Integr. Equ. Oper. Theory
DOI 10.1007/s00020-016-2313-2

Integral Equations and Operator Theory

Finite-Codimensional Compressions of Symmetric and Self-Adjoint Linear Relations in Krein Spaces

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Dedicated to Heinz Langer with congratulations for his 80th birthday and the honorary doctorates from Stockholm University and the Technical University Dresden

Abstract. Theorems due to Stenger (Bull Am Math Soc 74:369–372, 1968) and Nudelman (Int Equ Oper Theory 70:301–305, 2011) in Hilbert spaces and their generalizations to Krein spaces in Azizov and Dijksma (Int Equ Oper Theory 74(2):259–269, 2012) and Azizov et al. (Linear Algebra Appl 439:771–792, 2013) generate additional questions about properties a finite-codimensional compression T_0 of a symmetric or self-adjoint linear relation T may or may not inherit from T. These questions concern existence of invariant maximal nonnegative subspaces, definitizability, singular critical points and defect indices.

Mathematics Subject Classification. 47B50, 47B25, 47B44.

Keywords. Hilbert space, Pontryagin space, Krein space, linear relation, dissipative, maximal dissipative, symmetric, maximal symmetric, self-adjoint, compression, dense set, codimension, definitizability, invariant nonnegative subspace, singular critical point, defect number.

1. Introduction

Let $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ be a Krein space, let $\mathcal{G}$ be a Krein subspace of $\mathcal{K}$ with finite codimension $\operatorname{codim}\mathcal{G}$ and let $P = P_{\mathcal{G}}$ be the projection in $\mathcal{K}$ onto $\mathcal{G}$. Let T be a linear relation in $\mathcal{K}$. Then the *finite-codimensional compression* of T to $\mathcal{G}$ is the linear relation T_0 in $\mathcal{G}$ defined by

$$T_0 := PT|_{\mathcal{G}} = PT|_{\mathcal{G}\cap\operatorname{dom}T} = \{\{f; Pg\} : \{f; g\} \in T, f \in \mathcal{G}\}.$$

The research of Tomas Azizov was supported by Grant RFBR 15-01-05315-a.

We have the sad task to inform the reader that our coauthor Tomas Yakovlevich Azizov died on January 23, 2016. He was an exceptionally creative mathematician, a highly valued close friend and a gentle person. We shall miss him.

Birkhäuser

Published online: 13 September 2016

The index 0 attached to a linear relation will always denote a finite-codimensional compression of this relation, the spaces $\mathcal{K}$ and $\mathcal{G}$ involved being clear from the context. We assume that the reader is familiar with Krein and Pontryagin spaces and operators and linear relations (or multi-valued operators) on these spaces such as (maximal) dissipative, (maximal) symmetric and self-adjoint linear relations as treated in for example [5,8,11,12,25]. In this paper by a subspace we mean a closed linear subset. The symbol $\dot{+}$ stands for "direct sum" and the symbol $\oplus$ for "orthogonal direct sum". Alternatively, we use $\begin{bmatrix} \mathcal{L} \\ \mathcal{M} \end{bmatrix}$ to denote the direct sum $\mathcal{L}\dot{+}\mathcal{M}$ of linear spaces $\mathcal{L}$ and $\mathcal{M}$.

Lemma 1.1. *Let $\mathcal{K}$ be a Krein space and let $\mathcal{G}$ be a Krein subspace of finite codimension. If T is a closed linear relation (operator) in $\mathcal{K}$, then its compression T_0 to $\mathcal{G}$ is a closed linear relation (operator) in $\mathcal{G}$.*

Proof. We first prove the statement: *If $\mathcal{L}$ is a subspace of a Krein space $\mathcal{H}$ and Q is a projection in $\mathcal{H}$ onto a Krein subspace of finite codimension, then $Q\mathcal{L}$ is a subspace of $\mathcal{H}$.* Indeed, since $\mathcal{L} \cap Q\mathcal{H}$ is a subspace, there is a subspace $\mathcal{N}$ such that $\mathcal{L} = (\mathcal{L} \cap Q\mathcal{H})\dot{+}\mathcal{N}$, direct sum. For example, take $\mathcal{N}$ equal to the orthogonal complement of $\mathcal{L} \cap Q\mathcal{H}$ in $\mathcal{L}$ with respect to the Hilbert space inner product induced by a fundamental symmetry on $\mathcal{H}$. Assume $\dim \mathcal{N} > \dim(Q\mathcal{H})^{\perp}$. Then there is a nonzero element $x \in \mathcal{N} \subset \mathcal{L}$ such that $x \perp (Q\mathcal{H})^{\perp}$, that is,

$$0 \neq x \in (\mathcal{L} \cap Q\mathcal{H}) \cap \mathcal{N} = \{0\}.$$

This contradiction implies that $\dim \mathcal{N} \leq \dim(Q\mathcal{H})^{\perp}$. We have $Q\mathcal{L} = (\mathcal{L} \cap Q\mathcal{H}) + Q\mathcal{N}$ and hence, being the sum of two subspaces one of which is finite dimensional, $Q\mathcal{L}$ is a subspace (see [16, Theorem I.4.12]). This proves the statement. Take $\mathcal{L}$ equal to T considered as a subspace of $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ and $Q = \operatorname{diag}\{I; P_{\mathcal{G}}\}$. Then the statement implies that $P_{\mathcal{G}}T$ is a closed linear relation in $\mathcal{K}$ and hence its restriction T_0 to $\mathcal{G}$ is a closed linear relation in $\mathcal{G}$. ☐

For the next result see [15, Lemma 2.1].

Lemma 1.2. (Gohberg-Krein (1959)) *Let a Banach space $\mathcal{B}$ be decomposed into the direct sum of a subspace $\mathcal{R}$ and a finite dimensional subspace $\mathcal{N}$: $\mathcal{B} = \mathcal{R}\dot{+}\mathcal{N}$. If $\mathcal{D}$ is a dense linear subset of $\mathcal{B}$, then:*

(i) *$\mathcal{D} \cap \mathcal{R}$ is dense in $\mathcal{R}$, and*
(ii) *there is subspace $\mathcal{N}' \subset \mathcal{D}$ with $\dim \mathcal{N}' = \dim \mathcal{N}$ such that $\mathcal{B} = \mathcal{R}\dot{+}\mathcal{N}'$.*

Corollary 1.3. *Let $\mathcal{K}$ be a Krein space and let $\mathcal{G}$ be a Krein subspace of $\mathcal{K}$ of finite codimension.*

(i) *If T is a densely defined linear relation (operator) in $\mathcal{K}$, then its compression T_0 to $\mathcal{G}$ is a densely defined linear relation (operator) in $\mathcal{G}$.*
(ii) *Let T be a closed densely defined operator in a Krein space $\mathcal{K}$. Then T is bounded in $\mathcal{K}$ if and only if T_0 is bounded in $\mathcal{G}$.*

Proof. By Lemma 1.2(i) with $\mathcal{B} = \mathcal{K}$, $\mathcal{R} = \mathcal{G}$ and $\mathcal{D} = \operatorname{dom} T$, we have $\operatorname{dom} T_0 = \mathcal{D} \cap \mathcal{G}$ is dense is $\mathcal{G}$. This proves (i).

The "only if" part of (ii) is straightforward. For the "if" part assume that T is a closed densely defined operator in $\mathcal{K}$ and that T_0 is bounded in $\mathcal{G}$. By Lemma 1.2 there exists a finite dimensional subspace $\mathcal{F} \subset \operatorname{dom} T$ such that $\mathcal{K} = \mathcal{G} \dot{+} \mathcal{F}$. With respect to this direct sum T has the following block operator matrix representation

$$T = \begin{bmatrix} T_0 & U \\ V & W \end{bmatrix} : \begin{bmatrix} \mathcal{G} \\ \mathcal{F} \end{bmatrix} \to \begin{bmatrix} \mathcal{G} \\ \mathcal{F} \end{bmatrix},$$

where $U : \mathcal{F} \to \mathcal{G}$, $V : (\operatorname{dom} T) \cap \mathcal{G} \to \mathcal{F}$ and $W : \mathcal{F} \to \mathcal{F}$. As linear operators defined on $\mathcal{F}$, U and W are bounded. Since T is closed and T_0 is bounded, V is closed. As V is of finite rank it must be bounded. Thus all four blocks in the block operator representation of T are bounded, yielding that T is bounded as well. □

Recall the following definitions for a linear relation T in a Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,])$. T is *dissipative* if $\operatorname{Im}[g, f] \geq 0$ for all $\{f; g\} \in T$, and it is *maximal dissipative* if it is dissipative and not properly contained in another dissipative linear relation in $\mathcal{K}$. T is *symmetric* if $\operatorname{Im}[g, f] = 0$ for all $\{f; g\} \in T$, or, equivalently, $T \subset T^*$ and it is *maximal symmetric* if it is symmetric and not properly contained in another symmetric linear relation in $\mathcal{K}$. Finally, T is *self-adjoint* if $T = T^*$. These metric notions are related in the following way (see [5, Statement 2.3.7]).

Lemma 1.4. *In the above notation:*

(i) *T is symmetric $\Leftrightarrow$ T and $-T$ are dissipative.*

(ii) *T is maximal symmetric $\Leftrightarrow$ T is symmetric and T or $-T$ is maximal dissipative.*

(iii) *T is self-adjoint $\Leftrightarrow$ T and $-T$ are maximal dissipative.*

Finally, we recall that a maximal dissipative linear relation T is closed and that it is densely defined if and only if T is an operator, that is, $T(0) = \{0\}$, where $T(0) := \{g : \{0; g\} \in T\}$ is the multi-valued part of T.

The following theorems concern the metric properties that are shared by T and its finite-codimensional compression T_0. They are taken from [28, Lemma 1], [26, Theorem 2.2], [3, Theorems 3.3 and 3.4], and [4, Theorems 3.1 and 4.1].

Theorem 1.5. (Stenger (1968)) *If T is a self-adjoint operator in a Hilbert space $\mathcal{H}$, then T_0 is a self-adjoint operator in $\mathcal{G}$.*

Theorem 1.6. (Nudelman (2011)) *If T is a maximal dissipative operator in a Hilbert space $\mathcal{H}$, then T_0 is a maximal dissipative operator in $\mathcal{G}$.*

Theorem 1.7. (Azizov-Dijksma (2012)) *Let T be a closed densely defined dissipative (symmetric) operator in a Hilbert space $\mathcal{H}$. Then T is maximal dissipative (maximal symmetric, self-adjoint) if and only if T_0 is maximal dissipative (maximal symmetric, self-adjoint) in $\mathcal{G}$.*

Theorem 1.8. (Azizov-Dijksma-Wanjala (2013)) *Let T be a closed dissipative (symmetric) linear relation in a Krein space $\mathcal{K}$. Then T is maximal dissipative* (*maximal symmetric, self-adjoint*) *if and only if* $T(0) = T^*(0)$ *and* T_0 *is maximal dissipative* (*maximal symmetric, self-adjoint*) *in* $\mathcal{G}$.

Strictly speaking the maximal symmetric cases in the preceding two theorems are not considered in the papers [3] and [4], but they follow from the dissipative cases in these theorems. By Lemma 1.4, Theorem 1.8 $\Rightarrow$ Theorem 1.7 $\Rightarrow$ Theorem 1.6 $\Rightarrow$ Theorem 1.5.

