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ANTIPERIODIC SOLUTIONS FOR nTH-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this work, we establish the existence and uniqueness of antiperiodic solution for a class of nth-order functional differential equations with infinite delay. The main tool in our study is the coincidence degree theory. An example is presented to illustrate the results obtained.

1. Introduction

Given a positive number T, we say that a continuous function $x : \mathbb{R} \to \mathbb{R}$ is T-antiperiodic on $\mathbb{R}$ if

$$x(t+T) = x(t) \quad \text{and} \quad x\Big(t+\frac{T}{2}\Big) = -x(t) \quad \text{for all } t \in \mathbb{R}.$$

In this work, we consider the nth-order functional differential equation with infinite delay

$$x^{(n)}(t) = f\Big(t, x_t^{(n-1)}, x_t^{(n-2)}, \dots, x_t', x_t\Big), \quad t \in \mathbb{R}, \tag{1.1}$$

where
⋆ f is continuous and real defined on $\mathbb{R} \times \underbrace{C_B((-\infty,0],\mathbb{R}) \times \cdots \times C_B((-\infty,0],\mathbb{R})}_{n \text{ times}}$,
and T-periodic in the first argument, where $C_B((-\infty,0],\mathbb{R})$ represents the space of the bounded continuous functions $\phi : (-\infty,0] \to \mathbb{R}$ endowed with the norm $\|\phi\|_\infty = \sup_{\tau\in(-\infty,0]} |\phi(\tau)|$;
⋆ x_t denotes the mapping $x_t : (-\infty,0] \to \mathbb{R}$ defined by $x_t(\tau) = x(t+\tau)$ for $\tau \in (-\infty,0]$, where $t \in \mathbb{R}$;
⋆ $f\big(t+\frac{T}{2}, -\varphi_1, -\varphi_2, \dots, -\varphi_n\big) = -f\big(t, \varphi_1, \varphi_2, \dots, \varphi_n\big)$, for $\varphi_i \in C_B((-\infty,0],\mathbb{R})$, $i = 1, \dots, n$;
⋆] x_t' is the derivative of the unknown function x_t and $x_t^{(j)}$ is the derivative of jth-order of x_t, $j = 2, \dots, n-1$.

Functional differential equations with delay have an important role in modelling of natural processes in those which is necessary to consider the influence of past effects for a better understanding of their evolution. For example, in population dynamics, the gestation times is a natural source of delays, since present birth rates

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is strongly dependent on the number of individuals at fecundation (see [18, 20, 25]). In classical physics, realistic models must take in account the time-delays due to the finite propagation speed of the classical fields (see [17, 26]). More examples of real phenomena with delay effects in physiology, epidemiology, engineering, economics, neural networks, automatic control, etc, can be found in [12, 16, 27]. Furthermore of importance in applications, the functional differential equations with delay have several distinct mathematical properties of ordinary and partial differential equations, which also provides them with a purely mathematical interest.

Arising from problems in applied sciences, antiperiodic problems of nonlinear differential equations have been extensively studied by many authors during the past twenty years. We can cite [1]-[6], [8, 10, 15, 19] and references therein.

To the best of our knowledge there is not none work dedicated to study of the existence and uniqueness of antiperiodic solution for nth-order functional differential equations with infinite delay with level of generality of equation (1.1). We will obtain sufficient and necessary conditions for the existence and uniqueness of an antiperiodic solution on $\mathbb{R}$ of equation (1.1) via the coincidence degree theory.

This article is organized as follows. In Section 2 we obtain results ensuring existence and uniqueness of a T-antiperiodic solution for equation (1.1). An illustrative example is given in Section 3.

