20 April 2005 **Name:** Solutions

ECE 438 **Exam No. 3** **Spring 2005**

- You have 50 minutes to work the following four problems.
- Be sure to show all your work to obtain full credit.
- The exam is closed book and closed notes.
- Calculators are permitted.

1. (25 pts.) Let θ be a random variable that is uniformly distributed on the interval $(0, 2\pi)$.

   a. (7) Find the mean and variance of θ.

   Now, let $X = e^{j\theta}$ be a new random variable, where θ is defined as before.

   b. (6) Find $E\{X\}$ and $E\{|X|^2\}$, where $E\{\bullet\}$ denotes statistical expectation.

   Next, let $Y = A_1 e^{j\theta_1} + A_2 e^{j\theta_2}$ be a new random variable, where θ_1 and θ_2 are both uniformly distributed on the interval $(0, 2\pi)$, A_1 and A_2 are identically distributed with mean 0 and variance σ_A^2, and all four random variables θ_1, θ_2, A_1, and A_2 are mutually independent.

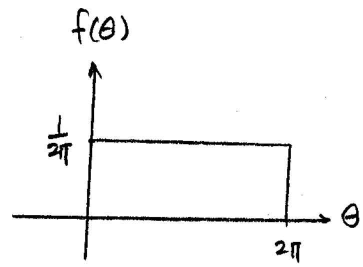
   c. (6) Find $E\{Y\}$ and $E\{|Y|^2\}$.

   Finally, let X_n, $n = ...,-2,-1,0,1,2,...$, be a wide-sense stationary random process with mean zero and autocorrelation $r_{xx}[n] = E\{X_m X_{m+n}\}$. Define a new random process $Y(\omega)$, $-\pi \le \omega \le \pi$, according to

$$Y(\omega) = \sum_{n=-N}^{N} X_n e^{-jwn}.$$

   d. (6) Find $E\{|Y(\omega)|^2\}$. Simplify your answer as much as possible, and discuss the significance of your result.

1a.



$$E[\theta] = \int_{-\infty}^{\infty} \theta \cdot f(\theta)\, d\theta$$

$$= \int_0^{2\pi} \theta \cdot \frac{1}{2\pi}\, d\theta$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2}\theta^2 \Big|_0^{2\pi}$$

$$= \pi$$

$$\text{Variance} = E[(\theta - E[\theta])^2]$$

$$= E[\theta^2] - (E[\theta])^2$$

1. (continued)

1a (Continue...)

$$E[\theta^2] = \int_0^{2\pi} \theta^2 \cdot \frac{1}{2\pi}\, d\theta$$
$$= \frac{1}{2\pi} \cdot \frac{1}{3}\theta^3 \Big|_0^{2\pi}$$
$$= \frac{4}{3}\pi^2$$

$\therefore$ Variance $= \frac{4}{3}\pi^2 - \pi^2 = \frac{1}{3}\pi^2$

1b.
$$E[X] = E[e^{j\theta}]$$
$$= \int_0^{2\pi} e^{j\theta} \cdot \frac{1}{2\pi}\, d\theta$$
$$= \frac{1}{2\pi} \cdot \frac{1}{j} e^{j\theta} \Big|_0^{2\pi}$$
$$= 0$$

$$E[|X|^2] = E[|e^{j\theta}|^2]$$
$$= E[1]$$
$$= 1$$

1c.
$$E[y] = E[A_1 e^{j\theta_1} + A_2 e^{j\theta_2}]$$
$$= E[A_1]\cdot E[e^{j\theta_1}] + E[A_2]\cdot E[e^{j\theta_2}]$$
$$= 0$$

because $\theta_1, \theta_2, A_1, A_2$ are mutually independent.

$$|y|^2 = |A_1\cos\theta_1 + jA_1\sin\theta_1 + A_2\cos\theta_2 + jA_2\sin\theta_2|^2$$
$$= (A_1\cos\theta_1 + A_2\cos\theta_2)^2 + (A_1\sin\theta_1 + A_2\sin\theta_2)^2$$
$$= A_1^2\cos^2\theta_1 + A_2^2\cos^2\theta_2 + 2A_1A_2\cos\theta_1\cos\theta_2 + A_1^2\sin^2\theta_1 + A_2^2\sin^2\theta_2 + 2A_1A_2\sin\theta_1\sin\theta_2$$

$$\therefore E[|y|^2] = E[A_1^2 + A_2^2 + 2A_1A_2\cos(\theta_1-\theta_2)]$$
$$= E[A_1^2] + E[A_2^2]$$
$$= 2\sigma_A^2$$