Lemma 1.1, Corollary 1.3 and Theorems 1.5, 1.6, 1.7, 1.8 are answers to easily formulated questions such as: Is the finite-codimensional compression of a closed linear relation again closed?, etc. In the same vein in [4] the following two questions were raised and answered:

(a) Is the finite-codimensional compression of the soft (hard) extension of a nonnegative relation S in a Hilbert space the soft (hard) extension of the finite-codimensional compression S_0 of S?
(b) Is the finite-codimensional compression of the minimal self-adjoint dilation of a maximal dissipative relation T equal to the minimal self-adjoint dilation of the finite-codimensional compression T_0 of T?

In this note our results are centered around the following five questions:

(1) If a self-adjoint operator in a Krein space has an invariant maximal nonnegative subspace, does its finite-codimensional compression, which by Theorem 1.8 is self-adjoint, have one? (See Sect. 2.)
(2) Is the finite-codimensional compression of a definitizable operator in a Krein space definitizable? (See Sect. 3.)
(3) If a self-adjoint operator in a Krein space is similar to a Hilbert space self-adjoint operator, does its finite-codimensional compression have the same property? (See Sect. 4.)
(4) If both are definitizable and if the point ∞ is a singular critical point for one of them, is ∞ a singular critical point for the other? (See Sect. 4.)
(5) What is the connection between the defect index of a closed symmetric linear relation in a Hilbert space and the defect index of its finite-codimensional compression? (See Sect. 5.)

We do not claim that we answer these questions in full, but each section in this note contains new results which lead to at least a partial answer to these questions.

Some of the results mentioned in this paper have been presented at the one-day workshop held at Stockholm University on September 28, 2015, in honor of Heinz Langer.

2. Invariant Maximal Nonnegative Subspaces

Let A be a self-adjoint operator in a Krein space $\mathcal{K}$ and let A_0 be the compression of A to a Krein subspace $\mathcal{G}$ of $\mathcal{K}$ with finite $\operatorname{codim}\mathcal{G}$. We consider the following question:

Question 2.1. *If* A *has an invariant maximal nonnegative subspace in* $\mathcal{K}$, *does* A_0 *have an invariant maximal nonnegative subspace in* $\mathcal{G}$?

The question is not relevant if $\mathcal{K}$ is a Pontryagin space. This follows from Pontryagin's theorem which states that every self-adjoint operator in a Pontryagin space has an invariant maximal nonnegative subspace (see [27] and, for a different proof, [25, Theorem 12.1′]).

In the rest of this section we assume that $\mathcal{K}$ is a Krein space with a fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ in which $\dim \mathcal{K}^+ = \infty$ and $\dim \mathcal{K}^- = \infty$; this then holds for every fundamental decomposition of $\mathcal{K}$. In this case we do not know the answer to Question 2.1. But we can answer a related question about a sufficient condition that ensures the existence of an invariant maximal nonnegative subspace. This condition is due to Langer [21, Satz II.2 and Bemerkung p. 80] and [22]. By $\mathfrak{S}_\infty(\mathcal{L}, \mathcal{M})$ we denote the class of compact operators from the Hilbert space $\mathcal{L}$ to the Hilbert space $\mathcal{M}$. We also use the following notation: if $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ is a Krein space with fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$, then $|\mathcal{K}^-|$ stands for the linear space $\mathcal{K}^-$ equipped with the inner product $-[\,\cdot\,,\cdot\,]$. Hence $\mathcal{K}^+$ and $|\mathcal{K}^-|$ are Hilbert spaces.

Theorem 2.2. (Langer (1962)) *Let A be a self-adjoint operator in $\mathcal{K}$. If there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ with projections $P^\pm : \mathcal{K} \to \mathcal{K}^\pm$ such that*

$$(*) \quad \mathcal{K}^+ \subset \operatorname{dom} A \quad \textit{and} \quad (C) \quad P^+AP^- \in \mathfrak{S}_\infty(|\mathcal{K}^-|, \mathcal{K}^+),$$

then A has an invariant maximal nonnegative subspace.

In particular, $(*)$ holds if A is bounded and for this case the next theorem provides the answer to the question:

Question 2.3. *If A satisfies (C), does A_0 satisfy (C)?*

Theorem 2.4. *If A is a bounded self-adjoint operator in $\mathcal{K}$, then A has property (C) in $\mathcal{K}$ if and only if A_0 has property (C) in $\mathcal{G}$.*

The theorem implies that if either A or A_0 has an invariant maximal nonnegative subspace thanks to having property (C), then the other one also has an invariant maximal nonnegative subspace. To prove the theorem we apply the following lemma.

Lemma 2.5. *Let A be a bounded self-adjoint operator in $\mathcal{K}$. Decompose A as a block operator matrix according to the fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ of $\mathcal{K}$:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{K}^+ \\ \mathcal{K}^- \end{bmatrix} \to \begin{bmatrix} \mathcal{K}^+ \\ \mathcal{K}^- \end{bmatrix}. \tag{2.1}$$

Then the following statements are equivalent:

(1) *There exists a fundamental decomposition of $\mathcal{K}$ for which A satisfies (C).*
(2) *There is a uniform contraction $K : \mathcal{K}^+ \to |\mathcal{K}^-|$ (that is, $\|K\| < 1$), such that*

$$S(A;K) := KA_{11} + KA_{12}K + A_{12}^* - A_{22}K \in \mathfrak{S}_\infty(\mathcal{K}^+, |\mathcal{K}^-|). \tag{2.2}$$

The operator A_{12}^* in formula (2.2) and in the proof below is the adjoint of A_{12} considered as a mapping from $|\mathcal{K}^-|$ to $\mathcal{K}^+$ and then $A_{21} = -A_{12}^*$.

Proof of Lemma 2.5. By [5, Theorem 1.8.17] there exists a one-to-one correspondence between the fundamental decompositions $\mathcal{K} = \mathcal{K}_1^+ \oplus \mathcal{K}_1^-$ of $\mathcal{K}$ with corresponding projections $P_1^\pm : \mathcal{K} \to \mathcal{K}_1^\pm$ and the uniform contractions $K : \mathcal{K}^+ \to |\mathcal{K}^-|$ given by the block operator matrix representation:

$$P_1^+ = \begin{bmatrix} (I-K^*K)^{-1} & -K^*(I-KK^*)^{-1} \\ K(I-K^*K)^{-1} & -KK^*(I-KK^*)^{-1} \end{bmatrix} : \begin{bmatrix} \mathcal{K}^+ \\ |\mathcal{K}^-| \end{bmatrix} \to \begin{bmatrix} \mathcal{K}^+ \\ |\mathcal{K}^-| \end{bmatrix},$$

$$P_1^- = I - P_1^+ = \begin{bmatrix} -K^*K(I-K^*K)^{-1} & K^*(I-KK^*)^{-1} \\ -K(I-K^*K)^{-1} & (I-KK^*)^{-1} \end{bmatrix}.$$

It follows that $P_1^+AP_1^- = G^{-1}BG^{-1}$, where G is the invertible operator

$$G = \begin{bmatrix} I-K^*K & 0 \\ 0 & I-KK^* \end{bmatrix}$$

and

$$B = \begin{bmatrix} I & -K^* \\ K & -KK^* \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ -A_{12}^* & A_{22} \end{bmatrix} \begin{bmatrix} -K^*K & K^* \\ -K & I \end{bmatrix} = \begin{bmatrix} -B_{12}K & B_{12} \\ -KB_{12}K & KB_{12} \end{bmatrix}$$

with

$$B_{12} := A_{11}K^* + K^*A_{12}^*K^* + A_{12} - K^*A_{22} = S(A;K)^*.$$

Since the class of compact operators is closed under multiplication from the left and from the right by bounded operators and under taking adjoints, we conclude that $P_1^+AP_1^- \in \mathfrak{S}_\infty(|\mathcal{K}^-|, \mathcal{K}^+)$ if and only if there is a uniform contraction $K : \mathcal{K}^+ \to |\mathcal{K}^-|$ such that $S(A;K) \in \mathfrak{S}_\infty(\mathcal{K}^+, |\mathcal{K}^-|)$. Hence the statements in the lemma are equivalent. □

Proof of Theorem 2.4. Choose fundamental decompositions: $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^-$ and $\mathcal{G}^\perp = \mathcal{F}^+ \oplus \mathcal{F}^-$ and set $\mathcal{K}^+ = \mathcal{F}^+ \oplus \mathcal{G}^+$, $\mathcal{K}^- = \mathcal{G}^- \oplus \mathcal{F}^-$. Then relative to the fundamental decomposition

$$\mathcal{K} = \mathcal{F}^+ \oplus \mathcal{G}^+ \oplus \mathcal{G}^- \oplus \mathcal{F}^-$$

A has the block operator matrix representation

$$A = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix} : \begin{bmatrix} \mathcal{F}^+ \\ \mathcal{G}^+ \\ \mathcal{G}^- \\ \mathcal{F}^- \end{bmatrix} \to \begin{bmatrix} \mathcal{F}^+ \\ \mathcal{G}^+ \\ \mathcal{G}^- \\ \mathcal{F}^- \end{bmatrix}.$$

Thus, if A is decomposed as in (2.1), then the A_{ij}'s have the block operator matrix representation

$$A_{11} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} B_{13} & B_{14} \\ B_{23} & B_{24} \end{bmatrix}, \quad A_{21} = \begin{bmatrix} B_{31} & B_{32} \\ B_{41} & B_{42} \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} B_{33} & B_{34} \\ B_{43} & B_{44} \end{bmatrix}$$

and the compression A_0 has the block operator matrix representation

$$A_0 = \begin{bmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{bmatrix} : \begin{bmatrix} \mathcal{G}^+ \\ \mathcal{G}^- \end{bmatrix} \to \begin{bmatrix} \mathcal{G}^+ \\ \mathcal{G}^- \end{bmatrix}.$$

If $K : \mathcal{K}^+ \to |\mathcal{K}^-|$ is any bounded operator with block operator matrix representation

$$K = \begin{bmatrix} K_{31} & K_{32} \\ K_{41} & K_{42} \end{bmatrix} : \begin{bmatrix} \mathcal{F}_+ \\ \mathcal{G}_+ \end{bmatrix} \to \begin{bmatrix} |\mathcal{G}_-| \\ |\mathcal{F}_-| \end{bmatrix},$$

then after some calculations we find that the block operator matrix representation of $S(A;K)$ in (2.2) takes the form

$$S(A;K) = \begin{bmatrix} * & S(A_0;K_{32}) + * \\ * & * \end{bmatrix} \tag{2.3}$$

in which the nonspecified operators indicated by a $*$ are finite dimensional and hence compact. Now we apply Lemma 2.5 four times:

If A satisfies (C), then there exists a uniform contraction K such that $S(A;K)$ is compact. It follows that $\|K_{32}\| \leq \|K\| < 1$, that is, K_{32} is a uniform contraction, and that, on account of (2.3), $S(A_0;K_{32})$ is compact. Hence A_0 satisfies (C).

