Supplement. Proclus's Commentary on Eudemus' *History of Geometry*

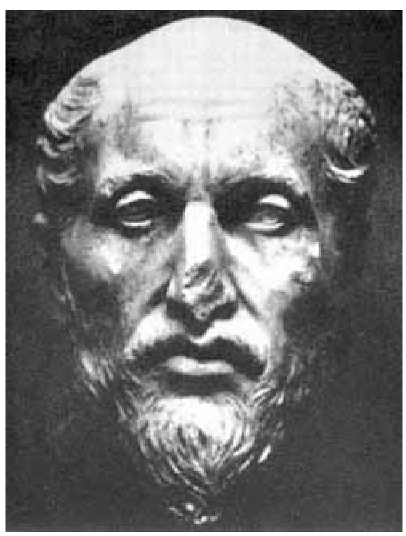
Note. Eudemus of Rhodes (circa 350 BCE–290 BCE) is known to have written three works on the history of mathematics: *History of Arithmetic*, *History of Astronomy*, and *History of Geometry*. In fact, each is now lost and only known today from references to the works by others whose writings did survive (see the MacTutor History of Mathematics biography of Eudemus webpage for more details; accessed 2/10/2023). Our interest in this supplement lies with the "History of Geometry" and references to it by Proclus. In Volume 1 of Thomas Heath's translation of *The Thirteen Books of EUCLID'S ELEMENTS* (second edition with revised additions), Cambridge University Press 1926 (also in print by Dover Publications since 1956) appears the following statement (on page 35):

> "The loss of Eudemus' history is one of the gravest which fate inflicted upon us, for it cannot be doubted that Eudemus had before him a number of the actual works of earlier geometers, which, as before observed, seem to have vanished completely when they were superseded by the treatises of Euclid, Archimedes, and Apollonius. As it is, we have to be thankful for the fragments from Eudemus which such writers as Proclus have preserved us." (page 35)

Heath's book includes a 150 page, 9 chapter introduction. It is an exceptional source! Heath is responsible for translations into English of many of the Greek works in mathematics (and astronomy). For a presentation on Euclid's *Elements* in general, see my online presentation Euclid's Elements—A 2,500 Year History.

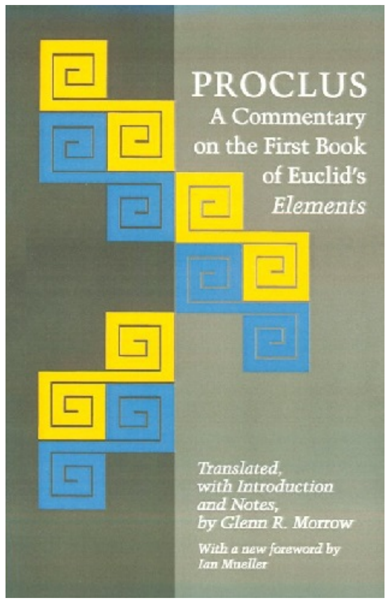
In particular, a section of the presentation concerning contributions of Heath is: *The Elements*—Heiberg and Heath.

Note. Proclus Diadochus (circa 411–April 17, 485), born in Constantinople, Byzantium (today, Istanbul, Turkey), was a Greek philosopher who became head of Plato's Academy. His contributions are commentaries on the work of other mathematicians, in particular a commentary on Euclid's *Elements*. His commentary on book I of the *Elements* is likely a written version of presentations he gave to geometry students in Athens (as claimed by Ian Mueller in his Foreword to the 1992 edition of *Proclus: A Commentary on the First Book of Euclid's Elements*, translated by Glenn R. Morrow). This commentary includes a summary of the now-lost "History of Geometry" by Eudemus. The image below and this biographical information is based from the MacTutor History of Mathematics Archive biography of Proclus (accessed 2/10/2023).



Proclus Diadochus (circa 411–April 17, 485)

Note. Princeton University Press published *Proclus: A Commentary on the First Book of Euclid's Elements*, translated by Glenn R. Morrow, in 1970. It was reissued in 1992 with a new 24 page foreword by Ian Mueller. This new edition includes a summary of Eudemus' "History of Geometry" on pages 52–57 (along with footnotes). This summary is included below.



We now quote from this source Proclus' description of Eudemus' "History of Geometry."

