Bayesian Decision Theory

Chapter 2 (Duda, Hart & Stork)

CS 7616 - Pattern Recognition

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Bayesian Decision Theory

- Design classifiers to recommend decisions that minimize some total expected ”risk”.
  - The simplest risk is the classification error (i.e., costs are equal).
  - Typically, the risk includes the cost associated with different decisions.

Terminology

- State of nature ω *(random variable):*
  - e.g., ω_1 for sea bass, ω_2 for salmon

- Probabilities $P(\omega_1)$ and $P(\omega_2)$ *(priors):*
  - e.g., prior knowledge of how likely is to get a sea bass or a salmon

- Probability density function $p(x)$ *(evidence):*
  - e.g., how frequently we will measure a pattern with feature value x (e.g., x corresponds to lightness)

Terminology (cont'd)

- Conditional probability density $p(x/\omega_j)$ (*likelihood*) :
  - e.g., how frequently we will measure a pattern with feature value *x* given that the pattern belongs to class ω_j

e.g., lightness distributions between salmon/sea-bass populations

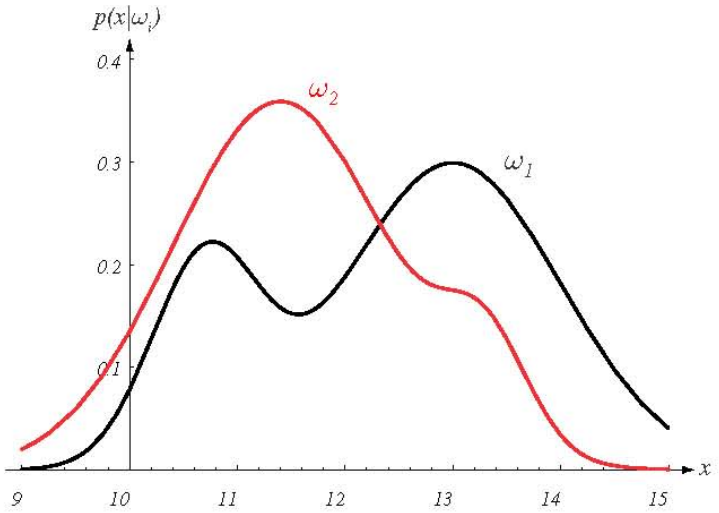


FIGURE 2.1. Hypothetical class-conditional probability density functions show the probability density of measuring a particular feature value *x* given the pattern is in category ω_i. If *x* represents the lightness of a fish, the two curves might describe the difference in lightness of populations of two types of fish. Density functions are normalized, and thus the area under each curve is 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons,

Terminology (cont'd)

- Conditional probability $P(\omega_j/x)$ *(posterior)* :
  - e.g., the probability that the fish belongs to class ω_j given measurement x.

Decision Rule Using Prior Probabilities

Decide ω_1 if $P(\omega_1) > P(\omega_2)$; otherwise **decide** ω_2

$$P(error) = \begin{cases} P(\omega_1) \ \ if \ we \, decide \, \omega_2 \\ P(\omega_2) \ if \ we \, decide \, \omega_1 \end{cases}$$

or $P(error) = min[P(\omega_1), P(\omega_2)]$

- Favours the most likely class.
- This rule will be making the same decision all times.
  - i.e., optimum if no other information is available

Decision Rule Using Conditional Probabilities

- Using Bayes' rule, the posterior probability of category ω_j given measurement x is given by:

$$P(\omega_j / x) = \frac{p(x/\omega_j)P(\omega_j)}{p(x)} = \frac{likelihood \times prior}{evidence}$$

where $p(x) = \sum_{j=1}^{2} p(x/\omega_j)P(\omega_j)$ (i.e., scale factor – sum of probs = 1)

Decide ω_1 if $P(\omega_1/x) > P(\omega_2/x)$; otherwise **decide** ω_2

or

Decide ω_1 if $p(x/\omega_1)P(\omega_1) > p(x/\omega_2)P(\omega_2)$ otherwise **decide** ω_2

Decision Rule Using Conditional pdf (cont'd)

$p(x/\omega_j)$ $\quad P(\omega_1) = \frac{2}{3} \quad P(\omega_2) = \frac{1}{3}$ $\quad P(\omega_j/x)$

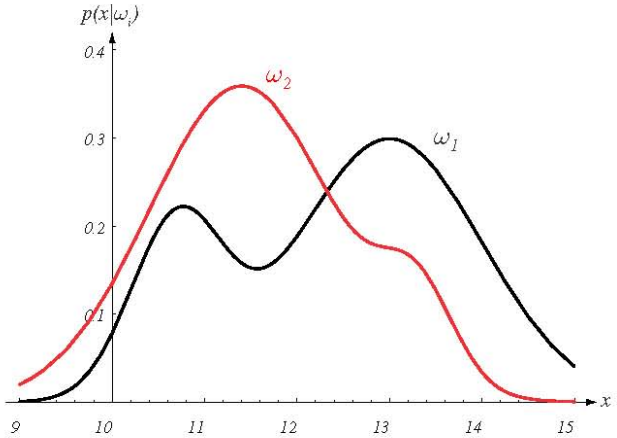


FIGURE 2.1. Hypothetical class-conditional probability density functions show the probability density of measuring a particular feature value x given the pattern is in category ω_i. If x represents the lightness of a fish, the two curves might describe the difference in lightness of populations of two types of fish. Density functions are normalized, and thus the area under each curve is 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons,

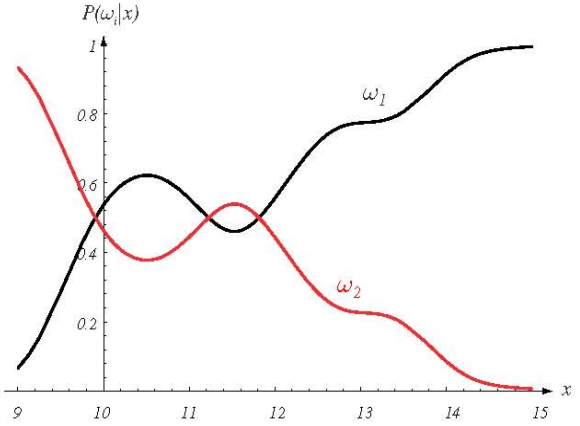


FIGURE 2.2. Posterior probabilities for the particular priors $P(\omega_1) = 2/3$ and $P(\omega_2) = 1/3$ for the class-conditional probability densities shown in Fig. 2.1. Thus in this case, given that a pattern is measured to have feature value $x = 14$, the probability it is in category ω_2 is roughly 0.08, and that it is in ω_1 is 0.92. At every x, the posteriors sum to 1.0. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

Probability of Error

- The probability of error is defined as:

$$P(error/x) = \begin{cases} P(\omega_1/x) & if\ we\ decide\ \omega_2 \\ P(\omega_2/x) & if\ we\ decide\ \omega_1 \end{cases}$$

or $P(error/x) = min[P(\omega_1/x), P(\omega_2/x)]$

- What is the average probability error?

$$P(error) = \int_{-\infty}^{\infty} P(error, x)dx = \int_{-\infty}^{\infty} P(error/x)\,p(x)dx$$

- The Bayes rule is optimum, that is, it minimizes the average probability error!

Where do Probabilities Come From?

- There are two competitive answers to this question:

  (1) **Relative frequency** (objective) approach.
  - Probabilities can only come from experiments.

  (2) **Bayesian** (subjective) approach.
  - Probabilities may reflect degree of belief and can be based on opinion.

Example (objective approach)

- Classify cars whether they are more or less than $50K:
  - Classes: C_1 if price > $50K, C_2 if price <= $50K
  - Features: x, the height of a car

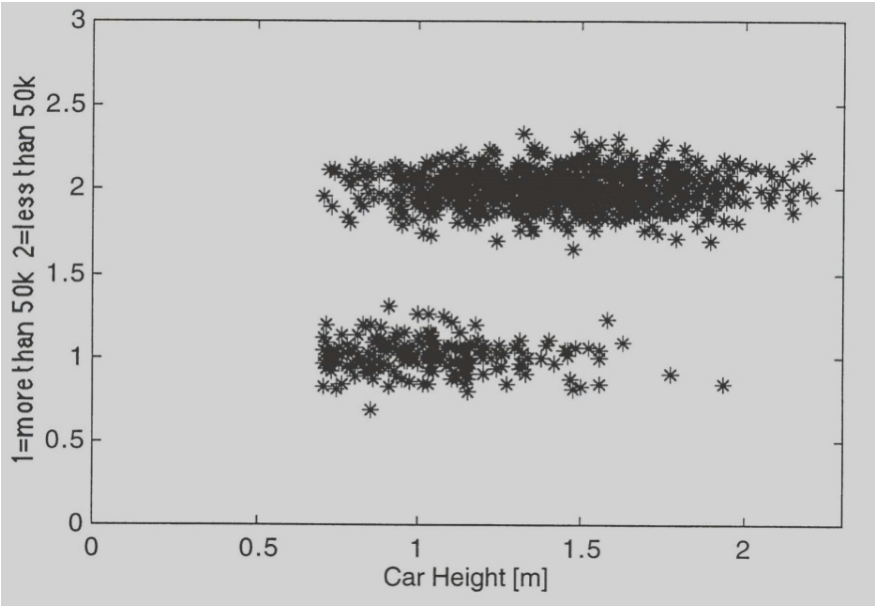
- Use the Bayes' rule to compute the posterior probabilities:

$$P(C_i / x) = \frac{p(x / C_i)P(C_i)}{p(x)}$$

- We need to estimate $p(x/C_1)$, $p(x/C_2)$, $P(C_1)$, $P(C_2)$

Example (cont'd)

- Collect data
  - Ask drivers how much their car was and measure height.
- Determine prior probabilities $P(C_1), P(C_2)$
  - e.g., 1209 samples: #C_1=221 #C_2=988



