on a partout supposé que la valuation de K soit non-triviale. Le théorème suivant cependant reste vrai pour une valuation triviale de K.

Théorème 7. *Soit E un espace linéaire totalement-non-archimédien satisfaisant à*

1. *N_E n'admet que le point 0 comme point d'accumulation;*
2. *E est complet;*
3. *soit $0 \ldots < C_i < C_{i+1} < \ldots$ la suite des nombres appartenant à N_E (N_E est dénombrable en vertu de la supposition 1); supposons alors que le nombre de classes modulo $\| x \| < C_i$ dans le groupe $\| x \| \leqq C_i$ est fini, quelle que soit la valeur de i. Soit $\xi_1^{(i)}, \xi_2^{(i)}, \ldots, \xi_{n_i}^{(i)}$ un système de représentants des classes dans $\| x \| \leqq C_i$, différentes entre eux et de la classe-zéro (le nombre des classes est alors $n_i + 1$). Alors chaque x en E peut être développé d'une façon unique en une série*

$$x = \xi_{j_i}^{(i)} + \varepsilon_{i-1}\, \xi_{j_{i-1}}^{(i-1)} + \ldots + \varepsilon_{i-k}\, \xi_{j_{i-k}}^{(i-k)} + \ldots$$

si $\| x \| = C_i$. Les ε_{i-k} ($k = 1, 2, \ldots$) ne prennent que les valeurs 0 et 1, en fonction de x.

Démonstration. Si $\| x \| = C_i$, les décompositions suivantes sont possibles:

$$x = \xi_1^{(i)} + \eta_1$$
$$x = \xi_2^{(1)} + \eta_2$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x = \xi_{n_i}^{(i)} + \eta_{n_i}.$$

Il en résulte

$$\| x - \eta_k \| = C_i \quad (k = 1, \ldots, n_i)$$
$$\| \eta_k - \eta_l \| = \| \xi_l^{(i)} - \xi_k^{(i)} \| = C_i \quad (k \neq l).$$

On ne peut avoir $\| \eta_k \| = C_i$ pour $k = 1, \ldots, n_i$. En effet, autrement les éléments $\theta, x, \eta_1, \ldots, \eta_{n_i}$ constitueraient un système de $n_i + 2$ représentants de classes différentes, en contradiction avec la supposition. Il existe donc au moins un η_λ tel que $\| \eta_\lambda \| < C_i$ et on a la décomposition

$$x = \xi_{j_i}^{(i)} + \eta_\lambda.$$

Ici j_i et η_λ sont uniquement déterminés. Supposons en effet

$$x = \xi^{(i)}_{j_i} + \eta_\lambda, \quad x = \xi^{(i)}_{j'_i} + \eta_{\lambda'}, \quad \|\eta_\lambda\| < C_i, \|\eta_{\lambda'}\| < C_i.$$

Alors

$$\|\xi^{(i)}_{j_i} - \xi^{(i)}_{j'_i}\| = \|\xi^{(i)}_{j_i} - x + x - \xi^{(i)}_{j'_i}\| \leqq \max (\|\xi^{(i)}_{j_i} - x\|, \|x - \xi^{(i)}_{j'_i}\|) =$$
$$= \max(\|\eta_\lambda\|, \|\eta_{\lambda'}\|) < C_i.$$

Il en résulte, la norme de cette différence étant $= C_i$ si $j_i \neq j'_i$

$$\xi^{(i)}_{j_i} = \xi^{(i)}_{j'_i}, \; j_i = j'_i, \; \eta_\lambda = \eta_{\lambda'}.$$

Continuant ainsi on trouve la décomposition

$$x = \xi^{(i)}_{j_i} + \varepsilon_{i-1}\, \xi^{(i-1)}_{j_{i-1}} + \ldots + \varepsilon_{i-k}\, \xi^{(i-k)}_{j_{i-k}} + \eta$$

avec $\|\eta\| < C_{i-k}$.

Pour les valeurs suffisamment grandes de k on a $\|\eta\| < \delta$ donc

$$\|x - \sum_k \varepsilon_{i-k}\, \xi^{(i-k)}_{j_{i-k}}\| < \delta$$

et par suite

$$x = \sum_{k=0}^{\infty} \varepsilon_{i-k}\, \xi^{(i-k)}_{j_{i-k}}, \; \varepsilon_i = 1.$$

Inversement chaque suite de cette forme représente un point de E. Posons pour cela

$$x_{k+1} = \varepsilon_i\, \xi^{(i)}_{j_i} + \ldots + \varepsilon_{i-k}\, \xi^{(i-k)}_{j_{i-k}}, \; x_{k+1} \in E.$$

Pour les valeurs suffisamment grandes de k et pour $p > 0$ on voit que $\| x_{k+p+1} - x_{k+1} \| < \delta$. La suite $\{ x_k \}$ est donc une suite fondamentale dans E et, E étant supposé complet, elle a une limite x et

$$x = \sum_{k=0}^{\infty} \varepsilon_{i-k}\, \xi^{(i-k)}_{j_{i-k}}.$$

Remarques. 1. On n'a pas supposé que le corps K soit complet.

2. Le théorème 7 reste vrai si la valuation de K est triviale. Dans ce cas il est possible que la suite $\{ C_i \}$ ne contient pas des nombres arbitrairement petits. En ce cas, on voit que chaque point de l'espace peut être représenté par une combinaison linéaire finie des $\xi^{(i)}_j$. Tous les points de l'espace sont alors isolés, de sorte que l'espace est alors localement compact.

3. Il résulte du théorème 2 que, si la valuation de K est non triviale, le théorème 7 s'applique aux espaces localement compacts.

Faisons de nouveau les restrictions que la valuation de K est non triviale et que K est complet et considérons les espaces localement compacts. Les résultats du paragraphe 3 s'appliquent. On peut donner dans ce cas aux séries du théorème 7 une forme plus simple. Remarquons d'abord que dans ce cas le nombre n_i du théorème 7 est une constante, l'indice de l'espace. Il y a deux cas.

a) L'indice de E est égal à celui de K, soit $= n + 1$.

Soit, dans ce cas ξ_i $(-\infty < i < +\infty)$ un point de E avec $\| \xi_i \| = C_i$ et soit $\lambda_0 = 0, \lambda_1, \ldots, \lambda_n$ $(\lambda_i \in K)$ un système S de représentants des classes modulo $|a| < 1$ dans le groupe $|a| \leqq 1$. Alors les éléments

$$\lambda_0 \xi_i, \lambda_1 \xi_i, \ldots, \lambda_n \xi_i$$

de E constituent un système de représentants des classes modulo $\| x \| < C_i$. En appliquant le théorème 7 on trouve:

Chaque point x d'un espace totalement-non-archimédien localement compact (valuation non triviale) dont l'indice $n + 1$ est égal à celui de K, peut être représenté d'une façon unique dans la forme

$$x = \sum_{k=0}^{\infty} \lambda_{i-k} \xi_{i-k}, \quad \|x\| = C_i, \quad \lambda_{i-k} \in S. \quad . \quad . \quad . \quad . \quad (5)$$

b) L'indice de E est plus grand que celui de K. Ce cas se réduit au cas précédent au moyen du théorème 5, qui s'applique aux espaces localement compacts. Si les points x de E sont représentés par

$$x = a_1 x_1 + \ldots + a_r x_r,$$

considérons, à côté de E, les espaces linéaires E_i des points

$$x = a x_i \; (a \in K; \; i = 1, \ldots, r).$$

Les E_i n'ont que le point θ en commun, de sorte que E est la somme des E_i. Les E_i sont totalement-non-archimédiens et localement compacts (puisque K est localement compact); ils sont complets puisque K est complet. De plus, on voit que l'indice de chaque E_i est égal à celui de K, de sorte qu'on peut appliquer le cas précédent. La valuation de K étant discrète, N_K consiste des nombres ϱ^i où $\varrho > 1$ et i entier. Si

$$\|x_l\| = C^{(l)} \, (l = 1, \ldots, r; \, C^{(l)} \in \{C_i\})$$

l'ensemble N_{E_l} consiste donc des nombres $\varrho^t C^{(l)}$ $(-\infty < t < +\infty)$.

Introduisons les points

$$\xi_i^{(l)} \, (-\infty < i < +\infty; \, l = 1, \ldots, r), \, \|\xi_i^{(l)}\| = \varrho^i \, C^{(l)}.$$

On obtient alors l'énoncé général:

Chaque point x d'un espace totalement-non-archimédien localement compact (valution non triviale) peut être représenté d'une facon unique dans la forme

$$x = \sum_{k=0}^{\infty} \lambda_{i_1-k}^{(1)} \xi_{i_1-k}^{(1)} + \sum_{k=0}^{\infty} \lambda_{i_2-k}^{(2)} \xi_{i_2-k}^{(2)} + \ldots + \sum_{k=0}^{\infty} \lambda_{i_r-k}^{(r)} \xi_{i_r-k}^{(r)}. \quad . \quad . \quad (6)$$

$$\lambda_{i_j-k}^{(j)} \in S.$$

Ici r est une constante, dépendant de l'espace. Inversement, chaque pareille combinaison de suites représente un point de l'espace.

On peut simplifier cette représentation encore un peu en introduisant dans E une métrique telle que $N_E \subset N_K$, ce qui est toujours possible, comme nous avons vu, dans les espaces localement compacts. On peut s'arranger que, dans la représentation $x = a_1 x_1 + \ldots + a_r x_r$, les x_j ont tous la même norme. Si $\| x_j \| = \varrho^m$, on introduit alors les points

$$\xi_i \, (-\infty < i < +\infty), \; \| \xi_i \| = \varrho^{i+m}$$

et on obtient la représentation

$$x = \sum_{k=0}^{\infty} \lambda^{(1)}_{i_1-k} \, \xi_{i_1-k} + \sum_{k=0}^{\infty} \lambda^{(2)}_{i_2-k} \, \xi_{i_2-k} + \ldots + \sum_{k=0}^{\infty} \lambda^{(r)}_{i_r-k} \, \xi_{i_r-k}$$

ou, par une addition des termes correspondants, ce qui est permis,

$$x = \mu_i \, \xi_i + \mu_{i-1} \, \xi_{i-1} + \ldots$$

où les coefficients des ξ_{i-k} sont maintenant des sommes d'au plus r éléments de S, uniquement déterminées.

Remarques. 1. Par un raisonnement analogue à celui valable pour les corps métrisés (voir 4)), on voit que pour les espaces dont l'indice est égal à celui de K, l'ensemble des points x de l'espace pour lesquels $\| x \|$ est constant, est homéomorphe à l'ensemble parfait non-dense de CANTOR.

2. Comparer les séries précédentes avec une série qui se présente dans la théorie des anneaux topologiques [7]).

Le théorème suivant est une conséquence immédiate des développements en série précédents, si l'on introduit aux espaces la métrique $\max\limits_{1 \leqq i \leqq r} | a_i |$, définie au fin du § 3.

Théorème 8. *Soient E et F deux espaces totalement-non-archimédiens localement compacts. La valuation des corps K_E et K_F des coefficients est supposée non triviale. Supposons: a) le nombre r du théorème 5 est le même pour E et F; b) les corps finis des classes modulo $|a| < 1$ (voir le fin du § 4), définis en K_E et K_F sont isomorphes. Alors il existe une transformation linéaire, biunivoque et bicontinue de E en F.*

Ce théorème exprime ce que l'on aperçoit immédiatement des séries: ce ne sont qu'un nombre fini d'éléments de K, à valeurs 0 ou 1, qui interviennent effectivement. Ces éléments constituent un corps fini, si on calcule avec eux modulo $|a| < 1$ (p. ex. les nombres 0, 1, ..., $P-1$ dans le corps des nombres P-adiques).

Une autre conséquence est la suivante. Ci-dessus nous avons considéré

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[7]) D. VAN DANTZIG. Zur topologischen Algebra II. Compositio Mathematica 2, 201—223 (1935) en particulier p. 220.

les espaces de la forme $x = a x_i$ $(a \in K)$; c'étaient des espaces dont l'indice est égal à celui de K. Or, c'est la forme générale de ces espaces. Soit donc E un espace dont l'indice est égal à celui de K. Introduisons en E une métrique telle que $N_E = N_K$ (voir § 3, p. 1054). Soit alors ξ un point de E tel que $\| \xi \| = 1$ et soient a_i $(-\infty < i < +\infty)$ des éléments de K tels que $|a_i| = \varrho^i$ $(-\infty < i < +\infty)$. Pour les points ξ_i de la série (5) on peut alors prendre les points $\xi_i = a_i \xi$. La série (5), qui est la forme générale de E, prend alors la forme

$$\sum_k \lambda_{i-k} \, a_{i-k} \, \xi = \xi \sum_k \lambda_{i-k} \, a_{i-k}$$

qui se réduit, à $a\,\xi$, puisque K est supposé complet.

§ 5. Dans la seconde partie, sauf le théorème 7, on a partout supposé que le corps K soit complet, bien que ceci n'est pas une condition nécessaire pour les espaces localement compacts (voir le théorème 2). De plus l'espace était partout complet. Remarquons que c'est bien possible que E soit complet sans que K a cette propriété.

Exemple: E est le corps des nombres P-adiques et K le corps des nombres rationnels munit d'une valuation P-adique.

Cependant, E étant complet, et la multiplication par les éléments de K étant définie, il est toujours possible de définir une multiplication par les éléments de $\overline{K}$ (le plus petit corps complet contenant K), sans qu'il soit nécessaire d'adjoindre de nouveaux points à E. Soit $a \in \overline{K}$ et $a_n \to a$, $a_n \in K$. Pour x fixé en E, la suite $\{ a_n x \}$ est une suite fondamentale de E, et, E étant complet, elle converge. La limite $\lim_{n \to \infty} a_n x$ existe donc; nous définissons

$$a\, x = \lim_{n \to \infty} a_n \, x.$$

L'unicité de cette définition est immédiate puisque si $b_n \to a$, la suite $\{ (a_n - b_n) x \}$ converge vers θ. On voit que les conditions de linéarité sont vérifiées. La condition que K est complet ne signifie donc pas, dans tous ce qui précède, une restriction de la généralité.

§ 6. Faisons quelques remarques sur le cas de la valuation triviale de K, que nous avons partout exclue, sauf au théorème 7. Ce ne sont que des remarques, car je n'ai pas réussi à élucider complètement ce cas.

1. Il ne résulte pas de la compacité locale que E et K possèdent un indice fini. *Exemple:* E soit un corps infini à valuation triviale et K un sous-corps infini (éventuellement $K \equiv E$).

2. Ils existent des espaces localement compacts qu'on ne peut pas représenter dans la forme $x = a_1 x_1 + \ldots + a_r x_r$, ni dans la forme des séries (5) ou (6). *Exemple:* K soit un corps fini, sur lequel on ne peut donc définir qu'une valuation triviale. E soit un sur-corps infini à valuation triviale.

3. Les espaces des deux exemples précédents ne contiennent que des points isolés et par là ils sont localement compacts. Ils ne sont donc pas très intéressants.

Faisons donc la restriction que N_E contiendra des nombres arbitrairement petits. Soit E localement compact. Alors K (à valuation triviale) est un corps fini. Soit, pour cela, $\varepsilon > 0$ donné et soit x_0 un point de E tel que $\| x_0 \| \leqq \varepsilon$. Si maintenant $\lambda_1, \lambda_2, \ldots$ étaient des éléments de K en nombre infini, différents l'un de l'autre, il suivrait $\| \lambda_i x_0 \| = \| x_0 \| \leqq \varepsilon$ et $\| \lambda_i x_0 - \lambda_j x_0 \| = | \lambda_i - \lambda_j | \cdot \| x_0 \| = \| x_0 \|$, de sorte que le voisinage $\| x \| \leqq \varepsilon$ de θ contiendrait la suite non-compacte $\{ \lambda_i x_0 \}$ en contradiction avec la compacité locale de E.

C'est une conséquence immédiate que l'espace, qui n'est pas fini puisque N_E n'est pas fini, ne peut pas s'écrire comme somme d'un nombre fini d'espaces, comme c'est le cas si la valuation de K est non triviale (théorème 5). Un pareil espace contient pour chaque x_0 une infinité de points x pour lesquels on a $\| x \| = \| x_0 \|$. En effet, pour tous les points du voisinage $\| x - x_0 \| < \varepsilon < \| x_0 \|$ on a $\| x \| = \| x_0 \|$; si ce nombre de points x était fini, cet ε-voisinage était fini et chaque point de E était isolé, en contradiction avec la supposition que N_E contient des nombres arbitrairement petits.

Comme exemple d'un pareil espace signalons le suivant. K soit un corps fini et $\overline{K}(x)$ le plus petit corps complet contenant une extension simplement transcendante de K. $\overline{K}(x)$ consiste des séries de puissances d'un indéfini x:

$$a = \lambda_s x^s + \lambda_{s+1} x^{s+1} + \ldots$$

où $\lambda_i \in K$; $| \lambda_i | = 1$ si $\lambda_i \neq 0$; $| a | = | x |^s$; $| x | < 1$. $\overline{K}(x)$ est localement compact.

L'analogie de cette série à la série (5) fait présumer que la série (6) donne la forme générale des espaces totalement-non-archimédiens localement compacts, même si la valuation de K est triviale, dans ce dernier cas sous l'hypothèse que N_E contient des nombres arbitrairement petits. Remarquons que dans (6) n'interviennent qu'un nombre fini d'éléments de K. Cette présomption est soutenue par le théorème 8. Je n'ai pas réussi à la démontrer.

En remplaçant la condition de la compacité locale par celle de la compacité, on montre, si la valuation de K est triviale, la nécessité des conditions suivantes:

1. N_E n'a que 0 comme point d'accumulation;
2. K est fini;
3. E est complet;
4. pour chaque $C \in N_E$, E ne contient qu'un nombre fini de points e_i tels que

$$\| e_i \| = C, \quad \| e_i - e_j \| = C \quad (i \neq j).$$

Une inversion ne m'est pas connue.

Signalons enfin le problème suivant. Dans la théorie des corps métriques (normés) on sait: si l'ensemble des normes N admet un point d'accumulation $\neq 0$, alors chaque point (nombre réel) est point d'accumulation de N. Vue l'analogie entre la théorie des corps métriques et la théorie des espaces linéaires totalement-non-archimédiens, a-t-on la même propriété pour l'ensemble N_E?

Mathematics. — *On the G-function.* VII. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of October 26, 1946.)

Theorem 17. Assumptions: *m, p and q are integers with*

$$0 \leqq p \leqq q-2 \text{ and}^{57)}\ p+1 \leqq m \leqq q-1;$$

the number z satisfies the inequality

$$-(m-p+1)\pi < \arg z < (m-p+1)\pi \quad . \quad . \quad . \quad . \quad (181)$$

Assertions: 1. *The function $G^{m,0}_{p,q}(z)$ possesses for large values of $|z|$ with $0 < \arg z < (m-p+1)\pi$ the asymptotic expansion*

$$G^{m,0}_{p,q}(z) \backsim A^{m,0}_{\ \ q} H_{p,q}(z\, e^{(q-m)\pi i}).$$

2. *The function $G^{m,0}_{p,q}(z)$ possesses for large values of $|z|$ with $-(m-p+1)\pi < \arg z < 0$ the asymptotic expansion*

$$G^{m,0}_{p,q}(z) \backsim \overline{A}^{m,0}_{\ \ q} H_{p,q}(z\, e^{(m-q)\pi i}).$$

3. *The function $G^{m,0}_{p,q}(z)$ possesses for large values of z with $\arg z = 0$ the asymptotic expansion*

$$G^{m,0}_{p,q}(z) \backsim A^{m,0}_{\ \ q} H_{p,q}(z\, e^{(q-m)\pi i}) + \overline{A}^{m,0}_{\ \ q} H_{p,q}(z\, e^{(m-q)\pi i}).$$

Proof. From definition 7 it follows that $\Omega^{m,0}_{\ \ q}(s)$ vanishes if $s > q-m$. Formula (148) reduces therefore for $n=0$ to

$$G^{m,0}_{p,q}(z) = A^{m,0}_{\ \ q} \sum_{s=0}^{q-m} \Omega^{m,0}_{\ \ q}(s)\, G^{q,0}_{p,q}(z\, e^{(q-m-2s)\pi i}). \quad . \quad . \quad (182)$$

On the right-hand side of this relation there are $q-m+1$ functions of the type $G^{q,0}_{p,q}(\zeta)$, the values of $\arg \zeta$ being

$$\arg z + (m-q)\pi,\ \arg z + (m-q+2)\pi, \ldots, \arg z + (q-m)\pi. \quad (183)$$

Because of $m \geqq p+1$ we have $q-m+1 \leqq q-p$. So the number of the functions $G^{q,0}_{p,q}$ on the right of (182) is at most equal to $q-p$. Now it follows from (181) that the values (183) all lie between $-(q-p+1)\pi$ and $(q-p+1)\pi$. The functions $G^{q,0}_{p,q}$ in (182) satisfy therefore the condition [58] (37) with $n=0$ and so they are fundamental

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[57]) The case with $m=q$ needs no consideration, since the asymptotic expansion $G^{q,0}_{p,q}(z) \backsim H_{p,q}(z)$ has already been given in theorem C.

[58]) The number ε occurring in (37) is 1, since $q \geqq p+2$.

solutions of the differential equation satisfied by $G^{m,0}_{p,q}(z)$. Considering the values (183) we deduce from (26) and (25):

If $\arg z = 0$ and $z \to \infty$, there are on the right of (182) two dominant functions $G^{q,0}_{p,q}$, viz. the function $G^{q,0}_{p,q}(z e^{(m-q)\pi i})$ and the function $G^{q,0}_{p,q}(z e^{(q-m)\pi i})$.

If $0 < \arg z < (m-p+1)\pi$, there is on the right of (182) for large values of $|z|$ only one dominant function, viz. the function $G^{q,0}_{p,q}(z e^{(q-m)\pi i})$.

If $-(m-p+1)\pi < \arg z < 0$, there is on the right of (182) for large values of $|z|$ only one dominant function, viz. the function $G^{q,0}_{p,q}(z e^{(m-q)\pi i})$.

It is further obvious:

If $\arg z = 0$, the terms with $s=0$ and $s=q-m$ of the sum $\sum\limits_{s=0}^{q-m}$ supply the dominant functions. If $0 < \arg z < (m-p+1)\pi$, the term with $s=0$ is the dominant term. The term with $s=q-m$ is the dominant term if $-(m-p+1)\pi < \arg z < 0$.

Now the coefficients of the dominant functions are $A^{m,0}_{\ \ q}\, \Omega^{m,0}_{\ \ q}(0)$ and $A^{m,0}_{\ \ q}\, \Omega^{m,0}_{\ \ q}(q-m)$. We have however by definition 7 $\Omega^{m,0}_{\ \ q}(0) = 1$ and $\Omega^{m,0}_{\ \ q}(q-m) = (-1)^{q-m}\, e^{2(b_{m+1}+\dots+b_q)\pi i}$. From the second of these relations and (45) and (46) it appears $A^{m,0}_{\ \ q}\, \Omega^{m,0}_{\ \ q}(q-m) = \overline{A}^{m,0}_{\ \ q}$. Hence the coefficients are $A^{m,0}_{\ \ q}$ and $\overline{A}^{m,0}_{\ \ q}$. The two dominant terms on the right of (182) possess therefore in virtue of (26) the asymptotic expansions

$$A^{m,0}_{\ \ q}\, H_{p,q}(z e^{(q-m)\pi i}), \text{ respect. } \overline{A}^{m,0}_{\ \ q}\, H_{p,q}(z e^{(m-q)\pi i}).$$

The assertions of theorem 17 may now at once be verified.

Theorem 18. Assumptions: *m, n, p and q are integers with*

$$0 \leqq n \leqq p \leqq q-2 \text{ and } p+1 \leqq m+n \leqq \tfrac{1}{2}p + \tfrac{1}{2}q \quad [59];$$

the number z satisfies the inequality

$$-(m+n-p+1)\pi < \arg z < (m+n-p+1)\pi; \quad . \quad . \quad (184)$$

the numbers $a_1, \dots, a_n$ *and* $b_1, \dots, b_m$ *fulfil the condition* (1) *in the formulae* (185), (186) *and* (187); *in formula* (188) *I assume that they satisfy the conditions* (1) *and* (20).

Assertions: 1. *The function* $G^{m,n}_{p,q}(z)$ *possesses for large values of* $|z|$ *with* $0 < \arg z < (m+n-p+1)\pi$ *the asymptotic expansion*

$$G^{m,n}_{p,q}(z) \backsim A^{m,n}_{\ \ q}\, H_{p,q}(z e^{(q-m-n)\pi i}). \quad . \quad . \quad . \quad . \quad (185)$$

2. *The function* $G^{m,n}_{p,q}(z)$ *possesses for large values of* $|z|$ *with* $-(m+n-p+1)\pi < \arg z < 0$ *the asymptotic expansion*

$$G^{m,n}_{p,q}(z) \backsim \overline{A}^{m,n}_{\ \ q}\, H_{p,q}(z e^{(m+n-q)\pi i}). \quad . \quad . \quad . \quad . \quad (186)$$

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[59]) From $0 \leq n \leq p \leq q-2$ and $p+1 \leq m+n \leq \frac{1}{2}p + \frac{1}{2}q$ it follows $1 \leq m \leq \frac{1}{2}p + \frac{1}{2}q \leq q-1$.

3. *The function* $G_{p,q}^{m,n}(z)$ *with* $p+1 \leqq m+n < \frac{1}{2}p+\frac{1}{2}q$ *possesses for large values of* z *with* $\arg z = 0$ *the asymptotic expansion*

$$G_{p,q}^{m,n}(z) \backsim A^{m,n}_{\ \ q} H_{p,q}(z e^{(q-m-n)\pi i}) + \overline{A}^{m,n}_{\ \ q} H_{p,q}(z e^{(m+n-q)\pi i}). \quad (187)$$

4. *The function* $G_{p,q}^{m,n}(z)$ *with* $m+1 = \frac{1}{2}p+\frac{1}{2}q$ *possesses for large values of* z *with* $\arg z = 0$ *the asymptotic expansion*

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) \backsim\, & A^{m,n}_{\ \ q} H_{p,q}(z e^{(q-m-n)\pi i}) + \overline{A}^{m,n}_{\ \ q} H_{p,q}(z e^{(m+n-q)\pi i}) \\ & + \sum_{t=1}^{n} e^{(m+n-q-1)\pi i a_t} \triangle^{m,n}_{\ \ q}(t)\, E_{p,q}(z e^{(q-m-n+1)\pi i} \,\|\, a_t). \end{aligned}\right\} . \quad (188)$$

Proof. The proof resembles that of theorem 17. We apply theorem 14, i.e. we use formula (150). The number ε occurring in formula (166) of theorem 14 is 1 because of $q \geqq p+2$.

Since $\frac{1}{2}p+\frac{1}{2}q-m-n \geqq 0$, it follows from (168)

$$\arg z < 2r\pi < (2q-2m-2n+2)\pi + \arg z.$$

From this inequality it appears:

If $\arg z = 0$, then $r > 0$ and $q-m-n-r+1 > 0$.

If $\arg z > 0$, then $r > 0$.

If $\arg z < 0$, then $q-m-n-r+1 > 0$.

Consequently:

If $\arg z = 0$, then both of the sums $\sum_{s=0}^{r-1}$ and $\sum_{\tau=0}^{q-m-n-r}$ in (150) contain at least one term, i.e. the terms with $s=0$ and $\tau=0$ occur certainly.

If $\arg z > 0$, then the sum $\sum_{s=0}^{r-1}$ contains at least one term, i.e. the term with $s=0$ occurs certainly.

If $\arg z < 0$, then the sum $\sum_{\tau=0}^{q-m-n-r}$ contains at least one term, i.e. the term with $\tau=0$ occurs certainly.

On the right-hand side of (150) there occur $q-m-n+1$ functions of the type $G_{p,q}^{q,0}(\zeta)$, the values of $\arg \zeta$ being

$$\arg z + (m+n-q)\pi,\ \arg z + (m+n-q+2)\pi, \ldots,\ \arg z + (q-m-n)\pi;$$

according to (184) these values all lie between $-(q-p+1)\pi$ and $(q-p+1)\pi$.

It is now easily seen, on account of (26) and (25):

If $\arg z = 0$ and $z \to \infty$, there are among the functions $G_{p,q}^{q,0}$ on the right of (150) two dominant functions, viz. the function $G_{p,q}^{q,0}(z e^{(m+n-q)\pi i})$ and the function $G_{p,q}^{q,0}(z e^{(q-m-n)\pi i})$; hence the terms with $\tau=0$, respect. $s=0$ of the sums $\sum_{\tau=0}^{q-m-n-r}$ and $\sum_{s=0}^{r-1}$ supply in this case the leading functions $G_{p,q}^{q,0}$.

If $0 < \arg z < (m+n-p+1)\pi$ and $|z| \to \infty$, there is one dominant function $G_{p,q}^{q,0}$, viz. the function $G_{p,q}^{q,0}(z e^{(q-m-n)\pi i})$; i.e. the term with $s=0$ of the sum $\sum_{s=0}^{r-1}$ supplies in this case the leading function $G_{p,q}^{q,0}$.

If $-(m+n-p+1)\pi < \arg z < 0$ and $|z| \to \infty$, there is one dominant function $G_{p,q}^{q,0}$, viz. the function $G_{p,q}^{q,0}(z\, e^{(m+n-q)\pi i})$; i.e. the term with $\tau = 0$ of the sum $\sum_{\tau=0}^{q-m-n-r}$ supplies in this case the leading function $G_{p,q}^{q,0}$.

Since $m+n \leqq \frac{1}{2}p + \frac{1}{2}q$, it follows besides from (26) and (25) that these leading functions $G_{p,q}^{q,0}$ are always exponentially infinite, except when $m+n = \frac{1}{2}p + \frac{1}{2}q$ and at the same time $\arg z = 0$; in the last case the two leading functions behave like $\exp\,(\pm\, i\,(p-q)\, z^{\frac{1}{q-p}})\, z^{\vartheta}$.

Starting from (150) we can now in the various cases by means of (26), (50), (18) and (15) write down the asymptotic expansions of the function $G_{p,q}^{m,n}(z)$. The expansions of algebraic order, which are caused by the functions $G_{p,q}^{q,1}(z\, e^{(q-m-n-2r+1)\pi i} \,\|\, a_t)$ on the right of (150), have the same form as in theorem 16; they may always be neglected in comparison with the exponential expansions, save in the above mentioned exceptional case.

With this the formulae (185), (186), (187) and (188) have been proved.

Remark. In (150) we have supposed that no two of the numbers $a_1, \ldots, a_n$ are equal or differ by an integer (condition (20)). If one or more of the differences $a_j - a_t$ ($j = 1, \ldots, n; t = 1, \ldots, n; j \neq t$) is equal to $0, \pm 1, \pm 2, \ldots$, the sum $\sum_{t=1}^{n}$ on the right-hand side of (150) assumes an undetermined form and must be replaced by its limit. Consequently we have supposed in (188) that the numbers $a_1, \ldots, a_n$ satisfy not only (1) but also (20). If one or more of the differences $a_j - a_t$ is equal to $0, \pm 1, \pm 2, \ldots$, the sum $\sum_{t=1}^{n}$ on the right of (188), which is the asymptotic expansion of the sum $\sum_{t=1}^{n}$ in (150), must be replaced by an expression involving logarithmic terms.

A similar remark holds with regard to the formulae (191), (192), (197), (203), (207) and (208).

Theorem 19. Assumptions: *m, n, p and q are integers with*

$$0 \leqq n \leqq p < q, \quad 1 \leqq m \leqq q \quad and \quad m+n > \tfrac{1}{2}p + \tfrac{1}{2}q;$$

the number z satisfies the inequality

$$(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\,\pi \leqq |\arg z| < (m+n-p+\varepsilon)\,\pi;$$

the numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ fulfil the conditions (1) *in the assertions* 1 *and* 2 (*formulae* (185) *and* (186)); *in the assertions* 3 *and* 4 (*formulae* (191) *and* (192)) *I assume that they satisfy the conditions* (1) *and* (20).

Assertions: *The function $G_{p,q}^{m,n}(z)$ possesses for large values of $|z|$ with*

$$(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\,\pi < \arg z < (m+n-p+\varepsilon)\,\pi \quad . \quad . \quad (189)$$

the asymptotic expansion (185).

2. *The function* $G_{p,q}^{m,n}(z)$ *possesses for large values of* $|z|$ *with*

$$-(m+n-p+\varepsilon)\pi < \arg z < -(m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi$$

the asymptotic expansion (186).

3. *The function* $G_{p,q}^{m,n}(z)$ *possesses for large values of* $|z|$ *with*

$$\arg z = (m+n-\tfrac{1}{2}p-\tfrac{1}{2}q)\pi. \quad . \quad . \quad . \quad . \quad . \quad (190)$$

the asymptotic expansion

$$G_{p,q}^{m,n}(z) \backsim A^{m,n}_{q}\, H_{p,q}(z\, e^{(q-m-n)\pi i}) + \sum_{t=1}^{n} e^{(m+n-q-1)\pi i a_t} \triangle^{m,n}_{q}(t)\, E_{p,q}(z\, e^{(q-m-n+1)\pi i} \| a_t). \quad (191)$$

4. *The function* $G_{p,q}^{m,n}(z)$ *possesses for large values of* $|z|$ *with* $\arg z = -(m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$ *the asymptotic expansion*

$$G_{p,q}^{m,n}(z) \backsim \overline{A}^{m,n}_{q}\, H_{p,q}(z\, e^{(m+n-q)\pi i}) + \sum_{t=1}^{n} e^{(q-m-n+1)\pi i a_t} \triangle^{m,n}_{q}(t)\, E_{p,q}(z\, e^{(m+n-q-1)\pi i} \| a_t). \quad (192)$$

Proof. Proof of the assertions 1 and 3. We may distinguish two cases:

First case: $m+n \geqq q+1$. We apply theorem 12A, i.e. we use formula (148). The sum $\sum_{t=1}^{n}$ on the right of (148) furnishes the same asymptotic expansion of algebraic order as in theorem 16; the sum $\sum_{t=1}^{n}$ in (191) appears therefore in this manner. We will yet investigate the sum $\sum_{s=0}^{r-1}$ in (148). Now it follows from (189), respect. (190)

$$(\tfrac{1}{2}p+\tfrac{1}{2}q-m-n)\pi + \arg z \geqq 0.$$

The integer r which satisfies the inequalities (158) and (159) is therefore >0. Hence the sum $\sum_{s=0}^{r-1}$ on the right-hand side of (148) contains at least one term, i.e. the term with $s=0$ occurs certainly.

Now we have if $\arg z$ satisfies (190)

$$\arg(z\, e^{(q-m-n)\pi i}) = (\tfrac{1}{2}q-\tfrac{1}{2}p)\pi,$$

and if $\arg z$ satisfies (189)

$$(\tfrac{1}{2}q-\tfrac{1}{2}p)\pi < \arg(z\, e^{(q-m-n)\pi i}) < (q-p+\varepsilon)\pi.$$

Besides it follows from (159)

$$-(\tfrac{1}{2}q-\tfrac{1}{2}p)\pi < \arg(z\, e^{(q-m-n-2r+2\,\pi i)}).$$

On comparing these relations with (26) and (25) we see that for $|z| \rightarrow \infty$ the term with $s=0$ is the dominant term of the sum $\sum_{s=0}^{r-1}$ on he right of (148). It appears at the same time that this dominant term is exponentially infinite if $\arg z$ satisfies (189); the dominant term behaves like $\exp(i(p-q)|z|^{\frac{1}{q-p}})|z|^{\vartheta}$ if $\arg z = (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$.

Taking account of (50) and (26) we can now immediately write down the following asymptotic expansion of the sum $A^{m,n}_{q} \sum_{s=0}^{r-1}$ in (148)

$$A^{m,n}_{q} \sum_{s=0}^{r-1} \Omega^{m,n}_{q}(s)\, G^{q,0}_{p,q}(z e^{(q-m-n-2s)\pi i}) \backsim A^{m,n}_{q} H_{p,q}(z e^{(q-m-n)\pi i}). \quad (193)$$

If $\arg z = (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$, we get the asymptotic expansion of $G^{m,n}_{p,q}(z)$ by adding the expansions of algebraic order to the expansion on the right of (193); we then find (191).

If $\arg z$ satisfies (189), the expansions of algebraic order are negligible compared with the exponential expansion (193); hence we then find (185).

Second case: $\frac{1}{2}p+\frac{1}{2}q < m+n \leqq q+1$. The proof is similar to that of the first case. We need only apply theorem 13 A instead of theorem 12 A.

Proof of the assertions 2 and 4. These assertions may be proved by means of the theorems 12 B and 13 B.

Theorem 20. Assumptions: *m, n, p and q are integers with*

$$0 \leqq n \leqq p \leqq q-2 \text{ and}^{60)} \quad 1 \leqq m \leqq q;$$

if $m+n \leqq p+1$, *then* λ *is an arbitrary integer; if* $m+n \geqq p+2$, *then* λ *is either an arbitrary integer* $\geqq 0$ *or an arbitrary integer* $\leqq p-m-n$; *the number z satisfies the inequality*

$$(m+n-p+2\lambda-1)\pi \leqq \arg z < (m+n-p+2\lambda+1)\pi;$$

the numbers $a_1, \ldots, a_p$ *and* $b_1, \ldots, b_m$ *fulfil the conditions* (1) *and* (38) *in formula* (197); *in the formulae* (195) *and* (196) *I assume that they satisfy the condition* (1) [61]).

Assertions: 1. *The function* $G^{m,n}_{p,q}(z)$ *possesses for large values of* $|z|$ *with*

$$(m+n-p+2\lambda-1)\pi < \arg z < (m+n-p+2\lambda+1)\pi \quad (194)$$

the asymptotic expansion

$$G^{m,n}_{p,q}(z) \backsim D^{m,n}_{p,q}(\lambda)\, H_{p,q}(z e^{(q-m-n-2\lambda)\pi i}). \quad (195)$$

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[60]) The case with $m=0$ needs no consideration, since $G^{0,n}_{p,q}(z) = 0$.

[61]) If one or more of the differences $a_j - a_t$ $(j=1,\ldots,p;\ t=1,\ldots,p;\ j \neq t)$ is equal to $0, \pm 1, \pm 2, \ldots$, the sum $\sum_{\tau=1}^{p}$ on the right-hand side of (197) must be replaced by an expression involving powers of $\log z$ (comp. the Remark at the end of the proof of theorem 18).

2. *The function* $G_{p,q}^{m,n}(z)$ *with* $q > p+2$ *possesses for large values of* $|z|$ *with* $\arg z = (m+n-p+2\lambda-1)\pi$ *the asymptotic expansion*

$$\left.\begin{aligned} G_{p,q}^{m,n}(z) &\backsim D_{p,q}^{m,n}(\lambda)\, H_{p,q}(z\, e^{(q-m-n-2\lambda)\pi i}) \\ &+ D_{p,q}^{m,n}(\lambda-1)\, H_{p,q}(z\, e^{(q-m-n-2\lambda+2)\pi i}). \end{aligned}\right\} \quad . \quad . \quad . \quad (196)$$

3. *The function* $G_{p,p+2}^{m,n}(z)$ *possesses for large values of* $|z|$ *with* $\arg z = (m+n-p+2\lambda-1)\pi$ *the asymptotic expansion*

$$\left.\begin{aligned} G_{p,p+2}^{m,n}(z) &\backsim D_{p,p+2}^{m,n}(\lambda)\, H_{p,p+2}(z\, e^{(p-m-n-2\lambda+2)\pi i}) \\ &+ D_{p,p+2}^{m,n}(\lambda-1)\, H_{p,p+2}(z\, e^{(p-m-n-2\lambda+4)\pi i}) \\ + \sum_{\sigma=1}^{p} e^{-2(\lambda+1)\pi i a_\sigma}\, & T_{p,p+2}^{m,n}(\sigma;\lambda)\, E_{p,p+2}(z\, e^{(p-m-n+3)\pi i} \,\|\, a_\sigma). \end{aligned}\right\} \quad . \quad (197)$$

Remark 1. If we make λ run through the sequence of the allowed integers [62], we find by means of (195), (196) and (197) asymptotic expansions of $G_{p,q}^{m,n}(z)$ with $q \geqq p+2$ in the three following cases:

$m+n \leqq p+1$, all values of $\arg z$,

$m+n \geqq p+2$, $\arg z \geqq (m+n-p-1)\pi$,

$m+n \geqq p+2$, $\arg z < -(m+n-p-1)\pi$.

Theorem 20 does not give informations concerning the asymptotic behaviour of $G_{p,q}^{m,n}(z)$ with $q \geqq p+2$ if

$m+n \geqq p+2$ and $-(m+n-p-1)\pi \leqq \arg z < (m+n-p-1)\pi$.

This case may be treated by means of the simple theorems 16—19.

Remark 2. From the definitions 12, 5, 6 and 8 it appears

If $m+n \geqq p+1$, then $D_{p,q}^{m,n}(0) = A^{m,n}_{q}$.

If $m+n \geqq p+1$, then

$$D_{p,q}^{m,n}(p-m-n) = (-1)^{p-q+1} \exp\left\{2\pi i \left(\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right)\right\} \overline{A}^{m,n}_{q}.$$

If $m+n > p+1$, then $D_{p,q}^{m,n}(-1) = 0$.

If $m+n = p+1$, then

$$D_{p,q}^{m,n}(-1) = (-1)^{p-q+1} \exp\left\{2\pi i \left(\sum_{h=1}^{p} a_h - \sum_{h=1}^{q} b_h\right)\right\} \overline{A}^{m,n}_{q}.$$

It further follows from (86), (87) and (54):

If $m+n \geqq p+1$ and $1 \leqq \sigma \leqq n$, then $T_{p,q}^{m,n}(\sigma;0) = e^{(m+n-p-1)\pi i a_\sigma} \triangle^{m,n}_{q}(\sigma)$.

62) See the second assumption of theorem 20.

If $m+n \geqq p+1$ and $n+1 \leqq \sigma \leqq p$, then $T_{p,q}^{m,n}(\sigma;0)=0$.

Besides we have by (27)

$$H_{p,q}(z e^{(q+m+n-2p)\pi i}) = (-1)^{p-q+1} \exp\left\{2\pi i\left(\sum_{h=1}^{q} b_h - \sum_{h=1}^{p} a_h\right)\right\} H_{p,q}(z e^{(m+n-q)\pi i}).$$

It is now easily seen:

If $m+n \geqq p+1$ and $\lambda=0$, then (195) reduces to (185).

If $m+n \geqq p+1$ and $\lambda=p-m-n$, then (195) reduces to (186).

If $m+n > p+1$ and $\lambda=0$, then (196) reduces to (185).

If $m+n=p+1$ and $\lambda=0$, then (196) reduces to (187) with $m+n=p+1$.

If $m+n > p+1$ and $\lambda=0$, then (197) reduces to (191) with $q=p+2$.

If $m+n=p+1$ and $\lambda=0$, then (197) reduces to (188) with $q=p+2$.

Proof of theorem 20. The formulae (195), (196) and (197) may be established by means of the theorems 15 and 10. Since $q \geqq p+2$ the number ε occurring in condition (169) of theorem 15 is equal to 1. From (170) and $\frac{1}{2}q-\frac{1}{2}p-1 \geqq 0$ it follows

$$(m+n-p+2\lambda-1)\pi - \arg z < 2\mu\pi < (m+n-p+2\lambda-1)\pi + (2q-2p)\pi - \arg z.$$

From this inequality it appears:

If $\arg z > (m+n-p+2\lambda-1)\pi$, then $q-p-\mu > 0$.

If $\arg z = (m+n-p+2\lambda-1)\pi$, then $\mu > 0$ and $q-p-\mu > 0$.

Thus we find:

If $\arg z > (m+n-p+2\lambda-1)\pi$, then the sum $\sum_{h=1}^{q-p-\mu}$ in (152) contains at least one term, i.e. the term with $h=1$ occurs certainly.

If $\arg z = (m+n-p+2\lambda-1)\pi$, then both of the sums $\sum_{h=1}^{q-p-\mu}$ and $\sum_{\varkappa=1}^{\mu}$ in (152) contain at least one term, i.e. the terms with $h=1$ and $\varkappa=1$ occur certainly.

We will now determine the dominant term of the sums $\sum_{h=1}^{q-p-\mu}$ and $\sum_{\varkappa=1}^{\mu}$ in (152). In the first place we suppose that $\arg z$ satisfies (194); then we have

$$(q-p-2h+1)\pi < \arg(z e^{(q-m-n-2h-2\lambda+2)\pi i}) < (q-p-2h+3)\pi \quad (198)$$

and

$$(p-q+2\varkappa-1)\pi < \arg(z e^{(2p-q-m-n+2\varkappa-2\lambda)\pi i}) < (p-q+2\varkappa+1)\pi.$$

From these inequalities it appears in view of (26) and (25) that for large values of $|z|$ the function $G_{p,q}^{q,0}(z e^{(q-m-n-2\lambda)\pi i})$, i.e. the term with

$h=1$ of the sum $\sum\limits_{h=1}^{q-p-\mu}$. is the dominant function; it follows further from (198) with $h=1$, (26) and (25) that this dominant function is exponentially infinite.

In the second place we suppose $\arg z = (m+n-p+2\lambda-1)\pi$; then we have

$$\arg\ (z\, e^{(q-m-n-2h-2\lambda+2)\pi i}) = (q-p-2h+1)\,\pi\ . \quad . \quad . \quad (199)$$

and

$$\arg\ (z\, e^{(2p-q-m-n+2\varkappa-2\lambda)\pi i}) = (p-q+2\varkappa-1)\,\pi\ . \quad . \quad . \quad (200)$$

We now deduce from (26) and (25) that there are on the right of (152) for large values of $|z|$ two dominant functions $G^{q,0}_{p,q}$, viz. the function $G^{q,0}_{p,q}(z\, e^{(q-m-n-2\lambda)\pi i})$ and the function $G^{q,0}_{p,q}(z\, e^{(2p-q-m-n-2\lambda+2)\pi i})$; these functions occur in the terms with $h=1$, respect. $\varkappa=1$ of the sums $\sum\limits_{h=1}^{q-p-\mu}$ and $\sum\limits_{\varkappa=1}^{\mu}$.

Besides it appears from (199) with $h=1$, (200) with $\varkappa=1$, (26) and (25) that the two dominant functions $G^{q,0}_{p,q}$ are exponentially infinite if $q > p+2$; the two dominant functions behave like $\exp(\mp 2i\,|z|^{\frac{1}{2}})\,|z|^{\vartheta}$ if $q=p+2$.

We can now easily write down the asymptotic expansion for the right-hand side of (152). The expansions of algebraic order, which are produced by the sum $\sum\limits_{\sigma=1}^{p}$, may always be neglected in comparison with the exponential expansions, except when $q=p+2$ and at the same time $\arg z = (m+n-p+2\lambda-1)\pi$.

If $\arg z$ satisfies (194), we need only take account of the term with $h=1$ of the sum $\sum\limits_{h=1}^{q-p-\mu}$. The asymptotic expansion of this term has according to (91) and (26) the form

$$D^{m,n}_{p,q}(\lambda)\, H_{p,q}(z\, e^{(q-m-n-2\lambda)\pi i});$$

so we find formula (195).

Now the coefficient $D^{m,n}_{p,q}(\lambda)$ vanishes identically if [63])

$$p-m-n < \lambda < 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (201)$$

In this case formula (195) does not hold [64]) and for this reason we have supposed: if $m+n \geqq p+2$, then either $\lambda \geqq 0$ or $\lambda \leqq p-m-n$.

[63]) See the second Remark in § 8.

[64]) If we wish to determine by means of theorem 15 the asymptotic behaviour of the function $G^{m,n}_{p,q}(z)$, when the relations (201) and (194) are satisfied, a further investigation of the various terms on the right-hand side of (152) is necessary. This investigation is rather complicated, since we must distinguish several cases. We have therefore proved the theorems 16, 17, 18 and 19 by means of the simple theorems 11, 12, 13 and 14 instead of the difficult theorem 15.

If $\arg z = (m+n-p+2\lambda-1)\pi$ and $q > p+2$, we must not only consider the term with $h=1$ of the sum $\sum_{h=1}^{q-p-\mu}$, but also the term with $\varkappa=1$ of the sum $\sum_{\varkappa=1}^{\mu}$. The asymptotic expansion of the last term is because of (92) and (26)

$$(-1)^{p-q+1} \exp \left\{ 2\pi i \left(\sum_{h=1}^{q} b_h - \sum_{h=1}^{p} a_h \right) \right\} D_{p,q}^{m,n}(\lambda-1)\, H_{p,q}(z\, e^{(2p-q-m-n-2\lambda+2)\pi i});$$

in view of (27) this expansion may also be written in the form

$$D_{p,q}^{m,n}(\lambda-1)\, H_{p,q}(z\, e^{(q-m-n-2\lambda+2)\pi i});$$

so formula (196) has been proved.

If $\arg z = (m+n-p+2\lambda-1)\pi$ and $q = p+2$, we must besides take account of the sum $\sum_{\sigma=1}^{p}$ in (152), that supplies the expansions of algebraic order. These expansions have because of (18) (with $q=p+2$) and (15) (with $\gamma = \mu - \lambda - 2$) the form

$$\sum_{\sigma=1}^{p} e^{(2-2\mu)\pi i a_\sigma}\, T_{p,p+2}^{m,n}(\sigma;\lambda)\, G_{p,p+2}^{p+2,1}(z\, e^{(p-m-n-2\lambda+2\mu-1)\pi i} \,\|\, a_\sigma)$$

$$\backsim \sum_{\sigma=1}^{p} e^{-2(\lambda+1)\pi i a_\sigma}\, T_{p,p+2}^{m,n}(\sigma;\lambda)\, E_{p,p+2}(z\, e^{(p-m-n+3)\pi i} \,\|\, a_\sigma).$$

Hence formula (197) has also been proved.

Biochemistry. — *Active and inactive calcium in the animal organism. Importance as to diagnostics and therapeutics.* *) By L. SEEKLES and E. HAVINGA. (The Laboratory for Veterinary Biochemistry, State University of Utrecht, Netherlands.) (Communicated by Prof. G. KREDIET.)

(Communicated at the meeting of October 26, 1946.)

On account of the research work performed of late years in this laboratory and in other institutes, concerning deviations of the calcium content of body fluids and tissues in man and different animals, we made a classification into two groups of a number of the principal forms of these disorders.

First group.

1. Man and different experimental animals: condition after removal of the parathyroid glands.
2. Cow: milkfever, grass tetany.
3. Man: the syndrome, characterized by tetany, epilepsy and turbidity of the eye lens.
4. Dog: canine hysteria (?).
5. Man: certain forms of tetany, for example in children.
6. Man and different animals: certain forms of rickets.
7. Pig: epileptic attacks in young pigs, with slight symptoms of rickets, observed during the period of dentition, at the age of about 9 weeks. (?)

Second group.

1. Man, dog (and cat?): condition in case of adenomes of the parathyroid glands (ostitis deformans Recklinghausen = osteitis fibrosa generalisata).
2. Cow: condition, in which the colloids of milk flocculate spontaneously, or on boiling, or after the addition of an equal volume of 70 per cent alcohol, the acid degree being unchanged.

It may be assumed that in the first group of deviations the activity of the calcium ions in the body liquids is lowered, whereas in the second group it may be raised.

In connection with this *provisional and by no means complete* classification [1]) we want to point out, that in the cases mentioned the modified

*) Read in June 1942 during "The Veterinary Week, Utrecht"; published in dutch in 1944 by the "Maatschappij voor Diergeneeskunde" (J. van Boekhoven, Utrecht).

[1]) In case of doubt as to the correctness of this classification, we put a question mark.

activity of the calcium ions is not the only deviation in the chemical constellation of body liquids (and tissues), nor need be so. Our results up to now rather support the opinion that in several of the deviations mentioned before, great importance — and may be sometimes a greater importance — must be attached to other chemical disease symptoms, c.q. to changes in the activity of other ions. We leave this problem, however, out of discussion and rather concentrate our attention in the first place on the problems mentioned in the title of this treatise.

Activity of the calcium ions.

Bloodserum of normal cows contains about 10 mg of calcium in 100 cm^3. It appears that 40—50 % is bound to the colloids of the serum, chiefly to protein.

The rest can move freely through ultrafilters, contrary to the calcium bound to protein.

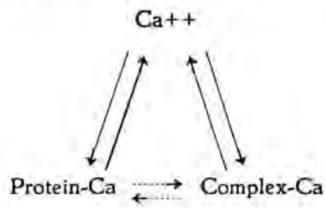
The "free" calcium is found in at least two forms, namely as calcium ions and as complexes in which calcium is bound to certain organic componds e.g. citric acid.

So we distinguish for instance:

Calcium, total	10 mg %
Calcium, bound to protein	4 „
Calcium "free" (ultrafiltrable)	6 „
Calcium ions	2 „
Calcium, in complex compounds	4 „

It is generally assumed, that in the first place it is the calcium ions which cause the physiological activities of "the Calcium" and therefore are "physiologically active". The calcium bound to protein and the calcium in complex compounds form so to say a stock out of which possible replenishment of the calcium ions takes place or to which a possible surplus of calcium ions runs back according to the need of the normal organism. On the other hand it goes without saying, that to exert their physiological activity the calcium ions most probably have to unite with some constituent of the organism, e.g. building up a membrane or an enzyme.