(p.t.o)

1d. $$|Y(\omega)|^2 = \left|\sum_{n=-N}^{N} X_n e^{-j\omega n}\right|^2$$

$$= \left|\sum_{n=-N}^{N} X_n \cos\omega n - j\sum_{n=-N}^{N} X_n \sin\omega n\right|^2$$

$$= \left(\sum_{n=-N}^{N} X_n \cos\omega n\right)^2 + \left(\sum_{n=-N}^{N} X_n \sin\omega n\right)^2$$

$$\left(\sum_{n=-N}^{N} X_n \cos\omega n\right)^2 = \sum_{n=-N}^{N} X_n \cos\omega n \cdot \sum_{k=-N}^{N} X_k \cos\omega k$$

$$= \sum_{n=-N}^{N}\sum_{k=-N}^{N} X_n X_k \cos\omega n \cdot \cos\omega k$$

Similarly,

$$\left(\sum_{n=-N}^{N} X_n \sin\omega n\right)^2 = \sum_{n=-N}^{N}\sum_{k=-N}^{N} X_n X_k \sin\omega n \cdot \sin\omega k$$

$$\therefore |Y(\omega)|^2 = \sum_{n=-N}^{N}\sum_{k=-N}^{N} X_n X_k \cos\omega n \cdot \cos\omega k + \sum_{n=-N}^{N}\sum_{k=-N}^{N} X_n X_k \sin\omega n \cdot \sin\omega k$$

$$= \sum_{n=-N}^{N}\sum_{k=-N}^{N} X_n X_k \cos(\omega(n-k))$$

$$E[|Y(\omega)|^2] = \sum_{n=-N}^{N}\sum_{k=-N}^{N} E[X_n X_k] \cos(\omega(n-k))$$

$$\cancel{(2N+1)\cdot r_{XX}[0]}$$

$$= \sum_{n=-N}^{N}\sum_{k=-N}^{N} r_{XX}[k-n] \cos(\omega(n-k))$$

$E[|Y(\omega)|^2]$ is basically the average power of the random process $Y(\omega)$, so it should be a positive quantity.

1c. (better solution)

$$E\{|Y|^2\} = E\{(A_1 e^{j\theta_1} + A_2 e^{j\theta_2})(A_1 e^{-j\theta_1} + A_2 e^{-j\theta_2})\}$$

$$= E\{A_1^2 + A_2^2 + A_1 A_2 e^{j(\theta_1 - \theta_2)} + A_1 A_2 e^{j(\theta_2 - \theta_1)}\}$$

$$= E\{A_1^2\} + E\{A_2^2\} + \underbrace{E\{A_1\}E\{A_2\}E\{e^{j\theta_1}\}E\{e^{-j\theta_2}\} + E\{A_1\}E\{A_2\}E\{e^{-j\theta_1}\}E\{e^{j\theta_2}\}}_{\text{all terms} = 0}$$

$$= 2\sigma_2^2$$

1d. $$E\{|Y(\omega)|^2\} = E\left\{\left(\sum_{m=-N}^{N} X_m e^{-j\omega m}\right)\left(\sum_{n=-N}^{N} X_n e^{+j\omega n}\right)\right\}$$

$$= \sum_{m=-N}^{N} \sum_{n=-N}^{N} E\{X_m X_n\} e^{-j\omega(m-n)}$$

$$= \sum_{m=-N}^{N} \sum_{n=-N}^{N} r_{xx}[m-n] e^{-j\omega(m-n)}$$

let $l = m - n$ $\qquad n = m - l$

$$E\{|Y(\omega)|^2\} = \sum_{m=-N}^{N} \sum_{l=m-N}^{m+N} r_{xx}[l] e^{-j\omega l}$$

$$= \sum_{m=-N}^{N} \sum_{l=-\infty}^{\infty} w[l-m]\, r_{xx}[l] e^{-j\omega l}$$

where $w[l] = \begin{cases} 1, & -N \le l \le N \\ 0, & \text{else} \end{cases}$

now

$$\sum_{l=-\infty}^{\infty} w[\cancel{m+l}\ l-m]\, r_{xx}(l)\, e^{-j\omega l} \quad \text{is DTFT}$$