Conversely, if A_0 satisfies (C), then there is a uniform contraction $K_{32} : \mathcal{G}^+ \to |\mathcal{G}^-|$ such that $S(A_0;K_{32})$ is compact. Define $K : \mathcal{K}^+ \to |\mathcal{K}^-|$ by

$$K = \begin{bmatrix} 0 & K_{32} \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{F}_+ \\ \mathcal{M}_+ \end{bmatrix} \to \begin{bmatrix} |\mathcal{M}_-| \\ |\mathcal{F}_-| \end{bmatrix}.$$

Then $\|K\| = \|K_{32}\| < 1$ and, by (2.3), $S(A;K)$ is compact, that is, A has property (C). □

3. Definitizable Operators

The following definition is due to Langer [23,24]: An operator A in a Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ is called *definitizable* if it is self-adjoint, its resolvent set $\rho(A) \neq \emptyset$ and there exists a real polynomial p of degree n, say, such that $[p(A)x, x] \geq 0$, $x \in \operatorname{dom}(A^n)$. A polynomial with this property is called a *definitizing polynomial* for A. It is shown in [17,23] that for a definitizable operator A the spectrum $\sigma(A)$ of A is symmetric with respect to the real axis and the set $\sigma(A)\backslash\mathbb{R}$ is either empty or finite.

Let A be a self-adjoint operator in a Krein space $\mathcal{K}$ and let A_0 be the compression of A to a Krein subspace $\mathcal{G}$ of $\mathcal{K}$ with finite $\operatorname{codim}\mathcal{G}$. In this section we consider the following question:

Question 3.1. *If A is definitizable in $\mathcal{K}$, is A_0 also definitizable in $\mathcal{G}$?*

When $\mathcal{K}$ is a Pontryagin space the question is not relevant, because every self-adjoint operator in a Pontryagin space is definitizable, see [25, Theorem 6.1, Note 2] and [24, pp. 11, 12]. In what follows we shall assume that for some and then for every fundamental decomposition $\mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^-$ of the Krein space $\mathcal{K}$ both subspaces $\mathcal{K}^+$ and $\mathcal{K}^-$ are infinite dimensional. In this case there exist examples which show that a self-adjoint operator on

$(\mathcal{K}, [\,\cdot\,,\cdot\,])$ may have an empty resolvent set, see, for instance, the example on [13, p. 6]. Now an answer to Question 3.1 almost immediately follows from [2, Theorem 2.2] or [1, Theorem 3.1] which we repeat below.

As in [2], for a closed symmetric relation S in a Krein space $\mathcal{K}$ which possesses a self-adjoint extension A in $\mathcal{K}$ we define its defect def S by $\operatorname{def} S = \dim(A/S)$. Notice that the number $\operatorname{def} S$ is either infinite or a nonnegative integer and it is independent of the choice of the self-adjoint extension A of S.

Theorem 3.2. (Azizov-Behrndt-Trunk (2008)) *Let A and A_1 be self-adjoint operators in a Krein space and assume that the symmetric operator*

$$S = A \cap A_1 = \big\{\{x; Ax\} \,:\, x \in \operatorname{dom} A \cap \operatorname{dom} A_1, Ax = A_1 x\big\}$$

has a finite defect. Then A is definitizable if and only if A_1 is definitizable.

For a self-adjoint operator A and its compression A_0 we set

$$A_1 := \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{G} \\ \mathcal{G}^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{G} \\ \mathcal{G}^{\perp} \end{bmatrix}. \tag{3.1}$$

Then $\rho(A_1) = \rho(A_0)\backslash\{0\}$, hence $\rho(A_1) \neq \emptyset$ if and only if $\rho(A_0) \neq \emptyset$. Also, A_1 is definitizable if and only if A_0 is definitizable. It follows that Theorem 3.2 implies part (i) of the following theorem, which is an answer to Question 3.1.

Theorem 3.3. *Let A be a self-adjoint operator in a Krein space $\mathcal{K}$ and let A_0 be the compression of A to a Krein subspace $\mathcal{G}$ of $\mathcal{K}$ with finite* $\operatorname{codim} \mathcal{G}$*. Define A_1 by* (3.1) *and set $S = A \cap A_1$.*

(i) *If S has finite defect, then A is definitizable in $\mathcal{K}$ if and only if A_0 is definitizable in $\mathcal{G}$.*
(ii) *In particular, if $\rho(A) \neq \emptyset$ and $\rho(A_0) \neq \emptyset$, then A is definitizable in $\mathcal{K}$ if and only if A_0 is definitizable in $\mathcal{G}$.*

The proof of part (ii) will be given below.

Theorem 3.2 is a generalization of [20, Theorem 1]:

Theorem 3.4. (Jonas-Langer (1979)) *Assume A and A_1 are self-adjoint operators in a Krein space $\mathcal{K}$ with $\rho(A) \cap \rho(A_1) \neq \emptyset$. If A is definitizable and the resolvent operators $R_A(\lambda) = (A - \lambda)^{-1}$ and $R_{A_1}(\lambda)$ of A and A_1 satisfy*

$$\dim \operatorname{ran}\big(R_A(\lambda) - R_{A_1}(\lambda)\big) = m < \infty$$

for some (*and then for all*) $\lambda \in \rho(A) \cap \rho(A_1)$*, then A_1 is definitizable.*

The crux in the proof of Theorem 3.2 is the proof of the implication

$$A \text{ definitizable and } \operatorname{def} S < \infty \Rightarrow \rho(A_1) \neq \emptyset.$$

For the rest of the proof of Theorem 3.2 one applies Theorem 3.4 after observing that, by the above implication, $\rho(A) \cap \rho(A_1) \neq \emptyset$, that the direct sum decomposition in $\mathcal{K}^2$

$$A \dot{+} (S^* \cap \lambda I) = S^*$$

holds for all $\lambda \in \rho(A)$ and that the equality

$$\ker(S^* - \lambda) = \overline{\operatorname{ran}}\big(R_A(\lambda) - R_{A_1}(\lambda)\big)$$

and the equalities

$$\begin{aligned} \dim(A/S) &= \dim(S^*/A) \\ &= \dim(S^* \cap \lambda I) \\ &= \dim\ker(S^* - \lambda) \\ &= \dim\operatorname{ran}\big(R_A(\lambda) - R_{A_1}(\lambda)\big) \end{aligned} \tag{3.2}$$

(in the sense that if one of these numbers is finite then all numbers are finite and equal) hold for all $\lambda \in \rho(A) \cap \rho(A_1)$.

The chain of equalities (3.2) is also used in the following proof.

Proof of part (ii) *of Theorem* 3.3. Assume that $\rho(A) \neq \emptyset$, $\rho(A_0) \neq \emptyset$ and that A (A_0, respecitvely) is definitizable in $\mathcal{K}$ ($\mathcal{G}$, respectively). Since the interior of the spectrum of a definitizable operator is empty, these assumptions imply that $\rho(A) \cap \rho(A_0) \neq \emptyset$. Consequently, $\rho(A) \cap \rho(A_1) \neq \emptyset$. We claim that for all $\lambda \in \rho(A) \cap \rho(A_1)$ we have

$$\dim\operatorname{ran}\left(R_A(\lambda) - R_{A_1}(\lambda)\right) \leq 2 \operatorname{codim}\mathcal{G}. \tag{3.3}$$

This claim and the chain of equalities (3.2) imply that $\operatorname{def} S = \dim(A/S) < \infty$ and hence part (ii) of the theorem follows from part (i).

It remains to prove the claim. Let $g \in \mathcal{G}$ and $h \in \mathcal{G}^\perp$. Then

$$(A_1 - \lambda)^{-1}(g + h) = (A_0 - \lambda)^{-1} g - \frac{1}{\lambda} h$$

and hence with $k = \big((A - \lambda)^{-1} + \frac{1}{\lambda}\big)h$

$$\begin{aligned} &\big((A - \lambda)^{-1} - (A_1 - \lambda)^{-1}\big)(g + h) \\ &= (A - \lambda)^{-1} g - (A_0 - \lambda)^{-1} g + k \\ &= (A - \lambda)^{-1}\big((A_0 - \lambda) - (A - \lambda)\big)(A_0 - \lambda)^{-1} g + k \\ &= -(A - \lambda)^{-1}(I - P_{\mathcal{G}})A(A_0 - \lambda)^{-1} g + k. \end{aligned}$$

The inequality (3.3) follows from the equality $\dim\operatorname{ran}(I - P_{\mathcal{G}}) = \operatorname{codim}\mathcal{G}$ and from the fact that with $h \in \mathcal{G}^\perp$ also k varies over a subspace of $\mathcal{K}$ of dimension $\leq \operatorname{codim}\mathcal{G}$. □

Since a bounded operator has a nonempty resolvent set, Theorem 3.3 and Corollary 1.3(ii) yield the following corollary.

Corollary 3.5. *Let A be a closed densely defined operator in a Krein space $\mathcal{K}$. Then A is a bounded definitizable operator in $\mathcal{K}$ if and only if A_0 is a bounded definitizable operator in $\mathcal{G}$.*

In case A is unbounded the following theorem provides a sufficient condition for A_0 to be definitizable.

Theorem 3.6. *If A is a definitizable operator in $\mathcal{K}$ with definitizing polynomial $p(z)$ of degree $n \geq 1$ and $\mathcal{G}^\perp \subset \operatorname{dom} A^{n-1}$, then A_0 is a definitizable operator in $\mathcal{G}$.*

In the proof of the theorem we use the following result, see [3, Theorem 2.2].

Lemma 3.7. *Let T be a linear operator on a Banach space $\mathcal{B}$. Let P be a projection in $\mathcal{B}$ such that* $\operatorname{codim}\operatorname{ran} P$ *is finite and let T_0 be the compression of T to* $\operatorname{ran} P$*, that is, $T_0 = PT|_{\operatorname{ran} P \cap \operatorname{dom} T}$. Then*

$$\rho(T) \backslash \sigma_p(T_0) \subset \rho(T_0).$$

Proof of Theorem 3.6. Assume that A is a definitizable operator in $\mathcal{K}$ with definitizing polynomial $p(z)$ of degree $n \geq 1$ and that $\mathcal{G}^\perp \subset \operatorname{dom} A^{n-1}$.

If $n = 1$, then we may assume without loss of generality that $A \geq 0$. Then also $A_0 \geq 0$ and, by [5, Corollary 2.3.25], $\sigma_p(A_0) \subset \mathbb{R}$. Since A is definitizable $\rho(A) \backslash \mathbb{R} \neq \emptyset$, so Lemma 3.7 implies that $\rho(A_0) \neq \emptyset$. Hence A_0 is definitizable.

From now on we assume $n \geq 2$. Set $m = \dim \mathcal{G}^\perp$ and consider the subspace

$$\mathcal{M} := \operatorname{span}\bigl\{ A^k \mathcal{G}^\perp : k \in \{0, \ldots, n-1\} \bigr\}.$$

Then $\mathcal{M}$ is finite dimensional and $\dim \mathcal{M} \leq mn$.

By Theorem 3.3 we only have to show that $\rho(A_0) \neq \emptyset$. We proceed by contradiction and assume $\rho(A_0) = \emptyset$. Now Lemma 3.7 yields that $\rho(A) \subset \sigma_p(A_0)$ and, since A is definitizable, $\sigma_p(A_0)$ contains the set $\mathbb{C} \backslash \mathbb{R}$ except for at most finitely many points.