So in the blood serum an equilibrium exists between the three abovementioned forms in which calcium is found [2]).



[2]) As it is not an established fact that a direct transition — without mediation of the calcium ions — takes place between protein-Ca and complex-Ca, the arrows between fractions last mentioned are dotted.

The state of this equilibrium — expressed by the percentage of each of the three compounds — depends on the temperature and the chemical condition of the serum. So, by supply of hydrogen-ions, sodium ions etc., part of the calcium bound to protein gets displaced and transformed into diffusible calcium. A supply of certain other compounds, e.g. citric acid, causes forming of a higher percentage of complex bound calcium at the cost of the other two fractions [3]). How great the latter influence can be, follows from a measurement we performed some time ago.

When, all other conditions remaining unchanged, 0.01 grammol. of sodium citrate per litre is added to a liquid containing 4 mg % of calcium ions — which is a higher content than that of normal bloodserum — the calcium ions content of this liquid falls to about 0.2 mg %. (A quantity of 0.01 grammol. of sodium citrate added to 1 litre of blood prevents its coagulation.)

So the activity of the calcium ions falls to about 5 % of the original value by adding the quantity of citrate mentioned, which is of the same order of size as the citrate content of normal milk. It is easy to understand that by such a change of the calcium ions activity the physiological properties of the liquid undergo a very considerable change, remembering what we communicated before as to the physiological properties of the calcium ions.

The importance of our knowledge of the activity of the calcium ions with respect to diagnostics.

In connection with the importance already mentioned of the calcium ions as bearers of the physiological activity of calcium, it is of great importance to diagnostics to know the activity of the calcium ions in the body-liquids. A parallelism between the chemical activity of the calcium ions and the physiological activity is stated to exist. A striking practical example may illustrate the correctness of this opinion. Patient in point is a very old Frisian cow, showing some days after parturition symptoms of non-typical milkfever (paresis without sopor, decreasing appetite), as had been the case for several years after each parturition. The animal's condition having remained fairly stationary for several days, the calcium content of the blood appears to amount to 6.8 mg %. So we state a moderate hypocalcaemia: the calcium content has fallen to about 2/3 of the normal level. The concentration of calcium ions amounts here to 0.50 mg %, whereas the calcium ions-content of normal cow serum was fixed on an average of 1.60 mg % [4]).

This explains atony of the muscles (paresis), although the total calcium content has fallen only in a relatively slight degree. Now para-

[3]) L. SEEKLES. Arch. néerl. de Physiol. **21**, 526 (1936); **22**, 93 (1937).

[4]) The method applied formerly for the determination of calcium ions is not very exact. It yields rather low values for the concentration of calcium ions, but this is not in the way of a comparison of values obtained according to this method.

thyroid extract is injected, which is known to possess the ability of mobilising and removing the calcium found in the depots. After an hour and a half the paresis disappears. An examination of the blood, made immediately after this, yields a value of 6.3 mg % for the calcium content of the bloodserum; so this means a slight fall compared to the concentration originally determined (6.8 mg %). The content of calcium ions, now, however, comes to 1.04 %, that means to say *the chemical "activity of calcium" has risen to double the value and this was attended with a parallel rise of the physiological activity of the calcium leading into the disappearing of the atony of the muscles (paresis).*

From the state of equilibrium mentioned before between calcium ions on the one hand and the calcium store — the calcium bound to protein and the calcium bound as complex — on the other hand, other facts can also be explained in an unrestrained manner.

From information, given by veterinary practitioners, it appeared to us to have been stated tens of years ago, that the symptoms of milkfever can be caused to disappear by injection of large quantities (1—2 litres) of a physiological solution of sodium chloride (0.9 % NaCl). This fact can be explained by the ability of the sodium ions (brought into the body in large quantities), of eliminating part of the calcium bound to protein. By this the liberated calcium comes into circulation as calcium ions and so raises the abnormally low activity of the calcium ions adherent to milk fever. By this apparently we can reach the physiological threshold-value, essential to cause the milkfever symptoms to disappear.

In the course of years we repeatedly found samples of bloodserum in cases of milkfever, the calcium-content of which was shown not to be lowered. Nevertheless these patients reacted normally upon an intravenous administration of calcium chloride. The next case is a striking example of this.

The bloodserum of a cow, showing typical symptoms of milkfever during the first 24 hours after parturition and getting cured instantly after an intravenous injection of calcium chloride, appears to contain 5,8 mg % of Ca, 3,0 mg % of inorg. P and 1,9 mg % of Mg. So we find a very distinct hypocalcaemia and hypophosphataemia. After 10 hours the animal relapses, exactly the same clinical picture developing again. Once more the patient recovers as readily as the first time after an intravenous administration of calcium chloride. However, the blood taken before the second injection was shown to contain 10,2 mg % of Ca, 4,1 mg % of inorg. P and 1,5 mg % of Mg, that is to say the mineral content of the bloodserum was almost normal and particularly the symptom of hypocalcaemie was missing here.

From the example given it may be concluded that the total calcium content of a patient's bloodserum not always produces an indication for applying the calcium therapy. In the meantime it is quite possible, that in this case the equilibrium between the three calcium fractions of the blood had been shifted, at the cost of the calcium ions. The therapeutic effect of the injection of calcium chloride may be ascribed to a rise of the activity of the calcium ions caused by the liquid injected. We can consider as a second possibility that the activity of the calcium ions of the blood was normal in the case taken as an example. The symptoms of milkfever may have been caused by a diminised irritability of the neuromuscular apparatus and of the other organs, the disfunction of which contributes to causing symptoms of milkfever [5]).

Incidentally it may be remarked, that lately we succeeded in showing that in certain cases of excitation and tetany in cows during the stall period conditions according to magnesium may be found in the bloodserum, as are sometimes found in cases of milkfever with regard to the calcium element.

So we found for instance Ca 10,7 mg %; P 7,0 mg %; Mg 3,0 mg %, Na 336 mg %, Cl 348 mg %, K 24 mg % in the bloodserum of such a patient, therefore a magnesium-content not lowered — as in case of typical grass tetany — but raised. Nevertheless these patients react favourably upon a subcutaneous injection of for instance 30 g of crystallised magnesium chloride ($Mg\,Cl_2 \cdot 6\,H_2O$) in 300 cm^3 of water, by which treatment the symptoms of tetany and excitation disappear [6]).

From the examples mentioned it may have become clear that, from a diagnostic point of view, it must be considered to be very important to have a method at our disposal enabling us to determine the activity of the calcium ions. This being attained, it will be possible to gain an insight into the nature and by this into the therapy of the non-typical cases of paresis and tetany in cows and other animals, as well as of a number of disorders, some principal forms of which have been mentioned in groups 1 and 2 at the beginning of this paper.

The importance of the activity of calcium ions of injection liquids with respect to therapeutics.

Some time ago we called attention to the fact that from experiments formerly made in this laboratory [7]) it appeared that solutions of calcium chloride and of calcium gluconate cause a very different local effect when subcutaneously administered.

A solution of calcium chloride, subcutaneously injected with all usual precautions, causes a painful swelling and in the end extensive phlegmones, even if the osmotic concentration of the solution does not exceed that of a physiological solution of sodium chloride. Necrosis and sloughing of the skin over an extensive region may occur which can cause serious complications. Contrary to this a solution of calcium gluconate administered in the same way causes only a painful swelling as a rule, which disappears in the course of some days to a week, without any injurious consequences. A solution of calcium lactate ranges, as far as the irritating effect on the subcutis is concerned, between the solutions of calcium chloride and of calcium gluconate.

Some time ago we succeeded in making it probable that the difference in irritating effect between equivalent solutions of calcium chloride, calcium lactate and calcium gluconate gets determined by the difference in con-

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5) L. SEEKLES. Tijdschrift v. Diergeneesk. **67**, N⁰. 2 (1940).

6) We hope to communicate further particulars as to these special cases of tetany.

7) B. SJOLLEMA, L. SEEKLES, F. C. VAN DER KAAY, Tijdschrift voor Diergeneesk. **58**, 254 (1931).

centration of the calcium ions in the liquids mentioned. In solutions of simple compounds an impression of the concentration of the ions can be obtained by measuring the electric conductivity of these liquids. In figure 1 we see the equivalent conductivity: $\left(\frac{\Lambda_c}{\Lambda_0}\right)$ plotted against the concentrations as we determined for some salts.

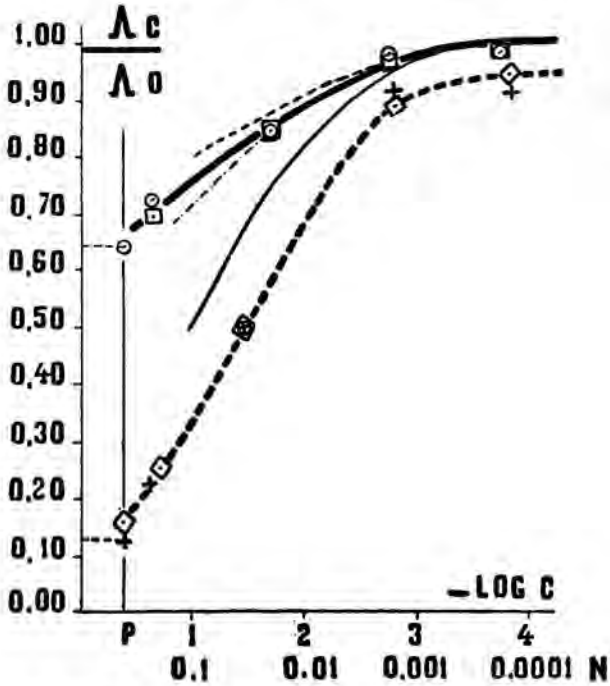

Fig. 1. Equivalent conductivity of salt solutions at different concentrations, temperature 20° and 40° C (calcium chloride, calcium gluconate and calcium lactate, in comparison with sodium gluconate and sodium lactate).

—·—·—·—·— sodium lactate
▬▬▬▬▬ calcium chloride
- - - - - - - - sodium gluconate
———— calcium lactate
■■■■■■■ calcium gluconate
P: concentration for subcutaneous injection.

With respect to calcium chloride the equivalent conductivity decreases very slowly only in proportion as the concentration increases. So the splitting into ions remains very considerable even in concentrated solutions. Calcium gluconate, however, behaves in a quite different way: the equivalent conductivity decreases strongly in proportion as the concentration of the solution increases. So we see from figure 1 that in the case of a 10 % solution of calcium gluconate as used for injection (marked by point P) $\frac{\Lambda_c}{\Lambda_0} = 0.15$. For an equivalent solution of $CaCl_2$ (about 5 %, also marked by point P) this value amounts to 0.65.

The most plausible explanation is, that the splitting into ions for calcium gluconate in more concentrated solution is far less than in the case of calcium chloride.

We may assume that the splitting of calcium gluconate molecules into ions takes its course in two consecutive steps.

1) Ca (gluconate)$_2$ $\rightleftarrows$ Ca gluconate+ + gluconate$^-$

2) Ca gluconate+ $\rightleftarrows$ Ca++ + gluconate$^-$

Especially by the second step, yielding the calcium ions, the dissociation-equilibrium in concentrated solution is moved up to the left. The relatively slight conductivity of a solution of calcium gluconate as it is used for injections therefore points in this train of thought to a small content of calcium ions.

Calcium lactate too — although perhaps a little better split into ions than calcium gluconate — forms concentrated solutions of a considerable less molecular conductivity than $CaCl_2$. In this case the concentration of calcium ions evidently also is slight. Finally from measurements of the conductivity at 20° and 40° the influence of the temperature on the splitting of the salts appears to be slight.

The results of the chemical examination of the solutions of calcium chloride, calcium lactate and calcium gluconate are in agreement with the results of the injection-experiments mentioned before, with respect to the physiological activity — irritation of the subcutis — exerted by these salts.

Finally it may be remembered that by building on systematically on the basis as stated above, we succeeded lately in preparing a calcium containing injection liquid possessing the advantages of the calcium gluconate-solution without certain disadvantages: the solution of sodium-calcium borogluconate [8]).

Determination of the calcium ions activity by means of the spreading method [9]).

It is not possible to estimate the concentration of the calcium ions from conductivity measurements in biological liquids such as blood serum, milk, etc. containing, apart from calcium ions, several other ions in large quantities. Neither can other electrometrical methods be applied for the determination of the activity of calcium ions. During the last few years we succeeded in working out a method, which for the first time offers an opportunity of measuring the activity of the calcium ions or at least a value analogous to this.

For this purpose we make so to say a print, a snap-shot of the ionic condition of the liquid. A very small quantity of a fatty acid (stearic acid) solved in a volatile, indifferent solvent (gasoline of low boiling point) is placed on the surface of the liquid to be examined which is kept in a flat basin (spreading-basin). The solution spreads itself over the surface and

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[8]) L. SEEKLES, E. HAVINGA, J. DE WAEL, Tijdschr. v. Diergeneesk. 69, 179 (1942).

[9]) A more ample description of the method will be published elsewhere in due time. For a provisional communication we refer to E. HAVINGA, Chem. Weekblad 39, 266 (1942).

after the solvent is evaporated, under circumstances aptly chosen, the stearic acid is left on the surface as a monomolecular film. The hydro-polar character of the fatty acid is manifested in the immersion of the carboxyl-group, the paraffine rest of the fatty acid "protruding above the water".

The carboxyl groups immersed ionize partly, the liberated hydrogen-ions diffusing into the underlying solution (hypophasis). On the other hand: part of the positive ions from the hypophasis occupy a number of places near to the ionized carboxyl groups which originally were occupied by hydrogen ions. It is particularly the calcium ions which show a strong tendency to salt formation with the carboxyl groups and an equilibrium is formed between film and hypophasis, dependent on the ionic condition of the hypophasis. So the composition of the film reflects the ionic condition in the liquid underneath. *The quantity of ions absorbed is so small that practically no change in the equilibrium between the different fractions of calcium and other elements in the hypophasis occurs.*

To check matters, solutions of known ionic strenghth are prepared, similar to the possible ionic concentration of the unknown liquid, the activity of calcium ions of which we wish to determine.

An example of such a test liquid comparable with the ultrafiltrate of blood serum is the following one.

Ca^{++} 2 mg % Mg^{++} 0,25 mg % Na^{+} 330 mg %	In addition to the negative Cl^- and HCO_3^--ions, not reacting with the ionised carboxyl groups of the film.

pH (20° C) = 7,50 (measure for the activity of hydrogen ions).

After the equilibrium between film and hypophasis has been established we collect the film quantitatively by compressing it and skimming it off from the surface and next we determine its content of the different ions as well as the "free" carboxyl groups — that is to say not bound to the ions.

The results with the "hypophasis" mentioned above as an example were as follows:

Bound by Ca: 65 % " " Mg: 2,3 % " " Na: 8 %	of the carboxyl groups originally present in the film.

Free carboxyl groups: 24,7 %.

Figure 2 represents, in the form of a verification curve, some of our results, obtained with simplified "synthetic serum ultrafiltrates".

Verification curves having been determined with hypophases of known composition, the spreading of stearic acid can be performed on the surface of a liquid the calcium ions-activity of which we wish to state.

The calcium ions content, *i.e.* the calcium ions activity of the liquid, then follows immediately from the calcium content of the film, by reading the graphic. Critical examination shows, that the possibility of applying this method is based upon the simultaneous determination of the activity in the hypophasis and the "concentration" in the surface layer for at least one ionic species. Evidently this can be done for the hydrogen ions (determination of pH by means of the glass electrode and of the COOH groupings of the monomolecular layer).

Finally just a few words as to the difficulties, experienced when performing the spreading-method. They are principally of an analytical chemical nature.

The extremely small quantities of the different ions contained by the removed film, cause the application of special methods of research and of special apparatuses. Therefore, before we were able to analyse these monolayers we were obliged to work out special ultra-microchemical

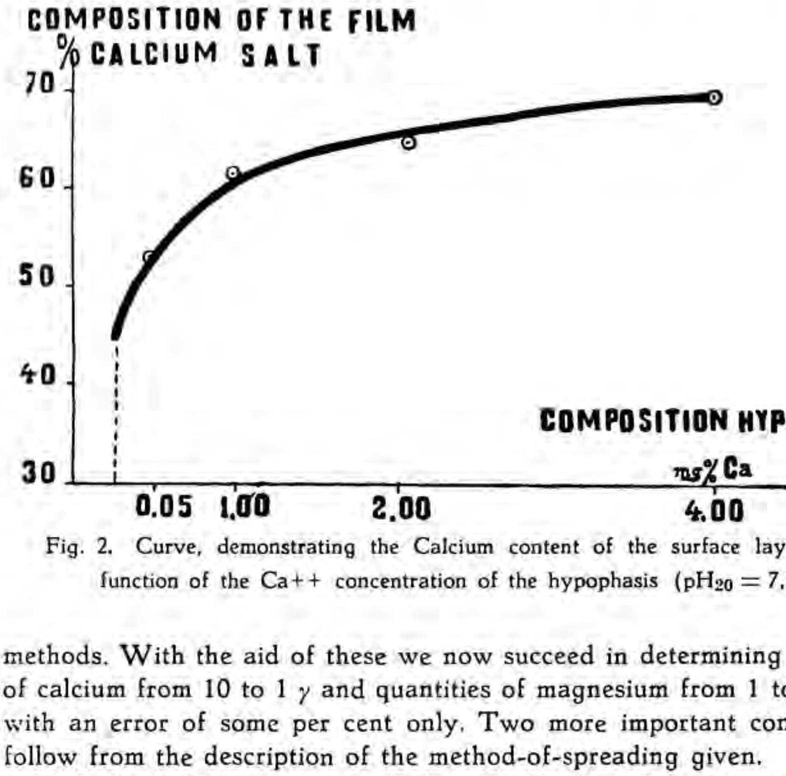

Fig. 2. Curve, demonstrating the Calcium content of the surface layer as a function of the Ca++ concentration of the hypophasis ($pH_{20} = 7{,}50$).

methods. With the aid of these we now succeed in determining quantities of calcium from 10 to 1 γ and quantities of magnesium from 1 to 0.1 γ [10]) with an error of some per cent only. Two more important consequences follow from the description of the method-of-spreading given.

In the first place the nature of the method raises the expectation that in principle it may also be applied for the determination of the activity of other ions.

The property of stearic acid, of reacting in the first place with calcium ions, renders this substance pre-eminently suitable for the determination of the activity of calcium ions. We hope that in future it may appear possible to use other substances for spreading which combine preferably with other ions and so enable us to determine the activity of these ions. In principal the method of spreading is a universal method of research for the determination of the activity of ions in (biological) liquids and of the

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[10]) E. HAVINGA, A. F. K. BUYS BALLOT; Rec. Trav. Chim. **61**, 849 (1942).

antagonism of ions in interfaces. So we may cherish the hope that in years to come numbers of biological problems which up till now could not or could not efficiently be studied, may further be solved. It is a great difficulty, however, that according to our experience gathered up till now, the presence of protein has a disturbing influence. Therefore in any case we shall have to work with ultrafiltrates. For the rest it may have become clear that the practical performing of the method of spreading research requires great technical skill and relatively a great deal of biological material. This is an objection to its being generally applied in laboratory and clinic. We therefore now carry out experiments in order to find a method feasible by simple means and by small quantities of material — therefore practical — which is to be verified by the universal, exact — but very complicated-method of working described before.

Summary.

The authors give a brief survey of the diseases in man and animals which are characterized by abnormal values of calcium in the body fluids. The different forms in which calcium occurs in blood and the importance of calcium ions activity for physiological phenomena, diagnostics and therapeutics are mentioned. In simple solutions of calcium salts the ionic activity has been estimated by means of conductivity measurements. It was shown that solutions of calcium lactate and particularly of calcium gluconate contain less calcium ions than equimolar solutions of calcium chloride. In consequence a subcutaneous injection of calcium chloride solutions causes considerable lesions in the subcutaneous tissue, as compared with solutions of calcium gluconate. The activity of (calcium) ions in liquids of complicated composition is estimated by means of a new method in which a monomolecular layer (of stearic acid) is brought on the surface of the liquid. Then the film is skimmed off from the surface and the contents of calcium (and other cations) and of free carboxyl groups are estimated by means of ultra-microchemical methods. From the calcium value of the film the calcium ion activity of the liquid can be calculated by comparison with films spreaded on solutions of known ionic concentrations.

Zoology. — *Pharynx regeneration in postpharyngeal fragments of Polycelis nigra (Ehrbg.).* By K. VAN ASPEREN. (From the Zoological Laboratory, University of Utrecht.) (Communicated by Prof. CHR. P. RAVEN.)

(Communicated at the meeting of October 26, 1946.)

This investigation deals with the problem, how and where the pharynx is formed in the regeneration process of postpharyngeal fragments of *Polycelis nigra*. It may be considered as a continuation of the experiments of DRESDEN (1940). His results may be shortly summarized as follows: In the regeneration process two phases can be distinguished, a first phase of r e g e n e r a t i o n, in which the formation of a regeneration blastema by dedifferentiation and shift of body cells takes place, and a second phase, in which the m o r p h a l l a x i s is most conspicuous. In DRESDEN's experiments the animals were cut more anteriorly, at the base of the pharynx, so that the pharynx sheath is present in the posterior fragment, but the pharynx itself is lacking. We may be sure, that this empty pharynx sheath has a great influence on the place, where the new pharynx will develop. Shortly after the closure of the wound a regeneration blastema is formed in front of the pharynx sheath. After a few days this regeneration blastema begins to grow out into the old pharynx sheath. At first, the differentiation in the pharynx is slight, only later on the different tissues are clearly distinguished. The prepharyngeal intestinal trunk appears rather early in the regeneration bud and has differentiated already, when the pharynx begins to grow out. It may be concluded that the place, where the new pharynx will be formed, is determined by the empty pharynx sheath. The cells taking part in the formation of the pharynx cannot be formed on the spot by de- and redifferentiation of body cells (metaplasia). This fact is significant in connection with our own investigations.

Literature.

The regeneration processes in Triclads are controlled by the action of gradients and the organizing powers of certain parts of the body (CHILD). BRØNDSTED (1939, 1942), working with *Bdellocephala punctata*, only obtained regeneration of a head with eyes by the posterior fragment of a transversely cut animal, when the cut was situated at some distance before the pharynx. The percentage of cases giving a complete regeneration of head with eyes shows a steady decrease according as the cut was made more posteriorly. The "head-frequency curve", obtained in this way, dropped to zero before the level of the pharynx. This is, however, not at all the case in all Turbellaria. SIVICKIS (1930) arranged the triclads into 5 groups with regard to the head-frequency curve. Unpublished investigations of T. MELTZER–SLUITER showed that *Polycelis nigra* belongs to the third group, the Phagocata group. This group regenerates heads from all segments of the body, but the curve slopes towards the tail, indicating that the ability to regenerate heads from posterior segments is not always present.

Several authors, a.o. STEINMANN (1926), LUS (1926), GOLDSMITH (1940) and BRØNDSTED (1942) point to the part played by the regeneration cells (Regenerationszellen, formative cells) in the regeneration process. These are undifferentiated cells present in the body of the adult animal, possessing the potency to develop into very different kinds of tissues, and which, in this way, are ready to take part in the formation of an indifferent regeneration blastema. It is also possible, however, that cells, which have been differentiated already to some extent, dedifferentiate and are transformed into regeneration cells.

GOLDSMITH (1940) experimented with *Dugesia tigrina* and *Procotyla* spec. He made small wounds in the prepharyngeal region by electro- or thermo-cautery or by making small radial incisions, and investigated the potency for the formation of an outgrowth from the wound, which in certain cases even contained eyes.

This potency is very strong in *Dugesia*, but is lacking completely in *Procotyla*; the author explains this difference by supposing that in *Dugesia* many formative cells are present in the prepharyngeal part in the normal animal too, whereas they are missing completely in *Procotyla*. The presence of these regeneration cells, according to GOLDSMITH, is related to the presence or release of sulfhydril-compounds.

Another problem is, how the place of the different organs and tissues is determined in a regeneration blastema. Some authors point, in this connection, to the great influence of the nervous system in the old tissue. So WILSON (1941) describes the formation of eyes in a regenerating head under the influence of nerves, growing out from the old nerve cord into the new tissue. This is incompatible with the opinion of STEINMANN (1926), after which the new and the old tissue should be connected only later on by morphallactic processes. Many investigators suppose a relation between the organizing power of a grafted piece of the body and the presence of nervous tissue in it. MILLER (1938), however, denies this influence. SILBER and HAMBURGER (1939) experimented with *Euplanaria tigrina*. The animals were split in the median plane from the tip of the tail to a point between the auricles and the anterior end of the pharynx, and the head was removed by a transverse cut in front of the crotch. Regeneration resulted in the formation of two heads, one at the anterior surface and one in the crotch, but furthermore one or two lateral heads were formed at the median wound margins in some cases. When the trunks and tails of the resulting duplicitas cruciata were split again, the percentage of lateral heads was greatly increased; these lateral heads developed only on the half-tails, which consisted of old tissue, the place of formation being determined by the presence of trunks of the old nervous system.

For the interpretation of the results of my own experiments, the following data from the literature are of great interest with respect to the problem, where and how the new organs are formed in the regenerating animal and how the new formations are brought into the normal proportional relations with the old tissue.

BRØNDSTED (1942) gives as his opinion that first an indifferent regeneration blastema is formed and afterwards morphallaxis takes place in the regenerate, whereas the old tissue is not transformed. After dedifferentiation in the old tissue, the newly formed formative cells are transported to the regeneration blastema, where they differentiate into new tissue. According to this opinion, the formation of a new pharynx should be expected in our own experiments in the new tissue only, consequently not by redifferentiation of the preexisting tissues (metaplasia) in the old posterior fragment. BRØNDSTED experimented with *Bdellocephala punctata*. KAHL (1936), working with *Planaria gonocephala*, got results proving that those of BRØNDSTED with *B. punctata* cannot be generalized and applied to all Triclads. KAHL investigated the regeneration of postpharyngeal pieces, when he made lateral incisions at varying distances from the wound surface. Both on the anterior and on the lateral wound surface a head with a pharynx was formed in most cases, but when the incision was made at a very short distance from the anterior surface, only one pharynx developed in such a way, as though it belonged to the anterior regenerating head and were induced by it. This pharynx was situated for a large part or completely posteriorly to the incision, consequently of necessity localized in the old tissue. Previous data on this subject were given by THACHER (1902) and STEVENS (1907). THACHER (1902), from his experiments with *Planaria maculata*, drew the conclusion that the pharynx in postpharyngeal pieces develops in the old tissue.

STEVENS (1907) found that in *P. simplicissima* the pharynx is regenerated in the new part, but on the border of the old one, in *P. maculata* in the old tissue and in *P. morgani* in the old tissue, but on the border of the new one. So no general conclusion can be drawn from the data in literature.

On the differentiation and morphallactic processes, which must take place in order to make the regenerating animal to a normal functioning organism, opinions diverge widely. After STEINMANN (1926) first a dedifferentiation takes place in the old tissue, then a transport of the dedifferentiated cells to the wound surface and the formation of a regeneration blastema. Only when sufficient indifferent material is present in the regeneration bud differentiation starts, and only later on morphallaxis and the connection of the newly formed organs with those in the old tissue takes place. WILSON (1941) defends quite another view. According to this author, nothing points to a general dedifferentiation of the old tissue near the wound and nothing confirms the opinion that the new head develops on the basis of a newly organized regeneration blastema and is connected with the old part only later on. He thinks that the new organs are formed by outgrowth of the corresponding organs from the old into the newly formed part, and that especially the nervous system plays an important part in this process.

From all these data it may be clear that many factors play a more or less important part in the building up and transformation of the tissues and organs in the regeneration process and that the different often conflicting experimental results for a large part may be imputed to differences in behaviour between the different species.

Material and methods.

The experiments were performed with *Polycelis nigra*, a fresh-water triclad, which could be collected very easily in ditches in the neighbourhood of Utrecht.

The animals were kept in aquaria containing some plant material, and were fed every few days. Just before the beginning of the experiment a number of animals were transferred from the aquaria to Petri-dishes. After being narcotized, the animals were divided with the aid of a very sharp small knife into two pieces by a transverse cut just behind the mouth. The posterior pieces were kept in tapwater in Petri-dishes; every day a number of these were fixed and slides were prepared of these partly regenerated fragments. In this way, the regeneration process was studied till the 27th day after the operation.

It may be stressed that the material used for the experiments was not homogeneous as to age, pigmentation and size of animals; therefore, the data about the velocity of the regeneration process are not mutually comparable.

Description of the regeneration process.

A short time after the operation the wound contracts strongly, the laterally situated epidermis being drawn inward. In this way, a minimum surface of the wound is exposed to contact with the surrounding medium, which, probably, may be seen as a temporary measure to restrict the injurious influences of the outer world to the utmost. After that, we often see a mucous cover appearing round the pieces, which STEINMANN (1926), like the phenomenon of wound contraction, considers as a temporary measure against damage. A distinct formation of mucous cover was often followed by death. This does not prove, however, that STEINMANN's opinion is wrong.

Soms days after the operation, the wound has healed again, by closure with low epithelial cells. The normal epidermis cells are many times higher and clearly provided with rhabdites. Now, during some days large quantities of completely undifferentiated cells begin to accumulate just behind this newly formed epidermal layer. In this way, a regeneration bud is formed at the extreme anterior end of the piece. About 6 to 10 days after the operation this regeneration blastema is divided into two separate parts one after the other, both characterized by a strong accumulation of cell nuclei, without showing a clear differentiation of the cell structures. These two parts are separated by parenchyma, which mostly is only slightly differentiated. Very soon the gut makes its first appearance in the regeneration bud or between its two parts (fig. 1). It consists of a lumen sur-

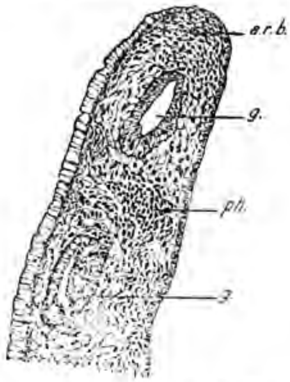

Fig. 1. *Polycelis* 82, 9 days after operation. Formation of gut (g) between anterior regeneration bud (a.r.b.) and pharynx bud (ph).

rounded by large cells, which differentiate at a very early stage, so that this organ is completely re-established soon afterwards. Now, both parts of the regeneration bud grow out, the anterior one in a cranial direction and the posterior bud in a caudal direction. The latter will form the new pharynx. Figure 2 shows, that a cavity is formed in the old tissue at the caudal side of this pharynx regeneration bud, which sometimes is only small in size, in other cases rather large; this forms the new pharynx sheath, so that the course, which will be followed by the outgrowing pharynx, can easily be predicted. The place, where this cavity will communicate with the outer world, may be seen already very early, as the pigment layer is missing here. As a matter of fact, this layer is displaced inward, forming the covering of the canal-shaped mouth opening. This opening, which is very narrow, often appears at a very early stage, when the pharynx bud is at the beginning of its development, sometimes at a somewhat later stage (Cf. fig. 3). The inward displacement of the pigment layer points to a shift of cells in an inward direction, so that we may assume that at least part of the mouth canal is covered by epidermis cells. The pharynx sheath is formed by fission of the parenchymous cells; soon, it is partly covered with high epithelial cells.

Shortly after the formation of the pharynx sheath, a small canalicular lumen develops in the regeneration bud of the pharynx, which communicates with the above-mentioned prepharyngeal intestinal lumen and with the lumen of the pharynx sheath after some time (fig. 4 and 5). Now,

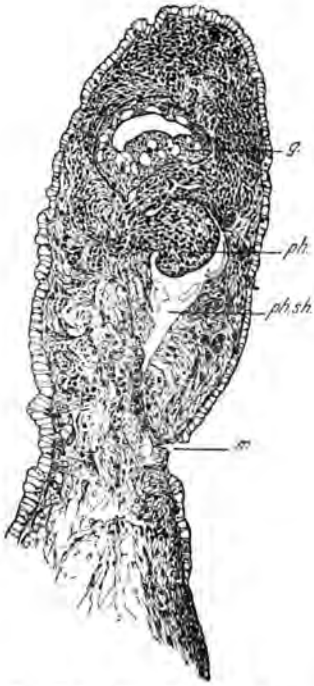

Fig. 2. *Polycelis* 65, 6 days after operation. Formation of pharynx sheath (ph. sh.) behind pharynx bud (ph). m = mouth. g = gut.

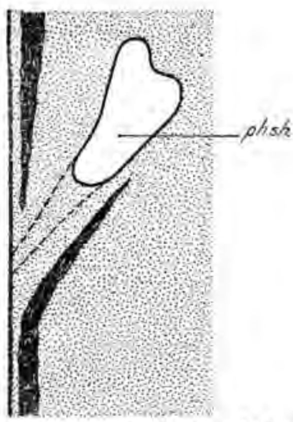

Fig. 3. Diagrammatic sketch of pharynx sheath (ph. sh.) and mouth anlage. Displacement of pigment layer.

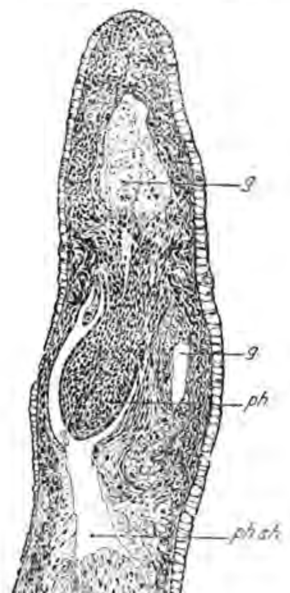

Fig. 4. *Polycelis* 67, 6 days after operation. Formation of pharynx lumen. g = gut. ph. = pharynx. ph. sh. = pharynx sheath.

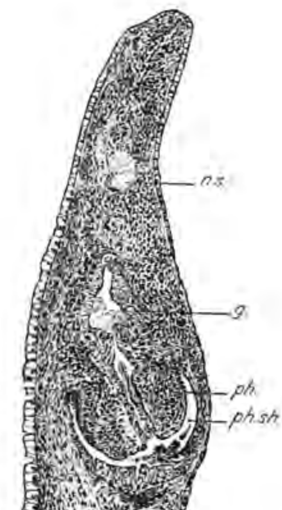

Fig. 5. *Polycelis* 88, 11 days after operation. Pharynxlumen communicates with gut (g) and pharynx sheath (ph. sh.). n.s. = nervous system in regenerated head.

the intestinal system is reestablished in principle. The differentiation of the pharynx, beginning rather late, now goes on at a fast rate. The cavities of the pre- and postpharyngeal gut communicate and, in this way, the animal disposes again of an apparatus for food-intake and -digestion, necessary for life. In the meantime, in the anterior regeneration bud the processes leading to the formation of nerve cord and eyes have taken place.

Discussion.

We have to consider the question, where the pharynx is formed, in the regenerating posterior fragment or in the newly formed regeneration blastema. It is very difficult to give a completely satisfying answer. In order to decide whether the pharynx regeneration bud develops in the old tissue or in the regeneration blastema, one needs a criterion, where the old tissue passes into the regeneration blastema. This transition, however, is not sharp and, moreover, in consequence of the wound contraction just after the operation, a deformation takes place in the region of the regeneration bud; the epidermis and the just underlying tissue of the regenerating piece are drawn over the wound and, in this way, form the covering of the newly formed regeneration blastema at a later stage. From the nature and form of these superficially situated tissue elements no definitive conclusion can be drawn as to the nature of the tissue, which is situated internally. However, soon a prepharyngeal intestinal trunk develops in the originally simple regeneration bud; caudally to this trunk, the regeneration tissue forms a pharynx bud.

Hence, the question arises, whether this pharynx bud must be considered as the caudal part of the newly formed head regeneration blastema or is formed in situ in the old tissue. In the first case, dedifferentiated cells of the old tissue near the wound or preformed regeneration cells will be displaced towards the wound surface; afterwards, they grow out in a posterior direction into the old tissue. In the second case, the dedifferentiated or preformed regeneration cells will accumulate directly within the old tissue; hence, they are to be displaced over much smaller distances only. Therefore, it comes to the point to what extent a transport of cell materials takes place, and whether the material, taking part in the formation of the pharynx, is derived from different parts of the fragment or arises at the place, where the pharynx bud is formed. It is clear that this last possibility cannot be true. The pharynx sheath in most cases is formed at an early stage, when the pharynx bud is still small; the latter then grows out into its cavity. As no mitoses or amitoses could be detected in this bud, evidently it does not grow by proliferation; hence, its cells must be derived from other regions. The same applies to the earliest rudiment of the pharynx bud. The observation that the genital organs in the posterior fragment disappear in the first days after the operation, proves that a breakdown of the old tissues occurs; therefore, it is probable that the cells of the regeneration bud arise at least partly by dedifferentiation of tissue cells. The pharynx regeneration bud is formed very near the place, where old and new tissue merge. Considering the vagueness of this limit, it is of little importance to determine whether this accumulation of cells takes place in the old or in the new tissue. An organization of old tissue by influences from the regenerating head does not occur, as far as the formation of the pharynx itself is concerned.

On the other hand, the pharynx sheath certainly is formed in the old tissue, partly by fission of the parenchyma and partly by an inward displacement of superficial cells in the region of the mouth opening.

Comparing our results with those of DRESDEN (1940), we can state a considerable agreement. Whereas the pharynx sheath was retained in his experiments, in our experiments it is formed anew at a rather early stage in the old posterior fragment, after which the process shows a great resemblance to that described by DRESDEN.

Summary.

In posterior fragments of *Polycelis nigra,* obtained by a transverse cut in the postpharyngeal region, the regeneration of the pharynx occurs in the following way: Indifferent or dedifferentiated cells are transported from the old tissue to the wound surface. There a regeneration blastema is formed. In the regeneration blastema, a prepharyngeal intestinal trunk develops. In front of this trunk, an anterior regeneration bud grows out and forms a head with nervous tissue and eyes. The part of the regeneration blastema, situated behind the gut, forms a pharynx regeneration bud. In the meantime, a pharynx sheath is induced in the old tissue at the caudal side of this bud, which is gradually filled by the growing pharynx. Not before the pharynx has reached its definitive size, the differentiation of the tissues in it begins. Finally, the proportional relations are normalized by morphallactic processes, which perhaps begin already at an earlier stage, but only after the organs are laid down.

The writer wishes to acknowledge his indebtedness to Prof. Dr. CHR. P. RAVEN and L. H. BRETSCHNEIDER for their direction and criticism.

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Anatomy. — *Multipele hyperodontie in boven- en onderkaak.* By TH. E. DE JONGE. (Communicated by Prof. M. W. WOERDEMAN.)

(Communicated at the meeting of September 21, 1946.)

Toen wij voor enkele jaren in de „*Proceedings*" dezer Akademie (1 en 2) verslag uitbrachten over een geval van hoektandverdubbeling in melkdentitie en blijvende reeks, was Professor KLIMES, hoogleeraar aan de Universiteit te BRNO (Tsjecho-Slowakije) zoo vriendelijk, onze aandacht te vestigen op den merkwaardigen casus van hoektandverdubbeling, welke zijn assistent SIMEK in 1943 bij een zeventienjarigen jongeman gevonden had en waarvan deze in eene gevallenbeschrijving in de CESKA STOMATOLOGIE (3) mededeeling deed. Naar aanleiding daarvan verzocht hij ons om eene epicrise, waartoe hij ons nà herstel der postverbindingen het desbetreffende materiaal (gipsafgietsels, geëxtraheerde tanden, photographieën en RÖNTGENogrammen) ter beschikking stelde.

Aldus konden wij aan de hand der modellen (fig. 1) in de onderkaak de aanwezigheid van dertien blijvende elementen vaststellen: ter linkerzijde bleek geen tweede præmolaris tot eruptie gekomen te zijn. In de bovenkaak troffen wij zeventien blijvende elementen aan, bovendien een melkcuspidatus. Ectoscopisch konden wij derhalve een totaal aantal van dertig blijvende tanden tellen, daarenboven de persistentie van één lactaal element vaststellen.

RÖNTGENoscopisch onderzoek leidde echter tot eene verrassende ontdekking (fig. 2 en 3): immers, verdeeld over boven- en onderkaak, konden niet minder dan tien geïmpacteerde elementen aan het licht gebracht worden, te weten: vijf præmolares en vijf molares. Het werkelijke aantal aanwezige gebitselementen bedraagt derhalve eenenveertig (veertig blijvende tanden en één melkelement, hetwelk verder nochtans buiten beschouwing kan blijven).

Hoe nu zijn de acht elementen, waarmede deze gebitsstructuur klaarblijkelijk boven het normale aantal uitgaat, verdeeld over boven- en onderkaak? In de onderkaak is een overtollige præmolaris bilateraal-symmetrisch wel tot (gedeeltelijke) ontwikkeling, niet echter tot doorbraak gekomen. Hetzelfde geldt in de bovenkaak voor een overtolligen bicuspidatus, welks ontwikkeling eveneens een bilateraal-symmetrisch karakter draagt en voor den overtolligen molaris links, welke achter den derden molaris in de kaak ingebed ligt.

In het gebied der bovenfronttanden, incisivi en cuspidati, tellen wij, den melkcuspidatus buiten beschouwing latende, niet minder dan negen elementen: zes snijtanden en drie hoektanden. Ofschoon deze, in tegenstelling met de postcanine gebitselementen, alle geheel dan wel ten deele doorgebroken zijn, vormt hier de diagnosestelling een niet zoo eenvoudig probleem.

Maar hoe dan ook: vast staat, dat tezamen met onze onder den titel van „*Tegengestelde ontwikkelingstendenties in 's menschen gebit*" beschreven resp. geciteerde gebitsformaties (4) deze casus *tot de allerzeldzaamste der literatuur behoort.* Vooreerst immers vormt overschrijding van het normale aantal van cuspidati — om het even of men daarbij van overtollige hoektanden dan wel van hoektandverdubbeling spreekt — een phænomeen, welks uitzonderlijke zeldzaamheid eene welsprekende bevestiging vindt in SIMEK's simpele mededeeling, dat dit geval het eenige was op een totaal van 41.260 door hem onderzochte gebitten. Doch afgezien daarvan is ons geen enkel ander voorbeeld van multipele hyperodontie bekend, *waarin bij alle tandgroepen zonder uitzondering* — te weten derhalve bij incisivi, cuspidati, præmolares en molares — *het normale getal overschreden wordt.*

Na deze korte afdwaling vraagt de interpretatie der voortanden onze aandacht; zij komt in feite neer op de vraag: hebben wij bij deze met tandverdubbeling te doen dan wel met de ontwikkeling van overtollige elementen?

Ofschoon op dit punt geenszins nog eenstemmigheid heerscht, schijnt ons nochtans voor de snijtanden het antwoord nauwelijks twijfelachtig. Men weet, dat BOLK (5 en 6) onderscheid maakt tusschen den gewoonlijk ter hoogte van de mediaanlijn gelegen *mesiodens* en de op klieving van den oorspronkelijken tandaanleg (*schizodontie*) berustende tweelingstanden resp. dubbeltanden (*schizogene variatie*-BOLK), welke laatste zich in tegenstelling tot den mesiodens kenmerken door hun normalen incisiviformen bouw. Wel is deze verklaringsmodus, vaak op grond van aanvechtbare, niet zelden zelfs onjuiste motiveering, van verschillende zijde ernstig bestreden [MATHIS (7), MARTA DE BOER (8) e.a.] — anderzijds echter hebben de onderzoekingen der latere jaren [BENNEJEANT (9 en 10), DE JONGE (11) e.a.] de nauwkeurigheid van BOLK's observaties en de juistheid zijner conclusies op zóó ondubbelzinnige wijze bevestigd, dat wij gelooven, dat zijne theorie der schizogene variaties nauwelijks meer een hypothetisch karakter draagt. Voor de præcanine elementen van onzen casus schijnt ons de beantwoording der bovengestelde vraag derhalve aan geenerlei twijfel onderhevig: wij hebben hier een bilateralen vorm van snijtandverdubbeling voor ons.

Wanneer wij niettemin óók ten aanzien van de drie cuspidati eenzelfde opvatting huldigen, dan zijn wij ons bewust, de theorie der schizogene variaties, door BOLK destijds alléén voor de snijtanden der bovenkaak opgesteld, hiermede dusdanige uitbreiding te geven, dat eene nadere motiveering niet achterwege mag blijven. Feitelijk hebben wij ons standpunt ten deze reeds in onze bovenvermelde studie over de verdubbeling van den hoektand (1 en 2) bepaald en zouden wij er derhalve mede kunnen volstaan, naar onze toenmalige beschouwingen te verwijzen. De zeldzaamheid der waarneming geeft ons echter aanleiding, er nog éénmaal op terug te komen.

De probleemstelling — overtollige hoektand of hoektandverdubbeling — is eenvoudig genoeg. Niet echter de oplossing. Overschrijding toch van het normale aantal vormt bij den cuspidatus een zóó uittermate zeldzaam phænomeen, dat, wanneer wij in de oudere literatuur bijvoorbeeld de vraag tegenkomen, of óók bij den hoektand overschrijding van het normale aantal mogelijk is, wij deze vraag gemeenlijk in dier voege beantwoord vinden, dat deze mogelijkheid ontkend dan wel betwijfeld wordt — in het gunstigste geval wordt de manifestatie van een overtolligen cuspidatus als hooge uitzondering erkend. Daarmede is echter allerminst eene verklaring gegeven. Eenerzijds toch weten wij, dat nòch bij de thans levende nòch bij de uitgestorven primaten of andere zoogdieren vormen bekend zijn met meer dan éénen hoektand [1]). Vast staat derhalve, dat wij ons op atavisme niet beroepen kunnen. Anderzijds: ofschoon BASTYR te Praag reeds in 1889 eene gevallenbeschrijving gaf, welke het voorkomen van meer dan éénen normalen cuspidatus boven elken twijfel verhief (12), bleef men ten onrechte niettemin — slechts enkele uitzonderingen daargelaten — vrijwel unaniem vasthouden aan het dogma, dat overschrijding van het normale aantal niet voorkomt. Zoo schreef b.v. MORAL (14) in zijne *„Einführung in die Klinik der Zahn- und Mundkrankheiten"* (1928): „Ueberzählige Eckzähne kommen nicht vor".

Sedertdien zijn echter een aantal gevallen beschreven, welke de juistheid van de waarneming van BASTYR op klare wijze in het licht stellen: met name vermelden wij slechts de uitnemend gedocumenteerde verhandelingen van GOTTARDI (15), KOWARSKY (16) en SIMEK. Zij vormen een fraaie illustratie op ADLOFF's reeds eerder gedane uitspraak (17): „Dass im übrigen die überzähligen Zähne nicht immer atavistischen Ursprungs sein können, geht ohne weiteres allein schon daraus hervor, dass auch überzählige Eckzähne vorkommen, für die natürlich Atavismus nicht in Betracht kommen kann" [2]). En wanneer dezelfde schrijver slechts enkele regels verder zijn standpunt ten deze aldus bepaalt: „Es können Abspaltungen oder gar Teilung einer Anlage in einem so frühen Entwicklungsstadium eintreten, dass aus jedem Teil ein normaler Zahn entstehen kann", dan kunnen wij in deze formuleering moeilijk iets anders zien dan de korte samenvatting van de door BOLK in zijne theorie der schizogene variaties gegeven en met klem van argumenten toegelichte interpretatie. Trouwens, reeds eerder had RIHA (1910), eveneens de mogelijkheid van atavisme van de hand wijzende, zich in gelijken zin uitgesproken: „Aus STETTENHEIMER's Zusammenstellung möchte ich noch den einwandfreien Nachweis des Vorkommens von überzähligem Eckzahn herausgreifen: Durch diese Entdeckung gewinnt die vielseits angefochtene, nach meinem Dafürhalten sehr plausible Hypothese von der Möglichkeit einer Zwillingsbildung des Kaninus eine gewichtige Fundierung" (18) [3]).

[1]) Slechts bij enkele fossiele theriodonten welke, gelijk bekend, als de oudste stamvormen der zoogdieren beschouwd worden, komen in de maxilla meerdere hoektanden tot ontwikkeling en doorbraak [BROOM (13, fig. 5 en 7) en ABEL (19, pag. 589)].

[2]) Op. cit. pag. 708.

[3]) Op. cit. pag. 494.

Welnu, wanneer wij, deze zienswijze deelende, van oordeel zijn, dat BOLK's theorie ons ook hier eene afdoende verklaring in handen geeft, dan erkennen wij gaarne, dat onze bewijsvoering ten deele niettemin op eene redeneering per analogiam steunt. Nu is dit alleszins begrijpelijk: vormt verdubbeling van den hoektand als zoodanig reeds een phænomeen van uitzonderlijke zeldzaamheid, in nog hooger mate geldt zulks voor de overgangsvormen, welke zijne volledige verdubbeling a.h.w. moeten voorbereiden. Enkele gevallen nochtans zijn bekend: wij beschreven en beeldden ze reeds in onze eerste studie over dit onderwerp af [4]) en merken daarbij op, dat één ervan op het melkgebit betrekking had. Gelijk bij de snijtanden bleek echter ook hier de uiteindelijke verdubbeling voorafgegaan door reductie resp. aplasie van het middelste kroontuberculum (*P*).

En BENNEJEANT, die zich in zijne uitnemende monographie „*Anomalies et variations dentaires chez les primates*" uitvoerig met dit onderwerp bezighoudt, geeft op pag. 182 (afb. 162) een buitengewoon instructieve afbeelding van aplasie van dit tuberculum bij den melkcuspidatus van *rhinopithecus roxellanna* (MILNE-EDWARDS).

Zoo concludeeren wij ook in dit geval tot verdubbeling (schizodontie) van den hoektand: in die richting wijst trouwens ook de identieke verdubbeling der beide snijtanden. De omstandigheid, dat de orale cuspidatus in al zijne afmetingen eenigszins ten achter blijft bij zijnen vestibulairen naamgenoot, vermag aan deze uitspraak geen afbreuk te doen: ook in het gebied der incisivi blijken dubbeltanden onderling niet zelden in grootte te verschillen!

Samenvatting.

Beschrijving eener gebitsformatie, welker multipele hyperodontie zich daardoor kenmerkte en tevens onderscheidde van de enkele tot nog toe beschreven specimina, *dat in ieder der vier groepen van tanden — derhalve bij incisivi, cuspidati, praemolares en molares — het normale aantal overschreden bleek.*

Résumé.

Description d'une formation de denture, de laquelle la hyperodontie multiple se caractérisa et à la fois se distingua des specimens, décrits jusqu'ici, par le fait, *que dans toutes les quatre groupes de dents — par conséquent chez les incisives, canines, prémolaires et molaires — le nombre normal se révéla d'être excédé.*

Zusammenfassung.

Beschreibung einer Gebissformation, deren multipele Hyperodontie sich dadurch kennzeichnete und sich zugleich unterscheidete von den bisher beschriebenen Spezimina, *dass in jeder der vier Gruppen von Zähnen — folglich bei Inzisiven, Kuspidaten, Praemolaren und Molaren — die normale Zahl überschritten war.*

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[4]) Ib. afb. 3, 4, 5 en 7.

Fig. 1.

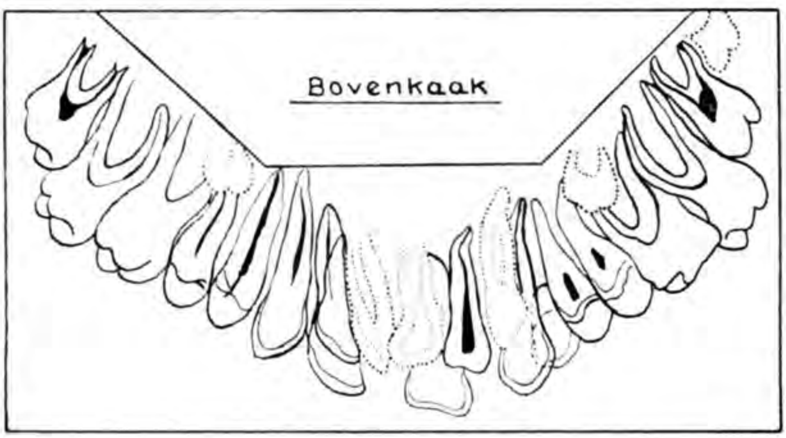

Fig. 2.

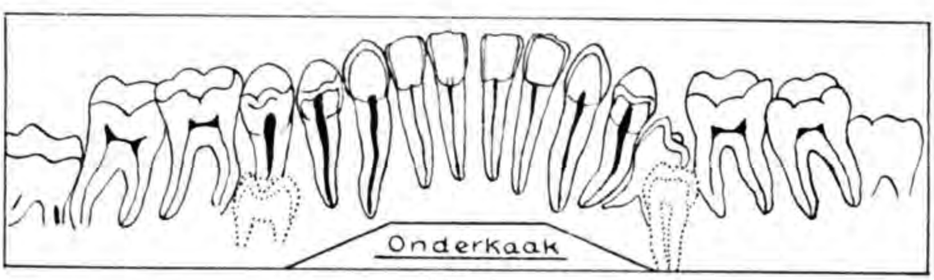

Fig. 3.

Summary.