2. Existence and uniqueness of antiperiodic solution

We adopt the following notation:

$$C_T^k = \{x \in C^k(\mathbb{R},\mathbb{R}); x \text{ is } T\text{-periodic}\}, \quad k \in \{0,1,2,\dots\},$$

$$\|x\|_2 = \Big(\int_0^T |x(t)|^2 dt\Big)^{1/2}, \quad \|x\|_\infty = \max_{t\in[0,T]} |x(t)|, \quad \text{for } x \in C_T^0,$$

$$\|x^{(k)}\|_\infty = \max_{t\in[0,T]} |x^{(k)}(t)|_\infty, \quad \text{for } x \in C_T^k.$$

Definition 2.1. A function $x : \mathbb{R} \to \mathbb{R}$ is said to be a *T-antiperiodic solution* of equation (1.1) if the following conditions are fulfilled:

(i) $x^{(n)}(t) = f\Big(t, x_t^{(n-1)}, x_t^{(n-2)}, \dots, x_t', x_t\Big)$ for each $t \in \mathbb{R}$;

(ii) x is T-antiperiodic on $\mathbb{R}$.

We use the following assumption:

(H1) There are positive constants $a_1, \dots, a_n$ such that $\sum_{i=1}^n a_i\Big(\frac{T}{2\pi}\Big)^i < 1$, and

$$|f(t,\varphi_1,\dots,\varphi_n) - f(t,\psi_1,\dots,\psi_n)| \le \sum_{i=1}^n a_i|\varphi_i(0) - \psi_i(0)|,$$

for each $t \in \mathbb{R}$ and $\varphi_i, \psi_i \in C_B((-\infty,0],\mathbb{R})$, $i = 1,\dots,n$.

In the next lines, our goal is to prove the following result.

Theorem 2.2. *If* (H1) *holds, then* (1.1) *has at least one T-antiperiodic solution.*

To prove Theorem 2.2, we start by recalling some concepts in the next lemma, which is crucial in the arguments of this section. Let X and Y be real normed vector spaces. A linear operator $L : \operatorname{Dom} L \subset X \to Y$ is a Fredholm operator if $\ker L$ and $Y \setminus \operatorname{Img} L$ are finite-dimensional and $\operatorname{Img} L$ is closed in Y. The *index* of L is defined by $\dim\ker L - \operatorname{codim}\operatorname{Img} L$. If L is a Fredholm operator of index zero, it is possible to prove (see [28]) that if L is a Fredholm operator of index zero, then there exist continuous linear and idempotent operators $P : X \to X$ and $Q : Y \to Y$ such that

$$\ker L = \operatorname{Img} P \quad \text{and} \quad \operatorname{Img} L = \ker Q. \tag{2.1}$$

The first equality in (2.1) implies that the restriction of L to $\operatorname{Dom} L \cap \ker P$, which we will denote by L_P, is an isomorphism onto its image. Indeed, by supposing $\ker L = \operatorname{Img} P$ and taking $x \in \operatorname{Dom} L \cap \ker P$ such that $L_P(x) = 0$, we have that $x \in \operatorname{Img} P$, that is, there exists $y \in X$ such that $Py = x$. Since P is idempotent and $x \in \ker P$, the last equality implies $x = Py = Px = 0$.

By assuming that $L : \operatorname{Dom} L \subset X \to Y$ is a Fredholm operator of index zero and P and Q are the aforementioned operators, we say that a continuous operator $N : X \to Y$ is L-compact on $\overline{\Omega}$, where $\Omega \subset X$ is open and bounded, if $QN(\overline{\Omega})$ is bounded and the operator $(L_P)^{-1}(I - Q)N : \overline{\Omega} \to X$ is compact.

To prove Theorem 2.2 we need the following result, whose proof can be found in [23].

Lemma 2.3. *Let X, Y be Banach spaces, $\Omega \subset X$ a bounded open set symmetric with $0 \in \Omega$. Suppose $L : \operatorname{Dom} L \subset X \to Y$ is a Fredholm operator of index zero with $\operatorname{Dom} L \cap \overline{\Omega} \neq \emptyset$ and $N : X \to Y$ is a L-compact operator on $\overline{\Omega}$. Assume, moreover, that*

$$Lx - Nx \neq -\lambda(Lx + N(-x)),$$

for all $x \in \operatorname{Dom} L \cap \partial\Omega$ and all $\lambda \in (0, 1]$, where $\partial\Omega$ is the boundary of Ω with respect to X. Under these conditions, the equation $Lx = Nx$ has at least one solution on $\operatorname{Dom} L \cap \overline{\Omega}$.