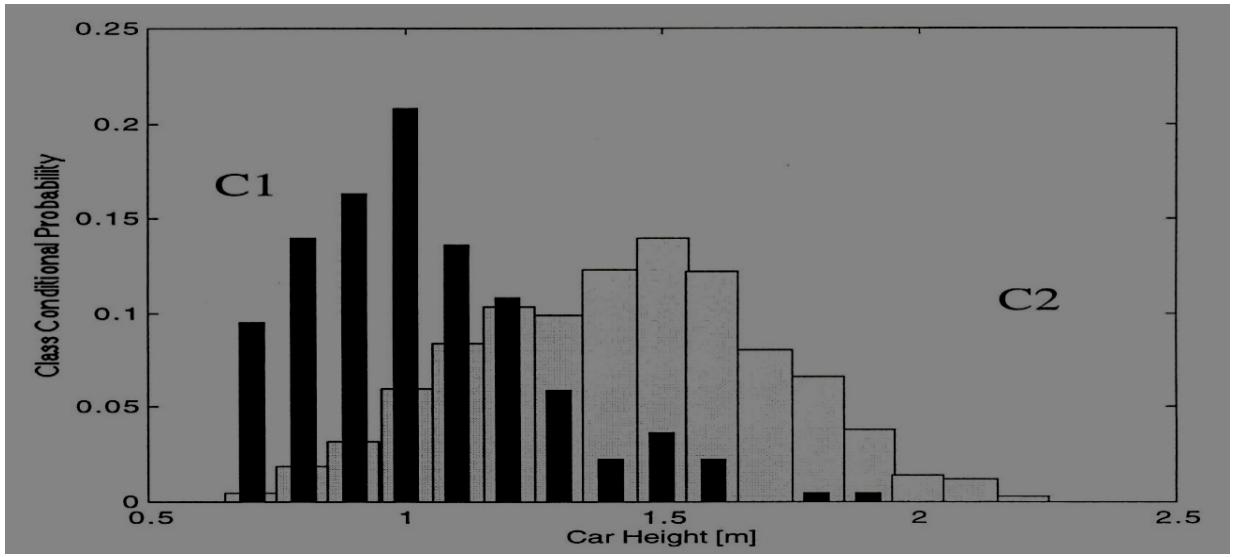
$$P(C_1) = \frac{221}{1209} = 0.183$$

$$P(C_2) = \frac{988}{1209} = 0.817$$

Example (cont'd)

- Determine class conditional probabilities (*likelihood*)
  - Discretize car height into bins and use normalized histogram

$p(x / C_i)$



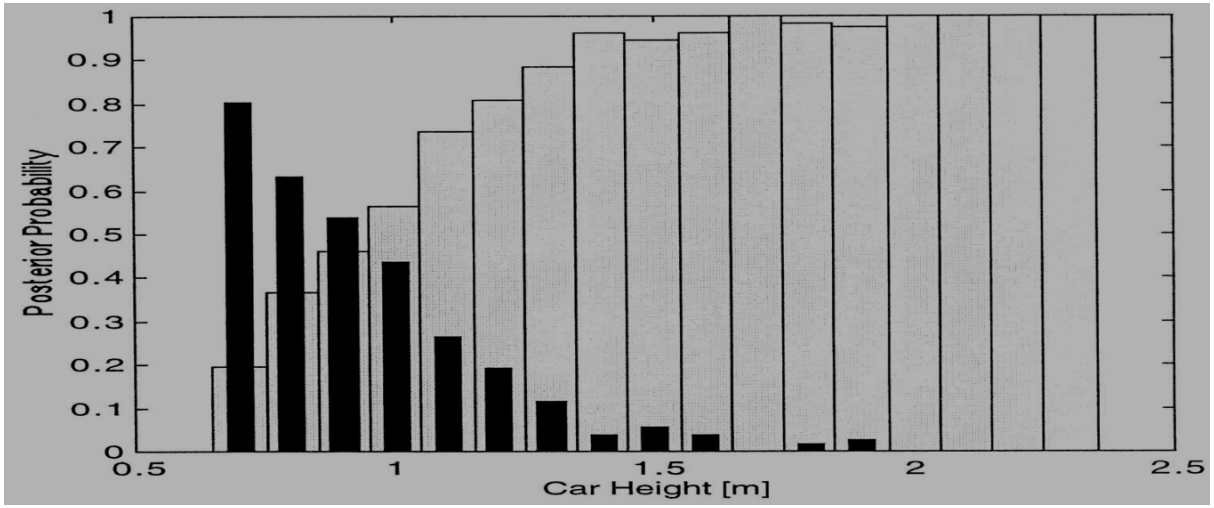
Example (cont'd)

- Calculate the posterior probability for each bin:

$$P(C_1 / x = 1.0) = \frac{p(x = 1.0 / C_1) P(C_1)}{p(x = 1.0 / C_1) P(C_1) + p(x = 1.0 / C_2) P(C_2)} =$$

$$= \frac{0.2081 * 0.183}{0.2081 * 0.183 + 0.0597 * 0.817} = 0.438$$

$P(C_i / x)$



A More General Theory

- Use more than one features.
- Allow more than two categories.
- Allow actions other than classifying the input to one of the possible categories (e.g., rejection).
- Employ a more general error function (i.e., "risk" function) by associating a "cost" ("loss" function) with each error (i.e., wrong action).

Terminology

- Features form a vector $\mathbf{x} \in R^d$
- A finite set of *c* categories $\omega_1, \omega_2, \ldots, \omega_c$
- Bayes rule (i.e., using vector notation):

$$P(\omega_j / \mathbf{x}) = \frac{p(\mathbf{x} / \omega_j) P(\omega_j)}{p(\mathbf{x})}$$

$$where \quad p(\mathbf{x}) = \sum_{j=1}^{c} p(\mathbf{x} / \omega_j) P(\omega_j)$$

- A finite set of *l* actions $\alpha_1, \alpha_2, \ldots, \alpha_l$
- A *loss* function $\lambda(\alpha_i / \omega_j)$
  - the cost associated with taking action α_i when the correct classification category is ω_j

Conditional Risk (or Expected Loss)

- Suppose we observe **x** and take action α_i
- Suppose that the cost associated with taking action α_i with ω_j being the correct category is $\lambda(\alpha_i / \omega_j)$
- The **conditional risk** (or **expected loss**) with taking action α_i is:

$$R(a_i / \mathbf{x}) = \sum_{j=1}^{c} \lambda(a_i / \omega_j) P(\omega_j / \mathbf{x})$$

Overall Risk

- Suppose $\alpha(\mathbf{x})$ is a general decision rule that determines which action $\alpha_1, \alpha_2, \ldots, \alpha_l$ to take for every **x**; then the overall risk is defined as:

$$R = \int R(a(\mathbf{x}) / \mathbf{x}) p(\mathbf{x}) d\mathbf{x}$$

- The optimum decision rule is the *Bayes rule*

Overall Risk (cont'd)

- The *Bayes decision rule* minimizes *R* by:
  (i) Computing $R(\alpha_i/\mathbf{x})$ for every α_i given an **x**
  (ii) Choosing the action α_i with the minimum $R(\alpha_i/\mathbf{x})$

- The resulting minimum overall risk is called *Bayes risk* and is the best (i.e., optimum) performance that can be achieved:

$$R^* = \min R$$

Example: Two-category classification

- Define
  - α_1: decide ω_1
  - α_2: decide ω_2 (c=2)
  - $\lambda_{ij} = \lambda(\alpha_i / \omega_j)$

- The conditional risks are:

$$R(a_i / \mathbf{x}) = \sum_{j=1}^{c} \lambda(a_i / \omega_j) P(\omega_j / \mathbf{x})$$

$$R(a_1/\mathbf{x}) = \lambda_{11} P(\omega_1/\mathbf{x}) + \lambda_{12} P(\omega_2/\mathbf{x})$$
$$R(a_2/\mathbf{x}) = \lambda_{21} P(\omega_1/\mathbf{x}) + \lambda_{22} P(\omega_2/\mathbf{x})$$

Example: Two-category classification (cont'd)

- Minimum risk decision rule:

$$\textbf{Decide } \omega_1 \text{ if } R(a_1/\mathbf{x}) < R(a_2/\mathbf{x}); \text{ otherwise decide } \omega_2$$

or

$$\textbf{Decide } \omega_1 \text{ if } (\lambda_{21} - \lambda_{11})P(\omega_1/\mathbf{x}) > (\lambda_{12} - \lambda_{22})P(\omega_2/\mathbf{x}); \text{ otherwise decide } \omega_2$$

or (i.e., using likelihood ratio)

$$\textbf{Decide } \omega_1 \text{ if } \frac{p(\mathbf{x}/\omega_1)}{p(\mathbf{x}/\omega_2)} > \frac{(\lambda_{12} - \lambda_{22})}{(\lambda_{21} - \lambda_{11})} \frac{P(\omega_2)}{P(\omega_1)}; \text{ otherwise decide } \omega_2$$

likelihood ratio threshold

Special Case:
Zero-One Loss Function

- Assign the same loss to all errors:

$$\lambda(a_i/\omega_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}$$

- The conditional risk corresponding to this loss function:

$$R(a_i/\mathbf{x}) = \sum_{j=1}^{c} \lambda(a_i/\omega_j) P(\omega_j/\mathbf{x}) = \sum_{i \neq j} P(\omega_j/\mathbf{x}) = 1 - P(\omega_i/\mathbf{x})$$

Special Case: Zero-One Loss Function (cont'd)

- The decision rule becomes:

$$\textbf{Decide } \omega_1 \text{ if } R(a_1/\mathbf{x}) < R(a_2/\mathbf{x}); \text{ otherwise decide } \omega_2$$

or $$\textbf{Decide } \omega_1 \text{ if } 1 - P(\omega_1/\mathbf{x}) < 1 - P(\omega_2/\mathbf{x}); \text{ otherwise decide } \omega_2$$

or $$\textbf{Decide } \omega_1 \text{ if } P(\omega_1/\mathbf{x}) > P(\omega_2/\mathbf{x}); \text{ otherwise decide } \omega_2$$

- In this case, the overall risk is the average probability error!

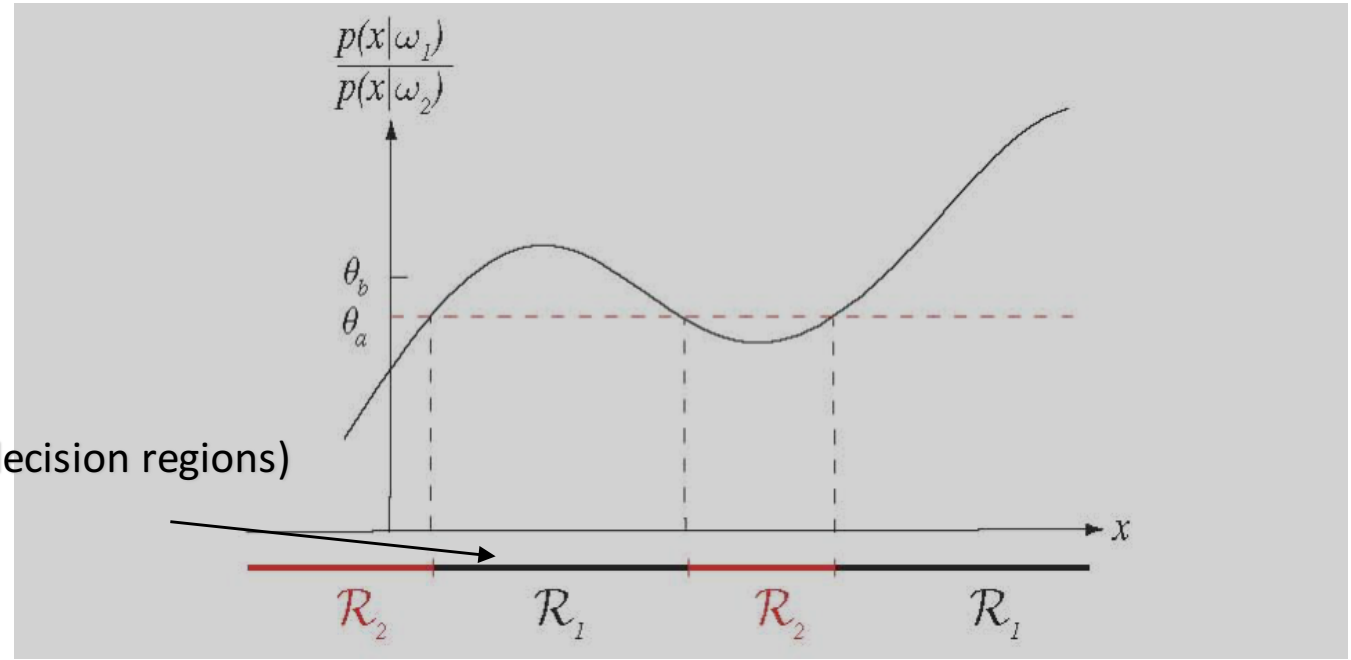
Example

Assuming general loss:

$$\textbf{Decide } \omega_1 \text{ if } \frac{p(\mathbf{x}/\omega_1)}{p(\mathbf{x}/\omega_2)} > \frac{(\lambda_{12}-\lambda_{22})}{(\lambda_{21}-\lambda_{11})}\frac{P(\omega_2)}{P(\omega_1)}; \text{ otherwise decide } \omega_2$$