Description of a teethformation, the multiple hyperodontia of which is characterized and at the same time distinguished from the specimens, described until now by the fact, *that in each of the four groups of teeth — consequently with incisors, canines, premolars and molars — the usual number turned out to be exceeded.*

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Botany. — *Notes on the Oxidation of Oxalic Acid by Mosses.* By C. J. NIEKERK–BLOM. (From the Botanical Laboratory, University of Leiden.) (Communicated by Prof. L. G. M. BAAS BECKING.)

(Communicated at the meeting of June 29, 1946.)

There exists a vast literature on the topic of oxalic-acid formation in plants, the extent of which is inversely proportional to our knowledge of the subject. While there is only little work done on the decomposition of oxalic acid, this work has already yielded many points of great interest, both to biochemists and to botanists and, while much remains to be done, it may be safely said that this field looks most promising.

When we confine ourselves to those oxidations in which atmospheric oxygen is a necessary component of the reaction, there are only ZALESKI [1]), who discovered the aerobic decomposition of oxalic acid as early as 1911 in wheat-flower, BASALIK [2]), who studied bacterial action, STAEHELIN [3]), who worked with higher plants, HOUGET, MAYER and PLANTEFOL [4]) [5]) who worked with mosses and finally the most modern research of FRANKE and his collaborators [6]) [7]), who studied the oxidation in mosses and bacteria as well as in higher plants.

There are at least three possible ways in which oxalic acid may be oxidized:

1) with internal formation of hydrogen peroxide, carbon monoxide is liberated, which may be oxidized by atmospheric oxygen (e.g. in a "Wielend" fashion).
2) Not the hydroxyl is split-off, but the terminal hydrogens are oxidized to H_2O_2, carbon dioxide remaining.
3) The terminal hydrogens are oxidized to water, carbon dioxide remaining.

Bio-oxidation of oxalic acid.

Hydrogen Peroxide formed (RQ = 2) principle thermostable, indifferent to HCN, CO, H_2S and urethane	water formed (RQ = 4) principle thermolabile influenced by HCN, CO, H_2S

less thermostable	extremely thermostable	free enzyme
desmoenzyme	free enzyme	opt. pH 6.0—12.0
opt. pH 4.7—7.0	opt. pH 2.5—3.0	Pseudomonas
higher plants	land-mosses	extorquens
(1, 3, 6, 7)	(4, 5, 6, 7)	(2, 7)

It must be stated at the outset that in the mosses studied in this paper, no carbon monoxide could be detected during the oxidation of oxalic acid, although the sensitive hemoglobin-method was used.

FRANKE [7]) assumes that the agon of the enzymatic agens is metalfree and that the enzymes of higher plants and of mosses differ only in their pheron.

Where this paper is confined to mosses it may be stated that these plants contain an enzymatic principle which is able to oxidize oxalic acid by means of atmospheric oxygen according to:

$$(COOH)_2 \pm O_2 = {}_2CO_2 + H_2O_2.$$

This principle is thermostable to a high degree, deploys maximal activity in acid milieu and is insensitive to heavy metal anticatalyzers. While both HOUGET and FRANKE studied the reaction by means of gasometric methods, it seemed promising to investigate the process by means of chemical analysis. To this end the oxalic was determined by means of precipitation as calcium oxalate and subsequent titration with permanganate, according to KRAMER and TISDALE [8]) while hydrogen peroxide was determined, also titrimetrically, by means of thiosulphate. While the results are preliminary, they indicate promising lines for future approach.

While FRANKE and collaborators worked with purified enzyme preparations, it seemed interesting to observe the combined action of the various enzymatic principles in macerated mosses.

As the oxalic acid oxydase in mosses is thermostable to a high degree, the catalase action could be excluded by "blanching" (submersion in boiling water) for between 8 and 10 minutes.

Enzyme solutions of sunflower leaves were also prepared, but as the oxalic acid oxydase is much less thermostable in this case, the experiments could only be carried out with unblanched preparations, in which the catalase activity remains unimpaired. Although it seems probable that no peroxide was formed in the sunflower preparations this point could not be proved. It has to be stated that the experiments were carried out under German occupation in the year 1944 when conditions, due to the absence of heating facilities, gas and, finally, electricity became increasing primitive and adverse, so that further experiments had to be stopped in November.

Polytrichum commune and, later Brachythecium rutabulum were used, in portions of 3—6 grammes to 100 cc liquid. The mosses were cut into very small parts prior to experimentation. Sufficient aeration was maintained by means of an air current. A temperature of about 34° C was maintained throughout.

1. *Oxalic acid concentration.*

While FRANKE c.s. found, with Hylocomium triquetrum, an optimal oxalic acid concentration of 0.02 N and HOUGET c.s. used 0.04 N with a variety of mosses in our experience 0.01 N oxalic acid showed optimal reaction.

2. *Temperature.*

With blanched and unblanched preparations an optimum action was observed at a little over 40° C (see figure 1).

Here the times needed for 50 % decomposition of 100 cc 0.01 N oxalic acid solution are given. FRANKE c.s. found that up to 60° C the enzymatic action is unimpaired in Hylocomium triquetum.

Brachythecium rutabulum: time needed for 50 % decomposition in 0.01 N oxalic acid at:

	unblanched	blanched
30° C	79 minutes	56 minutes
40° C	21 ,,	6 ,,
50° C	51 ,,	27 ,,

3. H_2O_2.

In a solution of 0.0125 N. H_2O_2 60 % of the oxalic acid was decomposed in 30 minutes at 34° C, while in 0.025 N. H_2O_2 the enzymatic action was stopped after 30 minutes (Brach. rect.)

4. *pH.*

Phosphate buffers were used throughout. Figure 2 and the following table show the result. In an agreement with FRANKE c.s. optimal action

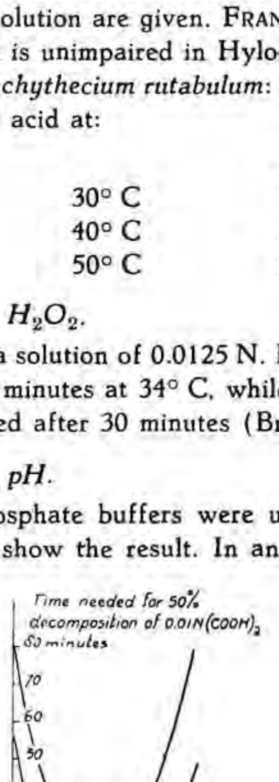

Figure 1

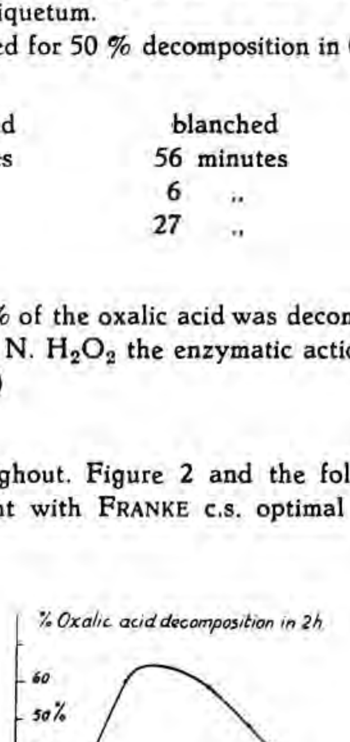

Figure 2

occurs at pH 3 (Brach. rect.) but as in their experiments the enzymatic action stops at pH 5, our enzyme remains active at pH 7.3.

Brachythecium rutabulum 0.01 oxalic acid

phosphate buffer % decomposition.

pH	2.3	2.9	6.2	7.3
15′	6	12	28	25
30′	28	—	30	—
60′	38	27	43	18
120′	52	71	60	49

at pH 7.3 no H_2O_2 was formed.

5. *The catalase* activity was investigated as a function of pH. The process followed a monomolecular reaction and the reaction constant:

$$\frac{1}{t}\,\frac{a}{a-x} \text{ was calculated (Figure 3)}$$

pH	K	(Polytrichum commune)
2.2	5 10^{-5}	
3.8	7.3 ,,	
5.4	8.0 ,,	
7.0	62 ,,	
8.0	65 ,,	

The curious fact remains that the catalase activity continues, with about one tenth of its activity, in solution of pH < 5.

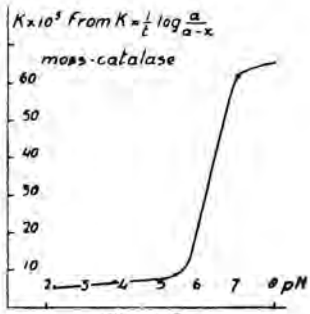

Figure 3

6. *0.01 KCN* had little effect on the enzymatic action of Brachytecium as in one hour 45 % of the oxalic acid was decomposed. No H_2O_2 was formed, however.

7. *Aeration* seems to be limiting factor, as soon as this because a minimum, the hydrogen peroxide formation was stopped, while the oxalic acid decomposition continued.

8. *Balance.* In the preparation without catalase (blanched) the decomposition of oxalic acid and the formation of peroxide and of CO_2 was determined simultaneously under optimal condition (pH 3,34° C 0.01 N oxalic acid) it seems that the equation $(COOH)_2 + O_2 = H_2O_2 + 2CO_2$ is fulfilled.

Equivalent amounts of thiosulphate and of potassium permanganate used by 6 grammes Brachythecium rutabulum in 100 cc 0.01 N oxalic acid, well aerated.

	Time	Thio	K-perm.	date
1)	5h	3.39	3.68	10–10–44
2)	12h	3.85	3.73	17–10–44
3)	4h	3.20	3.35	19–10–44

An average of 96.6 % of the H_2O_2, formed theoretically, was found back, which is well within the error of the experiment (5 %). The required amounts of CO_2 were found back (weighing in KOH solution) e.g. in experiment 1) found 129 mgr CO_2, calculated 128.2 mgr CO_2.

The law for monomolecular reaction was followed within the limits of the experiment. In Exp. 1) (10–10–44) we found $K = \frac{1}{t} \log \frac{a}{a-x}$

1h 3.5 10^{-3}; 2h 3.0 10^{-3}; 3h 3.3 10^{-3}; 4h 2.7 10^{-3}; 5h 3.0 10^{-3}.

As FRANKE's work was performed with purified enzyme solution his results cannot be directly compared to the data given above. Apparent contradictions may be caused by our use of the macerated organisms rather than enzyme preparations. A few outstanding facts may be mentioned, however.

1) Contrary to the data mentioned in the literature the catalase activity in mosses persists in very acid milieu (pH 2—3).

2) Oxalic acid decomposition in mosses has its optimum at pH 3.0. If accumulation of noxious peroxide within the cell has to be avoided it stands to reason that a catalase should be able to deploy its activity in the same range of H^+ concentration. The possibility remains, however, that the oxalic acid oxidase is a "vacuole" enzyme and that the peroxide formed permeates into the protoplasm through the tonoplast to be decomposed by the catalase, a "protoplasmic" enzyme.

3) Under optimal condition the oxalic acid is decomposed according to $(COOH)_2 + O_2 = H_2O_2 + 2\,CO_2$: As soon as conditions are adverse (0.01 N. KCN, insufficient aeration or pH 7.3), no H_2O_2 could be detected, while the rate of acid-decomposition is diminished. As no carbon monoxide is formed, the reaction might proceed as: $2(COOH)_2 + O_2 = 2\,H_2O +$ $+ 4\,CO_2$, which reaction also seems to hold in the case of Pseudomonas extorquens BASSALIK, a curious organism which BASSALIK [2]) isolated from rainworm-excrement. In the experiment described above it was repeatedly found that, under non-optimal conditions, the decomposition followed a reaction-scheme of a higher order (bi-trimolecular), which seems to substantiate the above suggestion.

The consequences of this preliminary work shall not be discussed here as more experiment is needed for the further elucidation of the phenomena described.

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Chemistry. — *The formation of two-dimensional nuclei by ionic molecules.* By J. H. DE BOER.

(Communicated at the meeting of November 30, 1946.)

§ 1. The growth of ionic crystals from solutions has been a subject of many investigations, both from the experimental and from the theoretical angle [1]). The importance of the differences of the potential energies of addition of ions to different positions on the various crystallographic planes has been very clearly pointed out by all authors. The addition of ions to edges or corners is favoured energetically, mainly because the electrostatic contributions to the total amount of the addition energies are, in these cases, far more important than the contributions by the VAN DER WAALS' forces. In practically all these cases more attention was given to the differences in rate of growth of crystallographic planes of already existing nuclei, or of small crystals which had already been formed beforehand, than to the formation of the nuclei themselves.

The present study deals with the formation of nuclei and of very small — sub-microscopical — crystals from the vapour phase. There are, therefore, some very important differences with respect to the investigations mentioned above. Firstly the formation of the nuclei and their growth do not occur from the ionic, but from the molecular state. Secondly neither the constituents nor the surfaces of the forming crystals are contaminated with molecules of a solvent. It simplifies the problem considerably, from an experimental as well as from a theoretical point of view, that solvation-energies need not be taken into account.

§ 2. When a salt like NaCl is evaporated in vacuo under such conditions that no collisions of salt molecules take place before the molecules reach the wall of the vessel, the question arises which structures will result from the agglomeration of these molecules on the wall and how the nuclei will develop when more molecules impinge upon them.

We will take NaCl as an example for our theoretical considerations, but it will be clear that, apart from the numerical values, the results which we obtain will hold good for similar compounds, as e.g. all alkali- and alkaline earth halides.

We may consider a molecule of NaCl as an ionic molecule [2]). The

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[1]) For a survey of the work of ARTEMJEFF, VALETON, KOSSEL, STRANSKI, SCHNORR, NEUHAUS and others, see A. E. VAN ARKEL and J. H. DE BOER, Chemische Binding, 1930, or the reviewed French translation: "La Valence et l'Electrostatique", Paris, Félix Alcan, 1936, pages 284—297.

[2]) E. J. W. VERWEY and J. H. DE BOER, Rec. Trav. Chim., 59, 633 (1940).

distance between Na^+ and Cl^- in the molecule is 2.51×10^{-8} cm at 1200° K [3]) and 2.47×10^{-8} cm at 0° K [2]). In solid NaCl the distance between two neighbouring ions is greater, viz. 2.814×10^{-8} cm at room temperature ($2.79^5 \times 10^{-8}$ cm at 0° K). In the following considerations we are dealing with agglomerates of 2, 3 or 4 molecules; the interionic distances will, therefore, have values between those in the molecules and those in the crystal. For the sake of convenience we will insert in our calculations only one invariable value, viz: $d = 2.5 \times 10^{-8}$ cm for the distance between two neighbouring ions in the single molecules as well as in the agglomerates.

In all calculations the electrostatic part of the energy has been calculated by straightforward summation of all attractive and repulsive parts. As it has been proved previously [2]) that the short range repulsion resulting from the interpenetration of the reciprocal electronic clouds may best be expressed by the term $\frac{b}{d^n}$, where $n = 12$, the electrostatic energy has been corrected for this repulsion just by subtracting $^1/_{12}$ part from the value obtained by summation.

The contributions to the potential energy resulting from the VAN DER WAALS' attraction forces have been evaluated from the dipole-dipole parts of these forces only, thus using only the terms $\frac{c}{d^6}$. The other contributing terms have been neglected, because they are practically counter-balanced by the part resulting from the repulsive forces [4]). Zero-vibration energy, polarisation energy and dipole interaction have also been neglected because, although they may have some influence on the numerical values, their contributions do not alter the importance of the qualitative results which are obtained from the following considerations; they would moreover only give contributions in such a way as to assist us to make the deductions. The following figures [5]) have been used in the evaluation of the VAN DER WAALS' forces:

$C_{++} = 1.68 \times 10^{-60}$ erg $\times$ cm^6 (action of Na^+ on another Na^+),
$C_{+-} = 11.2 \times 10^{-60}$ erg $\times$ cm^6 (action of a Na^+ on a Cl^-) and
$C_{--} = 116 \times 10^{-60}$ erg $\times$ cm^6 (action of a Cl^- on another Cl^-).

§ 3. When two molecules of NaCl (combination A in fig. 1) form an agglomerate the VAN DER WAALS' forces would tend to form a regular tetrahedron B. The energy of addition of the two molecules, however, would be small as in this configuration the electrostatic terms are just in balance. The VAN DER WAALS' forces lead to an energy of combination of

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3) L. R. MAXWELL, S. B. HENDRICKS and V. M. MOSLEY, Phys. Rev., 52, 968 (1937).
4) J. H. DE BOER, Trans. Far. Soc., 32, 10 (1936).
5) J. E. MAYER, J. Chem. Physics, 1, 270 (1933).

about -0.6×10^{-12} erg per double-molecule. Configuration C is favoured by the electrostatic forces and the total contribution of these and of the VAN DER WAALS' forces leads to an energy of combination of -5.0×10^{-12}

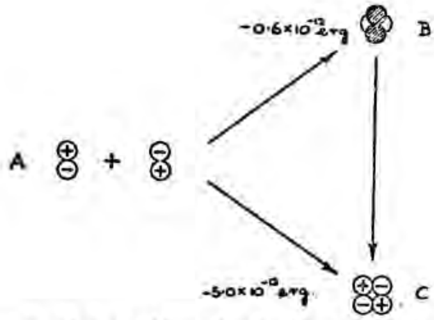

Fig. 1. Formation of a double-molecule from two single molecules.

erg per double molecule. C is by far the most stable combination, B is unstable and would always convert into C. It may be remarked that in a combination like C the individual character of the original molecules has already disappeared.

When a third molecule is added to this structure (fig. 2) a similar consideration leads to the conclusion that configuration B is unstable and is easily converted into C, which in turn is also unstable and passes into structure D, which is the only stable one. The resulting configuration, therefore, is a two-dimensional one in which all constituent ions are situated in the same plane.

§ 4. The addition of a fourth molecule to the stable configuration of D in fig. 2 represents a far more interesting case. In fig. 3 the relation between

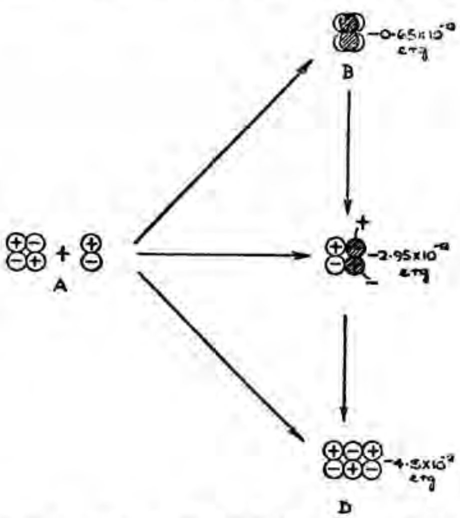

Fig. 2. Addition of a third molecule to the stable configuration of a double-molecule. (In all figures shaded circles mean ions situated on top of the first layer.)

all possible configurations which may form with a decrease of potential energy is given. The energy figures represent the energy of addition of the fourth molecule to the existing combination of three.

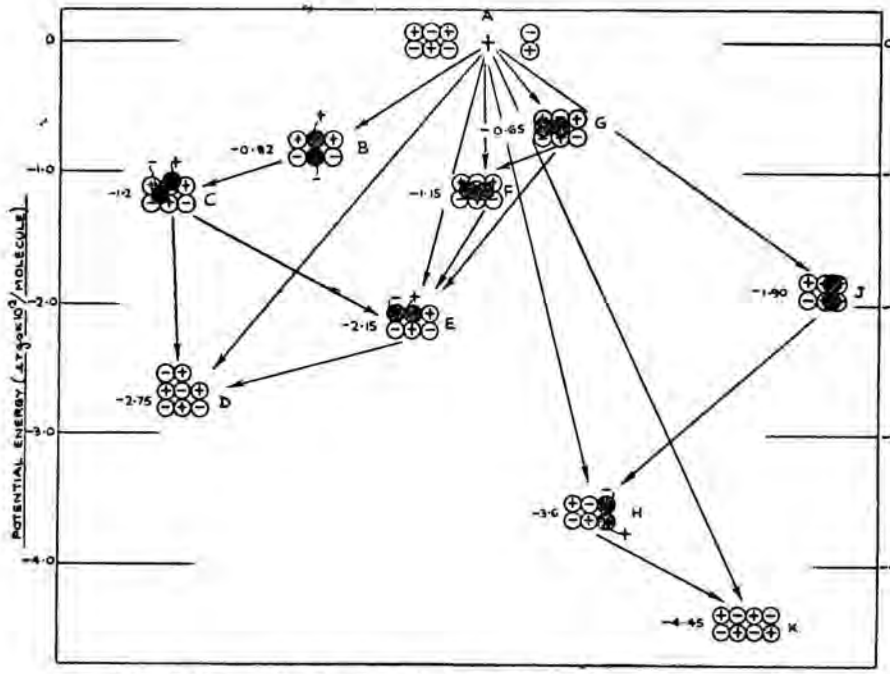

Fig. 3. Addition of a fourth molecule to the stable configuration of three.

Configuration *B* is quite stable with respect to sideward movements of the fourth molecule which is situated on top of the middle one of the three original ones. The orientation of this fourth molecule is such that its negative ion is on top of the positive ion of the underlying one and its positive ion on top of the negative ion of the lower molecule. This configuration, therefore, cannot be converted into *J*, where the orientation of the fourth molecule is just reversed.

Although configuration *B* is unstable with respect to configuration *D*, it would not be easy to convert *B* into *D* if there were no VAN DER WAALS' forces present. If only electrostatic forces were acting a considerable amount of activation energy would be necessary for this conversion. By the action of the VAN DER WAALS' forces, however, the negative ion of the fourth (top) molecule is easily shifted to a position on four direct neighbours, two of which are also negative ions. The electrostatic energy of this ion with respect to the whole underlying structure is zero; the VAN DER WAALS' energy, however, is large. The positive ion of the top molecule has to be pushed up a little, which results in a decrease of the electrostatic energy, which however is still negative.

A direct shift from *B* into *F* is not excluded, but both *C* and *F* are unstable and are ultimately converted into *D*.

The relation between J, H and K is similar to the relation of the configurations of fig. 2.

From all the configurations mentioned in fig 3, therefore, all but two are unstable and converted into others. Only configurations D and K are not converted into others. Both D and K are characterised by the fact that all constituent ions are situated in one single plane. These nuclei are again two-dimensional nuclei.

§ 5. D and K, however, are not the most stable configurations. The stable configuration to be formed from four of these ionic molecules (eight ions) is the regular cube. Starting from configuration K a cube could be formed by moving the end-molecules of the row of four round until they meet on top of the middle two, as is indicated in fig. 4. This figure gives the change in the electrostatic part of the potential energy when both end-

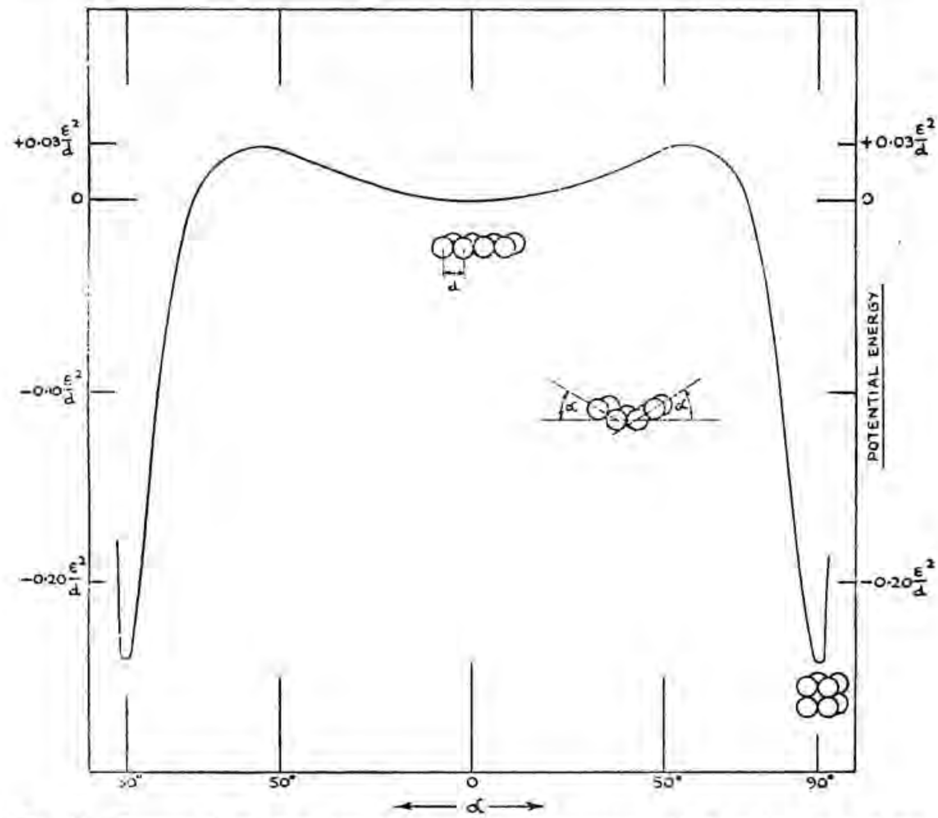

Fig. 4. Energetical relation between the flat configuration of 4 molecules and the cubical one; potential energy as a function of the angle α.

molecules are turned simultaneously over an angle α as is indicated in the figure. Obviously configuration K is not an unstable one, but only meta stable with respect to the cubic configuration ($\alpha = 90°$). An activation energy of $0.03 \frac{\varepsilon^2}{d}$ is required to transform configuration K into the cubical one. In our example of NaCl, with $d = 2.5 \times 10^{-8}$ cm this means 0.275×10^{-12} erg, a value nearly seven times the value of kT at room temperature, viz. $1.38 \times 10^{-16} \times 290 = 0.04 \times 10^{-12}$ erg. It will be easily seen that if the two end-molecules are not turned simultaneously the activation energy will be higher still. It will also be obvious that the VAN DER WAALS' contribution to the potential energy will not alter the result appreciably. Consequently the result of these considerations is that configuration K, though not the most stable form of this agglomerate of four molecules, does not easily transform into the stable cubical configuration. It may be remarked that the reciprocal relation of these two configurations has some resemblance to steric isomers in organic chemistry.

The transformation of D into K or into the cubical form is also not possible without passing an energy maximum of activation energy. The ultimate result of all these considerations is, therefore, that the addition of a fourth molecule to the combination of three also leads to a configuration with all molecules in the same plane. In other words the two-dimensional nucleus grows only two-dimensionally.

§ 6. Similar results will be obtained with the continuation of this process, hence in the further addition of molecules there is a strong tendency of the nucleus to grow in two dimensions only. Molecules which impinge upon the two-dimensional plane will migrate over this plane until they reach the edge, whereupon they will turn over the edge and contribute to the growth of the two-dimensional structure. It may, however, be expected that at higher temperatures these two-dimensional structures will transform into more stable three-dimensional cubical sub-microscopic crystals.

§ 7. In all our considerations so far we have dealt with ionic molecules and it is largely due to the electrostatic repulsion forces that the formation of two-dimensional structures will be favoured. When the vapour molecules do not consist of ions but of atoms bound together by homopolar forces we will get another picture. Even in the case of metallic oxides, which in the solid state may give ionic crystals but the molecules of which in the gaseous state are to be considered as atomic molecules (with a small dipole), we may expect a different behaviour. In these cases, the VAN DER WAALS' forces will largely govern the formation of the nuclei and consequently compact structures will result.

§ 8. The results obtained above give an excellent explanation of the experimental facts. Extensive studies of adsorption phenomena of different substances on salt films, obtained by sublimation in a high vacuum, revealed that these salt layers have a lamellar structure and consequently a highly developed surface [6]). The crystalline lamellae have an average thickness of only a few molecules. These results were completely confirmed by electron diffraction measurements on films of CaF_2 obtained by sublimation in a high vacuum. The molecules form flat plates perpendicular to the vapour beam and parallel to the under layer [7]).

This behaviour is shown by all alkali-halides and alkaline earth halides, as well as by several other fluorides, such as lead fluoride, or by complex fluorides such as potassium zirconium fluoride ($K_2\,Zr\,F_6$). In all these cases lamellar structures are obtained when the salts are deposited on a cool wall (room temperature) by sublimation in a high vacuum.

When, however, substances such as $Si\,O_2$, $Al_2\,O_3$, $Zr\,O_2$ or several other oxides, or salts like the silver halides are sublimated in a similar way, only compact layers are obtained which show no signs of formation of two-dimensional lamellae and show no capacity to adsorb substances by VAN DER WAALS' forces [8]). It is known that these latter substances, though they may form ionic crystals in the solid state (Ag Cl) are nevertheless to be considered as atomic-molecules in the gaseous state.

Obviously the experimental results are in harmony with the theoretical considerations, as described above, and it may be assumed that by sublimation in a high vacuum those substances which have ionic molecules in the gaseous state form two-dimensional deposits with a highly developed lamellar surface structure, showing a well-developed capacity for adsorption by VAN DER WAALS' forces. Substances with atomic molecules in the gaseous state, however, under similar conditions form compact layers, on which only electrostatic adsorption on active spots will be possible.

§ 9. One of the experimental conditions for the formation of these films is a high vacuum, such that no collisions take place before the molecules reach the wall. If the sublimation takes place in a diluted inert atmosphere, agglomeration takes place in the gas phase and a loosely built layer is formed [8]).

When the lamellar structure is heated a sintering takes place. In this sintering process the lamellae are only bound together by VAN DER WAALS' forces. Adsorption by suitable substances, such as caesium, gives a seperation of these layers again, restoring the original surface structure. This process is comparable with the swelling of other lamellar structures such as the swelling of graphite [9]). It is only at still higher temperatures (depending on

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[6]) J. H. DE BOER, Z. physikal. Chem., B **13**, 134 (1931); B **14**, 149 (1931); J. H. DE BOER and C. J. DIPPEL, Z. physikal. Chem., B **21**, 198 (1933); C. J. DIPPEL and J. H. DE BOER, Rec. Trav. Chim. **57**, 277 (1938); See also J. H. DE BOER, "Electron Emission and Adsorption Phenomena", Cambridge 1935, p. 184 et seq.

[7]) W. G. BURGERS and C. J. DIPPEL, Physica, **1**, 549 (1934).

[8]) J. H. DE BOER and J. F. H. CUSTERS, Physica, **4**, 1017 (1937).

[9]) C. J. DIPPEL and J. H. DE BOER, Rec. Trav. Chim., **57**, 277 (1938).

the salt) and when favoured by the presence of water vapour that a real recrystallisation to minute cubical crystals sets in. In this latter process the films lose their transparency and become opaque.

§ 10. It may be remarked that several metals also tend to form films of a character similar to that described above as being characteristic of substances with ionic molecules [10]). The well-known experiment of VOLMER and ESTERMANN [11]) with mercury may be mentioned here. These authors found that mercury atoms migrated over the surface of the hexagonal basic plane and contributed to the two-dimensional development of these planes, giving rise to the formation of a lamellar structure.

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13th November 1946.

[10]) Cf. e.g. V. VAND, Proc. Phys. Soc., **55**, 222 (1943); O. E. BEECK, A. E. SMITH and A. WHEELER, Proc. Roy. Soc., A **177**, 62 (1940); D. D. ELEY, Nature, **158**, 449 (1946).

[11]) M. VOLMER and I. ESTERMANN, Z. Physik., **7**, 13 (1921).

Mathematics. — *Determinants and quadratic forms.* (Second communication.) By J. G. VAN DER CORPUT and H. J. A. DUPARC.

(Communicated at the meeting of November 30, 1946.)

If a quadratic form $Q(u_1, \ldots, u_n)$ is negative definite, then

$$Q \leqq \frac{D}{D_k} u_k^2,$$

for $-Q$ is positive definite and if Q is replaced by $-Q$, then $\frac{D}{D_k}$ becomes $-\frac{D}{D_k}$.

If $a < 2$ and a is not equal to an integer, then the quadratic form

$$Q_2 = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{u_r u_s}{\Gamma(a-r-s)}$$

is positive definite or negative definite according to whether $[a]$ is even or odd.[1]) Hence

$$Q_2 \geqq \frac{D}{D_t} u_t^2 \text{ or } \leqq \frac{D}{D_t} u_t^2 \qquad (t = 1, \ldots, n)$$

according to whether $[a]$ is even or odd.

By formulae (8) and (9), applied with

$$x_r = r; \;\; y_r = a - 1 - r; \;\; \varphi_r(y) = 1; \;\; a_r = 1; \;\; e_{rs} = \frac{1}{\Gamma(a-r-s)}$$
$$(r = 1, \ldots, n; s = 1, \ldots, n),$$

we obtain

$$\frac{D_1}{D} = \frac{\dfrac{q(x_1)}{x_1 - y_1}}{e_{11} q'(y_1)} = (-1)^{n-1} \frac{\Gamma(a-2)\,\Gamma(a-3)}{\Gamma(a-n-2)\,\Gamma(a)},$$

$$\frac{D_2}{D} = \frac{\dfrac{q(x_2)}{x_2 - y_2} - \dfrac{q(x_1)}{x_1 - y_2}}{e_{21}(x_2 - x_1)\, q'(y_2)} = (-1)^{n-1} \frac{\Gamma(a-5)\,\Gamma(a-3)}{\Gamma(n-1)\,\Gamma(a-n-2)} \{(n-1)a - (4n-3)\}$$

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[1]) Confer C. P., Theorem 5, p. 626.

and by formula (10)

$$\frac{D_{n-1}}{D} = \frac{p(y_{n-1})}{e_{n-1,1}\, q'(y_{n-1})} = (-1)^{n-2} \frac{(n^2 - n\,\alpha + \alpha)\, \Gamma(\alpha - n)}{\Gamma(n-1)}$$

and

$$\frac{D_n}{D} = \frac{1}{e_{n1}\, q'(y_n)} = (-1)^{n-1} \frac{\Gamma(\alpha - n - 1)}{\Gamma(n)}.$$

Hence

$$Q_2 \geqq \text{or} \leqq (-1)^{n-1} \frac{\Gamma(\alpha - n - 2)\, \Gamma(n)}{\Gamma(\alpha - 3)\, \Gamma(\alpha - 2)}\, u_1^2$$

$$\geqq \text{or} \leqq (-1)^{n-1} \frac{\Gamma(\alpha - n - 2)\, \Gamma(n-1)}{\Gamma(\alpha - 5)\, \Gamma(\alpha - 3)} \frac{u_2^2}{(n-1)\,\alpha - (4n-3)}$$

$$\geqq \text{or} \leqq (-1)^{n-2} \frac{\Gamma(n-1)}{\Gamma(\alpha - n)} \frac{u_{n-1}^2}{n^2 - n\,\alpha + \alpha}$$

$$\geqq \text{or} \leqq (-1)^{n-1} \frac{\Gamma(n)}{\Gamma(\alpha - n - 1)}\, u_n^2 \quad ^{2)}$$

according to whether $[\alpha]$ is even or odd.

Applying these results on the quadratic form

$$Q_3 = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{u_r\, u_s}{\binom{2\alpha - r - s}{\alpha - r}}$$

we find, if $2\alpha < 1$ and if 2α is not equal to an integer,

$$Q_3 \geqq \text{or} \leqq (-1)^{n-1} \frac{u_1^2}{\binom{2\alpha - 3}{n-1} \binom{2\alpha - 2}{\alpha - 1}}$$

$$\geqq \text{or} \leqq (-1)^{n-1} \frac{1}{\binom{2\alpha - 4}{2\alpha - n - 2} \binom{2\alpha - 4}{\alpha - 2}} \frac{2\alpha - 4}{2\alpha - 3} \frac{u_2^2}{2\alpha(n-1) - (3n-2)}$$

$$\geqq \text{or} \leqq (-1)^{n-1} \frac{1}{\binom{2\alpha - 2n + 2}{\alpha - n + 1} \binom{2\alpha - n}{2\alpha - 2n + 2}} \frac{u_{n-1}^2}{2\alpha(n-1) - (n^2 - n + 1)}$$

$$\geqq \text{or} \leqq (-1)^{n-1} \frac{u_n^2}{\binom{2\alpha - 2n}{\alpha - n} \binom{2\alpha - n - 1}{2\alpha - 2n}}$$

according to whether $[2\alpha]$ is odd or even.

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[2]) Confer C. P., Theorem 5, p. 626.

If $a < \frac{1}{2}$ and a is no integer, then from C. P. theorem 8 page 767 we know

$$Q_4 = \sum_{r=1}^{n} \sum_{s=1}^{n} \Gamma(a+r-s)\, \Gamma(a-r+s)\, u_r\, u_s$$

to be positive definite. Hence

$$Q_4 \geqq \frac{D}{D_t}\, u_t^2 \qquad (t=1, \ldots, n).$$

By formulae (8) and (9), applied with

$$y_r = a-r;\; x_r = -r;\; \varphi_r(y) = \frac{\Gamma(2a-y-r)}{\Gamma(2a-y-n)};\; a_r = 1;\; e_{rs} = \Gamma(a-r+s)\Gamma(a+r-n)$$
$$(r=1, \ldots, n;\; s=1, \ldots, n),$$

we obtain

$$\frac{D_1}{D} = \frac{\dfrac{q(x_1)}{x_1-y_1}}{e_{11}\, \varphi_1(x_1)\, q'(y_1)} = (-1)^{n-1} \frac{\Gamma(2a-n+1)}{\Gamma(n)\, \Gamma(2a)\, \Gamma^2(a-n+1)}$$

and

$$\frac{D_2}{D} = \frac{\varphi_1(x_1)\dfrac{q(x_2)}{x_2-y_2} - \varphi_1(x_2)\dfrac{q(x_1)}{x_1-y_2}}{e_{21}\, \varphi_1(x_1)\, \varphi_2(x_2)\, (x_2-x_1)\, q'(y_2)}$$
$$= (-1)^{n-1} \frac{\Gamma(2a-n+1)\{(n-1)a^2-2a+n-1\}}{\Gamma(n-1)\, \Gamma(2a)\, \Gamma^2(a-n+2)}.$$

Hence

$$Q_4 \geqq (-1)^{n-1} \frac{\Gamma(n)\, \Gamma(2a)\, \Gamma^2(a-n+1)}{\Gamma(2a-n+1)}\, u_1^2$$

and

$$\geqq (-1)^{n-1} \frac{\Gamma(n-1)\, \Gamma(2a)\, \Gamma^2(a-n+2)\, u_2^2}{\Gamma(2a-n+1)\{(n-1)a^2-2a+n-1\}}.$$

Applying these results on the quadratic form

$$Q_5 = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{u_r\, u_s}{\dbinom{2a}{a+r-s}}$$

we find, if $2a$ is not an integer and $2a < -1$

$$Q_5 \geqq \text{or} \leqq (-1)^{n-1} \frac{2a+1}{a+1} \frac{u_1^2}{\dbinom{2a-n+2}{a-n+1}\dbinom{a}{a-n+1}}$$

and

$$\geqq \text{or} \leqq (-1)^{n-1} \frac{2a+1}{(n-1)a^2+2a(n-2)+2(n-2)} \frac{u_2^2}{\dbinom{2a-n+2}{a-n+2}\dbinom{a}{a-n+2}}$$

according to whether $[2a]$ is odd or even.

If $a > \frac{1}{2}$ is no integer, then from C. P. theorem 12 p. 775 we know

$$Q_6 = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{u_r u_s}{\Gamma(a+r-s)\,\Gamma(a-r+s)}$$

to be definite positive. Hence

$$Q_6 \geqq \frac{D}{D_t} u_t^2 \qquad (t = 1, \ldots, n).$$

By formulae (8) and (9), applied with

$$x_r = r;\ y_r = a+r-1;\ \varphi_r(y) = \frac{\Gamma(2a+n-y-1)}{\Gamma(2a+r-y-1)};\ a_r = 1;\ e_{rs} = \frac{1}{\Gamma(a+r-s)\,\Gamma(a-r+n)}$$

$$(r = 1, \ldots, n;\ s = 1, \ldots, n),$$

we obtain

$$\frac{D_1}{D} = \frac{\dfrac{q(x_1)}{x_1 - y_1}}{e_{11}\,\varphi_1(x_1)\,q'(y_1)} = \frac{\Gamma(2a-1)\,\Gamma^2(a+n-1)}{\Gamma(2a+n-2)\,\Gamma(n)}$$

and

$$\frac{D_2}{D} = \frac{\varphi_1(x_1)\dfrac{q(x_2)}{x_2 - y_2} - \varphi_1(x_2)\dfrac{q(x_1)}{x_1 - y_2}}{e_{21}\,\varphi_1(x_1)\,\varphi_2(x_2)\,(x_2 - x_1)\,q'(y_2)}$$

$$= \frac{\Gamma(2a-1)\,\Gamma^2(a+n-2)\,\{a^2(n-1) - 2a(n-2) + 2(n-2)\}}{\Gamma(2a+n-2)\,\Gamma(n-1)}.$$

Hence

$$Q_6 \geqq \frac{\Gamma(2a+n-2)\,\Gamma(n)}{\Gamma(2a-1)\,\Gamma^2(a+n-1)}\, u_1^2$$

and

$$\geqq \frac{\Gamma(2a+n-2)\,\Gamma(n-1)}{\Gamma(2a-1)\,\Gamma^2(a+n-2)}\, \frac{u_2^2}{a^2(n-1) - 2a(n-2) + 2(n-2)}.$$

Applying these results on the quadratic form

$$Q_7 = \sum_{r=1}^{n} \sum_{s=1}^{n} \binom{2a}{a+r-s} u_r u_s$$

we find, if $a > -\frac{1}{2}$ is not an integer,

$$Q_7 \geqq \frac{\dbinom{2a+n-1}{a+n-1} u_1^2}{\dbinom{a+n-1}{a}}$$

and

$$\geqq \frac{\dbinom{2a+n-1}{a+n-2}}{\dbinom{a+n-2}{a}} \frac{(a+1)\,u_2^2}{a^2(n-1) + 2a + n - 1}.$$

Other inequalities occurring in the mentioned paper of C. P. may be generalized in this manner, but the results become complicated and not interesting.

Geophysics. — *On the propagation of seismic waves.* I. By J. G. SCHOLTE.
(Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of November 30, 1946.)

§ 1. *Introduction.*

In 1904 LAMB [1]) published his well-known paper on the propagation of tremors along the surface of a homogeneous solid body; in this paper the vibrations registered by a seismograph are calculated, assuming these vibrations to be excited by a variable normal pressure at a point of the surface. In such a body no dispersion can occur as there is no finite length with which the wave length can be compared. The various dispersive features shown by seismograms could not be explained by this theory and LAMB therefore added the remark that the theory of propagation of seismic waves should be extended to an inhomogeneous medium.

Some years afterwards LOVE [2]), followed in later years by many other geophysicists [3]) [4]), investigated the effect of a surface layer on the phase velocity of these waves. This research dealt with harmonic vibrations existing during an infinite long time ("stationnary waves"), with amplitudes which depend only on the distance to the free surface of the layer. The pressure distribution associated with these wave systems has obviously the same functional character and is therefore completely different from the actual conditions existing during an earthquake, as seismic waves are caused by the sudden application of a normal or shearing stress (or both) in some small part of the medium during a very short time. The propagation of these waves can only be investigated by taking into account both the stratification of the Earth and the pressure distribution at the time of their origin.

Many questions are closely connected with this problem; for instance the theoretical basis of the seismic determination of geological strata as well as its better application rest upon its solution.

In this paper an attempt has been made to calculate the movement of the free surface of a mono-stratified body caused by a normal pressure at a small region of this surface. A generalization of this investigation is easy to obtain; no new mathematical difficulty arises if we assume the existence of a double layer or that the hearth is not situated in the upper surface but at a certain depth.

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1) K. LAMB, Phil. Trans. A **203**, 1—42 (1904).

2) A. LOVE, Some problems of geodynamics.

3) K. SEZAWA and K. KANAI, Bull. Earthq. Res. Inst. **18**, 1—9 (1940).

4) H. JEFFREYS, M.N.R.A.S., Geoph. Suppl. **4**, 571—578 (1939).

§ 2. *Waves excited by a normal pressure.*

Consider the vibrations of a layer of finite and uniform thickness d, resting on a semi-infinite body: both layer and underlying medium are supposed to be homogeneous. Starting with the most simple case we assume that these vibrations are caused by a normal pressure exerted on the whole surface ($z = 0$) of the layer and that this pressure can be expressed by

$$T_{zz} = Z \,.\, e^{i(\nu t - \xi x)}, \quad \ldots \ldots \ldots \quad (1)$$

where x and y are rectangular coordinates in the plane $z = 0$, and Z, ν and ξ are constants.

The waves existing in the layer are then ($z < d$)

two longitudinal waves:

$$A_1\, e^{i(\nu t - hx \sin\alpha - hz\cos\alpha)} \text{ and } A_2\, e^{i(\nu t - hx\sin\alpha + hz\cos\alpha)}$$

two transversal waves:

$$B_1\, e^{i(\nu t - kx\sin\beta - kz\cos\beta)} \text{ and } B_2\, e^{i(\nu t - kx\sin\beta + kz\cos\beta)}$$

where

$$h = \frac{\nu}{V},\ k = \frac{\nu}{\mathfrak{V}},\ \text{with } V = \sqrt{\frac{\lambda + 2\mu}{\varrho}} \text{ and } \mathfrak{V} = \sqrt{\frac{\mu}{\varrho}};\ h\sin\alpha = k\sin\beta = \xi.$$

In the subjacent medium the waves are ($z > d$)

$$A'\, e^{i(\nu t - h'x\sin\alpha' - h'z\cos\alpha')} \text{ and } B'\, e^{i(\nu t - k'x\sin\beta' - k'z\cos\beta')};$$

quantities connected with this medium are denoted by an accent.

The amplitudes A and B are determined by the boundary conditions: in the first place the normal pressure at $z = 0$ has to be equal to T_{zz}, or

$$-ik\mu\,\{A_1\, n\cos 2\beta + A_2\, n\cos 2\beta - B_1\sin 2\beta - B_2\sin 2\beta\} = Z, \text{ with } n = V/\mathfrak{V};$$

secondly the tangential pressure at $z = 0$ is equal to zero:

$$A_1\sin 2\alpha - A_2\sin 2\alpha + B_1\, n\cos 2\beta - B_2\, n\cos 2\beta = 0.$$

At the boundary between the two media the horizontal and vertical components of both pressure and movement are to be continuous; we obtain therefore:

$$A_1\, n\cos 2\beta\, e^{-i\varphi} + A_2\, n\cos 2\beta\, e^{i\varphi} - B_1\sin 2\beta\, e^{-i\psi} - B_2\sin 2\beta\, e^{i\psi} =$$
$$= A'\, mn'\cos 2\beta'\, e^{-i\varphi'} - B'\, n\sin 2\beta'\, e^{-i\psi'}$$
$$A_1\sin 2\alpha\, e^{-i\varphi} - A_2\sin 2\alpha\, e^{i\varphi} + B_1\, n\cos 2\beta\, e^{-i\psi} - B_2\, n\cos 2\beta\, e^{i\psi} =$$
$$= A'\, m\, n/n'\sin 2\alpha'\, e^{-i\varphi'} + B'\, mn\cos 2\beta'\, e^{-i\psi'}$$
$$A_1\sin\alpha\, e^{-i\varphi} + A_2\sin\alpha\, e^{i\varphi} + B_1\cos\beta\, e^{-i\psi} + B_2\cos\beta\, e^{i\psi} =$$
$$= A'\sin\alpha'\, e^{-i\varphi'} + B'\cos\beta'\, e^{-i\psi'}$$
$$A_1\cos\alpha\, e^{-i\varphi} - A_2\cos\alpha\, e^{i\varphi} - B_1\sin\beta\, e^{-i\psi} + B_2\sin\beta\, e^{i\psi} =$$
$$= A'\cos\alpha'\, e^{-i\varphi'} - B'\sin\beta'\, e^{-i\psi'}$$

where $\varphi = hd\cos\alpha$, $\psi = kd\cos\beta$ and $m = \dfrac{\varrho'\,\mathfrak{V}'}{\varrho\,\mathfrak{V}}$.

The solution of these six equations can be reduced to

$$A_1 = -i\frac{Z}{2\mu k\cos\varphi\cos\psi\cdot N}\, n\cos 2\beta \left\{ i\sin\psi\, e^{i\varphi}(P_1+Q_2+R_1)+ \right.$$
$$\left. +\cos\psi\, e^{i\varphi}(P_2+Q_1+R_2)-\frac{\sqrt{\sin 2\alpha\sin 2\beta}}{n\cos 2\beta}S\right\}$$

$$A_2 = -i\frac{Z}{2\mu k\cos\varphi\cos\psi\cdot N}\, n\cos 2\beta \left\{ -i\sin\psi\, e^{-i\varphi}(P_1+Q_2-R_1)+ \right.$$
$$\left. +\cos\psi\, e^{-i\varphi}(P_2+Q_1-R_2)-\frac{\sqrt{\sin 2\alpha\sin 2\beta}}{n\cos 2\beta}S\right\}$$

$$B_1 = +i\frac{Z}{2\mu k\cos\varphi\cos\psi\cdot N}\sin 2\alpha \left\{ i\sin\varphi\, e^{i\psi}(P_2+Q_1+R_1)+ \right.$$
$$\left. +\cos\varphi\, e^{i\psi}(P_1+Q_2+R_2)-\frac{n\cos 2\beta}{\sqrt{\sin 2\alpha\sin 2\beta}}S\right\}$$

$$B_2 = +i\frac{Z}{2\mu k\cos\varphi\cos\psi\cdot N}\sin 2\alpha \left\{ -i\sin\varphi\, e^{-i\psi}(P_2+Q_1-R_1)+ \right.$$
$$\left. +\cos\varphi\, e^{-i\psi}(P_1+Q_2-R_2)-\frac{n\cos 2\beta}{\sqrt{\sin 2\alpha\sin 2\beta}}S\right\}$$

with

$$\left.\begin{aligned} N = {} & (P_1+Q_2)(n^2\cos^2 2\beta\,\mathrm{tg}\,\varphi\,\mathrm{tg}\,\psi-\sin 2\alpha\sin 2\beta)- \\ & -(P_2+Q_1)(n^2\cos^2 2\beta-\sin 2\alpha\sin 2\beta\,\mathrm{tg}\,\varphi\,\mathrm{tg}\,\psi) \\ & -i R_1(n^2\cos^2 2\beta\,\mathrm{tg}\,\psi+\sin 2\alpha\sin 2\beta\,\mathrm{tg}\,\varphi)- \\ & -i R_2(n^2\cos^2 2\beta\,\mathrm{tg}\,\varphi+\sin 2\alpha\sin 2\beta\,\mathrm{tg}\,\psi)+ \\ & +2S\, n\cos 2\beta\sqrt{\sin 2\alpha\sin 2\beta}\cdot\frac{1}{\cos\varphi\cos\psi} \end{aligned}\right\} \quad . \quad . \quad (2)$$

We have used the following abbreviations:

$$P_1 = n^2 n'^2(\mu/\mu'\cdot 2\sin^2\beta'\cos 2\beta-2\sin^2\beta\cos 2\beta')^2$$
$$P_2 = \sin 2\alpha\sin 2\beta\sin 2\alpha'\sin 2\beta'(1-\mu/\mu')^2$$
$$Q_1 = n'^2\sin 2\alpha\sin 2\beta(\cos 2\beta'+2\mu/\mu'\sin^2\beta')^2$$
$$Q_2 = n^2\sin 2\alpha'\sin 2\beta'(\mu/\mu'\cos 2\beta+2\sin^2\beta)^2$$
$$R_1 = \mu/\mu'\, n'^2\sin 2\alpha\sin 2\beta',\quad R_2 = \mu/\mu'\, n^2\sin 2\alpha'\sin 2\beta$$

and $S = \sqrt{P_1 Q_1} - \sqrt{P_2 Q_2}$.