Proclus' Commentary on the First Book of Euclid's *Elements*

Chapter IV. The Origin and Development of Geometry (pages 52–57)

"Thales, who had travelled to Egypt, was the first to introduce this science into Greece. He made many discoveries himself and taught the principles for many others to his successors, attacking some problems in a general way and others more empirically. Next after him Mamercus, brother of the pet Stesichorous, is remembered as having applied himself to the study of geometry; and Hippias of Elis records that he acquired a reputation in it. Following upon these men, Pythagoras transformed mathematical philosophy into a scheme of liberal education, surveying its principles from the highest downwards and investigating its theorems in an immaterial and intellectual manner. He it was who discovered the doctrine of proportionals and the structure of the cosmic figures. After him Anaxagoras of Clazomenae applied himself to many questions in geometry, and so did Oenopides of Chios, who was a little younger than Anaxagoras. Both these men are mentioned by Plato in the *Erastae* as having got a reputation in mathematics. Following them Hippocrates of Chios, who invented the method of squaring lunules, and Theodorus of Cyrene became eminent in geometry. For Hippocrates wrote a book on elements [of geometry], the first of whom we have any record who did so.

Plato, who appeared after them, greatly advanced mathematics in general and geometry in particulat because of his zeal for these studies. It is well known that hsi writings are thickly sprinkled with mathematical terms and that he everywhere tries to arouse admiration for mathematics among students of philosophy. At this time also lived Leodamas of Thasos, Archytas of Tarentum, and Theaetetus of Athens, by whom the theorems were increased in number and brought into a more scientific arrangement. Younger than Leodamas were Neoclides and his pupil Leon, who added many discoveries to those of their predecessors, so that Leon was able to compile a book of elements more carefully designed to take account of the number of propositions that had been proved and of their utility. He also discovered *diorismi*, whose purpose is to determine when a problem under investigation is capable of solution and when it is not. Eudoxus of Cnidus, a little later than Leon and a member of Plato's group, was the first to increase the number of the so-called general theorems; to the three proportionals already known he added three more and multiplied the number of propositions concerning the "section" which had their origin in Plato, employing the method of analysis for their solution. Amyclas of Heracleia, one of Plato's followers, Menaechmus, a student of Eudoxus who also was associated with Plato, and his brother Dinostratus made the whole of geometry still mroe perfect. Theudius of Magnesia had a reputation for excellence in mathematics as in the rest of philosophy, for he produced an admirable arrangement of the elements and made many partial theorems more general. There was also Athenaeus of Cyzicus, who lived about this time and became eminent in other branches of mathematics and most of all in geometry. There men lived together in the Academy, making their inquiries in common. Hermotimus of Colophon pursued further the investigations already begun by Eudoxus and Theatetus, discovered many propositions in the *Elements*, and wrote some things about locus-theorems. Philippus of Mende, a pupil whom Plato had encouraged to study mathematics, also carried on his investigations according to Plato's instructions and set himself to study all the problems that he thought would contribute to Plato's philosophy.

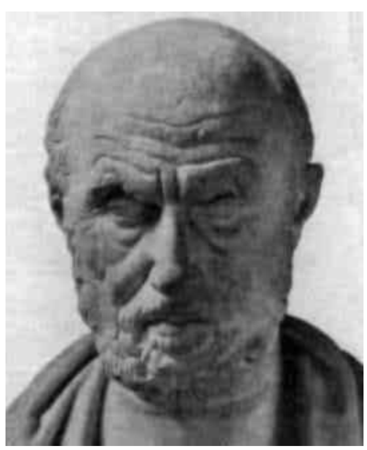
All those who have written histories bring to this point their account of the development of this science. Not long after these men came Euclid, who brought together the *Elements*, systematizing many of the theorems of Eudoxus, perfecting many of those of Theatetus, and putting in irrefutable demonstrable form propositions that had been rather loosely established by his predecessors. He lived in the time of Ptolemy the First, for Archimedes, who lived after the time of the first Ptolemy, mentions Euclid. It is also reported that Ptolemy once asked Euclid if there was not a shorter road to geometry than through the *Elements*, and Euclid replied that there was no royal road to geometry. He was therefore later than Plato's group but earlier than Eratosthenes and Archimedes, for those two men were contemporaries, as Eratosthenes somewhere says. Euclid belonged to the persuasion of Plato and was at home in this philosophy; and this is why he thought the goal of the *Elements* as a whole to be the construction of the so-called Platonic figures."