Next, we construct an equation $Lx = Nx$ that appropriately mirrors problem (1.1) and so that all the conditions of Lemma 2.3 are fulfilled.

Define the sets

$$X = \big\{x \in C^n_T; x\big(t + \frac{T}{2}\big) = -x(t), t \in \mathbb{R}\big\},$$
$$Y = \big\{x \in C^{n-1}_T; x\big(t + \frac{T}{2}\big) = -x(t), t \in \mathbb{R}\big\}.$$

By equipping X and Y with the norms

$$\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty, \dots, \|x^{(n)}\|_\infty\},$$
$$\|x\|_Y = \max\{\|x\|_\infty, \|x'\|_\infty, \dots, \|x^{(n-1)}\|_\infty\},$$

respectively, we obtain two Banach spaces.

Define the operators $L : X \to Y$ and $N : X \to Y$ by

$$Lx(t) = x^{(n)}(t), \quad t \in \mathbb{R}, \tag{2.2}$$
$$Nx(t) = f\Big(t, x_t^{(n-1)}, x_t^{(n-2)}, \dots, x'_t, x_t\Big), \quad t \in \mathbb{R}. \tag{2.3}$$

To prove Theorem 2.2, it is sufficient to show that condition (H1) implies that the assumptions of Lemma 2.3 are satisfied when L and N are defined as in (2.2) and (2.3). It is easy to verify that

$$\ker L = 0 \quad \text{and} \quad \operatorname{Img} L = \{x \in Y; \int_0^T x(s)ds = 0\} = Y.$$

Then $\dim \ker L = 0 = \operatorname{codim} \operatorname{Img} L$ and L is a linear Fredholm operator of index zero.

Proposition 2.4. *The operator N is L-compact on any bounded open set $\Omega \subset X$.*

Proof. Let us consider the operators P and Q given by

$$Px = \frac{1}{T}\int_0^T x(t)dt,\ x \in X \quad \text{and} \quad Qy = \frac{1}{T}\int_0^T y(t)dt,\ y \in Y.$$

Thus $\operatorname{Img} P = \ker L$ and $\ker Q = \operatorname{Img} L$. Denoting by $L_P^{-1} : \operatorname{Img} L \to X \cap \ker P$ the inverse of $L_{|X\cap \ker P}$, one can observe that L_P^{-1} is a compact operator. Besides, it is not difficult to show that, for any open bounded set $\Omega \subset X$, the set $QN(\overline{\Omega})$ is bounded and, using the Arzelà-Ascoli's Theorem, the operator $L_P^{-1}(I-Q)N : \overline{\Omega} \to X$ is compact. Therefore, N is L-compact on Ω. $\square$

The next lemma will be used later. Its proof can be found in [14].

Lemma 2.5. *If $v : \mathbb{R} \to \mathbb{R}$ is a T-periodic absolutely continuous function such that $\int_0^T v(t)dt = 0$ and $\int_0^T v'(t)^2 dt \in \mathbb{R}$, then*

$$\int_0^T v(t)^2 dt \le \frac{T^2}{4\pi^2}\int_0^T v'(t)^2 dt.$$

Proposition 2.6. *If condition* (H1) *holds, then there exists a positive number D, which does not depend on λ such that, if*

$$Lx - Nx = -\lambda[Lx + N(-x)], \quad \lambda \in (0,1], \tag{2.4}$$

then $\|x\|_X \le D$.