Assuming zero-one loss:

Decide ω_1 if $p(x/\omega_1)/p(x/\omega_2) > P(\omega_2)/P(\omega_1)$ otherwise **decide** ω_2



(decision regions)

$$\theta_a = P(\omega_2)/P(\omega_1)$$

$$\theta_b = \frac{P(\omega_2)(\lambda_{12}-\lambda_{22})}{P(\omega_1)(\lambda_{21}-\lambda_{11})}$$

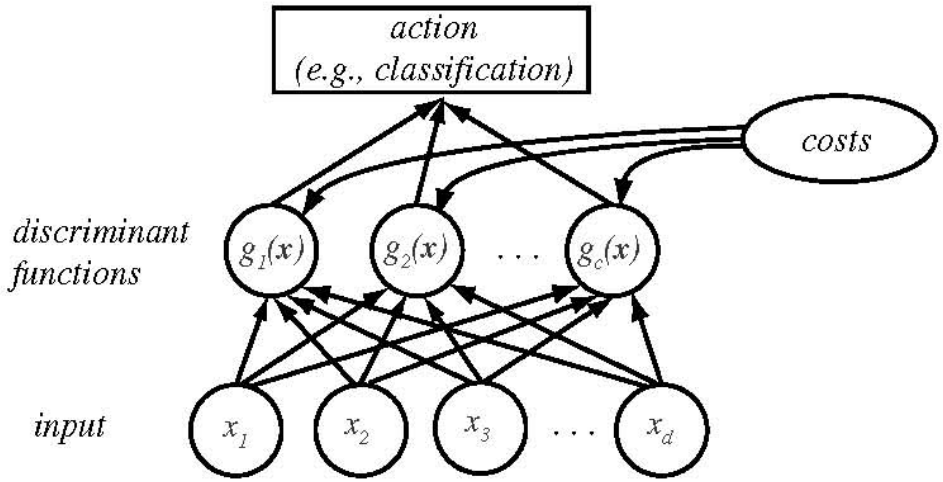
assume: $\lambda_{12} > \lambda_{21}$

Discriminant Functions

- A useful way to represent classifiers is through

discriminant functions $g_i(x)$, i = 1, . . . , c, where a feature vector **x** is assigned to class ω_i if:

$$g_i(\mathbf{x}) > g_j(\mathbf{x}) \text{ for all } j \neq i$$



Discriminants for Bayes Classifier

- Assuming a general loss function:

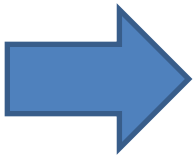
$$g_i(\mathbf{x}) = -R(\alpha_i / \mathbf{x})$$

- Assuming the zero-one loss function:

$$g_i(\mathbf{x}) = P(\omega_i / \mathbf{x})$$

Discriminants for Bayes Classifier (cont'd)

- Is the choice of g_i unique?
  - Replacing $g_i(\mathbf{x})$ with $f(g_i(\mathbf{x}))$, where $f()$ is monotonically increasing, does not change the classification results.

$g_i(x)=P(\omega_i/x)$ 

$$g_i(\mathbf{x}) = \frac{p(\mathbf{x}/\omega_i)P(\omega_i)}{p(\mathbf{x})}$$

$$g_i(\mathbf{x}) = p(\mathbf{x}/\omega_i)P(\omega_i)$$

$$g_i(\mathbf{x}) = \ln p(\mathbf{x}/\omega_i) + \ln P(\omega_i)$$

we'll use this form extensively!

Case of two categories

- More common to use a single discriminant function (*dichotomizer*) instead of two:

$$g(\mathbf{x}) = g_1(\mathbf{x}) - g_2(\mathbf{x})$$

Decide ω_1 if $g(\mathbf{x}) > 0$; otherwise decide ω_2

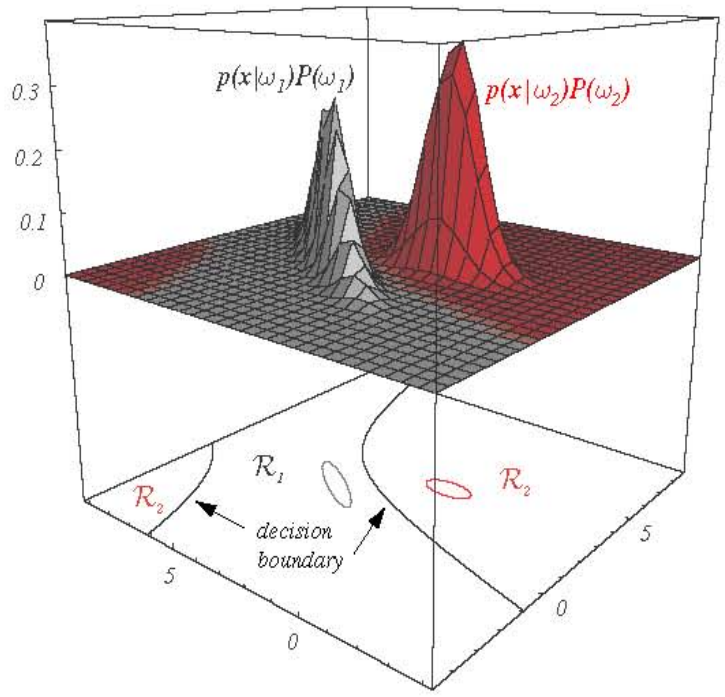
- Examples:

$$g(\mathbf{x}) = P(\omega_1 / \mathbf{x}) - P(\omega_2 / \mathbf{x})$$

$$g(\mathbf{x}) = \ln \frac{p(\mathbf{x} / \omega_1)}{p(\mathbf{x} / \omega_2)} + \ln \frac{P(\omega_1)}{P(\omega_2)}$$

Decision Regions and Boundaries

- Decision rules divide the feature space in *decision regions* $R_1, R_2, \ldots, R_c$, separated by *decision boundaries*.



decision boundary is defined by:

$$g_1(\mathbf{x}) = g_2(\mathbf{x})$$

Discriminant Function for Multivariate Gaussian Density

$$N(\mu,\Sigma) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} exp[-\frac{1}{2}(\mathbf{x}-\mu)^t\Sigma^{-1}(\mathbf{x}-\mu)]$$

- Consider the following discriminant function:

$$g_i(\mathbf{x}) = \ln p(\mathbf{x}/\omega_i) + \ln P(\omega_i)$$

- If $p(\mathbf{x}/\omega_i) \sim N(\mu_i, \Sigma_i)$, then

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^t\Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{d}{2}ln2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

Multivariate Gaussian Density:

Case I

- $\Sigma_i=\sigma^2$(diagonal)

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^t\Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{d}{2}ln2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

  - Features are statistically independent
  - Each feature has the same variance

- If we disregard $\frac{d}{2}ln2\pi$ and $\frac{1}{2}\ln|\Sigma_i|$ (constants):

$$g_i(\mathbf{x}) = -\frac{\|\mathbf{x}-\mu_i\|^2}{2\sigma^2} + \ln P(\omega_i)$$

where $\|\mathbf{x}-\mu_i\|^2 = (\mathbf{x}-\mu_i)^t(\mathbf{x}-\mu_i)$

favours the a-priori more likely category

- Expanding the above expression:

$$g_i(\mathbf{x}) = -\frac{1}{2\sigma^2}[\mathbf{x}^t\mathbf{x} - 2\mu_i^t\mathbf{x} + \mu_i^t\mu_i] + \ln P(\omega_i)$$

Multivariate Gaussian Density: Case I (cont'd)

- Disregarding $\mathbf{x}^t\mathbf{x}$ (constant), we get a linear discriminant:

$$\boxed{g_i(\mathbf{x}) = \mathbf{w}_i^t\mathbf{x} + w_{i0}}$$

where $\mathbf{w}_i = \frac{1}{\sigma^2}\mu_i$, and $w_{i0} = -\frac{1}{2\sigma^2}\mu_i^t\mu_i + \ln P(\omega_i)$

- Decision boundary is determined by hyperplanes; setting $g_i(\mathbf{x}) = g_j(\mathbf{x})$:

$$\boxed{\mathbf{w}^t(\mathbf{x} - \mathbf{x_0}) = 0}$$

where $\mathbf{w} = \mu_i - \mu_j$, and $\mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{\sigma^2}{\|\mu_i - \mu_j\|^2}\ln\frac{P(\omega_i)}{P(\omega_j)}(\mu_i - \mu_j)$

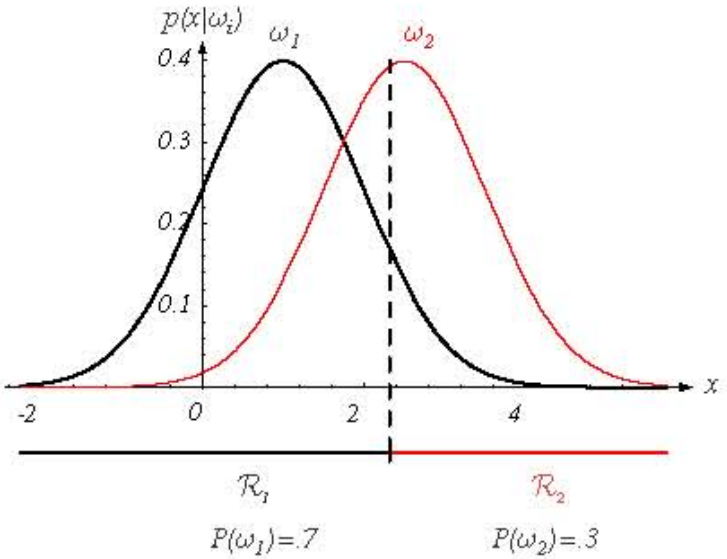
Multivariate Gaussian Density: Case I (cont'd)