The horizontal (U_1) and vertical (W_1) component of the movement of the free surface are therefore:

$$U_1 = i\frac{Z}{k\mu}\sin\alpha\frac{L}{N}e^{i(\nu t-\xi x)} \quad , \quad W_1 = i\frac{Z}{k\mu}n\cos\alpha\frac{M}{N}e^{i(\nu t-\xi x)}$$

with

$$\left.\begin{aligned}L = &+ (P_1 + Q_2)(n \cos 2\beta \operatorname{tg} \varphi \operatorname{tg} \psi + 2 \cos \alpha \cos \beta) - \\ &- (P_2 + Q_1)(n \cos 2\beta + 2 \cos \alpha \cos \beta \operatorname{tg} \varphi \operatorname{tg} \psi) - \\ &- iR_1 (n \cos 2\beta \operatorname{tg} \psi - 2 \cos \alpha \cos \beta \operatorname{tg} \varphi) - \\ &- iR_2 (n \cos 2\beta \operatorname{tg} \varphi - 2 \cos \alpha \cos \beta \operatorname{tg} \psi) + \\ &+ \sqrt{n \cos 2\beta . 2 \cos \alpha \cos \beta}\left(\sqrt{\frac{\cos 2\beta}{2 \sin^2 \beta}} - \sqrt{\frac{2 \sin^2 \beta}{\cos 2\beta}}\right)\frac{S}{\cos \alpha \cos \beta}\end{aligned}\right\} \quad (3a)$$

$$M = i(P_1 + Q_2)\operatorname{tg} \psi + i(P_2 + Q_1)\operatorname{tg} \varphi - R_1 \operatorname{tg} \varphi \operatorname{tg} \psi + R_2 . \quad . \quad . \quad (3b)$$

By separating these expressions into a part which is independent from the underlying medium and the part which depends on both media we obtain the more understandable forms:

$$\left.\begin{aligned}\frac{L}{N} = &\frac{n \cos 2\beta - 2 \cos \alpha \cos \beta}{n^2 \cos^2 2\beta + \sin 2\alpha \sin 2\beta} + \frac{4 n^2 \cos 2\beta \cos \alpha \cos \beta}{N(n^2 \cos^2 2\beta + \sin 2\alpha \sin 2\beta)} \cdot \\ &\cdot \Big[\{(P_1 + Q_2 - R_1) - (P_2 + Q_1 - R_2)\}\frac{e^{-2i\varphi}}{1 + e^{-2i\varphi}} + \\ &+ \{(P_1 + Q_2 - R_2) - (P_2 + Q_1 - R_1)\}\frac{e^{-2i\psi}}{1 + e^{-2i\psi}} - \\ &- 2S\frac{n^2 \cos^2 2\beta - \sin 2\alpha \sin 2\beta}{n \cos 2\beta \sqrt{\sin 2\alpha \sin 2\beta}}\frac{e^{-i(\varphi+\psi)}}{(1 + e^{-2i\varphi})(1 + e^{-2i\psi})} - \\ &- 2(P_1 + Q_2 - P_2 - Q_1)\frac{e^{-2i(\varphi+\psi)}}{(1 + e^{-2i\varphi})(1 + e^{-2i\psi})}\Big]\end{aligned}\right\} . \quad (4a)$$

$$\left.\begin{aligned}\frac{M}{N} = &-\frac{1}{n^2 \cos^2 2\beta + \sin 2\alpha \sin 2\beta} + \frac{2}{N(n^2 \cos^2 2\beta + \sin 2\alpha \sin 2\beta)} \cdot \\ &\cdot \Big[n^2 \cos^2 2\beta \{(P_1 + Q_2 - R_1) - (P_2 + Q_1 - R_2)\}\frac{e^{-2i\varphi}}{1 + e^{-2i\varphi}} - \\ &- \sin 2\alpha \sin 2\beta \{(P_1 + Q_2 - R_2) - (P_2 + Q_1 - R_1)\}\frac{e^{-2i\psi}}{1 + e^{-2i\psi}} + \\ &+ 4 S n \cos 2\beta \sqrt{\sin 2\alpha \sin 2\beta}\frac{e^{-i(\varphi+\psi)}}{(1 + e^{-2i\varphi})(1 + e^{-2i\psi})} - \\ &- 2\{n^2 \cos^2 2\beta (P_1 + Q_2 - R_1) + \sin 2\alpha \sin 2\beta (P_2 + Q_1 - R_1)\}\frac{e^{-2i(\varphi+\psi)}}{(1 + e^{-2i\varphi})(1 + e^{-2i\psi})}\Big]\end{aligned}\right\} . \quad (4b)$$

This movement is caused by the normal pressure $Ze^{i(\nu t - fx)}$ distributed uniformly over the free surface; it is our purpose however to calculate the vibrations originating from a small region of this surface. Consequently we have to change this expression into a form which represents a pressure concentrated in a point. Following the usual way to achieve this transformation we introduce the circular coordinates $r = (x^2 + y^2)^{1/2}$ and $\varepsilon = \operatorname{arc} \operatorname{tg} y/x$. The horizontal U_1 is composed of a radial part $= U_1 \cos \varepsilon$ and a tangential component $= U_1 \sin \varepsilon$.

As

$$I_0(x) = \frac{1}{\pi}\int_0^\pi e^{-ix\cos\varepsilon}\, d\varepsilon$$

expression (1) will be transformed into $T_{zz} = Z\, e^{i\nu t} I_0(\xi r)$ by application of the operator $1/\pi \int_0^\pi d\varepsilon$. Therefore the movement corresponding to this radial-symmetrical pressure distribution is obtained by applying this operator to $U_1 \cos \varepsilon$, $U_1 \sin \varepsilon$ and W_1. The tangential component disappears; the other components are:

$$U_2 = \frac{Z}{k\mu} \sin\alpha \frac{L}{N} e^{i\nu t} I_1(\xi r) \quad \text{and} \quad W_2 = i\frac{Z}{k\mu} n \cos\alpha \frac{M}{N} e^{i\nu t} I_0(\xi r).$$

A pressure T_{zz} concentrated in the point $r = 0$ is expressed by the FOURIER integral

$$T_{zz} = Z e^{i\nu t} \cdot \frac{1}{2\pi}\int_0^\infty I_0(\xi r) \cdot \xi\, d\xi;$$

substituting $\xi = h \sin\alpha$ we get

$$T_{zz} = Z e^{i\nu t} \cdot \frac{1}{2\pi}\int_0^{p+i\infty} h^2 \sin\alpha \cos\alpha\, I_0(h r \sin\alpha)\, d\alpha \text{ with } 0 < p < \pi.$$

Hence the movement excited by a normal pressure $Ze^{i\nu t}$ exerted in the point $r = 0$ is

$$\left.\begin{aligned} U &= \frac{Z}{2\pi n \mu} \frac{\nu}{V} e^{i\nu t} \int_0^{p+i\infty} \sin^2\alpha \cos\alpha \frac{L}{N} I_1(h r \sin\alpha)\, d\alpha \\ W &= i\frac{Z}{2\pi\mu} \frac{\nu}{V} e^{i\nu t} \int_0^{p+i\infty} \sin\alpha \cos^2\alpha \frac{M}{N} I_0(h r \sin\alpha)\, d\alpha. \end{aligned}\right\} \quad . \quad . \quad (5)$$

§ 3. *Calculation of the wave systems.*

Introducing the functions of HANKEL by means of the relations

$I_0(x) = {}^1/_2 \{H_0^{(2)}(-x) - H_0^{(2)}(x)\}$ and $I_1(x) = -{}^1/_2 \{H_1^{(2)}(-x) + H_1^{(2)}(x)\}$

integrals (5) can be reduced to

$$\left\{\begin{aligned} U &= -\frac{Z}{4\pi n \mu} \frac{\nu}{V} e^{i\nu t} \int_{p_1 \pm i\infty}^{p_2 \pm i\infty} \sin^2\alpha \cos\alpha \frac{L}{N} H_1^{(2)}(h r \sin\alpha)\, d\alpha \\ W &= -i\frac{Z}{4\pi\mu} \frac{\nu}{V} e^{i\nu t} \int_{p_1 \pm i\infty}^{p_2 \pm i\infty} \sin\alpha \cos^2\alpha \frac{M}{N} H_0^{(2)}(h r \sin\alpha)\, d\alpha, \\ &\qquad \text{with } 0 > p_1 > -\pi \text{ and } 0 < p_2 < \pi. \end{aligned}\right.$$

We choose $\alpha = -{}^1/_2\pi - i\infty$ as the starting point of the integration and follow a curve I (fig. 1) which passes the point $\alpha = 0$ (which is a singular point of the H functions) at the positive real side and which terminates at $\alpha = +{}^1/_2\pi + i\infty$. The evaluation of the integrals will be carried out in the assumption $hr \gg 1$; this restriction does not seriously impair the practical utility of the results as observations at points nearer to the hearth hardly if ever occur.

The HANKEL functions can now be expanded in an asymptotic series

$$H_\nu^{(2)}(x) = e^{-i\left(x - \frac{2\nu+1}{4}\pi\right)} \sqrt{\frac{2}{\pi x}} \{1 + \sum_{n=1} a_n x^{-n}\}.$$

Using only the first term we get in first approximation

$$\left.\begin{aligned} U &= \frac{Z}{4\pi n\mu}\sqrt{\frac{2h}{\pi r}}\, e^{i\nu t - {}^1/_4\pi i} \int_I \sin^{3/2}\alpha \cos\alpha \frac{L}{N} e^{-ihr\sin\alpha}\, d\alpha \\ W &= \frac{Z}{4\pi\mu}\sqrt{\frac{2h}{\pi r}}\, e^{i\nu t - {}^1/_4\pi i} \int_I \sin^{1/2}\alpha \cos^2\alpha \frac{M}{N} e^{-ihr\sin\alpha}\, d\alpha \end{aligned}\right\} \quad . \quad . \quad (6)$$

A. The direct waves.

The first parts of U and W are

$$U_0 = \frac{Z}{4\pi n\mu}\sqrt{\frac{2h}{\pi r}}\, e^{i\nu t - {}^1/_4\pi i} \int_I \sin^{3/2}\alpha \cos\alpha \frac{n\cos 2\beta - 2\cos\alpha\cos\beta}{n^2\cos^2 2\beta + \sin 2\alpha \sin 2\beta} e^{-ihr\sin\alpha}\, d\alpha$$

$$W_0 = -\frac{Z}{4\pi\mu}\sqrt{\frac{2h}{\pi r}}\, e^{i\nu t - {}^1/_4\pi i} \int_I \frac{\sin^{1/2}\alpha\cos^2\alpha}{n^2\cos^2 2\beta + \sin 2\alpha\sin 2\beta} e^{-ihr\sin\alpha}\, d\alpha.$$

As these expressions are independent from the underlying medium they represent waves which travel directly along the free surface from the hearth to the seismograph.

Using the saddle-point method we have to integrate along the line of steepest descent. This curve is determined by the following two conditions:

1. on this line the imaginary part of the exponent must be constant.
2. it has to pass through the saddle-point α_s of the exponent, defined by $\partial/\partial\alpha$ (exponent) $= 0$.

In this case the saddle-point is $\alpha_s = {}^1/_2\pi$; as the exponent is equal to $-ihr$ if $\alpha = {}^1/_2\pi$ the equation of the line of steepest descent is

$$Im\{-ihr\sin\alpha\} = -ihr$$

with $\alpha = p + qi$ this becomes:

$$\sin p \cosh q = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By transforming the chosen path of integration (curve I) into this curve we pass two singular points of the integrand; firstly the pole α_0 satisfying the RAYLEIGH equation

$$n^2\cos^2 2\beta + \sin 2\alpha \sin 2\beta = 0$$

and secondly the point $\alpha_1 = \text{arc} \sin n$. In this point $\cos\beta = 0$; it is therefore a branch point. It follows that the integrals U_0 and W_0

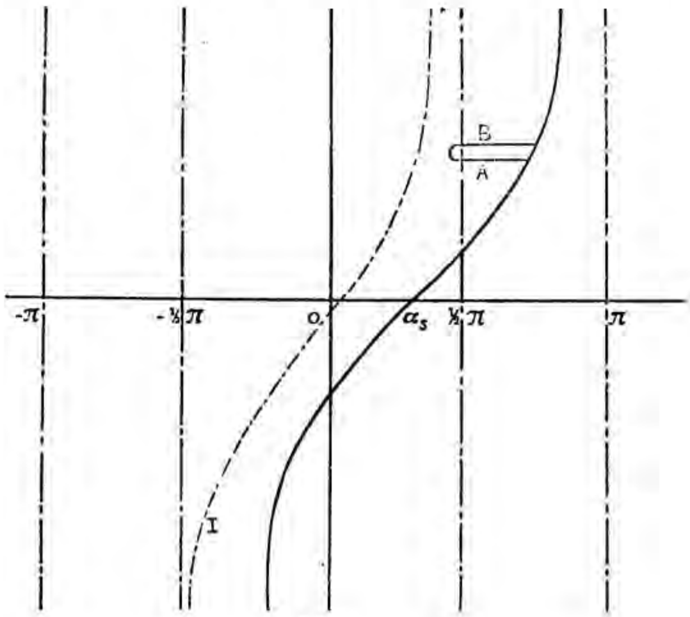

FIG. 1

consist of three parts. The first part $U_0(h)$ and $W_0(h)$ is obtained by the integration along curve (7); the second part $U_0(R)$ and $W_0(R)$ is the residue at $\alpha = \alpha_0$ and the third one is found by the integrations along the lines A and B joining the line of steepest descent with α_1.

Omitting the constant factors we obtain by the substitution $\alpha = {}^1/_2\pi + \eta$:

$$U_0(h) = -\int \cos^{3/2}\eta \sin\eta \, \frac{n\cos 2\beta + 2\sin\eta\cos\beta}{n^2\cos^2 2\beta - \sin 2\eta \sin 2\beta} \, e^{-ihr\cos\eta} \, d\eta.$$

On the line of steepest descent the exponent is equal to

$$-ihr + hr\cos p \,\text{sh}\, q \quad \text{or} \quad -ihr - hr \,\text{sh}\, q \,\text{tgh}\, q,$$

which decreases rapidly with increasing q. Hence the integrand differs sensibly from zero only for small values of q (or η). Integrating in the neighbourhood of $\eta = 0$ we get approximatedly

$$-\int_{-\eta_0}^{0} \cos^{3/2}\eta \sin\eta \, \frac{n\cos 2\beta + 2\sin\eta\cos\beta}{n^2\cos^2 2\beta - \sin 2\eta\sin 2\beta} \, e^{-ihr\cos\eta} \, d\eta -$$

$$-\int_{0}^{\eta_0} \cos^{3/2}\eta \sin\eta \, \frac{n\cos 2\beta - 2\sin\eta\cos\beta}{n^2\cos^2 2\beta + \sin 2\eta\sin 2\beta} \, e^{-ihr\cos\eta} \, d\eta$$

where $|\eta_0|$ satisfies $e^{-ihr\cos\eta} \ll 1$, and $\sin\eta > 0$.

Or:

$$\int_0^{\eta_0} \cos^{5/2}\eta \sin\eta \frac{-4n^2\cos 2\beta \sin\eta \cos\beta}{n^4\cos^4 2\beta - \sin^2 2\eta \sin^2 2\beta} e^{-ihr\cos\eta} d\eta$$

$$\approx \left(\frac{-4n^2\cos 2\beta \cos\beta \cos^{5/2}\eta}{n^4\cos^4 2\beta - \sin^2 2\eta \sin^2 2\beta}\right)_{\eta=0} e^{-ihr} \int_0^{\eta_0} \sin^2\eta \, e^{1/2 ihr\eta^2} d\eta$$

$$\approx -\frac{4n^3\sqrt{n^2-1}}{(n^2-2)^3} e^{-ihr} \int_0^{\eta_0} e^{1/2 ihr\eta^2} \eta^2 d\eta$$

Putting $\eta = \frac{u}{\sqrt{1/2\, h r}} e^{1/4 \pi i}$ the integral becomes $\frac{e^{3/4 \pi i}}{(1/2 hr)^{3/2}} \int_0^{u_0} e^{-u^2} u^2 du$.

As $e^{-u_0^2} \ll 1$ we make only a small error if we change the upper limit u_0 into ∞; the value of the last integral is therefore approximately $1/4 \sqrt{\pi}$. Hence the total result is

$$\left.\begin{aligned} U_0(h) &= -i \frac{Z}{\pi\mu} \frac{n^2\sqrt{n^2-1}}{(n^2-2)^3} \frac{e^{i(\nu t - hr)}}{hr^2} \text{ and} \\ W_0(h) &= -i \frac{Z}{\pi\mu} \frac{n^2}{2(n^2-2)^2} \frac{e^{i(\nu t - hr)}}{hr^2} \end{aligned}\right\} \quad \ldots \ldots (7)$$

(the last expression is found in the same way). This wave system will be denoted by $A_0(h)$.

By means of the residue theorem the terms contributed by the pole α_0 are readily found to be

$$\left.\begin{aligned} U_0(R) &= i\frac{Z}{\mu} h \sin\alpha_0 \left\{\frac{\sin\alpha\,(n\cos 2\beta - 2\cos\alpha\cos\beta)}{n\,\partial/\partial\sin\alpha\,(n^2\cos^2 2\beta + \sin 2\alpha \sin 2\beta)}\right\}_{\alpha=\alpha_0} \frac{e^{i\nu t - ihr\sin\alpha_0 - 1/4\pi i}}{\sqrt{2\pi h r \sin\alpha_0}} \\ W_0(R) &= -\frac{Z}{\mu} h \sin\alpha_0 \left\{\frac{i\cos\alpha}{n\,\partial/\partial\sin\alpha\,(n^2\cos^2 2\beta + \sin 2\alpha \sin 2\beta)}\right\}_{\alpha=\alpha_0} \frac{e^{i\nu t - ihr\sin\alpha_0 - 1/4\pi i}}{\sqrt{2\pi h r \sin\alpha_0}} \end{aligned}\right\} . (8)$$

These expressions represent the RAYLEIGH surface waves $A_0(R)$: they are identical with the results obtained by LAMB.

The integral along the lines A and B (fig. 1) is — the constant factors are again omitted —

$$\int_A \sin^{5/2}\alpha \cos\alpha \frac{n\cos 2\beta - 2\cos\alpha\cos\beta}{n^2\cos^2 2\beta + \sin 2\alpha \sin 2\beta} e^{-ihr\sin\alpha} d\alpha +$$

$$+ \int_B \sin^{5/2}\alpha \cos\alpha \frac{n\cos 2\beta + 2\cos\alpha\cos\beta}{n^2\cos^2 2\beta - \sin 2\alpha \sin 2\beta} e^{-ihr\sin\alpha} d\alpha$$

with $\cos a > 0$. Substituting $a = a_1 + \eta$ we get in first approximation

$$\frac{4(n^2-1)}{\sqrt{n}} \int_A e^{-ikr-hr\sqrt{n^2-1}\,\eta+\frac{1}{4}\pi i} \sqrt[4]{\frac{4(n^2-1)}{n^2}}\,\sqrt{\eta}\,d\eta.$$

After some reductions similar to those used above we obtain the wave system $A_0(k)$, expressed by

$$U_0(k) = \frac{Z}{\pi\mu}\,\frac{\sqrt{n^2-1}}{n}\,\frac{e^{i(\nu t-kr)}}{kr^2} \text{ and } W_0(k) = -i\,\frac{Z}{\pi\mu}\,\frac{2(n^2-1)}{n^2}\,\frac{e^{i(\nu t-kr)}}{kr^2}. \quad (9)$$

B. The reflected longitudinal waves.

The second part of (6) is

$$U_\varphi = \frac{Z}{\pi\mu}\,k\sqrt{\frac{2}{\pi hr}}\,e^{i\nu t-\frac{1}{4}\pi i}.$$

$$\int \frac{\sin^{3/2} a\cos^2 a\cos\beta\cos 2\beta\{(P_1+Q_2-R_1)-(P_2+Q_1-R_2)\}}{N(n^2\cos^2 2\beta+\sin 2a\sin 2\beta)}\cdot\frac{e^{-ihr\sin a-2i\varphi}}{1+e^{-2i\varphi}}\,da$$

$$W_\varphi = \frac{Z}{\pi\mu}\,k\sqrt{\frac{2}{\pi hr}}\,e^{i\nu t-\frac{1}{4}\pi i}.$$

$$\int \frac{n\sin^{1/2} a\cos^2 a\cos^2 2\beta\{(P_1+Q_2-R_1)-(P_2+Q_1-R_2)\}}{2N(n^2\cos^2 2\beta+\sin 2a\sin 2\beta)}\cdot\frac{e^{-ihr\sin a-2i\varphi}}{1+e^{-2i\varphi}}\,da.$$

Assuming a small damping factor, which in actual conditions always exists, to be incorporated in the exponential functions, the fraction $e^{-2i\varphi}/1+e^{-2i\varphi}$ can be expanded into the uniformly convergent series $\sum_{s=0}^{\infty}(-e^{-2i\varphi})^s$. Each of the corresponding integrals contains an exponent of the form

$-ihr\sin a - 2is\varphi$ or: $-ih\sqrt{r^2+(2sd)^2}\cos(a-\theta)$ with $\theta = \operatorname{arc\,tg} r/2sd$.

The saddlepoint is then $a_s = \theta$ and the line of steepest descent appears to be

$$\cos(p-\theta)\cosh q = 1. \quad \ldots\ldots\ldots \quad (10)$$

This curve intersects the real axis only in $p=\theta$ and meets the line $p = \frac{1}{2}\pi$ in the point $a^* = \frac{1}{2}\pi + iq^*$ where $\cosh q^* = 1/\sin\theta$. Its asymptotes are the lines $p = \theta \pm \frac{1}{2}\pi$.

With increasing distance r the point a_s travels along the real axis from 0 to $\frac{1}{2}\pi$, while a^* descends the line $p = \frac{1}{2}\pi$ from $\frac{1}{2}\pi + i\infty$ to $\frac{1}{2}\pi$. Therefore every point in the complex a plane where $\sin a$ has a positive real value (and $0 < p \leqq \frac{1}{2}\pi$) will be met by the line of steepest descent if r varies from a small value to ∞. Consequently this curve will at certain values of $r/2d$ pass through the poles of the integrand and through three of its four branch-points, namely the points where $\cos\beta$,

$\cos\alpha'$ or $\cos\beta'$ is equal to zero. Each time the path of integration (10) is moved beyond one of these points a new residue term or an integral is added to the value of U_φ and W_φ.

If $r/2d$ is sufficiently small not one of these singular points will be met by the path of integration; as the exponent on this curve is equal to

$$-ih\sqrt{r^2+4s^2d^2}-h\sqrt{r^2+4s^2d^2}\, sh\, q \operatorname{tgh} q$$

the integration has to be carried out only for small values of q.

Each integral is of the form

$$I=\int F e^{-ih\sqrt{r^2+4s^2d^2}\cos(\alpha-\theta)}\, d\alpha$$

or

$$I=\int \sum_{j=0,1,2\ldots} \frac{F_0^{(j)}}{j!} e^{-ih\sqrt{r^2+4s^2d^2}(1-{}^1/_2\eta^2+\ldots)}\eta^j\, d\eta,$$

where $\eta=\alpha-\theta$ and $F_0^{(j)}=\left(\frac{\partial^j F}{\partial\eta^j}\right)_{\eta=0}$.

Neglecting η^4 and following terms in the exponent we obtain by integration along the tangent $\eta=u\,e^{{}^1/_4\pi i}$ to the line of steepest descent in $\eta=0$:

$$I=\sum_{j=0,1,2\ldots}\frac{F_0^{(j)}}{j!} e^{-ih\sqrt{r^2+4s^2d^2}}\left(e^{{}^1/_4\pi i}\right)^{j+1}\int_{-u_0}^{+u_0} e^{-{}^1/_2hu^2\sqrt{r^2+4s^2d^2}}\,u^j\,du.$$

Now u_0 is a value of $|\eta|$ at which the exponential function is small enough to be neglected; by changing u_0 into ∞ we therefore add only very small terms to the integral. The result is then

$$I=\frac{e^{-ih\sqrt{r^2+4s^2d^2}+{}^1/_4\pi i}}{({}^1/_2h\sqrt{r^2+4s^2d^2})^{1/2}}\sum_{j=0,2,4\ldots}\frac{F_0^{(j)}}{j!}\cdot$$

$$\cdot\frac{1.3.5\ldots.(j-1)}{2^{1/2 j+1}}\left(\frac{e^{{}^1/_2\pi i}}{{}^1/_2h\sqrt{r^2+4s^2d^2}}\right)^{1/2 j}\sqrt{\pi}.$$

After some reduction we finally obtain the components of the wave system $A_\varphi(h)$:

$$U_\varphi(h) \text{ or } W_\varphi(h)=$$

$$\left.=\frac{Z}{\pi\mu}\frac{n}{\sqrt{\sin\theta}}\frac{e^{i(\nu t-h\sqrt{r^2+4s^2d^2})}}{\sqrt{r^2+4s^2d^2}}\sum_{j=0,1,2\ldots}\frac{F_0^{(2j)}}{2^j\cdot j!}\left(\frac{i}{h\sqrt{r^2+4s^2d^2}}\right)^j\right\} . \quad (11)$$

These expressions represent longitudinal waves which reach the point $r=2d\operatorname{tg}\theta$ after suffering one or more reflections at the surface $z=d$. Obviously only the terms $j=0$ and $j=1$ will be observed at any distance where $hr \gg 1$.

The line of steepest descent passes through the branch point $\cos\beta = 0$ if $a_1 = \arcsin n$ satisfies equation (10); we readily obtain $\sin\theta = 1/n$ or $r = \dfrac{2sd}{\sqrt{n^2-1}}$.

For larger values of r we have to integrate along the two lines A and B (fig. 1).

Putting $F = F_1 + F_2 \cos\beta$, where F_1 and F_2 are even functions of $\cos\beta$, this integral is equal to

$$I_1 = \int\limits_A (F_1 + F_2\cos\beta)\, e^{-ihR\cos(a-\theta)}\, da + \int\limits_B (F_1 - F_2\cos\beta)\, e^{-ihR\cos a-\theta)}\, da$$

with $R = (r^2 + 4s^2d^2)^{1/2}$. As the exponential function decreases with increasing value of $\eta = a - a_1$ we have in first approximation:

$$\cos(a-\theta) \approx \cos(a_1-\theta) - \eta\sin(a_1-\theta)$$

$$\cos\beta \approx e^{1/4\pi i}\sqrt{2\eta\sqrt{1-1/n^2}};$$

hence

$$I_1 \approx 2\, e^{-ihR\cos(\alpha_1-\theta)+1/4\pi i}\sqrt{2\sqrt{1-1/n^2}}\, F_2(a_1)\int e^{ihR\eta\sin(\alpha_1-\theta)}\sqrt{\eta}\, d\eta.$$

The factor $\sin(a_1-\theta) = n\cos\theta + i\sqrt{n^2-1}\sin\theta = (n^2 - \sin^2\theta)^{1/2} e^{i\delta}$ where $\operatorname{tg}\delta = (1-1/n^2)^{1/2}\operatorname{tg}\theta$; accordingly we have to integrate along the line $\eta = u^2 e^{(1/2\pi-\delta)i}$;

$$I_1 = -2\, e^{-ihR\cos(\alpha_1-\theta)+1/4\pi i+3/2(1/2\pi-\delta)i}\, F_2(a_1)\sqrt{2\sqrt{1-1/n^2}}\int\limits_0^{u_0} e^{-hRu^2\sqrt{n^2-\sin^2\theta}}\,.\,2u^2\, du$$

or

$$I_1 = -e^{-ihR\cos(\alpha_1-\theta)+1/4\pi i+3/2(1/2\pi-\delta)i}\, F_2(a_1)\cdot\frac{\sqrt{2\pi}\,(1-1/n^2)^{1/4}}{(hR)^{3/2}(n^2-\sin^2\theta)^{3/4}}.$$

The wave system $A_\varphi(k)$ appears to be expressed by

$$\left.\begin{aligned} U_\varphi(k) \text{ or } W_\varphi(k) = -\frac{Z}{\pi\mu}\left\{\frac{F(\cos\beta)-F(-\cos\beta)}{\cos\beta}\right\}_{\cos\beta=0}\cdot \\ \cdot\frac{\{n\sin\delta\,.\,e^{(1/2\pi-\delta)i}\}^{3/2}}{\sqrt{n^2-1}}\cdot\frac{e^{i(\nu t-kr)-2hd\sqrt{n^2-1}}}{kr^2}\end{aligned}\right\}. \quad (12)$$

Next we consider the branch point $\cos a' = 0$ or $\sin a_2 = V/V'$ $(=m_1)$; in actual conditions $V < V'$. Then a_2 is real and the path of integration strikes the branch point a_2 if $\theta = a_2$ or $r = \dfrac{2sd}{\sqrt{1/m_1^2-1}}$.

Using the same method as before we obtain the following expression for the wave system $A_\varphi(h')$:

$$U_\varphi(h') \text{ or } W_\varphi(h') = i\,\frac{Z}{\pi\mu}\left\{\left\{\frac{F(\cos\alpha') - F(-\cos\alpha')}{\cos\alpha'}\right\}_{\cos\alpha'=0} \cdot \frac{m_1\, n\, \sqrt[4]{1/m_1^2 - 1}}{\sin^{1/2}\theta\,\sin^{3/2}(\theta-\alpha_2)} \cdot \frac{e^{i\,(\nu t - h' r - 2hsd\sqrt{1-m_1^2})}}{h'\,(r^2 + 4s^2 d^2)}\right\}. \tag{13}$$

These waves start in the hearth as longitudinal waves in the direction $\alpha_2 = \arc\sin m_1$, traverse the depth of the layer $s-1$ times up and down, travel along the surface between the two media and reach the free surface again in the direction α_2. During this voyage the waves are always longitudinal.

The branch point $\cos\beta' = 0$ yields in a similar way the wave system $A_\varphi(k')$.

Finally we have to calculate the residue terms which arise when the line of steepest descent meets one of the poles of the integrand; these are the roots $\alpha_N^i\,(i = 1, 2, \ldots)$ of $N = 0$ and the root α_0 of the RAYLEYGH equation. Only one of the roots α_N^i is imaginary; this root is connected with the root of the RAYLEYGH equation of the underlying medium. A second imaginary root, associated with the STONELEY wave system, exists only in circumstances which virtually never occur [5]).

The residues are of course proportional to

$$\frac{Z}{\mu}\,h\sin\alpha_N^i\,\frac{e^{i\left(\nu t - hr\sin\alpha_N^i - 2hsd\cos\alpha_N^i\right)}}{\sqrt{2\pi h r\sin\alpha_N^i}}$$

and are therefore the expressions of surface waves. They are observed at distances $r > 2\,s\,d\,\mathrm{tg}\,\alpha_N^i$; it follows that at larger distances more surface waves will be registered and that seismograms will be more complicated. Of these wave systems the least important are those which correspond to the imaginary roots as the amplitudes of these waves decrease exponentially with exp. $(-2\,h\,s\,d\,|\cos\alpha|)$.

(To be continued).

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[5]) J. G. SCHOLTE, Proc. Ned. Akad. v. Wetensch., A'dam, **45**, 380—387, 449—457, 516—524 (1942).

Mathematics. — *Negationless intuitionistic mathematics.* By G. F. C. GRISS.
(Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of November 30, 1946.)

Introduction.

In these proceedings I already gave a sketch of some parts of *negationless intuitionistic mathematics* [1]). In the following I will treat of this subject more fully and systematically.

On philosophic grounds I think the use of the negation in intuitionistic mathematics has to be rejected. Proving that something is not right, i.e. proving the incorrectness of a supposition, is no intuitive method. For one cannot have a clear conception of a supposition that eventually proves to be a mistake. Only construction without the use of negation has some sense in intuitionistic mathematics.

But I am not going any further into the philosophical side of this question. *In intuitionistic mathematics one can consider the construction of negationless intuitionistic mathematics as a pure mathematical problem.*

To elucidate my intention I'll first give some examples, which will show what questions arise.

1. In elementary geometry there are often given proofs with the aid of the negation, whereas a proof without the use of the negation is simpler, e.g.

If a line in a triangle cuts off from two sides parts that are proportional to those sides, that line is parallel with the third side.

Let be given in $\triangle ABC$

$$CA : CB = CD : CE,$$

then we must prove that DE is parallel with AB.

Proof with negation:

If DE was not parallel with AB, we could draw a line, different from DE, that would cut BC in a point F different from E. Then we should have

$$CA : CB = CD : CF.$$

So CE and CF would be equal and E and F would coincide, which is impossible. The supposition, that DE would not be parallel with AB, is absurd, consequently $DE \parallel AB$.

Proof without negation:

Draw a line through D parallel with AB, that cuts BC in E'; then

$$CA : CB = CD : CE'.$$

So $CE = CE'$, so that E and E' coincide. So DE coincides with DE', whereas $DE' \parallel AB$, so that $DE \parallel AB$.

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1) Vol. 53 (1944).

The second demonstration is really simpler than the first. If, however, one wishes to avoid negation consistently, one must give a positive definition of parallel lines. Instead of saying: parallel lines (in a plane) are lines which do not intersect, one must define: parallel lines are such lines, that any point of one of them differs from any point of the other one. *And this, again, presupposes a positive definition* (i.e. a definition without negation) *of difference relative to points.*

If one wants to draw the line DE' in the second demonstration, one most probably draws a line different from DE, though DE' and DE prove to coincide. The figure has perhaps been one of the causes of the use of negation.

Another cause is the practice of questioning. One asks for instance: Which rational numbers satisfy $x^2 - 2 = 0$? The answer must be: No rational number satisfies. The question has been put in the wrong way. The fact is that $x^2 - 2$ differs positively from zero for every rational number.

2. Has the equation $ax + by = 0$ a solution for x and y, different from zero, i.e. a solution with at least x or y different from zero? The letters represent real numbers.

In intuitionistic mathematics they make a distinction between positively and negatively different with regard to real numbers. Two real numbers differ positively, if there can be indicated two approximating intervals which lie outside one another; they differ negatively, if it is impossible that they are equal; you can only divide by a real number if it differs positively from zero. In negationless mathematics the idea negatively different is, of course, omitted. Therefore we mean henceforth by different positively different.

If now in the equation $ax + by = 0$ a is different from zero, we can divide by a and find $x = -\frac{b}{a} y$, so among others $x = -\frac{b}{a}$ and $y = 1$ is a solution different from zero. If b is different from zero $x = 1$ and $y = -\frac{a}{b}$ is a solution different from zero. If a and b are both zero, then $x = 1$ and $y = 1$ is a solution different from zero. The result is:

$ax + by = 0$ has a solution different from zero, if at least one of the coefficients a and b differs from zero or if both are zero.

By using the negation this positive result can be formulated negatively: *It is impossible that no solution different from zero exists.* If namely there were no solution different from zero, a would be zero, for if a differs from zero, there would be a solution different from zero; likewise b would be zero; but then a and b were both zero, so there would still be a solution different from zero. This is impossible according to our supposition, so it is impossible that no solution different from zero exists.

The negative formulation is shorter, but distorted, and *the details of the positive result are lost.*

In non-intuitionistic mathematics $ax + by = 0$ has always a resolution different from zero. In this formulation the positive result has vanished entirely.

3. If $ax + b \neq 0$ for each value of x, then $a = 0$.

Proof without negation: Take $c \neq 0$ (this means c is different from zero) and determine x_1, in such a way that $cx_1 + b = 0$.

As $ax_1 + b \neq 0$, it follows that $(a - c)\ x_1 \neq 0$, which gives $a \neq c$. So $a \neq c$ for each value $c \neq 0$, so $a = 0$.

In this proof we used among others the proposition: If a differs from c for each value c that differs from b, then $a = b$.

This proposition replaces the negative proposition: If a does not differ from b, then $a = b$.

The positive proposition has to be proved for the different sorts of numbers, to begin with the natural numbers. But therefore again it proves to be necessary *to construct the whole of negationless intuitionistic mathematics from the beginning.*

4. We prove first:

Two triangles are congruent, if they have equal two pair of sides and the angles opposite the first pair, while the sum of the angles opposite the other pair differs from 180°.

Proof: Let be given in $\triangle ABC$ and $\triangle A'B'C'$: $AB = A'B'$, $BC = B'C'$, $\angle C = \angle C'$ and $\angle A + \angle A' \neq 180°$.

According to the sinus-rule $\sin A = \sin A'$, so that

$$2 \sin \tfrac{1}{2} (A - A') \cos \tfrac{1}{2} (A + A') = 0.$$

The latter factor differs from zero, so $\sin \frac{1}{2} (A - A') = 0$, so that $\angle A = \angle A'$. So $\triangle ABC \cong \triangle A'B'C'$.

I remark, that by using the negation $\angle A + \angle A' \neq 180°$ can be replaced by $\angle A + \angle A'$ is not equal to zero. However, no example is known where the second formulation could be applied but not the first, for no real number is known, of which is shown, that it is not possibly equal to zero, whereas it is not shown, that it differs (positively) from zero. It seems as if in intuitionistic mathematics the only way to show that two real numbers are not equal consists in showing that the numbers are (positively) different.

Two triangles are congruent, if they have equal one side, the angle opposite that side and the sum of the two other sides, while of one of the adjacent angles is known that they are either equal or different.

We need only prove the case in which the adjacent angles differ.

Let be given in $\triangle ABC$ and $\triangle A'B'C'$: $AC = A'C'$; $\angle B = \angle B'$ $AB + BC = A'B' + B'C'$, while $A \neq C'$.

Produce AB with $BD = BC$ and $A'B'$ with $B'D' = B'C'$, then $\triangle ACD = \triangle A'C'D'$, for $AD = A'D'$, $AC = A'C'$, $\angle D = \angle D'$, while $\angle ACD + \angle A'C'D' = (\angle C + \frac{1}{2} \angle B) + (\angle C' + \frac{1}{2} \angle B') =$

$= \angle C + \angle B + \angle C' \neq \angle C + \angle B + \angle A$, so $\angle ACD + \angle A'C'D' \neq 180^\circ$. From $\triangle ACD \cong \triangle A'C'D'$ it follows immediately $\triangle ABC \cong \triangle A'B'C'$.

Bij using negations one finds:

If two triangles have equal one side, the angle opposite that side and the sum of the two other sides, it is impossible, that they are not congruent.

Compare this with the second example.

We recapitulate the results we have obtained with these examples. In some cases it is simpler to avoid the use of the negation (ex. 1). Positive properties can sometimes be formulated more briefly in a negative way, but details get lost (ex. 2). The parts of intuitionistic mathematics which in a positive construction are disposed of are less important, for probably examples cannot be constructed for which a negative property could be applied and a corresponding positive property could not (ex. 4). To construct negationless mathematics one must begin with the elements and a positive definition of difference must be given instead of a negative one (ex. 1 and 3).

But even from a general intuitionistic point of view a positive construction of the theory of natural numbers must be given: one cannot define 2 is not equal to 1 (i.e. it is impossible that 2 and 1 are equal), for from this one could never conclude that 2 and 1 differ positively. Conversely one could define in a positive way negation by means of difference, e.g. not equal means different, etc., but, for the present, this seems unfit.

We finally consider the property: If a and b are elements of the set of natural numbers, and if $a \neq b$, then $a < b$ or $a > b$ for each element a of the set. If we apply this property to $b = 1$, we get: For each element $a \neq 1$ of the set of natural numbers we have $a < 1$ or $a > 1$. $a < 1$, however, has not any sense in negationless mathematics. If we say: $a < b$ or $a > b$ for each a of a set, we mean 1) that for each a at least one of these conditions is fulfilled, 2) that conversively at least one element fulfils the condition $a < b$ and another one the condition $a > b$. Properties which do not hold for any element do not occur. The combination of two properties (a natural number is even, a natural number is odd) need not be a property. Suppositions, of which it is not certain that they can be realized by a mathematic system, are not made. In axiomatic mathematics we must also stick to this.

Negationless intuitionistic logic will differ much from the usual intuitionistic logic [2]) by the absence of the negation and the altered meaning of the disjunction. Yet I'll not begin by giving such a system: intuitionistic mathematics has also been studied before a formal system was given.

"Affirmative" mathematics [3]) is something quite different from the negationless intuitionistic mathematics I'll treat of.

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[2]) A. HEYTING, Die formalen Regeln der intuitionistischen Logik. Preuss. Akad. der Wissenschaften 1930.

[3]) D. VAN DANTZIG, On the principles of intuitionistic mathematics. Rev. hispan. 1941?

§ 1. *The natural number.*

1. *Construction of the natural numbers.*

Imagine an object, e.g. 1. It remains the same [4]), 1 is the same as 1, in formula $1 = 1$.

Imagine another object, remaining the same, and distinguishable [4]) from 1, e.g. 2; $2 = 2$; 1 and 2 are distinguishable (from one another), in formula $1 \neq 2$, $2 \neq 1$.

They form the set $\{1, 2\}$; 1 and 2 belong to the set. If conversely an object belongs to this set, it is 1 or 2. If it is distinguishable from 1, it is 2; if it is distinguishable from 2, it is 1.

Imagine again another object (element), remaining the same and distinguishable from 1 and from 2, e.g. 3; $3 = 3$; 1 and 3 are distinguishable, 2 and 3 too, in formula $1 \neq 3$, $3 \neq 1$, $2 \neq 3$, $3 \neq 2$.

They form the set $\{1, 2, 3\}$. If an element belongs to $\{1, 2, 3\}$, it belongs to 1, 2 or it is 3. If it is distinguishable from each element of $\{1, 2\}$, it is 3; if it is distinguishable from 3, it is an element of $\{1, 2\}$.

If, in this way, we have proceeded to $\{1, 2, \ldots, n\}$, we can, again, imagine an element n', *remaining the same*, $n' = n'$, *and distinguishable from each element* p *of* $\{1, 2, \ldots, n\}$, *in formula* $n' \neq p$, $p \neq n'$.

They form the set $\{1, 2, \ldots, n'\}$. If an element belongs to $\{1, 2, \ldots, n'\}$, it belongs to $\{1, 2, \ldots, n\}$ or it is n'. If it is distinguishable from each element of $\{1, 2, \ldots, n\}$, it is n'; if it is distinguishable from n', it is an element of $\{1, 2, \ldots, n\}$.

We can *cease* with the mth element and get a finite set $\{1, 2, \ldots, m\}$; we can also *proceed* unlimitedly and get the countably infinite set $\{1, 2, \ldots\}$.

If one continuously chooses an arbitrary symbol as a new object, this soon leads to difficulties. One can try to prevent this by proceeding systematically in the choice of new symbols. So you can take I, II, III, etc.; after proceeding to a certain symbol you get a new symbol by adding a I. This is not practical either. It is better to use a numerative system, though, in principle, the same difficulties arise.

2. *Properties of the relations "the same" and "different".*

Proposition: *Two elements of the set* $\{1, 2, \ldots, m\}$ *are the same or distinguishable.*

Proof: For $\{1, 2\}$ the proposition holds. Let the proof have proceeded to the set $\{1, 2, \ldots, n\}$. Denote two elements of $\{1, 2, \ldots, n'\}$ by a and b. a is an element of $\{1, 2, \ldots, n\}$ or it is n', likewise b. There are 4 possibilities: 1) $a = n'$ and $b = n'$, so $a = b$; 2) $a = n'$ and b belongs to $\{1, 2, \ldots, n\}$, so $a \neq b$; 3) a belongs to $\{1, 2, \ldots, n\}$ and $b = n'$, so $a \neq b$; 4) a and b belong to $\{1, 2, \ldots, n\}$, then $a = b$ or $a \neq b$.

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[4]) Here, again, I do not enter into philosophical questions. Cf. my philosophical sketch "Idealistische filosofie" (Van Loghum Slaterus, Arnhem, 1946).

Proposition: *If for two elements a and b of* $\{1, 2, ..., m\}$ *holds:* $a \neq c$ *for each* $c \neq b$*, then* $a = b$.

Proof: For $\{1, 2\}$ the proposition holds. Let the proof have proceeded to $\{1, 2, ..., n\}$. There are two possibilities: In $\{1, 2, ..., n'\}$ $b = n'$ or b belongs to $\{1, 2, ..., n\}$. 1) If $b = n'$, then a is distinguishable from each element of $\{1, 2, ..., n\}$, so $a = n'$ and $a = b$. 2) b belongs to $\{1, 2, ..., n\}$; take $c = n'$, then a also belongs to $\{1, 2, ..., n\}$, so $a = b$.

We can also formulate these propositions in the following way, though we anticipate the general theory of sets we hope to treat of in a following paragraph.

If we denote the complementary set of the element a of the set $\{1, 2, ..., m\}$ by A, the complement of A is a and the sum of a and A is $\{1, 2, ..., m\}$.

In this connection I mention the so-called main proposition of arithmetic.

If there is a one to one reciprocal correspondence between $\{1, 2, ..., m\}$ and $\{1, 2, ..., p\}$, then $m = p$.

I do not repeat proofs which have been already given without using the negation.

For the elements of the set $\{1, 2, ..., m\}$ the following propositions hold now:

I $a = a$

II $a = b \rightarrow b = a$

III $a = b$ and $b = c \rightarrow a = c$

IV $a \neq b \rightarrow b \neq a$

V $a = b$ and $b \neq c \rightarrow a \neq c$

VI $a = b$ or $a \neq b$

VII $a \neq c$ for each $c \neq b \rightarrow a = b$.

VI replaces the negative proposition: Two natural numbers are the same or not, which in non-intuitionistic mathematics holds in virtue of the principium tertii exclusi, but which in intuitionistic mathematics must be proved.

VII runs with negation: If it is impossible, that a is not the same as b, then a is the same as b.

I briefly enumerate the negative propositions concerning the relations "the same" and "different" which have been replaced in a positive theory.

different $\rightleftarrows$ not the same.

the same $\rightleftarrows$ not different.

the same and different exclude one another.

two natural numbers are either the same or different.

3. *The order-relation.*

We define the relation *a precedes b*, $a < b$, which has the same meaning as *b follows a*, $b > a$, and the relation *a immediately precedes b* (*b immediately follows a*).

For $\{1, 2\}$ we have $1 < 2$. If a and b are elements of $\{1, 2\}$ and if $a < b$, then $a = 1$ and $b = 2$. 1 immediately precedes 2.

For $\{1, 2, 3\}$ we have $2 < 3$ and $1 < 3$. If a and b are elements of $\{1, 2, 3\}$ and if $a < b$, then $b = 3$ and a belongs to $\{1, 2\}$ or a and b belong to $\{1, 2\}$. 2 immediately precedes 3.

If, in this way, we have proceeded to $\{1, 2, \ldots, n\}$, for $\{1, 2, \ldots, n'\}$ we have $p < n'$ for each p of $\{1, 2, \ldots, n\}$. If a and b are elements of $\{1, 2, \ldots, n'\}$ and if $a < b$, then $b = n'$ and a belongs to $\{1, 2, \ldots, n\}$ or a and b belong to $\{1, 2, \ldots, n\}$. n immediately precedes n'.

If for $\{1, 2, \ldots, m\}$ $a < b$, *then* $a \neq b$.

Proof: For $\{1, 2\}$ the proposition holds. Let the proof have proceeded to $\{1, 2, \ldots, n\}$. If a and b are elements of $\{1, 2, \ldots, n'\}$ and if $a < b$, then $b = n'$ and a belongs to $\{1, 2, \ldots, n\}$, so that $a \neq b$ or a and b belong to $\{1, 2, \ldots, n\}$, so that $a \neq b$.

Property: If for $\{1, 2, \ldots, m\}$ $(m > 2)$ $a < b$ and $b < c$, then $a < c$.

Proof: $\{1, 2, 3\}$: as $a < b$ b belongs to $\{2, 3\}$ and as $b < c$ b belongs to $\{1, 2\}$. So $b = 2$, $a = 1$ and $c = 3$, so that $a < c$. Let the proof has proceeded to $\{1, 2, \ldots, n\}$. For the elements of $\{1, 2, \ldots, n'\}$ is $a < b$ and $b < c$. $c = n'$ or c belongs to $\{1, 2, \ldots, n\}$; in both cases b belongs to $\{1, 2, \ldots, n\}$, so a too and $a < c$.

Property: If $a \neq 1$ is an element of $\{1, 2, \ldots, m\}$, then $1 < a$.

Property: If $a \neq m$ is an element of $\{1, 2, \ldots, m\}$, then $a < m$.

Property: If a and b $(b \neq 1$ and $b \neq m)$ are elements of $\{1, 2, \ldots, m\}$, for each element a that differs from b holds $a < b$ or $a > b$.

These properties, just as the following ones, can easily be proved by induction. The condition $b \neq 1$ (also $b \neq m$) is necessary in the last property, for if $b = 1$, there is no element a which would satisfy $a < b$. This cannot be allowed in negationless mathematics (Cf. Introduction). We return to this subject in the theory of sets.

If $a < b$ and if for each $c < b$ and $c \neq a$ $c < a$ holds, then b immediately follows a.

If $a < b$ and if for each $c > a$ and $c \neq b$ $c > b$ holds, then b immediately follows b.

If b immediately follows a $(a \neq 1)$, then for each $c < b$ and $c \neq a$ holds $c < a$.

If b immediately follows a $(b \neq m)$, then for each $c > a$ and $c \neq b$ holds $c > b$.

Finally:

$a \neq b$ and $a \neq c$ for each $c < b$ $(b \neq 1) \rightarrow a > b$.

$a \neq b$ and $a \neq c$ for each $c > b$ $(b \neq m) \rightarrow a < b$.

From the preceding properties follows, if we define

$$a \leqq b \text{ as } a = b \text{ or } a < b \text{ and likewise } a \geqq b.$$

$a \neq c$ for each $c < b$ $(b \neq 1) \rightarrow a \geqq b$.

$a \neq c$ for each $c > b$ $(b \neq m) \rightarrow a \leqq b$.

$a \geqq 1$ and $a \leqq m$.

Mathematics. — *Sur les espaces linéaires normés.* III. Par A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 30, 1946.)

§ 7. *Opérations et fonctionnelles linéaires.* — Soient E et F deux espaces linéaires normés (archimédiens ou non-archimédiens) dont les coefficients appartienent à un même corps K, sur lequel on s'est donnée une valuation.

Si à chaque point x de E on fait correspondre un point $U(x)$ de F, $U(x)$ s'appelle une *opération* dans E.

L'opération s'appelle *continue au point* x_0, lorsque pour toute suite $x_n \to x_0$ (c. à. d. $\| x_0 - x_n \| < \varepsilon$ pour $n > N(\varepsilon)$) on a $U(x_n) \to U(x_0)$.

$U(x)$ s'appelle *continue dans* E lorsqu'elle l'est en tout point de E.

L'opération $U(x)$ s'appelle *additive* lorsqu'on a pour tout x et y de E

$$U(x+y) = U(x) + U(y),$$

et *homogène* lorsque pour pour tout $x \in E$ et $a \in K$

$$U(a\,x) = a\,U(x).$$

Une opération additive homogène continue s'appelle *linéaire*.

Théorème 9. *La valuation de K est supposée non triviale. Soit $U(x)$ une opération homogène et additive dans E. Alors il existe un nombre $M > 0$ tel que pour tout x en E on a*

$$\| U(x) \| \leqq M \| x \|. \qquad (7)$$

Démonstration. Il y a deux cas:

1. La valuation de K est non-archimédienne. Alors il y a de nouveau deux cas.

a) La valuation de K est discrète. Alors N_K est l'ensemble des puissances ϱ^i $(-\infty < i < +\infty)$ d'un nombre $\varrho > 1$. Supposons qu'un nombre M n'existe pas. Il existerait alors une suite $\{x_n\}$, $x_n \in E$, et une suite de nombres $\{M_n\}$ avec $M_n \to \infty$, telles que

$$\| U(x_n) \| > M_n \| x_n \|.$$

Ils existent en K des éléments a_n $(n = 1, 2, \ldots)$ tels que

$$\frac{1}{\varrho}\,\frac{1}{M_n \| x_n \|} \leqq | a_n | \leqq \frac{1}{M_n \| x_n \|},$$

puisqu'on peut supposer $\| x_n \| \neq 0$. Posons $y_n = a_n x_n$. On a

$$\| y_n \| = | a_n | \,.\, \| x_n \| \leqq \frac{1}{M_n} \to 0,$$

donc $y_n \to \theta$. De plus

$$\| U(y_n) \| = \| a_n U(x_n) \| = | a_n | \,.\, \| U(x_n) \| \geqq \frac{1}{\varrho} \frac{1}{M_n} \frac{\| U(x_n) \|}{\| x_n \|} \geqq \frac{1}{\varrho}.$$

Cependant il suit de $y_n \to \theta$ et de la continuité que $U(y_n) \to \theta$: contradiction (nous ne distinguons pas entre l'élément θ de E et celui de F).

b) La valuation de K est non discrète. Alors N_K est partout dense dans le corps des nombres réels. Pour chaque $\varepsilon > 0$ ils existent donc en K des éléments a_n tels que

$$\frac{1}{M_n \| x_n \|} - \varepsilon \leqq | a_n | \leqq \frac{1}{M_n \| x_n \|}.$$

On achève comme au cas *a*).

2. La valuation de K est archimédienne. On peut alors prendre pour K un corps de nombres réels ou complexes avec la valuation absolue, de sorte que K contient les nombres rationnels. Pour un nombre rationnel $a_n > 0$, convenablement choisi (donc $| a_n | = a_n$) on peut satisfaire à l'inégalité précédente; on achève de même.

Théorème 10. *Soit $U(x)$ une opération additive pour laquelle existe un nombre M tel qu'on a (7) pour tout x en E. Alors $U(x)$ est continue.*

Démonstration. C'est trivial (voir BANACH l.c. 2) p. 54).

A côté de la continuité on peut considérer la *continuité topologique* qu'on définit ainsi: l'opération $U(x)$ s'appelle topologiquement continue au point x_0 si pour tout $\varepsilon > 0$ donné il existe un $\delta > 0$ tel que

$$\| U(x) - U(x_0) \| < \varepsilon$$

pour $\| x - x_0 \| < \delta$. Chaque opération topologiquement continue est continue, de sorte que le théorème 9 reste vrai quand on y remplace la continuité par la continuité topologique. C'est une conséquence de l'additivité de $U(x)$ que l'opération du théorème 10 est aussi topologiquement continue: la continuité topologique en θ résulte immédiatement de (7), la continuité topologique partout suit de l'additivité.

Pour les opérations linéaires il n'y a donc pas lieu de distinguer entre la continuité et la continuité topologique.

Dans $M2$ se trouve une démonstration *directe* du théorème 9 pour les opérations additives homogènes topologiquement continues, en supposant $N_E \subset N_K$. Sans cette dernière supposition la même méthode ne conduit qu'à la propriété suivante: il existe un nombre $M > 0$ tel qu'on a

$$\| U(x) \| \leq M a$$

pour tout $\| x \| \leq a$. On ne peut pas la réduire à (7) puisque maintenant il n'est pas possible en général de normer un vecteur (voir la partie I).