Proof. Assume (H1) and that $x \in X$ satisfies (2.4). Then, by using the definitions of operators L and N, given in (2.2) and (2.3), respectively, we obtain

$$\begin{aligned} x^{(n)}(t) &= \frac{1}{1+\lambda} f\Big(t, x_t^{(n-1)}, x_t^{(n-2)}, \dots, x_t', x_t\Big) \\ &\quad - \frac{\lambda}{1+\lambda} f\Big(t, -x_t^{(n-1)}, -x_t^{(n-2)}, \dots, -x_t', -x_t\Big). \end{aligned}$$

Thereby, considering $F(t,x) = f\big(t, x_t^{(n-1)}, x_t^{(n-2)}, \dots, x_t', x_t\big)$, we have

$$x^{(n)}(t) = \frac{1}{1+\lambda}F(t,x) - \frac{\lambda}{1+\lambda}F(t,-x).$$

Multiplying both sides of this equality by $x^{(n)}(t)$ and subsequently integrating it from 0 to T and using the triangle inequality, we obtain

$$\begin{aligned} \|x^{(n)}\|_2^2 &\le \frac{1}{1+\lambda}\int_0^T |F(t,x)||x^{(n)}(t)|dt + \frac{\lambda}{1+\lambda}\int_0^T |F(t,-x)||x^{(n)}(t)|dt \\ &\le \frac{1}{1+\lambda}\Big[\int_0^T |F(t,x) - F(t,0)||x^{(n)}(t)|dt + \int_0^T |F(t,0)||x^{(n)}(t)|dt\Big] \\ &\quad + \frac{\lambda}{1+\lambda}\Big[\int_0^T |F(t,-x) - F(t,0)||x^{(n)}(t)|dt + \int_0^T |F(t,0)||x^{(n)}(t)|dt\Big]. \end{aligned}$$

Therefore,

$$\|x^{(n)}\|_2^2 \le \int_0^T \max\{|F(t,x) - F(t,0)|, |F(t,-x) - F(t,0)|\}|x^{(n)}(t)|dt$$

$$+\int_0^T |F(t,0)\|x^{(n)}(t)|dt.$$

This, assumption (H1), Lemma 2.5 and Hölder inequality, imply

$$\begin{aligned}\|x^{(n)}\|_2^2 &\le a_1\|x^{(n)}\|_2\|x^{(n-1)}\|_2 + a_2\|x^{(n)}\|_2\|x^{(n-2)}\|_2 + \dots\\ &\quad + a_n\|x^{(n)}\|_2\|x\|_2 + R\sqrt{T}\|x^{(n)}\|_2,\\ &\le a_1\frac{T}{2\pi}\|x^{(n)}\|_2^2 + a_2\Big(\frac{T}{2\pi}\Big)^2\|x^{(n)}\|_2^2 + \dots + a_n\Big(\frac{T}{2\pi}\Big)^n\|x^{(n)}\|_2^2\\ &\quad + R\sqrt{T}\|x^{(n)}\|_2,\end{aligned}$$

where $R = \max_{t\in[0,T]}|F(t,0)|$. Thus we obtain

$$\|x^{(n)}\|_2 \le K, \quad \text{where } K = \frac{R\sqrt{T}}{1-\sum_{i=1}^n a_i\Big(\frac{T}{2\pi}\Big)^i}, \tag{2.5}$$

since, by hypothesis (H1), $\sum_{i=1}^n a_i\Big(\frac{T}{2\pi}\Big)^i < 1$. Then the inequalities

$$\|x^{(j)}\|_2 \le K\Big(\frac{T}{2\pi}\Big)^{n-j}, \quad j = 1,\dots,n, \tag{2.6}$$

follow from (2.5) and Lemma 2.5.