Note. Several geometers are mentioned in Proclus' commentary. Footnotes are given in Morrow's translation which give brief descriptions and mention other references. We now give some details on some of these other figures (Proclus' brief mention is the only known reference to some of the figures). Thales is mentioned first, and his contributions are explored in Section 3.1. Birth of Demonstrative Mathematics of these notes. Hippias of Elis (circa 460 BCE– circa 400 BCE) is the inventor of the *quadratrix*, a curve used for trisecting of an angle and also for squaring the circle. More compass and straight-edge constructions will be explored in "Chapter 4. Duplication, Trisection, and Quadrature." These constructions are also discussed in my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 1.8. Three Famous Problems of Greek Geometry. The use of the quadratrix to trisect an angle is given in notes for Introduction to Modern Geometry on Section 4.1. The Conchoid of Nicomedes, The Trisection of an Angle (see Theorem 4.1.A). A "modern" resolution of the compass and straight-edge constructions is given in Introduction to Modern Algebra 2 (MATH 4127/5127); see my online notes on Section VI.32. Geometric Constructions. I have a 24 minute

YouTube video addressing this titled "Compass and Straightedge Constructions." The topic is also addressed in our graduate level Modern Algebra 2 (MATH 5420) in Section V.1. Appendix. Ruler and Compass Constructions.

Note. Proclus mentions Pythagoras of Samos (circa 570 BCE–490 BCE), and we will study him and his work in the next four sections. Oenopides of Chios (circa 490 BCE–420 BCE) is credited (elsewhere in Proclus' commentary) with giving proofs of Propositions 12 and 23 of Euclid's Book I. These state (in the clunky wording of the *Elements*): "To draw a perpendicular to a given straight line from a point outside it" and "On a given straight line and at a given point on it to construct a rectilineal angle equal to a given rectilineal angle" (respectively). The proofs given in the *Elements* very much reflect a compass and straight-edge approach to these (and other problems in Book I). For some additional insight on the content of Book I of the *Elements*, see my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 2.1. Book I. We also devote a chapter in this course to the *Elements*, "Chapter 5. Euclid and His Elements." Thomas Heath in his *A History of Greek Mathematics, Volume I. From Thales to Euclid* (Clarendon Press, Oxford, 1921) credits Oenopides with making compass and straight-edge constructions central geometry: "It may therefore be that Oenopides's significance lay in improvements of method from the point of view of theory; he may, for example, have been the first to lay down the restriction of the means permissible in constructions to the ruler and compasses which became a canon of Greek geometry for all 'plane' constructions, i.e. for all problems involving the equivalent of the solution of algebraical equations of degree not higher than the second." (See his pages 175 and 176.) **Some editorializing from your instructor:** WOW! In my opinion, this makes Oenopides the second most influential geometer (after Euclid), until the appearance of Descartes in the 17th century with his analytic geometry.

Note. Next, Proclus mentions Hippocrates of Chios (circa 470 BCE–circa 410 BCE). (This Hippocrates is not to be confused with the physician, Hippocrates of Kos.) Latter in his commentary Proclus states that Hippocrates reduced the problem of doubling the cube to finding two mean proportions. We'll see below that Menaechmus introduced conic sections to deal with mean proportions. Hippocrates used the Pythagorean Theorem to find areas of 'lunes." For more details, see my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 1.8. Three Famous Problems of Greek Geometry and the Geometry Before Euclid part of my online presentation on Euclid's Elements—A 2,500 Year History. The elements of geometry book by Hippocrates which Proclus references is lost, but as Proclus states, his is the first book on the elements of geometry for which we have a record.



Heath in his *A History of Greek Mathematics, Volume I* states (see page 182): "…Simplicius has preserved in his commentary on the *Physics* of Aristotle a fragment from Eudemus's *History of Geometry* giving an account of Hippocrates's quadratures of certain 'lunules' or lines." Heath's book includes a 20 page description of Hippocrates' work on lunes (see Heath's pages 183–202). The image of Hippocrates (above) is from MacTutor History of Mathematics Archive biography of Hippocrates (accessed 2/11/2023).