Si K est un corps de nombres réels et si la valuation de K est archimédienne, l'homogénéité est une conséquence de l'additivité et de la continuité. La démonstration se fait comme dans BANACH l.c. 2) p. 36, puisque K contient les nombres rationnels. La question se pose si cette propriété reste vraie si la valuation de K est non-archimédienne. Si K est le corps des nombres P-adiques, on peut suivre la même démonstration puisque chaque nombre P-adique est la limite P-adique d'une suite de nombres rationnels. La question générale reste ouverte.

Si les points de F sont identiques aux éléments de K et $\| a \| = | a |$, on parle de *fonctionnelles*; une fonctionnelle est donc une opération dont les valeurs appartiennent à K.

Les théorèmes 9 et 10 ne sont démontrés que pour le cas d'une valuation non triviale de K. Cependant, j'ai pu démontrer le théorème 9 pour les fonctionnelles dans le cas de la valuation triviale.

Théorème 11. *La valuation du corps K est supposée triviale. Soit $F(x)$ une fonctionnelle topologiquement continue dans E. On ne suppose pas que $F(x)$ est additive et homogène. Alors il existe en chaque point de E un voisinage de ce point où $F(x)$ est constant.*

Démonstration. Soit x_0 un point arbitraire de E. Il résulte de la continuité topologique que pour tout $\varepsilon > 0$ donné, il existe un $\delta > 0$ tel que

$$| F(x) - F(x_0) | \leqq \varepsilon \text{ pour } \| x - x_0 \| \leqq \delta(\varepsilon).$$

Puisque $| F(x) |$ n'admet que les valeurs 0 et 1, il s'ensuit que pour $\varepsilon < 1$ on a $| F(x) - F(x_0) | = 0$, donc $F(x) = F(x_0)$ pour $\| x - x_0 \| \leqq \delta(\varepsilon)$.

Remarque. On ne peut pas conclure de cette propriété que $F(x)$ est partout constant dans E. Remarquons pour cela d'abord que, si $F(x)$ est additive, le nombre $\delta(\varepsilon)$, qui détermine le voisinage dans lequel F est constant, ne dépend pas du point x_0; on le voit en déterminant le voisinage pour θ et en appliquant alors la transformation $y = x - x_0$, qui transforme E en soi-même. Si alors, par exemple, E est totalement-non-archimédien, on a: si les voisinages $\| x - x_0 \| \leqq \delta(\varepsilon)$ et $\| x - x_1 \| \leqq \delta(\varepsilon)$ ont un point commun, ils coïncident. En effet, supposons que le point x soit commun. Il en résulte

$$\| x_0 - x_1 \| = \| x_0 - x + x - x_1 \| \leqq \max (\| x_0 - x \|, \| x - x_1 \|) < \delta.$$

Pour tous les points vérifiants $\| x - x_0 \| \leqq \delta(\varepsilon)$ on a donc

$$\| x - x_1 \| = \| x - x_0 + x_0 - x_1 \| \leqq \max (\| x - x_0 \|, \| x_0 - x_1 \|) < \delta,$$

et inversement; les voisinages sont donc identiques. On en voit qu'on ne peut pas conclure à la constance de $F(x)$ en général.

Théorème 12. *Mêmes suppositions qu'au théorème* 11. *Il existe un* $M > 0$ tel que

$$| F(x) | \leqq M \| x \| \qquad (x \in E).$$

Démonstration. Il suffit de prendre $M = \dfrac{1}{\delta(\varepsilon)}$ où $\delta(\varepsilon)$ est le δ du théorème 11 et $\varepsilon < 1$. En effet, pour $\| x \| \leqq \delta(\varepsilon)$ on a $F(x) = 0$. Pour $\| x \| > \delta(\varepsilon)$, on a $| F(x) | \leqq 1$ donc

$$| F(x) | \leqq \frac{\| x \|}{\delta(\varepsilon)} = M \cdot \| x \|.$$

Théorème 13. *La valuation de K est supposée non triviale. Soient E et F complets. Soit $U(x)$ une opération linéaire de E en F. Alors l'ensemble des $U(x)$ pour x en E est un ensemble de première catégorie, ou sinon, est identique à F.*

Démonstration. Nous renverrons à la démonstration du théorème analogue chez BANACH l.c. 2) p. 38. On peut la suivre à condition de remplacer la suite des nombres naturels, qu'on y emploie pour la définition des ensembles notés là par G_n, par une suite $\lambda_1 = 1, \lambda_2, \ldots$ d'éléments de K telle que $| \lambda_i | \to \infty$. Une telle suite existe en vertu de la supposition que la valuation de K soit non triviale.

Théorème 14. *Suppositions comme au théorème 13. $U(x)$ transforme E en F tout entier d'une façon biunivoque. Soit $U(x)$ linéaire. Alors $U(x)$ est bicontinu.*

Démonstration. Comme chez BANACH, l.c. 2) p. 40. Remarquons seulement que chaque espace métrique complet, qu'il soit archimédien ou non, est de deuxième catégorie en lui-même.

Remarquons que les théorèmes 13 et 14 restent vrais dans une classe d'espaces plus générale. Les conditions suivantes sont suffisantes:

Soient E et F des espaces linéaires; la valuation du corps K, auquel appartiennent les coefficients, est supposée non triviale. Soient E et F métriques et complets; on a

1°. $(x, y) = (x - y, \theta)$.

2°. de $\lim\limits_{n \to \infty} a_n = 0$ $(a_n \in K)$, il suit $\lim a_n x = \theta$ pour tout $x \in E\,(F)$.

3°. de $\lim\limits_{n \to \infty} x_n = \theta$, il suit $\lim\limits_{n \to \infty} a\, x_n = \theta$ pour tout $a \in K$.

La borne inférieure des nombres M pour lesquels on a (7), s'appelle la *norme* $\| U \|$ *de l'opération.*

§ 8. Dans ce paragraphe nous étudions l'existence de fonctionnelles linéaires dans les espaces totalement-non-archimédiens.

Théorème 15. *Soit E un espace totalement-non-archimédien. K est supposé complet; valuation triviale de K est permise. Supposons que l'ensemble N_E a 0 comme seul point d'accumulation, s'il y en a. Soient V un sous-espace linéaire fermé de E et $f(x)$ une fonctionnelle linéaire, définie sur V. Alors il existe dans E une fonctionnelle linéaire $F(x)$ telle que*

$$F(x) = f(x) \text{ pour } x \in V$$
$$\| F \| = \| f \|_V.$$

Démonstration. Soient y_0 un élément de $E - V$ et V_{y_0} l'ensemble des points

$$z = x + a\, y_0 \qquad (x \in V, a \in K).$$

Nous allons construire d'abord une extension linéaire de $f(x)$ dans V_{y_0}. Soit $V_{y_0}^{(1)}$ la variété des points $x + y_0$ avec $x \in V$; $V_{y_0}^{(1)}$ est fermé et, comme on l'a vu au cours de la démonstration du théorème 4, θ a une plus petite distance $\neq 0$ à $V_{y_0}^{(1)}$. Il y a des points en $V_{y_0}^{(1)}$ où cette plus petite distance à θ est atteinte; soit $\bar{y}_0$ un tel point. Comme dans la démonstration du théorème 4, on voit que V_{y_0} est identique à l'ensemble des points $x + a\bar{y}_0$ $x \in V$, $a \in K)$ et on a

$$\| x + a\, \bar{y}_0 \| = \max\, (\| x \|, \| a\, \bar{y}_0 \|). \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Si maintenant z de V_{y_0} admet la représentation unique (puisque y_0 n'est pas en V)

$$z = x + a\, \bar{y}_0$$

nous posons

$$F(z) = f(x).$$

On a

$$| F(z) | = | f(x) | \leqq \| x \| \,.\, \| f \|_V \leqq \max\, (\| x \|, \| a\, \bar{y}_0 \|) =$$
$$= \| x + a\, \bar{y}_0 \| \,.\, \| f \|_V = \| z \| \,.\, \| f \|_V.$$

Il en résulte $\| F \|_{V_{y_0}} \leqq \| f \|_V$ et puisque $F = f$ sur V

$$\| F \|_{V_{y_0}} = \| f \|_V.$$

Remarquons que V_{y_0} est fermé (voir le théorème 4). Il suffit alors de bien ordonner $E - V$, de construire les fermetures successives et les extensions partielles pour arriver à l'extension désirée de f dans E.

Théorème 16. *Suppositions concernant E comme au théorème* 15. *Il existe dans E des fonctionnelles linéaires non identiquement nulle.*

Démonstration. Appliquons le théorème précédent à l'espace linéaire V des points

$$x = a\, x_0$$

où $x_0 \neq \theta$ est un point fixé de E et $a \in K$. V est fermé puisque K est complet. Soit $C \neq 0$ une constante de K et posons $f(x) = a\, C$. On a

$$| a\, C | = | a | \,.\, | C | = \frac{| C |}{\| x_0 \|} \,.\, \| x \|,$$

donc

$$\| f \|_V = \frac{| C |}{\| x_0 \|}$$

L'existence d'une fonctionnelle linéaire $F(x)$ telle que

$$\| F \| = \frac{|C|}{\| x_0 \|}.$$

résulte alors du théorème précédent.

Remarque. Le théorème de ZERMELO, exprimant que chaque ensemble peut être bien ordonné d'une possibilité idéale, peut s'éviter si l'on suppose que l'espace E soit séparable. Soit, pour démontrer cela, $x_1, x_2, \ldots$ un ensemble partout dense dans E. Construisons alors les extensions successives, comme au théorème 15 en faisant parcourir y_0 la suite $\{x_n\}$. Soit alors x un point de E où F n'est pas défini. Il existe une suite $\{x_n\}$ telle que $x = \lim x_n$ donc $\|x_n - x_m\| < \varepsilon$ pour n et $m > \mu$. On a

$$| F(x_n) - F(x_m) | = | F(x_n - x_m) | \leqq \| F \| . \| x_n - x_m \| \to 0,$$

de sorte que la suite $\{F(x_n)\}$ est une suite fondamentale de K. K étant supposé complet, elle converge vers une limite en K, que nous posons $= F(x)$. On voit immédiatement que la fonctionnelle ainsi définie en E est linéaire et satisfait aux conditions.

Je n'ai pas réussi à démontrer le théorème 15 sans la condition que N_E a 0 comme seul point d'accumulation. Je n'y ai réussi qu'au cas où E a une dimension finie en ce sens que E peut être représenté dans la forme

$$x = a_1 x_1 + \ldots + a_r x_r,$$

où $x_1, \ldots, x_r$ sont des points fixés, et même alors avec une restriction [8]). On obtient:

Théorème 15a. *Soit E un espace totalement-non-archimédien de dimension finie r. K soit complet; valuation triviale de K est permise. Soient V un sous-espace linéaire fermé de E et $f(x)$ une fonctionnelle linéaire, définie sur V. Soit $0 < \lambda < 1$ une constante. Alors il existe dans E une fonctionnelle linéaire $F(x)$ telle que*

$$F(x) = f(x) \quad \textit{pour} \quad x \in V$$

$$\| f \|_V \leqq \| F \| \leqq \frac{1}{\lambda} \| f \|_V.$$

Démonstration. On peut suivre la démonstration du théorème 15. Changement est nécessaire puisque maintenant il n'est pas certain que $V_{y_0}^{(1)}$

[8]) Dans l'article: Over een integraal van een functie waarvan de waarden elementen zijn van een niet-archimedisch gewaardeerd lichaam (Proc. Ned. Akad. v. Wetensch. 53 (1944)) j'ai appliqué le théorème 15*a* au cas de dimension non-finie. Il faut donc se restreindre dans cet article au cas où N_E a 0 comme seul point d'accumulation, qui se réduit au cas d'une valuation discrète de K. Remarquons que les résultats y démontrés restent vrais au cas général dès que l'on a obtenu le théorème 15*a* au cas général (dimension non-finie).

contient des points où la plus petite distance de θ à $V_{y_0}^{(1)}$ est atteinte. Il existe un point $\bar{y}_0 \in V_{y_0}^{(1)}$ tel que pour tout $x \in V$

$$\lambda \| \bar{y}_0 \| \leqq \| x + \bar{y}_0 \|,$$

et plus général pour $x \in V$, $a \in K$

$$\lambda \| a \bar{y}_0 \| \leqq \| x + a \bar{y}_0 \|.$$

Au lieu de (8) on obtient les inégalités

$$\| x + a \bar{y}_0 \| \leqq \max (\| x \|, \| a \bar{y}_0 \|),$$
$$\| x + a \bar{y}_0 \| \geqq \lambda \max (\| x \|, \| a \bar{y}_0 \|).$$

V_{y_0} est identique à l'ensemble des points $x + a \bar{y}_0$.

Si $z = x + a \bar{y}_0$, on pose

$$F(z) = f(x).$$

On a

$$| F(z) | = | f(x) \leqq \| x \| \cdot \| f \|_V \leqq \max (\| x \|, \| a \bar{y}_0 \|) \cdot \| f \|_V \leqq$$
$$\leqq \frac{1}{\lambda} \| x + a \bar{y}_0 \| \cdot \| f \|_V = \frac{\| z \|}{\lambda} \| f \|_V.$$

où l'on tire

$$\| f \|_V \leqq \| F \|_{V_{y_0}} \leqq \frac{1}{\lambda} \| f \|_V.$$

On achève par un nombre fini d'extensions (pour λ il faut substituer $\lambda^{\frac{1}{r}}$).

On voit qu'on ne réussit pas par cette voie s'il faut un nombre infini de points x pour épuiser E. Seulement si l'on savait démontrer qu'on peut poser $\lambda = 1$ au théorème 15a, on pourrait arriver au cas général.

L'extension du théorème 16 est évidente.

§ 9. Etudions enfin les espaces archimédiens, donc le cas où la valuation de K est archimédienne (le cas A de la première partie). Les théorèmes 15 et 16 restent vrais dans ce cas. La condition que K est complet est maintenant superflue (on peut se demander si l'on peut se passer de cette condition aussi au § 8).

Supposons que K est un corps de nombres réels à valuation absolue. On a le théorème suivant.

Théorème 17. *Soit E un espace linéaire dont les coefficients appartiennent à un corps K de nombres réels. Supposons qu'on a donné:*

1. *une fonctionnelle $p(x)$, dont les valeurs n'appartiennent pas nécessairement à K, sauf au troisième point ci-dessous, définie en E, telle qu'on a pour tout x et y de E*

$$p(x + y) \leqq p(x) + p(y)$$
$$p(t x) = t\, p(x) \qquad \text{pour tout } t \geqq 0 \text{ de } K.$$

2. *une fonctionnelle additive homogène $f(x)$, de valeurs en K, définie sur un sous-espace linéaire V de E, telle qu'on a*

$$f(x) \leqq p(x).$$

3. *si au point x de E on a $p(x) = -p(-x)$, alors on a $p(x) \in K$. Alors il existe en E une fonctionnelle additive homogène $F(x)$, de valeurs en K, telle que*

$$F(x) \leqq p(x) \quad \text{pour} \quad x \in E$$
$$F(x) = f(x) \quad \text{pour} \quad x \in V.$$

Démonstration. Nous renvoyons le lecteur au livre de BANACH, l.c. 2) p. 28. On peut suivre la marche de la démonstration là donnée. Remarquons seulement qu'il faut choisir pour le nombre r_0, là défini, un nombre rationnel convenable, ou bien on applique la troisième donnée (K contient les nombres rationnels). Dans la définition de l'ensemble G_0 par les éléments $y = x + t\,x_0$, il faut faire parcourir t le corps K.

On n'a pas supposé que E est un espace topologique de sorte qu'il n'a pas de sens de dire que V est fermé.

En supposant que E est un espace normé (la valuation de K est supposée absolue), on peut prendre pour $p(x)$ la fonction

$$p(x) = \| x \| \cdot \| f \|_V.$$

Evidemment les conditions sont vérifiées; quant à la troisième puisque $\| x \| = \| -x \|$, donc $p(x) = p(-x)$. Il existe alors une extension linéaire de $f(x)$ en E avec conservation de norme.

On montre l'existence de fonctionnelles linéaires non identiquement nulle comme ci-dessus.

Mathematics. — *Sur les espaces linéaires normés.* IV. Par A. F. MONNA. (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of November 30, 1946.)

III. *Décomposition des vecteurs; projecteurs.*

§ 10. Les espaces que nous allons considérer sont assujettis aux conditions suivantes:

1. Les espaces sont totalement-non-archimédiens;
2. Le corps K est complet;
3. L'ensemble N_E des normes a 0 comme seul point d'accumulation;
4. Les espaces sont complets.

Valuation triviale de K est permise.

Ces conditions sont remplies pour la classe bien plus spéciale des espaces localement compacts pourvu que K est complet (ce qui n'est pas, comme nous avons vu, une restriction de la généralité).

Il est remarquable que, comme nous allons le voir, on peut montrer pour ces espaces des théorèmes tout à fait analogues aux théorèmes bien connus de l'espace de HILBERT concernant la décomposition des vecteurs, les variétés complémentaires et les projecteurs. Il n'est pas même besoin de supposer que les espaces sont séparables. Seulement si l'on veut éviter le théorème de ZERMELO, il faut faire cette restriction [9]).

Soit V un sous-espace linéaire fermé de E. Un point arbitraire x de $E - V$ a une plus petite distance $d \neq 0$ à V. Puisque N_E a 0 comme seul point d'accumulation, ils existent en V des points ξ_x tels que

$$\| x - \xi_x \| = d.$$

Chaque vecteur x de E peut donc s'écrire comme somme de deux vecteurs

$$x = (x - \xi_x) + \xi_x, \quad \ldots \ldots \ldots \quad (9)$$

où un des vecteurs appartient à V. Cette décomposition n'est pas uniquement déterminée, puisque le vecteur ξ_x n'est pas unique; $\xi_x = \theta$ si $x \in V$

Définition. *Le vecteur ξ_x dans la décomposition* (9) *sera appelé une projection de x sur V.*

Théorème 18. *Pour chaque projection ξ_x de x sur V on a*

$$\| \xi_x \| \leqq \| x \|.$$

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[9]) Pour les espaces de HILBERT le corps K des coefficients est localement compact et $N_E \subset N_K$. Pour les espaces ici considérés ces deux conditions conduisent à la troisième des conditions que doivent remplir ces espaces. Cette circonstance fait paraître de plus l'analogie.

Démonstration. Pour tout $y \in V$ on a

$$\| x - \xi_x \| \leqq \| x - y \|. \quad \ldots \ldots \ldots \quad (10)$$

donc pour $y = \theta$

$$\| x - \xi_x \| \leqq \| x \|.$$

Il s'ensuit

$$\| \xi_x \| \leqq \max (\| x - \xi_x \|, \| x \|) = \| x \|.$$

Nous étudierons dans ce qui suit l'existence d'un sous-espace linéaire fermé V^* de E tel que chaque x de E peut s'écrire d'une façon *unique* dans la forme $x = \xi_x + \eta_x$ où ξ_x est une projection de x sur V et $\eta_x \in V^*$.

Définition. *Une opération linéaire P qui transformé chaque $x \in E$ en une projection uniquement déterminée Px de x sur une variété linéaire fermée V sera appelée un projecteur.*

Etant donné un projecteur P — nous montrerons qu'il existent des projecteurs — chaque x de E peut s'écrire d'une façon unique dans la forme

$$x = P x + (x - P x). \quad \ldots \ldots \ldots \quad (11)$$

Les vecteurs $x - Px = \eta_x$ constituent un espace linéaire V^*. En effet, si $a \in K$ on a $ax = Pax + (ax - Pax) = aPx + a(x - Px)$, de sorte que avec η_x aussi $a\eta_x$ appartient à V^*. De plus, si $y = Py + (y - Py)$ il suit de la linéarité de P

$$x + y = P x + P y + (x + y - P x - P y) = P(x + y) + (x + y - P(x + y)).$$

de sorte que

$$\eta_{x+y} = x + y - P x - P y = \eta_x + \eta_y.$$

Donc, avec η_x et η_y, $\eta_x + \eta_y$ est dans V^*.

V^* est identique à la variété linéaire des vecteurs pour lesquels $Px = \theta$. Il suit de (11) qu'un vecteur x tel que $Px = \theta$, appartient à V^*. Si inversement $\eta \in V^*$, donc η de la forme $x - Px$, alors $P\eta = Px - P^2x$. Puisqu'un vecteur dans V a pour projection le vecteur lui-même — dans ce cas spécial la projection est uniquement déterminée — on a $P^2x = Px$, de sorte que $P\eta = \theta$.

V^* est fermé. Soit $\eta^{(n)} \to \eta$, $\eta^{(n)} \in V^*$; donc

$$\| \eta^{(n)} - \eta \| < \varepsilon$$

pour $n > N(\varepsilon)$. En vertu du théorème 18 donc $\| P(\eta^{(n)} - \eta) \| < \varepsilon$. Puisque $P\eta^{(n)} = \theta$, il suit $\| P\eta^{(n)} - P\eta \| = \| P\eta \| < \varepsilon$; ε étant arbitraire, on a $P\eta = \theta$ de sorte que $\eta \in V^*$.

La relation entre V et V^* est réciproque, c. à d. dans la décomposition (11) le vecteur $x - Px$ est une projection de x sur V^*, uniquement déterminée en vertu de l'unicité de la décomposition (11). Pour cela il faut montrer qu'on a

$$\| x-(x-Px) \| = \| Px \| \leqq \| x-\eta \| \quad . \quad . \quad . \quad . \quad (12)$$

pour tout $\eta \in V^*$ [10]).

Puisque $PV^* = 0$, on a

$$\| \eta \| \leqq \| \eta - y \| \text{ pour tout } y \in V, \eta \in V^*.$$

Substituons $\eta - (x - Px)$ pour η et Px pour y:

$$\| \eta-(x-Px) \| \leqq \| \eta-(x-Px)-Px \|$$

donc

$$\| (\eta - x) + Px \| \leqq \| \eta - x \|.$$

$$\| Px \| \leqq \max \left(\| \eta - x \|, \| (\eta - x) - Px \| \right) \leqq \| \eta - x \|.$$

q. e. d.

Enfin V et V^* n'ont que θ en cummun.

Ces propriétés justifient d'appeler V^* une variété complémentaire de V.

Nous montrerons ensuite qu'il existent des projecteurs dans E.

Jusqu'ici il n'était pas nécessaire de supposer que E est complet. Dans la démonstration de l'existence de projecteurs nous utilisons au contraire que E est complet.

Soient x_1 un vecteur de $E - V$ et ξ_{x_1} une projection arbitraire, restant fixée dans ce qui suit, de x_1 sur V. Soit V_1 l'espace linéaire fermé des éléments $x + ax_1$ ($x \in V$, $a \in K$) ou, ce qui est équivalent, l'espace linéaire déterminé par V et $x_1 - \xi_{x_1}$. A chaque $x \in V_1$ correspondent de façon unique un vecteur $\xi_x \in V$ et un élément $a \in K$ tels que

$$x = \xi_x + a(x_1 - \xi_{x_1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

En effet, si $x = \xi'_x + a'(x_1 - \xi_{x_1})$, on aurait $\xi'_x - \xi_x = (a - a')(x_1 - \xi_{x_1})$ d'où, puisque $x_1 - \xi_{x_1}$ appartient à $E - V$ et $\xi'_x - \xi_x$ à V à cause de la linéarité de V, $a = a'$ et $\xi'_x = \xi_x$.

Dans la décomposition (13), ξ_x est une projection de x sur V. On a

$$\| x_1 - \xi_{x_1} \| \leqq \| x_1 - y \| \text{ pour tout } y \in V.$$

Substituons ici pour y : $\frac{y}{c} + \xi_{x_1}$, où c appartient à K, et multiplions par c. On obtient

$$\| c(x_1 - \xi_{x_1}) \| \leqq \| y - c(x_1 - \xi_{x_1}) \|.$$

Il en résulte pour tout $c \in K$ et $y \in V$:

$$\| y \| \leqq \max \left(\| y - c(x_1 - \xi_{x_1}) \|, \| c(x_1 - \xi_{x_1}) \| \right) \leqq \| y - c(x_1 - \xi_{x_1}) \|.$$

[10]) Cette démonstration et quelques autres, qui sont plus simples que celles que j'avais données originellement, m'ont été communiquées par Prof. H. FREUDENTHAL.

donc

$$\| (y + a(x_1 - \xi_{x_1})) - a(x_1 - \xi_{x_1}) \| \leqq \| (y + a(x_1 - \xi_{x_1})) - c'(x_1 - \xi_{x_1}) \|.$$

Il s'ensuit que

$$P^*(y + a(x_1 - \xi_{x_1})) = a(x_1 - \xi_{x_1})$$

est une projection des vecteurs de V_1 sur l'espace V_1^* des éléments $a(x_1 - \xi_{x_1})$, et les vecteurs de V ont une projection θ. En vertu de la réciprocité, démontrée ci-dessus, on voit donc que

$$P(y + a(x_1 - \xi_{x_1})) = y$$

est une projection de V_1 sur V pour laquelle les vecteurs de V_1^* ont une projection θ; q.e.d.

Soit

$$y = \xi_y + a(x_1 - \xi_{x_1})$$
$$z = \xi_z + b(x_1 - \xi_{x_1}).$$

On tire de l'unicité de la décomposition (13) pour les vecteurs de V_1

$$y + z = \xi_y + \xi_z + (a + b)(x_1 - \xi_{x_1}) = \xi_{y+z} + (a + b)(x_1 - \xi_{x_1}).$$

Donc les vecteurs-non-de-V dans cette décomposition constituent une variété linéaire.

Bien ordonnons alors $E - V$:

$$x_1, x_2, \ldots, x_\omega, x_{\omega+1}, \ldots \qquad (x_\alpha \in E - V) \quad . \quad . \quad . \quad . \quad (14)$$

Après V_1 formons successivement les espaces linéaires fermés $V_2, V_3, \ldots$ où V_n est l'espace déterminé par V_{n-1} et x_n, pourvu que x_n ne soit pas dans un V_i pour $i < n$. Si x_n est dans un V_i avec $i < n$, nous supprimons cet élément de la suite (14). Soit V_ω le plus petit espace linéaire fermé contenant les vecteurs qui appartiennent à un quelconque des

$$V_n \ (n = 1, 2, \ldots).$$

L'application transfinie de ce procédé fournit une suite croissante d'espaces fermés

$$V \subset V_1 \subset V_2 \ldots \subset V_\omega \subset V_{\omega+1} \subset \ldots$$

telle que chaque $x \in E$ se trouve dans un V_α à partir d'un certain ordre.

Après avoir fixé comme ci-dessus une projection unique sur V pour les vecteurs de V_1, on a la définition récurrente suivante pour la projection sur V d'un vecteur arbitraire de E:

1) Supposons qu'on a fixé une projection unique sur V pour les vecteurs de V_α. Fixons pour le vecteur $x_{\alpha+1}$, par lequel on déduit $V_{\alpha+1}$ de V_α par combinaison linéaire, une projection $\xi^{(\alpha)}_{x_{\alpha+1}}$ sur V_α. Alors, comme ci-dessus pour V_1, une projection unique $\xi^{(\alpha)}_x$ sur V_α est déterminée pour les vecteurs de $V_{\alpha+1}$. Admettons pour projection ξ_x sur V du vecteur x de

$V_{\alpha+1}$ la projection de $\xi_x^{(\alpha)}$ sur V. Il faut justifier cette définition: il faut montrer que le vecteur ξ_x ainsi déterminé est une projection sur V. Cela résulte de la propriété suivante:

Soient $V_1 \subset V_2 \subset V_3$ des espaces linéaires fermés, P_2 un projecteur de V_3 sur V_2 et P_1 un projecteur de V_2 sur V_1. Alors nous verrons que P_1P_2 est un projecteur de V_3 sur V_1. Soit z un élément arbitraire tel que $P_1P_2z = \theta$. En vertu du théorème 18 on a

$$\| x - z \| \geqq \| P_2 x - P_2 z \|.$$

La projection par P_1 du vecteur P_2z est θ, donc

$$\| x' - P_2 z \| \geqq \| P_1 x' \|$$

pour tout $x' \in V_2$, en particulier pour $x' = P_2x$. Donc

$$\| P_2 x - P_2 z \| \geqq \| P_1 P_2 x \|,$$

et, d'après ce qui précède

$$\| x - z \| \geqq \| P_1 P_2 x \|$$

pour tout z tel que $P_1P_2z = \theta$. Il s'ensuit, puisque $(P_1P_2)^2 = P_1P_2$, que P_1P_2x est une projection de x sur V_1 (comparer les théorèmes 20 et 23). Soient x et y deux vecteurs de $V_{\alpha+1}$; on montre par induction qu'on a pour les projections ξ_x, ξ_y, ξ_z, si $z = x + y$,

$$\xi_z = \xi_x + \xi_y.$$

En effet, on a

$$x = \xi_x^{(\alpha)} + a\,(x_{\alpha+1} - \xi_{x_{\alpha+1}}^{(\alpha)})$$
$$b = \xi_y^{(\alpha)} + b\,(x_{\alpha+1} - \xi_{x_{\alpha+1}}^{(\alpha)}),$$

et cette décomposition est unique. Donc

$$x + y = \xi_x^{(\alpha)} + \xi_y^{(\alpha)} + (a + b)\,(x_{\alpha+1} - \xi_{x_{\alpha+1}}^{(\alpha)}),$$

d'où il suit

$$\xi_z^{(\alpha)} = \xi_x^{(\alpha)} + \xi_y^{(\alpha)}.$$

Pour la détermination des projections sur V à l'aide de la définition, on peut appliquer, en vertu de l'hypothèse, la propriété additive au membre à droite. Donc

$$\xi_z = \xi_z + \xi_y.$$

2) Supposons α n'a pas un précédent immédiat; il faut définir la projection pour les vecteurs qui sont limites d'une suite de vecteurs de $V_i (i < \alpha)$ et qui n'appartiennent pas eux-mêmes à un $V_i (i < \alpha)$. Soit donc

$$x = \lim_{i \to \alpha;\, i < \alpha} x_i.$$

En vertu de l'hypothèse de l'induction, une projection ξ_{x_i} sur V est uniquement déterminée pour x_i et on a la propriété additive, c. à d. $\xi_{x_i} - \xi_{x_j}$ est la projection de $x_i - x_j$. Il suit du théorème 18

$$\| \xi_{x_i} - \xi_{x_j} \| \leqq \| x_i - x_j \| < \varepsilon$$

à partir d'une valeur convenable de i. L'espace E étant supposé complet, la limite $\lim \xi_{x_i}$ existe; nous définions cette limite comme projection de x. Pour cela il faut montrer que $\| x - \xi_x \|$ est la plus petite distance de x à V. C'est une conséquence du:

Lemme. *Si la suite* $\{z_i\}$ *converge vers* $z \neq \theta$, *donc si* $\| z_i - z \| < \varepsilon$ *pour* $i > I$, *alors on a* $\| z_i \| = \| z \|$ *pour les valeurs suffisamment grandes de* i.

En effet, on a alors

$$\| z_n \| = \max (\| z_n - z \|, \| z \|) = \| z \|.$$

Si maintenant $\| x - \xi_x \|$ n'était pas la plus petite distance, il existait un η tel que $\| x - \xi_x \| > \| x - \eta \|$. Alors pour des valeurs convenables de i

$$\| x_i - \eta \| = \| x_i - x + x - \eta \| = \| x - \eta \| < \| x - \xi_x \| = \| x_i - \xi_{x_i} \|,$$

en contradiction avec l'hypothèse selon laquelle $\| x_i - \xi_{x_i} \|$ sera la plus petite distance de x_i à V (le cas trivial $x = \xi_x$, dans lequel la distance à V est zéro, est ici exclu).

Si l'on a aussi $x = \lim y_i$, alors $\| \xi_{x_i} - \xi_{y_i} \| \leqq \| x_i - y_i \| < \varepsilon$ pour valeurs convenables de i; il s'ensuit que la projection est uniquement déterminée.

On montre la propriété additive de la projection comme il suit: soit $\lim \xi_{x_i} = \xi_x$, $\lim \xi_{y_i} = \xi_y$, alors

$$\xi_x + \xi_y = \lim \xi_{x_i} + \lim \xi_{y_i} = \lim (\xi_{x_i} + \xi_{y_i}) = \lim \xi_{x_i + y_i} = \xi_{x+y}.$$

Soient x et y deux vecteurs arbitraires de E; on a les décompositions uniques

$$x = \xi_x + \varphi,$$
$$x = \xi_y + \psi,$$

où ξ_x et $\xi_y \in V$ désignent les projections de x et y sur V. Pour $x + y$ on a

$$x + y = \xi_{x+y} + \chi,$$

et aussi

$$x + y = \xi_x + \xi_y + \varphi + \psi.$$

Or $\xi_x + \xi_y = \xi_{x+y}$, et il suit alors de l'unicité de la décomposition

$$\varphi + \psi = \chi.$$

Il s'ensuit que dans cette décomposition les vecteurs-non-de-V constituent un espace linéaire; c'est évident que $\lambda\varphi$ appartient à cet espace avec φ: on considère λx. Cet espace est un espace complémentaire de V. On montre comme au début que cet espace est fermé et qu'il y a réciprocité.

L'existence de projecteurs est ainsi démontré. On a donc le théorème:

Théorème 19. *Soient E un espace satisfaisant aux conditions du début du § 10 et V un sous-espace linéaire fermé de E. Alors ils existent en E des projecteurs P relativement à V, c. à d. des opérations linéaires qui transforment tout $x \in E$ en une projection uniquement déterminée Px de x sur V. Chaque $x \in E$ se décompose alors en deux vecteurs Px et η tels que*

$$x = Px + \eta,$$

et les vecteurs η constituent un sous-espace linéaire fermé V^* de V, une variété complémentaire de V.

Etant donné un projecteur P, deux variétés linéaires et fermées sont déterminées: V et V^*. En général un projecteur P n'est pas uniquement déterminé par un espace V. Pour cette raison pous désignons parfois un projecteur par P_{V,V^*}.

Les projecteurs possédent des propriétés tout à fait analogue à celles de l'espace hilbertien.

1. Un projecteur P est défini partout dans E.
2. Un projecteur P est borné. Selon le théorème 1 on a

$$\| Px \| \leqq \| x \|.$$

Puisque $Px = x$ pour les vecteurs de V, la norme d'un projecteur est 1.

3. Pour chaque projecteur on a

$$P^2 = P$$

4. V consiste des éléments pour lesquels

$$P_{V,V^*} x = x.$$

5. V^* consiste des éléments pour lesquels

$$P_{V,V^*} x = \theta.$$

6. Si I désigne l'opération identique, on a

$$I - P_{V,V^*} = P_{V^*,V}.$$

C'est une conséquence immédiate de la réciprocité entre V et V^*.

7. Un projecteur $\neq I$ n'a pas d'inverse. L'équation $Px = y$ avec $y \in V$ n'a pas une solution unique x dans E: la solution générale est $x = y + z$ avec z arbitraire dans V^*.
8. Un projecteur est une opération fermée, c. à d. si $x_n \to x$ et $Px_n \to \xi$ on a $\xi = Px$.

Pour $n > N'(\varepsilon)$ on a $\|x_n - x\| < \varepsilon$ et, avec le théorème 1,

$$\| Px_n - Px \| < \varepsilon.$$

Pour $n > N''(\varepsilon)$ on a $\| Px_n - \xi \| < \varepsilon$. Si N est le maximum de N' et N'', les deux inégalités sont vraies pour $n > N(\varepsilon)$. Il s'ensuit avec l'inégalité triangulaire $\| Px - \xi \| < \varepsilon$, donc $\xi = Px$.

9. Si V_1^* et V_2^* désignent deux espaces complémentaires de V, on a

$$P_{V,V_2^*} \cdot P_{V,V_1^*} = P_{V,V_1^*},$$
$$P_{V,V_1^*} \cdot P_{V,V_2^*} = P_{V,V_2^*}.$$

La propriété 3 en est un cas spécial.

10. Les projecteurs P_{V,V^*} et $P_{V^*,V}$ sont orthogonaux, c. à d.

$$P_{V^*,V} \cdot P_{V,V^*} = P_{V,V^*} \cdot P_{V^*,V} = 0.$$

Plus général on a

$$P_{V_1,V_1^*} \cdot P_{V_2,V_2^*} = 0$$

si et seulement si $V_1^* \supset V_2$. C'est puisqué la projection d'un vecteur vaut θ si et seulement si le vecteur appartient à l'espace complémentaire dont on fait usage pour la projection.

Remarquons qu'on ne peut pas conclure de $P_{V_1,V_1^*} \cdot P_{V_2,V_2^*} = 0$ qu'on a $P_{V_2,V_2^*} \cdot P_{V_1,V_1^*} = 0$, comme il est possible dans l'espace de HILBERT. Si p. ex. $V_1 \equiv V_2^*$ et V_2 diffère de V_1^*, on a $P_2P_1 = 0$ mais $P_1P_2 \neq 0$.

Théorème 20. *Soit A une opération linéaire partout définie dans E avec les propriétés suivantes:*

1. *A est fermé,*
2. *$A^2 = A$,*
3. *$\| x - Ax \| \leqq \| x - y \|$ pour tout $x \in E$ et pour tout y tel que $y = Ay$.*

Alors A est un projecteur.

Démonstration. Soit V l'ensemble des éléments x pour lesquels $Ax = x$. Il suit de la linéarité de A que V est linéaire:

$$Ax_1 + Ax_2 = A(x_1 + x_2) = x_1 + x_2.$$

De $x_n \to x$ ($x_n \in V$) il suit $Ax_n = x_n \to x$, donc en vertu de 1, $Ax = x$; V est donc fermé. Pour tout $x \in E$ on a $Ax \in V$ puisque $A(Ax) = A^2x = Ax$. Avec 3 il suit que Ax est une projection de x sur V. A est alors un projecteur sur V dont la variété complémentaire est définie par $Ax = \theta$.

Ce théorème est l'analogue du théorème selon lequel une opération linéaire hermitienne, partout définie dans l'espace de HILBERT, qui est égale à son carré, est un projecteur.

Théorème 21. *Soient P_1 et P_2 deux projecteurs, correspondants respectivement à V_1, V_1^* et V_2, V_2^*. Une condition nécessaire et suffisante pourque $P_1 - P_2$ soit un projecteur est que $P_1P_2 = P_2P_1 = P_2$.*

Démonstration. 1. Si $P_1 - P_2$ est un projecteur on a

$$(P_1 - P_2)^2 = P_1 - P_2,$$

d'où il suit

$$P_1 - P_1 P_2 - P_2 P_1 + P_2 = P_1 - P_2,$$

ou

$$P_1 P_2 + P_2 P_1 = 2 P_2. \quad \ldots \ldots \ldots (15)$$

Il s'ensuit

$$P_1 (P_1 P_2 + P_2 P_1) = 2 P_1 P_2,$$

ou

$$P_1 P_2 P_1 = P_1 P_2.$$

De plus

$$(P_1 P_2 + P_2 P_1) P_1 = 2 P_2 P_1,$$

ou

$$P_1 P_2 P_1 = P_2 P_1.$$

Donc

$$P_1 P_2 = P_2 P_1 = P_1 P_2 P_1$$

et donc avec (15)

$$P_1 P_2 = P_2 P_1 = P_2.$$

2. Soit donné $P_1P_2 = P_2P_1 = P_2$. Il s'ensuit

$$(P_1 - P_2)^2 = P_1 - P_1 P_2 - P_2 P_1 + P_2 = P_1 - P_2.$$

$P_1 - P_2$ étant fermé avec P_1 et P_2, il ne reste que de montrer (voir théorème 20) que

$$\| x - (P_1 - P_2) x \| \leqq \| x - y \| \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

pour tout y tel que $y = (P_1 - P_2) y$.

On a

$$\| x - (P_1 - P_2) x \| \leqq \max (\| x - P_1 x \|, \| P_2 x \|)$$

ou, P_1 étant un projecteur,

$$\| x - (P_1 - P_2) x \| \leqq \max (\| x - z \|, \| P_2 x \|) \quad . \quad . \quad . \quad (17)$$

pour tout z tel que $z = P_1 z$. Chaque y tel que $y = (P_1 - P_2) y$ satisfait à la relation $z = P_1 z$.

En effet

$$P_1 y = P_1 (P_1 - P_2) y = (P_1 - P_1 P_2) y = (P_1 - P_2) y = y.$$

A fortiori à (17) est satisfait pour $z = y$.

La relation désirée (16) suit alors de (17) si nous montrons que

$$\| x - y \| \geqq \| P_2 x \|.$$

On a

$$P_2 (x - y) = P_2 x - P_2 y = P_2 x - P_2 (P_1 - P_2) y =$$
$$= P_2 x - P_2 P_1 y + P_2 y = P_2 x - P_2 y + P_2 y = P_2 x.$$

Il suit alors du théorème 18

$$\| x - y \| \geqq \| P_2 (x - y) \| = \| P_2 x \|.$$

Le théorème est ainsi montré. Remarquons que la condition

$$P_1 P_2 = P_2 P_1 = P_2$$

exprime que $V_1 \supset V_2$; $P_1 - P_2$ projette alors sur l'espace linéaire qui donne V_1 par combinaison linéaire avec V_2.

Théorème 22. *Notation comme dans le théorème 21. Pour que $P_1 + P_2$ soit un projecteur il faut et suffit que $P_1P_2 = P_2P_1 = 0$; $P_1 + P_2$ projette alors sur $V_1 + V_2$.*

Démonstration. Si $P_1 + P_2$ est un projecteur, on a

$$(P_1 + P_2)^2 = P_1 + P_2,$$

ou

$$P_1 P_2 + P_2 P_1 = 0$$

Alors

$$0 = P_1 (P_1 P_2 + P_2 P_1) = P_1 P_2 + P_1 P_2 P_1$$
$$0 = (P_1 P_2 + P_2 P_1) P_1 = 2 P_1 P_2 P_1,$$

d'où il suit $P_1P_2 = P_2P_1 = 0$.

Soit ensuite donné $P_1P_2 = P_2P_1 = 0$. Or, $P_1 + P_2$ est un projecteur si et seulement si $I - (P_1 + P_2) = (I - P_1) - P_2$ est un projecteur (voir la propriété 6). Selon le théorème 21 ceci est le cas, puisque

$$(I - P_1) P_2 = P_2 - P_1 P_2 = P_2,$$
$$P_2 (I - P_1) = P_2 - P_2 P_1 = P_2.$$

Remarque. Soient données deux variétés linéaires fermées V_1 et V_2 et soient P_1 et P_2 deux projecteurs sur V_1 resp. V_2; alors le dernier théorème exprime en particulier:

Soient V_1 et V_2 orthogonaux — nous entendons par cela la propriété $P_1P_2 = P_2P_1 = 0$ — et soit x un vecteur tel que $P_1x = \theta$, $P_2x = \theta$, donc soit (x, θ) la plus petite distance de x à V_1 et à V_2, alors la plus petite distance de x à la variété linéaire $V_1 + V_2$ déterminée par V_1 et V_2, est également (x, θ). Comparer cette propriété avec celle de la géométrie euclidienne selon laquelle une ligne droite, qui est perpendiculaire sur deux lignes droites non-parallèles d'un plan, est perpendiculaire sur chaque ligne de ce plan. Remarquons que l'orthogonalité de V_1 et V_2 n'est pas une propriété spécifique de V_1 et V_2, puisque de $P_{V_1, V_1^*} \cdot P_{V_2, V_2^*} = 0$ on ne peut pas conclure que $P_{V_1, V_1^{*\prime}} \cdot P_{V_2, V_2^{*\prime}} = 0$ (voir la propriété 10).

Théorème 23. *Pourque P_1P_2 soit un projecteur il suffit que*

$$P_1P_2 = P_2P_1.$$

Démonstration. P_1 étant un projecteur, on a

$$\| P_1 x \| \leqq \| x - y \|$$

pour tout $x \in E$ et tout y tel que $P_1y = \theta$.

Substituons pour x: P_2x $(x \in E)$:

$$\| P_1 P_2 x \| \leqq \| P_2 x - y \|.$$

Soit alors z tel que $P_1P_2z = \theta$ et posons $y = P_2z$. On obtient

$$\| P_1 P_2 x \| \leqq \| P_2 x - P_2 z \|,$$

et donc en vertu du théorème 18

$$\| P_1 P_2 x \| \leqq \| x - z \|$$

pour tout z tel que $P_1P_2z = \theta$. Cela s'écrit

$$\| x - (I - P_1 P_2) x \| \leqq \| x - z \|$$

tandis que $(I - P_1P_2)z = z - P_1P_2z = z$.

Puisque $P_1P_2 = P_2P_1$ on a

$$(I - P_1 P_2)^2 = I - P_1 P_2$$

et il suit donc du théorème 20 que $I - P_1P_2$ est un projecteur; de même donc P_1P_2.

La condition $P_1P_2 = P_2P_1$ a été posée d'après l'analogie de l'espace de HILBERT. On voit cependant que la condition $P_2P_1P_2 = P_1P_2$ est déjà suffisante.

La condition n'est pas nécessaire comme il résulte des exemples suivants.

a) Supposons que les espaces V_1 et V_2, déterminés par $x = P_1x$ resp. $x = P_2x$ satisfont à $V_1 \supset V_2$. Alors $P_1P_2 = P_2$ ou $(I - P_1)P_2 = 0$. Cependant il ne suit pas nécessairement $P_2(I - P_1) = 0$ ou $P_2P_1 = P_2$. Donc P_1P_2 peut être un projecteur sans que $P_1P_2 = P_2P_1$ (comparer la propriété 10). On sait que dans l'espace de HILBERT il suit de $P_1P_2 = 0$ qu'on a aussi $P_2P_1 = 0$.

b) Si $V_1^* \supset V_2^*$ on a $P_1P_2 = P_1$. En effet, on a pour les projecteurs $\overline{P}_1 = I - P_1$, $\overline{P}_2 = I - P_2$

$$\overline{V}_1 = V_1^*,\ \overline{V}_2 = V_2^*,\ \overline{V}_1 \supset \overline{V}_2$$

de sorte qu'on a selon l'exemple *a*)

$$\overline{P}_1 \overline{P}_2 = \overline{P}_2$$

$$(I - P_1)(I - P_2) = I - P_2$$

ou $P_1P_2 = P_1$. Cependant on ne peut pas conclure que $P_2P_1 = P_1$, ni $P_2P_1P_2 = P_1P_2$.

Mathematics. — *Ueber den Aussagen- und den engeren Prädikatenkalkül.* I. By J. RIDDER. (Communicated by Prof. W. VAN DER WOUDE.)

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Durch Hinzufügung der hier folgenden Axiome V und VI (siehe § 2) zu den Axiomen des RUSSELL-WHITEHEAD'schen Aussagenkalküls [1]) entsteht ein System, dessen Gleichwertigkeit mit einer BEELE'schen Algebra in einem Felde von abzählbar unendlich vielen Elementen zeigt, dass der Namen von zweiwertigem Kalkül, mit welchem der R.—W. Aussagenkalkül oft angedeutet wird, nicht ganz richtig gewählt ist; denn dieser lässt sich, ebense wie der in der angegebenen Weise erweiterte Kalkül, auch als mehrwertiger Kalkul auffassen. Beide Kalküle lassen sich intuitiv deuten als Aussagenkalküle einer elementaren Wahrscheinlichkeitsrechnung. [2])

Völlig gleichartige Bemerkungen gelten für den engeren Prädikatenkalkül. [3])

Erster Aufbau des Aussagenkalküls.

§ 1. Von den im folgenden durch grosse lateinische Buchstaben [4]) angedeuteten *elementaren Aussagen* (*elementaren Kalkülformeln*) wird nicht angenommen, dass sie entweder wahr oder falsch sind. Wir lassen sowohl die Möglichkeit von endlich- wie die von abzählbar unendlich vielen elementaren Aussagen zu.

Als undefinierte *Grundverknüpfungen* nehmen wir *oder* und *nicht*- (*Komplement von*), und deuten diese bzw. durch + und durch ein Akzent an. Also $X+Y$ lese man: X oder Y; A' lese man: nicht- (non-) A oder Komplement von A.

Einsetzungsregel E (erster Teil). Aus einer *Kalkülformel* erhält man wieder eine Kalkülformel, wenn man einen in ihr auftretenden grossen lateinischen Buchstaben durch eine Kalkülformel ersetzt (gleichgestaltete Buchstaben durch gleichgestaltete Formeln); dabei sind die grossen lateinischen Buchstaben, λ und ν als Kalkülformeln anzusehen, ferner mit $\mathfrak{R}$ und $\mathfrak{S}$ auch $\mathfrak{R}+\mathfrak{S}$ und $\mathfrak{R}'$, $\mathfrak{S}'$. [5])

Definition. $\mathfrak{A} \to \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}'+\mathfrak{B}$. [5])

Definition. $\doteq \bar{\nu}$ ist *ein Assertionszeichen*.

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[1]) Siehe D. HILBERT und W. ACKERMANN, Grundzüge der theoretischen Logik, 2e Aufl. New York, Dover Publ. 1946, Kap. 1.

[2]) Siehe etwa A. KOLMOGOROFF, Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin 1933, Kap. 1.

[3]) Siehe ihre Darstellung bei HILBERT und ACKERMANN, loc. cit. 1), Kap. 3.

[4]) Also durch $A, B, C, \ldots$; doch auch $A_1, B_1, C_1 \ldots$; $A_2, B_2, \ldots$; $A_n, \ldots$ (n eine natürliche Zahl).

[5]) Grosse deutsche Buchstaben werden immer Kalkülformeln andeuten.

Definition. Ein Ausdruck, bestehend aus einer Kalkülformel gefolgt durch $\doteq \bar{\nu}$, und entstanden nach endlichmaliger Anwendung von Axiomen, Einsetzungsregel E oder (und) Schlussschema S, ist *ein Theorem*.

So sind die Axiome I bis IV als Theoreme zu betrachten.

Einsetzungsregel E (zweiter Teil). Ist $\mathfrak{A} \doteq \bar{\nu}$ ein Theorem, und $\mathfrak{B}$ eine neue, aus $\mathfrak{A}$ mittels der Einsetzungsregel E (erster Teil) hervorgehende Kalkülformel, so liefert auch $\mathfrak{B} \doteq \bar{\nu}$ ein Theorem.

Axiom I. $[(X+X) \rightarrow X] \doteq \bar{\nu}$.

Axiom II. $[X \rightarrow (X+Y)] \doteq \bar{\nu}$.

Axiom III. $[(X+Y) \rightarrow (Y+X)] \doteq \bar{\nu}$.

Axiom IV. $[(Y \rightarrow Z) \rightarrow \{(X+Y) \rightarrow (X+Z)\}] \doteq \bar{\nu}$.

Schlussschema S. Sind

$$\mathfrak{A} \doteq \bar{\nu} \text{ und } [\mathfrak{A} \rightarrow \mathfrak{B}] \doteq \bar{\nu}$$

Theoreme, so ist auch

$$\mathfrak{B} \doteq \bar{\nu}$$

ein Theorem. [6])

Definition. $\mathfrak{A} . \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A}' + \mathfrak{B}')'$.

Definition. $\mathfrak{A} \leftrightarrow \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A} \rightarrow \mathfrak{B}) . (\mathfrak{B} \rightarrow \mathfrak{A})$.

Definition. $\mathfrak{A} \subset \mathfrak{B}$ ist eine andere Schreibweise von $(\mathfrak{A} \rightarrow \mathfrak{B}) \doteq \bar{\nu}$.

Satz 1. *a*) $\mathfrak{A} \subset \mathfrak{A}$ ist ein Theorem;

β) (Schlussschema) hat man als Theoreme $\mathfrak{A} \subset \mathfrak{B}$, $\mathfrak{B} \subset \mathfrak{C}$, so lässt sich auch schreiben $\mathfrak{A} \subset \mathfrak{C}$.

Beweis von *a*). $[\mathfrak{A} \rightarrow (\mathfrak{A} + \mathfrak{A})] \doteq \bar{\nu}$. (Axiom II u. Einsetzungsregel E)

$[(\mathfrak{A} + \mathfrak{A}) \rightarrow \mathfrak{A}] \doteq \bar{\nu}$. (Axiom I u. Einsetzungsregel E)

Also, nach H.-A., loc. cit. 1), S. 26 (Regel V),

$$[\mathfrak{A} \rightarrow \mathfrak{A}] \doteq \bar{\nu},$$

was sich auch schreiben lässt:

$$\mathfrak{A} \subset \mathfrak{A}.$$

Beweis von *β*). Statt $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{C}$ lässt sich auch schreiben:

$$(\mathfrak{A} \rightarrow \mathfrak{B}) \doteq \bar{\nu} \text{ bzw. } (\mathfrak{B} \rightarrow \mathfrak{C}) \doteq \bar{\nu}.$$

Also, nach H.-A., S. 26 (Regel V),

$$(\mathfrak{A} \rightarrow \mathfrak{C}) \doteq \bar{\nu},$$

oder

$$\mathfrak{A} \subset \mathfrak{C}.$$

Satz 2. *a*) $\mathfrak{A} . \mathfrak{B} \subset \mathfrak{A}$ und $\mathfrak{A} . \mathfrak{B} \subset \mathfrak{B}$ (sind Theoreme);

β) (Schlussschema) aus $\mathfrak{C} \subset \mathfrak{A}$ und $\mathfrak{C} \subset \mathfrak{B}$ folgt, dass sich auch schreiben lässt $\mathfrak{C} \subset \mathfrak{A} . \mathfrak{B}$.