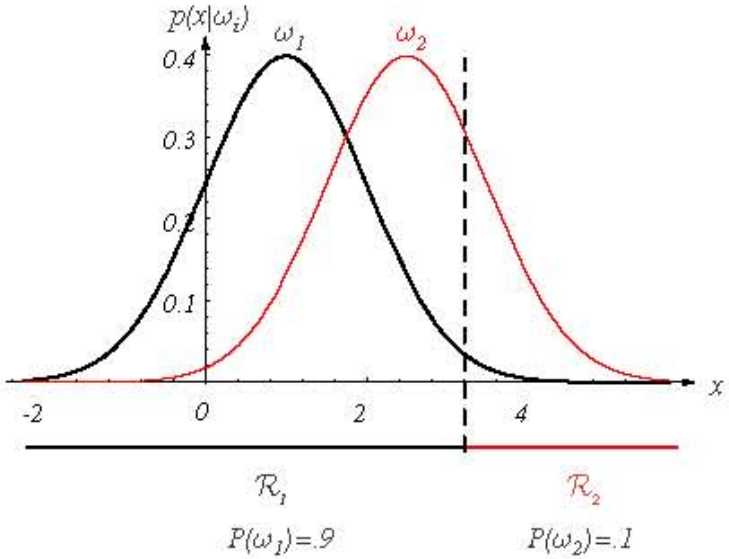
- Properties of decision boundary:
  - It passes through $\mathbf{x_0}$
  - It is orthogonal to the line linking the means.
  - What happens when $P(\omega_i) = P(\omega_j)$?
  - If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.
  - If σ is very small, the position of the boundary is insensitive to $P(\omega_i)$ and $P(\omega_j)$

$$\mathbf{w}^t(\mathbf{x} - \mathbf{x_0}) = 0$$

$$\text{where } \mathbf{w} = \mu_i - \mu_j, \text{ and } \mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{\sigma^2}{\|\mu_i - \mu_j\|^2} \ln \frac{P(\omega_i)}{P(\omega_j)}(\mu_i - \mu_j)$$

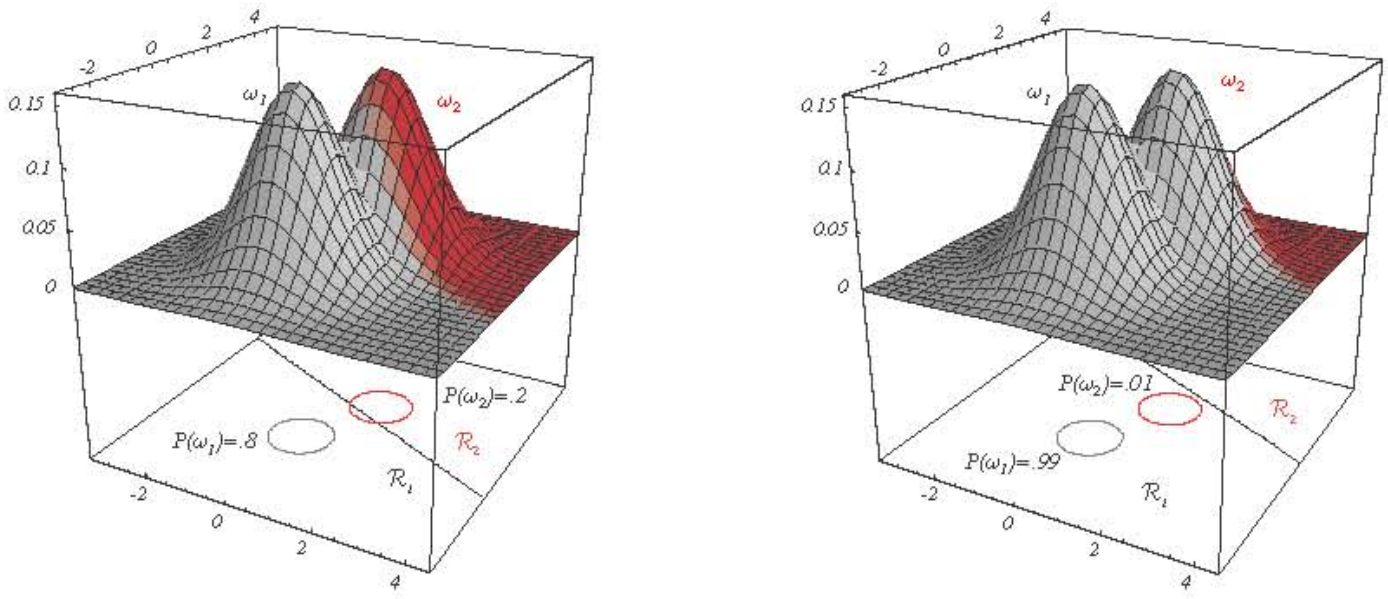
Multivariate Gaussian Density: Case I (cont'd)





If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.

Multivariate Gaussian Density: Case I (cont'd)



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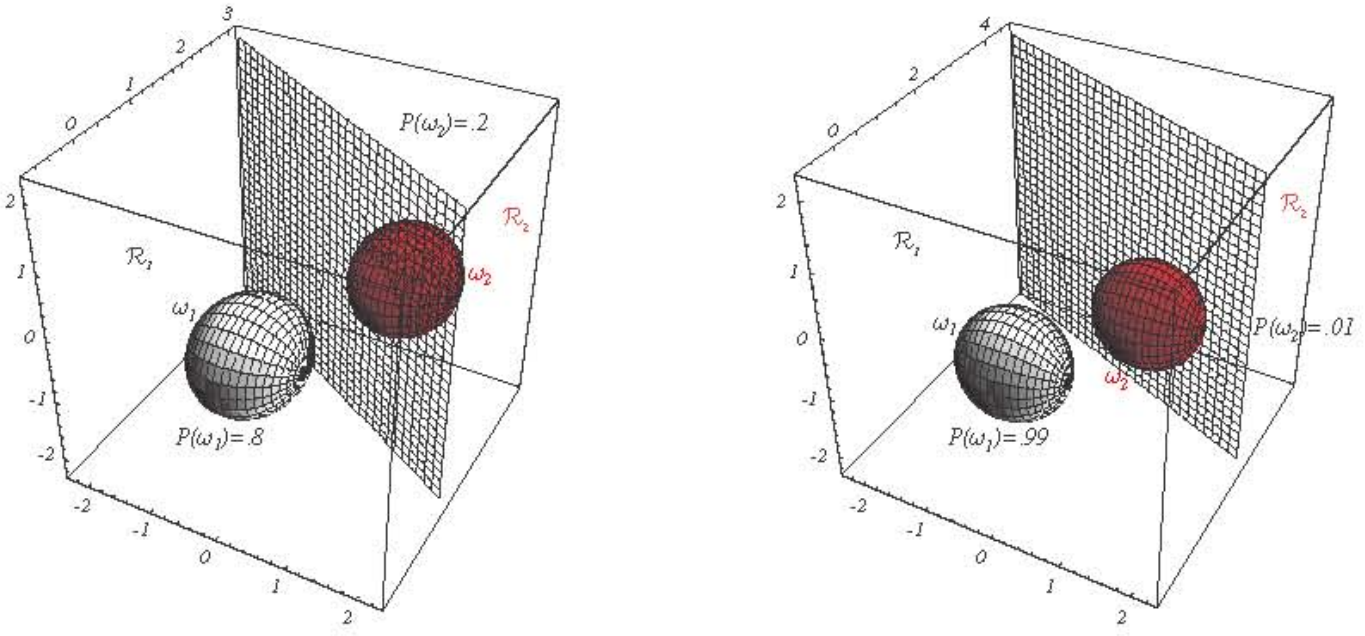
Multivariate Gaussian Density: Case I (cont'd)

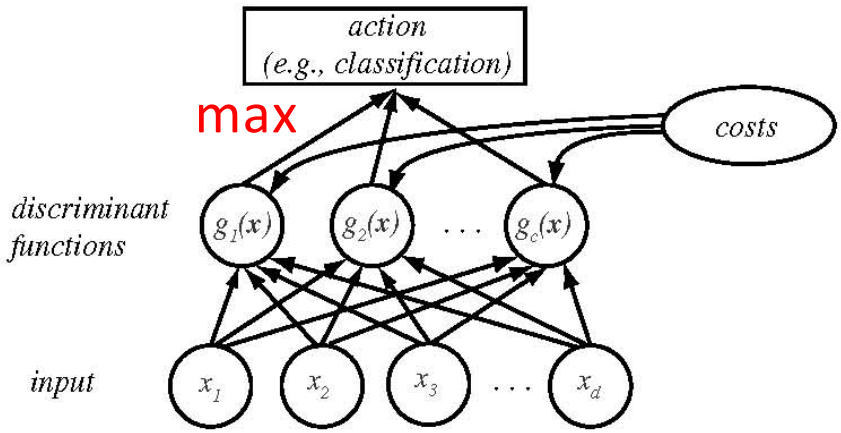


If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.

Multivariate Gaussian Density: Case I (cont'd)

- Minimum distance classifier
  - When $P(\omega_i)$ are equal, then:

$$g_i(\mathbf{x}) = -\frac{\|\mathbf{x}-\mu_i\|^2}{2\sigma^2} + \ln P(\omega_i) \quad \Rightarrow \quad g_i(\mathbf{x}) = -\|\mathbf{x}-\mu_i\|^2$$



Multivariate Gaussian Density:

Case II

- $\Sigma_i = \Sigma$ $$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^t\Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{d}{2}ln2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

- The clusters have hyperellipsoidal shape and same size (centered at μ).