of product $w[\cancel{m+l}\ l-m]$ & $r_{xx}(l)$

$$\Rightarrow \quad = W(\omega) e^{-j\omega m} R_{xx}(\omega)$$

where $w[m] \xleftrightarrow{DTFT} W(\omega)$

$r_{xx}[m] \xleftrightarrow{DTFT} R_{xx}(\omega)$

so

$$E\{|Y(\omega)|^2\} = \sum_{m=-N}^{N} W(\omega) R_{xx}(\omega) e^{-j\omega m}$$

$$= W(\omega) R_{xx}(\omega) \sum_{m=-N}^{N} e^{-j\omega m}$$

$$= |W(\omega)|^2 R_{xx}(\omega)$$

Since $W(\omega)$ is real & even

Significance: $E\{|Y(\omega)|^2\}$ provides DTFT of autocorrelation $r_{xx}[n]$ weighted by magnitude squared of ~~window~~ DTFT of window $w[n]$.

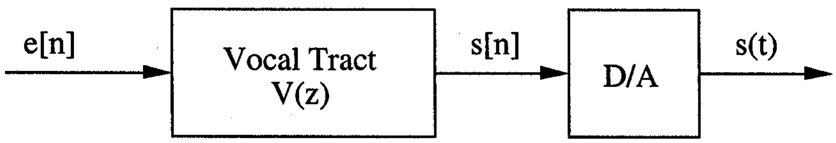
This is an estimate of power spectrum of ~~[illegible]~~ signal. The method is called a "periodogram".

2. (25 pts.) A voiced speech waveform has pitch 200 Hz, and three formant frequencies at 0.5 kHz, 2 kHz, and 4 kHz, which decrease in amplitude with increasing frequency.

   a. (7) Sketch a wideband spectrogram for this waveform. Be sure to label and dimension all important quantities.

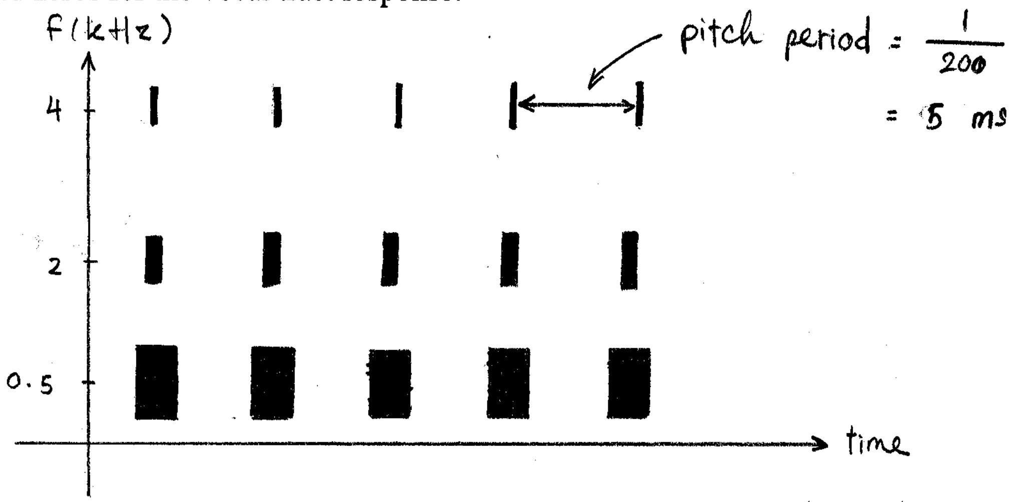
   b. (7) Sketch a narrowband spectrogram for this waveform. Be sure to label and dimension all important quantities.

   We will use the digital system shown below to synthesize this speech waveform, where the excitation is given by $e[n] = \sum_{k=-\infty}^{\infty} \delta[n - Nk]$. Assume that the system operates at a 20 kHz sampling rate.