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[6]) Die Axiome I—IV mit Einzetzungsregel E und Schlussschema S findet man in HILBERT–ACKERMANN, loc. cit. 1), Kap. 1 als Grundlage für den Aufbau des RUSSELL–WHITEHEAD'schen zweiwertigen Aussagenkalküls. Da die Zweiwertigkeit bei der Ableitung von Theoremen und Schlussschemata im zit. Kapitel nirgends benutzt wird, gelten diese auch hier; nur ist unsere Schreibweise eine andere.

B e w e i s von α). Nach H.-A., S. 29, Formel (12) und Formel (13) (mit unserer Einsetzungsregel E) gilt

$$\{(\mathfrak{A}\,.\,\mathfrak{B}) \rightarrow \mathfrak{A}\} \doteq \bar{\nu} \text{ bzw. } \{(\mathfrak{A}\,.\,\mathfrak{B}) \rightarrow \mathfrak{B}\} \doteq \bar{\nu},$$

oder in anderer Schreibweise:

$$\mathfrak{A}\,.\,\mathfrak{B} \subset \mathfrak{A} \text{ bzw. } \mathfrak{A}\,.\,\mathfrak{B} \subset \mathfrak{B}.$$

B e w e i s v o n β). Nach Annahme lässt sich schreiben:

$$\mathfrak{C} \subset \mathfrak{A} \text{ oder } (\mathfrak{C} \rightarrow \mathfrak{A}) \doteq \bar{\nu},$$

und

$$\mathfrak{C} \subset \mathfrak{B} \text{ oder } (\mathfrak{C} \rightarrow \mathfrak{B}) \doteq \bar{\nu}.$$

Nach H.-A., S. 31 u. 12, b 3) folgt

$$\{(\mathfrak{C} \rightarrow \mathfrak{A})\,.\,(\mathfrak{C} \rightarrow \mathfrak{B})\} \doteq \bar{\nu}. \quad \ldots \ldots \quad (1)$$

Auch hat man das Theorem:

$$[\{(\mathfrak{C} \rightarrow \mathfrak{A})\,.\,(\mathfrak{C} \rightarrow \mathfrak{B})\} \rightarrow \{\mathfrak{C} \rightarrow (\mathfrak{A}\,.\,\mathfrak{B})\}] \doteq \bar{\nu}.\text{[7]} \quad \ldots \quad (2)$$

Aus (1) und (2) folgt, nach Schema S,

$$\{\mathfrak{C} \rightarrow (\mathfrak{A}\,.\,\mathfrak{B})\} \doteq \bar{\nu}, \text{ oder } \mathfrak{C} \subset \mathfrak{A}\,.\,\mathfrak{B}.$$

S a t z 3. α) $\mathfrak{A} \subset \mathfrak{A} + \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A} + \mathfrak{B}$;
β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset \mathfrak{C}$ und $\mathfrak{B} \subset \mathfrak{C}$, so auch $\mathfrak{A} + \mathfrak{B} \subset \mathfrak{C}$.

B e w e i s v o n α).

$\{\mathfrak{A} \rightarrow (\mathfrak{A} + \mathfrak{B})\} \doteq \bar{\nu}$. (Axiom II u. Einsetzungsregel E) . (3)

$\{\mathfrak{B} \rightarrow (\mathfrak{B} + \mathfrak{A})\} \doteq \bar{\nu}$. (Axiom II u. Einsetzungsregel E)

$\{(\mathfrak{B} + \mathfrak{A}) \rightarrow (\mathfrak{A} + \mathfrak{B})\} \doteq \bar{\nu}$. (Axiom III u. Einsetzungsregel E)

$\{\mathfrak{B} \rightarrow (\mathfrak{A} + \mathfrak{B})\} \doteq \bar{\nu}$. (H.-A., S. 26 (Regel V)) . . . . . . (4)

(3) und (4) liefern α).

B e w e i s v o n β). Aus $\mathfrak{A} \subset \mathfrak{C}$ oder $(\mathfrak{A} \rightarrow \mathfrak{C}) \doteq \bar{\nu}$, und $\mathfrak{B} \subset \mathfrak{C}$ oder $(\mathfrak{B} \rightarrow \mathfrak{C}) \doteq \bar{\nu}$ folgt, nach H.-A., S. 31 u. 12, b 3),

$$\{(\mathfrak{A} \rightarrow \mathfrak{C})\,.\,(\mathfrak{B} \rightarrow \mathfrak{C})\} \doteq \bar{\nu}. \quad \ldots \ldots \quad (5)$$

Auch hat man das Theorem:

$$[\{(\mathfrak{A} \rightarrow \mathfrak{C})\,.\,(\mathfrak{B} \rightarrow \mathfrak{C})\} \rightarrow \{(\mathfrak{A} + \mathfrak{B}) \rightarrow \mathfrak{C}\}] \doteq \bar{\nu}.\text{[7]} \quad \ldots \quad (6)$$

Aus (5) und (6) folgt, nach Schema S,

$$\{(\mathfrak{A} + \mathfrak{B}) \rightarrow \mathfrak{C}\} \doteq \bar{\nu}, \text{ oder } \mathfrak{A} + \mathfrak{B} \subset \mathfrak{C}.$$

D e f i n i t i o n. $\mathfrak{A} = \mathfrak{B}$ ist eine kürzere Schreibweise von: $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$.

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[7]) Zum Beweise bringe man den Ausdruck [— — — — — — —] in die konjunktive Normalform. Siehe H.–A., loc. cit. 1), S. 31 und 10.

Satz 4. $[\mathfrak{A} . (\mathfrak{B} + \mathfrak{C})] = [(\mathfrak{A} . \mathfrak{B}) + (\mathfrak{A} . \mathfrak{C})]$.

Beweis. Nach H.-A., S. 31 ist

$$[\{\mathfrak{A} . (\mathfrak{B} + \mathfrak{C})\} \leftrightarrow \{(\mathfrak{A} . \mathfrak{B}) + (\mathfrak{A} . \mathfrak{C})\}] \doteq \bar{\nu}.$$

Nach Schema S und H.-A., S. 29, Formeln (12) u. (13) folgt daraus

$$[\{\mathfrak{A} . (\mathfrak{B} + \mathfrak{C})\} \rightarrow \{(\mathfrak{A} . \mathfrak{B}) + (\mathfrak{A} . \mathfrak{C})\}] \doteq \bar{\nu} \quad \ldots \quad (7)$$

und

$$[\{(\mathfrak{A} . \mathfrak{B}) + (\mathfrak{A} . \mathfrak{C})\} \rightarrow \{\mathfrak{A} . (\mathfrak{B} + \mathfrak{C})\}] \doteq \bar{\nu}; \quad \ldots \quad (8)$$

statt (7) und (8) lässt sich auch schreiben:

$$[\mathfrak{A} . (\mathfrak{B} + \mathfrak{C})] \subset [(\mathfrak{A} . \mathfrak{B}) + (\mathfrak{A} . \mathfrak{C})],$$

bzw.

$$[(\mathfrak{A} . \mathfrak{B}) + (\mathfrak{A} . \mathfrak{C})] \subset [\mathfrak{A} . (\mathfrak{B} + \mathfrak{C})].$$

Daraus folgt der Satz.

Satz 5. $(\mathfrak{A} + \mathfrak{A}') \doteq \bar{\nu}$,
und $(\mathfrak{A} . \mathfrak{A}')' \doteq \bar{\nu}$.

Beweis. Das erste Theorem ist eine andere Schreibweise von H.-A., S. 27, Formel (3); das zweite folgt aus dem ersten mit H.-A., S. 31 u. 10, a 3).

§ 2. Wir erweitern nun den RUSSELL-Whitehead'schen Aussagenkalkül durch Hinzufügung der hier folgenden Axiome V und VI.

Axiom V. $[\lambda \rightarrow X] \doteq \bar{\nu}$.

Satz 6. $\lambda \subset \mathfrak{A}$.

Das folgt mit Einsetzungsregel E aus Axiom V.

Axiom VI. $[X \rightarrow \nu] \doteq \bar{\nu}$.

Satz 7. $\mathfrak{A} \subset \nu$.

Satz 8. Lässt sich schreiben $\mathfrak{A} \doteq \bar{\nu}$, so auch $\mathfrak{A} = \nu$, und umgekehrt.

Beweis. Nach Satz 3 lässt sich mit willkürlicher Kalkülformel $\mathfrak{A}$ schreiben:

$$\mathfrak{A} \subset (\nu' + \mathfrak{A}), \text{ oder } \mathfrak{A} \subset (\nu \rightarrow \mathfrak{A}), \text{ oder } \{\mathfrak{A} \rightarrow (\nu \rightarrow \mathfrak{A})\} \doteq \bar{\nu}.$$

Ist nun auch $\mathfrak{A} \doteq \bar{\nu}$, so folgt mit Schema S

$$(\nu \rightarrow \mathfrak{A}) \doteq \bar{\nu}, \text{ oder } \nu \subset \mathfrak{A},$$

also, wegen Satz 7, auch

$$\mathfrak{A} = \nu.$$

Ist $\mathfrak{K} \doteq \bar{\nu}$ ein Theorem [man nehme für $\mathfrak{K}$ etwa $(X + X) \rightarrow X$], so folgt daraus und aus dem (mit Einsetzungsregel E) aus Axiom VI ableitbaren Theorem $[\mathfrak{K} \rightarrow \nu] \doteq \bar{\nu}$, dass sich schreiben lässt:

$$\nu \doteq \bar{\nu}. \quad \ldots \quad (9)$$

Nehmen wir nun $\mathfrak{A} = \nu$ an, so ist auch

$$\nu \subset \mathfrak{A}, \text{ oder } (\nu \rightarrow \mathfrak{A}) \doteq \bar{\nu}.$$

Dies und (9) liefern nach Schema S

$$\mathfrak{A} \doteq \bar{\nu}.$$

Satz 9. $\mathfrak{A}''$ oder $(\mathfrak{A}')' = \mathfrak{A}$. [8])

Beweis. Wegen H.-A., S. 27, Formeln (4) u. (5) und der Einsetzungsregel E ist immer

$$(\mathfrak{A}'' \rightarrow \mathfrak{A}) \doteq \bar{\nu} \text{ und } (\mathfrak{A} \rightarrow \mathfrak{A}'') \doteq \bar{\nu},$$

oder in anderer Schreibweise:

$$\mathfrak{A}'' \subset \mathfrak{A} \text{ und } \mathfrak{A} \subset \mathfrak{A}'',$$

somit

$$\mathfrak{A}'' = \mathfrak{A}.$$

Satz 10. Lässt sich schreiben $\mathfrak{A} = \mathfrak{B}$, so auch $\mathfrak{A}' = \mathfrak{B}'$. [8])

Beweis. $\mathfrak{A} = \mathfrak{B}$, oder $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$, oder

$$(\mathfrak{A} \rightarrow \mathfrak{B}) \doteq \bar{\nu} \text{ und } (\mathfrak{B} \rightarrow \mathfrak{A}) \doteq \bar{\nu}.$$

Daraus folgt nach H.-A., S. 27, Formel (6) und dem Schlussschema S

$$(\mathfrak{B}' \rightarrow \mathfrak{A}') \doteq \bar{\nu} \text{ und } (\mathfrak{A}' \rightarrow \mathfrak{B}') \doteq \bar{\nu}$$

oder

$$\mathfrak{B}' \subset \mathfrak{A}' \text{ und } \mathfrak{A}' \subset \mathfrak{B}',$$

oder

$$\mathfrak{A}' = \mathfrak{B}'.$$

Satz 11. $\lambda' = \nu$; $\nu' = \lambda$.

Beweis. Nach Axiom V und Einsetzungsregel E ist

$$[\lambda \rightarrow \lambda] \doteq \bar{\nu},$$

also nach Satz 8 auch

$$[\lambda \rightarrow \lambda] = \nu, \text{ oder } (\lambda' + \lambda) = \nu. \quad (10)$$

Nach Satz 3 ist

$$\lambda' \subset (\lambda' + \lambda); \quad (11)$$

nach den Sätzen 1 und 6 ist

$$\lambda' \subset \lambda' \text{ und } \lambda \subset \lambda',$$

also nach Satz 3 auch

$$(\lambda' + \lambda) \subset \lambda', \quad (12)$$

Aus (11) und (12) folgt

$$\lambda' = \lambda' + \lambda. \quad (13)$$

Da das Gleichheitszeichen, wie sich leicht mit Satz 1 beweisen lässt, die Eigenschaft der Transitivität hat, folgt aus (10) und (13)

$$\lambda' = \nu.$$

[8]) Dieser Satz wird bewiesen ohne Benutzung der Axiome V und VI; er gilt somit schon in dem Aussagenkalkül des § 1.

Auch ist nun

$$\lambda'' = \lambda \text{ (Satz 9) und } \lambda'' = \nu' \text{ (Satz 10), somit auch } \nu' = \lambda''.$$

Daraus folgt schliesslich

$$\nu' = \lambda.$$

Satz 5 bis. $(\mathfrak{A} + \mathfrak{A}') = \nu$
und

$$(\mathfrak{A} . \mathfrak{A}') = \lambda.$$

Beweis. Nach den Sätzen 5 und 8 ist $(\mathfrak{A} + \mathfrak{A}') = \nu$, und

$$(\mathfrak{A} . \mathfrak{A}')' = \nu,$$

somit nach Satz 10

$$(\mathfrak{A} . \mathfrak{A}')'' = \nu'.$$

Nach Satz 9 ist

$$(\mathfrak{A} . \mathfrak{A}')'' = (\mathfrak{A} . \mathfrak{A}') \text{ oder } (\mathfrak{A} . \mathfrak{A}') = (\mathfrak{A} . \mathfrak{A}')''.$$

Die Transitivität des Gleichheitszeichens liefert

$$(\mathfrak{A} . \mathfrak{A}') = \nu',$$

und, mit Satz 11, auch

$$(\mathfrak{A} . \mathfrak{A}') = \lambda.$$

Satz 12. Lässt zich schreiben $\mathfrak{A} \subset \mathfrak{B}$, und sind $\mathfrak{D}$ und $\mathfrak{E}$ neue, aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ mittels der Einsetzungsregel E (erster Teil) hervorgehende Kalkülformeln, wobei sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende gleichgestaltete Buchstaben in beiden nicht oder in beiden an allen Stellen in gleicher Weise (d.h. durch gleichgestaltete Kalkülformeln) ersetzt sind, so hat man auch $\mathfrak{D} \subset \mathfrak{E}$.

Beweis. $\mathfrak{A} \subset \mathfrak{B}$ ist eine andere Schreibweise von

$$\mathfrak{A}' + \mathfrak{B} \doteq \bar{\nu}.$$

Nach der Einsetzungsregel E (2er Teil) lässt sich nun auch schreiben

$$\mathfrak{D}' + \mathfrak{E} \doteq \bar{\nu}, \text{ oder } \mathfrak{D} \subset \mathfrak{E}.$$

Zweiter Aufbau des Aussagenkalküls.

§ 3. *Elementare Aussagen* (*elementare Kalkülformeln*) sollen wieder durch grosse lateinische Buchstaben angedeutet werden [4]).

Als undefinierte *Grundverknüpfungen* nehmen wir *die Inklusion* $\subset$ und die Verknüpfungen ‚*oder*', ‚*und*' und ‚*nicht-* (*Komplement von*)', bzw. angedeutet durch + , . und ein Akzent; die Relationen, welche man zwischen diesen Verknüpfungen annehmen soll, sind in den nachfolgenden Axiomen enthalten.

Einsetzungsregel E^* (erster Teil) habe den gleichen Wortlaut wie Einsetzungsregel E (erster Teil) mit dem Unterschiede, dass mit $\mathfrak{R}$ und $\mathfrak{S}$ (neben $\mathfrak{R} + \mathfrak{S}$, $\mathfrak{R}'$, $\mathfrak{S}'$) auch $\mathfrak{R} . \mathfrak{S}$ als Formel betrachtet werden soll.

Den Inhalt von Satz 12 nehmen wir als zweiten Teil von Regel E^* an. Also:

Einsetzungsregel E^* (zweiter Teil). Lässt sich (auf Grund der nachfolgenden Axiome) schreiben $\mathfrak{A} \subset \mathfrak{B}$, mit $\mathfrak{A}$ und $\mathfrak{B}$ Kalkülformeln, und sind $\mathfrak{D}$ und $\mathfrak{E}$ aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ mittels der Einsetzungsregel E^* (erster Teil) hervorgehende Kalkülformeln, wobei sowohl in $\mathfrak{A}$ wie in $\mathfrak{B}$ vorkommende gleichgestaltete Buchstaben in beiden nicht oder in beiden an allen Stellen durch gleichgestaltete Kalkülformeln etsetzt sind, so lässt sich auch schreiben $\mathfrak{D} \subset \mathfrak{E}$ [5]).

Axiom 1°. Das Inklusionszeichen soll die folgenden Eigenschaften haben:

α) es lässt sich schreiben $X \subset X$;

β) (Schlussschema) lässt zich schreiben $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{C}$, so auch $\mathfrak{A} \subset \mathfrak{C}$.

Definition. $\mathfrak{A} = \mathfrak{B}$ ist eine kürzere Schreibweise von: $\mathfrak{A} \subset \mathfrak{B}$ und $\mathfrak{B} \subset \mathfrak{A}$.

Satz 13. $\mathfrak{A} = \mathfrak{A}$ (die Gleichheitsrelation ist *reflexiv*).

Satz 14 (Schlussschema). Lässt sich schreiben $\mathfrak{A} = \mathfrak{B}$, so auch $\mathfrak{B} = \mathfrak{A}$ (die Relation ist *symmetrisch*).

Satz 15 (Schlussschema). Lässt sich schreiben $\mathfrak{A} = \mathfrak{B}$, $\mathfrak{B} = \mathfrak{C}$, so auch $\mathfrak{A} = \mathfrak{C}$ (die Relation ist *transitiv*).

Axiom 2°. Die Konjunktion soll die folgenden Eigenschaften haben:

α) $(X . Y) \subset X$ und $(X . Y) \subset Y$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{C} \subset \mathfrak{A}$ und $\mathfrak{C} \subset \mathfrak{B}$, so auch $\mathfrak{C} \subset (\mathfrak{A} . \mathfrak{B})$.

Axiom 3°. Die Disjunktion soll die folgenden Eigenschaften haben:

α) $X \subset (X + Y)$ und $Y \subset (X + Y)$;

β) (Schlussschema) lässt sich schreiben $\mathfrak{A} \subset \mathfrak{C}$ und $\mathfrak{B} \subset \mathfrak{C}$, so auch $(\mathfrak{A} + \mathfrak{B}) \subset \mathfrak{C}$.

Satz 16 (Schlussschemata). Lässt sich schreiben $\mathfrak{A} = \mathfrak{A}_1$ und $\mathfrak{B} = \mathfrak{B}_1$, so auch $(\mathfrak{A} . \mathfrak{B}) = (\mathfrak{A}_1 . \mathfrak{B}_1)$ und $(\mathfrak{A} + \mathfrak{B}) = (\mathfrak{A}_1 + \mathfrak{B}_1)$.

Satz 17.

$$[(\mathfrak{A} . \mathfrak{B}) . \mathfrak{C}] = [\mathfrak{A} . (\mathfrak{B} . \mathfrak{C})] \text{ und } [(\mathfrak{A} + \mathfrak{B}) + \mathfrak{C}] = [\mathfrak{A} + (\mathfrak{B} + \mathfrak{C})].$$

Satz 18. $(\mathfrak{A} . \mathfrak{B}) = (\mathfrak{B} . \mathfrak{A})$ und $(\mathfrak{A} + \mathfrak{B}) = (\mathfrak{B} + \mathfrak{A})$.

Satz 19. $(\mathfrak{A} . \mathfrak{A}) = \mathfrak{A}$ und $(\mathfrak{A} + \mathfrak{A}) = \mathfrak{A}$.

Satz 20. (Schlussschemata). Lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$, so auch $(\mathfrak{A} . \mathfrak{B}) = \mathfrak{A}$; und umgekehrt. Lässt sich schreiben $\mathfrak{B} \subset \mathfrak{A}$, so auch $\mathfrak{A} + \mathfrak{B} = \mathfrak{A}$; und umgekehrt.

Satz 21 (Schlussschemata). Lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$, so gibt es eine Kalkülformel $\mathfrak{A}_1$ mit $(\mathfrak{A} + \mathfrak{A}_1) = \mathfrak{B}$; und umgekehrt. Lässt sich schreiben $\mathfrak{B} \subset \mathfrak{A}$, so gibt es eine Kalkülformel $\mathfrak{A}_2$ mit $(\mathfrak{A} . \mathfrak{A}_2) = \mathfrak{B}$; und umgekehrt.

Axiom 4°. $[X . (Y + Z)] = [(X . Y) + (X . Z)]$.

Satz 22. $[\mathfrak{A} + (\mathfrak{B} . \mathfrak{C})] = [(\mathfrak{A} + \mathfrak{B}) . (\mathfrak{A} + \mathfrak{C})]$ [9]).

Axiom 5°. $\lambda \subset X$.

Axiom 6°. $X \subset \nu$.

Satz 23 (Schlussschemata). Lässt sich schreiben $(\mathfrak{A} . \mathfrak{B}) = \lambda$ und $\mathfrak{A} \subset \mathfrak{B}$, so auch $\mathfrak{A} = \lambda$. Lässt sich schreiben $(\mathfrak{A} + \mathfrak{B}) = \nu$ und $\mathfrak{B} \subset \mathfrak{A}$, so auch $\mathfrak{A} = \nu$.

Satz 24 (Schlussschemata). Lässt sich schreiben $(\mathfrak{A} . \mathfrak{B}) = \lambda$ und $\mathfrak{C} \subset \mathfrak{B}$, so auch $\mathfrak{A} . \mathfrak{C} = \lambda$. Lässt sich schreiben $(\mathfrak{A} + \mathfrak{B}) = \nu$ und $\mathfrak{B} \subset \mathfrak{C}$, so auch $\mathfrak{A} + \mathfrak{C} = \nu$.

Axiom 7°. $(X . X') = \lambda$ und $(X + X') = \nu$.

Dualitätsprinzip. Zu jedem mit den Axiomen 1°—7° und Einsetzungsregel E^* ableitbaren Satz erhält man einen dualen, wenn: α) in jeder vorkommenden Kalkülformel + durch . , λ durch ν, und umgekehrt, ersetzt werden; β) $\mathfrak{A} \subset \mathfrak{B}$ durch $\overline{\mathfrak{B}} \subset \overline{\mathfrak{A}}$ ersetzt wird; dabei sollen $\overline{\mathfrak{A}}$ und $\overline{\mathfrak{B}}$ die gemäss α) aus $\mathfrak{A}$ bzw. $\mathfrak{B}$ hervorgehenden Kalkülformeln sein [10]).

Satz 25 (Schlussschema). Lässt sich schreiben $(\mathfrak{A} . \mathfrak{B}) = \lambda$ und $(\mathfrak{A} + \mathfrak{B}) = \nu$, so auch $\mathfrak{A}' = \mathfrak{B}$ [11]).

Satz 26 (Schlussschema). Lässt sich schreiben $(\mathfrak{C} . \mathfrak{B}) = \lambda$ und $\mathfrak{C} \subset (\mathfrak{A} + \mathfrak{B})$, so auch $\mathfrak{C} \subset \mathfrak{A}$.

Beweis. Lässt sich schreiben:

$$\mathfrak{C} \subset (\mathfrak{A} + \mathfrak{B}),$$

so auch:

$$[\mathfrak{C} . (\mathfrak{A} + \mathfrak{B})] = \mathfrak{C}. \qquad \text{(Satz 20)}$$

Und:

$$[(\mathfrak{C} . \mathfrak{A}) + (\mathfrak{C} . \mathfrak{B})] = [\mathfrak{C} . (\mathfrak{A} + \mathfrak{B})]. \qquad \text{(Ax. 4°, Satz 14)}$$

Also auch:

$$[(\mathfrak{C} . \mathfrak{A}) + (\mathfrak{C} . \mathfrak{B})] = \mathfrak{C}. \qquad \text{(Satz 15)} \quad . \; . \; . \quad (14)$$

Lässt sich ausserdem schreiben:

$$(\mathfrak{C} . \mathfrak{B}) = \lambda,$$

so auch:

$$[(\mathfrak{C} . \mathfrak{A}) + (\mathfrak{C} . \mathfrak{B})] = [(\mathfrak{C} . \mathfrak{A}) + \lambda]. \qquad \text{(Satz 13, 16)}$$

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9) Zum Beweise vergleiche man V. GLIVENKO, Théorie générale des structures, Act. Sci. Paris 1938, S. 32.

10) $\mathfrak{A} = \mathfrak{B}$ geht somit in $\overline{\mathfrak{A}} = \overline{\mathfrak{B}}$ über.

11) Zum Beweise vergl. GLIVENKO, loc. cit. 9), S. 33, 34.

Und:

$$[(\mathfrak{C} . \mathfrak{A}) + \lambda] = (\mathfrak{C} . \mathfrak{A}). \qquad (\text{Ax. } 3^\circ, 1^\circ, 5^\circ)$$

Also auch:

$$[(\mathfrak{C} . \mathfrak{A}) + (\mathfrak{C} . \mathfrak{B})] = (\mathfrak{C} . \mathfrak{A}). \qquad (\text{Satz } 15) \quad . \; . \quad (15)$$

(14) und (15) führen zu

$$(\mathfrak{C} . \mathfrak{A}) = \mathfrak{C}, \qquad (\text{Satz } 14, 15)$$

wodurch

$$\mathfrak{C} \subset \mathfrak{A}. \qquad (\text{Satz } 20)$$

Mit dem Dualitätsprinzip folgt aus Satz 26 der

Satz 26′ (Schlussschema). Lässt sich schreiben $\mathfrak{C} + \mathfrak{B} = \nu$ und $\mathfrak{A} . \mathfrak{B} \subset \mathfrak{C}$, so auch $\mathfrak{A} \subset \mathfrak{C}$.

Satz 27. $\mathfrak{A}''$ oder $(\mathfrak{A}')' = \mathfrak{A}$.

Beweis. $(\mathfrak{A} + \mathfrak{A}') = \nu$ und $(\mathfrak{A} . \mathfrak{A}') = \lambda$, $\qquad$ (Ax. 7°)
somit auch:

$$(\mathfrak{A}' + \mathfrak{A}) = \nu \text{ und } (\mathfrak{A}' . \mathfrak{A}) = \lambda. \qquad (\text{Satz } 18, 15) \quad . \; . \quad (16)$$

Aus (16) folgt mit Satz 25

$$(\mathfrak{A}')' = \mathfrak{A}.$$

Satz 28 (Schlussschema). Lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$, so auch $\mathfrak{B}' \subset \mathfrak{A}'$ [12]).

Folgerung (Schlussschema). Lässt sich schreiben $\mathfrak{A} = \mathfrak{B}$, so auch $\mathfrak{A}' = \mathfrak{B}'$.

Satz 29. $(\mathfrak{A} + \mathfrak{B})' = (\mathfrak{A}' . \mathfrak{B}')$ und $(\mathfrak{A} . \mathfrak{B})' = (\mathfrak{A}' + \mathfrak{B}')$ [13]).

Folgerung. $(\mathfrak{A} . \mathfrak{B}) = (\mathfrak{A}' + \mathfrak{B}')'$.

§ 4. Definition. $\mathfrak{A} \rightarrow \mathfrak{B}$ ist eine andere Schreibweise von $\mathfrak{A}' + \mathfrak{B}$.

Satz 30 (Schlussschema). Lässt sich schreiben $\mathfrak{A} = \nu$ und $(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu$, so auch $\mathfrak{B} = \nu$.

Beweis. Aus $\mathfrak{A} = \nu$ und $(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu$ oder $(\mathfrak{A}' + \mathfrak{B}) = \nu$ folgt, mit Satz 16,

$$[\mathfrak{A} . (\mathfrak{A}' + \mathfrak{B})] = (\nu . \nu).$$

$$(\nu . \nu) = \nu. \qquad (\text{Satz } 19)$$

Also auch:

$$[\mathfrak{A} . (\mathfrak{A}' + \mathfrak{B})] = \nu. \qquad (\text{Satz } 15)$$

Es lässt sich schreiben

$$[(\mathfrak{A} . \mathfrak{A}') + (\mathfrak{A} . \mathfrak{B})] = [\mathfrak{A} . (\mathfrak{A}' + \mathfrak{B})]. \qquad (\text{Ax. } 4^\circ, \text{Satz } 14)$$

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12) Zum Beweise vergl. GLIVENKO, loc. cit. 9), S. 35, 36.

13) Zum Beweise vergl. GLIVENKO, loc. cit. 9), S. 36.

Also auch:

$$[(\mathfrak{A} . \mathfrak{A}') + (\mathfrak{A} . \mathfrak{B})] = \nu. \qquad \text{(Satz 15)}$$

$$[\lambda + (\mathfrak{A} . \mathfrak{B})] = \nu. \qquad \text{(Ax. 7°, Satz 13, 16, 15)}$$

$$(\mathfrak{A} . \mathfrak{B}) = \nu. \qquad \text{(Ax. 3°, 1°, 5°, Satz 15)}$$

$$(\nu . \mathfrak{B}) = \nu. \qquad \text{(Satz 13, 16, 14, 15)}$$

somit:

$$\mathfrak{B} = \nu. \qquad \text{(Ax. 6°, Satz 20, 18, 15, 14)}$$

S a t z 31 (Schlussschemata). Lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$, so auch $(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu$, und umgekehrt.

B e w e i s. Nach Axiom 7° und Satz 14 ist

$$\nu = (\mathfrak{A} + \mathfrak{A}').$$

Lässt sich schreiben $\mathfrak{A} \subset \mathfrak{B}$, so auch

$$(\mathfrak{A} + \mathfrak{A}') \subset (\mathfrak{B} + \mathfrak{A}'). \qquad \text{(Ax. 3°, 1°)}$$

und dadurch auch

$$\nu \subset (\mathfrak{B} + \mathfrak{A}'). \qquad \text{(Ax. 1°)}$$

Nach Axiom 6°:

$$(\mathfrak{B} + \mathfrak{A}') \subset \nu,$$

also

$$(\mathfrak{B} + \mathfrak{A}') = \nu,$$

und

$$(\mathfrak{A}' + \mathfrak{B}) = \nu, \text{ oder } (\mathfrak{A} \rightarrow \mathfrak{B}) = \nu. \qquad \text{(Satz 18, 15)}$$

Umgekehrt, lässt sich schreiben $(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu$, oder $(\mathfrak{A}' + \mathfrak{B}) = \nu$, so folgt aus $(\mathfrak{A} . \mathfrak{A}') = \lambda$ (Ax. 7°) und $\mathfrak{A} \subset \nu$ (Ax. 6°), dass

$$\mathfrak{A} \subset \mathfrak{B}. \qquad \text{(Ax. 1°, Satz 18, 26)}$$

D e f i n i t i o n. $\mathfrak{A} \longleftrightarrow \mathfrak{B}$ ist eine andere Schreibweise von

$$(\mathfrak{A} \rightarrow \mathfrak{B}) . (\mathfrak{B} \rightarrow \mathfrak{A}).$$

S a t z 32 (Schlussschemata). Lässt sich schreiben $(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu$ und $(\mathfrak{B} \rightarrow \mathfrak{A}) = \nu$, so auch $(\mathfrak{A} \longleftrightarrow \mathfrak{B}) = \nu$, und umgekehrt.

B e w e i s. Lässt sich schreiben $(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu$ und $(\mathfrak{B} \rightarrow \mathfrak{A}) = \nu$, so auch

$$[(\mathfrak{A} \rightarrow \mathfrak{B}) . (\mathfrak{B} \rightarrow \mathfrak{A})] = \nu, \text{ d.i. } (\mathfrak{A} \longleftrightarrow \mathfrak{B}) = \nu. \qquad \text{(Satz 16, 19, 15)}$$

Umgekehrt, lässt sich schreiben $(\mathfrak{A} \longleftrightarrow \mathfrak{B}) = \nu$, so auch:

$$\nu \subset [(\mathfrak{A} \rightarrow \mathfrak{B}) . (\mathfrak{B} \rightarrow \mathfrak{A})],$$

und dadurch weiter:

$$\nu \subset (\mathfrak{A} \rightarrow \mathfrak{B}) \text{ und } \nu \subset (\mathfrak{B} \rightarrow \mathfrak{A}). \qquad \text{(Ax. 2°, 1°)}$$

Mit Axiom 6° liefert dies

$$(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu \text{ und } (\mathfrak{B} \rightarrow \mathfrak{A}) = \nu.$$

Satz 33 (Schlussschemata). Lässt sich schreiben $\mathfrak{A} = \mathfrak{B}$, so auch $(\mathfrak{A} \longleftrightarrow \mathfrak{B}) = \nu$, und umgekehrt.

Beweis. Lässt sich schreiben $\mathfrak{A} = \mathfrak{B}$, so auch

$$\mathfrak{A} \subset \mathfrak{B}, \ \mathfrak{B} \subset \mathfrak{A},$$

und, nach Satz 31,

$$(\mathfrak{A} \rightarrow \mathfrak{B}) = \nu, \ (\mathfrak{B} \rightarrow \mathfrak{A}) = \nu,$$

endlich, nach Satz 32,

$$(\mathfrak{A} \longleftrightarrow \mathfrak{B}) = \nu.$$

Diese Ableitung lässt sich umkehren.

§ 5. Satz 34. $[(\mathfrak{A} + \mathfrak{A}) \rightarrow \mathfrak{A}] = \nu.$

Beweis.

$$[(\mathfrak{A} + \mathfrak{A})' + \mathfrak{A}] = [(\mathfrak{A}' \,.\, \mathfrak{A}') + \mathfrak{A}] = (\mathfrak{A}' + \mathfrak{A}) = (\mathfrak{A} + \mathfrak{A}') = \nu,$$
(Satz 29, 13, 16, 19, 18, Ax. 7°)

also

$$[(\mathfrak{A} + \mathfrak{A})' + \mathfrak{A}] = \nu, \text{ oder } [(\mathfrak{A} + \mathfrak{A}) \rightarrow \mathfrak{A}] = \nu. \qquad \text{(Satz 15)}$$

Satz 35. $[\mathfrak{A} \rightarrow (\mathfrak{A} + \mathfrak{B})] = \nu.$

Beweis.

$$[\mathfrak{A}' + (\mathfrak{A} + \mathfrak{B})] = [(\mathfrak{A}' + \mathfrak{A}) + \mathfrak{B}] = [(\mathfrak{A} + \mathfrak{A}') + \mathfrak{B}] = (\nu + \mathfrak{B}) = \nu,$$

also

$$[\mathfrak{A}' + (\mathfrak{A} + \mathfrak{B})] = \nu. \qquad \text{(Satz 17, 14, 18, 13, 16, Ax. 7°, 6°, 3°, Satz 15)}$$

oder

$$[\mathfrak{A} \rightarrow (\mathfrak{A} + \mathfrak{B})] = \nu.$$

Satz 36. $[(\mathfrak{A} + \mathfrak{B}) \rightarrow (\mathfrak{B} + \mathfrak{A})] = \nu.$

Beweis. $(\mathfrak{A} + \mathfrak{B}) \subset (\mathfrak{B} + \mathfrak{A})$, (Satz 18)

also auch

$$[(\mathfrak{A} + \mathfrak{B}) \rightarrow (\mathfrak{B} + \mathfrak{A})] = \nu. \qquad \text{(Satz 31)}$$

Satz 37. $[(\mathfrak{B} \rightarrow \mathfrak{C}) \rightarrow \{(\mathfrak{A} + \mathfrak{B}) \rightarrow (\mathfrak{A} + \mathfrak{C})\}] = \nu.$

Beweis. Es lässt sich schreiben:

$$(\mathfrak{B}' + \mathfrak{C}) = [(\mathfrak{B}' + \mathfrak{C}) \,.\, \nu] = [\nu \,.\, (\mathfrak{B}' + \mathfrak{C})] = [(\nu \,.\, \mathfrak{B}') + (\nu \,.\, \mathfrak{C})] =$$
$$= [\{(\mathfrak{A} + \mathfrak{A}') \,.\, \mathfrak{B}'\} + \mathfrak{C}] =$$
$$= [\{(\mathfrak{A} \,.\, \mathfrak{B}') + (\mathfrak{A}' \,.\, \mathfrak{B}')\} + \mathfrak{C}] \subset [\{\mathfrak{A} + (\mathfrak{A}' \,.\, \mathfrak{B}')\} + \mathfrak{C}],$$
(Ax. 6°, Satz 20, 18, Ax. 4°, 7°, Satz 13, 16, Ax. 2°, 3°, 1°)

oder auch

$$(\mathfrak{B}' + \mathfrak{C}) \subset [(\mathfrak{A}' . \mathfrak{B}') + (\mathfrak{A} + \mathfrak{C})]. \qquad \text{(Ax. 1°, Satz 18, 13, 16, 17)}$$

oder

$$(\mathfrak{B}' + \mathfrak{C}) \subset [(\mathfrak{A} + \mathfrak{B})' + (\mathfrak{A} + \mathfrak{C})]. \qquad \text{(Satz 29, 14, 13, 16, Ax. 1°)}$$

oder

$$[(\mathfrak{B} \rightarrow \mathfrak{C}) \rightarrow \{(\mathfrak{A} + \mathfrak{B}) \rightarrow (\mathfrak{A} + \mathfrak{C})\}] = \nu. \qquad \text{(Satz 31)}$$

S a t z 38. $\lambda' = \nu$ und $\nu' = \lambda$.

B e w e i s. Nach den Axiomen 2°, 5°, 3°, 6° ist

$$(\lambda + \nu) = \nu, \ (\lambda . \nu) = \lambda, \text{ und } (\nu + \lambda) = \nu, \ (\nu . \lambda) = \lambda.$$

Satz 25 zeigt, dass sich auch schreiben lässt

$$\lambda' = \nu \quad \text{und} \quad \nu' = \lambda.$$

S a t z 39. $(\lambda \rightarrow \mathfrak{A}) = \nu$.

B e w e i s. $\lambda' + \mathfrak{A} = \nu + \mathfrak{A} = \nu$, (Satz 38, 13, 16, Ax. 3°, 6°)

oder

$$\lambda' + \mathfrak{A} = \nu. \qquad \text{(Satz 15)}$$

oder

$$(\lambda \rightarrow \mathfrak{A}) = \nu.$$

S a t z 40. $(\mathfrak{A} \rightarrow \nu) = \nu$.

B e w e i s. $\mathfrak{A}' + \nu = \nu$, (Ax. 3°, 6°)

oder

$$(\mathfrak{A} \rightarrow \nu) = \nu.$$

S a t z 41. Lässt sich schreiben $\mathfrak{A} = \nu$, und ist $\mathfrak{B}$ eine mittels der Einsetzungsregel E^* (erster Teil) aus $\mathfrak{A}$ hervorgehende Kalkülformel, so darf man auch schreiben: $\mathfrak{B} = \nu$.

B e w e i s. Lässt sich schreiben $\mathfrak{A} = \nu$, so auch

$$\mathfrak{A} \subset \nu, \quad \nu \subset \mathfrak{A}.$$

Durch Anwendung von Einsetzungsregel E^* (zweiter Teil) folgt:

$$\mathfrak{B} \subset \nu, \quad \nu \subset \mathfrak{B},$$

somit

$$\mathfrak{B} = \nu.$$

D e f i n i t i o n (Einführung des Assertionszeichens $\doteq \bar{\nu}$). Die Schreibweisen $\mathfrak{A} \doteq \bar{\nu}$ und $\mathfrak{A} = \nu$ sollen einander ersetzen können.

Mathematics. — *On the G-function.* VIII. By C. S. MEIJER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 30, 1946.)

Theorem 21. Assumptions: *m, n and p are integers with* [65]

$$p \geqq 1,\ 0 \leqq n \leqq p \text{ and } 1 \leqq m \leqq p+1;$$

λ is an arbitrary integer;
the number z satisfies the inequality

$$(m+n-p+2\lambda-\tfrac{3}{2})\pi \leqq \arg z < (m+n-p+2\lambda+\tfrac{1}{2})\pi;$$

the numbers $a_1, \ldots, a_p$ and $b_1, \ldots, b_m$ fulfil the conditions (1) *and* (38) *in the formulae* (203), (207) *and* (208) *(assertions* 1, 2, 3, 6, 7, 8 *and* 9); *in formula* (206) *(assertions* 4 *and* 5) *I assume that they satisfy the condition* (1).

Assertions: 1. *The function $G^{m,n}_{p,p+1}(z)$ possesses for large values of $|z|$ with*

$$(m+n-p+2\lambda-\tfrac{3}{2})\pi < \arg z < (m+n-p+2\lambda-\tfrac{1}{2})\pi \tag{202}$$

the asymptotic expansion

$$G^{m,n}_{p,p+1}(z) \backsim \sum_{\sigma=1}^{p} e^{-(2\lambda+1)\pi i a_\sigma}\, T^{m,n}_{p,p+1}(\sigma;\lambda)\, E_{p,p+1}(z\, e^{(p-m-n+2)\pi i} \,||\, a_\sigma). \tag{203}$$

Formula (203) *does not hold when $n=\lambda=0$ and $m=p+1$* [66].

2. *The expansion* (203) *is also true if the following conditions are satisfied:*

$$m+n \geqq p+2,\ p-m-n < \lambda < 0,$$

$$(m+n-p+2\lambda-\tfrac{1}{2})\pi \leqq \arg z < (m+n-p+2\lambda+\tfrac{1}{2})\pi. \tag{204}$$

3. *The expansion* (203) *is further valid if the following conditions a e satisfied:*

$$m+n \geqq p+2,\ p-m-n+1 < \lambda < 1,\ \arg z = (m+n-p+2\lambda-\tfrac{3}{2})\pi.$$

4. *If $m+n \geqq p+2$ and λ is either an arbitrary integer $\geqq 0$ or an arbitrary integer $\leqq p-m-n$, then for large values of $|z|$ with*

$$(m+n-p+2\lambda-\tfrac{1}{2})\pi < \arg z < (m+n-p+2\lambda+\tfrac{1}{2})\pi \tag{205}$$

[65] We need not give attention to the cases with $p=0$ and $m=0$, since $G^{1,0}_{0,1}(z\,|\,\lambda) = z^\lambda e^{-z}$ and $G^{0,n}_{p,p+1}(z) = 0$.

[66] The asymptotic expansion of $G^{p+1,0}_{p,p+1}(z)$ for $|\arg z| < \frac{1}{2}\pi$ is $G^{p+1,0}_{p,p+1}(z) \backsim H_{p,p+1}(z)$ (see theorem C).

the following asymptotic expansion holds

$$G^{m,n}_{p,p+1}(z) \backsim D^{m,n}_{p,p+1}(\lambda)\, H_{p,p+1}(z\, e^{(p-m-n-2\lambda+1)\pi i}). \qquad (206)$$

5. *The asymptotic expansion* (206) *is also valid if the following conditions are satisfied*:

$m+n \leqq p+1$, λ *is an arbitrary integer*, arg z *satisfies* (205).

6. *If* $m+n \geqq p+2$ *and* λ *is either an arbitrary integer* $\geqq 0$ *or an arbitrary integer* $\leqq p-m-n$, *then for large values of* $|z|$ *with* $\arg z = (m+n-p+2\lambda-\frac{1}{2})\pi$ *the following asymptotic expansion holds*

$$\left.\begin{aligned} &G^{m,n}_{p,p+1}(z) \backsim D^{m,n}_{p,p+1}(\lambda)\, H_{p,p+1}(z\, e^{(p-m-n-2\lambda+1)\pi i}) \\ &+ \sum_{\sigma=1}^{p} e^{-(2\lambda+1)\pi i a_\sigma}\, T^{m,n}_{p,p+1}(\sigma;\lambda)\, E_{p,p+1}(z\, e^{(p-m-n+2)\pi i} \,\|\, a_\sigma). \end{aligned}\right\}. \qquad (207)$$

7. *The asymptotic expansion* (207) *is also valid if the following conditions are satisfied*:

$m+n \leqq p+1$, λ *is an arbitrary integer*, $\arg z = (m+n-p+2\lambda-\frac{1}{2})\pi$

8. *If* $m+n \geqq p+2$ *and* λ *is either an arbitrary integer* $\geqq 1$ *or an arbitrary integer* $\leqq p-m-n+1$, *then for large values of* $|z|$ *with* $\arg z = (m+n-p+2\lambda-\frac{3}{2})\pi$ *the following asymptotic expansion holds*

$$\left.\begin{aligned} &G^{m,n}_{p,p+1}(z) \backsim D^{m,n}_{p,p+1}(\lambda-1)\, H_{p,p+1}(z\, e^{(p-m-n-2\lambda+3)\pi i}) \\ &+ \sum_{\sigma=1}^{p} e^{-(2\lambda+1)\pi i a_\sigma}\, T^{m,n}_{p,p+1}(\sigma;\lambda)\, E_{p,p+1}(z\, e^{(p-m-n+2)\pi i} \,\|\, a_\sigma). \end{aligned}\right\}. \qquad (208)$$

9. *The asymptotic expansion* (208) *is also valid if the following conditions are satisfied*:

$m+n \leqq p+1$, λ *is an arbitrary integer*, $\arg z = (m+n-p+2\lambda-\frac{3}{2})\pi$.

Remark. Precisely as by theorem 20 there are by theorem 21 certain cases wherein an expansion coincides with an expansion given in one of the previous theorems.

For instance: Formula (180) with $q=p+1$ is a particular case of (203). It follows namely from (86), (87) and (54):

If $1 \leqq \sigma \leqq n$ and $p-m-n+1 \leqq \lambda \leqq 0$, then

$$T^{m,n}_{p,p+1}(\sigma;\lambda) = e^{(m+n-p+2\lambda-1)\pi i a_\sigma}\, \triangle^{m,n}{}_{p+1}(\sigma).$$

If $n+1 \leqq \sigma \leqq p$ and $p-m-n+1 \leqq \lambda \leqq 0$, then $T^{m,n}_{p,p+1}(\sigma;\lambda) = 0$.

Hence, formula (203) with $n \geqq 1$, $m+n \geqq p+1$ and $p-m-n+1 \leqq \lambda \leqq 0$ is equivalent to (180) with $q=p+1$.

Proof of theorem 21. This proof rests, like that of theorem 20, on an application of the theorems 15 and 10. The number ε, occurring in condition (169) of theorem 15 is now equal to $\frac{1}{2}$, since $q=p+1$.

The inequality (170) reduces for $q=p+1$ to

$$(m+n-p+2\lambda-\tfrac{3}{2})\pi - \arg z < 2\mu\pi < (m+n-p+2\lambda+\tfrac{3}{2})\pi - \arg z. \qquad (209)$$

Hence it is easily seen:

If $\arg z$ satisfies (204), then $\mu=0$.

If $\arg z=(m+n-p+2\lambda-\frac{3}{2})\pi$, then $\mu=1$.

If $\arg z$ fulfils the condition (202), then the inequality (209) is satisfied by both $\mu=0$ and $\mu=1$.

Now on the right of (152) there occurs for $q=p+1$ only one function $G_{p,p+1}^{p+1,0}$, viz. the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$, the coefficient of this function being

$$R_{p,p+1}^{m,n}(1;\lambda) \text{ or } \overline{R}_{p,p+1}^{m,n}(1;p-m-n-\lambda+1)$$

according as $\mu=0$ or $\mu=1$; these coefficients may because of (91) and (92) also be written in the form $D_{p,p+1}^{m,n}(\lambda)$, respect.

$$\exp\left\{2\pi i\left(\sum_{h=1}^{p+1} b_h-\sum_{h=1}^{p} a_h\right)\right\} D_{p,p+1}^{m,n}(\lambda-1).$$

On account of the just determined values of μ we see that the coefficient in question is equal to

$$D_{p,p+1}^{m,n}(\lambda) \text{ if } \arg z \text{ satisfies (204)} \quad . \quad . \quad . \quad . \quad . \quad (210)$$

and equal to

$$\exp\left\{2\pi i\left(\sum_{h=1}^{p+1} b_h-\sum_{h=1}^{p} a_h\right)\right\} D_{p,p+1}^{m,n}(\lambda-1) \text{ if } \arg z=(m+n-p+2\lambda-\tfrac{3}{2})\pi; \ (211)$$

the coefficient depends on the choice of μ if $\arg z$ fulfils the condition (202).

Now if $\arg z$ satisfies (205), then

$$\tfrac{1}{2}\pi<\arg(z\,e^{(p-m-n-2\lambda+1)\pi i})<\tfrac{3}{2}\pi;$$

hence it follows from (26) and (25) that the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ is exponentially infinite for $|z|\to\infty$.

If $\arg z$ satisfies (202), then

$$-\tfrac{1}{2}\pi<\arg(z\,e^{(p-m-n-2\lambda+1)\pi i})<\tfrac{1}{2}\pi;$$

in this case the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ tends exponentially to zero for $|z|\to\infty$. If

$$\arg z=(m+n-p+2\lambda-\tfrac{1}{2})\pi \text{ or } \arg z=(m+n-p+2\lambda-\tfrac{3}{2})\pi,$$

the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ behaves like $e^{\mp i|z|}|z|^{\vartheta}$ for $|z|\to\infty$.

Hence, if $\arg z$ satisfies (205), we may, in writing down the asymptotic expansion of the right-hand side of (152) with $q=p+1$, neglect the asymptotic expansions of algebraic order which are caused by the functions $G_{p,p+1}^{p+1,1}(z\,e^{(p-m-n-2\lambda+2\mu)\pi i}\,||\,a_\sigma)$ $(\sigma=1,\ldots,p)$; we need only consider the exponential expansion of the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$, provided that the coefficient of this function does not vanish.

On the other hand, if $\arg z$ satisfies (202), the expansion of the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$, that is exponentially zero for $|z|\to\infty$, may be neglected; unless the coefficients of all the functions

$G_{p,p+1}^{p+1,1}(z\,e^{(p-m-n-2\lambda+2\mu)\pi i}\,||\,a_\sigma)$ vanish, we need only take account of the algebraic expansions of these functions.

If $\arg z = (m+n-p+2\lambda-\frac{1}{2})\pi$ or $\arg z = (m+n-p+2\lambda-\frac{3}{2})\pi$, we must take account of both the exponential and the algebraic expansions.

Now the coefficients $D_{p,p+1}^{m,n}(\lambda)$ and $D_{p,p+1}^{m,n}(\lambda-1)$ vanish identically if [67]) $p-m-n<\lambda<0$, respect. $p-m-n+1<\lambda<1$. In these cases the coefficient of the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ is zero, so that this function does not at all occur on the right-hand side of (152) with $q=p+1$: asymptotic expansions which are exponentially infinite are then certainly impossible.

Hence we see in virtue of (210) and (211) that the asymptotic expansion of the right-hand side of (152) with $q=p+1$ contains no expansions which are exponentially infinite but only algebraic expansions in the two following cases:

1. $m+n \geqq p+2$, $p-m-n<\lambda<0$, $\arg z$ satisfies (204).
2. $m+n \geqq p+2$, $p-m-n+1<\lambda<1$, $\arg z = (m+n-p+2\lambda-\frac{3}{2})\pi$.

Of course, the algebraic expansions are also predominant in the above mentioned case that $\arg z$ satisfies (202).

Since the coefficients $\Theta_p^{m,n}(l;\lambda)$ are zero for $\lambda=-1,-2,-3,\ldots$, it follows from (87) that the coefficients $T_{p,q}^{m,n}(l;\lambda)$ of the second type $(n+1 \leqq l \leqq p)$ vanish identically if $p-m-n+1 \leqq \lambda \leqq 0$. On the other hand we deduce from (86) that the coefficients $T_{p,q}^{m,n}(l;\lambda)$ of the first type $(1 \leqq l \leqq n)$ are in general not zero if $p-m-n+1 \leqq \lambda \leqq 0$, since $\triangle^{m,n}_{q}(l)$ is only zero if the parameters $b_{m+1},\ldots,b_q$ and a_l satisfy a certain equation. If $n=0$, there occur no coefficients $T_{p,q}^{m,n}(l;\lambda)$ of the first type. Hence, if $p-m+1 \leqq \lambda \leqq 0$, the coefficients $T_{p,q}^{m,0}(l;\lambda)$ vanish for $1 \leqq l \leqq p$. In the case before us the coefficients of the functions $G_{p,p+1}^{p+1,1}(z\,e^{(p-m-n-2\lambda+2\mu)\pi i}\,||\,a_\sigma)$ on the right-hand side of (152) with $q=p+1$ are equal to $e^{(1-2\mu)\pi i a_\sigma}\,T_{p,p+1}^{m,n}(\sigma;\lambda)$; we further have $m \leqq p+1$. So we see that the coefficients of all the functions $G_{p,p+1}^{p+1,1}$ on the right-hand side of (152) with $q=p+1$ are zero if we take $n=0$, $m=p+1$ and $\lambda=0$. The function $G_{p,p+1}^{p+1,0}(z)$ possesses therefore for $-\frac{1}{2}\pi<\arg z<\frac{1}{2}\pi$ no expansion of algebraic order, but instead of that an expansion which is exponentially zero for $|z|\to\infty$. The expansion in question is $G_{p,p+1}^{p+1,0}(z) \backsim H_{p,p+1}(z)$ (see theorem C).

