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D. McFadden
Yunbog Lee

Additional Sample Problems

1. The following table gives the investment rate X_i in 8 developed countries. Some summary statistics are $\sum X_i = 157.7$, $\sum X_i^2 = 3271.47$, $\overline{X} = 19.71$, $\sum(X_i - \overline{X})^2 = 162.81$, $s^2 = \sum(X_i - \overline{X})^2/7 = 23.26$, $s = \sqrt{s^2} = 4.82$, $\sqrt{8} = 2.828$, $\sqrt{7} = 2.646$. Assume that X_i are i.i.d. draws from a normally distributed population.

Country	Ratio of Gross Fixed Capital Formation to GDP (Pct., 1993)
Japan	30.1
Germany	22.7
Netherlands	19.7
France	18.9
Canada	18.2
Italy	17.1
U.S.A.	16.1
U.K.	14.9

   (a) How would you test the hypothesis that the population mean investment rate is no greater than 17.0 using a significance level of 95 percent? Be specific about the test statistic, its distribution under the null, and the critical level you would use, and give numerical values if possible.

   (b) Describe how you would compute the power of this test against the alternative that the mean is 20.0. Be specific about the distribution that would be used for the power calculation, and if possible give numerical values for the parameters of this distribution, substituting s for σ if necessary. Do not attempt to give a numerical value for the power.

2. An individual searching for a job draws wage rates at random from the same continuous cumulative distribution function of wages, $F(w)$.

   (a) What is the probability that the first draw is larger than the next 4 draws?

   (b) If the individual continues to draw until obtaining a wage exceeding a reservation level w^*, what is the expected total number of draws, given that the first N are unsuccessful? Assume that $0 < p \equiv F(w^*) < 1$.
   [Hint: If $|r| < 1$, then $S(r) \equiv \sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$, and $\frac{dS(r)}{dr} = \sum_{i=1}^{\infty} i r^{i-1} = \frac{1}{(1-r)^2}$.]

3. Let $X_1, \ldots, X_N$ be a random sample from a uniform distribution on $[0, 1/\beta]$, where β is an unknown positive parameter. Consider the estimator $\widehat{\beta} = \frac{1}{\max_{n \leq N}(X_n)}$. Is this estimator unbiased? Is it consistent? Sufficient for β?

Solutions

1. Using the given numbers...

   (a) If the population mean is μ, then $\frac{\overline{X}-\mu}{s/\sqrt{8}} \sim t(7)$. The null hypothesis is $\mu \leq 17$. Under our null, our statistic $T \equiv \frac{\overline{X}-17}{s/\sqrt{8}}$ will tend to be negative. If $\mu > 17$, $\frac{\overline{X}-17}{s/\sqrt{8}}$ will tend to be positive. So reject the null if $\frac{\overline{X}-17}{s/\sqrt{8}} > t_c$, where t_c is the 95% quantile of the t distribution with 7 degrees of freedom. So $t_c = 1.8946$. And

$$\begin{aligned}\frac{\overline{X}-17}{s/\sqrt{8}} &= \frac{19.71-17}{4.82/\sqrt{8}} \\ &= 1.5903\end{aligned}$$

   Therefore, we do not reject the null hypothesis.
   Under the null hypothesis that the population mean is μ, our test statistic has the distribution

$$\begin{aligned}\frac{\overline{X}-17}{s/\sqrt{8}} &= \frac{\frac{\overline{X}-\mu}{\sigma/\sqrt{8}} + \frac{\mu-17}{\sigma/\sqrt{8}}}{\sqrt{\frac{7s^2}{7\sigma^2}}} \\ &\sim \text{noncentral } t(7) \text{ with noncentrality parameter } \frac{\mu-17}{\sigma/\sqrt{8}}.\end{aligned}$$

   (b) The power under the given alternative is

$$\begin{aligned}\mathrm{P}\left(\frac{\overline{X}-17}{s/\sqrt{8}} > 1.8946 \,\middle|\, \frac{\overline{X}-20}{s/\sqrt{8}} \sim t(7)\right) &= \mathrm{P}\left(\frac{\overline{X}-20}{s/\sqrt{8}} + \frac{3}{s/\sqrt{8}} > 1.8946 \,\middle|\, \frac{\overline{X}-20}{s/\sqrt{8}} \sim t(7)\right) \\ &= \mathrm{P}\left(\frac{\overline{X}-20}{s/\sqrt{8}} > 1.8946 - \frac{3}{s/\sqrt{8}} \,\middle|\, \frac{\overline{X}-20}{s/\sqrt{8}} \sim t(7)\right)\end{aligned}$$

   If $Z \equiv \frac{\overline{X}-20}{s/\sqrt{8}}$, then $Z \sim t(7)$. So the 'estimated' power is

$$\mathrm{P}\left(\mathrm{Z} > 1.8946 - \frac{3}{4.82/\sqrt{8}} \doteq 0.1342\right),$$

   where $Z \sim t(7)$.
   Alternatively, we could use noncentral t distribution with 7 degrees of freedom and noncentrality parameter $\frac{3}{4.82/\sqrt{8}}$ using the fact that $T \sim$noncentral $t(7)$ with noncentrality parameter $\frac{3}{\sigma/\sqrt{8}}$. Here, the exact noncentrality parameter is $\frac{3}{\sigma/\sqrt{8}}$, but since we don't know what σ is, we substitute s for σ and get an estimated value of power: $1 - \text{nctcdf}(1.8946, 7, \frac{3}{4.82/\sqrt{8}})$, where `nctcdf` is the noncentral t CDF function.

2. Look how simple the answers are!

(a) Each of the 5 draws has an equal chance of being largest, so the probability that the first is largest is 1/5.

(b) The probability that $N + k$ draws are needed is $p^{k-1}(1-p)$. Hence, the expected number of draws is $N + \sum_{k=1}^{\infty} kp^{k-1}(1-p) = N + \frac{1}{1-p}$.

3. First, $\max(X_n) < \frac{1}{\beta}$, so $\widehat{\beta} > \beta$ with probability one. Therefore, $\mathrm{E}\widehat{\beta} > \beta$, and the estimator is biased.
For consistency, the only way $\widehat{\beta}$ can stay some distance above β is if every observation X_n stays some distance below $1/\beta$. The probability of this event for each observation is less than one, and for N observations is the product of these probabilities, and so approaches zero.
For sufficiency, let's look at the likelihood function $f(\mathbf{x}, \beta)$:

$$f(\mathbf{x}, \beta) = \begin{cases} \beta^N & \text{if } 0 \le x_n \le \frac{1}{\beta}\ \forall n \\ 0 & \text{otherwise} \end{cases}.$$

So for $x_n \ge 0,\ \forall n$, $f(\mathbf{x}, \beta)$ is either non-zero or zero depending on the maximum value of $x_n,\ n = 1, \dots, N$. Let's define a new function $h(\max_{n \le N} x_n, \beta)$ as follows:

$$h(\max_{n \le N} x_n, \beta) = \begin{cases} 1 & \text{if } \max_{n \le N} x_n \le \frac{1}{\beta} \\ 0 & \text{otherwise} \end{cases}.$$

Then we can rewrite the likelihood function like this: for $x_n \ge 0,\ \forall n$,

$$f(\mathbf{x}, \beta) = \beta^N h(\max_{n \le N} x_n, \beta).$$

Thus $\max_{n \le N} X_n$ is a sufficient statistic for $\frac{1}{\beta}$ by the factorization criterion. Therefore, $\widehat{\beta}$ is sufficient for β.