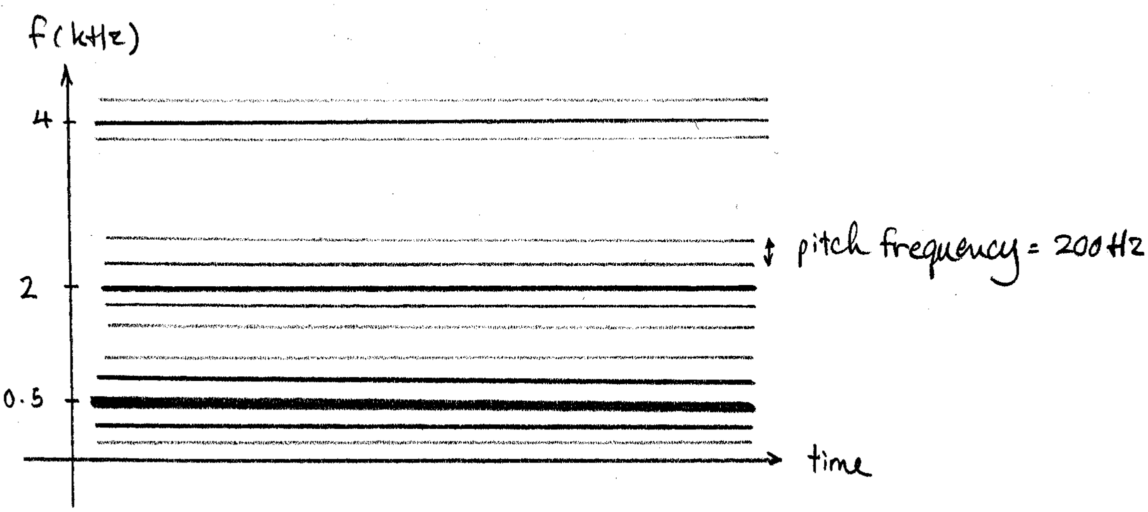


   c. (11) Specify the value for the parameter N and the approximate location of all poles and zeros for the vocal tract response.

2a.

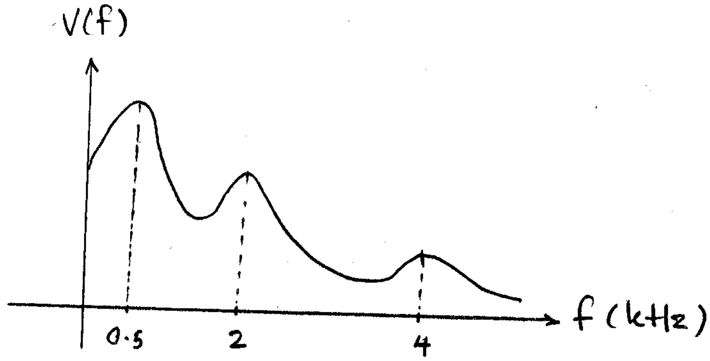


2b.



2. (continued)

2c.



We can model the vocal tract response as an all-pole system since (assuming) $V(f) \neq 0 \ \forall f$.
Therefore, the vocal tract response will have the form

$$V(z) = \frac{K}{(z - Ae^{j\omega_1})(z - Ae^{-j\omega_1})(z - Be^{j\omega_2})(z - Be^{-j\omega_2})(z - Ce^{j\omega_3})(z - Ce^{-j\omega_3})}.$$

where A, B, C, K are constants and $1 > A > B > C > 0$.

Thus, ω_1 corresponds to the formant frequency $f_1 = 500$ Hz (since amplitude at this frequency is the largest).
From sampling theory, we know that $\omega = \frac{2\pi f}{f_s}$.

$$\therefore \quad f_1 = 500 \text{ Hz} \longrightarrow \omega_1 = \frac{2\pi \cdot 500}{20k} = \frac{\pi}{20}$$

$$f_2 = 2000 \text{ Hz} \longrightarrow \omega_2 = \frac{2\pi \cdot 2000}{20k} = \frac{\pi}{5}$$

$$f_3 = 4000 \text{ Hz} \longrightarrow \omega_3 = \frac{2\pi \cdot 4000}{20k} = \frac{2\pi}{5}$$

Now, a 20 kHz sampling rate generates 20000 samples/sec.
Since the pitch period is $\frac{1}{200} = 5$ ms, we then need $N = 5\text{ms} \times 20000 = 100$ to make the excitation $e[n]$ periodic w/ the right pitch period. $\therefore N = 100$

3. (25 pts.) The short-time discrete-time Fourier transform of the signal $x[k]$ is defined according to the equation

$$X(\omega,n)=\sum_k x[k]w[n-k]e^{-j\omega k},$$

Assume that we have a rectangular window

$$w[n]=\begin{cases}1, & |n|<10\\ 0, & \text{else}\end{cases}$$

and that our signal is given by

$$x[n]=\begin{cases}\cos(\pi n/3), & |n|<100\\ 0, & \text{else}\end{cases}$$

Find an exact expression for $X(\omega,n)$ for the following cases:

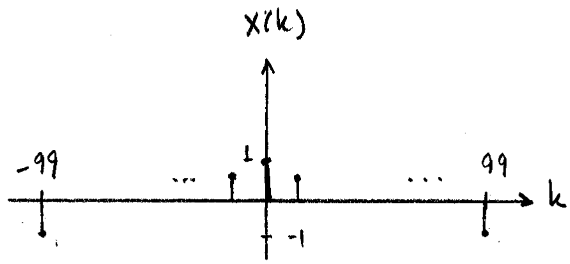
a. (8) $|n|<90$

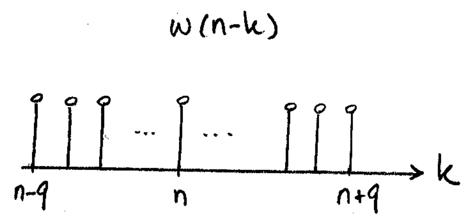
b. (8) $|n|=100$

c. (1) $|n|>110$

d. (8) Based on your answers above, approximately sketch the complete spectrogram for all n.

3a.





$|n|<90$: there is complete overlap of the window in this region.

$$\therefore\quad X(\omega,n)=\sum_{k=n-9}^{n+9}\cos\left(\frac{\pi k}{3}\right)\cdot 1\, e^{-j\omega k}$$

$$=\sum_{k=n-9}^{n+9}\frac{1}{2}\left[e^{j\frac{\pi k}{3}}+e^{-j\frac{\pi k}{3}}\right]e^{-j\omega k}$$

$\ell=k-(n-9)$

$$=\sum_{\ell=0}^{18}\frac{1}{2}\left[e^{j\frac{\pi}{3}(\ell+(n-9))}+e^{-j\frac{\pi}{3}(\ell+(n-9))}\right]e^{-j\omega(\ell+(n-9))}$$

$$=\ldots$$

3. (continued)

3a (continue ...)

$$X(\omega,n) = \frac{1}{2}e^{-j(\omega-\frac{\pi}{3})n} \cdot \frac{\sin((\omega-\frac{\pi}{3})\cdot\frac{19}{2})}{\sin((\omega-\frac{\pi}{3})\frac{1}{2})} + \frac{1}{2}e^{-j(\omega+\frac{\pi}{3})n} \cdot \frac{\sin((\omega+\frac{\pi}{3})\cdot\frac{19}{2})}{\sin((\omega+\frac{\pi}{3})\cdot\frac{1}{2})}$$

Note:

$$\sum_{k=N_1}^{N_2} e^{j\omega_0 k} e^{-j\omega k} = e^{-j(\omega-\omega_0)(\frac{N_2+N_1}{2})} \cdot \frac{\sin((\omega-\omega_0)\frac{N_2-N_1+1}{2})}{\sin((\omega-\omega_0)\cdot\frac{1}{2})}$$

3b. $|n| = 100$: there is partial overlap of the window

Take $n=100$.

$$X(\omega,100) = \sum_{k=91}^{99} \frac{1}{2}[e^{j\frac{\pi k}{3}} + e^{-j\frac{\pi k}{3}}]e^{-j\omega k}$$

$$= \frac{1}{2}e^{-j(\omega-\frac{\pi}{3})95} \cdot \frac{\sin((\omega-\frac{\pi}{3})\cdot\frac{9}{2})}{\sin((\omega-\frac{\pi}{3})\cdot\frac{1}{2})} + \frac{1}{2}e^{-j(\omega+\frac{\pi}{3})95} \cdot \frac{\sin((\omega+\frac{\pi}{3})\cdot\frac{9}{2})}{\sin((\omega+\frac{\pi}{3})\frac{1}{2})}$$

Take $n=-100$.