Now the expansions of algebraic order, which are caused by the sum $\sum\limits_{\sigma=1}^{p}$ in (152), have according to (18) (with $q=p+1$) and (15) (with $\gamma=\mu-\lambda-1$) the form

$$\sum_{\sigma=1}^{p} e^{-(2\lambda+1)\pi i a_\sigma}\,T_{p,p+1}^{m,n}(\sigma;\lambda)\,E_{p,p+1}(z\,e^{(p-m-n+2)\pi i}\,||\,a_\sigma).$$

[67]) Comp. the second Remark in § 8.

The assertions 1, 2 and 3 have therefore been proved.

The asymptotic expansion of the function $G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i})$ is in virtue of (26)

$$G_{p,p+1}^{p+1,0}(z\,e^{(p-m-n-2\lambda+1)\pi i}) \backsim H_{p,p+1}(z\,e^{(p-m-n-2\lambda+1)\pi i}).$$

Hence the exponential expansion of the right-hand side of (152) with $q=p+1$ is

$$D_{p,p+1}^{m,n}(\lambda)\,H_{p,p+1}(z\,e^{(p-m-n-2\lambda+1)\pi i})$$

or

$$\exp\left\{2\pi i\left(\sum_{h=1}^{p+1} b_h - \sum_{h=1}^{p} a_h\right)\right\} D_{p,p+1}^{m,n}(\lambda-1)\,H_{p,p+1}(z\,e^{(p-m-n-2\lambda+1)\pi i}),$$

according as we find ourselves in the case (210) or (211). The second expansion may because of (27) with $q=p+1$ be written in the simpler form

$$D_{p,p+1}^{m,n}(\lambda-1)\,H_{p,p+1}(z\,e^{(p-m-n-2\lambda+3)\pi i}).$$

We can now easily verify the assertions 4, 5, 6, 7, 8 and 9.

§ 19. The analytic continuation of $G_{p,p}^{m,n}(z)$ (general case).

The function $G_{p,p}^{m,n}(z)$ satisfies the differential equation (34) with $q=p$. As I have proved in § 4 the p functions (43) form, provided that the conditions (41), (42) and (38) are satisfied [68], a system of fundamental solutions in the vicinity of $z=\infty$.

In this § I will give the expression of $G_{p,p}^{m,n}(z)$ in terms of these fundamental solutions. From this expression we may derive by means of theorem F the analytic continuation of $G_{p,p}^{m,n}(z)$ outside the circle $|z|=1$.

The result runs as follows:

Theorem 22. Assumptions: *m, n and p are integers with*

$$p \geqq 1,\ 0 \leqq n \leqq p \text{ and } 0 \leqq m \leqq q;$$

the numbers $a_1,\ldots,a_p$ and $b_1,\ldots,b_m$ satisfy the conditions (1) *and* (38); *λ is an arbitrary integer.*

Assertions: 1. *The function $G_{p,p}^{m,n}(z)$ can in the sector*

$$(m+n-p+2\lambda-2)\pi < \arg z < (m+n-p+2\lambda)\pi \quad . \quad (212)$$

by means of

$$G_{p,p}^{m,n}(z) = \sum_{\sigma=1}^{p} T_{p,p}^{m,n}(\sigma;\lambda)\, G_{p,p}^{p,1}(z\,e^{(p-m-n-2\lambda+1)\pi i}\,\|\,a_\sigma) \quad . \quad . \quad (213)$$

be expressed in terms of fundamental solutions valid near $z=\infty$.

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[68]) Comp. also the Remark at the end of § 4.

2. *The function $G_{p,p}^{m,n}(z)$ possesses in the sector* (212) *an analytic continuation outside the circle* $|z|=1$ *which can be expressed in the form*

$$G_{p,p}^{m,n}(z)=\sum_{\sigma=1}^{p} e^{-2\lambda\pi i a_\sigma}\, T_{p,p}^{m,n}(\sigma;\lambda)\, E_{p,p}(z\, e^{(p-m-n+1)\pi i}\,||\,a_\sigma). \quad (214)$$

Proof. If we put $\mu=0$ and $q=p$ in (152), we find (213). The functions $G_{p,p}^{p,1}$ on the right of (213) satisfy because of (212) the conditions (41) and (42). Hence they are fundamental solutions.

The analytic continuation of $G_{p,p}^{p,1}(z\, e^{(p-m-n-2\lambda+1)\pi i}\,||\,a_\sigma)$ outside the circle $|z|=1$ is in virtue of (33)

$$G_{p,p}^{p,1}(z\, e^{(p-m-n-2\lambda+1)\pi i}\,||\,a_\sigma)=E_{p,p}(z\, e^{(p-m-n-2\lambda+1)\pi i}\,||\,a_\sigma)$$

and this relation is on account of (15) equivalent to

$$G_{p,p}^{p,1}(z\, e^{(p-m-n-2\lambda+1)\pi i}\,||\,a_\sigma)=e^{-2\lambda\pi i a_\sigma} E_{p,p}(z\, e^{(p-m-n+1)\pi i}\,||\,a_\sigma).$$

Formula (214) follows therefore from (213), so that the theorem has been proved.

I now consider the particular case with $m+n\geqq p+1$ of formula (214) and I suppose that λ satisfies the inequality $p-m-n+1\leqq\lambda\leqq 0$. Then it is clear on account of (86), (87) and (54)

$$T_{p,p}^{m,n}(\sigma;\lambda)=e^{(m+n-p+2\lambda-1)\pi i a_\sigma}\,\Delta^{m,n}{}_{p}(\sigma) \text{ for } 1\leqq\sigma\leqq n$$

and

$$T_{p,p}^{m,n}(\sigma;\lambda)=0 \text{ for } n+1\leqq\sigma\leqq p.$$

It follows therefore from (214): If $m+n\geqq p+1$, $|\arg z|<(m+n-p)\pi$ and

$$\arg z\neq(p-m-n+2)\pi,\ (p-m-n+4)\pi,\ \dots,\ (m+n-p-2)\pi,$$

then the analytic continuation of $G_{p,p}^{m,n}(z)$ outside the circle $|z|=1$ has the form

$$G_{p,p}^{m,n}(z)=\sum_{\sigma=1}^{n} e^{(m+n-p-1)\pi i a_\sigma}\,\Delta^{m,n}{}_{p}(\sigma)\, E_{p,p}(z\, e^{(p-m-n+1)\pi i}\,||\,a_\sigma).$$

This formula is equivalent to formula (32) of theorem E*. But the present result is slightly less general than that of theorem E*, since we now have excluded the values

$$\arg z=(p-m-n+2)\pi,\ (p-m-n+4)\pi,\ \dots,\ (m+n-p-2)\pi,$$

which are not omitted in theorem E*.

§ 20. The asymptotic expansion of the generalized hypergeometric function ${}_pF_q(z)$ $(q\geqq p)$.

Preliminary Remarks.

Remark 1. It is easily seen in view of (27) with $q+1$ instead of q that the expressions

$$(-2\pi i)^{p-q}\exp\left\{\pi i\left(\sum_{h=1}^{p} a_h-\sum_{h=1}^{q+1} b_h\right)\right\} H_{p,q+1}(z\, e^{(q-p+1)\pi i}) \quad . \quad (215)$$

and

$$(2\pi i)^{p-q} \exp\left\{\pi i\left(\sum_{h=1}^{q+1} b_h - \sum_{h=1}^{p} a_h\right)\right\} H_{p,q+1}(z\, e^{(p-q-1)\pi i}) \quad . \quad (216)$$

are equal one to another.

We now put

$$\left.\begin{array}{ll} a_j = 1-\alpha_j & (j=1,\ldots,p), \\ b_1 = 0 \text{ and } b_j = 1-\beta_{j-1} & (j=2,\ldots,q+1). \end{array}\right\} \quad . \quad . \quad (217)$$

It follows from (25) and (23) that the expressions (215) and (216) then reduce to

$$\exp\left\{(q-p+1)\, z^{\frac{1}{q-p+1}}\right\} z^{\gamma} \left\{\frac{(2\pi)^{\frac{p-q}{2}}}{\sqrt{q-p+1}} + \frac{N_1}{z^{\frac{1}{q-p+1}}} + \frac{N_2}{z^{\frac{2}{q-p+1}}} + \ldots\right\}, (218)$$

where

$$\gamma = \frac{1}{q-p+1}\left\{\tfrac{1}{2}(q-p) + \sum_{h=1}^{p} \alpha_h - \sum_{h=1}^{q} \beta_h\right\}; \quad . \quad . \quad . \quad (219)$$

the values of the coefficients N can be deduced from those of the corresponding coefficients M in (25).

The expression (218), where γ is defined by (219), will, for brevity, be denoted by $K_{p,q}(z)$.

It is now easily seen, on account of (45) and (46), that the expressions

$$A^{1,p}_{q+1}\, H_{p,q+1}(z\, e^{(q-p+1)\pi i})$$

and

$$\overline{A}^{1,p}_{q+1}\, H_{p,q+1}(z\, e^{(p-q-1)\pi i})$$

also reduce to $K_{p,q}(z)$ when we make the substitution (217).

Remark 2. We consider

$$\sum_{t=1}^{p} e^{(p-q-1)\pi i a_t}\, \Delta^{1,p}_{q+1}(t)\, E_{p,q+1}(z\, e^{(q-p+1)\pi i} \,\|\, a_t) \quad . \quad . \quad (220)$$

and

$$\sum_{t=1}^{p} e^{(q-p+1)\pi i a_t}\, \Delta^{1,p}_{q+1}(t)\, E_{p,q+1}(z\, e^{(p-q-1)\pi i} \,\|\, a_t). \quad . \quad . \quad (221)$$

These sums are equal one to another because of (15) with $q+1$ instead of q and $\gamma = q-p+1$.

After being transformed by means of the substitution (217) the sums (220) and (221) will be denoted by $L_{p,q}(z)$. It follows from (17) and (13) that (220) and (221), after the substitution (217) has been applied, may be written in the form

$$\sum_{t=1}^{p} \frac{z^{-\alpha_t} \prod\limits_{\substack{j=1 \\ j\neq t}}^{p} \Gamma(\alpha_j-\alpha_t)}{\prod\limits_{j=1}^{q} \Gamma(\beta_j-\alpha_t)} \sum_{h=0}^{\infty} \frac{\{(-1)^{q-p} z\}^{-h}\, \Gamma(\alpha_t+h) \prod\limits_{j=1}^{q} \{(1+\alpha_t-\beta_j)(2+\alpha_t-\beta_j)\ldots(h+\alpha_t-\beta_j)\}}{h!\prod\limits_{\substack{j=1 \\ j\neq t}}^{p} \{(1+\alpha_t-\alpha_j)(2+\alpha_t-\alpha_j)\ldots(h+\alpha_t-\alpha_j)\}}.$$

This expression will therefore be denoted by $L_{p,q}(z)$.

The behaviour of the generalized hypergeometric function ${}_pF_q(z)$ $(q \geqq p)$ for large values of $|z|$ has already been investigated in various ways and by several authors. I mention here the researches of STOKES, BARNES, WATSON, FOX, WRINCH and WRIGHT [69]).

Now the function ${}_pF_q(z)$ is a special case of the function $G^{m,n}_{p,q+1}(z)$ and so it must be possible to deduce asymptotic expansions for ${}_pF_q(z)$ from those of $G^{m,n}_{p,q+1}(z)$. Indeed it follows from (7) [70])

$$\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j)}{\prod\limits_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;-z\end{matrix}\right)=G^{1,p}_{p,q+1}\left(z\,\middle|\,\begin{matrix}1-\alpha_1,\ldots,1-\alpha_p\\ 0,1-\beta_1,\ldots,1-\beta_q\end{matrix}\right) \quad (222)$$

and

$$\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j)}{\prod\limits_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;z\end{matrix}\right)=G^{1,p}_{p,q+1}\left(z\,e^{\pi i}\,\middle|\,\begin{matrix}1-\alpha_1,\ldots,1-\alpha_p\\ 0,1-\beta_1,\ldots,1-\beta_q\end{matrix}\right). \quad (223)$$

I will now write down the asymptotic expansions of the function ${}_pF_q(z)$. I suppose that the parameters $\alpha_1,\ldots,\alpha_p$ fulfil the condition

$$\alpha_j \neq 0, -1, -2, \ldots \qquad (j=1,\ldots,p);$$

in the formulae (224), (225), (226) and (227) I will assume that they satisfy besides the condition

$$\alpha_j-\alpha_t \neq 0, \pm 1, \pm 2, \ldots \qquad (j=1,\ldots,p;\, t=1,\ldots,p;\, j\neq t).$$

We now distinguish two cases:

F i r s t c a s e: $0 \leqq p \leqq q-1$. We apply theorem 18.

If $-\pi < \arg z < \pi$, it follows from (223) and (185) (with $m=1$, $n=p$, $q+1$ instead of q and $z e^{\pi i}$ instead of z) on account of Remark 1

$$\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j)}{\prod\limits_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;z\end{matrix}\right) \backsim K_{p,q}(z).$$

If $0 \leqq p < q-1$ and $z > 0$, it follows from (222) and (187) (with $m=1$, $n=p$ and $q+1$ instead of q) on account of Remark 1

$$\frac{\prod\limits_{j=1}^{p}\Gamma(\alpha_j)}{\prod\limits_{j=1}^{q}\Gamma(\beta_j)}\,{}_pF_q\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;-z\end{matrix}\right) \backsim K_{p,q}(z\,e^{-\pi i}) + K_{p,q}(z\,e^{\pi i}).$$

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[69]) STOKES, [28]; BARNES, [3], 68, 83, 115 and [4]; WATSON [30], 37; FOX, [9]; WRINCH, [37], [38] and [39]; WRIGHT, [35]; comp. also WRIGHT, [36].

[70]) Comp. footnote [4]).

If $p \geqq 0$ and $z > 0$, it follows from (222) (with $q = p+1$) and (188) (with $m = 1$, $n = p$ and $q = p+2$) on account of the Remarks 1 and 2

$$\frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{p+1} \Gamma(\beta_j)} {}_pF_{p+1}\begin{pmatrix} a_1, \ldots, a_p; \\ \beta_1, \ldots, \beta_{p+1}; -z \end{pmatrix} \backsim K_{p,p+1}(z e^{-\pi i}) + K_{p,p+1}(z e^{\pi i}) + L_{p,p+1}(z). \quad (224)$$

Second case: $q = p \geqq 1$. We apply the theorems 16 and 19.

If $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$, it follows from (222) (with $q = p$) and (180) (with $m = 1$, $n = p$ and $q = p+1$) on account of Remark 2

$$\frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{p} \Gamma(\beta_j)} {}_pF_p\begin{pmatrix} a_1, \ldots, a_p; \\ \beta_1, \ldots, \beta_p; -z \end{pmatrix} \backsim L_{p,p}(z). \quad . \quad . \quad . \quad (225)$$

If $-\frac{1}{2}\pi < \arg z < \frac{1}{2}\pi$, it follows from (223) and the first assertion of theorem (19) (formula (185) with $m = 1$, $n = p$, $q = p+1$ and $z e^{\pi i}$ instead of z) on account of Remark 1

$$\frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{p} \Gamma(\beta_j)} {}_pF_p\begin{pmatrix} a_1, \ldots, a_p; \\ \beta_1, \ldots, \beta_p; z \end{pmatrix} \backsim K_{p,p}(z).$$

If $\arg z = \frac{1}{2}\pi$, it follows from (222) and (191) (with $m = 1$, $n = p$ and $q = p+1$) on account of the Remarks 1 and 2

$$\frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{p} \Gamma(\beta_j)} {}_pF_p\begin{pmatrix} a_1, \ldots, a_p; \\ \beta_1, \ldots, \beta_p; -z \end{pmatrix} \backsim K_{p,p}(z e^{-\pi i}) + L_{p,p}(z). \quad (226)$$

If $\arg z = -\frac{1}{2}\pi$, it follows from (222) and (192) (with $m = 1$, $n = p$ and $q = p+1$) on account of the Remarks 1 and 2

$$\frac{\prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{p} \Gamma(\beta_j)} {}_pF_p\begin{pmatrix} a_1, \ldots, a_p; \\ \beta_1, \ldots, \beta_p; -z \end{pmatrix} \backsim K_{p,p}(z e^{\pi i}) + L_{p,p}(z). \quad . \quad (227)$$

Closing Remark. If it is desirable to express the function ${}_pF_q(z)$ in terms of fundamental solutions near $z = \infty$ of the differential equation satisfied by it, we may use the formulae (222) and (223) in connection with the theorems 13 and 14.

§ 21. The asymptotic expansion of $W_{k,m}(z)$.

As another application of the theorems of § 18 I will write down the asymptotic expansions of WHITTAKER's function $W_{k,m}(z)$. The well-known expansion [71])

$$W_{k,m}(z) \backsim e^{-\frac{1}{2}z} z^k \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right) \quad . \quad (228)$$

holds only when $|\arg z| < \frac{3}{2}\pi$. This formula is because of (143) equivalent to

$$G^{2,0}_{1,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}-k \\ m, -m \end{matrix}\right) \backsim e^{-z} z^{k-\frac{1}{2}} \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right).$$

Taking account of (26) the last expression may be interpreted as follows:

If $a_1 = \frac{1}{2}-k$, $b_1 = m$ and $b_2 = -m$, then

$$H_{1,2}(z) = e^{-z} z^{k-\frac{1}{2}} \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right).$$

We now suppose $(2\lambda+\frac{1}{2})\pi < \arg z < (2\lambda+\frac{3}{2})\pi$, where λ is an arbitrary integer. Then we have for large values of $|z|$ because of (143), the fifth assertion of theorem 21 and lemma 17

$$W_{k,m}(z) \backsim (-1)^{\lambda} e^{-2\lambda k\pi i} e^{-\frac{1}{2}z} z^k \left\{\frac{\sin 2(\lambda+1)m\pi}{\sin 2m\pi} + e^{-2k\pi i}\frac{\sin 2\lambda m\pi}{\sin 2m\pi}\right\} \times$$
$$\times \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right).$$

This formula may also be proved by means of (144) and (228); it reduces for $\lambda = 0$ and $\lambda = -1$ to (228).

We now suppose $(2\lambda-\frac{1}{2})\pi < \arg z < (2\lambda+\frac{1}{2})\pi$, where λ is an arbitrary positive or negative integer ($\lambda \neq 0$). Then it follows from (143), the first assertion of theorem 21 and lemma 18

$$W_{k,m}(z) \backsim (-1)^{\lambda-1} e^{2\lambda k\pi i} \, 2\pi i e^{\frac{1}{2}z} z^{-k} \times$$
$$\times \frac{\sin 2\lambda m\pi}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)\sin 2m\pi} \, {}_2F_0\left(\tfrac{1}{2}+k+m, \tfrac{1}{2}+k-m; \frac{1}{z}\right).$$

The asymptotic expansion of $W_{k,m}(z)$ when $\arg z = (2\lambda+\frac{1}{2})\pi$ or $\arg z = (2\lambda-\frac{1}{2})\pi$, where λ is any integer, can be deduced from the seventh, respect. the ninth assertion of theorem 21. These expansions run as follows

$$W_{k,m}(z) \backsim (-1)^{\lambda} e^{-2\lambda k\pi i} e^{-\frac{1}{2}z} z^k \left\{\frac{\sin 2(\lambda+1)m\pi}{\sin 2m\pi} + e^{-2k\pi i}\frac{\sin 2\lambda m\pi}{\sin 2m\pi}\right\} \times$$
$$\times \, {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right) + (-1)^{\lambda-1} e^{2\lambda k\pi i} \, 2\pi i \, e^{\frac{1}{2}z} z^{-k} \times$$
$$\times \frac{\sin 2\lambda m\pi}{\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)\sin 2m\pi} \, {}_2F_0\left(\tfrac{1}{2}+k+m, \tfrac{1}{2}+k-m; \frac{1}{z}\right)$$

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71) WHITTAKER and WATSON, [32], § 16.3—16.4.

respect.

$$W_{k,m}(z) \backsim (-1)^{\lambda-1} e^{-2(\lambda-1)k\pi i} e^{-\frac{1}{2}z} z^k \left\{ \frac{\sin 2\lambda m\pi}{\sin 2m\pi} + e^{-2k\pi i} \frac{\sin 2(\lambda-1)m\pi}{\sin 2m\pi} \right\} \times$$

$$\times {}_2F_0\left(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m; -\frac{1}{z}\right) + (-1)^{\lambda-1} e^{2\lambda k\pi i}\, 2\pi i\, e^{\frac{1}{2}z} z^{-k} \times$$

$$\times \frac{\sin 2\lambda m\pi}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(\frac{1}{2}-k-m)\sin 2m\pi}\, {}_2F_0\left(\tfrac{1}{2}+k+m, \tfrac{1}{2}+k-m; \frac{1}{z}\right).$$

Mathematics. — *On the dissection of rectangles into squares.* (First communication.) By C. J. BOUWKAMP. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of November 30, 1946.)

1. *Introduction.*

Many mathematical problems originate from early puzzling and recreation. For instance, the problem of magic squares. Less known is the puzzle problem treated below, namely that of the dissection of a rectangle of commensurable sides into a finite number of non-overlapping unequal squares, by means of straight line segments drawn parallel to the sides of the rectangle. The latter problem is, unlike that of magic squares, of very recent times only.

The first example of a rectangle, divided into (nine) incongruent squares, was apparently given by MOROŃ in 1925. It was also published in various books on amusement-mathematics, such as KRAITCHIK's "La Mathématique des Jeux".

From the present author's own experience it may be concluded that skilful puzzlers do not encounter great difficulties in constructing such a squared rectangle, containing a small number (10, say) of squares. The question to construct all the possible rectangles with 10 squares, however, will not be so easily answered. In this connection it may be remarked that another nine-squares-solution was published in 1940 only [1]).

Already DEHN remarked that the difficulty of the problem is the semi-topological one of characterizing how the various squares fit together. This difficulty has been completely overcome by the authors of paper A. Curiously they succeeded to associate a squared rectangle with the flowing of electric currents in certain networks. This provides a typical example of how to overcome mathematical difficulties by adequate physical reasoning.

The afore-mentioned authors have proved that 9 is the minimum number of two by two unequal squares that can completely build up a rectangle without overlapping one another. Moreover they have shown that there exist two ninth-order [2]) solutions, apart from those obtained by trivial transformations, such as reflections, rotations. Furthermore they found

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[1]) R. L. BROOKS, C. A. B. SMITH, A. H. STONE and W. T. TUTTE, The dissection of rectangles into squares, Duke Math. J. **7**, 312—340 (1940).

Henceforth this fundamental paper will be quoted by the letter A; special attention may be drawn to the bibliography at the end of A.

[2]) The finite number of squares in a dissection is called the *order* of the squaring.

6 solutions of order 10, 22 of order 11, and 67 of order 12. Only for squared rectangles of order 9 and 10 were the elements [3]) of the dissection explicitly given, whilst for squarings of order 11 only the "full" sides were specified in paper A.

Basing ourselves on the fundamental paper A, we have constructed the squarings of order less than 14. We found their total number amounting to 311; 214 of them are of order 13, the remainder being as specified above.

The following definitions are frequently used in A as well as in the present article. A squaring is called *perfect* if all the squares of the dissection are two by two unequal. Otherwise the squaring is called *imperfect*. We are mostly interested in perfect squared rectangles, though a certain type of imperfect rectangles will be considered too, namely the so-called *non-trivially imperfect* ones. The latter may contain equal elements; only in such a sense, however, that equal elements have never a side in common; nor must it be possible to get them in such a position by a trivial displacement of some of the elements. The remaining imperfect squarings, called *trivially imperfect*, are not investigated here.

A squaring is called *compound* if it is possible, by suitably omitting of some (not all, but at least one) dissecting line segments, to get the original rectangle dissected (by the remaining segments) into rectangles, not necessarily squares. If this is not possible, the squaring is called *simple*.

Furthermore, a compound squaring is called *trivially compound* if one of the elements has a side equal to one of the sides of the squared rectangle under consideration; when this element is omitted, a squaring of order one less can be obtained. Conversely, once a squaring of order n is given, one can readily obtain two different trivially compound squarings of order $n+1$, by merely introducing an extra $(n+1)$-th square whose side equals one of the two sides of the given n-th order rectangle. Compound squarings that are not trivially compound are called *non-trivially compound*.

It will be clear that in case of simple squarings the character of imperfection is only non-trivial.

As already stated, our investigation is not restricted to *perfect* squarings. We rather consider all the *simple* ones, whether they are perfect or imperfect. In addition to the numbers of squarings given previously, which are in fact the *simple perfect* ones, there are 43 *simple imperfect* squarings, namely 1, 0, 0, 9, 33, of order 9, 10, 11, 12, 13, respectively.

Our main aim is then to classify those 354 simple squarings of order less than 14.

Of course, there still remain some other perfect rectangles, namely the compound ones. It can be shown that all the compound perfect squarings of order less than 14 are trivially compound, with only one exception. The

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[3]) The squares of a squared rectangle are called *elements* of the squaring.

exceptional case, which is thus non-trivially compound, consists of a rectangle dissected into 4 squares and 1 rectangle, the latter in its turn being dissected into 9 unequal squares (details will be given in section 5). The trivially compound perfect squarings can be derived in a trivial manner from the simple perfect ones of lower order, namely by introduction of a new element whose side is equal to one of the sides of the original simple perfect squaring. There are 4, 16, 60, 195, compound perfect squarings of order 10, 11, 12, 13, respectively.

In addition to the number of 275 compound perfect squarings above, there are similarly 28 compound non-trivially imperfect ones, namely 2, 2, 2, 22, of order 10, 11, 12, 13, respectively. They can be derived in a trivial manner from the simple imperfect squarings of order less than 13, with only two exceptions which will be given in section 5.

Only the trivially imperfect (which are compound too) squarings are not yet enumerated. Unfortunately these squarings cannot be found in some trivial manner. Furthermore, without the restriction on imperfection, the total number of squared rectangles would become too large for adequate classification. For instance, there are already 1, 1, 2, 5, 11, 29, trivially imperfect squared rectangles of order 1, 2, 3, 4, 5, 6, respectively. These are the reasons why we omit the trivially imperfect rectangles in our further investigation.

A complete specification of the various types of squared rectangles of order not exceeding 13 is given in table I.

TABLE I.

Numbers of squared rectangles of different type, and of order less than 14. Only the trivial imperfections are excluded (the latter show equal elements lying aside, and thus belong to the compound type).

Type \ Order	9	10	11	12	13	Total
Simple, perfect	2	6	22	67	214	311
Simple, imperfect	1	0	0	9	33	43
Trivially compound, perfect	0	4	16	60	194	274
Non-trivially compound, perfect	0	0	0	0	1	1
Trivially comp., non-trivially imperfect	0	2	2	2	20	26
Non-triv. compound, non-triv. imperf.	0	0	0	0	2	2
Perfect	2	10	38	127	409	586
Non-trivially imperfect	1	2	2	11	55	71
Total	3	12	40	138	464	657

It is interesting to note that there is only one *square* amongst the total number of 657 squared rectangles so far obtained. It is of the order 13;

although it is simple, it is to a high degree imperfect, as it contains 5 pairs of equal elements. *It provides the simplest example of a simple squared square,* in so far that 13 is the minimum possible order of such a square. This remarkable square is drawn out in fig. 1; the various numbers correspond to the relative linear sizes of the elements.

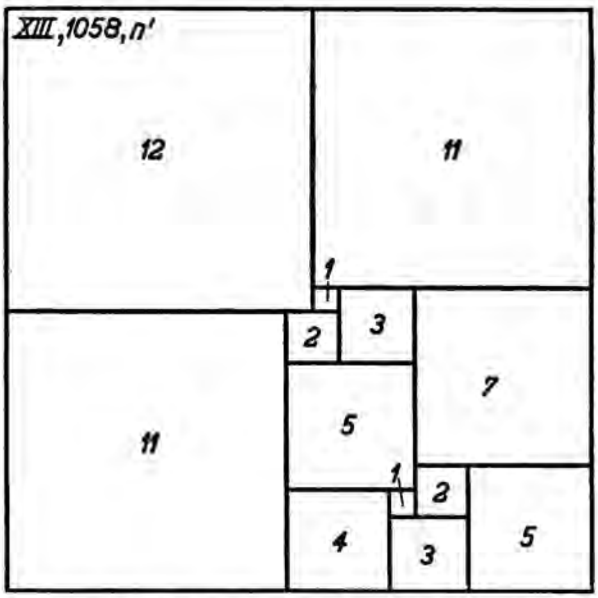

Fig. 1. The simplest example of a simple squared square.

It is of course impossible to draw out in this journal all the 657 squared rectangles as specified in table I. Even the total number of 354 simple squarings would require too much space. The difficulty, however, can be easily overcome by a suitable *coding system.*

First we suppose the rectangle to be drawn out in such a manner that its largest sides are horizontal. Then, the element in the upper-left corner should not be smaller than the three remaining corner elements. Now, for orders not exceeding 13, the geometrical configuration of the simple squarings appears to be unique except in the case of the square of fig. 1. Henceforth we will always "orient" a squared rectangle in the above sense before the proper coding starts, though this is not necessary.

Now the given oriented rectangle is squared by horizontal and vertical line segments. Consider the group of elements with their upper horizontal sides in a common horizontal segment. The individual elements of this group are conveniently ordered by a reading from left to right. The various groups themselves are ordered according to upwards-downwards-reading, starting with the upper horizontal side of the given rectangle. If necessary, line segments at the same horizontal level are ordered from left to right too. In the written code the various groups are separated by parentheses, the elements of a group by commas. Thus the code of the squared square of fig. 1 reads as follows

$$(12, 11)\ (1, 3, 7)\ (11, 2)\ (5)\ (2, 5)\ (4, 1)\ (3).$$

This system of coding provides us with a very simple method to pass over from code to figure, and vice versa. So MORON's solution (cf. fig. 4 γ) is coded as

$$(18, 15)\ (7, 8)\ (14, 4)\ (10, 1)\ (9),$$

and the other perfect nine-squares-rectangle is (see fig. 4 β)

$$(36, 33)\ (5, 28)\ (25, 9, 2)\ (7)\ (16).$$

The coding of compound squarings is similar. As an example we may give the code of a *perfect squared square* of order 26, published in paper A, fig. 9, p. 333, namely

$$(231, 136, 123, 118)\ (5, 113)\ (20, 108)\ (95, 34, 7)\ (27, 61)$$
$$(209, 205, 194)\ (11, 183)\ (44, 172)\ (168, 41)\ (1, 43)\ (42, 85).$$

This squared square is compound, consisting of one square and two rectangles which in their turn are dissected into 12 and 13 squares, respectively. Probably it is the simplest (that means of minimum order) perfect squared square, known at present.

The few examples above clearly demonstrate the usefulness of our coding system.

2. *Squared rectangles from an electrical point of view* [4]).

Let us suppose an oriented squared rectangle to be drawn upon a thin metal plate. The upper and lower side of the rectangle be electrodes of infinitely conducting material. Next a downwards flow I of electric current may exist through this plate, and due to a potential difference V across the electrodes. By a suitable choice of units we can suppose the number I of the total current to be equal to the value of the horizontal side of the rectangle. In the same manner V may be taken equal to the height of the rectangle. The flow is homogeneous; stream lines are vertical, equipotential lines are horizontal. Let us now make infinitely thin cuts along the *vertical* line segments; this does not influence the flowing. The various squares still remain connected to one another by means of the *horizontal* line segments.

We have now obtained an electrical "network", in which the current I is streaming through "wires", each of which is a thin metal plate. Every such "wire" is flown through by a certain current i, while its ends show a potential difference v. On account of our choise of units, the ratio v/i is

[4]) We are indebted to Prof. VAN DER POL and Mr VAN DER MARK for most valuable discussions.

equal to 1 for all the "wires"; that means their ohmic resistances are all equal to 1.

We want to have this "network" transformed into a more conventional one. Theretofore let us consider some "wire". We cut it along the upper and lower side, from left as well as from right, up to its middle vertical [5]). The freecoming left- and right-hand parts of the square plate are then rolled up towards the middle vertical, and afterwards melted together in the form of a true wire. The current through the "wire" is the same as that in the true wire. The same holds with regard to the potential difference v across it. The network still remains connected via the horizontal line segments, once the transformation is applied to all elements. Finally, we contract each infinitely conducting horizontal line segment to a single point. Especially, the horizontal sides of the rectangle are transformed into the terminals or *poles* of the network. Each element of the squared rectangle now corresponds to a *wire*, each horizontal line segment to a *vertex*, and finally each vertical line segment to a *mesh*, not containing other parts of the network in its interior.

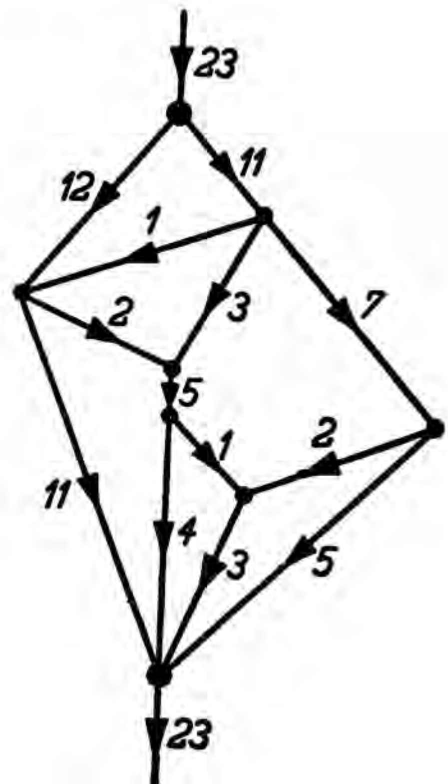

Fig. 2. Currents in the network corresponding to the squared square of fig. 1.

[5]) We suppose the horizontal line segments to be of perfectly conducting material.

The network corresponding to the squared square of fig. 1 is given in fig. 2. The arrows denote the directions of the currents, the numbers their relative magnitudes. If this network is realized by ohmic resistances of 1 ohm, then a potential difference of 23 volts between the poles causes a current of 23 amperes through the network. Therefore, the substitutional resistance of this special network is also equal to 1 ohm. In general the resulting resistance will be smaller, because for an oriented rectangle $I > V$. Hence, from an engineer's point of view, the network of fig. 2 is very remarkable: it provides us with the simplest planar [6]) network, without any series or parallel connection, that can be built up with equal resistances, all showing non-zero currents, such that the resulting resistance is equal to that of the individual wires.

It is evident that the energy $V.I$, delivered to the network by the source, is transformed into Joule heat $\Sigma\, vi$; indeed, the area of the rectangle is equal to the sum of the areas of the individual squares.

This association of linear graphs with squared rectangles was first derived in the quoted paper A, though not so physically as we did above. We showed that the squaring is a real network; the wires are only square metal plates.

It is convenient to join the poles of the network by an extra wire that contains the exterior source. The network obtained in this manner is planar, as is obvious from the construction above. A further simplification can be arranged by projecting the network upon a sphere because then all the wires, including that containing the source, are topologically equivalent as far as the concepts of "interior" and "exterior" are concerned.

Thus, starting with an oriented squared rectangle of order n, we obtain, in a unique way, a network on the sphere containing $n + 1$ wires which do not cross each other. It will be clear that a similar construction holds if one starts with a rectangle, dissected into rectangles that are not necessarily squares.

Conversely, starting with a planar network on the sphere, containing $n + 1$ wires (having general values of resistance), and after placing a source in one of them (no matter which one), we are led to an electrical network in which the currents and voltages are calculable by the laws of KIRCHHOFF. This network can be drawn in a plane in such a way that the poles lie at the outside, and are joined by the source-containing wire, which will be omitted further on. The resulting "polar" network, together with its currents and voltages, can be moulded into the unconventional form consisting of rectangular plates. The final result is therefore a rectangled rectangle.

In case of equal resistances, the rectangle becomes squared; its order is *n or less*. The latter addition is necessary because, accidently, some wires may be current-free, due, for instance, to properties of symmetry (in elec-

[6]) A network is called *planar* if it can be drawn in a plane without crossings.

tricity, wires with zero-currents are called "conjugate" to that containing the source). Obviously, a zero-current wire corresponds to a square of vanishing dimensions. This reduction of the order may also occur in case of unequal resistances, though in general n will be the actual number of elements in the rectangled rectangle.

It is clear that the difficulty to describe how the squares fit together has now been overcome completely. The question how to obtain the possible squarings of order n is reduced to a merely combinatorial problem: how to obtain the possible topologically different connected networks, involving $n+1$ wires, that can be drawn on a sphere without crossings. Every such network will then give rise to a *class* of squared rectangles. The number in a class is at most $n+1$, corresponding to as many places for the exterior source.

3. *Duality. Bounds for the numbers of vertices and meshes in terms of the numbers of wires.*

In electricity the interchanging of voltages and currents is governed by the *principle of duality*. Its interpretation in terms of rectangled rectangles is as follows. Instead of taking the electrodes along the *horizontal* sides of the rectangle, we may take them along the *vertical* ones. Accordingly, stream lines and equipotential lines have interchanged [7]; the same holds with regard to the vertices and meshes. The ohmic resistances of the respective polar networks are obviously reciprocal; so are the resistances of corresponding wires. Thus, these two polar networks are *electrically dual*. The corresponding "completed" (after joining the poles by the extra wire) networks N, N' on the sphere (we ignore the numerical values of the resistances) are said to be *topologically dual*. Unless otherwise stated, we use here the concept of duality in a topologically sense only.

Dual networks can be drawn in such a manner that the vertices of either of them lie inside the corresponding meshes of the other, whilst corresponding wires, and only these, cross each other [8]).

Roughly spoken, only one half of the total number of networks needs be investigated, as a pair of dual networks leads to one class of rectangled rectangles. Similarly, in case of self-dual networks, only half the number of wires needs insertion of an exterior source.

Our aim is to obtain the possible squarings up to a certain order with the only restriction that, in case of imperfection, equal elements do not lie aside, i.e. have no sides in common. Therefore, in the polar networks, simple

[7]) "Cross-points", which are common to four elements of the rectangling, should be first removed by suitable small displacements of some of the line segments; if not, there remains an ambiguity in the "cutting" process.

[8]) Cf. B. D. H. TELLEGEN, Geometrical configurations and duality of electrical networks, Philips tech. Rev. 5, 324—330 (1940).

————, Meetkundige configuraties en dualiteit van electrische netwerken, Tijdschr. Nederl. Radiogenootschap 9, 37—60 (1941).

wires in series or parallel connection are forbidden. Furthermore, we may suppose that even the "completed" network on the sphere does not contain such connections because it would give either forbidden imperfect rectangles or trivially compound ones; in the latter case the largest element of the squaring would lie along the whole of one of the sides of the rectangles. As we already saw, the trivially compound squarings are not very interesting because they can be obtained in a trivial manner from squarings of lower order.

Let N be a planar network, containing T wires, K vertices, and M meshes. On account of EULER's polyhedron formula one has

$$K + M = T + 2. \quad \quad (1)$$

As N does not contain any pair of wires in series, at least three wires come together in each vertex. Therefore, twice the number of wires cannot be less than three times the number of vertices:

$$K \leqslant 2T/3. \quad \quad (2)$$

Together with (1), this yields for the number of meshes

$$M \geqslant 2 + T/3. \quad \quad (3)$$

Let N' be the dual network of N [9]). Its numbers of wires, vertices and meshes are $T' = T$, $K' = M$ and $M' = K$, respectively.

The inequalities (2), (3) apply also to N'; therefore, interchanging of K, M is allowed. We thus obtain for K, M the same upper and lower bounds:

$$\frac{T+6}{3} \leqslant (K, M) \leqslant \frac{2T}{3}. \quad \quad (4)$$

The conditions (4) are only *necessary;* they do not guarantee the planarity of the network satisfying (4). Although we may certainly confine ourselves to networks fulfilling (4), many among them appear non-planar. Of course, the latter cannot be used in the construction of squared rectangles.

We would like to mention an interesting interpretation of EULER's polyhedron formula (1). If n_1, n_2, denote the numbers of the horizontal and vertical line segments, respectively, lying inside the rectangled rectangle of order n, then $n_1 = K - 2$, $n_2 = M - 2$, $n = T - 1$, and hence $n = n_1 +$

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[9]) Two polar networks N_1 (poles p_1, q_1) and N_2 (poles p_2, q_2) can be united to a "compound" network in two different ways by suitable coalescence of the respective poles, namely either by $p_1 \leftrightarrow p_2$, $q_1 \leftrightarrow q_2$ or by $p_1 \leftrightarrow q_2$, $q_1 \leftrightarrow p_2$. These two distict (in a topological sense) networks are *electrically equivalent.* If equivalent networks are identified, there is only one network N' dual to N, as was shown by WHITNEY. Cf. the second of TELLEGEN's papers quoted above. Squarings corresponding to equivalent networks are not essentially different as they can be transformed into one another by trivial geometrical displacements (reflection, rotation, translation) of some of the elements.

$n_2 + 1$. Therefore, the total number of elements in any dissection is one more than is the total number of dissecting line segments. Cross-points, however, must be first removed.

In table II the possible combinations of K, M, T are given as far as squarings of order less than 14 are concerned. It also shows the number of the vertical and horizontal line segments, which numbers are very characteristic for any given squared rectangle. On account of (4) they satisfy

$$\frac{n+1}{3} \leqslant (n_1, n_2) \leqslant \frac{2n-4}{3}. \quad \ldots \ldots \quad (5)$$

TABLE II.

Possible combinations of wires (T), vertices (K) and meshes (M) of the required networks. The same with regard to the order (n), horizontal (n_1) and vertical (n_2) line segments of the corresponding squarings.

T	K, M	n	n_1, n_2
6	4,4	5	2,2
7	—	6	—
8	5,5	7	3,3
9	5,6	8	3,4
10	6,6	9	4,4
11	6,7	10	4,5
12	6,8	11	4,6
12	7,7	11	5,5
13	7,8	12	5,6
14	7,9	13	5,7
14	8,8	13	6,6

4. *Construction of the required networks. Nine is the least possible order of a perfect squared rectangle.*

We will now briefly indicate how one can obtain all the required networks, such that certainly none of them has been overlooked. Again, we only consider networks without wires in series or parallel connection; the numbers of wires and vertices be T and K, respectively.

Henceforth a vertex is called a p_n if it is a junction of n wires ($n \geq 3$). The number of vertices p_n will be denoted by x_n (≥ 0). These numbers obviously satisfy

$$x_3 + x_4 + x_5 + \ldots = K,$$
$$3x_3 + 4x_4 + 5x_5 + \ldots = 2T.$$

For small values of K, T, the solutions of this diophantine system are easily obtained. Some simplification is gained by first eliminating x_3, viz.

$$x_4 + 2x_5 + 3x_6 + 4x_7 + \ldots = 2T - 3K. \quad \ldots \ldots \quad (6)$$

As an example, take $T = 13$, $K = 8$; then $2T - 3K = 2$. Obviously

$x_6, x_7, \ldots$ must be all zero. The only possible types of vertices are p_3, p_4, p_5. Moreover, there are either one p_5 or two p_4's, the remaining vertices being p_3's. Therefore, concerning the character of the vertices occurring in this example, only the following combinations are possible:

$$7p_3 + p_5 \; : \; 6p_3 + 2p_4.$$

If, given T, the numbers K, M are unequal, we may confine ourselves to $K > M$, as follows from the principle of duality treated in the preceding section. Of course, one could also confine oneself to $K < M$. We, however, prefer the former restriction because then the right-hand side of (6) is smaller, and so is consequently the number of possible combinations of p_n's to be investigated.

With this in mind, the possible combinations of the various types of vertices for squarings up to the order 13 are easily derived. They are given in table III.

TABLE III.

Possible combinations of vertices as a function of the numbers of wires and vertices.

T	K	Combinations of vertices
6	4	$4p_3$
7	—	—
8	5	$4p_3 + p_4$
9	6	$6p_3$
10	6	$5p_3 + p_5$: $4p_3 + 2p_4$
11	7	$6p_3 + p_4$
12	7	$6p_3 + p_6$: $5p_3 + p_4 + p_5$: $4p_3 + 3p_4$
12	8	$8p_3$
13	8	$7p_3 + p_5$: $6p_3 + 2p_4$
14	8	$7p_3 + p_7$: $6p_3 + p_4 + p_6$: $6p_3 + 2p_5$: $5p_3 + 2p_4 + p_5$: $4p_3 + 4p_4$
14	9	$8p_3 + p_4$

The first combination of table III (i.e. $4p_3$, $T = 6$, $K = 4$) only admits of the tetrahedron. This network is self-dual; cf. fig. 3*a*. In case of $T = 8$ the only possible network is the self-dual four-sided pyramid of fig. 3*b*. For $T = 9$ one obtains the self-dual three-sided prism (fig. 3*c*) together with the simplest non-planar network (fig. 3*d*), which can be omitted further on. For $T = 10$ our "sieve" yields 4 networks, one of which is non-planar. The combination $5p_3 + p_5$ gives rise to the self-dual five-sided pyramid of fig. 3*e*. The combination $4p_3 + 2p_4$ admits 3 networks of which 2 are planar.

The latter combination will be treated in some detail, as it may give an idea of how the higher networks were actually obtained.

Two distinct cases may occur as to whether or not the pair of p_4's are connected by a wire. If they are not connected, any of the 4 remaining p_3's must be joined to both of them. The two remaining wires can yet be placed in one way only, and the final result is a parallel (or series) connection of two Wheatstone bridges; cf. fig. 3*f*. In the other case, where the vertices p_4 are connected to one another, either three or two of the p_3's must be connected to each of them. In the first case the yet unconnected p_3 must be joined to the other three p_3's, giving the non-planar network of fig.

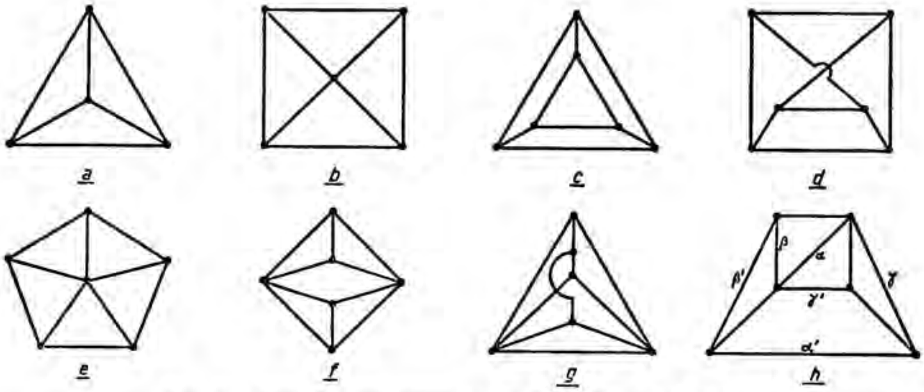

Fig. 3. The simplest networks up to $T = 10$ $(K \geq M)$.

3*g*. In the second case the two points p_3, which are connected to both p_4's, cannot be connected to one another; the other pair of p_3's must do so, however. Furthermore, the two wires, not yet used, can be placed in one

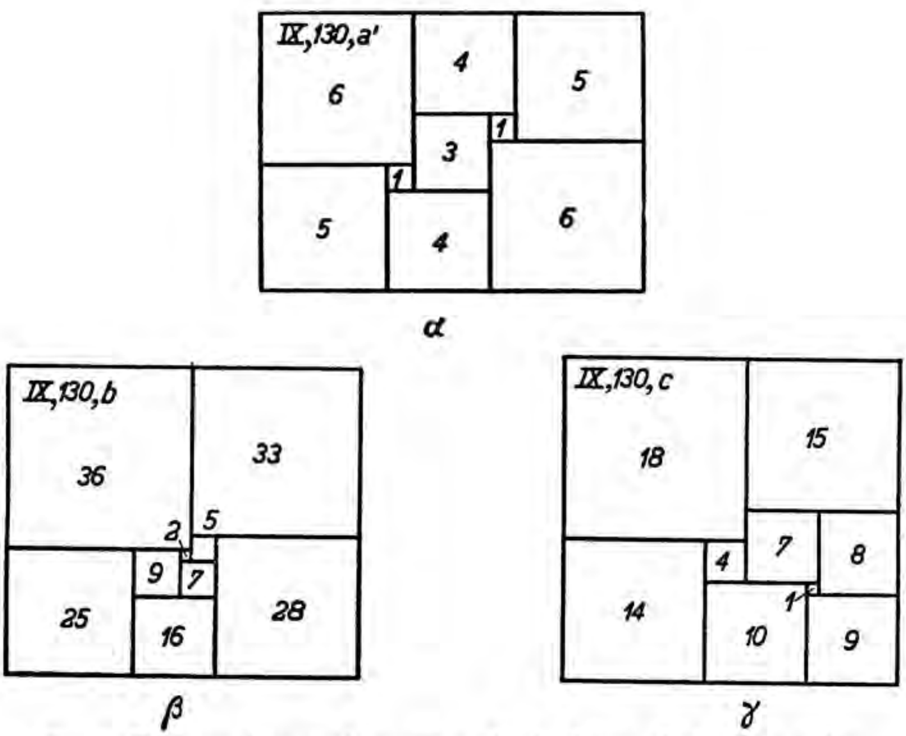

Fig. 4. There are three simple squarings of order 9, two of which are perfect.

way only, and the result is the self-dual network of fig. *3h*. Drawn in space, it is a three-sided prism with a diagonal in one of its uprising sides.

It is not difficult to see that out of the six planar networks in fig. 3 there is at most one (*3h*) that can be used in our problem because, on account of the large degree of symmetry, the other networks will show either (*i*) at least one zero-current or (*ii*) equal currents giving rise to trivial imperfection, after the insertion of a source in one of the wires (all having the same resistance).

Furthermore, in the network *3h* only 6 of the total number of 10 wires need be broken up on account of symmetry properties. This number is again reduced to 3 because the network is self-dual. In fig. *3h* the yet remaining wires are distinguished from each other by α, β, γ; their "duals" are primed correspondingly. When the source is inserted in α, the network gives a simple squaring of order 9. Although this squaring is imperfect, showing double elements, it is only non-trivially imperfect as equal elements do not lie aside; cf. fig. 4α. Two different perfect squarings of order 9 are obtained if the source is placed in β, γ, respectively. They have been drawn out in fig. 4β, γ.

We have thus shown in detail that 9 is the least possible order of a perfect squared rectangle; and there are only two distinct solutions involving nine elements.

Mathematics. — *On the elastic stability of thin plates, supported by a continuous medium.* By P. P. BIJLAARD. (Communicated by Prof. F. K. TH. VAN ITERSON.)

(Communicated at the meeting of November 30, 1946.)

In order to stabilise thin sheets of metal or plywood, as used in the construction of wings and bodies of aeroplanes, these sheets are sometimes laterally supported by a thicker layer of an elastic material with small specific weight. GOUGH, ELAM and DE BRUYNE [1]) developed a theory in order to compute the critical thrust of suchlike laterally supported sheets, in which it is assumed that the entire compressive load is taken by the sheets alone and that the strains of the sheets may be neglected. This theory was extended by VAN DER NEUT [2]) in order to eliminate the latter assumption.

Prior to these authors, the problem of the buckling of a plate, that is supported by an elastic layer was already examined by us [3]), in order to check our presumption, that wave formation in the surface of roads, as is often found in Java, is caused by horizontally directed compressive forces, originated in the road cover by rolling and traffic. We reported on the same subject to the Sixth Congress of Applied Mechanics in Paris, September 1946 [4]). In both papers we considered the case of a plate, supported by a semi-infinite elastic substratum, which is also one of the cases, considered by the above mentioned authors and which is identical with that of a plate, supported by an elastic medium of finite thickness with rigid back, if the thickness of that medium is somewhat larger than the half wave length of the corrugations.

As, shortly after our arrival in Europe, our attention was called to their communications, we saw that the most complete formula for the buckling stress, as found by VAN DER NEUT, did not agree with ours. The reason is, that these authors assume, that plate and supporting layer are connected in the middle plane of the plate and thus assume the plate to be infinitely thin, whilst we took the real connection at the bottom of the plate into account. By assuming the plate infinitely thin, however, the strains of the plate by its buckling, by which the vertical resistance of the substratum is

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[1]) GOUGH, ELAM and DE BRUYNE, The stabilisation of a thin sheet by a continuous supporting medium, Journal of the Royal Aeronautical Society, 1940.

[2]) VAN DER NEUT, Die Stabilität geschichteter Streifen. Nationaal Luchtvaart Laboratorium, Bericht S. 284, 1943.

[3]) BIJLAARD, Het plooien van een vlakke plaat, die op een elastisch-isotrope halfruimte is opgelegd, met een toepassing op de golfvorming in wegverhardingen. De Ingenieur in Ned. Indië, 1939, Nr. 9.

[4]) BIJLAARD, Plastic buckling of a plate on an elastic foundation.

somewhat enhanced, are neglected, as also the restraining action of the shearing stresses at its bottom on the plate, which both influences increase the buckling resistance of the plate.

The method followed in our papers differs somewhat from that of the above mentioned authors. Under here we will show, that also with their method, it is easy to take the real relations into account. Moreover we will derive the exact formula for so called sandwich plates, being two sheets of metal or plywood with an intermediate layer of light material. Finally we will derive a formula for the critical thrust of sandwich plates, according to a simple method, as developed by us for intricate buckling problems in general, giving practically the same result as the exact one. The exact critical thrust of sandwich plates may be of the order of (50 to 100) h/t % more than that computed by VAN DER NEUT, h and t being the thickness of the plate and that of the intermediate layer respectively.