Let r be an integer $\geq mn+1$. Let λ_{ij}, $i, j \in \{1, 2, \ldots, r\}$, be r^2 mutually distinct numbers from $\sigma_p(A_0) \cap \mathbb{C}^+$ such that $p(\lambda_{ij}) \neq 0$. Let $x_{ij} \neq 0$ be corresponding eigenvectors of A_0: $A_0 x_{ij} = \lambda_{ij} x_{ij}$, $i, j \in \{1, 2, \ldots, r\}$. As eigenvectors corresponding to distinct eigenvalues the vectors x_{ij}, $i, j \in \{1, 2, \ldots, r\}$, are linearly independent.

Denote by $\mathcal{X}$ the linear span of the x_{ij}'s. Then $\mathcal{X}$, being a subspace of

$$\operatorname{span}\bigl\{ \ker(A_0 - \lambda_{ij}) \,:\, i, j \in \{1, \ldots, r\} \bigr\},$$

is a neutral subspace of $\mathcal{K}$ with $\dim \mathcal{X} = r^2$. Since $\mathcal{X}$ is spanned by eigenvectors of A_0 we have $\mathcal{X} \subset \operatorname{dom} A_0$ and $\mathcal{X}$ is invariant under A_0. Consequently $p(A_0)$ is defined on $\mathcal{X}$, $\mathcal{X}$ is invariant under $p(A_0)$ and $p(A_0)|_{\mathcal{X}} = p\bigl(A_0|_{\mathcal{X}}\bigr)$. Hence, the eigenvalues of $p(A_0)|_{\mathcal{X}}$ are $p(\lambda_{ij}) \neq 0$, $i, j \in \{1, 2, \ldots, r\}$. Consequently, $p(A_0)|_{\mathcal{X}}$ is a bijection on $\mathcal{X}$.

Since $\dim \mathcal{M} < r$, for each $j \in \{1, 2, \ldots, r\}$ there exists an element $x_j \neq 0$ in the r-dimensional $\operatorname{span}\{x_{1j}, x_{2j}, \ldots, x_{rj}\}$ which is orthogonal to $\mathcal{M}$. Note that

$$x_1, x_2, \ldots, x_r \text{ are linearly independent.} \tag{3.4}$$

We shall derive a contradiction with this property of the x_j's.

First we prove that

$$\forall\, j \in \{1, 2, \ldots, r\} \quad \exists\, y_j \in \mathcal{M} \,:\, p(A) x_j = p(A_0) x_j + y_j. \tag{3.5}$$

For this we use induction and with the help of the block operator matrix decomposition

$$A = \begin{bmatrix} A_0 & A_{11} \\ A_{21} & A_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{G} \\ \mathcal{G}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{G} \\ \mathcal{G}^\perp \end{bmatrix}$$

we show that for all $s \in \{0, 1, \dots, n\}$

$$A^s x_j = A_0^s x_j + z_{s-1} + A z_{s-2} + \cdots A^{s-1} z_0,\ z_{s-1}, \dots, z_0 \in \mathcal{G}^\perp. \tag{3.6}$$

For $s = 0$ we have $x_j = x_j$ and for $s = 1$ we have $Ax_j = A_0 x_j + z_0$ with $z_0 = A_{21} x_j \in \mathcal{G}^\perp$. Assume (3.6) is true for some $s \in \{1, \dots, n-1\}$. Then with $z_s = A_{21} A_0^s x_j$

$$\begin{aligned} A^{s+1} x_j &= A A_0^s x_j + A z_{s-1} + A^2 z_{s-2} + \cdots A^s z_0 \\ &= A_0^{s+1} x_j + z_s + A z_{s-1} + A^2 z_{s-2} + \cdots A^s z_0. \end{aligned}$$

This proves (3.6) for $s+1$. Taking linear combinations of both sides of the equalities in (3.6) we obtain formula (3.5) in which y_j is a linear combination of the elements $A^s z_t$, $s + t \le n - 1$, from $\mathcal{M}$.

Next we observe that $x_j, p(A_0)x_j \in \mathcal{X}$, that $\mathcal{X}$ is neutral, that $x_j \in \mathcal{M}^\perp$ and $y_j \in \mathcal{M}$ and apply (3.5) to obtain

$$[p(A)x_j, x_j] = [p(A_0)x_j, x_j] + [y_j, x_j] = 0, \quad j = 1, 2, \dots, r.$$

Since $p(A) \ge 0$ the inner product $[p(A)\,\cdot, \cdot\,]$ is nonnegative on $\mathcal{K}$. Therefore the Cauchy-Bunyakovsky-Schwarz inequality (see [5, 1.1.16]) yields that for all $y \in \mathcal{K}$ and all $j \in \{1, \dots, r\}$ we have

$$|[p(A)x_j, y]|^2 \le [p(A)x_j, x_j][p(A)y, y] = 0.$$

This implies that $p(A)x_j = 0$ and hence, by (3.5),

$$p(A_0)x_j + y_j = 0, \quad j = 1, 2, \dots, r. \tag{3.7}$$

Since $\dim \mathcal{M} < r$, the the vectors $y_1, \dots, y_r \in \mathcal{M}$ are linear dependent. Now (3.7) yields that the vectors $p(A_0)x_j$, $j \in \{1, \dots, r\}$, are also linearly dependent. As $x_1, \dots, x_r \in \mathcal{X}$ and $p(A_0)|_{\mathcal{X}}$ is a bijection on $\mathcal{X}$, we deduce that $x_1, \dots, x_r$ are linearly dependent. This contradicts (3.4). □

Remark 3.8. The definition of definitizability also makes sense for self-adjoint linear relations in a Krein space. For the definition of definitizability, for a proof that the spectrum $\sigma(A)$ of a definitizable relation A is symmetric with respect to the real axis and that the set $\sigma(A)\backslash\mathbb{R}$ is either empty or finite and for a proof that a self-adjoint relation with a non-empty resolvent set in a Pontgryagin space is definitizable, see [12, Sections 4 and 5]. By [2, Remark 2.3], Theorem 3.2 and its corollary Theorem 3.3(i) do not hold for linear relations. Theorem 3.4 has been generalized by Behrndt [6, Theorem 2.2] not only to self-adjoint linear relations but also to the more general notion of local definitizability on certain domains Ω of $\overline{\mathbb{C}}$. Jonas [19, Theorem 4.7] showed that if $\Omega = \overline{\mathbb{C}}$, then this generalized notion coincides with definitizability as defined here. Finally, the proof of Theorem 3.3(ii) can easily be adapted to the case of linear relations. Indeed, using the same notation, we have that if $(A_0 - \lambda)^{-1} g = f$, say, then $\{f; \lambda f + g\} \in A_0 = P_{\mathcal{G}} A|_{\mathcal{G}}$ and hence there is an

$x \in \mathcal{G}^\perp$ such that $\{f; \lambda f + g + x\} \in A$, that is, $(A-\lambda)^{-1}g = f - (A-\lambda)^{-1}x$. It follows that

$$\big((A-\lambda)^{-1} - (A_1 - \lambda\,)^{-1}\big)(g+h) = (A-\lambda)^{-1}g - (A_0-\lambda)^{-1}g + k$$
$$= -(A-\lambda)^{-1}x + k.$$

This implies (3.3).

4. Similarity and Critical Points

We begin with the question:

Question 4.1. *If a self-adjoint operator in a Krein space is similar to a self-adjoint operator in a Hilbert space, does its finite-codimensional compressions have the same property?*

The answer to this question is negative as the following simple example shows.

Example 4.2. Let $\mathbb{C}^4$ be equipped with the Euclidean inner product $\langle\,\cdot\,,\cdot\,\rangle$ and let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal basis of $\mathbb{C}^4$. Let $\mathcal{H}_1 = \operatorname{span}\{e_1, e_2\}$ and $\mathcal{H}_2 = \operatorname{span}\{e_3, e_4\}$ and let J be the self-adjoint involution on $(\mathbb{C}^4, \langle\,\cdot\,,\cdot\,\rangle)$ whose matrix representation with respect to the decomposition $\mathbb{C}^4 = \mathcal{H}_1 \oplus \mathcal{H}_2$ is $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Then $[\,\cdot\,,\cdot\,] = \langle J\cdot,\cdot\,\rangle$ is an indefinite inner product on $\mathbb{C}^4$ and $(\mathbb{C}^4, [\,\cdot\,,\cdot\,])$ is a Krein space.

Let A be the operator on $\mathbb{C}^4$ whose matrix representation with respect to the decomposition $\mathbb{C}^4 = \mathcal{H}_1 \oplus \mathcal{H}_2$ is as follows:

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} \quad \text{with } A_{12} = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A_{21} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}.$$

Then, since A_{12} and A_{21} are symmetric matrices, A is selfadjoint in the Krein space $(\mathbb{C}^4, [\,\cdot\,,\cdot\,])$. The spectrum of A coincides with the spectrum of the quadratic pencil

$$L(\lambda) = \lambda^2 - D \quad \text{with } D = A_{21}A_{12} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$$

(obtained by considering the Schur complement of $A - \lambda$). The latter consists of 4 different real numbers, hence A is similar to a self-adjoint operator in $(\mathbb{C}^4, \langle\,\cdot\,,\cdot\,\rangle)$.

The subspace $\mathcal{H}_3 = \operatorname{span}\{e_1, e_3\}$ is a regular subspace of $(\mathbb{C}^4, [\,\cdot\,,\cdot\,])$ and its orthogonal complement in $(\mathbb{C}^4, [\,\cdot\,,\cdot\,])$ is $\mathcal{H}_4 = \operatorname{span}\{e_2, e_4\}$. The compression of A to $\mathcal{H}_3$ is

$$A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which is not similar to a self-adjoint operator on a Hilbert space.

Similarity of a Krein space self-adjoint operator to a Hilbert space self-adjoint operator is characterized by the following theorem. It will be applied in giving a partial answer to Question 4.5 below.

Theorem 4.3. *Let $(\mathcal{K},[\,\cdot\,,\cdot\,])$ be a Krein space and let J be a fundamental symmetry on $(\mathcal{K},[\,\cdot\,,\cdot\,])$. Let S be a self-adjoint operator in $(\mathcal{K},[\,\cdot\,,\cdot\,])$. Then S is similar to a self-adjoint operator in the Hilbert space $(\mathcal{K},[J\cdot\,,\cdot\,])$ if and only if there exists a fundamental symmetry on $(\mathcal{K},[\,\cdot\,,\cdot\,])$ which commutes with S.*

That the self-adjoint operator A in the Krein space $(\mathbb{C}^4,[\,\cdot\,,\cdot\,])$ in Example 4.2 is similar to a self-adjoint operator in the Hilbert space $(\mathbb{C}^4,\langle\,\cdot\,,\cdot\,\rangle)$ now also follows from the just stated theorem. To see this we observe that A commutes with the matrix

$$J_1 = \frac{1}{\sqrt{5+2\sqrt{2}}}\begin{bmatrix} 0 & 0 & 3+\sqrt{2} & -2 \\ 0 & 0 & -2 & 2+\sqrt{2} \\ 1+\sqrt{2} & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 1+\frac{3}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

and that J_1 is a fundamental symmetry in the Krein space $(\mathbb{C}^4,[\,\cdot\,,\cdot\,])$ because it is idempotent and JJ_1 is a positive self-adjoint matrix. The matrix J_1 was obtained using Mathematica but its asserted properties can easily be verified by direct calculations. Note that A is a self-adjoint operator in the positive definite inner product $\langle JJ_1\cdot,\cdot\,\rangle$, that is $JJ_1A = A^*JJ_1$.