- If we disregard $\frac{d}{2}ln2\pi$ and $\frac{1}{2}\ln|\Sigma_i|$ (constants):

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^t\Sigma^{-1}(\mathbf{x}-\mu_i) + \ln P(\omega_i)$$

- Expanding the above expression and disregarding the quadratic term:

$$\boxed{\begin{array}{c} g_i(\mathbf{x}) = \mathbf{w}_i^t\mathbf{x} + w_{i0} \\ (\textit{linear discriminant}) \end{array}}$$

where $\mathbf{w}_i = \Sigma^{-1}\mu_i$, and $w_{i0} = -\frac{1}{2}\mu_i^t\Sigma^{-1}\mu_i + \ln P(\omega_i)$

Multivariate Gaussian Density: Case II (cont'd)

- Decision boundary is determined by hyperplanes; setting $g_i(\mathbf{x}) = g_j(\mathbf{x})$:

$$\mathbf{w}^t(\mathbf{x} - \mathbf{x_0}) = 0$$

where $\mathbf{w} = \Sigma^{-1}(\mu_i - \mu_j)$ and $\mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{ln[P(\omega_i)/P(\omega_j)]}{(\mu_i - \mu_j)^t\Sigma^{-1}(\mu_i - \mu_j)}(\mu_i - \mu_j)$

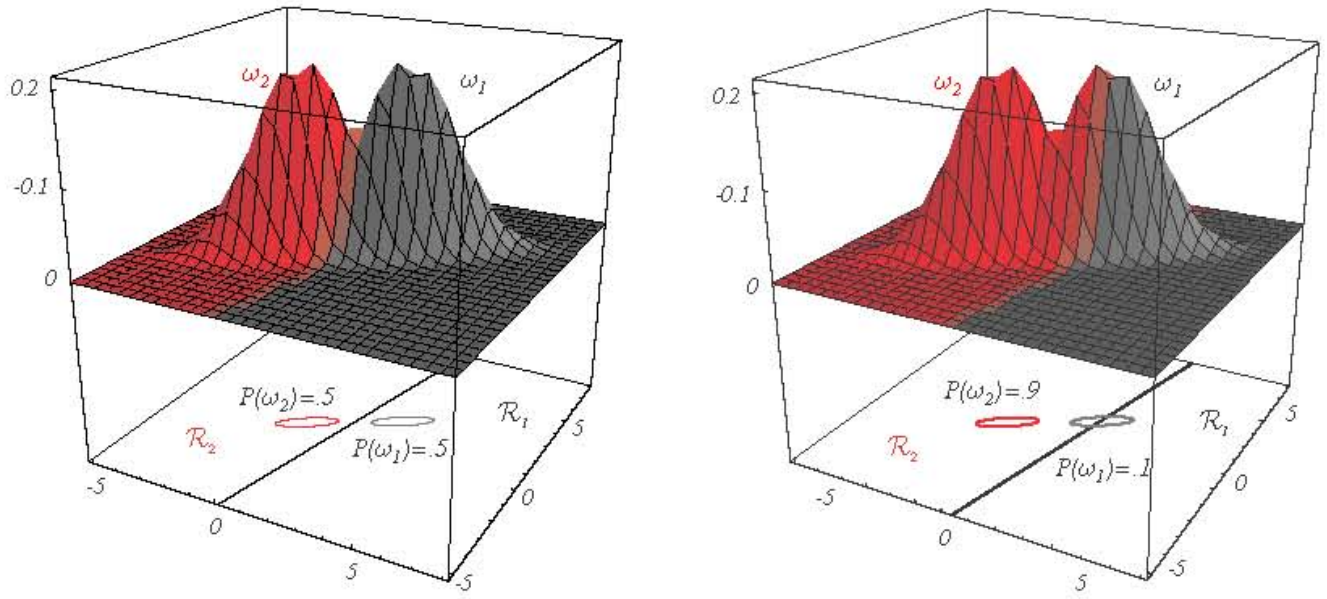
Multivariate Gaussian Density: Case II (cont'd)

- Properties of hyperplane (decision boundary):
  - It passes through $\mathbf{x_0}$
  - It is not orthogonal to the line linking the means.
  - What happens when $P(\omega_i) = P(\omega_j)$?
  - If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.

$$\mathbf{w}^t(\mathbf{x} - \mathbf{x_0}) = 0$$

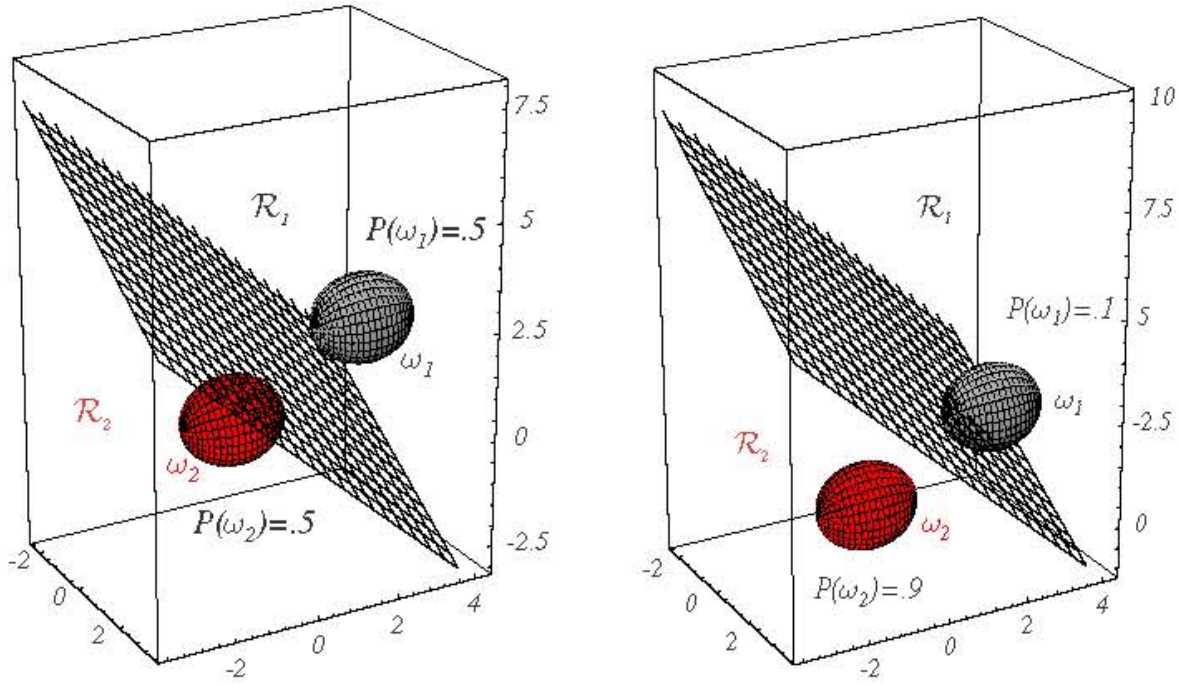
$$\text{where } \mathbf{w} = \Sigma^{-1}(\mu_i - \mu_j) \text{ and } \mathbf{x}_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{ln[P(\omega_i)/P(\omega_j)]}{(\mu_i - \mu_j)^t \Sigma^{-1}(\mu_i - \mu_j)}(\mu_i - \mu_j)$$

Multivariate Gaussian Density: Case II (cont'd)



If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.

Multivariate Gaussian Density: Case II (cont'd)



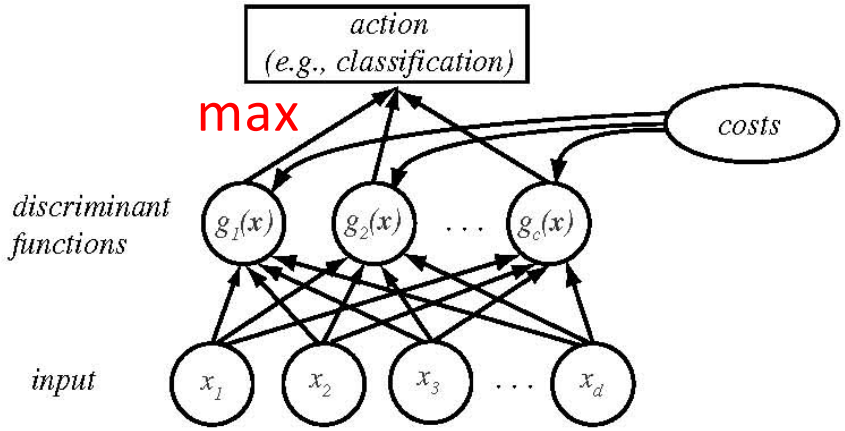
If $P(\omega_i) \neq P(\omega_j)$, then $\mathbf{x_0}$ shifts away from the most likely category.

Multivariate Gaussian Density: Case II (cont'd)

- Mahalanobis distance classifier
  - When $P(\omega_i)$ are equal, then:

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_\mathbf{i})^t\Sigma^{-1}(\mathbf{x}-\mu_\mathbf{i}) + \ln P(\omega_i)$$

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_\mathbf{i})^t\Sigma^{-1}(\mathbf{x}-\mu_\mathbf{i})$$



Multivariate Gaussian Density:

Case III

- Σ_i= arbitrary $$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^t\Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{d}{2}ln2\pi - \frac{1}{2}\ln|\Sigma_i| + \ln P(\omega_i)$$

  - The clusters have different shapes and sizes (centered at μ).

  - If we disregard $\frac{d}{2}ln2\pi$ (constant):

$$\boxed{g_i(\mathbf{x}) = \mathbf{x}^t\mathbf{W}_i\mathbf{x} + \mathbf{w}_i\mathbf{x} + w_{i0}}$$
(quadratic discriminant)

where $\mathbf{W}_i = -\frac{1}{2}\Sigma_i^{-1}$, $\mathbf{w}_i = \Sigma_i^{-1}\mu_i$, and $w_{i0} = -\frac{1}{2}\mu_i^t\Sigma^{-1}\mu_i - \frac{1}{2}ln|\Sigma_i| + \ln P(\omega_i)$

- Decision boundary is determined by hyperquadrics; setting $g_i(\mathbf{x}) = g_j(\mathbf{x})$

  e.g., hyperplanes, pairs of hyperplanes, hyperspheres, hyperellipsoids, hyperparaboloids etc.

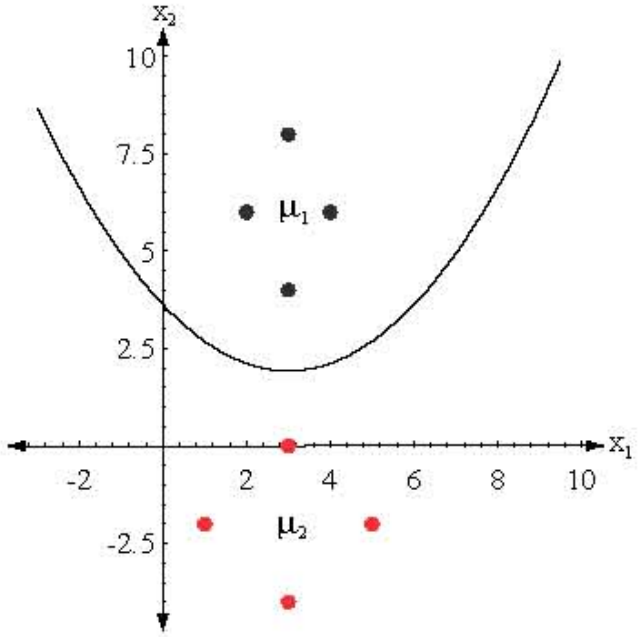
Example - Case III

$$\boldsymbol{\mu}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}; \quad \boldsymbol{\Sigma}_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } \boldsymbol{\mu}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}; \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