If a plate is supported by a semi-infinite layer (fig. 1), it will buckle in

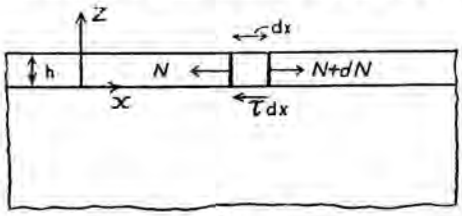

FIG. 1

rather short waves perpendicularly to the direction of the compressive force, so that with sufficient breadth of the plate with respect to the wave-length of the corrugations, as occurred in the case of the road cover, a state of plane strain will prevail in plate and layer. The stress distribution in the substratum is governed by the equation

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial z^2} + \frac{\partial^4 \Phi}{\partial z^4} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

in which Φ is the AIRY stress function, whilst the stresses are

$$\sigma_x = \frac{\partial^2 \Phi}{\partial z^2}, \qquad \sigma_z = \frac{\partial^2 \Phi}{\partial x^2}, \qquad \tau_{xz} = \tau = -\frac{\partial^2 \Phi}{\partial x \partial z} \quad . \; . \; . \quad (2)$$

Furthermore, the modulus of elasticity and POISSON's ratio of the substratum being E and ν respectively, we have, as the unit strain ε_y is zero,

$$E\,\varepsilon_x = (1-\nu^2)\,\sigma_x - \nu\,(1+\nu)\,\sigma_z$$

or, the displacements in X- and Z-direction being u and w,

$$E\frac{\partial u}{\partial x} = (1-\nu^2)\frac{\partial^2 \Phi}{\partial z^2} - \nu\,(1+\nu)\frac{\partial^2 \Phi}{\partial x^2} \quad . \; . \; . \; . \; . \quad (3)$$

At the other hand

$$G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \tau = -\frac{\partial^2 \Phi}{\partial x \partial z}$$

partial differentiation of which with respect to x, after expressing the modulus of rigidity G in E and ν, yields

$$E\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial z}\right)=-2(1+\nu)\frac{\partial^3 \Phi}{\partial x^2 \partial z}$$

By partial differentiation of eq. (3) with respect to z we find

$$E\frac{\partial^2 u}{\partial x\,\partial z}=(1-\nu^2)\frac{\partial^3 \Phi}{\partial z^3}-\nu(1+\nu)\frac{\partial^3 \Phi}{\partial x^2 \partial z}$$

introducing of which in the preceding equation yields

$$E\frac{\partial^2 w}{\partial x^2}=-(2-\nu)(1+\nu)\frac{\partial^3 \Phi}{\partial x^2 \partial z}-(1-\nu^2)\frac{\partial^3 \Phi}{\partial z^3} \quad . \quad . \quad . \quad (4)$$

We consider a strip of unit breadth. The excess unit strain ε_x of the plate bottom by its buckling consists of two parts, $(\varepsilon_x)_N$ by the excess normal forces N and $(\varepsilon_x)_B$ by its bending, so that

$$\varepsilon_x=(\varepsilon_x)_N+(\varepsilon_x)_B=\frac{N}{R}+\frac{h}{2}\frac{\partial^2 w}{\partial x^2}$$

$R=E_1\,h/(1-\nu_1^2)$ being the rigidity of the plate to normal forces, the constants E and ν of the plate being given the index 1. This unit strain of the plate has to equal that of the substratum according to eq. (3), which leads, using also eq. (4), to

$$\frac{\partial u}{\partial x}=\frac{1}{E}\left\{(1-\nu^2)\frac{\partial^2 \Phi}{\partial z^2}-\nu(1+\nu)\frac{\partial^2 \Phi}{\partial x^2}\right\}=$$
$$=\frac{N}{R}-\frac{h}{2E}\left\{(2-\nu)(1+\nu)\frac{\partial^3 \Phi}{\partial x^2\partial z}+(1-\nu^2)\frac{\partial^3 \Phi}{\partial z^3}\right\}$$

The equilibrium of an element hdx of the plate in X-direction requires (fig. 1)

$$dN=\tau\,dx \text{ or } dN/dx=\tau=-\partial^2\Phi/\partial x\,\partial z$$

so that differentiation of the preceding equation with respect to x yields, after ranging

$$\left.\begin{aligned}\frac{h}{2}\left\{(2-\nu)(1+\nu)\frac{\partial^4 \Phi}{\partial x^3\partial z}+(1-\nu^2)\frac{\partial^4 \Phi}{\partial x\partial z^3}\right\}-\nu(1+\nu)\frac{\partial^3 \Phi}{\partial x^3}+\\+(1-\nu^2)\frac{\partial^3 \Phi}{\partial x\partial z^2}+\frac{E}{R}\frac{\partial^2 \Phi}{\partial x\partial z}=0\end{aligned}\right\}. \quad (5)$$

being our boundary condition for Φ at $z=0$.

The equilibrium of a plate element demands furthermore (fig. 2)

$$dM-Q\,dx+P\,dw-{}^1\!/_2 h\tau\,dx=0 \text{ and } dQ=-\sigma_z\,dx$$

As $M = B\partial w^2/\partial x^2$, $B = E_1 h^3/12(1-\nu_1^2)$, being the bending rigidity of the plate, these equations lead, after elimination of Q, to the following differential equation of the plate

$$B\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} + (\sigma_z)_{z=0} + \frac{h}{2}\frac{\partial^3 \Phi}{\partial x^2 \partial z} = 0 \quad . \quad . \quad . \quad . \quad (6)$$

With

$$\Phi = Z \cos \pi x/a = Z \cos \lambda x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

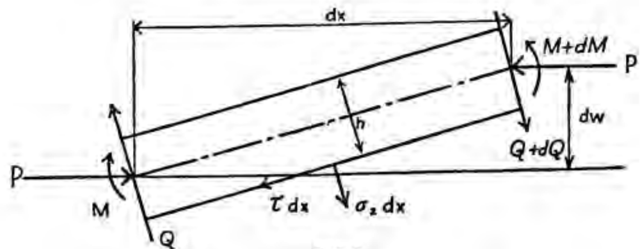

FIG. 2

Z being a function of z only, with $\zeta = \lambda z$ and $dZ/d\zeta = Z'$, equations (1) to (6) inclusive yield

$$Z'''' - 2Z'' + Z = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1')$$

$$\sigma_{x_0} = \frac{\sigma_x}{\cos \lambda x} = \lambda^2 Z'', \ \sigma_{z_0} = \frac{\sigma_z}{\cos \lambda x} = -\lambda^2 Z, \ \tau_0 = \frac{\tau}{\sin \lambda x} = \lambda^2 Z' . \quad (2')$$

$$u_0 = \frac{u}{\sin \lambda x} = \frac{\lambda}{E}\{(1-\nu^2) Z'' + \nu(1+\nu) Z\} . \quad . \quad . \quad . \quad (3')$$

$$w_0 = \frac{w}{\cos \lambda x} = \frac{\lambda}{E}\{(1-\nu^2) Z''' - (2-\nu)(1+\nu) Z'\} . \quad . \quad . \quad (4')$$

$$\left.\begin{aligned} \frac{h}{2}(1-\nu^2)\lambda^2 Z''' + (1-\nu^2)\lambda Z'' - \left\{\frac{h}{2}(2-\nu)(1+\nu)\lambda^2 - \frac{E}{R}\right\} Z' + \\ + \nu(1+\nu)\lambda Z = 0 \end{aligned}\right\} \quad (5')$$

$$B\lambda^4 - P\lambda^2 + \left(\frac{\sigma_{z_0}}{w_0}\right)_{\zeta=0} - \frac{h}{2}\left(\frac{Z'}{w_0}\right)_{\zeta=0} \lambda^3 = 0 \quad . \quad . \quad . \quad (6')$$

From eq. (6′) we compute the critical thrust $P = h\varrho_1$ of the plate

$$P = h\varrho_1 = B\lambda^2 + \frac{1}{\lambda^2}\left(\frac{\sigma_{z_0}}{w_0}\right)_{\zeta=0} - \frac{h\lambda}{2}\left(\frac{Z'}{w_0}\right)_{\zeta=0} \quad . \quad . \quad . \quad (8)$$

or, with the notations of our former papers

$$h\varrho_1 = B\lambda^2 + E'/2\lambda + \tfrac{1}{4}\psi h E' \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

so that, using also eqs. (2′) and (4′),

$$E' = \frac{2}{\lambda}\left(\frac{\sigma_{z_0}}{w_0}\right)_{\zeta=0} = 2\,E\left\{\frac{Z}{(2-\nu)(1+\nu)Z' - (1-\nu^2)Z'''}\right\}_{\zeta=0} \quad . \quad (10)$$

$$\psi = -\frac{2\lambda}{E'}\left(\frac{Z'}{w_0}\right)_{\zeta=0} = -\lambda^2\left(\frac{Z'}{\sigma_{z_0}}\right)_{\zeta=0} = \left(\frac{Z'}{Z}\right)_{\zeta=0} \quad . \quad . \quad (11)$$

The general solution of eq. (1′) is

$$Z = \alpha\,(\cosh\zeta + C_1\sinh\zeta + C_2\,\zeta\cosh\zeta + C_3\,\zeta\sinh\zeta) \quad . \quad . \quad (12)$$

but, as the buckling of the plate will not influence the stresses in the substratum at $\zeta = -\infty$, Z will have to equal zero there, demanding that it must be free of the function $e^{-\zeta}$, so that the coefficients of sinh and cosh must be equal, yielding $C_1 = 1$ and $C_3 = C_2 = C$, by which eq. (12) transforms in

$$Z = \alpha\,e^{\zeta}(1 + C\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Introducing this in our boundary condition (5′) at $\zeta = 0$, we find

$$C = \frac{h\,(1+\nu)\,\lambda^2 - 2\,(1+\nu)\,\lambda - 2\,E/R}{h\,(1-2\,\nu)(1+\nu)\,\lambda^2 + 4\,(1-\nu^2)\,\lambda + 2\,E/R} \quad . \quad . \quad . \quad (14)$$

so that eqs. (10) and (11) yield, with $E/R = 1/sh$

$$E' = \frac{(1+\nu)\{4\,(1-\nu) + (1-2\,\nu)\,\lambda h\}\,\lambda h s + 2}{(1+\nu)\{(1+\nu)(3-4\,\nu)\,\lambda h s + 2\,(1-\nu)\}}\,E \quad . \quad . \quad . \quad (15)$$

$$\psi = \frac{2\,(1+\nu)\{1-2\,\nu + (1-\nu)\,\lambda h\}\,\lambda h s}{(1+\nu)\{4\,(1-\nu) + (1-2\,\nu)\,\lambda h\}\,\lambda h s + 2} \quad . \quad . \quad . \quad (16)$$

in accordance with our second paper [4]). The strains of the plate by normal forces may, however, practically be neglected here, giving in the circumstances as prevail in aeroplane construction only a correction of at most 0,07 % [2]), so that s may be equaled to infinity, by which eqs. (15) and (16) transform in those of our first paper [3]).

Gough, Elam and de Bruyne and also van der Neut assume a state of plane stress, so $\sigma_y = 0$, in the substratum, which will occur if the breadth of the plate in Y-direction is small. In this case $E\varepsilon_x = \sigma_x - \nu\sigma_z$, by which our eqs. (3), (4) and (5) transform in

$$E\,\frac{\partial u}{\partial x} = \frac{\partial^2\Phi}{\partial z^2} - \nu\,\frac{\partial^2\Phi}{\partial x^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

$$E\,\frac{\partial^2 w}{\partial x^2} = -(2+\nu)\,\frac{\partial^3\Phi}{\partial x^2\,\partial z} - \frac{\partial^3\Phi}{\partial z^3} \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

$$\frac{h}{2}\left\{(2+\nu)\,\frac{\partial^4\Phi}{\partial x^3\,\partial z} + \frac{\partial^4\Phi}{\partial x\,\partial z^3}\right\} - \nu\,\frac{\partial^3\Phi}{\partial x^3} + \frac{\partial^3\Phi}{\partial x\,\partial z^2} + \frac{E}{R'}\,\frac{\partial^2\Phi}{\partial x\,\partial z} = 0 \quad (5a)$$

R' being now equal to $E_1 h$, which, using eq. (7), yield

$$u_0 = \frac{u}{\sin \lambda x} = \frac{\lambda}{E}(Z'' + \nu Z) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a')$$

$$w_0 = \frac{w}{\cos \lambda x} = \frac{\lambda}{E}\{Z''' - (2+\nu) Z'\} \quad . \quad . \quad . \quad . \quad (4a')$$

$$\frac{h}{2}\lambda^2 Z''' + \lambda Z'' - \left\{\frac{h}{2}(2+\nu)\lambda^2 - \frac{E}{R'}\right\} Z' + \nu \lambda Z = 0 \quad . \quad (5a')$$

so that, according to eqs. (2′) and (4a′) and to the first two members of eqs. (10)

$$E' = 2E\left\{\frac{Z}{(2+\nu)Z' - Z'''}\right\}_{\zeta=0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10a)$$

whilst $\psi = Z'/Z$ according to eq. (11).

Substitution of eq. (13) in eq. (5a′) yields now

$$C = \frac{(1+\nu) h \lambda^2 - 2(1+\nu)\lambda - 2E/R'}{(1-\nu) h \lambda + 4\lambda + 2E/R'} \quad . \quad . \quad . \quad . \quad (14a)$$

by which we find with eqs. (10a) and (11)

$$E' = \frac{\{4 + (1-\nu) h \lambda\}\lambda + 2E/R'}{(3-\nu)(1+\nu)\lambda + 2E/R'} E \quad . \quad . \quad . \quad . \quad . \quad (15a)$$

$$\psi = \frac{2(1-\nu+h\lambda)\lambda}{\{4+(1-\nu)h\lambda\}\lambda + 2E/R'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16a)$$

Introduction of these values in eq. (9) gives the critical thrust of the plate, if buckling with a half wave length $a = \pi/\lambda$. In differentiating eq. (9) with respect to λ, in order to find the wave length making ϱ_1 minimum, it is, with the ratio E/E_1 prevailing in aeroplane construction, sufficiently accurate to assume E' and ψ as constants, yielding

$$a = \pi (4B/E')^{1/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and after substitution of this value in eq. (9)

$$h \varrho_1 = \tfrac{3}{4}(4BE'^2)^{1/3} + \tfrac{1}{4} h \psi E' \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Furthermore E/R' may be equated to zero, so that in eq. (18)

$$E' = \frac{4 + (1-\nu) h \lambda}{(3-\nu)(1+\nu)} E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\psi = \frac{2(1-\nu+h\lambda)}{4+(1-\nu)h\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

If value h is equaled to zero in eqs. (9) and (15a), whilst $E/R' \neq \infty$, which coincides with the assumptions of VAN DER NEUT, we find

$$P = B\lambda^2 + \frac{E}{2\lambda}\,\frac{4\lambda + 2E/R'}{(3-\nu)(1+\nu)\lambda + 2E/R'} \quad . \quad . \quad . \quad . \quad (N_1)$$

which is indeed identical with his formula for the critical thrust. The result of GOUGH, ELAM and DE BRUYNE is obtained by equaling E/R' to zero in this formula. With plywooden plates, $E_1 = 110.000$ kg/cm², on a substratum of onazote, $E = 304$ kg/cm², and equaling ν and ν_1 to 0,25, we found that formula (N_1) gives values, that are about 5 to 6 % to low, so that in this case the difference is small.

In the case of two plates with an intermediate supporting layer, so called sandwich plates, with buckling of both plates in the same direction, the conditions at $\zeta = 0$ (fig. 3) are

$$u = 0 \quad \text{and} \quad \sigma_z = 0$$

so that according to eqs. (2′) and (3a′), at $\zeta = 0$

$$Z = 0 \quad \text{and} \quad Z'' = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

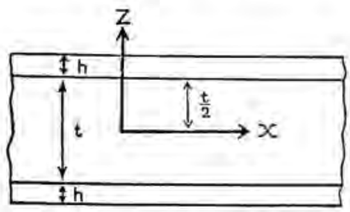

FIG. 3

After substitution of eq. (12) in these conditions, we find that a and aC_3 have to equal zero, so that, with $aC_1 = d$ and $aC_2 = dD$, we get

$$Z = d\,(\sinh\zeta + D\zeta\cosh\zeta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

As may be easily checked, a state of plane strain in plates and layer will only be possible here with rather broad plates, owing to the small modulus of rigidity of the layer. Therefore, except for the bending of the plates with respect to their own middle plane, a state of plane stress will be assumed here.

By substitution of eq. (22) in eq. (5a′), which is now our boundary condition at $\zeta = \lambda t/2 = \beta$, we get

$$D = \frac{\{(1+\nu)\lambda^2 h - 2E/R'\}\cosh\beta - 2(1+\nu)\lambda\sinh\beta}{\{(1-\nu)\lambda^2 h + 2(1+\nu)\lambda\beta + 2E/R'\}\cosh\beta - \{(1+\nu)\lambda^2 h\beta - 4\lambda - 2\beta E/R'\}\sinh\beta} \,. \quad (23)$$

E' and ψ are given here by similar equations as eqs. (10a) and (11), with the only difference, that the values between the brackets have now to be calculated for $\zeta = \beta$. In this way we obtain

$$E' = \frac{\{(1+\nu)\lambda^2 h - 2E/R'\}\alpha + \{(1-\nu)\lambda^2 h + 2E/R'\}\sinh\alpha + 4\lambda(\cosh\alpha - 1)}{(1+\nu)^2\lambda\alpha + 2(\cosh\alpha + 1)E/R' + (3-\nu)(1+\nu)\lambda\sinh\alpha}\,E \quad (24)$$

$$\psi = 2\,\frac{(1+\nu)\lambda\alpha + (1-\nu)\lambda\sinh\alpha + \lambda^2 h(\cosh\alpha + 1)}{\{(1+\nu)\lambda^2 h - 2E/R'\}\alpha + \{(1-\nu)\lambda^2 h + 2E/R'\}\sinh\alpha + 4\lambda(\cosh\alpha - 1)} \quad (25)$$

the critical thrust itself being given by eq. (9). In eqs. (24) and (25) value $\alpha = 2\beta = \lambda t$. By expansion of sinh α and cosh α in series, using at most 2 terms, eqs. (24) and (25) are, with $\lambda = \pi/a$, transformed in

$$E' = \frac{2\pi}{a} \frac{\pi^2 E_1 (t+h)\, th + (\pi^2/6) E t^3}{4(1+\nu)\pi^2 E_1\, th + 4E a^2 + \pi^2 E t^2} E \quad . \; . \; . \quad (26)$$

$$\psi = 2a/\pi t. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (27)$$

Substitution of these values in eq. (9) yields, after division of both members by h

$$\varrho_1 = \frac{\pi^2 E_1 h^2}{12(1-\nu_1^2) l^2} + \frac{\pi^2 E_1 (t+h)^2 + (\pi^2/6h) E (t+h) t^2}{4(1+\nu)\pi^2 E_1\, th + 4El^2 + \pi^2 E t^2} E. \quad . \quad (28)$$

In this formula, in which $\alpha = \pi t/a = \pi t/l$ is apparently assumed to be smaller than unity, the half wave length a has been substituted by the free supported length l of the strip, as it follows immediately from it, that the critical stress ϱ_1 is minimum, if a equals l.

We will show now, that also with an elementary method, a similar result may be obtained, by which moreover the physical meaning of this formula will be elucidated. To this end we use a method, as far as we know first used by ourselves [5]), by which the elasticity of a structure is assumed to be divided in two parts, say 1 and 2. In principle our reasoning was as follows. We assume first part 1 to have its normal elasticity, but part 2 to be absolutely rigid. At the critical thrust P_1, with an arbitrary deflection w_1, the inner moment M_1 will be equal to the outer moment $P_1 w_1$. Now we assume the rigidity of part 2 to decrease from infinite to its real value, during which the inner moment does not change. The deflection w_1 will now increase unto w, so that the equilibrium demands, that the critical thrust decreases to its real value P, whilst from the equality of inner and outer moment it follows that $P_1 w_1 = P w$. In the same way we assume now part 1 to be rigid and part 2 elastic, by which the equation $P_2 w_2 = P w$ is obtained. As $w = w_1 + w_2$ we obtain now the equation

$$w = w_1 + w_2 = P(P_1^{-1} + P_2^{-1})\, w$$

or [6])

$$P = (P_1^{-1} + P_2^{-1})^{-1}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (29)$$

With the sandwich plates, we assume first the intermediate layer to have an infinite modulus of rigidity G. The critical thrust P_M may now be

[5]) BIJLAARD, Nauwkeurige berekening van de plooispanning van hoekstalen, zoowel voor het elastische als voor het plastische gebied. De Ingenieur in Ned. Indië, 1939, Nr. 3.
BIJLAARD, Berekening van de knikspanning van gekoppelde profielen volgens een nieuwe methode. De Ingenieur in Ned. Indië, 1939, Nr. 3.

[6]) The same equation was obtained later by BUCKENS, Décomposition des coefficients d'influence dans les problèmes de vibration et de flambage. Publications International Association for Bridge and Structural Engineering. 7th Volume (1943/1944).

computed with EULER's formula $P_M = \pi^2 B_M / l^2$. In calculating the bending rigidity B_M, the proper moments of inertia $h^3/12$ of the plates are provisionally left out of account, as they cannot be assumed as rigid with our second assumption [7]). In this way we get

$$P_1 = P_M = (\pi^2/l^2)\{\tfrac{1}{2} E_1 (t+h)^2 h + \tfrac{1}{12} E t^3\} \quad . \quad . \quad . \quad . \quad (30)$$

We now assume the moduli of elasticity E_1 and E of plates and layer to be infinite, but the modulus of rigidity G of the layer to have its real value, whilst also in this case the plates themselves are assumed to have no bending resistance. Considering an element dx of the strip (fig. 4), we have

$$\tau = G\gamma = G(\alpha + \beta) = G\alpha(t+h)/t \quad . \quad . \quad . \quad . \quad . \quad (31)$$

whilst the equilibrium of one half of the element demands

$$\tfrac{1}{2}(t+h)\,\tau\,dx = \tfrac{1}{2} P_S\,\alpha\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

---

[7]) In a similar way we obtained a useful formula for the critical thrust of built-up timber columns, viz. two or more single struts, that are joined at two or more places by agency of wooden couplings. In the second mentioned publication of footnote 5 we considered the buckling of built-up steel columns, in which case it is sufficiently accurate to consider the couplings themselves, the so called batten plates, as infinitely rigid. With timber columns this assumption is not allowed. On request of ir. H. KRULL, expert in timber structures of the Department of Forestry in the Netherlands Indies, we tried to find a simple formula, in which, besides the elasticity of the couplings, also their length is taken into account. In the above indicated way we found for example, that the ideal slenderness ratio λ_i of a built-up column with two single struts is given approximately by the formula

$$\lambda_i^2 = \frac{\lambda_r^2}{1 + (\lambda_r^2 - \lambda_s^2)(r_s/l)^2}$$

in which

$$\lambda_r^2 = \lambda_t^2 + \lambda_s^2 + \lambda_c^2$$

and

$$\lambda_t = 2l/h, \;\; \lambda_s = c_0/r_s \quad \text{and} \quad \lambda_c^2 = 2\pi^2\, T A_s\, cv/h^2$$

in which l is the free supported length of the column, h is the distance of the axes of the single struts, c is the mutual distance of the couplings, $c_0 = c - 2a$, in which a is the theoretical length of the end couplings or half this length for the intermediate ones, the least of these two values to be taken into account, r_s is the radius of inertia of the single struts, A_s is the cross section of the single struts, T is the reduced modulus and v is the displacement of the single struts per unit of shearing force in the intermediate couplings or half of this displacement for the end couplings, the highest value to be taken into account. On request of and in cooperation with Ir. KRULL we executed several tests on built-up timber columns in the Laboratory for Testing Materials in Bandoeng, in order to check this formula and a similar one for columns of more than two struts and to determine the values v and $c_0 = c - 2a$ for couplings effected by wooden disks, bolts and nails, on which we will report occasionally in the technical papers.

combination of which equations yields [8])

$$P_2 = P_S = G\,(t+h)^2/t \quad . \; . \; . \; . \; . \; . \; . \; . \quad (33)$$

In order to find the real critical thrust of the strip, we have to add now

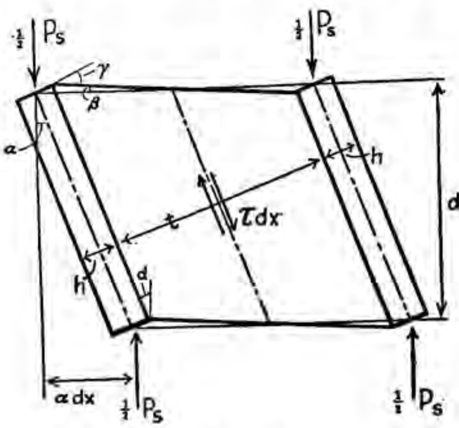

FIG. 4

the proper critical thrusts of the plates to the critical thrust according to eq. (29), by which we obtain

$$P = 2\,h\,\varrho_1 = 2\,\frac{\pi^2}{l^2}\,\frac{E_1\,h^3}{12\,(1-\nu_1^2)} + (P_M^{-1} + P_S^{-1})^{-1} \quad . \; . \; . \; . \quad (34)$$

Substitution of eqs. (30) and (33) in eq. (34) yields, after ranging and with $G = E/2(1+\nu)$, the critical stress

$$\varrho_1 = \frac{\pi^2 E_1 h^2}{12(1-\nu_1^2)\,l^2} + \frac{\pi^2 E_1 (t+h)^2 + (\pi^2/6\,h)\,E\,t^3}{4(1+\nu)\,\pi^2 E_1\, th + 4\,E\,l^2 + \frac{2}{3}(1+\nu)\,\pi^2 E\, t^4/(t+h)^2}\,E. \quad (35)$$

If, with metal strips, ϱ_1 surpasses the proportional limit, in the first term of the second member of eq. (35) $E_1/(1-\nu_1^2)$ should be replaced by $E_1 A$, according to our theory of plastic stability [9]), whilst in the second term E_1 has to be replaced by the reduced modulus T_1.

VAN DER NEUT obtained, after expansion of cosh and sinh in series, using our notations,

$$\varrho_1 = \frac{\pi^2 E_1 h^2}{12(1-\nu_1^2)\,l^2} + \frac{\pi^2 E_1\, E\,(t+h)^2}{4\,(1+\nu)\,\pi^2 E_1\,(t+h)\,h + 4\,E\,l^2} \quad . \; . \quad (N_2)$$

whilst in the same way the result of GOUGH, ELAM and DE BRUYNE would lead to

$$\varrho_1 = \frac{\pi^2 E_1\,h^2}{12\,(1-\nu_1^2)\,l^2} + \frac{E\,(t+h)}{4\,(1+\nu)\,h} \quad . \; . \; . \; . \; . \; . \quad (G)$$

[8]) In order to keep things simple, we neglect here small axial strains in the layer.

[9]) BIJLAARD, Theory of the plastic stability of thin plates, Publications International Association for Bridge and Structural Engineering. 6th Volume. (1940/41).

in which we have taken into account the fact, that all these authors measure the distance t between the middle planes of the plates, so that in their formulae, with our notations (fig. 3), t has to be replaced by $t + h$.

Although with small ratios h/t, for which it is meant, eq. (N_2) will be sufficiently accurate, it may nevertheless be useful, to dispose of the new formulae (28) or (35) for higher ratios h/t. We calculated for example the critical stress ϱ_1 for plates of plywood, $E_1 = 110.000$ kg/cm², with an intermediate layer of onazote, $E = 304$ kg/cm², $\nu_1 = \nu = 0{,}25$, for ratios $t/h = 5$ and $l/t = 10$. Our exact solution, eqs. (9), (24) and (25), with $a = l$, yielded $\varrho_1 =$ **431** kg/cm², our eqs. (28) and (35) both **432** kg/cm², whilst formulae (N_2) and (G) yielded **372** and **403** kg/cm² respectively, the correction to be made on the value of GOUGH c.s. being apparently of opposite sign than that made by VAN DER NEUT.

Apparently our reasoning in order to obtain eq. (29) holds also for plates with arbitrary boundary conditions, if inner and outer moment are replaced by restraining and deflecting force on an element $Hdxdy$ of the plate. An application to sandwich plates seems obvious.

Physiology. — *Influence of testosterone on the production of gonadotropic hormone in the hypophysis.* By J. H. GAARENSTROOM and S. E. DE JONGH. (Department of Pharmacology, University Leiden). (Communicated by Prof. G. G. J. RADEMAKER).

(Communicated at the meeting of November 30, 1946.)

In 1943 we [1]) could show that the effect of oestradiol benzoate on the hypophysis consists in a check on the production of the gonadotropic interstitium factor but not on that of the epithelium factor. The check on the production of the interstitium factor makes itself conspicuous in a decrease of the production of male hormone, which depends on the presence of this factor only. To prove the absence of a deleterious effect on the production of epithelium factor, a somewhat more complicated technique was required.

Hypophyses of ordinary rats and of rats treated with oestradiol benzoate were implanted in the abdomen of juvenile rats whose hypophyses had been removed and which from that very moment were exposed to large doses of chorionic gonadotrophine (interstitium factor!). After a week the effect of the implantations was estimated by a comparison of the testis weights. As no difference was observed between the testis weight of the rats in whose abdomen the hypophysis of a rat treated with oestradiol benzoate had been implanted and that of the rats which had received the hypophysis of an untreated animal, we came to the conclusion that in both cases equal amounts of epithelium factor must have been present.

We explained at that occasion what had led the earlier investigators to the tacit assumption (for which obviously no sufficient grounds were present) that the production of both factors was affected. With regard to the influence which testosterone exercises on the hypophysis, it has also been assumed that this comprises the production of both factors, and here too we were bound to ask whether this conclusion is justified. To decide this question we proceeded in the same way as before, i.e. we applied the technique of which a short description was given in the preceding paragraph. The test animals were juvenile rats of which the hypophysis had been extirpated. Each of them received a daily dose of 5 I.U. of pregnyl (chorionic gonadotr.). In 28 of the test animals a hypophysis was implanted which was obtained from an adult rat to which during the preceding fourteen days a dose of 1—2 mg testosterone propionate had been administered daily and 22 received instead a hypophysis from a rat which had ondergone no treatment or which had been treated with oil. Table 1 gives the weight of each of the test animals and also that of their donors, and further the weight of the right testicle, which was removed at the beginning of the experiment,

[1]) GAARENSTROOM and DE JONGH, Versl. Ned. Akad. v. Wetensch. 52, 446 (1943).

and that of the left one, which was taken at the end. The last column shows the difference in weight between the two testicles expressed as a percentage of that of the right one.

TABLE I.

TESTOSTERONE					CONTROL				
Weight of donor in g	Weight of recipient in g	Weight r. test. in mg	Weight l. test. in mg	Difference in per cent of initial weight	Weight of donor in g	Weight of recipient in g	Weight r. test. in mg	Weight l. test. in mg	Difference in per cent of initial weight
115	56	181	273	51	106	54	186	251	35
111	51	120	163	36	83	61	280	369	32
97	44	99	146	48	86	57	198	346	75
91	49	121	148	22	80	51	156	255	63
83	55	123	168	37	87	59	210	423	101
90	50	98	148	51	97	64	58	102	76
101	50	158	218	38	100	52	193	253	31
79	52	189	275	46	83	56	204	297	46
86	46	153	192	25	105	64	158	234	48
81	56	213	328	54	106	93	250	410	64
73	57	237	333	41	101	57	131	181	38
89	60	239	321	34	90	53	94	136	45
84	60	237	345	46	238	55	146	228	56
98	57	200	214	7	212	57	160	177	11
110	90	227	330	45	226	51	97	168	73
106	75	238	345	45	230	55	203	252	24
228	54	199	246	24	245	56	200	362	81
237	55	83	152	83	244	65	124	288	133
240	59	217	194	— 11	230	58	212	344	62
220	75	283	325	15	217	68	343	494	44
242	66	348	447	28	190	58	246	448	82
205	70	319	400	25	?	60	121	211	75
238	64	116	154	33					
262	57	390	505	30					
238	60	220	374	70					
239	60	192	350	82					
218	56	205	323	58					
198	66	208	364	75					
Average:		200	278	41 ± 4			180	312	59 ± 6

The average difference appears to be unequal in the two groups of test animals. Implantation of a normal hypophysis caused an increase of 59 ± 6 per cent, whereas implantation of a testosterone hypophysis caused an increase of but 41 ± 4 per cent. The significancy coefficient of the difference between the two values amounts to 2.5. Taking the large number of animals into account, this practically excludes the possibility that it might be accidental.

The preceding treatment with testosterone propionate, therefore, has reduced the gonadotropic potency of the hypophysis as shown by the implantation-effect. As the influence of the reduction in the amount of interstitium factor has been eliminated by the administration of an excess of this substance, this proves that the *amount* of epithelium factor too must have diminished. On the ground of general considerations this justifies the conclusion that the *production* of this factor has been reduced.

It might perhaps be argued that it is not allowed to base our conclusions on the relative increase in weight of the testicles when we are not sure that their initial weight may be left out of consideration. As the average weight of the removed right testicles in the two groups is unequal (200 mg in the rats in which the testosterone hypophyses were implanted, and 180 mg in the other ones), this factor, indeed, might have been a source of error.

To eliminate the possible error mentioned in the preceding paragraph we have calculated in table 2 the average increase in weight (expressed in

TABLE II.

Right testicles of		1—100	101—200	201—300	301—400 mg
Test. hyp.	Weight increase l. testis in per cent of r. testis	61	36	43	28
	Number of rats	3	11	11	3
Norm. hyp.	Weight increase l. testis in per cent of r. testis	65	59	59	44
	Number of rats	3	11	7	1

per cent of the initial weight) for four classes of test animals differing in initial testicle weight. The table shows that there exists a certain degree of correlation between the initial weight and the relative value of the increase, for the lighter testicles show a higher increase than the heavier ones. However, it appears also that a difference of 20 mg in the initial weight can not be considered of any importance. Moreover, in all classes, even in those containing but a small number of rats, the testosterone group showed a smaller increase than the control group.

Another argument in favour of our contention that the difference in the relative increase of the testicle weight can not be due to a difference in the average value of the initial weight, is found in the fact that the average of the absolute weight values reached by the testicles of the testosterone group is (notwithstanding the higher value of the initial weight) lower than that of the control group (cf. table 1).

A second objection which might be raised against our conclusion, concerns the question whether the doses of pregnyl (chorionic gonadotrophine) which we administered to both groups of test animals, really are large enough to be considered as an excess. Our conviction that this condition has been fulfilled, rests on two facts which were already mentioned in our former communication:

1. The daily pregnyl dose required to maintain in a juvenile male rat whose hypophysis has been removed the testosterone production on the same level as that of a normal rat of the same age, is 0.5 I.U. (the result is estimated by a comparison of the growth of the seminal vesicles). This amount must be aequivalent to the daily production of interstitium factor. As we administered 5 I.U. daily, i.e. ten times as much, our dose may justly be considered a *strong* one.

2. That the 5 I.U. must have been sufficient to overcome the effect of the diminution, or eventually the stopping, of the production of interstitium factor follows from our earlier experiments with oestradiol benzoate to which reference was made in our introductory remarks, for when the test animals in those experiments received 5 I.U. of pregnyl, there was found no difference between the effect of the implantation of oestradiol hypophyses and normal ones, although the first must have contained less interstitium factor than the others.

On account of these arguments the second objection too must in our opinion be rejected.

Summary.

A daily dose of 1—2 mg testosterone propionate administered to male rats, reduced the implantation effect in juvenile rats which have been deprived of their own hypophysis, notwithstanding the fact that these animals were injected with an excess of interstitium factor; as a gauge of the effect the increase in weight of the testicles of the recipients was used. This proves that the production of epithelium factor too must have been reduced. The effect of testosterone on the hypophysis, therefore, must consist in a check exercised on the production of *both* gonadotropic factors, the interstitium factor as well as the epithelium factor. Its action on the first had never been doubted, but with regard to its action on the second there was as yet no certainty.

Chemotherapy. — *L'action inhibitrice des métaux sur la croissance du B. tuberculeux. IV. Gallium, indium et thallium.* By ONG SIAN GWAN. (Communicated by Prof. E. GORTER.)

(Communicated at the meeting of November 30, 1946.)

1. Dans des mémoires précédents [1]) nous avons montré l'action inhibitrice sur la croissance du B. tuberculeux des éléments du quatrième groupe, sous-groupe *b*, du tableau périodique: germanium, étain et plomb et ceux du cinquième groupe, sous-groupe *b*: arsénic, antimoine et bismuth. L'action inhibitrice de Ge (32), Sn (50) et Pb (82) diminue à mesure que le nombre atomique augmente; le plomb n'a aucune action inhibitrice. Par contre, celle de As (33), Sb (51) et Bi (83) augmente avec le nombre atomique.

Nous examinons dans ce mémoire l'action inhibitrice des éléments du troisième groupe, sous-groupe *b*: gallium, indium et thallium. Parmi ces métaux examinés seul le thallium montre une action inhibitrice très marquée.

Les sels de thallium sont connus par leur action toxique sur l'organisme. Mais ceci n'est pas une raison pour abandonner des essais avec des composés de thallium. On sait que l'arsénic a également une action toxique, certains composés arsénicaux ont cependant une action puissante dans le traitement de la syphilis. Il est peut-être possible que la toxicité du thallium diminue si l'on utilise des composés organo-métalliques. Des essais thérapeutiques dans ce sens sur les animaux tuberculeux sont en ce moment en cours.

2. *Action inhibitrice de gallium, indium et thallium sur la croissance du B. tuberculeux.*

Dans toutes ces expériences, comme dans les expériences précédentes, on a utilisé une même souche bovine (souche Vallée) de B. tuberculeux, qui a subi plusieurs passages dans le milieu synthétique de Sauton. Les métaux employés sont les suivants: **1.** gallium des Vereinigte chemische Fabriken, Leopoldsthall, **2.** indium puriss. HEYL et **3.** thallium HILGER. Le tableau 1 représente les résultats obtenus, les colonnes indiquent successivement le métal utilisé, le nombre d'observations, le poids moyen $\bar{p}$ des bacilles desséchés, l'écart type σ du poids moyen, le coefficient de variation v en %, $v = 100 \frac{\sigma}{\bar{p}}$, la valeur de $z = \frac{1}{2} \ln \frac{\sigma_1^2}{\sigma_2^2}$, la probabilité obtenue de z,

[1]) ONG SIAN GWAN, Verslagen Ned. Akad. v. Wetensch., A'dam, Afd. Natuurkunde, **53**, 345 et 353 (1944); Proc. Kon. Ned. Akad. v. Wetensch., A'dam, **48**, 411 (1945).

la valeur de *t* et la probabilité obtenue de *t*. Le coefficient de variation peut être utilisé pour comparer la variation relative des poids moyens obtenus.

TABLEAU 1.

Action inhibitrice de gallium, d'indium et de thallium sur la croissance du B. tuberculeux.

Métal	Nombre d'observations	Poids moyen en mg	Écart type σ	$v = 100 \frac{\sigma}{p}$	z	P	t	P
A. 10 mg de métal/100 cc								
Témoins	4	484.0	116.9	24				
Gallium	4	714.5	56.5	8	0.7282	0.1	—3.550	0.01
Indium	4	792.1	65.0	8	0.5868	0.2	—4.604	0.003
Thallium	4	4.7	2.7	59	3.7523	<0.001	8.196	0.0001
B. 10 mg de thallium/100 cc								
Témoins	4	665.8	7.1	1				
Thallium	4	3.6	1.4	39	3.2773	<0.001	183.408	<0.00003
C. 10 mg de thallium/100 cc								
Témoins	4	462.0	90.2	20				
Thallium	4	1.9	1.1	58	4.4153	<0.001	10.227	0.00003
D. 10 mg de gallium/100 cc								
Témoins	3	700.5	81.4	12				
Gallium	2	692.1	52.2	8	0.4265	>0.2	0.126	>0.9
E. 20 mg de métal/100 cc								
Témoins	5	694.4	47.7	7				
Gallium	3	726.3	102.9	14	0.7691	0.1	—0.615	0.3
Indium	4	704.1	57.9	8	0.1934	>0.2	—0.275	0.4
Thallium	4	1.9						
F. 2,5 mg de $TlNO_3$/100 cc								
Témoins	4	745.6	12.1	2				
$TlNO_3$	4	561.0	88.7	16	1.9923	0.001	4.127	0.006

Le nombre *z* donne une comparaison de l'écart type obtenu par les cultures témoins et celui des cultures avec métal et enfin *t* est une mesure de la différence des poids moyens des cultures témoins et des cultures avec métal. Dans tous les cas on dirait que la différence est significative si la probabilité *P* pour que, *z* ou *t* ait une valeur supérieure à la valeur trouvée, est égale ou inférieure à 5 p. 100.

Expérience réalisée avec 10 mg de métal par 100 cc de milieu de culture (milieu synthétique de Sauton).

On constate dans tous les groupes que le thallium a donné une différence significative avec le témoin. Cette différence s'observe non seulement sur le poids moyen, mais aussi sur l'écart type et sur le coefficient de variation. Ce résultat montre clairement que la culture avec thallium et la culture témoin ne proviennent pas d'une même population. On constate également une différence significative entre le poids moyen des cultures témoins et celui des cultures avec gallium ou indium. Le poids moyen de ces dernières cultures est plus élevé que celui des cultures témoins. Ce résultat doit être attribué aux cultures témoins qui ont données une récolte moins abondante d'une cause indéterminée. En effet, en comparant le poids moyen de ces cultures témoins avec celui du groupe E (témoins) on constate que la différence est réelle; la différence des écarts types est dans ce cas également significative.

Dans le même groupe A on constate cependant que la différence n'est pas siginificative entre l'écart type des cultures témoins et celui des cultures avec gallium ou indium. De plus l'expérience répétée avec le gallium (groupe D) ne donne plus une différence significative entre ces cultures et les cultures témoins.

Expérience réalisée avec 20 mg de métal par 100 cc de milieu de culture.

On obtient avec 20 mg de métal le même résultat que celui avec 10 mg. Seul le thallium a donné une action inhibitrice très élevée. Nous n'avons pas pu déterminer séparément les poids des cultures avec thallium dans le groupe E, mais le calcul montre que la probabilité pour que, le poids des cultures témoins soit égal ou inférieur à 1,9 mg serait égale à 1 : 17000. La différence des poids moyens des cultures avec thallium et des cultures témoins est donc très significative.

Dans le même groupe une cinquième culture avec thallium, qui ne figure pas dans le tableau 1 a donné un poids assez élevé, $p = 221{,}5$ mg. En examinant cette culture on constate que la réaction du liquide à la fin de l'expérience est légèrement alcaline, on trouve $pH = 7{,}4$. Par contre, elle est légèrement acide, $pH = 6{,}7$, dans les autres cultures avec thallium, dans les cultures témoins et dans les cultures avec gallium ou indium. On sait que le thallium est pratiquement insoluble dans un milieu alcalin, le résultat que nous avons obtenu pourrait être ainsi expliqué.

Le mécanisme d'action de thallium doit être le même comme celui du bismuth et probablement d'autres métaux actifs. Nous avons montré dans les mémoires précédents que le bismuth métallique se dissout très peu dans le liquide de culture et que l'action inhibitrice pourrait être expliqué par l'absorption du bismuth par le B. tuberculeux. Dans le milieu de culture du groupe A nous avons ainsi déterminé la quantité de thallium à la fin de l'expérience par une méthode colorimétrique, on trouve 0,111 mg de Tl par cc de liquide. De plus, une expérience réalisée avec un sel soluble de thallium, le nitrate de thallium a donné le même effet inhibitrice (groupe F). En effet, pour diminuer le poids de récolte de B. tuberculeux de 25 p. 100 il faut une concentration de thallium de 1 : 52.000.

3. *Action inhibitrice de thallium en présence de sérum sanguin.*

On peut se demander si l'action inhibitrice de thallium est la même en présence de sérum sanguin. A cet effet une expérience a été réalisée comme suit. Dans 100 cc de milieu de SAUTON on ajoute 10 cc de sérum de cheval non chauffé. On fait ainsi quatre cultures témoins en présence de sérum et deux autres cultures témoins sans sérum. Dans les quatre cultures avec thallium on ajoute de plus 10 mg de thallium métallique. On ensemence tous les flacons avec une souche de B. tuberculeux agé de 10 jours et qui a subi sept passages sur milieu de SAUTON. Au bout de 27 jours l'expérience est terminée et les poids de récolte de B. tuberculeux obtenus sont reproduits dans le tableau 2.

TABLEAU 2.

Action inhibitrice de thallium en présence de sérum sanguin.

Métal	Poids en mg	Poids moyen en mg
Témoins	617.2 704.6	660.9
Témoins avec sérum	(79.2) (80.7) (117.1) 779.9	779.9
Thallium	—*) —*) —*) —*)	—*)

*) Pas de croissance.

On voit dans le tableau que l'action inhibitrice de thallium est la même en présence de sérum sanguin. On constate de plus que la présence de sérum sanguin dans les cultures témoins n'empêche nullement le développement normal de B. tuberculeux. Il faut cependant remarquer que dans trois flacons témoins avec sérum la voile de B. tuberculeux est tombée au cours de l'expérience, de sorte que le poids obtenu est trouvé trop petit. On sait que le B. tuberculeux est un aérobie strict, qui se procure dans le milieu nutritif ordinaire l'énergie nécessaire pour son métabolisme en se servant de l'oxydation directe, c'est à dire en n'utilisant que l'oxygène libre. Le quatrième flacon témoin avec sérum a cependant donné une culture normale et le poids obtenu est du même ordre de grandeur que celui des cultures témoins sans sérum.

La chute de la voile de bacilles au cours de la croissance doit être attribuée à l'abaissement de la tension superficielle du liquide de culture par le sérum. Ce phénomène s'observe également avec d'autres substances capillaro-actives ajoutées au milieu de culture.

4. *Culture de B. tuberculeux ayant été en contact avec le thallium.*

L'expérience a pour but de savoir, premièrement si le B. tuberculeux n'est pas tué après un séjour déterminé dans un milieu de culture avec thallium et deuxièmement s'il montre une lésion provoquée par le thallium. Dans ce dernier cas on devrait obtenir une diminution de croissance si l'on transporte le B. tuberculeux dans un milieu de culture normal. Pour cela trois groupes d'expériences ont été realisés, les résultats obtenus sont reproduits dans le tableau 3 et 4. Dans le premier groupe A le repiquage de B. tuberculeux au bout de trois jours de contact avec le thallium n'a pas donné une différence avec les cultures témoins du même âge et dans les même conditions, mais sans thallium. Le repiquage au bout de 12 jours ne permet pas de faire une comparaison entre les deux cultures puisque trois cultures témoins sont tombées au cours de la croissance; les poids obtenus sont placés entre parenthèses. Par contre, le repiquage après 44 jours ne donne plus de culture avec le B. tuberculeux ayant été en contact avec le thallium, il est donc tué.

L'expérience répétée avec 20 mg de thallium par 100 cc de milieu de culture (groupe B) a donné un résultat comparable. Cependant le repiquage au bout de 43 jours a seulement donné une culture négative sur cinq.

TABLEAU 3.

Culture de B. tuberculeux ayant été en contact avec 10 mg (groupe A) ou 20 mg (groupe B et C) de thallium par 100 cc

Repiquage après *n* jours	Durée de l'expérience en jours	Cultures témoins Poids en mg	Poids moyen en mg $\bar{p}$	Cultures á thallium Poids en mg	Poids moyen en mg $\bar{p}$
		A. 10 mg de Tl/100 cc			
3	64	746.0 735.5 748.5 779.0	752.3	753.0 771.0 719.8 779.0	747.0
12	64	654.0 (73.0) (2.5) (27.5)	—	688.0 662.0 65.0 70.0	371.4
44	63	794.0 797.0 801.0 776.0	792.0	—*) —*) —*) —*)	0.0
		B. 20 mg de Tl/100 cc			
7	62	808.5 742.0 783.0	777.8	810.5 779.5 791.8	793.9
25	61	(172.0) 719.5 785.0	752.3	864.5 634.0 469.0	649.8
43	68	813.5 841.5 829.0	828.0	—*) 808.5 772.5 821.5 857.0	651.9
		C. 20 mg de Tl/100 cc			
5	26	676.5 721.0 665.0 700.0	690.6	408.0 471.0 402.5 468.0	437.4
16	29	381.0 408.0 387.0 411.0	396.8	142.0 —*) 279.0 —*)	105.3
37	26	689.5 664.5 690.5 666.0	677.6	193.0 301.5 612.5 698.0	451.3
56	41	—*) —*) —*) +**)	—*)	52.9 84.1 —*) —*)	34.3

*) Pas de croissance. **) Croissance faible.

TABLEAU 4.

Comparaison des cultures de repiquage avec les cultures témoins

Repiquage après *n* jours	Cultures témoins Poids moyen en mg	Cultures à thallium Poids moyen en mg	*t*	*P*	F	P
A. 10 mg de Tl/100 cc						
3	752.3	747.0	0.374	0.7	1.30	>0.05
12	—	371.4				
44	792.0	0.0		<0.001		<0.01
B. 20 mg de Tl/100 cc						
7	777.8	793.9	0.287	0.8	37.72	<0.05
25	752.3	649.8	0.715	0.5	16.70	>0.05
43	828.0	651.9	0.759	0.5	679.6	<0.01
C. 20 mg de Tl/100 cc						
5	690.6	437.4	11.312	<0.001	2.22	>0.05
16	396.8	105.3	2.174	0.08	79.80	<0.01
37	677.6	451.3	1.866	0.1	286.61	<0.01
56	—	34.3	—	—	—	—

Dans le troisième groupe C, également réalisé avec 20 mg de thallium on constate une différence significative entre les poids moyens obtenus après un contact de 5 jours avec le thallium. Le poids moyen de B. tuberculeux ayant été en contact avec le thallium est moins élevé que celui des cultures témoins. De même les poids moyens obtenus par le B. tuberculeux ayant été en contact avec le thallium pendant 16 et 37 jours sont moins élevés que ceux obtenus par les cultures témoins, mais la différence n'est pas significative.

La différence de résultats obtenus avec cette expérience et avec les deux autres précedentes doit être attribuée à la durée de l'expérience, donc à l'âge de la culture. En effet, dans les groupes A et B on laisse le B. tuberculeux se developper pendant au moins 60 jours, tandis que dans le groupe C l'expérience est déjà terminée au bout de 26 et 29 jours. On peut donc en conclure que le B. tuberculeux est modifié après un contact avec le thallium, mais que cette modification est reversible.

Il est important de remarquer que certaines parties de la culture de B. tuberculeux en contact avec le thallium sont tuées tandis que d'autres parties sont encore vivantes (groupe C). On constate dans la culture d'origine avec thallium à la fin de l'expérience dans quelques flacons des colonies isolées surnageantes d'une couleur brun clair. Ce résultat se rapproche de celui obtenu par le Professeur CHAIN avec le staphylocoque résistant à la penicilline [2]).

[2]) E. CHAIN, Conférence donnée à l'Université de Leyde le 14 Octobre 1946.

5. *Addition de thallium au cours de la croissance de B. tuberculeux.*

L'addition de thallium au cours de la croissance de B. tuberculeux permet de savoir si la croissance est arrêtée ou bien si la croissance est ralentie. Pour savoir si la croissance est arrêtée on compare à la fin de l'expérience le poids moyen des cultures avec métal avec le poids moyens des cultures témoins au moment de l'addition de thallium. La diminution de croissance se révèle par une diminution de poids moyen à la fin de l'expérience, par rapport au poids moyen des cultures témoins. Le résultat de cette expérience est reproduit dans le tableau 5.

TABLEAU 5.

Addition de 10 mg de thallium/100 cc au cours de la croissance de B. tuberculeux

Temps en jours T_i	Cultures témoins		Cultures avec Tl		Comparaison avec $\bar{p}_i$		Comparaison avec $\bar{p}_{38}$	
	n	Poids moyen en mg $\bar{p}_i$	n	Poids moyen en mg	t	P	t	P
7	4	4.1	4	25.1	7.687	0.0002	8.831	<0.00003
15	4	94.2	4	83.4	0.653	0.55		
30	4	124.8	4	105.5	0.755	0.5		
38	6	223.1						

n, nombre d'observations

Il montre que l'addition de 10 mg de thallium par 100 cc de milieu de culture au bout de 7 jours a donné une différence très significative avec les cultures témoins au moment de l'addition. Par contre le poids moyen des cultures avec thallium à la fin de l'expérience est beaucoup plus petit que celui des cultures témoins, la différence obtenu est très significative. L'addition de thallium au bout de 7 jours n'arrête pas la croissance de B. tuberculeux, mais elle la ralentie considérablement. Par contre l'addition de thallium au bout de 15 et de 30 jours arrête complètement le développement du B. tuberculeux. Ce résultat montre que pendant la croissance logarithmique le B. tuberculeux est plus sensible à l'action nocive de thallium. Il confirme de plus le résultat obtenu avec le germanium [1]). Indépendamment de nous le Professeur CHAIN [2]) a observé le même résultat sur le staphylocoque avec la penicilline.

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