Proof of Theorem 4.3. In this proof the superscript * denotes the adjoint of an operator in the Hilbert space $(\mathcal{K},\langle\,\cdot\,,\cdot\,\rangle)$, where $\langle\,\cdot\,,\cdot\,\rangle = [J\cdot\,,\cdot\,]$. First assume that J_1 is a fundamental symmetry on $(\mathcal{K},[\,\cdot\,,\cdot\,])$ which commutes with S, that is, $J_1 \operatorname{dom} S = \operatorname{dom} S$ and $J_1Sx = SJ_1x$ for all $x \in \operatorname{dom} S$. As S is self-adjoint in $(\mathcal{K},[\,\cdot\,,\cdot\,])$, this assumption implies that S is self-adjoint in the inner product $\langle\,\cdot\,,\cdot\,\rangle_1 = [J_1\cdot\,,\cdot\,]$. Since $\langle JJ_1\cdot\,,\cdot\,\rangle = \langle\,\cdot\,,\cdot\,\rangle_1$, the operator JJ_1 is self-adjoint and uniformly positive, and with $T = \sqrt{JJ_1}$ the operator TST^{-1} is self-adjoint in $(\mathcal{K},\langle\,\cdot\,,\cdot\,\rangle)$.

Now assume that S is self-adjoint in $(\mathcal{K},[\,\cdot\,,\cdot\,])$ and similar to a self-adjoint operator in $(\mathcal{K},\langle\,\cdot\,,\cdot\,\rangle)$. Then there exists a bounded and boundedly invertible operator T on $\mathcal{K}$ such that S is self-adjoint in the inner product $(x,y) = \langle Tx,Ty\rangle$, $x,y \in \mathcal{K}$. Consequently, TST^{-1} is self-adjoint in $(\mathcal{K},\langle\,\cdot\,,\cdot\,\rangle)$. Let $U = (S-i)(S+i)^{-1}$ be the Cayley transform of S. Since S is self-adjoint in both $(\mathcal{K},(\,\cdot\,,\cdot\,))$ and $(\mathcal{K},[\,\cdot\,,\cdot\,])$, the operator U is unitary in both $(\mathcal{K},(\,\cdot\,,\cdot\,))$ and $(\mathcal{K},[\,\cdot\,,\cdot\,])$. Also, $V = TUT^{-1}$ is unitary in $(\mathcal{K},\langle\,\cdot\,,\cdot\,\rangle)$, that is, $V^{-*} = V$. Since U is unitary in $(\mathcal{K},[\,\cdot\,,\cdot\,])$ we have

$$J = U^*JU = T^*V^*T^{-*}JT^{-1}VT.$$

Set $G = T^{-*}JT^{-1}$. Then the preceding identity yields $G = V^*GV$ and consequently $VG = GV$. The operator G is self-adjoint, bounded and boundedly invertible in $(\mathcal{K},\langle\,\cdot\,,\cdot\,\rangle)$. Since G commutes with V, the operator $\operatorname{sgn} G$ also commutes with V. Therefore, the involution $J_1 = T^{-1}(\operatorname{sgn} G)T$ commutes with U. Next we show that J_1 is a fundamental symmetry on $(\mathcal{K},[\,\cdot\,,\cdot\,])$. Let $x \in \mathcal{K}$ be arbitrary and calculate

$$\begin{aligned}[J_1x,x] &= \big\langle (\operatorname{sgn} G)Tx, T^{-*}Jx\big\rangle \\ &= \big\langle (\operatorname{sgn} G)Tx, GTx\big\rangle \\ &= \big\langle T^*|G|Tx, x\big\rangle.\end{aligned}$$

Since the operator $T^*|G|T$ is self-adjoint, bounded and boundedly invertible in $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle)$, the inner product $[J_1\cdot,\cdot]$ is a Hilbert space inner product on $\mathcal{K}$. Hence J_1 is a fundamental symmetry on $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ which commutes with U. Consequently J_1 commutes with S, as well. □

The definitions of what is a regular or singular critical point for a definitizable operator are due to Langer in [23], see also [24]. They are given in terms of the spectral function of the definitizable operator. In the cited publications equivalent formulations can be found. We give another characterization and show in the "Appendix" that it is equivalent to the one in [24]. We only consider the point ∞.

Let A be a definitizable operator in a Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,])$. Then A admits a diagonal form (D) by which we mean that A can be represented as a diagonal operator $A = \operatorname{diag}\{A_b; A_\infty\}$ relative to an orthogonal decomposition $\mathcal{K} = \mathcal{K}_b \oplus \mathcal{K}_\infty$ into Krein subspaces $\mathcal{K}_b$ and $\mathcal{K}_\infty$ of $\mathcal{K}$ which are invariant under A and such that

$$\mathcal{K}_b \subset \operatorname{dom} A \quad \text{and} \quad 0 \in \rho(A_\infty).$$

The requirements imply that $A_b = A|_{\mathcal{K}_b}$ is a bounded definitizable operator: $\operatorname{dom} A_b = \mathcal{K}_b \to \mathcal{K}_b$ and that $A_\infty = A|_{\mathcal{K}_\infty}$ is a densely defined boundedly invertible definitizable operator: $\operatorname{dom} A_\infty = \mathcal{K}_\infty \cap \operatorname{dom} A \to \mathcal{K}_\infty$. The point ∞ is *not a critical point of* A if $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ can be chosen such that it is a Hilbert space or an anti-Hilbert space; otherwise it is called a *critical point of* A. If ∞ is a critical point of A, then it is called a *regular critical point* if $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ can be chosen so that it has a fundamental symmetry which commutes with A_∞ and A_∞ is uniformly definite on $\mathcal{K}_\infty$; otherwise it is called a *singular critical point*.

Reference [10, Theorem 3.2] contains three equivalent formulations of the statement that ∞ is a not a singular critical point, that is, ∞ is a regular critical point or not a critical point at all. A fourth characterization in the same spirit is given by the next theorem.

Theorem 4.4. *Let* $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ *be a Krein space and let* J *be a fundamental symmetry on* $(\mathcal{K}, [\,\cdot\,,\cdot\,])$. *Let* A *be a definitizable operator in* $(\mathcal{K}, [\,\cdot\,,\cdot\,])$. *Then* ∞ *is not a singular critical point of* A *if and only if there exists an operator* S *which is self-adjoint in* $(\mathcal{K}, [\,\cdot\,,\cdot\,])$, *similar to a self-adjoint operator in the Hilbert space* $(\mathcal{K}, [J\cdot\,,\cdot\,])$ *and such that* $\operatorname{dom} S = \operatorname{dom} A$.

Proof. First assume that A is a definitizable operator and that ∞ is not a singular critical point of A. Then, by Theorem 6.5 in the "Appendix", A has a diagonalization (D) such that A_∞ is similar to a self-adjoint operator in the Hilbert space $(\mathcal{K}_\infty, [J_1\cdot\,,\cdot\,])$, where J_1 is any fundamental symmetry on $\mathcal{K}_\infty$. Define $S = \operatorname{diag}\{J_b; A_\infty\}$, where J_b is a fundamental symmetry on $\mathcal{K}_b$. Then S is self-adjoint in $(\mathcal{K}, [\,\cdot\,,\cdot\,])$, $\widehat{J} := \operatorname{diag}\{J_b; J_1\}$ is a fundamental symmetry on

$\mathcal{K}$ and S is a similar to a self-adjoint operator in the Hilbert space $(\mathcal{K}, [\widehat{J}\cdot, \cdot])$. It follows that USU^{-1} is self-adjoint in the Hilbert space $(\mathcal{K}, [J\cdot, \cdot])$, where U is the unitary operator on $(\mathcal{K}, [\cdot, \cdot])$ such that $J = U\widehat{J}U^{-1}$. Finally, $\operatorname{dom} S = \mathcal{K}_b \oplus \operatorname{dom} A_\infty = \operatorname{dom} A$.

Next assume that A is a definitizable operator in $(\mathcal{K}, [\cdot, \cdot])$ and there exists an operator S which is self-adjoint in $(\mathcal{K}, [\cdot, \cdot])$, similar to a self-adjoint operator in $(\mathcal{K}, [J\cdot, \cdot])$ and such that $\operatorname{dom} S = \operatorname{dom} A$. By Theorem 4.3 there exists a fundamental symmetry J_1 which commutes with S. In particular $J_1 \operatorname{dom} S = \operatorname{dom} S$ and thus $J_1 \operatorname{dom} A = \operatorname{dom} A$. By [10, Theorem 3.2], ∞ is not a singular critical point of A. □

Question 4.5. *Let A be a self-adjoint operator in a Krein space $\mathcal{K}$ and let A_0 be the compression of A to a Krein subspace $\mathcal{G}$ of $\mathcal{K}$ with finite* $\operatorname{codim} \mathcal{G}$. *Assume A and A_0 are definitizable. If ∞ is a singular critical point for one, is it also a singular critical point for the other?*

The formulation of this question is very general and we have no complete answer to it. A partial answer is provided by Theorem 4.4.

Corollary 4.6. *Assume A and A_0 are definitizable operators and $\mathcal{G}^\perp \subset \operatorname{dom} A$. If ∞ is a singular critical point for A, then it is also a singular critical point for A_0.*

Proof. We prove the contra positive version of the statement in the corollary. Let J_0 and J_1 be fundamental symmetries for $\mathcal{G}$ and $\mathcal{G}^\perp$, then $J = J_0 + J_1$ is a fundamental symmetry for $\mathcal{K}$. We denote the Hilbert space inner products on $\mathcal{G}$ and $\mathcal{K}$ corresponding to J_0 and J by $\langle \cdot, \cdot \rangle_0$ and $\langle \cdot, \cdot \rangle$. If ∞ is not a singular critical point for A_0, then, by Theorem 4.4, there exists an operator S_0 which is self-adjoint in $(\mathcal{G}, [\cdot, \cdot])$, similar to a self-adjoint operator in $(\mathcal{G}, \langle \cdot, \cdot \rangle_0)$ and such that $\operatorname{dom} S_0 = \operatorname{dom} A_0$. Since $\mathcal{G}^\perp \in \operatorname{dom} A$, we have $\operatorname{dom} A = \operatorname{dom} A_0 \dot{+} \mathcal{G}^\perp$ and hence

$$S = \begin{bmatrix} S_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{G} \\ \mathcal{G}^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{G} \\ \mathcal{G}^\perp \end{bmatrix}$$

is a self-adjoint operator in $(\mathcal{K}, [\cdot, \cdot])$ which is similar to a self-adjoint operator in $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ and $\operatorname{dom} S = \operatorname{dom} A$. It follows that ∞ is not a singular critical point for A. □

Remark 4.7. Let $\mathcal{K}$ be a Krein space, $\mathcal{G}$ a Krein subspace of $\mathcal{K}$ and A a (densely defined) self-adjoint operator on $\mathcal{K}$ with $\rho(A) \neq \emptyset$. Assume that $\mathcal{G}$ is invariant under A, so that the restriction of A to $\mathcal{G}$ is a self-adjoint operator in $\mathcal{G}$. Then $(A - \lambda)^{-1} \mathcal{G}^\perp \subset \mathcal{G}^\perp$, $\lambda \in \rho(A)$. If, moreover, $\dim \mathcal{G}^\perp < \infty$, then $\mathcal{G}^\perp \subset \operatorname{dom} A$, hence A is a diagonal operator in $\mathcal{K} = \mathcal{G} \oplus \mathcal{G}^\perp$. (Indeed, the assumption implies that $(A - \lambda)^{-1} \mathcal{G} \subset \mathcal{G}$ and by taking inner products we obtain $(A - \lambda)^{-1} \mathcal{G}^\perp \subset \mathcal{G}^\perp$. If $\dim G^\perp < \infty$, then equality prevails and hence $\mathcal{G}^\perp \subset \operatorname{dom} A$.) In particular, the inclusion $\mathcal{G}^\perp \subset \operatorname{dom} A$ in the assumptions of Theorem 3.6 (with $n \geq 2$) and Corollary 4.6 holds if the finite-codimensional compression A_0 to $\mathcal{G}$ coincides with the restriction of A to $\mathcal{G}$.