On the other hand, by mean value theorem for integrals we conclude that, for each $j = 0,\dots,n-1$, there exists $\tau_j \in [0,T]$ such that $x^{(j)}(\tau_j) = 0$, because $\int_0^T x^{(j)}(t)dt = 0$. Hence, by Hölder inequality, for each $j = 0,\dots,n-1$, we have

$$|x^{(j)}(t)| = \Big|\int_{\tau_j}^t x^{(j+1)}(s)ds\Big| \le \int_0^T |x^{(j+1)}(s)ds| \le \sqrt{T}\|x^{(j+1)}\|_2, \quad t\in[0,T].$$

Consequently, $\|x^{(j)}\|_\infty \le \sqrt{T}\|x^{(j+1)}\|_2$ for $j = 0,\dots,n-1$. Now inequalities (2.6) imply

$$\|x\|_X = \max_{0\le j\le n-1}\|x^{(j)}\|_\infty \le D, \tag{2.7}$$

where $D = K\sqrt{T}\max_{1\le j\le n}\Big(\frac{T}{2\pi}\Big)^{n-j}$ and the statement follows. □

Proposition 2.7. *If condition* (H1) *is satisfied, then there is a bounded open set* $\Omega \subset X$ *such that*

$$Lx - Nx \ne -\lambda(Lx + N(-x)), \tag{2.8}$$

for all $x \in \partial\Omega$ *and all* $\lambda \in (0,1]$.

Proof. By (H1) and Proposition 2.6 there exists a positive constant D, which does not depend on λ such that, if x satisfies the equality $Lx - Nx = -\lambda(Lx + N(-x))$, $\lambda \in (0,1]$, then $\|x\|_X \le D$. Thus, if

$$\Omega = \{x \in X; \|x\|_X < M\}, \tag{2.9}$$

where $M > D$, we conclude that

$$Lx - Nx \ne -\lambda(Lx - N(-x)),$$

for every $x \in \partial\Omega = \{x \in X; \|x\|_X = M\}$ and $\lambda \in (0,1]$. □

Proof of Theorem 2.2. By (H1), clearly, the set Ω defined in (2.9) is symmetric, $0 \in \Omega$ and $X \cap \overline{\Omega} = \overline{\Omega} \neq \emptyset$. Furthermore, it follows from Proposition 2.7 that if condition (H1) is fulfilled then

$$Lx - Nx \neq -\lambda[Lx - N(-x)],$$

for all $x \in X \cap \partial\Omega = \partial\Omega$ and all $\lambda \in (0,1]$. This together with Lemma 2.3 imply that equation (1.1) has at least one T-antiperiodic solution. $\square$

Our purpose now is to show the following result.

Theorem 2.8. *If* (H1) *holds, then* (1.1) *has at most one T-antiperiodic solution.*

Proof. Assume (H1) and that x and y are T-antiperiodic solutions of (1.1). To obtain the result, we show that the function $z = x - y$ is identically zero. Then, whereas x and y are T-periodic, it is sufficient to prove that $z(t) = 0$ for all $t \in [0, T]$.

Since x and y are solutions of equation (1.1),

$$z^{(n)}(t) = f(t, x_t^{(n-1)}, x_t^{(n-2)}, \dots, x'_t, x_t) - f(t, y_t^{(n-1)}, y_t^{(n-2)}, \dots, y'_t, y_t). \tag{2.10}$$

Multiplying both sides of (2.10) by $z^{(n)}(t)$, integrating it from 0 to T, using (H1) and Hölder inequality, we obtain

$$\begin{aligned}
\|z^{(n)}\|_2^2 &= \int_0^T |z^{(n)}(t)||f(t, x_t^{(n-1)}, x_t^{(n-2)}, \dots, x'_t, x_t) \\
&\quad - f(t, y_t^{(n-1)}, y_t^{(n-2)}, \dots, y'_t, y_t)|dt \\
&\leq a_1 \int_0^T |z^{(n)}(t)||z^{(n-1)}(t)|dt + \dots + a_n \int_0^T |z^{(n)}(t)||z(t)|dt \\
&\leq a_1 \|z^{(n)}\|_2 \|z^{(n-1)}\|_2 + \dots + a_n \|z^{(n)}\|_2 \|z\|_2 \\
&= \|z^{(n)}\|_2 (a_1 \|z^{(n-1)}\|_2 + \dots + a_n \|z\|_2).
\end{aligned} \tag{2.11}$$