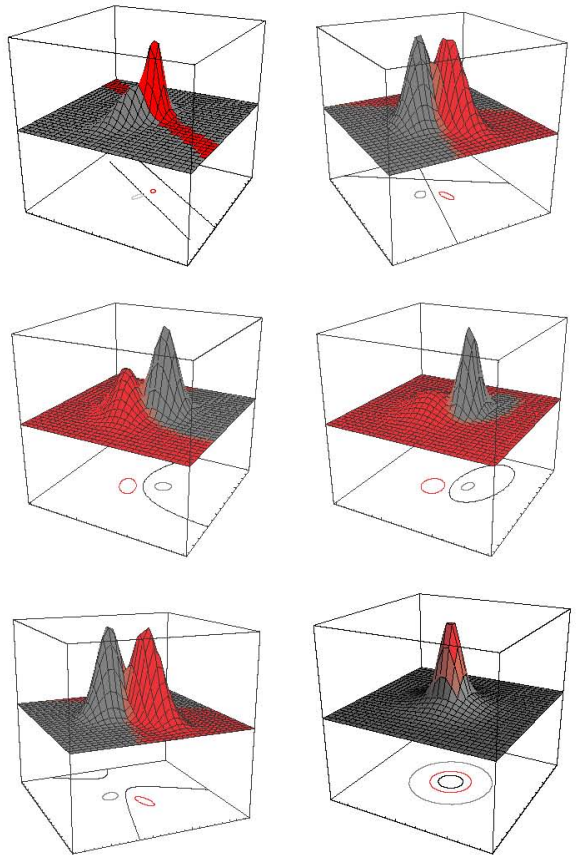
decision boundary: $x_2 = 3.514 - 1.125x_1 + 0.1875x_1^2.$

$P(\omega_1)=P(\omega_2)$

boundary does not pass through midpoint of μ_1,μ_2



Multivariate Gaussian Density: Case III (cont'd)

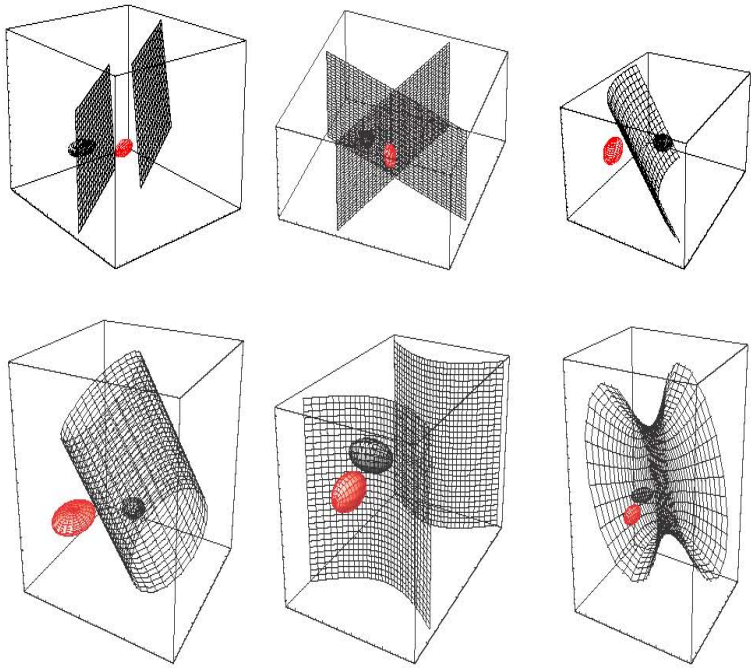


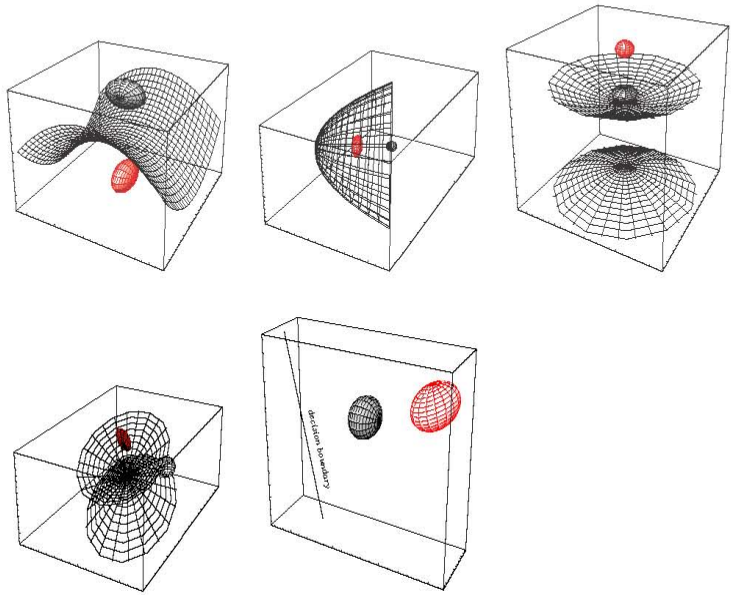
non-linear decision boundaries

FIGURE 2.14. Arbitrary Gaussian distributions lead to Bayes decision boundaries that are general hyperquadrics. Conversely, given any hyperquadric, one can find two Gaussian distributions whose Bayes decision boundary is that hyperquadric. These variances are indicated by the contours of constant probability density. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

Multivariate Gaussian Density: Case III (cont'd)

- More examples





URE 2.15. Arbitrary three-dimensional Gaussian distributions yield Bayes decision boundaries that are dimensional hyperquadrics. There are even degenerate cases in which the decision boundary is a line. n: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John ey & Sons, Inc.

Error Bounds

- Exact error calculations could be difficult – easier to estimate error bounds!

$$P(error) = \int P(error, \mathbf{x})\, d\mathbf{x} = \int P(error/\mathbf{x}) p(\mathbf{x})\, d\mathbf{x}$$

$$P(error/\mathbf{x}) = \begin{cases} P(\omega_1/\mathbf{x}) & \text{if we decide } \omega_2 \\ P(\omega_2/\mathbf{x}) & \text{if we decide } \omega_1 \end{cases}$$

or

$\min[P(\omega_1/x), P(\omega_2/x)]$

- Using the inequality:

$$min[a, b] \le a^{\beta} b^{1-\beta}, \qquad a, b \ge 0, 0 \le \beta \le 1$$

$$P(\text{error}) = \int min[p(\mathbf{x}/\omega_1)P(\omega_1),\ p(\mathbf{x}/\omega_2)P(\omega_2)]d\mathbf{x} \le$$

$$P^{\beta}(\omega_1)P^{1-\beta}(\omega_2) \int p^{\beta}(\mathbf{x}/\omega_1)\ p^{1-\beta}(\mathbf{x}/\omega_2)d\mathbf{x}$$

Error Bounds (cont'd)

- If the class conditional distributions are Gaussian, then

$$\int p^{\beta}(\mathbf{x}/\omega_1)\, p^{1-\beta}(\mathbf{x}/\omega_2)d\mathbf{x} = e^{-\kappa(\beta)}$$

where:

$$k(\beta) = \frac{\beta(1-\beta)}{2}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^t\left[(1-\beta)\boldsymbol{\Sigma}_1+\beta\boldsymbol{\Sigma}_2\right]^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2) + \frac{1}{2}\ln\frac{|(1-\beta)\boldsymbol{\Sigma}_1+\beta\boldsymbol{\Sigma}_2|}{|\boldsymbol{\Sigma}_1|^{1-\beta}|\boldsymbol{\Sigma}_2|^{\beta}}.$$

Error Bounds (cont'd)

- The *Chernoff* bound corresponds to β that minimizes $e^{-\kappa(\beta)}$
  - This is a 1-D optimization problem, regardless to the dimensionality of the class conditional densities.

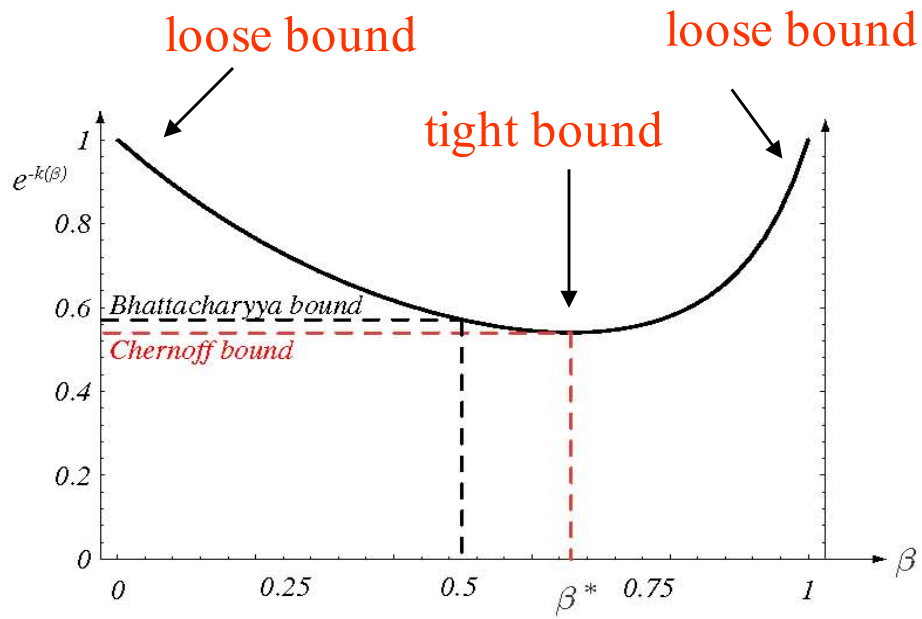


FIGURE 2.18. The Chernoff error bound is never looser than the Bhattacharyya bound. For this example, the Chernoff bound happens to be at $\beta^* = 0.66$, and is slightly tighter than the Bhattacharyya bound ($\beta = 0.5$). From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

Error Bounds (cont'd)

- *Bhattacharyya* bound
  - Approximate the error bound using β=0.5
  - Easier to compute than Chernoff error but looser.