Remark 4.8. There are closely connected results to Corollary 4.6 by Jonas [18, Theorems 3.6 and 3.10] and by Behrndt and Jonas [7, Theorem 3.4] on the regularity and singularity of the critical point ∞. It is not clear if these results imply Corollary 4.6, because the present situation differs slightly from the ones in the mentioned results: Since $\mathcal{G}^{\perp} \subset \operatorname{dom} A$, the operators A and

$$A_1 := \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{G} \\ \mathcal{G}^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{G} \\ \mathcal{G}^{\perp} \end{bmatrix}$$

differ by a finite dimensional operator and by a finite dimensional pertubation in resolvent sense, whereas in the results of Jonas and Behrndt-Jonas the perturbation is not finite rank but the form domains remain the same.

5. Defect indices

We recall that if S is a closed symmetric linear relation in a Hilbert space $\mathcal{H}$, then the numbers $d^{\pm} = d^{\pm}(S) := \dim \ker(S^* - z) = \dim(S^* \cap zI)$ are independent of $z \in \mathbb{C}^{\pm}$ and called the *defect numbers* of S. The pair $\{d^+; d^-\}$ is called the *defect index* of S. Furthermore, S has self-adjoint extensions in $\mathcal{H}$ if and only if there is a unitary map from $\ker(S^* - z)$ onto $\ker(S^* - z^*)$ for some (and then for all) $z \in \mathbb{C}\backslash\mathbb{R}$, that is, $d^+ = d^-(\leq \infty)$, and in this case we have $\operatorname{def} S = d^+ = d^-$. In the proof of the following theorem we use that

$$d^{\pm}(S) = 0 \quad \Leftrightarrow \quad \mathbb{C}^{\mp} \subset \rho(S).$$

Theorem 5.1. *Let S be a closed densely defined symmetric operator in a Hilbert space $\mathcal{H}$. Then S_0 is a closed densely defined symmetric operator in $\mathcal{G}$ and S and S_0 have the same defect index.*

Proof. S_0 is closed and densely defined by Lemma 1.1 and Corollary 1.3(i). S_0 is symmetric in $\mathcal{G}$, because for $x \in \operatorname{dom} S_0$

$$\operatorname{Im}[S_0 x, x] = \operatorname{Im}[P_{\mathcal{G}} S x, x] = \operatorname{Im}[S x, x] = 0.$$

Let $\{m; n\}$ and $\{p; q\}$ be the defect indices of S and S_0. The proof consists of two parts: we first show that $p = \infty$ ($q = \infty$) if and only if $m = \infty$ ($n = \infty$) and then we consider the case where the defect indices are all finite.

(I) We set $k = \operatorname{codim} \mathcal{G}$. We only prove $p = \infty \Leftrightarrow m = \infty$. Assume $p = \infty$ and let $\lambda \in \mathbb{C}^+$. Then there exist countably infinite linearly independent elements in $S_0^* \cap \lambda I$. Denote them by $\{u_j; \lambda u_j\}$. Since $S_0^* = P_{\mathcal{G}} S^*|_{\mathcal{G}}$, there are $h_j \in \mathcal{G}^{\perp}$ such that the elements $\{u_j; \lambda u_j + h_j\}$ belong to S^*. Group these elements successively in infinitely many disjoint sets of size $k+1$. Then in the linear span of these sets there is a nonzero element of the form $\{v; \lambda v\} \in S^* \cap \lambda I$. We choose one such element in each linear span. They are linearly independent and hence $m = \infty$.

Now we assume $p < \infty$ and show that $m < (2k^2 + 1)(p + 1)$. This implies that $m = \infty \Rightarrow p = \infty$. We apply Lemma 1.2 with $\mathcal{H}$ decomposed as $\mathcal{H} = \mathcal{G} \oplus \mathcal{G}^{\perp}$ and $\mathcal{D} = \operatorname{dom} S^*$. There is a subspace $\mathcal{F} \subset \operatorname{dom} S^*$ with $\dim \mathcal{F} = k$ such that $\mathcal{H} = \mathcal{G} \dot{+} \mathcal{F}$, direct sum. Taking the intersection of both sides with $\operatorname{dom} S^*$ we find

$$\operatorname{dom} S^* = \operatorname{dom} S_0^* \dot{+} \mathcal{F}, \quad \text{direct sum in } \mathcal{H}.$$

If $\{x;y\} \in S^*$, then $x = x_0 + f$ with $x_0 \in \operatorname{dom} S_0^*$ and $f \in \mathcal{F}$ and

$\{x;y\} = \{x_0; S_0^* x_0\} + \{0; (I - P_{\mathcal{G}}) S^* x_0\} + \{f; S^* f\} \in S_0^* + \big(\mathcal{F} \times (\mathcal{G}^\perp + S^* \mathcal{F})\big).$

We denote the linear set on the right hand side by T. Then $S^* \subset T$ and hence $m \le \dim(T \cap \lambda I)$, $\lambda \in \mathbb{C}^+$. We claim that

$$\dim(T \cap \lambda I) < (2k^2 + 1)(p + 1).$$

which implies the asserted inequality for m. We prove the claim by assuming it is not true and deriving a contradiction. Assume

$$\dim(T \cap \lambda I) \ge (2k^2 + 1)(p + 1).$$

Then $T \cap \lambda I$ contains $(2k^2+1)(p+1)$ linearly independent elements $\{u_j; v_j\}$, $j = 1, 2, \ldots, (2k^2 + 1)(p + 1)$. We group these elements successively in $p + 1$ disjoint sets of size $2k^2 + 1$. Since

$$\dim\big(\mathcal{F} \times (\mathcal{G}^\perp + S^* \mathcal{F})\big) \le 2k^2,$$

the linear span of each of these sets contains a nonzero element from $S_0^* \cap (T \cap \lambda I) \subset S_0^* \cap \lambda I$. We choose one such element in each linear span. Since these $p + 1$ elements are linearly independent, we arrive at the contradiction that $p = \dim(S_0^* \cap \lambda I) \ge p + 1$. The proof of the claim is complete.

(II) We now assume that the defect indices are finite and show that $p = m$ and $q = n$ in four steps:

Step 1. $m = n \Rightarrow p = q$. This follows from Theorem 1.5 and the fact that S (S_0) has equal defect numbers if and only if S (S_0) has a self-adjoint extension in $\mathcal{H}$ $(\mathcal{G})$.

Step 2. $m = n \Rightarrow m = n = p = q$. Let A be a self-adjoint extension of S in $\mathcal{H}$. Then A_0 is a self-adjoint extension of S_0 in $\mathcal{G}$. We apply Lemma 1.2 with $\mathcal{D} = \operatorname{dom} S$: There is a subspace $\mathcal{F} \subset \operatorname{dom} S \subset \operatorname{dom} A$ such that $\dim \mathcal{F} = \dim \mathcal{G}^\perp$ and

$$\mathcal{H} = \mathcal{G} \dot{+} \mathcal{F}.$$

We take the intersection of the spaces on the left and right with $\operatorname{dom} S$ and $\operatorname{dom} A$ and obtain

$$\operatorname{dom} S = \operatorname{dom} S_0 \dot{+} \mathcal{F}, \quad \operatorname{dom} A = \operatorname{dom} A_0 \dot{+} \mathcal{F}.$$

From $\operatorname{dom} S \subset \operatorname{dom} A$ and $\operatorname{dom} S_0 \subset \operatorname{dom} A_0$ it follows that

$$m = \dim(\operatorname{dom} A / \operatorname{dom} S) = \dim(\operatorname{dom} A_0 / \operatorname{dom} S_0) = p.$$

Step 3. $m + n = p + q$, in other words:

$$\dim(\operatorname{dom} S^* / \operatorname{dom} S) = \dim(\operatorname{dom} S_0^* / \operatorname{dom} S_0).$$

The closed densely defined symmetric operator $S \oplus (-S)$ in $\mathcal{H} \oplus \mathcal{H}$ has defect index $\{m + n; m + n\}$ and its finite-codimensional compression to $\mathcal{G} \oplus \mathcal{G}$ is the closed densely defined symmetric operator $S_0 \oplus (-S_0)$ which has defect index $\{p + q; p + q\}$. Then, by (2), $m + n = p + q$.

Step 4. $n - m = q - p$. We assume $n \ge m$, otherwise we should consider $-S$. Then S has a maximal symmetric extension T with defect index $\{0; n - m\}$. This implies that $\mathbb{C}^- \subset \rho(T)$. By Theorem 1.7, T_0 is maximal symmetric. By Lemma 3.7 and $\sigma_p(T_0) \cap \mathbb{C}_- = \emptyset$, $\mathbb{C}_- \subset \rho(T_0)$. Hence the defect index of

T_0 is of the form $\{0; t\}$ for some nonzero integer t. From $S_0 \subset T_0$ it follows that $q \geq p$ and $t = q - p$. By (3), $n - m = q - p$.

That $p = m$ and $q = n$ now follows from (3) and (4). □

Corollary 5.2. *Let S be a closed densely defined symmetric operator in a Krein space $\mathcal{K}$ and let S_0 be its compression to a finite-codimensional Krein subspace $\mathcal{G}$ of $\mathcal{K}$. Then* $\operatorname{def} S = \operatorname{def} S_0$ *in the sense that if S in $\mathcal{K}$ or S_0 in $\mathcal{G}$ has a self-adjoint extension then so does the other and their defects coincide.*

Proof. We shall use that if J is a fundamental symmetry in a Krein space $(\mathcal{R}, [\cdot\,,\cdot])$, then a linear relation T is symmetric or self-adjoint in this space if and only if JT is symmetric or self-adjoint in the Hilbert space $(\mathcal{R}, [J\cdot\,,\cdot])$

Let J_0 and J_1 be fundamental symmetries of $\mathcal{G}$ and $\mathcal{G}^\perp$. Then $J = J_0 + J_1$ is a fundamental symmetry in $\mathcal{K}$, $J_0 = P_{\mathcal{G}} J|_{\mathcal{G}}$ and $(JS)_0 = J_0 S_0$. By Theorem 5.1, $d^\pm(JS) = d^\pm(J_0 S_0)$. If S has a self-adjoint extension A in $\mathcal{K}$, then A_0 is a self-adjoint extension of S_0 in $\mathcal{G}$ and

$$\begin{aligned}\operatorname{def} S &= \dim(A/S) = \dim(JA/JS) = d^+(JS)\\ &= d^+(J_0S_0) = \dim(J_0A_0/J_0S_0) = \dim(A_0/S_0) = \operatorname{def} S_0.\end{aligned}$$

Conversely, if S_0 has a self-adjoint extension in the space $\mathcal{G}$, then the defect numbers of J_0S_0 are equal and coincide with those of JS, which are therefore also equal. Hence S has a self-adjoint extension in the space $\mathcal{K}$ and as before we have $\operatorname{def} S = \operatorname{def} S_0$. □

To extend Theorem 5.1 to a symmetric linear relation we use the following lemma. The first part of this lemma coincides with [9, Theorem 3.1]. We repeat the proof for completeness.