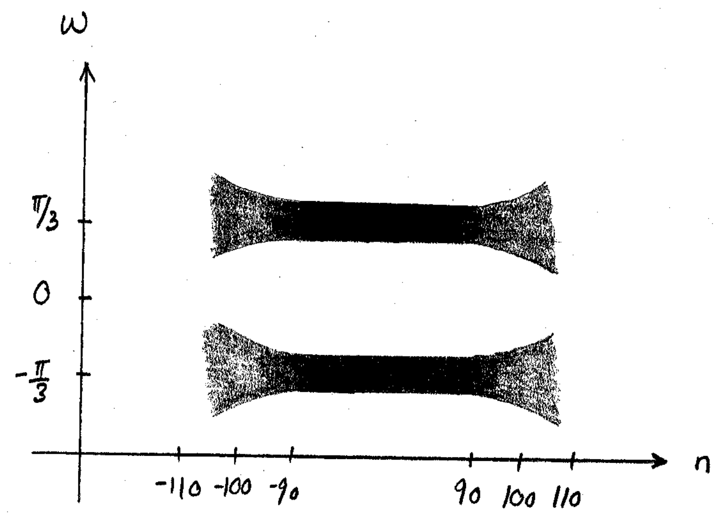
$$X(\omega,-100) = \sum_{k=-99}^{-91} \frac{1}{2}[e^{j\frac{\pi k}{3}} + e^{-j\frac{\pi k}{3}}]e^{-j\omega k}$$

$$= \frac{1}{2}e^{-j(\omega-\frac{\pi}{3})(-95)} \cdot \frac{\sin((\omega-\frac{\pi}{3})\frac{9}{2})}{\sin((\omega-\frac{\pi}{3})1/2)} + \frac{1}{2}e^{-j(\omega+\frac{\pi}{3})(-95)} \cdot \frac{\sin((\omega+\frac{\pi}{3})\frac{9}{2})}{\sin((\omega+\frac{\pi}{3})1/2)}$$

3c. $|n| > 110$: no overlap at all

$\therefore\ X(\omega,n) = 0$.

3d.



4. (25 pts.) We observe the following 3 point signal $x[n]$:

n	0	1	2
$x[n]$	0	1	3

We wish to fit a straight line of the form $\hat{x}[n] = a_0 + a_1 n$ to these data points, where the coefficients a_0 and a_1 are chosen to minimize the mean-squared error

$$\varepsilon = \sum_{n=0}^{2} |\hat{x}[n] - x[n]|^2$$

a) (20) Find the optimal values for a_0 and a_1.

b) (5) Find the value of the mean-squared error ε for the choice of a_0 and a_1 given by your answer to part b).

4a.

$$\varepsilon = \sum_{n=0}^{2} |\hat{x}[n] - x[n]|^2$$

$$= \sum_{n=0}^{2} (\hat{x}(n) - x(n))^2$$

because these data points are real-valued.

$$\frac{\partial \varepsilon}{\partial a_0} = \sum_{n=0}^{2} 2(\hat{x}(n) - x(n)) = 0$$

$$\Rightarrow \quad \sum_{n=0}^{2} (\hat{x}(n) - x(n)) = 0$$

$$(a_0 - x(0)) + (a_0 + a_1 - x(1)) + (a_0 + 2a_1 - x(2)) = 0$$

$$3a_0 + 3a_1 - 4 = 0$$

$$a_0 + a_1 = \frac{4}{3} \quad \text{— (1)}$$

$$\frac{\partial \varepsilon}{\partial a_1} = \sum_{n=0}^{2} 2n(a_0 + a_1 n - x(n)) = 0$$

$$\Rightarrow \quad 2(a_0 + a_1 - x(1)) + 4(a_0 + 2a_1 - x(2)) = 0$$

$$6a_0 + 10a_1 - 14 = 0$$

$$3a_0 + 5a_1 = 7 \quad \text{— (2)}$$

(2) − 3·(1):

$$3a_0 + 5a_1 - 3a_0 - 3a_1 = 7 - 4$$

$$2a_1 = 3$$

$$a_1 = \frac{3}{2}$$

4. (continued)

$$\therefore \quad a_0 = \frac{4}{3} - \frac{3}{2} = -\frac{1}{6}$$

Thus, the optimal values ~~for~~ are $a_0 = -\frac{1}{6}$ & $a_1 = \frac{3}{2}$.

4b.

$$\varepsilon = \sum_{n=0}^{2} \left(-\frac{1}{6} + \frac{3}{2}n - x(n)\right)^2$$

$$= \left(-\frac{1}{6} - 0\right)^2 + \left(-\frac{1}{6} + \frac{3}{2} - 1\right)^2 + \left(-\frac{1}{6} + 3 - 3\right)^2$$

$$= \frac{1}{36} + \frac{4}{36} + \frac{1}{36}$$

$$= \frac{1}{6}$$