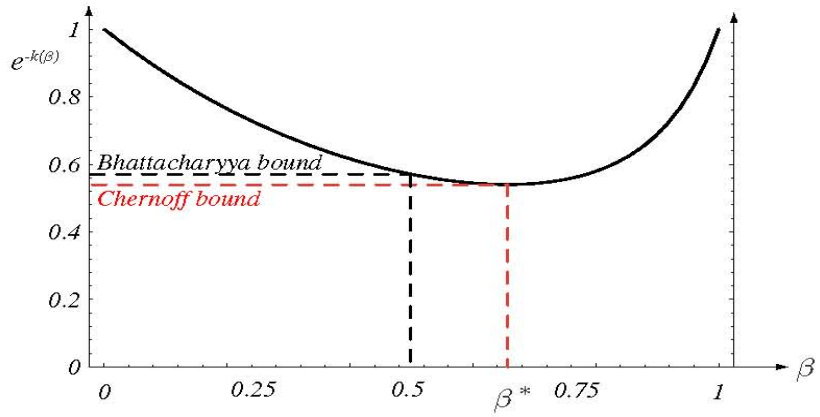
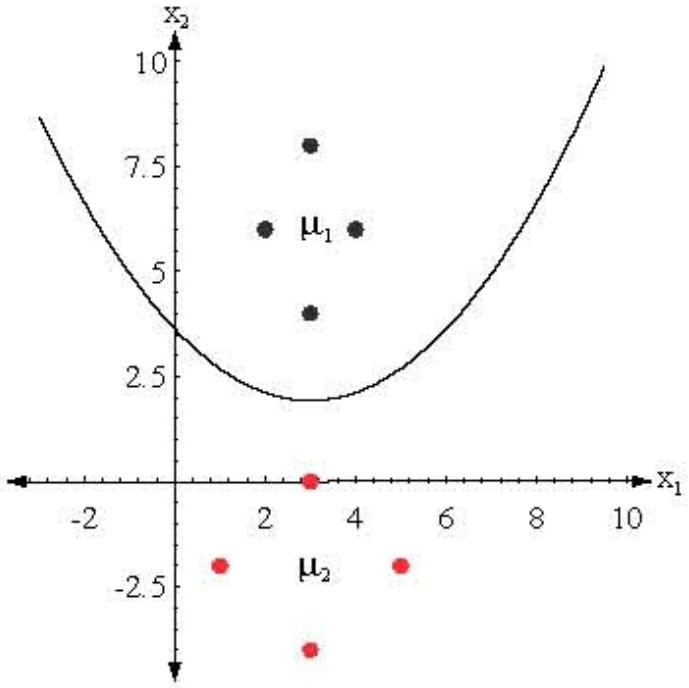


FIGURE 2.18. The Chernoff error bound is never looser than the Bhattacharyya bound. For this example, the Chernoff bound happens to be at $\beta^* = 0.66$, and is slightly tighter than the Bhattacharyya bound ($\beta = 0.5$). From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.

- The Chernoff and Bhattacharyya bounds will not be good bounds if the distributions are not Gaussian.

Example

$$k(\beta) = \frac{\beta(1-\beta)}{2}(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)^t[\beta\boldsymbol{\Sigma}_1+(1-\beta)\boldsymbol{\Sigma}_2]^{-1}(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)+\frac{1}{2}\ln\frac{|\beta\boldsymbol{\Sigma}_1+(1-\beta)\boldsymbol{\Sigma}_2|}{|\boldsymbol{\Sigma}_1|^{\beta}|\boldsymbol{\Sigma}_2|^{1-\beta}}.$$



$$\boldsymbol{\mu}_1=\begin{bmatrix}3\\6\end{bmatrix};\quad \boldsymbol{\Sigma}_1=\begin{pmatrix}1/2 & 0\\0 & 2\end{pmatrix}$$

$$\boldsymbol{\mu}_2=\begin{bmatrix}3\\-2\end{bmatrix};\quad \boldsymbol{\Sigma}_2=\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}.$$

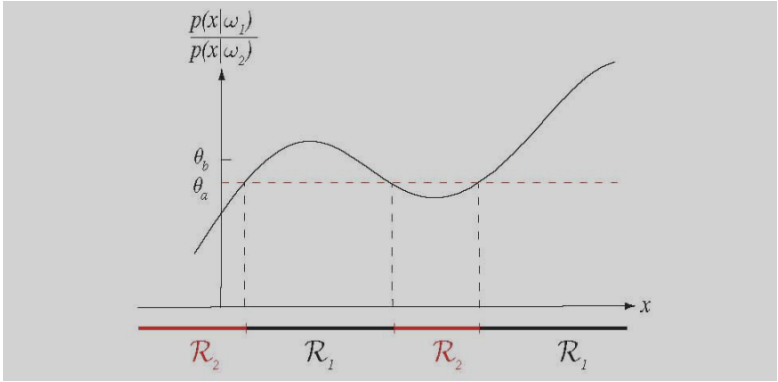
Bhattacharyya error:

k(0.5)=4.06

$$P(error) \le 0.0087$$

Receiver Operating Characteristic (ROC) Curve

- Every classifier employs some kind of a threshold.



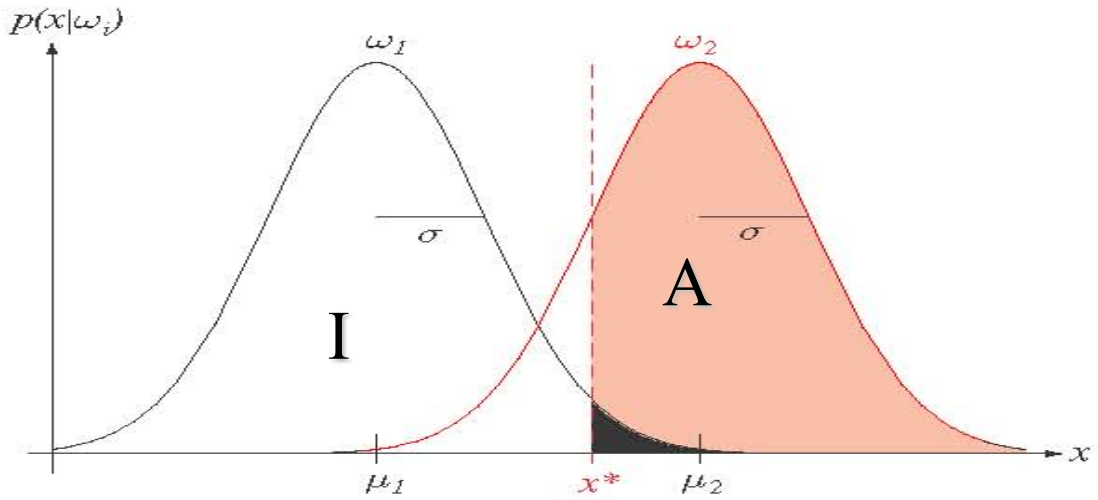
$$\theta_a = P(\omega_2)/P(\omega_1)$$

$$\theta_b = \frac{P(\omega_2)(\lambda_{12}-\lambda_{22})}{P(\omega_1)(\lambda_{21}-\lambda_{11})}$$

- Changing the threshold affects the performance of the system.
- ROC curves can help us evaluate system performance for different thresholds.

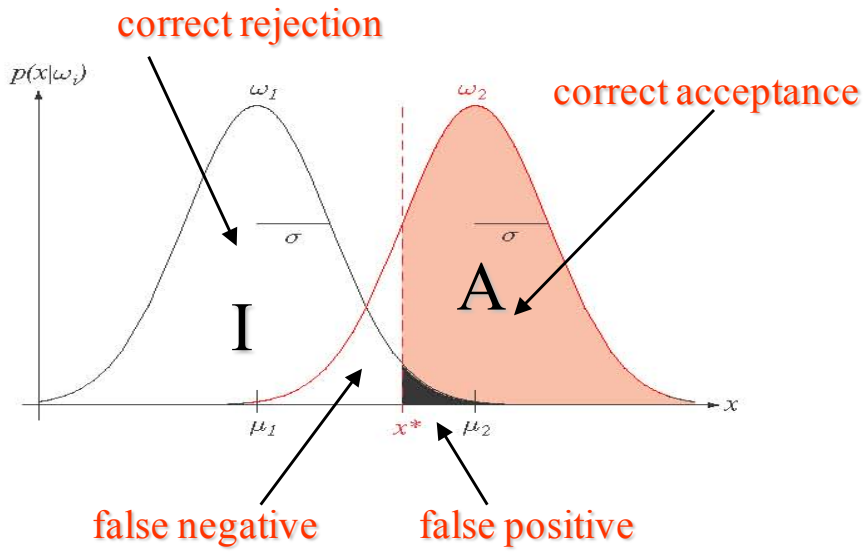
Example: Person Authentication

- Authenticate a person using biometrics (e.g., fingerprints).
- There are two possible distributions (i.e., classes):
  - *Authentic* (A) and *Impostor* (I)

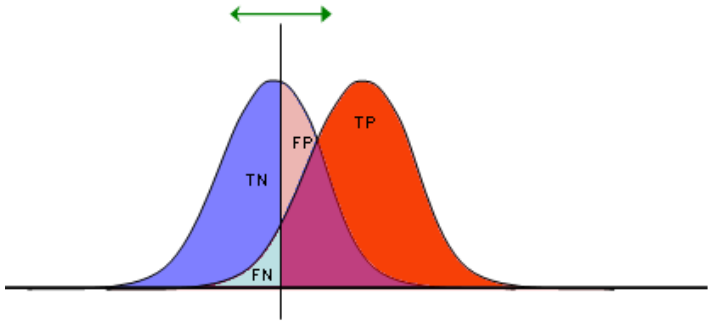


Example: Person Authentication (cont'd)

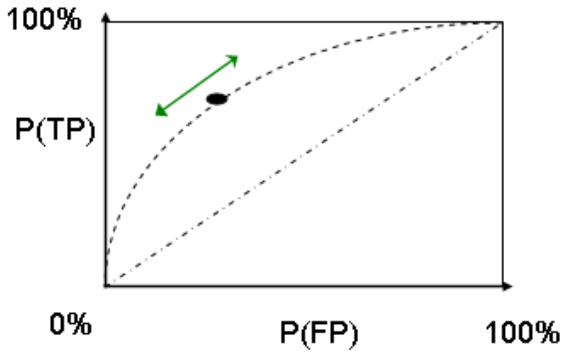
- Possible decisions:
  - (1) **correct acceptance** (true positive):
    - X belongs to A, and we decide A
  - (2) **incorrect acceptance** (false positive):
    - X belongs to I, and we decide A
  - (3) **correct rejection** (true negative):
    - X belongs to I, and we decide I
  - (4) **incorrect rejection** (false negative):
    - X belongs to A, and we decide I