Lemma 5.3. *Let T be a closed symmetric linear relation in $\mathcal{H}$ with finite defect numbers. Let B be a finite dimensional subspace of $\mathcal{H} \oplus \mathcal{H}$.*

(i) *Then $T \cap B^*$ is a closed symmetric relation in $\mathcal{H}$ and if $T^* \cap B = \{0; 0\}$, then its defect numbers are $d^\pm(T \cap B^*) = d^\pm(T) + \dim B$.*
(ii) *If $T + B \subset (T + B)^* = T^* \cap B^*$ and $T \cap B = \{0; 0\}$, then* $\dim B \leq \min\{d^+(T), d^-(T)\}$ *and $T + B$ is a closed symmetric relation with defect numbers $d^\pm(T + B) = d^\pm(T) - \dim B$.*

Proof. (i)Since $T \cap B^* \subset T$ and T is closed and symmetric, $T \cap B^*$ is closed and symmetric. Its adjoint is given by $T^* + B$, which is a closed linear relation as it is the sum of a closed subspace T^* and a finite dimensional subspace B in $\mathcal{H} \oplus \mathcal{H}$ (see [16, Theorem I.4.12]). Now assume $T^* \cap B = \{0; 0\}$, then $T^* + B$ is a direct sum. For $\mu \in \mathbb{C}\backslash\mathbb{R}$ we define the mapping $\mathbf{b} : (T^* + B) \cap \mu I \to B$ by: if $\{h; \mu h\} \in (T^* + B) \cap \mu I$ and $\{h; \mu h\} = \{f; g\} + \{\sigma; \tau\}$ with $\{f; g\} \in T^*$ and $\{\sigma; \tau\} \in B$, then $\mathbf{b}(\{h; \mu h\}) = \{\sigma; \tau\}$. Since the sum $T^* + B$ is a direct, $\mathbf{b}$ is well defined. Moreover, $\ker \mathbf{b} = T^* \cap \mu I$. We now show that $\mathbf{b}$ is surjective. Let $\{\sigma; \tau\} \in B$, then since $\operatorname{ran}(T^* - \mu) = \mathcal{H}$, there is an element $\{f; g\} \in T^*$ such that $g - \mu f = \mu\sigma - \tau$. Then

$$\{f; g\} + \{\sigma; \tau\} = \{f + \sigma; \mu(f + \sigma)\} \in (T^* + B) \cap \mu I$$

and $\mathbf{b}(\{f+\sigma;\mu(f+\sigma)\}) = \{\sigma;\tau\}$. The properties of $\mathbf{b}$ just established imply that for $\mu \in \mathbb{C}^{\pm}$

$$d^{\pm}(T \cap B^*) = \dim\ker\big((T^* + B) \cap \mu I\big) = d^{\pm}(T) + \dim B.$$

(ii) As T is closed and B is finite dimensional, $T + B$ is a closed linear relation. Since $T \cap B = \{0;0\}$, $\dim(T+B)/T = \dim B$. Hence there is a subspace C of $\mathcal{H}\oplus\mathcal{H}$ with $\dim C = \dim B$ such that $T = (T+B)\cap C^*$. From (i) it follows that

$$d^{\pm}(T + B) + \dim B = d^{\pm}(T + B) + \dim C = d^{\pm}(T),$$

hence $\dim B \le \min\{d^+(T), d^-(T)\}$ and $d^{\pm}(T + B) = d^{\pm}(T) - \dim B$. □

Theorem 5.4. *Let S be a closed symmetric linear relation in a Hilbert space $\mathcal{H}$ with defect index $\{m;n\}$. Let $\mathcal{G}_1$ be a subspace of $\overline{\operatorname{dom}}\,S$ of finite codimension in $\overline{\operatorname{dom}}S$, let $\mathcal{G}_2$ be a subspace of $S(0)$ and set $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$. Then $P_{\mathcal{G}}S|_{\mathcal{G}}$ is a closed linear relation in $\mathcal{G}$ with multivalued part $\mathcal{G}_2$ and defect index $\{m-r;n-r\}$ with $r = \dim(S^*(0)\ominus S(0))$.*

Proof. From $\overline{\operatorname{dom}}S = S^*(0)^{\perp} \subset S(0)^{\perp}$ we see that $\mathcal{G}_1 \perp \mathcal{G}_2$ and $P_{\mathcal{G}} = P_{\mathcal{G}_1} + P_{\mathcal{G}_2}$ Consider $\widehat{S} := S \oplus \big(\{0\}\times(S^*(0)\ominus S(0))\big)$. Then, by Lemma 5.3(ii), $\widehat{S}$ is a symmetric linear relation in $\mathcal{H}$ with defect index $\{m-r;n-r\}$ and $\widehat{S}(0) = S^*(0)$. Now $\widehat{S}$ can be decomposed as the orthogonal sum

$$\widehat{S} = \widehat{S}_s \oplus \widehat{S}_\infty,$$

in which $\widehat{S}_\infty := \{0\}\times S^*(0)$ is a self-adjoint relation in $S^*(0)$ and $\widehat{S}_s = \widehat{S}\ominus\widehat{S}_\infty$ is a closed densely defined symmetric operator in $S^*(0)^{\perp}$. The latter has defect index $\{m-r;n-r\}$. We find that $P_{\mathcal{G}}S|_{\mathcal{G}} = P_{\mathcal{G}}\widehat{S}|_{\mathcal{G}} = (\widehat{S}_s)_0 \oplus (\{0\}\times\mathcal{G}_2)$, where $(\widehat{S}_s)_0 = P_{\mathcal{G}_1}\widehat{S}_s|_{\mathcal{G}_1}$ is a closed symmetric operator in $\mathcal{G}_1$ and $\{0\}\times\mathcal{G}_2$ is a self-adjoint relation in $\mathcal{G}_2$. Hence $P_{\mathcal{G}}S|_{\mathcal{G}}$ is a closed linear relation in $\mathcal{G}$ with multi-valued part $\mathcal{G}_2$ and with defect index = the defect index of $(\widehat{S}_s)_0$ = the defect index of $\widehat{S}_s$ (by Theorem 5.1) $= \{m-r;n-r\}$. □

6. Appendix

In this appendix A is a definitizable operator on a Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,])$. The notions "$\infty$ is a critical point", "∞ is a regular critical point" and "∞ is a singular critical point" of A are due to Langer [23,24]. To recall these definitions we denote by E the spectral function for A as in [24, Theorem 3.1] (see also [5, Chapter 4, §1]). It is defined on the semiring R_A generated by the bounded intervals whose endpoints are not in $c(A)$ and their complements in $\mathbb{R}$; its values $E(\Delta)$, $\Delta \in R_A$, are orthogonal projections in $(\mathcal{K}, [\,\cdot\,,\cdot\,])$. Here $c(A)$ stands for the set of (finite) critical points of A defined by

$$c(A)=\big\{\lambda \in \sigma(A)\cap\mathbb{R} \,:\, p(\lambda)=0 \quad \text{for all definitizing polynomials } p \text{ of } A\big\}.$$

Definition 6.1. (Langer (1965)) ∞ is a *critical point* for A if for all unbounded $\Delta \in R_A$ the space $E(\Delta)\mathcal{K}$ is indefinite. It is a *regular critical point* for A if the limits

$$\lim_{\lambda \downarrow -\infty} E[\lambda, \lambda_0] \qquad \text{and} \qquad \lim_{\lambda \uparrow +\infty} E[\lambda_1, \lambda],$$

with $\lambda_0, \lambda_1 \in \mathbb{R}$, exist in the strong operator topology. If ∞ is a critical but not a regular critical point, then it is called a *singular critical point*.

We show that the definitions in Sect. 4 are equivalent to the ones given in Definition 6.1. For this we use the following notion: We say that A admits a *diagonal form* (D) if A can be represented as a diagonal operator $A = \operatorname{diag}\{A_b; A_\infty\}$ relative to an orthogonal decomposition $\mathcal{K} = \mathcal{K}_b \oplus \mathcal{K}_\infty$ into Krein subspaces $\mathcal{K}_b$ and $\mathcal{K}_\infty$ of $\mathcal{K}$ which are invariant under A and such that

$$\mathcal{K}_b \subset \operatorname{dom} A \quad \text{and} \quad 0 \in \rho(A_\infty).$$

The requirements imply that $A_b = A|_{\mathcal{K}_b}$ is a bounded definitzable operator defined on $\operatorname{dom} A_b = \mathcal{K}_b$ with the values in $\mathcal{K}_b$ and that $A_\infty = A|_{\mathcal{K}_\infty}$ is a densely defined boundedly invertible definitizable operator defined on $\operatorname{dom} A_\infty = \mathcal{K}_\infty \cap \operatorname{dom} A$ with the values in $\mathcal{K}_\infty$. If p is a definitizing polynomial for A, then it is also a definitizing polynomial for A_b and for A_∞.

Theorem 6.2. *For a definitizable operator A on a Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ the following statements are equivalent:*

(i) ∞ *is not a critical point for A.*
(ii) *A has a diagonalization (D) in which $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ is a Hilbert or an anti-Hilbert space.*

Proof. Assume (i) and let Δ be an unbounded set in R_A such that the Krein subspace $E(\Delta)\mathcal{K}$ is definite. Choose a closed unbounded set $\Delta_\infty \in R_A$ such that $\Delta_\infty \subset \Delta$ and 0 belongs to the open bounded set $\Delta_b = \mathbb{R} \backslash \Delta_\infty \in R_A$. Denote by E_0 the Riesz-Dunford projection on the nonreal spectrum $\sigma_0(A)$ of A. Define $\mathcal{K}_b := E(\Delta_b)\mathcal{K} \oplus E_0\mathcal{K}$ and $\mathcal{K}_\infty := E(\Delta_\infty)\mathcal{K}$. These subspaces are Krein subspaces of $\mathcal{K}$ which are invariant under A and $\mathcal{K} = \mathcal{K}_b \oplus \mathcal{K}_\infty$. Moreover, $\mathcal{K}_b \subset \operatorname{dom} A$ and $\mathcal{K}_\infty \subset E(\Delta)\mathcal{K}$, hence $\mathcal{K}_\infty$ is a definite subspace of $\mathcal{K}$. Finally, $0 \in \rho(A_\infty)$ because $\sigma(A_\infty) \subset \Delta_\infty$ and $0 \in \mathbb{R} \backslash \Delta_\infty$. This implies (ii).