Note. Theaetetus of Athens (circa 417 BCE–circa 369 BCE) is one of the two greatest Greek mathematicians of the 4th century BCE (along with Eudoxus, to be discussed next). His contributions to Euclid's *Elements* include his theory of irrationals, which appears in Euclid's Book X. In terms of contributions to geometry, Theaetetus considered constructions of the five regular solids, which makes up the content of Euclid's final book, Book XIII. Concerning this, Heath in his *A History of Greek Mathematics, Volume I* states (see page 212): "We have already mentioned (pp. 159, 162) the traditions that Theaetetus was the first to 'construct' or 'write upon' the five regular solids, and that his name was specially associated with the octahedron and the icosahedron. There can be little doubt that Theaetetus's 'construction' of, or treatise upon, the regular solids gave the theoretical constructions much as we find them in Euclid."

Note. Eudoxus of Cnidus (408 BCE–355 BCE) contributed significantly to the theory of proportions which make up much of Books V and VI of Euclid's *Elements*. As mentioned above when discussing Hippocrates, this study is related to the problem of doubling the cube. Eudoxus seems to be the first to successfully use the "method of exhaustion." In this approach, a quantity (often an area or volume) is shown to take on a particular value A, say, by first showing if the true value is less than A by a certain amount, say $A - a$, then we can inscribe shapes inside the object (the shapes used are often triangles of smaller and smaller area) which have a sum of areas (or volumes, as appropriate) greater than $A - a$, giving a contradiction. Second, it is shown that if the true value is greater than A by a certain amount, say $A + a$, then we can circumscribe shapes outside the object which have a sum of areas (or volumes) less than $A + a$, given another contradiction. Hence the true value must be exactly A. This argument probably, justifiably, reminds you of a modern-day epsilon argument. Archimedes would use this method with great success some 100 to 150 years later, as will be explored in Section 6.2. Archimedes (Archimedes also gives a numerical approximation of π, which is inspired by the method of exhaustion). It seems that Hippocrates was close to the method of exhaustion, but that it was Eudoxus who formalized it. Heath in *A History of Greek Mathematics, Volume I* states (see page 328): "Without therefore detracting from the merit of Hippocrates, whose argument may have contained the germ of the method of exhaustion, we do not seem to have any sufficient reason to doubt that is was Eudoxus who established this method as part of the regular machinery of geometry." It is likely that some of the content of Euclid's Book XII (namely, propositions 16, 17, and 18) were first proved by Eudoxus (see Heath's page 329). Archimedes himself ties Eudoxus to the method of exhaustion in the preface to Book I of his *On the Sphere and Cylinder* (including crediting Eudoxus with the formula that we use for the volume of a cone). The image below of Eudoxus is from MacTutor History of Mathematics Archive biography of Eudoxus (accessed

2/11/2023) and is credited as a drawing by Andreas Strick.

Note. Menaechmus (circa 380 BCE—320 BCE) was a student of both Eudoxus and Plato. Menaechmus is credited with "discovering" the conic sections! In Thomas Heath's *Apollonius of Perga: Treatise on Conic Sections* (Cambridge University Press, 1896) it is stated (on page *xix*): "Thus the evidence so far shows (1) Menaechmus (a pupil of Eudoxus and a contemporary of Plato) was the discoverer of the conic sections, and (2) that he used them as a means of solving the problem of the doubling of the cube." It is likely that Menaechmus concentrated on conic sections for the very reason that he could use them in solving the doubling of the cube problem. See my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Chapter 3. Conic Sections, in which it is explained in modern notation how Menaechmus used the intersection of conic sections (namely, a parabola and a hyperbola) in doubling the cube (and so in the construction of the irrational number $\sqrt[3]{2}$); see Note 3.A. We'll see Apollonius (circa 262 BCE–circa 190 BCE) again in Section 6.4. Apollonius and see that his work is the authority on conic sections in its time.

Note. Dinostratus (circa 390 BCE–320 BCE) was a brother of Meneachmus who appears to have used the quadratrix of Hippeas to square the circle. Heath in *A History of Greek Mathematics, Volume I* states (see page 225): "...these passages [from works of Pappus and Proclus] refer to the *quadratrix* invented by Hippias of Elis. [They] seem to imply that it was not used by Hippias himself for squaring the circle, but that is was Dinostratus...and other later geometers who first applied it to that purpose; Iamblichus and Pappus do not even mention the name of Hippias." Noting more is known about Theudius of Magnesia, other than the mention of him by Proclus. Heath links him to Aristotle and revewls that he also wrote a book on the elements of geometry when he states (see page 321): "It is probably, however, that the propositions, &c., in elementary geometry which are quoted by Aristotle were taken from the Elements of Thedius, which would no doubt be the text-book of the time just preceding Euclid."

Revised: 3/10/2023