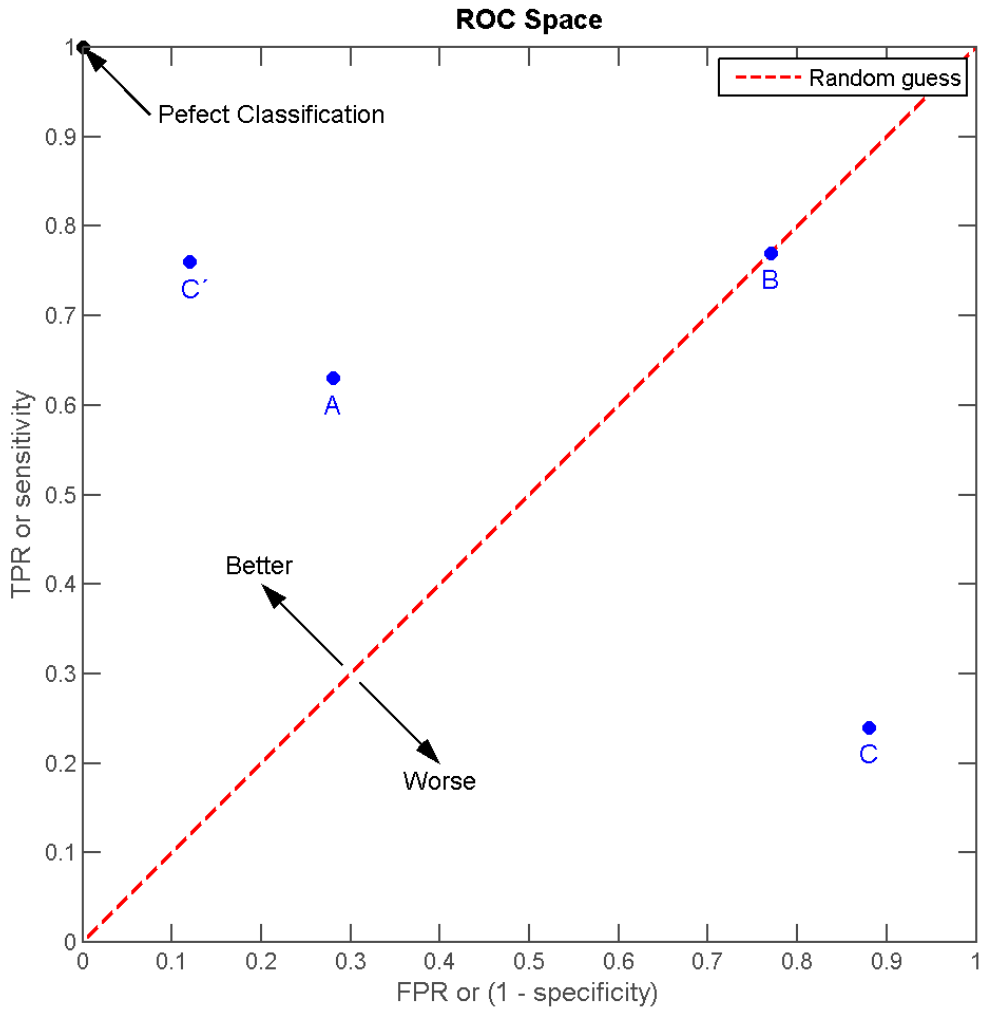
Error vs Threshold

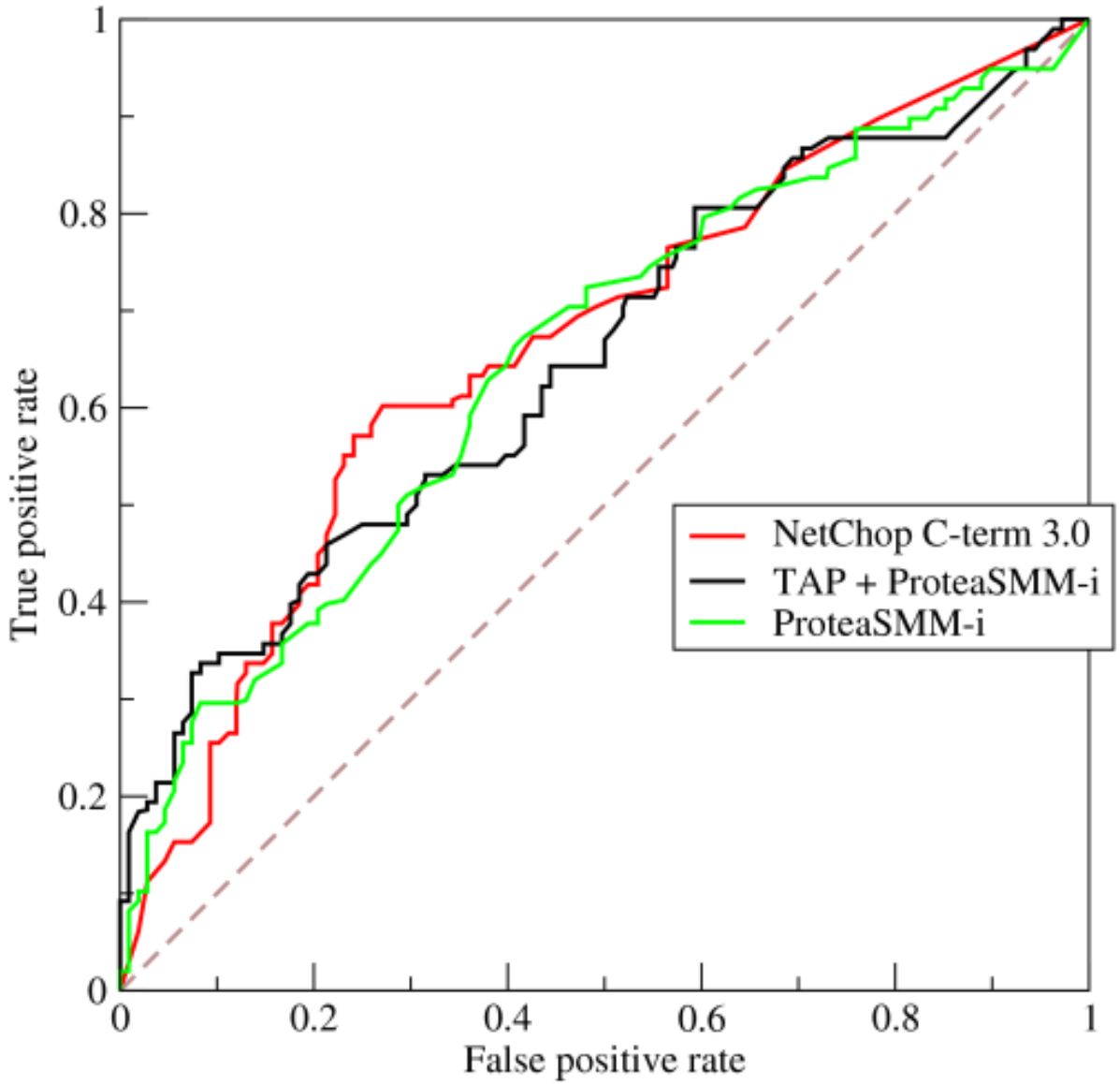


TP	FP
FN	TN
1	1



False Negatives vs Positives




True positive rate
False positive rate
0
0.2
0.4
0.6
0.8
1
NetChop C-term 3.0
TAP + ProteaSMM-i
ProteaSMM-i

Next Lecture

- Linear Classification Methods
  - Hastie et al, Chapter 4

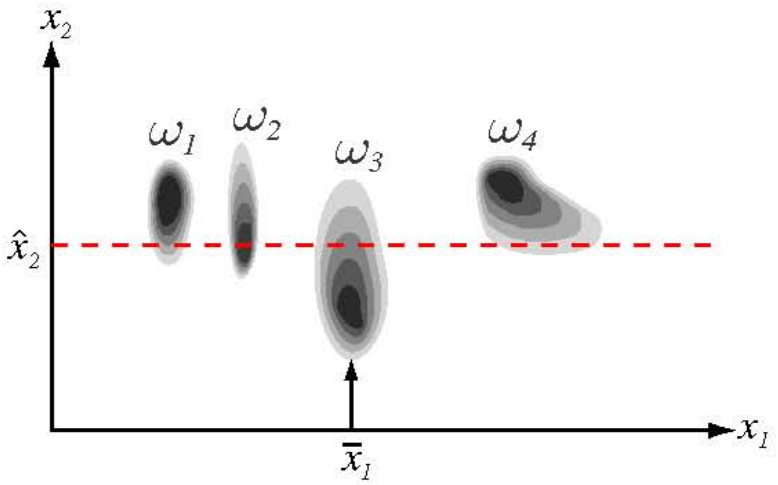
- Paper list will available by Weekend
  - Bidding to start on Monday

Bayes Decision Theory: Case of Discrete Features

- Replace $\int p(\mathbf{x}/\omega_j)d\mathbf{x}$ with $\sum_{\mathbf{x}} P(\mathbf{x}/\omega_j)$
- See section 2.9

Missing Features

- Consider a Bayes classifier using uncorrupted data.
- Suppose **x**=(x_1,x_2) is a test vector where x_1 is missing and the value of x_2 is $\hat{x}_2$ - how can we classify it?
  - If we set x_1 equal to the average value, we will classify **x** as ω_3
  - But $p(\hat{x}_2 / \omega_2)$ is larger; maybe we should classify **x** as ω_2 ?



Missing Features (cont'd)

- Suppose $\mathbf{x}=[\mathbf{x}_g,\mathbf{x}_b]$ ($\mathbf{x}_g$: good features, $\mathbf{x}_b$: bad features)
- Derive the Bayes rule using the good features:

$$P(\omega_i/\mathbf{x}_g) = \frac{\mathrm{p}(\omega_i,\mathbf{x}_g)}{p(\mathbf{x}_g)} = \frac{\int \mathrm{p}(\omega_i,\mathbf{x}_g,\mathbf{x}_b)d\mathbf{x}_b}{p(\mathbf{x}_g)} =$$

$$\frac{\int P(\omega_i/\mathbf{x}_g,\mathbf{x}_b)p(\mathbf{x}_g,\mathbf{x}_b)d\mathbf{x}_b}{p(\mathbf{x}_g)} = \frac{\int P(\omega_i/\mathbf{x}_g,\mathbf{x}_b)p(\mathbf{x})d\mathbf{x}_b}{\int p(\mathbf{x})d\mathbf{x}_b}$$

Marginalize posterior probability over bad features.

Decide ω_1 if $P(\omega_1/\mathbf{x}_g) > P(\omega_2/\mathbf{x}_g)$; otherwise decide ω_2

Compound Bayesian Decision Theory

- Sequential decision

  (1) Decide as each fish emerges.

- Compound decision

  (1) Wait for *n* fish to emerge.

  (2) Make all *n* decisions jointly.

  – Could improve performance when consecutive states of nature are **not** be statistically independent.

Compound Bayesian Decision Theory (cont'd)

- Suppose $\boldsymbol{\Omega}=(\omega(1), \omega(2), \ldots, \omega(n))$denotes the n states of nature where $\omega(i)$ can take one of c values $\omega_1, \omega_2, \ldots, \omega_c$ (i.e., c categories)
- Suppose $P(\boldsymbol{\Omega})$ is the prior probability of the n states of nature.
- Suppose $\mathbf{X}=(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ are n observed vectors.

Compound Bayesian Decision Theory (cont'd)

- Suppose p($\mathbf{X}/\boldsymbol{\Omega}$) is the conditional probability function for $\mathbf{X}$

$$\mathrm{P}(\boldsymbol{\Omega}/\mathbf{X})=\frac{p(\mathbf{X}/\boldsymbol{\Omega})P(\boldsymbol{\Omega})}{p(\mathbf{X})}$$

- The assumption $p(\mathbf{X}/\boldsymbol{\Omega})= \prod_{i=1}^{c} \mathrm{p}(\boldsymbol{x}_i/\omega(\mathrm{i}))$ might be acceptable.
- The assumption $\mathrm{P}(\boldsymbol{\Omega})= \prod_{i=1}^{c} \mathrm{P}(\omega(\mathrm{i}))$ is not acceptable!

i.e., consecutive states of nature may **not** be statistically independent!