Now assume (ii). Choose a bounded open interval $\Delta_b \in R_A$ such that

$$\{0\} \cup c(A) \cup (\sigma(A_b) \cap \mathbb{R}) \subset \Delta_b.$$

Set $\Delta_\infty = \mathbb{R} \backslash \Delta_b$. Then Δ_∞ is an unbounded set in R_A. Denote by E_0 and E_{0b} the Riesz-Dunford projections on the nonreal spectra $\sigma_0(A)$ and $\sigma_0(A_b)$ of A and A_b. Then $E_{0b}\mathcal{K}_b = E_0\mathcal{K}$ and $\mathcal{K}_b \ominus E_{0b}\mathcal{K}_b \subset E(\Delta_b)\mathcal{K}$. This follows from the defining formulas for E_0, E_{0b}, and $E(\Delta_b)$ as integrals over contours around $\sigma_0(A) = \sigma_0(A_b)$ and around Δ_b of $(A - z)^{-1}$, the diagonal form of A and the fact that A_∞ is self-adjoint in a Hilbert space and thus has no nonreal spectrum. The orthogonal decomposition

$$\mathcal{K} = E(\Delta_\infty)\mathcal{K} \oplus E(\Delta_b)\mathcal{K} \oplus E_0\mathcal{K}$$

and the inclusion $\mathcal{K}_b \subset E(\Delta_b)\mathcal{K} \oplus E_{0b}\mathcal{K}_b$ imply $E(\Delta_\infty)\mathcal{K} \subset K_b^\perp = \mathcal{K}_\infty$. As a subspace of the definite subspace $\mathcal{K}_\infty$, $E(\Delta_\infty)\mathcal{K}$ is a definite subspace. This yields (i). □

For the proof of Theorem 6.4 below we use the following lemma, see also [24, Proposition II 4.2].

Lemma 6.3. *Let $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ be a Krein space and let A be a self-adjoint operator on $\mathcal{K}$. Assume $A \geq 0$, $\ker A = \{0\}$ and $\sigma(A) \subseteq [0, +\infty)$. Then $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ is a Hilbert space.*

Proof. If A is bounded, the lemma is proved in [14, Theorem 6.7]. Assume A is unbounded and let E be the spectral function for A. Set $\Delta_1 = (-1, 1)$ and $\Delta_2 = \mathbb{R}\backslash\Delta_1$. Then A has the diagonal representation $A = \operatorname{diag}\{A_1; A_2\}$ relative to the orthogonal decomposition $\mathcal{K} = E(\Delta_1)\mathcal{K} \oplus E(\Delta_2)\mathcal{K}$: $A_j = A|_{E(\Delta_j)\mathcal{K}}$, $j = 1, 2$. The operators A_1 and A_2^{-1} are self-adjoint on $E(\Delta_1)\mathcal{K}$ and $E(\Delta_2)\mathcal{K}$, have the same properties as A in the lemma and are bounded. It follows that $(E(\Delta_1)\mathcal{K}, [\,\cdot\,,\cdot\,])$ and $(E(\Delta_2)\mathcal{K}, [\,\cdot\,,\cdot\,])$ are Hilbert spaces. Hence $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ is a Hilbert space. □

Theorem 6.4. *For a definitizable operator A on a Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ the following statements are equivalent:*

(i) *∞ is a regular critical point for A.*
(ii) *∞ is a critical point for A and A has a diagonalization (D) in which $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ is an indefinite Krein space and A_∞ commutes with a fundamental symmetry on $\mathcal{K}_\infty$.*

The diagonalization (D) in (ii) can be chosen so that A_∞ is even uniformly definite on $\mathcal{K}_\infty$.

Proof. Choose $-\infty < \lambda_0 < 0 < \lambda_1 < +\infty$ such that the interval $\Delta_b = (\lambda_0, \lambda_1)$ belongs to R_A, $c(A) \cup (\sigma(A_0) \cap \mathbb{R}) \subset \Delta_0$ and some definitizing polynomial p for A does not vanish on $\Delta_\infty = \mathbb{R}\backslash\Delta_b$. Denote by E_0 the Riesz-Dunford projection on the nonreal spectrum $\sigma_0(A)$ of A. Set $\mathcal{K}_b = E_0\mathcal{K} \oplus E(\Delta_b)\mathcal{K}$ and $K_\infty = \mathcal{K}_b^\perp = E(\Delta_\infty)\mathcal{K}$. Then A admits a diagonalization (D) relative to the fundamental decomposition $\mathcal{K} = \mathcal{K}_b \oplus \mathcal{K}_\infty$. (i) holds if and only if ∞ is a regular critical point for A_∞, because the spectral function for A_∞ in the Krein space $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ coincides with the spectral function E for A on all intervals and their complements from R_A which are contained in Δ_∞.

Assume (i). Then ∞ is a regular critical point for A_∞ and hence, by [24, Theorem II 5.7], there exist orthogonal and mutually orthogonal projections E_+ and E_- on $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ such that

(a) E_+ and E_- belong to the double commutant of $(A_\infty - z)^{-1}$,
(b) $I_{\mathcal{K}_\infty} = E_+ + E_-$, where $I_{\mathcal{K}_\infty}$ is the identity operator on $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$,
(c) $\sigma(A_\infty|_{E_\pm\mathcal{K}_\infty}) = \{\lambda \in \Delta_\infty : p(\lambda) \gtrless 0\}$.

By (a), $E_\pm$ commute with A, hence the Krein subspaces $\mathcal{K}_\infty^\pm = E_\pm\mathcal{K}_\infty$ are invariant under A_∞. On account of (c) and the spectral mapping theorem,

$$\sigma\left(\pm p(A_\infty)|_{\mathcal{K}_\infty^\pm}\right) = \pm p\left(\sigma(A_\infty)|_{\mathcal{K}_\infty^\pm}\right) \subseteq [0, \infty). \qquad (6.1)$$

We claim $\ker p(A_\infty) = \{0\}$. Indeed, assume the kernel is not trivial, then $0 \in \sigma(p(A_\infty) = p(\sigma(A_\infty))$ and this means that p has a zero in $\sigma(A_\infty) \subseteq \Delta_\infty$ contradicting the fact that p does not vanish on Δ_∞. This proves the claim. The inequality $p(A_\infty) \geq 0$ on $\mathcal{K}_\infty$, the claim and the inclusion (6.1) imply, by Lemma 6.3, that the subspaces $(\mathcal{K}_\infty^\pm, \pm[\,\cdot\,,\cdot\,])$ are Hilbert spaces. Thus, by (b), $\mathcal{K}_\infty = \mathcal{K}_\infty^+ \oplus \mathcal{K}_\infty^-$ is a fundamental decomposition of $\mathcal{K}_\infty$, $J_\infty := E_+ - E_-$ is the corresponding fundamental symmetry and J_∞ commutes with A. The proof of (ii) is complete. To prove the last part of the theorem, let $m = \min\{|p(\lambda_0)|, |p(\lambda_1)|\}$. Then $m > 0$ and, by (c),

$$\sigma(A_\infty|_{\mathcal{K}_\infty^\pm}) \subseteq [m, \infty), \quad \sigma(A_\infty|_{\mathcal{K}_\infty^\mp}) \subseteq (-\infty, -m] \quad \text{if} \lim_{\lambda \to +\infty} p(\lambda) = \pm\infty.$$

Hence

$$\pm[A_\infty x, x] \geq m[J_\infty x, x], \quad x \in \mathcal{K}_\infty,$$

depending on $\lim_{\lambda\to+\infty} p(\lambda) = \pm\infty$. This implies that A_∞ is uniformly definite on $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$.

We now prove (ii) $\Rightarrow$ (i). Assume (ii). Then ∞ is a critical point for A_∞. For if it is not a critical point for A_∞, then, by Theorem 6.2, A_∞ admits a diagonalization of the form (D) in which the space on which the unbounded operator acts is a Hilbert or an anti-Hilbert space. By combining the diagonalizations of A and A_∞ we obtain a new diagonalization of the form (D) for A in which $\mathcal{K}_\infty$ is a Hilbert or anti-Hilbert space. This cannot be, because ∞ is a critical point for A. This contradiction implies that ∞ is a critical point for A_∞. We now apply [10, Theorem 3.2] which states that ∞ is not a singular critical point of A_∞ if and only if there is a positive, bounded and boundedly invertible operator W on the Krein space $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ such that $\operatorname{dom} A_\infty \subseteq \operatorname{dom} W$ and $W \operatorname{dom} A_\infty \subseteq \operatorname{dom} A_\infty$. We use the if part of this theorem and choose for W the fundamental symmetry J_∞, say, on $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ which commutes with A_∞. It follows that ∞ is not a singular critical point for A_∞. Since ∞ is a critical point, hence it is a regular critical point for A_∞. It remains to show that this implies that ∞ is a critical point for A. This too follows from the if part of [10, Theorem 3.2] mentioned above, by chosing $W = \operatorname{diag}\{I_{\mathcal{K}_b}; J_\infty\}$, where $I_{\mathcal{K}_b}$ is the identity operator on $\mathcal{K}_b$. This proves (i). □

Theorem 6.5. *For a definitizable operator A on a Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,])$ the following statements are equivalent:*

(i) *∞ is not a singular critical point for A.*
(ii) *A has a diagonalization (D) in which A_∞ is similar to a self-adjoint operator in the Hilbert space $(\mathcal{K}_\infty, [J\cdot\,,\cdot\,])$ for some (and then for every) fundamental symmetry J on $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$.*

Proof. Assume (i). If ∞ is not a critical point, then in the diagonalization as in Theorem 6.2(ii) A_∞ is self-adjoint in the Hilbert space holds $(\mathcal{K}_\infty, [J\cdot\,,\cdot\,])$ with $J = \pm I_{\mathcal{K}_\infty}$, where $I_{\mathcal{K}_\infty}$ is the identity operator on $\mathcal{K}_\infty$. If ∞ is a regular critical point, then in the diagonalization as in Theorem 6.4(ii) A_∞ is self-adjoint in the Hilbert space $(\mathcal{K}_\infty, [J_\infty\cdot\,,\cdot\,])$ where J_∞ is the fundamental symmetry that commutes with A_∞. Let J be any fundamental symmetry on $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$. Then there is a unitary operator U on $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ such that $J = UJ_\infty U^{-1}$. It is the similarity operator we looked for: UAU^{-1} is selfadjoint in the Hilbert space $(\mathcal{K}_\infty, [J\cdot\,,\cdot\,])$.

Now assume (ii). If ∞ is not a critical point for A, then (i) holds. So assume ∞ is a critical point for A. We prove that it is a regular critical point for A. Let $U_\infty = (A_\infty - \mu^*)(A_\infty - \mu)^{-1}$ be the Cayley transform of A_∞ relative to a nonreal point $\mu \in \rho(A_\infty)$; it is unitary in the Krein space $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$. Let T be the similarity operator on $\mathcal{K}$ such that $TA_\infty T^{-1}$ is selfadjoint in the Hilbert space $(\mathcal{K}_\infty, [J\cdot\,,\cdot\,])$. Then $TU_\infty T^{-1}$ is unitary in the same Hilbert space. By [8, Theorem VIII 1.4] this implies that U_∞ is fundamentally reducible which means that there exists a fundamental decomposition of $(\mathcal{K}_\infty, [\,\cdot\,,\cdot\,])$ which reduces U_∞. According to [8, VIII Lemma 1.1], the corresponding fundamental symmetry commutes with U_∞ and hence with $A_\infty = (\mu U_\infty - \mu^*)(U_\infty - 1)^{-1}$. We apply Theorem 6.4 and conclude that ∞ is a regular critical point for A. □

Acknowledgements

We thank the referee for useful suggestions and references which led to the present version of Sect. 3, Remark 4.8 and Corollary 5.2.

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Received: February 20, 2016.
Revised: August 16, 2016.