On the other hand, by Lemma 2.5,

$$\|z^{(n-j)}\|_2 \leq \Big(\frac{T}{2\pi}\Big)^j \|z^{(n)}\|_2, \quad j = 1, \dots, n.$$

From this and (2.11), we obtain

$$\|z^{(n)}\|_2^2 \leq \|z^{(n)}\|_2^2 \Big[a_1\big(\frac{T}{2\pi}\big) + \dots + a_n\Big(\frac{T}{2\pi}\Big)^n\Big].$$

Then, since $\sum_{i=1}^n a_i \big(\frac{T}{2\pi}\big)^i < 1$, we conclude that $z^{(n)} \equiv 0$ and consequently $z^{(n-1)}$ is a constant function. Let us see now that $z^{(n-1)}$ is identically zero. Indeed, since $z^{(n-2)}(0) = z^{(n-2)}(T)$ then, by Mean Value Theorem, it follows that there is $\tau \in [0, T]$ such that $z^{(n-1)}(\tau) = 0$. Repeating this argument $n - 1$ times, we can conclude that $z \equiv 0$ and the proof is complete. $\square$

Finally, we present the main result of this work.

Theorem 2.9. *If* (H1) *holds, then* (1.1) *has a unique T-antiperiodic solution.*

The above theorem follows immediately from Theorems 2.2 and 2.8.

3. Example

Set $T = 2\pi$. Let us consider the equation

$$x'''(t) = f(t, x_t'', x_t', x_t), \ t \in \mathbb{R}, \tag{3.1}$$

where $f : \mathbb{R} \times C_B((-\infty, 0], \mathbb{R}) \times C_B((-\infty, 0], \mathbb{R}) \times C_B((-\infty, 0], \mathbb{R}) \to \mathbb{R}$ is given by

$$f(t, \varphi_1, \varphi_2, \varphi_3) = \frac{\sin^4 t}{28}\varphi_1(0) + \frac{\cos^2 t}{10}\varphi_2(0) + \frac{1}{4}\varphi_3(0),$$

for $t \in [0, +\infty)$ and $\varphi_1, \varphi_2, \varphi_3 \in C_B((-\infty, 0], \mathbb{R})$. Clearly, f is continuous and

$$f(t + \pi, -\varphi_1, -\varphi_2, -\varphi_3) = -f(t, \varphi_1, \varphi_2, \varphi_3).$$

Furthermore, condition (H1) is satisfied with $a_1 = \frac{1}{28}$, $a_2 = \frac{1}{10}$ and $a_3 = \frac{1}{4}$. Indeed, if $t \in \mathbb{R}$ and $\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3 \in C_B((-\infty, 0], \mathbb{R})$, we have

$$\begin{aligned}
&|f(t, \varphi_1, \varphi_2, \varphi_3) - f(t, \psi_1, \psi_2, \psi_3)| \\
&= \Big|\frac{\sin^4 t}{28}\left(\varphi_1(0) - \psi_1(0)\right) + \frac{\cos^2 t}{10}\left(\varphi_2(0) - \psi_2(0)\right) + \frac{1}{4}\left(\varphi_3(0) - \psi_3(0)\right)\Big| \\
&\le \frac{\sin^4 t}{28}|\varphi_1(0) - \psi_1(0)| + \frac{\cos^2 t}{10}|\varphi_2(0) - \psi_2(0)| + \frac{1}{4}|\varphi_3(0) - \psi_3(0)| \\
&\le \frac{1}{28}|\varphi_1(0) - \psi_1(0)| + \frac{1}{10}|\varphi_2(0) - \psi_2(0)| + \frac{1}{4}|\varphi_3(0) - \psi_3(0)|.
\end{aligned}$$

Then, by Theorem 2.9, equation (3.1) has precisely one T-antiperiodic solution on $\mathbb{R}$.

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